Lecture Notes: Feb 05, 2024

In the previous lecture notes I had discussed about the classification of states, and we have seen that the communication relation is an equivalent relation, and it divides the state space into disjoint equivalent classes. In this note we will be discussing some interesting properties of this equivalent classes.

First we talk about the periodicity of a state 'i'. The periodicity of a state 'i', is defined as

$$d(i) = \text{g.c.d}\{n : n \geq 1, p_{ii}^{(n)} > 0\}.$$

Here g.c.d. stands for greatest common divisor. It is very clear that if $p_{ii}^{(1)} > 0$, or $p_{ii}^{(n)} > 0$ and $p_{ii}^{(n+1)} > 0$, then $d(i) = 1$. Now it can be very easily shown (please see the video lecture), that if $i \leftrightarrow j$, then $d(i) = d(j)$. It implies that the periodicity is a class property, in the sense in an equivalent class all the states have the same period. It clearly indicates that if the Markov Chain is an irreducible Markov Chain, then all the states have the same period.

Example 1: Consider the following transition probability matrix of a finite Markov Chain, with state space $\{1, 2, \ldots, M\}$.

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

It is clear that all the states communicate with each other, hence it is an irreducible Markov Chain. Moreover, $d(1) = \ldots = d(M) = M$.

Example 2: Consider the following infinite Markov chain with state space

$\{0, 1, 2, \ldots\}$. Let the transition probability matrix be

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \ldots \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2} & 0 & \ldots & 0 & \ldots & \frac{1}{2} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Note that here all the states communicate with each other, hence it is an irreducible Markov Chain. Now since $p_{00}^{(2)} > 0$ and $p_{00}^{(3)} > 0$, hence, $d(0) = d(1) = \ldots = 1$. Now we will make a statement without proof. If a state has a period $d(i)$, it means there exists an integer N, such that for all $n \geq N$, $p_{ii}^{nd(i)} > 0$.

Now we will be discussing another important concept in a Markov Chain, and it is called recurrence class or recurrence state. Suppose we have a Markov Chain, finite or infinite, and consider a state 'i'. Let us define for each integer $n \geq 1$,

$$f_{ii}^{n} = f_{ii}^{(n)} = P(X_n = i, X_k \neq i, k = 1, 2, \ldots, n-1 | X_0 = 0).$$

It means f_{ii}^{n} is the probability that, starting from the state 'i', the first return to state 'i' occurs at the n-th transition. In the first example it is clear that $f_{11}^{M} = f_{22}^{M} = \ldots = f_{MM}^{M} = 1$. In the second example

$$\begin{aligned} & f_{00}^{1} = 0, f_{00}^{2} = \frac{1}{2}, f_{00}^{3} = \frac{1}{2^2}, \ldots, f_{00}^{k+1} = \frac{1}{2^k}, \ldots, \\ & f_{11}^{1} = 0, f_{11}^{2} = \frac{1}{2}, f_{11}^{3} = \frac{1}{2^2}, \ldots, f_{11}^{k+1} = \frac{1}{2^k}, \ldots, \\ & \qquad \vdots \\ & f_{ii}^{1} = 0, f_{ii}^{2} = \frac{1}{2}, f_{ii}^{3} = \frac{1}{2^2}, \ldots, f_{ii}^{k+1} = \frac{1}{2^k}, \ldots, \end{aligned}$$

In a Markov Chain a state 'i' is called a recurrent state, if and only if $\sum_{i=1}^{\infty} f_{ii}^{n} = 1$. This says that starting from state 'i', it comes back to the state 'i' in a finite number of steps with probability one. If a state is not a recurrent, it is called a transient state. Consider the transition matrix of the following finite Markov Chain, with state space $\{1, 2, 3, 4\}$.

$$\boldsymbol{P} = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{array} \right].$$

In this case

$$f_{11}^1 = 1, f_{11}^n = 0; \quad n > 1.$$

$$f_{22}^1 = 0, f_{22}^2 = \frac{1}{2^2}, f_{22}^n = 0; \quad n > 2.$$

$$f_{33}^1 = 0, f_{33}^2 = \frac{1}{2^2}, f_{33}^n = 0; \quad n > 2.$$

$$f_{44}^1 = 1, f_{44}^n = 0; \quad n > 1.$$

Hence, it is clear that the state '1' and '4' are recurrent states, and '2' and '3' are transient states. Note that in this we have three different equivalent classes, namely $\{1\}$, $\{2, 3\}$ and $\{4\}$. We will prove that in an equivalent class either all the states are transient or all the states are recurrent. So it is also a class property.

Now before discussing some theoretical issues, let us discuss another interesting example, a general version of Example 2. Consider the following infinite Markov Chain with the following transition probability matrix

$$\boldsymbol{P} = \left[\begin{array}{ccccccc} p_0 & 1-p_0 & 0 & 0 & 0 & 0 & \dots \\ p_1 & 0 & 1-p_1 & 0 & 0 & 0 & \dots \\ p_2 & 0 & 0 & 1-p_2 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_k & 0 & \dots & 0 & \dots & 1-p_k & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right].$$

Here $0 < p_k < 1$, for $k = 0, 1, 2, \dots$. Now observe that

$$\begin{aligned} f_{00}^1 &= p_0 = 1 - (1 - p_0) \\ f_{00}^2 &= (1 - p_0)p_1 \\ f_{00}^3 &= (1 - p_0)(1 - p_1)p_2 \end{aligned}$$

$$\begin{array}{ccc} \vdots & & \vdots \\ f_{00}^n & = & \left(\prod_{i=0}^{n-2}(1-p_i)\right)p_{n-1} \end{array}$$

Hence, for $n > 1$,

$$f_{00}^n = \left(\prod_{i=0}^{n-2}(1-p_i)\right)(1-(1-p_{n-1})) = \prod_{i=0}^{n-2}(1-p_i) - \prod_{i=0}^{n-1}(1-p_i).$$

Let us write

$$u_n = \begin{cases} \prod_{i=0}^{n}(1-p_i) & \text{if} \quad n \geq 0 \\ 1 & \text{if} \quad n = -1. \end{cases}$$

Hence,

$$\sum_{n=1}^{m+1} f_{00}^n = 1 - u_m.$$

Therefore, the state '0' will be transient or recurrent if $\lim_{m\to\infty} u_m > 0$ or $\lim_{m\to\infty} u_m = 0$. We have the following interesting result if $0 < p_i < 1$, for $i = 0, 1, \ldots$, then $u_m \to 0$, as $m \to \infty$, if and only if $\sum_{i=0}^{\infty} p_i = \infty$. We will show this result next time. Therefore, if p_i's are constant, then clearly all the states are recurrent. Similarly, if $p_n = \frac{1}{n}$, then all the states are recurrent, where as if $p_n = \frac{1}{n^2}$, then all the states are transient.