MTH211A: Theory of Statistics

End-semester Examination

Time: 90 minutes **Total marks:** 45

Name: ______________________ **Roll number:** ____________

1. *Answer all questions.*
2. *All notations used are as discussed in class.*

Q.1 Let $X_1, \ldots, X_n$ be a random sample from normal$(0, \theta^2)$. Find the expectation of $T = X_1 / \sum_{i=1}^n X_i^2$. [3]

Method 1.

$$E(T) = \frac{1}{(2\pi\theta^2)^{n/2}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \frac{u_1}{\sum_{i=1}^n u_i^2} \exp\left\{-\frac{1}{2\theta^2}\sum_{i=1}^n u_i^2\right\} du_1 \cdots du_n$$

$= g(\theta)$, say

Let $v_1 = -u_1$ then

$$E(T) = \frac{1}{(2\pi\theta^2)^{n/2}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \frac{v_1}{\sum_{i=2}^n u_i^2 + v_1^2} \exp\left\{-\frac{1}{2\theta^2}\left(v_1^2 + \sum_{i=2}^n u_i^2\right)\right\}$$

$(-dv_1)\, du_2 \cdots du_n$.

$= -g(\theta)$.

As $g(\theta) = -g(\theta)$. $\Rightarrow g(\theta) = 0 = E(T)$. [ANS]

Method 2: Let $S = \dfrac{x_1}{\sqrt{\sum_{i=1}^n x_i^2}}$. As $N(0,\theta^2)$ belongs to scale family, S is ancillary for θ.

Further, $\sum_{i=1}^n x_i^2 = W$ is CSS for θ.

So, $S \perp\!\!\!\perp g(W)$ for any function g of W.

$$\therefore E(T) = E\left(S \cdot \frac{1}{\sqrt{\sum_{i=1}^n x_i^2}}\right) = E(S)\, E\left[\frac{1}{\sqrt{\sum_{i=1}^n x_i^2}}\right], \quad — (1)$$

and $0 = E(x_1) = E\left(S \times \sqrt{\sum_{i=1}^n x_i^2}\right) = E(S)\, E\left(\sqrt{\sum_{i=1}^n x_i^2}\right) \Rightarrow E(S) = 0$ — (2).

$\Rightarrow E(T) = 0$. [ANS]

Q.2 Let X be a random variable, and we are interested in testing $H_0 : X \sim f_0$ against $H_1 : X \sim f_1$, where

x	-4	-3	0	1	2	5
$f_0(x)$	0.05	0.20	0.30	0.15	0.25	0.05
$f_1(x)$	0.15	0.30	0.05	0.05	0.25	0.20

Find an MP level-α test for testing H_0 against H_1, where $\alpha = 0.15$. [3]

Define $\lambda(\underset{\sim}{x}) = \dfrac{f_1(x)}{f_0(x)}$.

The MP level α test is :

$$\phi^*(x) = \begin{cases} 1 & \text{if} \quad \lambda(x) > k \\ \gamma & \text{if} \quad \lambda(x) = k \\ 0 & \text{if} \quad \lambda(x) < k , \end{cases} \quad - (*)$$

and k is s.t. $E_{f_0}\left[\phi^*(x)\right] = P(\lambda(x) > k) + \gamma P(\lambda(x) = k)$
$= \alpha$.

Here, $\lambda(x)$ is given by :

x	-4	-3	0	1	2	5
$\lambda(x)$	3	$3/2$	$1/6$	$1/3$	1	4

If we choose $k = 3/2$ then $P_{f_0}(\lambda(x) > k) =$

$= P_{f_0}(|x| \geq 4) = \cancel{0.05}\ 0.10$

and $P_{f_0}(\lambda(x) \geq k) = P_{f_0}(|x| \geq 3) = 0.25$.

So, $k = 3/2$ is the optimal choice.

Further, for $k = 3/2$,

$E_{f_0}\left[\phi^*(x)\right] = \alpha = 0.15 \quad \Rightarrow \quad \overset{0.1}{\cancel{0.05}} + \gamma\, 0.2 = 0.15$

$\Rightarrow \quad \gamma = \dfrac{5}{20} = \dfrac{1}{4}$.

