MTH211A: Theory of Statistics

Problem set 5

Methods of Interval Estimation

1. Find a pivotal quantity based on a random sample of size n from a $\texttt{normal}(\theta, \theta)$ distribution, where $\theta > 0$. Use the pivotal quantity to set up a $1 - \alpha$ confidence interval for θ.

2. Let X be a single observation from a $\texttt{beta}(\theta, 1)$ distribution.

   (a) Let $Y = -(\log X)^{-1}$. Evaluate the confidence coefficient, β, of the test $[y/2, y]$.

   (b) Find a pivotal quantity and use it to set up a confidence interval for θ with confidence coefficient β, where β is as in part (a).

   (c) Compare the two confidence intervals.

3. Consider the problem of finding a confidence set for the parameters (μ, σ^2) given a random sample of size n from $\texttt{normal}(\mu, \sigma^2)$ distribution. Use an appropriate function of the sufficient statistics $\left(\bar{X}_n, S_n^2\right)$ as a pivot, and hence find a confidence set $U_{\mathbf{X}}$ with confidence coefficient at least $1 - \alpha$.

4. Find a $1 - \alpha$ confidence interval for θ, given $X_1, \cdots, X_n$ iid with pdf

   (a) $f(x \mid \theta) = 1$, $\theta - 1/2 < x < \theta + 1/2$,

   (b) $f(x \mid \theta) = 2x/\theta^2$, $0 < x < \theta$.

5. Let $X_1, \cdots, X_n$ be a random sample from $\texttt{Poisson}(\lambda)$ distribution. Find the smallest possible interval of the form $[L(\mathbf{X}), U(\mathbf{X})]$, where both L and U are functions of $T(\mathbf{X}) = \sum_{i=1}^{n} X_i$, and $[L(\mathbf{X}), U(\mathbf{X})]$ satisfies $P_\lambda\left(L(\mathbf{X}) < \lambda < U(\mathbf{X})\right) \geq 1 - \alpha$.

   [Hint: You may use the following fact:

   Let X is distributed as $\texttt{Gamma}(\alpha, \beta)$, then for any $x > 0$

   $$P(X \leq x) = P(Y \geq \alpha)$$

   where Y is distributed as $\texttt{Poisson}(x/\beta)$.]

6. A confidence interval $[L(\mathbf{X}), U(\mathbf{X})]$ for the parameter θ with confidence coefficient at least $1 - \alpha$ is called *unbiased* if $P_\theta\left(L(\mathbf{X}) < \theta < U(\mathbf{X})\right) \geq 1 - \alpha$, and $P_\theta\left(L(\mathbf{X}) < \theta' < U(\mathbf{X})\right) \leq 1 - \alpha$ for all $\theta' \neq \theta$.

   Based on a random sample of size n from $\texttt{uniform}(0, \theta)$, find an unbiased confidence interval of θ, with the pivot $X_{(n)}/\theta$.