Mid-Sem-2024 (MTH212M) & Final - 2024 (MTH412A)
Applied Stochastic Process
Grading Schemes

Time Two Hours
Maximum Score: 80
Name and Roll Number

Instructions: You have to return this entire question paper along with the answer script. Write name and roll number in each page of the question paper. If you do not write your name and roll number in each page 2 marks will be deducted. This question paper has 6 questions. Questions 1 and 2 are subjective questions, and you should write the answers in the answer script provided to you. If you answer Questions 1 and 2 in sequential manner at the beginning in the answer script you will get two points. Questions 3 to 6 are multiple choice questions, where more than one answers might be correct. Each correct answer will give you two points and if you tick a wrong answer it will be negative two. If you can identify all the correct answers without ticking any wrong answer, you will get ten points in Questions 3 to 6. You need to write the answers of Questions 3 to 6, in this question paper only. Do not do any rough work in the question paper, then two marks will be deducted. After answering Questions 1 and 2, you may use the answer script provided to you for rough work.

Name and Roll Number

1. Let us consider two Markov chains $\mathcal{A}$ and $\mathcal{B}$, where the states are all non-negative integers. The transition probability matrices are as follows:

$$\mathbf{P} = \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ p_{20} & p_{21} & p_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \quad \text{and} \quad \widetilde{\mathbf{P}} = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ p_{20} & p_{21} & p_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

   (a) Prove or give a counter example that if $\mathcal{A}$ is irreducible and aperiodic, then $\mathcal{B}$ is also aperiodic.

   (b) Prove or give a counter example that if $\mathcal{A}$ is irreducible and transient, then the states $\{1, 2, 3, \ldots\}$ in $\mathcal{B}$ are also transient.

   (c) Prove or give a counter example that if the states $\{1, 2, 3, \ldots\}$ in $\mathcal{B}$ are recurrent, then they are recurrent in $\mathcal{A}$ also.

   (d) Prove or give a counter example that if $\mathcal{A}$ is irreducible and recurrent, then all the states in $\mathcal{B}$ are recurrent.

(5+5+5+5=20)

Solutions:

(a) FALSE (counter example)

$$\mathbf{P} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & \cdots \\ 1/2 & 0 & 1/2 & 0 & 0 & \cdots \\ 1/2 & 0 & 0 & 1/2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Note that $p_{00}^1 > 0$, all the states communicate with each other, hence in $\mathcal{A}$ all the states have period 1, where as in $\mathcal{B}$, $p_{ii}^n = 0$, for all $n \geq 1$ and for all $i \geq 1$.

(b) TRUE. If $\mathcal{A}$ is irreducible then $p_{ii}^n \geq \widetilde{p}_{ii}^n$ for all $i \geq 1$. It simply follows from the all possible paths. Hence, the result follows.

(c) TRUE. Same argument as in (b).

(d) FALSE. Same counter example as in (a). Note that all the states are recurrent as $f_{00}^n = 1/2^n$, for all $n \geq 1$. Hence $\sum_{n=1}^{\infty} f_{00}^n = 1$.

Grading Scheme: If some body does not show explicitly whether the states have period 1 or they are recurrect, marks have been deducted.

2. Suppose $\{X_n\}$ is a Markov chains with the state space $S = \{0, 1, \ldots, k\}$, and having the transition probability matrices $\mathbf{P}$, where $\mathbf{P}$ can be decomposed as follows

$$\mathbf{P} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{R} & \mathbf{Q} \end{bmatrix}.$$

Here $\mathbf{Q}$ is a $k \times k$ square matrix, $\mathbf{R}$ and $\mathbf{0}$ are k dimensional column and row vectors, respectively. Suppose $\tau = \min\{n \geq 1 : X_n = 1\}$, $m_i = E(\tau | X_0 = i)$, $v_i = Var(\tau | X_0 = i)$, for $i = 2, \ldots, k$, $\widetilde{\mathbf{m}} = (m_1, \ldots, m_k)^T$ and $\widetilde{\mathbf{e}} = (1, \ldots, 1)^T$. Here $\mathbf{I}$ denotes the $k \times k$ identity matrix.

(a) Show that $(\mathbf{I} - \mathbf{Q})^{-j}$ for any $j \geq 1$ exists.

(b) Show that $\widetilde{\mathbf{m}} = (\mathbf{I} - \mathbf{Q})^{-2}\mathbf{R}$. (Hint: Use the definition of expectation)

(c) Show that $\mathbf{R} = (\mathbf{I} - \mathbf{Q})\widetilde{\mathbf{e}}$, hence show $\widetilde{\mathbf{m}} = (\mathbf{I} - \mathbf{Q})^{-1}\widetilde{\mathbf{e}}$.