$\therefore$ The test $(*)$ with $k = 3/2$ and $\gamma = \frac{1}{4}$ is the MP test. $\boxed{\text{ANS}}$

Q.3 Let the distribution of X be given by

x	0	1	2	3
$P_\theta(X = x)$	θ	2θ	$0.9 - 2\theta$	$0.1 - \theta$

where $0 < \theta < 0.1$. For testing $H_0 : \theta = 0.05$ against $H_1 : \theta > 0.05$ at level $\alpha = 0.05$, which of the following tests (if any) is uniformly most powerful (UMP) at level α among these four tests? [3]

Test $x \rightarrow$	0	1	2	3
$\phi_1(x)$	1	0	0	0
$\phi_2(x)$	0	1	0	0
$\phi_3(x)$	0.5	0.25	0	0
$\phi_4(x)$	0	0	0	1

Power functions and sizes of $\phi_1 - \phi_4$ are given below:

$\beta_{\phi_1}(\theta) = P_\theta(X=0) = \theta$ and $\beta_{\phi_1}(0.05) = 0.05$ — (1)

$\beta_{\phi_2}(\theta) = P_\theta(X=1) = 2\theta$ and $\beta_{\phi_2}(0.05) = 0.1$ — (2)

$\left.\begin{array}{l} \beta_{\phi_3}(\theta) = 0.5 \times P_\theta(X=0) + 0.25 \times P_\theta(X=1) = \frac{\theta}{2} + \frac{\theta}{2} = \theta \\ \text{and } \beta_{\phi_3}(0.05) = 0.05 \end{array}\right\}$ — (3)

$\beta_{\phi_4}(\theta) = P_\theta(X=3) = 0.1 - \theta$, and $\beta_{\phi_4}(0.05) = 0.05$ — (4)

From (2), ϕ_2 is not a Level-α test.

From (1) and (3), ϕ_1 and ϕ_3 have same power fn.

From (1) and (4), $\beta_{\phi_1}(\theta) = \beta_{\phi_3}(\theta) > \beta_{\phi_4}(\theta)$

$\Leftrightarrow \theta > 0.1 - \theta \Leftrightarrow \theta > 0.05$. (i.e., under H_1)

So, $\underline{\phi_1 \text{ and } \phi_3 \text{ are UMP.}}$ [ANS]

Q.4 Let $X_1, \ldots, X_n$ be a random sample from Poisson(θ) distribution, and $E = \{0, 2, 4, \ldots\}$ be the the set of positive even integers. Find the UMVUE of $g(\theta) = P_\theta(X_1 \in E)$. [5]

$$P_\theta(X_1 \in E) = \sum_{x=0}^{\infty} e^{-\theta} \frac{\theta^{2x}}{(2x)!} = \frac{e^{-\theta}}{2}\left[\sum_{x=0}^{\infty} \frac{\theta^x}{x!} + \sum_{x=0}^{\infty} \frac{(-\theta)^x}{x!}\right]$$

$$= \frac{e^{-\theta}}{2}\left[e^{\theta} + e^{-\theta}\right] = \frac{(1+e^{-2\theta})}{2} = g(\theta)$$

Method 1 : Define $T_1 = \begin{cases} 1 & \text{if } X_1 = 0 \text{ and } X_2 = 0 \\ 0 & \text{ow.} \end{cases}$

$$E_\theta(T_1) = P(X_1 = 0, X_2 = 0) = e^{-2\theta}.$$

$\therefore \frac{(T_1+1)}{2}$ is an unbiased estimator of $g(\theta)$.

$T = \sum_{i=1}^n X_i$ is ~~th~~ a css for θ, $T \sim$ Poisson $(n\theta)$.

