Vandermonde's Identity

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Vandermonde's Identity states that:

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}$$

Intuitive Proof

- Consider a set of $m+n$ distinct objects, divided into two subsets:
  - A subset of m objects.
  - A subset of n objects.
- We want to select k objects from the total $m+n$ objects (order does not matter).
- **Left-Hand Side (LHS):** $\binom{m+n}{k}$ represents the total number of ways to choose k objects from the combined set of $m+n$ objects.
- **Right-Hand Side (RHS):** Consider how many objects are chosen from each subset:
  - Let j be the number of objects chosen from the subset of m objects. There are $\binom{m}{j}$ ways to do this.
  - Then, $k-j$ objects must be chosen from the subset of n objects. There are $\binom{n}{k-j}$ ways to do this.
  - For a specific j, the total number of ways is $\binom{m}{j}\binom{n}{k-j}$.
  - We need to sum this product for all possible values of j (from 0 to k) to consider all cases.
- The identity holds when $m \geq k$ and $n \geq k$. If this condition is not met, the identity does not hold.

Practical Applications

- Vandermonde's Identity can be used to decompose or simplify binomial coefficients.
- For example, $\binom{7}{2}$ can be expressed as $\binom{3+4}{2}$. Then, by Vandermonde's identity:

$$\binom{7}{2} = \binom{3+4}{2} = \sum_{j=0}^{2} \binom{3}{j}\binom{4}{2-j}$$

- Useful for transforming expressions that involve sums of products of combinations into a single binomial coefficient (or vice-versa).