Final
Elementary Stochastic Process (MTH-412A)

Name:
Roll Number:
Time 120 minutes
Maximum Marks: 40

Important Instruction: This question paper has 7 questions, and each question has 6 marks. Questions 1-3 are short answer types and they do not have any negative marks, Question 4-7 are multiple choice where more than one answers might be correct. Each correct answer will give you one mark and each wrong answer will give you -1 mark. If you can identify all the correct answers without any wrong answer, then you will get 6 points of that question. We have used the same notations as we have used in the class. You need to write your name and roll number in each page, if you do not write in any page 2 marks will be deducted. Do not do any rough work in these pages. If you do any rough work in these pages two marks will be deducted. Write your answers in this question paper, and return this question paper along with the booklet.

$1 \leftrightarrow 3 \to 5$

Question 1: Suppose $\{X_n; n \geq 1\}$ is a Markov Chain with the state space $S = \{1, 2, 3, 4, 5, 6\}$, and having the transition probability matrix

$\{1,3,5\}$ $\{2,4,6\}$

$\tau_1 = \lim_{n\to\infty} p_{11}^{(n)} = \frac{1}{2}\tau_1 + \frac{1}{4}\tau_3 + \frac{1}{4}\tau_5$

$2\tau_1 = \tau_3 + \tau_5$ $\quad \underbrace{\tau_1+\tau_3+\tau_5}_{=1}$

$2\tau_3 = \tau_1 + \tau_3$

$2\tau_5 = \tau_1 + \tau_3 = 1 - \tau_5$

$\tau_5 = 1/3 = \tau_1 = \tau_3$

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} \frac{1}{2} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{10} & \frac{3}{10} & \frac{1}{10} & \frac{3}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ \frac{3}{10} & \frac{1}{10} & \frac{3}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ \frac{3}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{3}{10} \end{array}\right] \end{array}.$$

Then (correct up to two decimal places or in fraction):

1. $\lim_{n\to\infty} p_{11}^{(n)} = [\ 1/3\ ]$, 2. $\lim_{n\to\infty} p_{31}^{(n)} = [\ 1/3\ ]$, 3. $\lim_{n\to\infty} p_{24}^{(n)} = [\ 0\ ]$,

4. $\lim_{n\to\infty} p_{25}^{(n)} = [\ 1/3\ ]$, 5. $\lim_{n\to\infty} p_{65}^{(n)} = [\ 1/3\ ]$, 6. $\lim_{n\to\infty} p_{66}^{(n)} = [\ 0\ ]$.

Question 2: Consider a sequence of tosses of a coin independently with the probability of head 1/3 and the probability of tail 2/3. Suppose X_n and Y_n denote the number of heads and number of tails, respectively, after n tosses. If $Z_n = |X_n - Y_n|$, then (up to two decimal points or in fraction) 49

$P(X_{100} - Y_{100} = 2 \mid Z_{100} = 2) =$

$P(\underline{Z_{101}} = 1|\underline{Z_{100}} = 2) = [\qquad\qquad]$

$Z_{100} = 2 \to X_{100} = t;\ Y_{100} = t - 2 \to Y{+}{+} \to Z_{101} = 1$ wp $= 2/3$

$Z_{100} = 2 \to X_{100} = t;\ Y_{100} = t + 2 \to X{+}{+} \to .$ wp $= 1/3$

Question 3: Let $\{X_n\}$ and $\{Y_n\}$ be two independent Markov chains with the same state space $S = \{0, 1\}$ and having the same transition probability matrix

$$P^n = P \qquad \mathbf{P} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}.$$

Given $Z_n = 1$ $= \max\{X_n, Y_n\}$ $\qquad Z_{n+1} = \max\{X_{n+1}, Y_{n+1}\}$

Suppose $Z_n = \max\{X_n, Y_n\}$. Then (correct up to two decimal places or in fraction). each

$\lim_{n\to\infty} P(Z_{n+1} = 1 | Z_n = 1) =$ [3/4] $\quad \frac{1}{3}$ $X_n = 1, Y_n = 0$ $\quad X_n = 1, Y_n = 1$ $\quad X_n = 0, Y_n = 1$

