Combinatorics: Choosing k items out of n

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We are considering a bag with n distinct items. We want to select k of these items. We will analyze four different scenarios based on whether we replace the items and if the order matters.

1. **Don't Replace, Order Matters**

   Selecting k items without replacement and order matters (e.g., 2, 1, 3 is different from 1, 2, 3). For the first item, there are n choices. For the second item, there are $n-1$ choices. And so on, until the k-th item, where there are $n-(k-1) = n-k+1$ choices. The total number of ways is given by $n \times (n-1) \times \cdots \times (n-k+1)$, which is equivalent to $\frac{n!}{(n-k)!}$. This is the permutation formula: $P(n,k) = \frac{n!}{(n-k)!}$.

2. **Don't Replace, Order Doesn't Matter**

   Selecting k items without replacement and where order doesn't matter (e.g., 2, 1, 3 is the same as 1, 2, 3), we start with the number of ways to select them if order mattered, which is $\frac{n!}{(n-k)!}$. Since there are $k!$ different ways to order any set of k items, we divide by $k!$ to find the number of ways when order doesn't matter. This results in the combination formula: $C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.

3. **Replace, Order Matters**

   Selecting k items with replacement and order matters. For each of the k picks, there are n options because the item is replaced. Total number of ways: $n \times n \times \cdots \times n$ (k times) $= n^k$

4. **Replace, Order Doesn't Matter**

   Selecting k items with replacement, but order doesn't matter, is the most complicated case. To visualize this, imagine representing the n items as n baskets and the k selections as k tennis balls. The problem then becomes equivalent to placing k identical balls into n distinct baskets. For instance, if there are 3 balls in the 2nd basket, it signifies choosing the 2nd item three times in the selection process.

   To solve this, consider a line of k balls and $n-1$ barriers (dividers). The number of ways to arrange these k balls and $n-1$ barriers corresponds to the number of ways to distribute the balls into the baskets. Since there are $k+(n-1)$ total objects, the number of ways to choose the positions for the k balls is given by $\binom{k+(n-1)}{k} = \frac{(n+k-1)!}{k!(n-1)!}$. This can also be expressed as $\binom{n+k-1}{k}$ or $\binom{n+k-1}{n-1}$.

Summary Table:

	Order Matters	**Order Doesn't Matter**
Don't Replace	$\frac{n!}{(n-k)!}$	$\frac{n!}{k!(n-k)!}$
Replace	n^k	$\frac{(n+k-1)!}{k!(n-1)!}$