Problem_set1_2024

Gamma(α,β): $\frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta x} x^{\alpha-1}$ (b) $: x \geq 0$

(1) $\mathbb{E}X = \int_0^\infty x f(x)\,dx = \sqrt{\frac{2}{\pi}} \int x e^{-\frac{x^2}{2}} dx \Rightarrow -\frac{x^2}{2} = t \Rightarrow -dt = x\,dx$; $\sqrt{\frac{2}{\pi}} \int e^{t}\, dt = \sqrt{\frac{2}{\pi}}\,[1] = \sqrt{\frac{2}{\pi}}$

(2). $\mathbb{E}X^2 = \int_0^\infty x^2 f(x)\,dx = \sqrt{\frac{2}{\pi}} \int_0^\infty x^2 e^{-\frac{x^2}{2}} dx \Rightarrow \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{2t}\, e^{-t}\, dt = \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) = \frac{2}{\sqrt{\pi}} \times \frac{1}{2} \times \sqrt{\pi}$ ($\frac{x^2}{2} = t$)

$\Rightarrow Var(X) = \left(1 - 2/\pi\right)$

$\star\ \boxed{\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx} = 1$

MTH211A: Theory of Statistics

Problem set 1

1. Let the random variable X have pdf

$$f(x) = \sqrt{\frac{2}{\pi}} \exp\{-x^2/2\}, \qquad x > 0.$$

   (a) Find $E(X)$ and $\mathrm{var}(X)$.
   (b) Find an appropriate transformation $Y = g(X)$ and α, β, so that $Y \sim \mathtt{Gamma}(\alpha, \beta)$.

2. Let X is distributed as $\mathtt{Gamma}(\alpha, \beta)$ distribution, $\alpha, \beta > 0$. Then show that the r-th order population moment

$$E\left(X^r\right) = \beta^{-r} \frac{\Gamma(r+\alpha)}{\Gamma(\alpha)}, \qquad r > -\alpha.$$

3. Let the bivariate random variable (X, Y) has a joint pdf

$$f_{X,Y}(x,y) = \begin{cases} C(x+2y) & \text{if } \ 0 < y < 1,\ 0 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the marginal distribution of Y.
   (b) Find the conditional distribution of Y given $X = 1$.
   (c) Compare the expectations of the above two distributions of Y.
   (d) Find the covariance between X and Y.
   (e) Find the distribution of $Z = 9/(2Y+1)^2$.
   (f) What is $P(X > Y)$?

4. Let $X \sim \mathtt{normal}(0,1)$. Define $Y = -X\mathbb{I}(|X| \leq 1) + X\mathbb{I}(|X| > 1)$.
   (a) Find the distribution of Y (Hint: Start by finding the CDF of Y).
   (b) Prove or disprove: The distribution of (X, Y) is bivariate normal. (Hint: Argue that the distribution of $X + Y$ is not continuous.)

5. Let $X \sim \mathtt{normal}(0,1)$. Define $Y = \mathrm{sign}(X)$ and $Z = |X|$. Here $\mathrm{sign}(\cdot)$ is a $\mathbb{R} \to \{0,1\}$ function such that $\mathrm{sign}(a) = 1$ if $a \geq 0$, and $\mathrm{sign}(a) = -1$ otherwise.
   (a) Find the marginal distributions of Y and Z.
   (b) Find the joint CDF of (Y, Z). Hence or otherwise prove that Y and Z are independently distributed.

6. Suppose the distribution of Y, conditional on $X = x_0$ is $\mathtt{normal}(x_0, x_0^2)$, and the marginal distribution of X is $\mathtt{uniform}(0,1)$. Show that $Z = Y/X$ and X are independently distributed. Find the distribution of Y/X.