$\frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \qquad 1 - \frac{1}{4} = \frac{3}{4} \qquad 1 - \frac{1}{4}$

Question 4: Consider a Markov Chain $\{X_n; n \geq 1\}$ with the following transition probability matrix: $P(X_{n+1} = i+1 | X_n = i) = P(X_{n+1} = i-1 | X_n = i) = P(X_{n+1} = i | X_n = i) = \frac{1}{3}$, for $i = 0, \mp 1, \mp 2, \ldots$. Then which of the following statement(s) is (are) correct for $n \geq 2$. $\frac{3}{4}$

1. $P(X_{3n+1} = 0 | X_n = 0) = 0$.
2. $P(X_{3n} = 0 | X_n = 0) > \binom{2n}{n} \left(\frac{1}{3}\right)^{2n}$.
3. $P(X_{3n} = 0 | X_n = 0) \leq \binom{2n}{n} \left(\frac{1}{3}\right)^{2n}$.
4. $P(X_{3n} = 0 | X_n = 0) < \left(\binom{2n}{n} + 1\right) \left(\frac{1}{3}\right)^{2n}$.
5. All the states communicate with each other.

${}^{2n}C_n \left(\frac{1}{3}\right)^n \left(\frac{1}{3}\right)^n + \left(\frac{1}{3}\right)^{2n} + \qquad n-1 \quad n-1 \qquad \frac{2n!}{(n-1)!(n-1)!} \times \left(\frac{1}{3}\right)^{2n}$

Then which is of the above statements are correct: [2, 5]

Question 5: Let us consider two Markov chains $\{X_n\}$ and $\{Y_n\}$, with the same states space $S = \{0, 1, 2, \ldots\}$. The transition probability matrices are as follows:

$$\mathbf{P} = \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ p_{20} & p_{21} & p_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \quad \text{and} \quad \widetilde{\mathbf{P}} = \begin{array}{cc} & 0 \\ 0 & \begin{bmatrix} 1 & 0 & 0 & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ p_{20} & p_{21} & p_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \end{array}$$

Consider the following statements:

1. If $\{X_n\}$ is irreducible and aperiodic, then $\{Y_n\}$ is also aperiodic.
2. If $\{X_n\}$ is irreducible and transient, then the states $\{1, 2, 3, \ldots\}$ in $\{Y_n\}$ are also transient.
3. If the states $\{1, 2, 3, \ldots\}$ in $\{Y_n\}$ are recurrent, then they are recurrent in $\{X_n\}$ also.
4. If all the states in $\{X_n\}$ are recurrent, then the states $\{1, 2, 3, \ldots\}$ in $\{Y_n\}$ are also recurrent.

Then which is of the above statements are correct: [ ]

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Question 6: A $M \times M$ matrix $\mathbf{P} = ((p_{ij}))$, is called a stochastic matrix if $p_{ij} \geq 0$, and $\sum_{j=1}^{M} p_{ij} = 1$, for $i = 1, \ldots, M$. A stochastic matrix $\mathbf{P}$ is called a doubly stochastic if $\sum_{i=1}^{M} p_{ij} = 1$, for $j = 1, \ldots, M$. Consider the following statements.

1. If $\mathbf{P}$ is a stochastic matrix then $\mathbf{P}^m$ is also a stochastic matrix for $m = 1, 2, \ldots$.
2. If $\mathbf{P}$ is a doubly stochastic matrix then $\mathbf{P}^m$ is also a doubly stochastic matrix for $m = 1, 2, \ldots$.
3. If $\mathbf{P}$ is a doubly stochastic matrix then for any $m = 1, 2, \ldots$, then there exists a doubly stochastic matrix $\mathbf{Q}_m$, such that $\mathbf{P} = (\mathbf{Q}_m)^m$. $m=2$
4. If $\mathbf{P}$ is a doubly stochastic matrix, and if $p_{ij} > 0$, for $1 \leq i, j \leq M$, then $\lim_{n\to\infty} p_{ij}^{(n)} = 0$.
5. If $\mathbf{P}$ is a doubly stochastic matrix, and if $p_{ij} > 0$, for $1 \leq i, j \leq M$, then $\lim_{n\to\infty} p_{ij}^{(n)} = \frac{1}{M}$.
6. If $\mathbf{P}$ is a stochastic matrix, and if $p_{ij} > 0$, for $1 \leq i, j \leq M$, then $\lim_{n\to\infty} p_{ij}^{(n)} = \frac{1}{M}$.