By Lehman- Scheffe, UMVUE is $E_{T_1|T}\left[2T_1 - 1 \mid T\right] = \left[E_{T_1|T}(T_1|T) + 1\right]/2$

$$E_{T_1|T=t}(T_1 \mid T=t) = P(X_1 = 0, X_2 = 0 \mid T = t)$$

$$= P(X_1 = 0, X_2 = 0, \sum_{i=1}^n X_i = t) \,/\, P(\sum_{i=1}^n X_i = t)$$

$$= P(X_1 = 0)\, P(X_2 = 0)\, P(\sum_{i=3}^n X_i = t) \,/\, P(\sum_{i=1}^n X_i = t)$$

$$= \frac{e^{-2\theta}\, e^{-(n-2)\theta}\, [(n-2)\theta]^t \; t!}{t!\; e^{-n\theta}\, (n\theta)^t} = \left(1 - \frac{2}{n}\right)^t.$$

$\therefore$ UMVUE

$$= \frac{1}{2}\left[1 + \left(1 - \frac{2}{n}\right)^T\right].$$

Method 2 : Define $T_2 = \begin{cases} 1 & \text{if } X_1 \in \{0, 2, 4, \ldots\} \\ 0 & \text{ow.} \end{cases}$

$\therefore E_\theta(T_2) = g(\theta)$

$\therefore$ UMVUE $= E_{T_2|T}\left[T_2 \mid T\right]$

$$E_{T_2|T=t}(T_2 \mid T=t) = \frac{P(X_1 \in E, \sum_{i=1}^n X_i = t)}{P(\sum_{i=1}^n X_i = t)} = \sum_{x=0}^{\lfloor t/2 \rfloor} \frac{P(X_1 = 2x, \sum_{i=2}^n X_i = t - 2x)}{P(T = t)}$$

$$= \sum_{x=0}^{\lfloor t/2 \rfloor} \frac{t!\,(n-1)^{t-2x}}{(2x)!\,(t-2x)!\; n^t}$$

Q.5 Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be two mutually independent random samples from $\texttt{normal}(\mu_1, \sigma^2)$ and $\texttt{normal}(\mu_2, \sigma^2)$, respectively, where σ is known, and $m = 2n$. Find the $(1-\alpha)100\%$ shortest length confidence interval of $\mu_1 - \mu_2$ involving a complete sufficient statistics of μ_1 and μ_2. Comment on the limiting length of the confidence interval when $n \to \infty$. [4+1]

$$X_i \overset{\text{IID}}{\sim} N(\mu_1, \sigma^2)\ ;\ i = 1, \ldots, n \implies \bar{X}_n \sim N\left(\mu_1, \frac{\sigma^2}{n}\right)$$

$$Y_i \overset{\text{IID}}{\sim} N(\mu_2, \sigma^2)\ ;\ i = 1, \ldots, m \implies \bar{Y}_m \sim N\left(\mu_2, \frac{\sigma^2}{m}\right)$$

ind.

$$\implies \bar{X}_n - \bar{Y}_m \sim N\left(\mu_1 - \mu_2, \sigma^2\left(\frac{1}{n} + \frac{1}{m}\right)\right).$$

$$\implies T(\underset{\sim}{X}, \underset{\sim}{Y}, \underset{\sim}{\theta}) = \frac{\bar{X}_n - \bar{Y}_m - (\mu_1 - \mu_2)}{\sigma\sqrt{\frac{1}{n} + \frac{1}{m}}} \sim N(0,1). \qquad \underset{\sim}{\theta} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}.$$

We choose $T(\underset{\sim}{X}, \underset{\sim}{Y}, \underset{\sim}{\theta})$ as the pivot, and find c_1, c_2 s.t.

$$P\left(c_1 < T(\underset{\sim}{X}, \underset{\sim}{Y}, \underset{\sim}{\theta}) \leq c_2\right) = (1-\alpha). \quad \text{— } (*)$$