(2) $X \sim$ Gamma(α,β) $: \alpha, \beta > 0$
r^{th} order population moment $= \mathbb{E}(X^r) = \mu_r'$

$\mu_r' = \int_0^\infty x^{(r+\alpha)-1} \times \frac{\beta^\alpha e^{-\beta x}}{\Gamma(\alpha)}\, dx = \frac{\Gamma(r+\alpha)}{\beta^r \Gamma(\alpha)}$ $\quad r > -\alpha$

(3) $\int_0^1 \int_0^2 c(x+2y)\, dx\, dy = c\int_0^1 (2+4y)\,dy = 1 \Rightarrow c = 1/4$ ✓

(a) $f_Y(y) = \int_0^2 \frac{1}{4}(x+2y)\,dx = \frac{1}{4}(2+4y)$ $\quad$ (b) Given $X=1$: $f_{Y|X=1}(y) = \frac{c\,(1+2y)}{c\,[2]} =$

$f_X(x) = \int_0^1 c(x+2y)\,dy = c\,[x+1]$ ✓

(5) (a) : $X \sim N(0,1)$: $Y = \mathrm{sign}(X)$

$S_Y = \{1, -1\}$

$f_Y(1) = \mathbb{P}(Y=1) = \mathbb{P}(X \geq 0) = 1/2$

$\mathbb{P}(Y=-1) = 1/2$

$\Rightarrow f(y) = \begin{cases} 1/2 & , y = -1 \\ 1/2 & , y = 1 \\ 0 & , \text{o.w.} \end{cases}$

5 (a) for (Z) : $Z = |X| \Rightarrow F_Z(z) = \mathbb{P}(Z \leq z) = \mathbb{P}(|X| \leq z) = \mathbb{P}(-z \leq X \leq z) = \int_{-z}^{z} f_X(a)\, da$

$\Rightarrow f_Z(z) = F_Z' = 2 f_X(z)$ $\quad$ $\underline{z \in [0, \infty)}$

(b) joint cdf of (Y, Z): $F_{Y,Z}(y,z) = \mathbb{P}(\mathrm{sign}(X) \leq y,\ |X| \leq z) = \begin{cases} 0 & y < -1 \\ \mathbb{P}(-z \leq X < 0) & 1 > y \geq -1 \\ \mathbb{P}(-z \leq X \leq z) & y \geq 1 \end{cases}$

$\Rightarrow F_{Y,Z}(y,z) = \begin{cases} 0 & y < -1 \\ \int_{-z}^{0} f_X(x)\,dx & 1 > y \geq -1 \\ \int_{-z}^{z} f_X(a)\,da & y \geq 1 \end{cases} =$

(6) $Y \mid x_0$ follows $N(x_0, x_0^2)$ $\Rightarrow$ $\because x_0 \in (0,1)$ $\Rightarrow$ $f_{Y,X}(y, x_0) = \frac{1}{x_0\sqrt{2\pi}} e^{-\frac{(y-x_0)^2}{2x_0^2}}$ $\quad \forall x_0 \in [0,1]$

joint pmf

X follows $U(0,1)$

Now, $(X, Y) \rightarrow (\overset{U}{X}, \overset{V}{\tfrac{Y}{X}})$

$x = u,\ y = vu$

$f_{U,V}(u,v) = f_{X,Y}(u, uv) \begin{vmatrix} 1 & 0 \\ v & u \end{vmatrix}$ $\quad |u|$

$f_{X, \frac{Y}{X}}(u,v) = \frac{\cancel{u}\ 1}{\cancel{u}\sqrt{2\pi}} e^{-\frac{(v-1)^2}{2}} = f_X(u) \times f_{\frac{Y}{X}}(v)$

Thus, independent distribution + $\frac{Y}{X} = Z \sim N(1,1)$

(7) (a) $(X,Y) \sim N_2(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$ $\leftarrow$ uncorrelated $\hookrightarrow \rho = 0$ $\quad$ To show: $X \perp Y$ are independent.

