Quiz-3

Elementary Stochastic Process (MTH-212A)

Name (Roll Number):

No extra sheet will be provided or collected, **Time 20 mins.**, **Max. Marks: 15**.

1. Suppose a finite Markov Chain with state space $\{1, 2, \ldots, M\}$ have the following transition probability matrix; $\boldsymbol{P} = ((p_{ij}))$, where $p_{ij} = \frac{1}{M}$, for $1 \leq i, j \leq M$. (a) Find the equivalent classes. (b) Find the period of each state. (c) Find $\sum_{n=2}^{\infty} f_{11}^{n}$. (d) Find $\lim_{n\to\infty} p_{11}^{n}$. (2+2+2+2 = 8)

Solution: Observe $\boldsymbol{P}^n = \boldsymbol{P}$. Since $p_{ij} > 0$, for all $1 \leq i, j \leq M$, hence all the states communicate with each other, and each has period one. $\lim_{n\to\infty} p_{11}^{n} = 1/M$. All the states are recurrent, hence $\sum_{n=2}^{\infty} f_{11}^{n} = 1 - 1/M$

$$P = \begin{bmatrix} \frac{1}{M} & \cdots & \\ \vdots & & \\ & & \frac{1}{M} \end{bmatrix}_{M\times M}$$

$(P^n = P)$

(a) Equivalent class: irreducible

1 $\{1, 2, \ldots, M\} \equiv S$

(b) period = 1

(c) $f_{11}^{(n)} = \mathbb{P}(X_n = 1, X_{n-1} \neq 1, \ldots, X_1 \neq 1 \mid X_0 = 1)$

$$\boxed{p_{ii}^{n} = \frac{1}{M}} \Rightarrow \sum_{n=1}^{\infty} p_{ii}^{n} = \infty \Rightarrow \text{all states are recurrent}$$

$$\Rightarrow \sum_{n=1}^{\infty} f_{ii}^{n} = 1$$

$$\Rightarrow \sum_{n=2}^{\infty} f_{ii}^{n} = 1 - \frac{1}{M}$$

2. Suppose a finite Markov Chain with state space $\{1, 2, \ldots, M\}$, where $M > 2$, have the following transition probability matrix;

Equivalence class $= \{i\}$ unique

all integers in a set.

$$\sum p_{ii}^{k} < \infty \qquad = \infty$$

calculate f_{ii}^{∞}

$$
\begin{array}{c|c}
 & \begin{matrix} 1 & 2 & \cdots & \cdots & M \end{matrix} \\
\begin{matrix} 1 \\ 2 \\ \vdots \\ \\ \\ M \end{matrix} &
\boldsymbol{P} = \begin{bmatrix}
\frac{1}{M} & \frac{1}{M} & \cdots & \frac{1}{M} & \frac{1}{M} \\
0 & \frac{1}{M-1} & \cdots & \frac{1}{M-1} & \frac{1}{M-1} \\
0 & 0 & \frac{1}{M-2} & \cdots & \frac{1}{M-2} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & \cdots & 0 & 1
\end{bmatrix}.
\end{array}
$$

Find the recurret and transient states. Find $\lim_{n\to\infty} p_{kk}^n$, for $k = 1, \ldots, M$. (3+4=7)

#M1

Solution: Note that $p_{kk}^n = 1/(M-k+1)^n$, for $k = 1, \ldots, M$. Hence, all the states are transient except the state 'M', as $\sum_{n=1}^{\infty} p_{kk}^n < \infty$, for $k = 1, \ldots, M-1$, and $\sum_{n=1}^{\infty} p_{MM}^n = \infty$. $\lim_{n\to\infty} p_{kk}^n = 0$ if $k = 1, \ldots, M-1$, and it is 1, for $k = M$.

#M2 Let us calculate

$$f_{22}^{1} = \frac{1}{M-1}; \quad f_{22}^{2} = 2 \to \underset{(i>2)}{i} \to 2 = 0 \Rightarrow f_{22}^{t} = 0 \ \forall t \geq 2$$

$$\begin{cases} f_{11}^{1} = \frac{1}{M} & 1 \to \underset{\neq 1}{i} \to 1 \\ f_{11}^{2} = 0 \Rightarrow f_{ii}^{t} = 0 \ \forall t \geq 2 \end{cases}$$

$$f_{ii} = \sum f_{ii}^{n} = \frac{1}{M} < \frac{1}{2} < 1 \Rightarrow \text{transient}$$

Thus, by telescoping: $f_{ii} = p_{ii} < 1$ for $i \neq M$

$\Rightarrow$ $i \neq M$: transient

$i = M$: recurrent.