Then, by solving for θ, and as T is ~~st~~ strictly monotonically decreasing w.r.t., we get:

$$P\Bigg[\underbrace{\bar{X}_n - \bar{Y}_m - c_2\sigma\sqrt{\frac{1}{m} + \frac{1}{n}}}_{L(\underset{\sim}{X}, \underset{\sim}{Y})} < \mu_1 - \mu_2 < \underbrace{\bar{X}_n - \bar{Y}_m - c_1\sigma\sqrt{\frac{1}{m} + \frac{1}{m}}}_{U(\underset{\sim}{X}, \underset{\sim}{Y})}\Bigg] = (1-\alpha)$$

Length of C.I $= U(\underset{\sim}{X}, \underset{\sim}{Y}) - L(\underset{\sim}{X}, \underset{\sim}{Y}) = (c_2 - c_1)\,\sigma\sqrt{\frac{1}{m} + \frac{1}{m}} = L^*(\underset{\sim}{X}, \underset{\sim}{Y})$, say.

As $L^*(\underset{\sim}{X}, \underset{\sim}{Y}) \propto (c_2 - c_1)$, ~~by~~ ~~theorem~~ it is enough to minimize $c_2 - c_1$.

By the theorem (i) $\phi(c_1) = \phi(c_2)$ where ϕ is the CDF of std. normal distn

(ii) $c_1 < 0 < c_2$, and $(*)$ should satisfy.

So, $c_2 = -c_1$ and by $(*)$ $c_2 = \tau_{\alpha/2}$: upper $\alpha/2$ point of $N(0,1)$.

Finaly, as $n \to \infty$, $m \to \infty$. $(m = 2n)$. and $L^*(\underset{\sim}{X}, \underset{\sim}{Y}) \to 0$ as $\sigma(c_2 - c_1)$ is bdd, and free of n.

Q.6 Let X belong to the scale family of distributions, i.e., $X = \theta W$, and the density of W be $f_W(\cdot)$. Show that the Fisher information of θ is proportional to θ^{-2}, and the proportionality constant is free on θ. [5]

$$f_X(x) = f_W\left(\frac{x}{\theta}\right)\left|\frac{\partial w}{\partial x}\right| = \frac{1}{\theta} f_W\left(\frac{x}{\theta}\right).$$

Log likelihood :

$$l_\theta(x) = -\log\theta + \log f_W\left(\frac{x}{\theta}\right).$$

Score fn. :

$$\frac{\partial}{\partial\theta} l_\theta(x) = s(x,\theta) = -\frac{1}{\theta} + \frac{\frac{\partial}{\partial(x/\theta)} f_W(x/\theta)}{f_W(x/\theta)} \cdot \left(-\frac{x}{\theta^2}\right)$$

$$I(\theta) = E_\theta\left[s^2(X;\theta)\right]$$

$$= \frac{1}{\theta^2}\left[1 + 2E_\theta\left\{\frac{X}{\theta}\cdot\frac{\frac{\partial}{\partial(X/\theta)} f_W(X/\theta)}{f_W(X/\theta)}\right\} + E_\theta\left\{\frac{X^2}{\theta^2}\cdot\frac{\left[\frac{\partial}{\partial(X/\theta)}\cdot f_W(X/\theta)\right]^2}{\left\{f_W(X/\theta)\right\}^2}\right\}\right]$$

(1) —

Now,

$$E_\theta\left[\frac{X}{\theta}\cdot\frac{\frac{\partial}{\partial(X/\theta)} f_W(X/\theta)}{f_W(X/\theta)}\right]$$

$$= \int_{\mathbb{R}} \frac{x}{\theta}\cdot\frac{\partial}{\partial(x/\theta)} f_W(x/\theta).\ \left\{f_W(\cancel{x/\theta})\right\}^{-1} f_W\left(\cancel{\frac{x}{\theta}}\right)\cdot\frac{1}{\theta}\,dx.$$

$u = x/\theta \Rightarrow du = dx/\theta.$; and the above integral

$$= \int_{\mathbb{R}} u \frac{\partial}{\partial u} f_W(u)\,du$$ is free of θ as f_W is free of θ.

