## MTH211A: Theory of Statistics

## Problem set 4

## Methods of Point Estimation

- 1. Let  $X_1, \ldots, X_n$  be a random sample from the following distribution. In each case, find the method of moments (MoM) estimator for  $g(\theta)$ :
  - (a)  $\operatorname{Gamma}(\alpha, \beta)$ , and  $g(\boldsymbol{\theta}) = (\alpha, \beta)^{\top}$ .  $\rightarrow \left(\begin{array}{c} \overline{\mathbf{X}}_{\mathbf{N}} \\ \overline{\mathbf{S}}_{\mathbf{n}} \end{array}\right)^{\top}$
  - (b) Beta $(\alpha, \beta)$  and  $g(\theta) = \alpha/\beta$ .
  - (c) Poisson( $\lambda$ ) and  $g(\theta) = \exp\{-\lambda\}$ . = exp(- $\frac{\pi}{2}$ )
  - (d) Location scale Exponential  $(\mu, \sigma)$  and and  $g(\theta) = (\mu, \sigma)$ .
- 2. Let  $X_1, \ldots, X_n$  be a random sample from the following distribution. In each case, find the maximum likelihood estimator (MLE) for  $g(\theta)$ :
  - (a) Binomial $(m, \theta)$ , and  $g(\theta) = \theta$ .
  - (b) Binomial $(\theta, p)$ , and  $g(\theta) = \theta$  when n = 1.
  - (c)  $\operatorname{Binomial}(m, \theta)$ , and  $g(\theta) = P(X_1 + X_2 = 0)$ .
  - (d) Hypergeometric $(m, r, \theta)$  with p.m.f.

$$f_X(x; m, r, \theta) = \frac{\binom{m}{x} \binom{\theta - m}{r - x}}{\binom{\theta}{x}}, \quad \theta = m + 1, m + 2, \dots; \quad \max\{0, r + m - \theta\} \le x \le \min\{m, r\},$$

 $g(\theta) = \theta$  and n = 1.

- (e) Double exponential: pdf  $f_X(x;\theta) = 2^{-1} \exp\{-|x-\theta|\}$ ; with  $x \in \mathbb{R}$  and  $\theta \in \mathbb{R}$ .
- (f) Uniform $(\alpha, \beta)$ , and  $g(\theta) = \alpha + \beta$ .
- (g) Normal $(\theta, \theta^2)$ , and  $g(\theta) = \theta$ .
- (h) Inverse Gaussian( $\theta_1, \theta_2$ ) and  $g(\boldsymbol{\theta}) = (\theta_1, \theta_2)$ .
- (i) Uniform $(\theta, \theta + |\theta|), \theta \in \mathbb{R}$  and  $g(\theta) = \theta$ .
- (i) Weibull $(\theta, k)$  distribution with pdf as follows

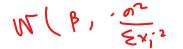
$$f_X(x;\theta,k) = \frac{k}{\theta} \left(\frac{x}{\theta}\right)^{k-1} e^{-(x/\theta)^k}, \qquad x \ge 0, \quad \theta, k > 0,$$

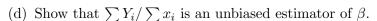
and  $q(\theta) = \theta^k$ .

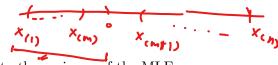
3. Suppose that the random variables  $Y_1, \ldots, Y_n$  satisfy  $Y_i = \beta x_i + \epsilon_i$ ,  $i = 1, \ldots, n$ , where  $x_1, \ldots, x_n$  are fixed constants, and  $\epsilon_1, \ldots, \epsilon_n$  are iid N  $(0, \sigma^2), \sigma^2$  unknown.

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- (a) Find a two-dimensional sufficient statistic for  $(\beta, \sigma^2)$ .
- (b) Find the MLE of  $\beta$ , and show that it is an unbiased estimator of  $\beta$ .
- (c) Find the distribution of the MLE of  $\beta$ .







- (e) Calculate the exact variance of  $\sum Y_i / \sum x_i$  and compare it to the variance of the MLE.
- (f) Show that  $\left[\sum_{i} (Y_i/x_i)\right]/n$  is also an unbiased estimator of  $\beta$ .
- (g) Calculate the exact variance of  $\left[\sum (Y_i/x_i)\right]/n$  and compare it to the variances of the estimators in the previous two estimates.
- 4. Suppose n independent observations are taken from a random variable X with distribution  $normal(\mu, 1)$ , but instead of recording all the observations, one notes only whether or not the observation is less than 0. If m observations are less than 0, then find the MLE of  $\mu$ .
- than 0. If m observations are less than 0, then find the MLE of  $\mu$ .

