Quiz-1

Elementary Stochastic Process (MTH-212A)

Name (Roll Number):

No extra sheet will be provided or collected, **Time 20 mins.**, **Max. Marks: 15**.

1. Two brothers (Avik and Bablu) are playing a friendly game with two fair dice. At the beginning Avik has Rs. 20 and Bablu has Rs. 40. At each game they throw their own dice and who ever has the larger number on the top wins, otherwise it is a tie. Whoever wins, gets Rs. 10 from the other. Whenever one is broke, they share the whole money equally and start playing it again. If X_n denotes the amount money Avik has after the n-th game, show that $\{X_n, n \geq 1\}$ is a Markov Chain. Find the transition probability matrix. [3+3=6]

Solution: Note that the state space is $\{0, 10, 20, 30, 40, 50, 60\}$. X_{n+1} can be written as $X_{n+1} = X_n + Y_{n+1}$, here $Y_{n+1} = 30$, if $X_n = 0$ or 60, else

A B

₹20 ₹40

Win : (+₹10) (−₹10)

If A=0 or B=0 → equal division (A=30; B=30)

$$Y_{n+1} = \begin{cases} -10 & \text{w.p } p \\ 0 & \text{w.p. } q \\ 10 & \text{w.p. } p \end{cases}$$

where $p = 15/36$, $q = 1/6$. Hence,

or p can be calculated by:

(1, –) (5, –)

$$\frac{1}{6}\times\frac{5}{6} + \frac{1}{6}\times\frac{4}{6} + \dots + \frac{1}{6}\times\frac{1}{6} = \frac{1}{6}\times\frac{5\times 6}{2\times 6} = \frac{5}{12}$$

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccccc} & & & 30 & & & \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ p & q & p & 0 & 0 & 0 & 0 \\ 0 & p & q & p & 0 & 0 & 0 \\ 0 & 0 & p & q & p & 0 & 0 \\ 0 & 0 & 0 & p & q & p & 0 \\ 0 & 0 & 0 & 0 & p & q & p \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array},$$

X_n # money of Avik

$S = \{0, 10, 20, \dots, 60\}$

$$X_{n+1} = \begin{cases} 30 & ; X_n = 0/60 \quad \text{wp } 1 \\ X_n + 10 & X_n \neq 0/60 \quad \text{wp } p \\ X_n - 10 & X_n \neq 0/60 \quad \text{wp } p \\ X_n & \quad \text{wp } q \end{cases}$$

$q = 1/6$ (same number)

$$p = \frac{\left(1 - \frac{6}{36}\right)}{2} = \frac{15}{36} = \frac{5}{12}$$

2. Let $\{X_n; n \geq 1\}$ be a sequence of independent and identically distributed random variables with $P(X_1 = 1) = P(X_1 = -1) = \frac{1}{2}$. Let us define $Y_n = X_1 \times \ldots \times X_n$, for $n \geq 1$. (a) Find the distribution of Y_n, (b) Show that $\{Y_n\}$ is a Markov Chain, (c) Find the corresponding transition probability matrix P. [3+3+3=9]

Solution: By induction it easily follows that Y_n has the same distribution as X_1. Note that $Y_{n+1} = Y_n \times X_{n+1}$. In this case the state space is $\{-1, 1\}$. Hence,

$\{-1, 1\}$

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$Y_1 = X_1$ (same distribution as X_1)

Let Y_i has same distribution as X

$(Y_n, X_{i+1}) \subseteq$

$Y_{i+1} = Y_i \times X_{i+1} \Rightarrow Y_{i+1} = 1$ iff $\{(-1, -1), (1, 1)\}$

Thus $Y_{i+1} = 1$ w.p. $\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$

similarly $Y_{i+1} = -1$ w.p. $\frac{1}{2}$

Thus Y_n follows X- distribution

(Y_n ✓ Markov chain)

$$P = \begin{array}{c|cc} & -1 & 1 \\ \hline -1 & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} \end{array}$$

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