Similarly, the last expectation in (1) is

$$\int_{\mathbb{R}} \frac{x^2}{\theta^2}\cdot\left(\frac{\partial}{\partial(x/\theta)} f_W(x/\theta)\right)^2 \left\{f_W(x/\theta)\right\}^{-2} f_W\left(\frac{x}{\theta}\right)\frac{1}{\theta}\,dx$$

$$= \int_{\mathbb{R}} u^2 \left(\frac{\partial}{\partial u} f_W(u)\right)^2 \left(f_W(u)\right)^{-1} du. \quad \text{is free of } \theta.$$

Thus, $\quad I(\theta) = \theta^{-2} \times k \quad$ where k is free of θ.

Q.7 Let $X_1, \ldots, X_n$ be a random sample from $\texttt{uniform}[\mu - 3\sqrt{\sigma}, \mu + 3\sqrt{\sigma}]$, $\mu \in \mathbb{R}$ and $\sigma > 0$. [3+3+1]

(a) Find maximum likelihood estimates (MLEs) of μ, σ and $\sqrt{\sigma}$.

(b) Find the UMVUEs of μ and $\sqrt{\sigma}$.
[You may assume that given a random sample $X_1, \ldots, X_n$ of size n from $\texttt{uniform}(\alpha, \beta)$ distribution, $\mathbf{T} = [X_{(1)}, X_{(n)}]$ is jointly complete and sufficient for (α, β).]

(c) Compare the MLEs and UMVUEs of μ and $\sqrt{\sigma}$ when $n \to \infty$.

(a) Define $\alpha = \mu - 3\sqrt{\sigma}$ and $\beta = \mu + 3\sqrt{\sigma}$.

Then $\mu = \dfrac{\alpha+\beta}{2}$ and $\sqrt{\sigma} = \dfrac{\beta-\alpha}{6}$.

Likelihood of α, β, $L\left(\binom{\alpha}{\beta}\right) = \dfrac{1}{(\beta-\alpha)^n} \mathbb{I}(x_{(1)} \geq \alpha)\, \mathbb{I}(x_{(n)} \leq \beta)$

Fixing $\overset{=\alpha_0}{\alpha}$, we get $L\left(\binom{\alpha_0}{\beta}\right)$ is a decreasing fn. of β over the range $[x_{(n)}, \infty)$ and $L\left(\binom{\alpha_0}{\beta}\right) = 0$ over $(-\infty, x_{(n)})$.

So, $\hat{\beta}_{MLE} = X_{(n)}$, irrespective of the choice of α.

Fixing, $\beta = \beta_0$, observe that $L\left(\binom{\alpha}{\beta_0}\right)$ is increasing over $(-\infty, x_{(1)}]$ and 0 over $(x_{(1)}, \infty)$.

So, $\hat{\alpha}_{MLE} = X_{(1)}$, irrespective of the choice of β.

Thus, by invariance property of MLE,

$$\left.\begin{aligned} \hat{\mu}_{MLE} = \frac{X_{(n)} + X_{(1)}}{2} \quad \text{and} \quad \widehat{(\sqrt{\sigma})}_{MLE} = \frac{X_{(n)} - X_{(1)}}{6} \end{aligned}\right\} - (*)$$

and $\hat{\sigma}_{MLE} = \left(\dfrac{X_{(n)} - X_{(1)}}{6}\right)^2$.

(b) [UMVUE]. As $[X_{(1)}, X_{(n)}]$ is jointly CSS for (α, β), it is CSS for $(\mu, \sqrt{\sigma})$ as well (bijective fn.)

Now, $E\left[X_{(n)} - \alpha\right] = n \displaystyle\int_\alpha^\beta \frac{(t-\alpha)^n}{(\beta-\alpha)^n}\, dt = n \int_0^1 \frac{u^n\, du}{(\beta-\alpha)^{-1}}$ where $u = \dfrac{t-\alpha}{\beta-\alpha}$.