$P=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $|P| = -1$

$\begin{pmatrix} \frac14 & \frac34 \\ \frac14 & 3/4 \end{pmatrix}$

Then which is of the above statements are correct: [1, 2, 5]

Let $P = \begin{bmatrix} \frac12 & \frac12 \\ \frac12 & \frac12 \end{bmatrix}$: $P^n = P$: $\frac12 = \lim_{n\to\infty} p_{ij}^{n}$

Question 7: Consider a Markov Chain $\{X_n\}$ having state space $S = \{0, 1, 2, 3, \ldots\}$, with the following transition probability matrix

$$\mathbf{P} = \begin{bmatrix} p & 1-p & 0 & 0 & 0 & 0 & \ldots \\ p & 0 & 1-p & 0 & 0 & 0 & \ldots \\ p & 0 & 0 & 1-p & 0 & 0 & \ldots \\ p & 0 & 0 & 0 & 1-p & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Here $0 < p < 1$. Define a random variable $N = \min\{n : n \geq 1, X_n = 0 | X_0 = 0\}$. $E(N)$ and $V(N)$ denote the mean and variance of the random variable N.

$E(N) = (I - Q)^{-1} e$

$(1-p)1 + (p-1)1$

$2-2p$

$A = \begin{bmatrix} p & p-1 & 0 & 0 & \ldots \\ p & 1 & p-1 & 0 \\ p & 0 & 1 & p-1 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$

$d = p + (p-1)d$

$d = p - pd + d$

$d = 1$

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Consider the following statements.

1. $\{X_n\}$ is an irreducible Markov Chain for all $0 < p < 1$.
2. $\{X_n\}$ is an irreducible Markov Chain only for $0.5 \leq p < 1$, and not for $0 < p < 0.5$.
3. $E(N)$ exists and it is $\frac{1}{p}$.
4. $E(N)$ exists and it is $\frac{1-p}{p}$.
5. $V(N)$ exists and it is $\frac{1-p}{p^2}$.
6. $V(N)$ exists and it is $\frac{(1-p)^2}{p}$.

Then which is of the above statements are correct: [ ].

All the best.

Solutions:

Question 1: Suppose $\{X_n; n \geq 1\}$ is a Markov Chain with the state space $S = \{1, 2, 3, 4, 5, 6\}$, and having the transition probability matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{10} & \frac{3}{10} & \frac{1}{10} & \frac{3}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ \frac{3}{10} & \frac{1}{10} & \frac{3}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ \frac{3}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{3}{10} \end{bmatrix}.$$

Provide the answers up to two decimal places:

1. $\lim_{n\to\infty} p_{11}^{(n)} = \frac{1}{3}$
2. $\lim_{n\to\infty} p_{31}^{(n)} = \frac{1}{3}$
3. $\lim_{n\to\infty} p_{24}^{(n)} = 0$
4. $\lim_{n\to\infty} p_{25}^{(n)} = \frac{1}{3}$
5. $\lim_{n\to\infty} p_{65}^{(n)} = \frac{1}{3}$
6. $\lim_{n\to\infty} p_{66}^{(n)} = 0.$