  5. Let  $X_1, \dots, X_n$  be a random sample from  $normal(0, \sigma^2)$  distribution, where  $\sigma^2 > 2$ . Find the MLE of  $\sigma^2$ . Is it a consistent estimator of  $\sigma^2$ ?
- σ². Is it a consistent estimator of σ²?
  6. Let X<sub>i,j</sub>, i = 1,...,s and j = 1,...,n be independent random variables, where X<sub>i,j</sub> is distributed as normal(μ<sub>i</sub>, σ²), i = 1,...,s.
  - (a) Find MLEs for  $\mu_1, \ldots, \mu_s$ , and  $\sigma^2$ .
  - (b) Show that the MLE for  $\sigma^2$  is not consistent as  $s \to \infty$ , if n is fixed.
- 7. Let  $X_1, \dots, X_n$  be a sample from the PDF

$$f_{\theta}(x) = \begin{cases} 2x/(\alpha\theta), & \text{if } 0 \le x \le \theta, \\ 2(\alpha - x)/\{\alpha(\alpha - \theta)\}, & \text{if } \theta \le x \le \alpha, \\ 0 & \text{otherwise,} \end{cases}$$

where  $\alpha > 0$  is known. Show that the MLE of  $\theta$  has to be one of the observations.

- 8. Let  $\mathbf{X}_1, \dots, \mathbf{X}_n$  be a random sample from a multivariate normal distribution  $N_k(\boldsymbol{\mu}, \sigma^2 I)$  where I is the identity matrix of order k. Find MLEs of  $\boldsymbol{\mu}$  and  $\sigma^2$ .
- 9. Let  $X_1, \dots, X_n$  be iid with one of the two pdfs. If  $\theta = 0$ , then

$$f(x \mid \theta) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

while if  $\theta = 1$ , then

$$f(x \mid \theta) = \begin{cases} 1/(2\sqrt{x}) & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find a MLE of  $\theta$ .

10. Let X and Y be independent random variables having exponential distributions with expectations  $\lambda$  and  $\mu$ , respectively. Define

$$Z = \min\{X, Y\},$$
 and  $W = \begin{cases} 1 & \text{if } Z = X, \\ 0 & \text{if } Z = Y. \end{cases}$ 

Let  $(Z_1, W_1), \dots, (Z_n, W_n)$  be n iid observations on (Z, W). Based on these samples, find MLEs of  $\lambda$  and  $\mu$ .

- 11. Given a random sample  $X_1, \dots, X_n$  from a population with pdf  $f(x \mid \theta)$ , show that maximizing the likelihood function,  $L(\theta \mid \mathbf{x})$ , as a function of  $\theta$  is equivalent to maximizing  $l(\theta \mid \mathbf{x}) = \log L(\theta \mid \mathbf{x})$ .
- 12. A density function  $f_x$  is called *unimodal* or *log-concave* if  $\log f_x$  is a concave function.

- (a) Let  $X_1, \dots, X_n$  be iid with density  $f(x \theta)$ . Show that the likelihood function has a unique root if f'(x)/f(x) is monotone, and the root is a maxima if f'(x)/f(x) is decreasing. Hence, the densities that are yield unique MLEs.
- (b) Let  $X_1, \dots, X_n$  be iid positive random variables (or, symmetrically distributed about zero) with common pdf  $f_X(x) = af(ax)$ , a > 0. Show that the likelihood equation has a unique maxima if xf'(x)/f(x) is strictly decreasing for x > 0.
- (c) If  $X_1, \dots, X_n$  are iid with density  $f(x_i \theta)$  where f is unimodal and the likelihood equation has a unique root. Show that the likelihood equation also has a unique root if the density of each  $X_i$  is  $af[a(x_i \theta)]$ , with a > 0 known.
- 13. If  $X_1, \dots, X_n$  are iid with density  $f(x_i \theta)$  or  $\sigma f(\sigma x_i)$ , and f is the logistic density as follows

$$f(u) = \frac{\exp\{-x\}}{[1 + \exp\{-x\}]^2}, \quad x \in \mathbb{R}.$$

Find the MLEs of  $\widehat{\theta}_{\text{ML}}$  and  $\widehat{\sigma}_{\text{ML}}$  of  $\theta$  and  $\sigma$ . Find the limiting distributions of  $\sqrt{n}\left(\widehat{\theta}_{\text{ML}} - \theta\right)$  and  $\sqrt{n}\left(\widehat{\sigma}_{\text{ML}} - \sigma\right)$ .

$$\frac{f'(0)}{f(0)} = \frac{-e^{-x}}{(1+e^{-x})^2} + \frac{2 \cdot e^{-x} \cdot e^{-x}}{(1+e^{-x})^3}$$

$$g(u) = \frac{f(4u)}{f(u)} - 1 + \frac{2e^{-x}}{(1+e^{-x})} = -1 + \frac{2}{1+e^{-x}}$$