$$= \frac{n}{n+1}(\beta-\alpha) \quad \Rightarrow \quad E[X_{(n)}] = \alpha + \frac{n}{n+1}(\beta-\alpha). \quad — (1)$$

$$E[X_{(1)}-\alpha] = n\int_{\alpha}^{\beta}(t-\alpha)\frac{(\beta-t)^{n-1}}{(\beta-\alpha)^n}\,dt$$

$$= n\int_0^{\beta-\alpha} v\left[\frac{\beta-\alpha-v}{\beta-\alpha}\right]^{n-1}\frac{1}{(\beta-\alpha)}\,dv \qquad v=(t-\alpha).$$

$$= n(\beta-\alpha)\int_0^1 u(1-u)^{n-1}\,du.$$

$$= n(\beta-\alpha)\,\text{Beta}(2,n) = (\beta-\alpha)\frac{1}{n+1}$$

$$\Rightarrow E[X_{(1)}] = \alpha + \frac{(\beta-\alpha)}{n+1}. \quad — (2).$$

$$\therefore E[X_{(n)}-X_{(1)}] = \frac{n-1}{n+1}(\beta-\alpha) = 6\left(\frac{n-1}{n+1}\right)\sqrt{\sigma}.$$

$$\Rightarrow T_1 = \frac{n+1}{6(n-1)}(X_{(n)}-X_{(1)})$$ is the UMVUE of $\sqrt{\sigma}$. — (2)

Further, $$E[X_{(n)}+X_{(1)}] = 2\alpha + \beta - \alpha = \beta+\alpha. = 2\mu$$

$$\Rightarrow T_2 = \frac{X_{(n)}+X_{(1)}}{2}$$ is the UMVUE of μ. — (3)

(c) From (*), (2) and (3) it is clear that,

$\hat{\mu}_{MLE} = T_2$ = UMVUE of μ.

and as $n\to\infty$, $\left(\frac{n-1}{n+1}\right)\to 1$, thus $\frac{\sqrt{\hat{\sigma}_{MLE}}}{T_1}\to 1$, w.p.1.

x——x

[3+3+2]

Q.8 Let $X_1, \ldots, X_n$ be a random sample from unifrom$(\theta, 0)$ distribution, $\theta < 0$.

(a) Find an UMP test for testing $H_0 : \theta \geq \theta_0$ against $H_1 : \theta < \theta_0$.

(b) Consider another test of the form

$$\phi^*(\mathbf{x}) = \begin{cases} 1 & \text{if } x_{(1)} < \theta_0, \\ \alpha & \text{if } x_{(1)} \geq \theta_0. \end{cases}$$

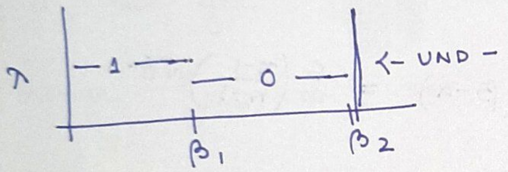
Find the size (maximum probability of type I error) and power function of ϕ^*.

(c) Is ϕ^* UMP at level α? Justify your answer.

(a) Define $Y_i = -X_i \overset{IID}{\sim} \mathcal{U}(0, -\theta)$, $i = 1, \ldots, n$, $-\theta = \beta$, say.

For $\beta_1 < \beta_2$,

$$\lambda(\underset{\sim}{y}, \beta_1, \beta_2) = \frac{f_{\beta_1}(\underset{\sim}{y})}{f_{\beta_2}(\underset{\sim}{y})} = \frac{\mathbb{I}(y_{(n)} < \beta_1)}{\mathbb{I}(y_{(n)} < \beta_2)} \cdot \left(\frac{\beta_2}{\beta_1}\right)^n$$



$\therefore$ $U(0, \beta)$ distn. has MLR in $Y_{(n)}$, and λ is monotonically non-increasing.

To test $H_0 : \beta \leq \beta_0$ vs $H_1 : \beta > \beta_0$ where $\beta_0 = -\theta_0$.