Question 2:

$$\begin{aligned} P(Z_{101} = 1||Z_{100} = 2) &= P(Z_{101} = 1, X_{100} - Y_{100} = 2|Z_{100} = 2) + \\ &\quad P(Z_{101} = 1, Y_{100} - X_{100} = 2|Z_{100} = 2) \\ &= P(Z_{101} = 1|X_{100} - Y_{100} = 2)P(X_{100} - Y_{100} = 2|Z_{100} = 2) + \\ &\quad P(Z_{101} = 1|Y_{100} - X_{100} = 2)P(Y_{100} - X_{100} = 2|Z_{100} = 2) \\ &\quad \frac{2}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{4}{5} = \frac{2}{5} = 0.40 \end{aligned}$$

Question 3: Let $\{X_n\}$ and $\{Y_n\}$ be two Markov chains with the same state space $S = \{0, 1\}$ and having the same transition probability matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

and they are independently distributed. Suppose $Z_n = \max\{X_n, Y_n\}$. Then $\lim_{n\to\infty} P(Z_{n+1} = 1|Z_n = 1) = \frac{3}{4} = 0.75$ (up to two decimal places).

Observe that $\mathbf{P}^n = \mathbf{P}$ and what ever be $P(X_1 = 0)$ and $P(Y_1 = 0)$, $P(X_n = 0) = P(X_n = 1) = P(Y_n = 0) = P(Y_n = 1) = \frac{1}{2}$. Therefore,

$$\begin{aligned} P(Z_{n+1} = 0|Z_n = 1) &= P(Z_{n+1} = 0, X_n = 0, Y_n = 1|Z_n = 1) + \\ &\quad P(Z_{n+1} = 0, X_n = 1, Y_n = 0|Z_n = 1) + \\ &\quad P(Z_{n+1} = 0, X_n = 1, Y_n = 1|Z_n = 1) = 3 \times \frac{1}{4} \times \frac{1}{3}. \end{aligned}$$

Question 4: Consider a Markov Chain $\{X_n; n \geq 1\}$ with the following transition probability matrix: $P(X_{n+1} = i+1|X_n = i) = P(X_{n+1} = i-1|X_n = i) = P(X_{n+1} = i|X_n = i) = \frac{1}{3}$, for $i = 0, \mp 1, \mp 2, \ldots$. Then which is of the following statement(s) is (are) correct.

1. $P(X_{3n+1} = 0|X_n = 0) = 0$. (F)
2. $P(X_{3n} = 0|X_n = 0) > \binom{2n}{n}\left(\frac{1}{3}\right)^{2n}$. (T)
3. $P(X_{3n} = 0|X_n = 0) \leq \binom{2n}{n}\left(\frac{1}{3}\right)^{2n}$. (F)
4. $P(X_{3n} = 0|X_n = 0) \leq \left(\binom{2n}{n} + 1\right)\left(\frac{1}{3}\right)^{2n}$. (F)
5. All the states communicate with each other. (T)

Then which is of the above statements are correct: [2, 5]

It follows by counting the paths the way it can come back to the same state after $2n$ or $2n+1$ steps.

Question 5:

Aperiodic.

1. False. Take

$$\boldsymbol{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2^2} & \frac{1}{2^3} & \frac{1}{2^4} & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$1 \to 0 \to 1$ $\boxed{\gcd = 1}$

$1 \to \underbrace{2 \to 0 \to 1}_{3}$

2. True. If $i, j \in \{1, 2, \ldots\}$, then $\tilde{p}_{ij}^n \leq p_{ij}^n$, for all n. Hence, $\sum_{n=1}^{\infty} \tilde{p}_{ij}^n \leq \sum_{n=1}^{\infty} p_{ij}^n < \infty$.
3. True. Follows from the same inequality as the previous one.

$\sum_{n=1}^{\infty} f_{00}^n = 1$

4. False. Same example as in (a). Note that $f_{00}^n = 2^{-n}$. Hence all the states are recurrent in $\mathcal{A}$.

$f_{00}^1 = 2^{-1}$; $f_{00}^2 = 2^{-3} + 2^{-4} + \ldots$

2^{-2}

Then which is of the above statements are correct: [2, 3]

Question 6: A $M \times M$ matrix $\mathbf{P} = ((p_{ij}))$, is called a stochastic matrix if $p_{ij} \geq 0$, and $\sum_{j=1}^{M} p_{ij} = 1$, for $i = 1, \ldots, M$. A stochastic matrix $\mathbf{P}$ is called a doubly stochastic if $\sum_{i=1}^{M} p_{ij} = 1$, for $j = 1, \ldots, M$. Then which of the following statements are always correct. Here $p_{ij}^{(n)}$ denotes the (i,j)th element of the matrix $\mathbf{P}^n$.

