Homework Problems

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Homework 1

1) $\Omega = [0,1]$ $\quad \mathcal{F} = 2^{\Omega}$ $\quad X(\omega) = \omega$

X is a RV iff $X^{-1}(-\infty, b) \in \mathcal{F} \ \forall\ b$

$$X^{-1}(-\infty, b] = \begin{cases} \phi & b < 0 \\ [0,b] & b \leq 1 \\ [0,1] & b > 1 \end{cases} \in \mathcal{F}.$$

(2)

Homework 2

1) $\{X_n\}_n$: iid Bernoulli $\quad \mathbb{P}(X_1 = 0) = p$, $\ \mathbb{P}(X_1 = 1) = 1-p$

$Y_n = \min\{M, X_1 + \dots + X_n\}$ for $n = 1, 2, \dots$

$Y_n \to$ Markov chain

$\because \sum_{i=1}^{n} X_i = Z \sim \text{Binomial}(n, 1-p)$

$\because Y_n = \min\{M, \sum_{i=1}^{n} X_i\} \Rightarrow Y_{n+1} = \min\{M, Y_n + X_{n+1}\}$

Thus Y_{n+1} is independent of all Y_{n-i}'s $\quad \underline{i \geq 1}$, $\ \underline{n-i \geq 1}$.

$\because \mathbb{P}(Y_{n+1} = i_{n+1} \mid Y_n = i_n, \dots, Y_1 = i_1) = \mathbb{P}(Y_{n+1} = i_{n+1} \mid Y_n = i_n)$

$\Rightarrow$ Markov chain

$$\Rightarrow Y_{n+1} = \begin{cases} M & \leftarrow Y_n = M, \ Y_n = M-1, X_{n+1} = 1 \\ Y_n + X_{n+1} \end{cases} \qquad S = \{0, 1, \dots, M\}$$

$$P = \begin{array}{c|cccccc} Y_n \backslash Y_{n+1} & 0 & 1 & 2 & 3 & \dots & M \\ \hline 0 & p & 1-p & 0 & 0 & \dots & \\ 1 & 0 & p & 1-p & 0 & \dots & \\ \vdots & & & & p & & 1-p \\ M & 0 & 0 & 0 & 0 & \dots & 1 \end{array}$$

(2) B - Black ; R - Red $\quad \because X_n$: # Black balls

A ball drawn, replaced by opposite color. $\{X_n\}_n$: Markov chain

P??

$$\Rightarrow X_{n+1} = \begin{cases} X_n - 1 & \text{wp } X_n / B+R \\ X_n + 1 & \text{wp. } 1 - \frac{X_n}{B+R} \end{cases} \qquad \text{Total \#B+R}$$

(3)

	A	B
₹	5	10

If head : A(+1) ; B(-1)

ow : A_{-1} ; B^{+1}

X_n : #A after n tosses $\qquad \{X_n\}_n$: Markov ✓

$S = \{0, 1, \dots, 15\}$ $\qquad$ P ??

$$X_n = \begin{cases} X_n - 1 & \text{wp } 1/2 \\ X_n + 1 & \text{wp } 1/2 \end{cases} \Bigg\} \text{ Thus, } \underline{\text{markov chain}}$$

$$P = \begin{array}{c} 0 \\ 1 \\ \vdots \\ 15 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & \dots & 15 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & \dots & & 0 \\ 1/2 & 0 & 1/2 & \dots & & \\ & & & & & \\ & & & & -1/2 & 0 \ \ 1/2 \\ 0 & 0 & \dots & & 0 & 1 \end{bmatrix} \end{array}$$

(4) Z_n : number of arrivals in the n-th period.

$\mathbb{P}(Z_n = k) = p_k$; $k = 0, 1, \dots$ $\quad \Big| \sum_{k=0}^{\infty} p_k = 1$

$Z_1, \dots, Z_n, \dots$ (independent)

X_n # customers waiting $\qquad \{X_n\}_n$: markov chain ✓

P : Transition probab matrix

* $X_{n+1} = \{ \quad 0 \quad X_n + Z_n = 0$ $\qquad \downarrow Z_n$

X_n # customers waiting $\{X_n\}_n$: Markov chain

P : Transition probab matrix

* $X_{n+1} = \begin{cases} 0 & X_n + Z_n = 0 \\ \to & X_n + Z_n - 1 \end{cases}$

$S = \{0, 1, \ldots\}$

$$P = \begin{array}{c|cccc} X_n \backslash X_{n+1} & 0 & 1 & 2 & 3 \\ \hline 0 & P_0+P_1 & P_2 & P_3 & \cdots \\ 1 & P_0 & P_1 & P_2 & P_3 \cdots \\ 2 & 0 & P_0 & P_1 & P_2 \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