Thus, $f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{-\frac{1}{2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right\} \Rightarrow f_X(x) f_Y(y)$

$X \sim N(\mu_x, \sigma_x^2)$, $Y \sim N(\mu_y, \sigma_y^2)$

$\because$ independence $\Rightarrow$ zero correlation (but reverse only in multivariate normal)

(b) $\underset{\sim}{X} \sim N_k(\underset{\sim}{\mu}, \Sigma)$ with $\Sigma = \mathrm{Diag}(\sigma_1^2, \dots, \sigma_k^2)$, $\forall \sigma_j > 0$ $\Rightarrow \rho_{ij} = 0\ \forall i \neq j$

(8) $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$ $\quad U := (a_1X_1 + \dots + a_nX_n)$ & $V := (b_1X_1 + \dots + b_nX_n)$ $\quad$ To show $Z := \alpha U + \beta V \sim N(\gamma, \delta)$

$\mathbb{E}e^{tX} = e^{\mu t + \frac{\sigma^2 t^2}{2}}$ (aX) using mgf

Now, Mgf $Z = \mathbb{E}e^{tZ} = \mathbb{E}e^{t(\alpha U + \beta V)} = \mathbb{E}\, e^{t\left(\sum_{i=1}^{n}(\alpha a_i + \beta b_i)X_i\right)}$ $\quad \because X_i$'s are iid $\Rightarrow$ Mgf$_Z$ =

$\Rightarrow Z \sim N(\mu K$

7. (a) Let (X, Y) is jointly distributed as $\mathtt{N}_2(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$. Suppose (X, Y) are uncorrelated in the sense that $\mathrm{cov}(X,Y) = E\left[\{X - E(X)\}\{Y - E(Y)\}\right] = 0$. Then show that X and Y are independently distributed.
   (b) Let $\mathbf{X} = (X_1, \cdots, X_k)^\top$ is distributed as a k-variate normal distribution with parameters $\boldsymbol{\mu}$ and $\Sigma = \mathrm{Diag}\left(\sigma_1^2, \cdots, \sigma_k^2\right)$, $\sigma_j > 0$ for all $j = 1, \ldots, k$. Show that $X_1, \cdots, X_k$ are mutually independent. ?

8. Let $X_i \overset{\text{IID}}{\sim}$ normal (μ, σ^2). Define $U = (a_1X_1 + \cdots + a_nX_n)$ and $V = (b_1X_1 + \cdots + b_nX_n)$.
   (a) Show that (U, V) jointly follow a bi-variate normal distribution. Identify the parameters of the distribution. (Hint: For any $\mathbf{c} \in \mathbb{R}^2$, show that $\mathbf{c}^\top \begin{bmatrix} U \\ V \end{bmatrix}$ has an univariate normal distribution.)
   (b) Find conditions on $\mathbf{a}$ and $\mathbf{b}$ such that U and V are independently distributed. (Hint: Apply problem 7.) $\left(\sum a_i b_i = 0\right)$

9. Suppose $X_1, \cdots, X_n \overset{\text{IID}}{\sim} \mathtt{normal}\,(\mu_x, \sigma^2)$, $Y_1, \cdots, Y_m \overset{\text{IID}}{\sim} \mathtt{normal}\,(\mu_y, \sigma^2)$, and all the random variables $\{X_1, \cdots, X_n, Y_1, \cdots, Y_m\}$ are mutually independent. Then find the distribution of $T := S_X^{*2}/S_Y^{*2}$, where S_X^{*2} and S_Y^{*2} are the unbiased sample variances of X and Y, respectively.

10. Let $X_1, \cdots, X_n$ be iid random variables with continuous CDF F_X, and suppose $E(X_1) = \mu$. Define the random variables $Y_1, \cdots, Y_n$ as follows:

$$Y_i = \begin{cases} 1 & \text{if} X_i > \mu \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find $E(Y_1)$.
   (b) Find the distribution of $\sum_{i=1}^{n} Y_i$.