1. If $\mathbf{P}$ is a stochastic matrix then $\mathbf{P}^m$ is also a stochastic matrix for $m = 1, 2, \ldots$. (T)
2. If $\mathbf{P}$ is a doubly stochastic matrix then $\mathbf{P}^m$ is also a doubly stochastic matrix for $m = 1, 2, \ldots$. (T)
3. If $\mathbf{P}$ is a doubly stochastic matrix then for any $m = 1, 2, \ldots$, there exists a doubly stochastic matrix $\mathbf{Q}_m$, such that $\mathbf{P} = (\mathbf{Q}_m)^m$. (F)
4. If $\mathbf{P}$ is a doubly stochastic matrix, and if $p_{ij} > 0$, for $1 \leq i, j \leq M$, then $\lim_{n\to\infty} p_{ij}^{(n)} = 0$. (F)
5. If $\mathbf{P}$ is a doubly stochastic matrix, and if $p_{ij} > 0$, for $1 \leq i, j \leq M$, then $\lim_{n\to\infty} p_{ij}^{(n)} = \frac{1}{M}$. (T)
6. If $\mathbf{P}$ is a stochastic matrix, and if $p_{ij} > 0$, for $1 \leq i, j \leq M$, then $\lim_{n\to\infty} p_{ij}^{(n)} = \frac{1}{M}$. (F)

Then which is of the above statements are correct: [1, 2, 5]

Question 7: Consider a Markov Chain $\{X_n\}$ having state space $S = \{0, 1, 2, 3, \ldots\}$, with the following transition probability matrix

$\sum p_{ii} = \cancel{\frac{1}{p}} < \infty \Rightarrow$ ~~transient (all)~~

$$\mathbf{P} = \begin{bmatrix} p & 1-p & 0 & 0 & 0 & 0 & \cdots \\ p & 0 & 1-p & 0 & 0 & 0 & \cdots \\ p & 0 & 0 & 1-p & 0 & 0 & \cdots \\ p & 0 & 0 & 0 & 1-p & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

$P_{00}^1 = p$
$P_{00}^2 = (1-p)p$
$P_{00}^3 = (1-p)^2 p$

Here $0 < p < 1$. Define a random variable $N = \min\{n : n \geq 1, X_n = 0 | X_0 = 0\}$. Consider the following statements.

1. $\{X_n\}$ is an irreducible Markov Chain for all $0 < p < 1$. [T]
2. $\{X_n\}$ is an irreducible Markov Chain only for $0.5 \leq p < 1$, and not for $0 < p < 0.5$. [F]
3. $E(N)$ exists and it is $\frac{1}{p}$. [T]

$S = p + 2(1-p)p + 3(1-p)^2 p + \cdots$

$(1-p)S = (1-p)p + 2(1-p)^2 p + \cdots$

$pS = p + (1-p)p + (1-p)^2 p$

$= \frac{p}{p} \Rightarrow S = 1/p = \underline{E(N)}$

$\cancel{V(N) = \frac{1-p}{p^2}}$

4. $E(N)$ exists and it is $\frac{1-p}{p}$. [F]

5. $V(N)$ exists and it is $\frac{1-p}{p^2}$. [T]

6. $V(N)$ exists and it is $\frac{(1-p)^2}{p}$. [F]

Then which is of the above statements are correct: $[\ 1, 3, 5]$

In this case $P(N = k) = p(1-p)^{k-1}$, for $k = 1, 2, \ldots$. It is a geometric random variable. $E(N) = \frac{1}{p}$ and $V(N) = \frac{1-p}{p^2}$.