(5) $P(H) = p$; $0 < p < 1$ $X_n = K$ (k - successes from last failure) $k = 0, 1, \cdots$

$H \equiv$ success $\{X_n\}_n$: markov chain ✓

P ✓

$$X_{n+1} = \begin{cases} X_n + 1 & : 1/2 \\ 0 & : 1/2 \end{cases}$$

$S = \{0, 1, 2, \ldots\}$

$$P = \begin{array}{c|cccc} X_n \backslash X_{n+1} & 0 & 1 & 2 & 3 \\ \hline 0 & \frac{1}{2} & 1/2 & 0 & 0 \cdots \\ 1 & 1/2 & 0 & \frac{1}{2} & 0 \cdots \\ 2 & 1/2 & 0 & 0 & 1/2 \cdots \\ 3 & 1/2 & 0 & 0 & 0 \quad 1/2 \cdots \\ & \vdots & \vdots & \vdots & \vdots \end{array}$$

(8) : $Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow P^2 = Q$ (does not exist)

(7) P is a stochastic matrix then $\sum_{j=1}^{\infty} P_{ij} = 1$
$\{P_{ij}\}$
Prove that ~~for~~ any row of $P^2 \Rightarrow$ (sum $= 1$)

(6) Later

Homework 3

(1) Consider $S = \{1, 2\}$ $f(1,1) = 0$ $f(1,2) = 1$
$f(2,2) = 0$ $f(2,1) = 1$
$\Downarrow$ Example

Reflexive : $f(a,a) = 1$

or simply $f(a,b) = \begin{cases} 0 & a = b \\ 1 & a \neq b \end{cases}$

(2) Same as above S : $f(1,2) = 0$ $f(2,1) = 1$

Symmetric : $f(a,b) = f(b,a)$

(3) $\{1, 2, 3\}$ $f(1,2) = 1$; $f(2,3) = 1$; $f(3,1) = 0$

(4) ?

(5) P $P(\{head\}) = p$

X_n : # heads - # tails

$$\Rightarrow X_{n+1} = \begin{cases} X_n - 1 & \text{wp } (1-p) \\ X_n + 1 & \text{wp } p \end{cases}$$

discrete space

$S = \{0, \pm 1, \pm 2, \ldots\}$

$$P = \begin{array}{c|cccccccc} X_n \backslash X_{n+1} & -n & -n+1 & \cdots & 0 & \cdots & +n-1 & n \\ \hline -n & 0 & p & & 0 & \cdots & & \\ -n+1 & 1-p & 0 & p & 0 & \cdots & & \\ \vdots & & & & & & & \\ 0 & & & & & & & \\ \vdots & & & & & & & \\ n-1 & 0 & \cdots & 0 & 0 & 1-p & 0 & p \\ n & 0 & \cdots & & & 0 & 1-p & 0 \cdots \end{array}$$

(6) LATER

(7) (1) Q = $\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array}$ $[0 \;\; 0 \;\; 1 \;\; 0]$ Equivalence class

(6) LATER

(7) (I)

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & & & & \\ 1 & & & & \\ 2 & & & & \\ 3 & & & & \end{array}\quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

Equivalence class

$0 \to 2$
$1 \to 0 \Rightarrow 1 \to 2$
$2 \to 0 ; 2 \to 1$
$3 \to 0, 1, 2$
$\{0,1,2\}$; $\{3\}$

Consider $\underline{0}$: $P_{00}^{1} = 0$
$P_{00}^{2} = 1/2$
$P_{00}^{3} = 0 \xrightarrow{1} 2 \xrightarrow{\frac{1}{2}} 1 \xrightarrow{1} 0 = 1/2 > 0$

$\boxed{d(i) = 1 \text{ for } i \in \{1,0,2\}}$

for $\underline{3}$: $P_{33}^{n} = 0 \quad \forall n \geq 1$
Thus periodicity $d(3) = \infty$

(II)

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \end{bmatrix} \end{array}$$

Equivalence class:
$0 \to 1 \Rightarrow 0 \to 3$
$1 \to 3$
$2 \to 1 \Rightarrow 2 \to 3$
$3 \to 0, 3 \to 2 \Rightarrow 3 \to 1, 3 \to 3$

$\left.\begin{matrix} 1 \leftrightarrow 3 \\ 2 \leftrightarrow 3 \\ 0 \leftrightarrow 3 \end{matrix}\right\}$ $\{0,1,2,3\} \equiv$ Equivalence set .

