Home Work 4
MTH-212M/MTH-412A
Elementary Stochastic Process

1. Consider the two dimensional random walk with the full infinite plane. Prove that if all the probabilities are equal, i.e. starting from any state going up, down, forward and backward are all $1/4$, then it is a recurrent Markov Chain.

2. Consider the following one dimensional random walk, where $S = \{0, \mp 1, \mp 2, \ldots\}$, and $P(X_{n+1} = i+1 | X_n = i) = P(X_{n+1} = i-1 | X_n = i) = 1/4$, and $P(X_{n+1} = i | X_n = i) = 1/2$. Find the equivalent classes. Show that it is a recurrent Markov Chain.

3. Consider the following transition probability matrix

$$\mathbf{P} = \begin{bmatrix} p_0 & 1-p_0 & 0 & 0 & 0 & 0 & \ldots \\ p_1 & 0 & 1-p_1 & 0 & 0 & 0 & \ldots \\ p_2 & 0 & 0 & 1-p_2 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Here $0 < p_i < 1$, for all i. Show that $\sum_{n=1}^{m+1} f_{00}^n = 1 - u_m$, where

$$u_m = \begin{cases} \prod_{i=0}^{n}(1-p_i) & \text{if} \quad n \geq 0 \\ 1 & \text{if} \quad n = -1 \end{cases}$$

Prove that $\prod_{i=0}^{n}(1-p_i) \leq e^{-\sum_{i=0}^{m} p_i}$. Show that if $\sum_{i=0}^{\infty} p_i = \infty$, then it is a recurrent Markov Chain.

4. Consider the following transition probability matrix with the state space $S = \{0, 1, 2, 3, 4, 5\}$

$$P = \begin{bmatrix} 1/3 & 2/3 & 0 & 0 & 0 & 0 \\ 2/3 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 3/4 & 0 & 0 \\ 0 & 0 & 1/5 & 4/5 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 & 1/4 & 1/4 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{bmatrix}$$

Find the recurrent and transient classes. Find the probability that starting from the transient state i, it is going to get absorbed in a given recurrent class C_j, for $j = 1, \ldots, K$.

5. Consider the following transition probability matrix with the state space $S = \{0, 1, 2, 3\}$

$$P = \begin{bmatrix} 1/6 & 1/3 & 1/2 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/6 & 1/3 & 1/2 & 0 \\ 0 & 1/6 & 1/3 & 1/2 \end{bmatrix}$$

Find the recurrent and transient classes. Find the probability that starting from the transient state i, it is going to get absorbed in a given recurrent class C_j, for $j = 1, \ldots, K$.

6. A Markov Chain on a finite state space, for which all the recurrent states are absorbing, is called an absorbing chain. Show that for a finite absorbing chain the transition probability matrix P can be written as follows:

$$P = \begin{bmatrix} I & 0 \\ R_1 & Q \end{bmatrix}$$

Find P^n.