A UMP test is given by

$$① \quad \phi^{**}(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } y_{(n)} \geq c, \text{ equivalently, } x_{(1)} \leq -c \\ 0 & \text{otherwise,} \end{cases}$$

and c is s.t. $E_{\beta_0}\left[\phi^{**}(\underset{\sim}{x})\right] = \alpha$

i.e., $P_{\beta_0}\left[y_{(n)} \geq c\right] = 1 - \left(\frac{c}{\beta_0}\right)^n = \alpha$

$\Rightarrow c = \beta_0 (1-\alpha)^{1/n}$. — ②.

Combining ① and ②, we get the UMP test.

(b) Power fn. of ϕ^{*} :

$$E_\theta\left[\phi^{*}(\underset{\sim}{x})\right] = P_\theta\left[X_{(1)} < \theta_0\right] + \alpha\, P_\theta\left[X_{(1)} \geq \theta_0\right]$$

$$= P_\beta\left[Y_{(n)} > \beta_0\right] + \alpha\, P_\beta\left[Y_{(n)} \leq \beta_0\right]$$

$$= 1 - \left(\frac{\beta_0}{\beta}\right)^n + \alpha\left(\frac{\beta_0}{\beta}\right)^n$$

$$= 1 - (1-\alpha)\left(\frac{\beta_0}{\beta}\right)^n = 1 - (1-\alpha)\left(\frac{\theta_0}{\theta}\right)^n.$$

Size :

$$\sup_{\theta \geq \theta_0}\left\{1-(1-\alpha)\left(\frac{\theta_0}{\theta}\right)^n\right\} = \sup_{\beta \leq \beta_0}\left\{1-(1-\alpha)\left(\frac{\beta_0}{\beta}\right)^n\right\}$$

$$= 1 - (1-\alpha)\inf_{\beta\leq\beta_0}\left(\frac{\beta_0}{\beta}\right)^n = \alpha.$$

(c) Power fn. of the UMP test in (a) is :

$$E_\theta\left[\phi^{**}(\underset{\sim}{x})\right] = P_\beta\left[Y_{(n)} \geq \beta_0(1-\alpha)^{1/n}\right]$$

$$= 1 - \left(\frac{\beta_0}{\beta}\right)^n(1-\alpha) = \text{power fn. of } \phi^{*}(\underset{\sim}{x}).$$

Thus, ϕ^{*} is also a UMP test.

x —— x

Q.9 Consider the problem of testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ $(\theta_1 > \theta_0)$ at level-α, given a random sample $X_1, \cdots, X_n$ from `normal`$(\theta, 1)$ distribution. Given a realization $\mathbf{x}$, derive the p-value, say $p(\mathbf{x})$, under the MP test setup. Hence, find the distribution of $p(\mathbf{X})$ under θ_0. [3+3]

Under the MP test setup, Ho is rejected if $\sum_{i=1}^{n} X_i$ is large.

Thus, p-value of the realization $\underset{\sim}{x}$ is :

$$P(\underset{\sim}{x}) = P_{\theta_0}\left(\sum_{i=1}^{n} X_i > n t_0\right) \quad \text{where} \quad t_0 = \frac{\sum_{i=1}^{n} x_i}{n} \text{ : realized mean}$$

$$= P_{\theta_0}\left(\bar{X}_n > t_0\right)$$

$$= P_{\theta_0}\left(\frac{\bar{X}_n - \theta_0}{1/\sqrt{n}} > \frac{t_0 - \theta_0}{1/\sqrt{n}}\right) = 1 - \Phi\left(\sqrt{n}\,(t_0 - \theta_0)\right) \quad - (1)$$

Now, let $P(\underset{\sim}{x}) = W$. Then CDF of W under θ_0,

$$P_{\theta_0}(W \le w) = P_{\theta_0}\left[\Phi\left(\sqrt{n}\left(\bar{X}_n - \theta_0\right)\right) \le w\right]$$

$$= P_{\theta_0}\left[\sqrt{n}\left(\bar{X}_n - \theta_0\right) \le \Phi^{-1}(w)\right] \quad \text{as } \Phi \text{ is strictly monotonically increasing}$$

$$= \Phi\left(\Phi^{-1}(w)\right) = w.$$

$\Rightarrow$ $W \sim$ Uniform $(0, 1)$.