Consider : $d(0) \equiv$ $P_{00}^{1} = 0$, $\underset{0\to1\to3}{P_{00}^{2}} = 0$, $\underset{0\to1\to3\to0}{P_{00}^{3}} = 1/3 > 0$, $\underset{0\to1\to3\to2\to1}{P_{00}^{4}} = 0$

$\boxed{P_{01}^{1} = 1}$

$\boxed{P_{10}^{2} > P_{1,3}^{1} P_{30} = \frac{1}{3}}$

$\Rightarrow P_{00}^{3} > \frac{1}{3} > 0$
$\underset{(0\to1\to3\to2\to1)}{P_{01}^{4} > 0} \Rightarrow P_{00}^{6} \geq P_{01}^{4} P_{10}^{2} > 0$
thus gcd $\{3, 6, \ldots\} = \boxed{3}$

(8) P is doubly stochastic $\Rightarrow P^{n}$ is doubly stochastic

If $x = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ $\boxed{\begin{matrix} Px = x \\ \& \ P^{T}x = x \end{matrix}}$

$Px = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = x$

$\to P^{2}x = Px = \underline{x}$

$P^{T}x = x$
$\downarrow$
$P^{T}P^{T}x = P^{T}x = x$
$(P^{2})^{T}x = \underline{x}$

$\Rightarrow P^{2}$ is doubly stochastic .

(9) r states $\quad j \to k \}$ in $r-1$ steps $\underline{\text{or less}}$

(10) $P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$ $\quad P^{n} = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^{n}}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}$

Homework 4

(1) $1/4$ each $\quad \{X_n\}_n$: recurrent markov chain

for any state i : $P_{ii}^{2n+1} = 0 \ \forall n \in \{0,1,\ldots\}$

$P_{ii}^{2n} = \sum_{i=0}^{n} \frac{(2n)!}{i!\,i!\,(n-i)!\,(n-i)!} \left(\frac{1}{4}\right)^{2n}$

By Stirling approx : $P_{ii}^{2n} \propto \frac{1}{n}$

$\Rightarrow \sum P_{ii}^{2n} = \infty$ (recurrent $\boxed{P_{ii} = 1}$ Markov chain)

(2) $i \leftrightarrow j \quad \forall i, j \in \mathbb{Z}$ Thus, equivalent class $\equiv \underline{S}$

$P_{ii}^{n} = \sum_{i=0}^{n} \left(\frac{1}{2}\right)^{i}$

Compare it with $\{Y_n\}_n$: 2 dimensional $\underline{RW}$ q_{ii}

Now, in 2-D : going up or down $\left(\frac{1}{4}\right) \Rightarrow \left(2k\right) \left(\frac{1}{4}\right)^{2k}$

HOME WORK 4
MTH-212M/MTH-412A
ELEMENTARY STOCHASTIC PROCESS

1. Consider the two dimensional random walk with the full infinite plane. Prove that if all the probabilities are equal, i.e. starting from any state going up, down, forward and backward are all 1/4, then it is a recurrent Markov Chain.

2. Consider the following one dimensional random walk, where $S = \{0, \mp 1, \mp 2, \ldots\}$, and $P(X_{n+1} = i+1 | X_n = i) = P(X_{n+1} = i-1 | X_n = i) = 1/4$, and $P(X_{n+1} = i | X_n = i) = 1/2$. Find the equivalent classes. Show that it is a recurrent Markov Chain.

3. Consider the following transition probability matrix

$$\mathbf{P} = \begin{bmatrix} p_0 & 1-p_0 & 0 & 0 & 0 & 0 & \ldots \\ p_1 & 0 & 1-p_1 & 0 & 0 & 0 & \ldots \\ p_2 & 0 & 0 & 1-p_2 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Here $0 < p_i < 1$, for all i. Show that $\sum_{n=1}^{m+1} f_{00}^{n} = 1 - u_m$, where

$$u_m = \begin{cases} \prod_{i=0}^{n}(1-p_i) & \text{if } n \geq 0 \\ 1 & \text{if } n = -1 \end{cases}$$

Prove that $\prod_{i=0}^{n}(1-p_i) \leq e^{-\sum_{i=0}^{m} p_i}$. Show that if $\sum_{i=0}^{\infty} p_i = \infty$, then it is a recurrent Markov Chain.

4. Consider the following transition probability matrix with the state space $S = \{0, 1, 2, 3, 4, 5\}$

compare it with $\{Y_n\}_n$: 2 dimensional RW.