(6+8+6=20)

Solutions: We have already proved it in the class.

Name and Roll Number

3. Suppose $\{X_n; n \geq 1\}$ is a Markov Chain with the state space $S = \{1, 2, 3, 4, 5, 6\}$, and having the transition probability martrix

$$\mathbf{P} = \begin{bmatrix} 1/4 & 0 & 0 & 0 & 0 & 3/4 \\ 0 & 2/3 & 0 & 1/3 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 0 & 0 & 0 & 1/2 \end{bmatrix}.$$

Then which of the following statements are correct.

(a) $\lim_{n\to\infty} p_{11}^{(n)} = 2/5$

(b) $\lim_{n\to\infty} p_{31}^{(n)} = 1/5$

(c) $\lim_{n\to\infty} p_{66}^{(n)} = 3/10$

(d) $\displaystyle\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} p_{36}^{(j)} = 1/2$

(e) $\displaystyle\lim_{n\to\infty} \sum_{j=1}^{n} p_{35}^{(j)} = 0$

The correct options are: (a),(b),(e)

4. Let $\{X_n\}$ and $\{Y_n\}$ be two Markov Chains with the same state space $\mathcal{S} = \{1, \ldots, M\}$, and $M < \infty$. Suppose X_n and Y_n have the transition probability matrix $\mathbf{P}$ and $\mathbf{Q}$, respectively. The Markov Chain $\{Z_n\}$ has the same state space $\mathcal{S}$ and it is defined as follows: Given $Z_n = i$, Z_{n+1} takes values in $\mathcal{S}$ following the transition probability matrix $\mathbf{P}$ with probability a and following the transition probability matrix $\mathbf{Q}$ with probability $1 - a$. Here $0 < a < 1$. Then which of the following statements are always correct.

(a) $\{Z_n\}$ is an aperiodic Markov Chain if and only if both $\{X_n\}$ and $\{Y_n\}$ are aperiodic Markov Chains.

(b) $\{Z_n\}$ is an irreducible Markov Chain if and only if both $\{X_n\}$ and $\{Y_n\}$ are irreducible Markov Chains.

(c) If $\{X_n\}$ is an irreducible Markov Chain then $\{Z_n\}$ is a recurrent Markov Chain.

(d) If $\{Y_n\}$ is an irreducible Markov Chain then $\{Z_n\}$ is a recurrent Markov Chain.

(e) If $\{X_n\}$ and $\{Y_n\}$ are both irreducible Markov Chains then $\{Z_n\}$ is a recurrent Markov Chain.

The correct options are: (c),(d),(e).

Name and Roll Number

5. Suppose N black balls and N white balls are placed in two urns so that each urn contains N balls. At each step one ball is selected at random from each urn, and the two balls are interchanged. Let X_n denote the number of white balls in the first urn at the n-th step. Let the transition probability matrix be $\mathbf{P}$ of the Markov Chain $\{X(n)\}$, and $\pi_j = \lim\limits_{n\to\infty} p_{ij}^{(n)}$ for $0 \le i, j \le N$. Then which of the following statements are always true.

   (a) $\sum\limits_{j=0}^{N} p_{j0} = 1$.

   (b) $\{X_n\}$ is a recurrent Markov Chain.

   (c) $\{X_n\}$ is an irreducible Markov Chain.

   (d) $\{X_n\}$ is an aperiodic Markov Chain.

   (e) When $N = 3$, $\pi_3 = 9\pi_2$.

   (f) When $N = 3$, $\pi_1 + \pi_2 = 0.75$.

   The correct options are: (b),(c),(d).

6. Suppose $\{X_n\}$ is a Markov chain with the state space $S = \{0, \pm 1, \pm 2, \ldots\}$ and the elements of the transition probability matrix is as follows:

$$p_{ij} = \begin{cases} a & \text{if} \quad j = i - 1 \\ 1 - a & \text{if} \quad j = i + 1 \\ 0 & \quad\quad \text{otherwise,} \end{cases}$$

   here $0 < a < 1$. Then which of the following staements are always correct.

   (a) The Markov Chain $\{X_n\}$ is an irreducible Markov Chain.

   (b) The Markov Chain $\{X_n\}$ is an aperiodic Markov Chain.

   (c) $\lim_{n\to\infty} p_{00}^{(n)}$ exists if and only if $a \ne \frac{1}{2}$.

   (d) $\lim_{n\to\infty} p_{00}^{(n)}$ exists for all $0 < a < 1$.

   (e) $\sum_{n=0}^{\infty} p_{00}^{(n)} = \infty$ if $a = \frac{1}{2}$.

   (f) If $\lim_{n\to\infty} n p_{00}^{(n)}$ exists for all $a \ne \frac{1}{2}$.

   The correct options are: (a),(d),(e),(f).