Now, in 2-D : going up or down $\left(\frac{1}{4}\right) \Rightarrow \binom{2k}{k}\left(\frac{1}{4}\right)^{2k}$

in this scenario : staying at same place $= \left(\frac{1}{2}\right)^{2k}$

Now, $p_{ii}^n \equiv c\left(\left(\frac{1}{2}\right)^{2k}\right) \Rightarrow \left(\frac{1}{4}+\frac{1}{4}\right)^{2k} > {}^{2k}C_k \left(\frac{1}{4}\right)^{2k}$

hence, $p_{ii}^n \geq q_{ii}^n \; \forall n \geq 0 \Rightarrow$ Recurrent Markov chain

(3) $$P = \begin{bmatrix} p_0 & 1-p_0 & 0 & 0 & \cdots \\ p_1 & 0 & 1-p_1 & 0 & \cdots \\ p_2 & & & 1-p_2 & \\ \vdots & & & & \ddots \end{bmatrix}$$

$f_{00}^1 = p_0$

$\underset{0\to1\to0}{f_{00}^2} = (1-p_0)p_1$

$\underset{0\to1\to2\to0}{f_{00}^3} = (1-p_0)(1-p_1)p_2$

$f_{00}^n = \mathbb{P}\left[X_n = 0, X_{n-1} \neq 0, \ldots, X_1 \neq 0 \mid X_0 = 0\right]$

$= \prod_{i=0}^{n-2}(1-p_i)\, p_{n-1}$

Now, $\sum_{n=1}^{m+1} f_{00}^n = \sum_{n=1}^{m+1} \prod_{i=0}^{n-2}(1-p_i)\left(1-(1-p_{n-1})\right) = \sum_{n=1}^{m+1}\left[\prod_{i=0}^{n-2}(1-p_i) - \prod_{i=0}^{n-1}(1-p_i)\right] = 1 - U_m$

$U_m = \begin{cases} \prod_{i=0}^{n}(1-p_i) & ; n \geq 0 \\ 1 & ; n = -1 \end{cases}$

To show, $(1-p_i) \leq e^{-p_i} = 1 - p_i + \frac{p_i^2}{2!} \ldots$ ∵ If $\sum p_i = \infty \Rightarrow \prod_{i=0}^{\infty}(1-p_i) = 0 = U_\infty \Rightarrow \left[\sum_{n=1}^{\infty} f_{00}^n = 1\right] \to$ recurrent

Prove that $\prod_{i=0}^{\infty}(1-p_i) \leq e^{-\sum_{i=0}^{\infty} p_i}$. Show that if $\sum_{i=0}^{\infty} p_i = \infty$, then it is a recurrent Markov Chain.

4. Consider the following transition probability matrix with the state space $S = \{0, 1, 2, 3, 4, 5\}$

$$P = \begin{bmatrix} 1/3 & 2/3 & 0 & 0 & 0 & 0 \\ 2/3 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 3/4 & 0 & 0 \\ 0 & 0 & 1/5 & 4/5 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{bmatrix}$$

Find the recurrent and transient classes. Find the probability that starting from the transient state i, it is going to get absorbed in a given recurrent class C_j, for $j = 1, \ldots, K$.

(4) $S = \{0, 1, 2, 3, 4, 5\}$

$\underset{C_1}{\{0 \leftrightarrow 1\}}$ $\quad \underset{C_2}{\{2 \leftrightarrow 3\}}$ $\quad \underset{S_T}{\{4 \leftrightarrow 5\}}$ $\quad p_{00}$

Consider C_1 :- $Q = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{6} & \frac{5}{6} \end{bmatrix}$; $(I-Q)^{-1} = \frac{12}{7}\begin{bmatrix} \frac{5}{6} & \frac{1}{4} \\ \frac{1}{6} & \frac{3}{4} \end{bmatrix}$

5. Consider the following transition probability matrix with the state space $S = \{0, 1, 2, 3\}$

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1/6 & 1/3 & 1/2 & 0 \\ 1 & 1/2 & 1/2 & 0 & 0 \\ 2 & 1/6 & 1/3 & 1/2 & 0 \\ 3 & 0 & 1/6 & 1/3 & 1/2 \end{array}$$

$P = $ the matrix above.

$\underset{R \equiv C_1}{\{0, 1, 2\}}$ $\quad \underset{T}{\{3\}}$

1

Find the recurrent and transient classes. Find the probability that starting from the transient state i, it is going to get absorbed in a given recurrent class C_j, for $j = 1, \ldots, K$.

6. A Markov Chain on a finite state space, for which all the recurrent states are absorbing, is called an absorbing chain. Show that for a finite absorbing chain the transition probability matrix P can be written as follows:

$$P = \begin{bmatrix} I & 0 \\ R_1 & Q \end{bmatrix}$$

Find P^n.

$\begin{bmatrix} \alpha_{41} \\ \alpha_{51} \end{bmatrix} = \underset{(I-Q)^{-1}}{\begin{bmatrix} & \end{bmatrix}} \times \begin{bmatrix} 1/4 \\ 2/6 \end{bmatrix}$ ✓

$4 \to C_1$, $5 \to C_1$, $4 \xrightarrow{X_1} C_1$, $5 \to C_1$, $6 \to X_1$

2