(9) $T := \frac{S_X^{*2}}{S_Y^{*2}}$

we know, $S_X^{*2} = \frac{n S_X^2}{n-1}$

$S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i^2 - \bar{x}_n^2)$

(b) U, V are independent iff $\mathrm{Corr}(U,V) = 0$ $\quad (\because (U,V)$

Now, $E(XY) = E(X)E(Y)$

$U \sim N(\mu \sum a_i,\ \sigma^2 \sum a_i^2) \rightarrow E(U) = \mu \sum a_i$

$V \sim N(\mu \sum b_i,\ \sigma^2 \sum b_i^2) \rightarrow E(V) = \mu \sum b_i$

$E(UV) = E\left(\left(\sum a_i x_i\right)\left(\sum b_i x_i\right)\right) = \sum a_i b_i + \sum_{i \neq j}$

Equating (green + blue) $\Rightarrow \sigma^2 \sum a_i b_i = 0 \Rightarrow \boxed{\sum a_i b_i = 0}$ $\quad = \sum a_i b_i (\mu$

$\frac{1+2y}{2}$

$= f_Y(y)$

$y(k)t + \frac{\lambda t^2 \sigma^2}{2}$

$, \sigma^2 \lambda)$

$\Rightarrow \mathrm{Cov}(U, V) = 0$

bivariate
normal)

$\mathbb{E}X_i^2$

$a_i b_j (\mathbb{E}X_i)(\mathbb{E}X_j)$

$^2 + \sigma^2) + \sum_{i \neq j} a_i b_j \mu^2$

Equating ⇒ (green + blue) $\sigma^2 \sum a_i b_i = 0 \Rightarrow \boxed{\sum a_i b_i = 0}$ $= \sum a_i b_i (\ldots$ $+ \sum_{i \neq j}$

10. Let $X_1, \cdots, X_n$ be iid random variables with continuous CDF F_X, and suppose $E(X_1) = \mu$. Define the random variables $Y_1, \cdots, Y_n$ as follows:

$$Y_i = \begin{cases} 1 & \text{if } X_i > \mu \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find $E(Y_1)$.
   (b) Find the distribution of $\sum_{i=1}^n Y_i$.

11. Let $X_1, \cdots, X_n \overset{\text{IID}}{\sim} \mathtt{normal}(\mu, \sigma^2)$, and S_n^2 be the sample variance. Find a function of S_n^2, say $g(S_n^2)$, which satisfies $E[g(S_n^2)] = \sigma$. (Hint: You may use problem 2.)

12. Let $\mathbf{X}_1, \cdots, \mathbf{X}_n$ be a random sample from a k-variate distribution F, where $\mathbf{X}_i = (Y_{1,i}, \cdots, Y_{k,i})^\top$. Let the expectation $E(\mathbf{X}_1) = \boldsymbol{\mu}$ and the variance covariance matrix of F, $\Sigma = E\left\{(\mathbf{X}_1 - \boldsymbol{\mu})(\mathbf{X}_1 - \boldsymbol{\mu})^\top\right\}$, have all finite components, and $S_n = n^{-1}\sum_{i=1}^n (\mathbf{X}_i - \bar{\mathbf{X}}_n)(\mathbf{X}_i - \bar{\mathbf{X}}_n)^\top$ be the sample variance-covariance matrix, where $\bar{\mathbf{X}}_n = n^{-1}\sum_{i=1}^n \mathbf{X}_i$. Show that
   (a) $\bar{\mathbf{X}}_n = (\bar{Y}_1, \cdots, \bar{Y}_k)^\top$
   (b) For all $j, l = 1, \ldots, k$, the (j,l)-th component of Σ is $\sigma_{j,l} = E[\{Y_j - E(Y_j)\}\{Y_l - E(Y_l)\}]$. Also, for all $j, l = 1, \ldots, k$, the (j,l)-th component of S_n is $S_{j,l} = n^{-1}\sum_{i=1}^n (Y_{j,i} - \bar{Y}_j)(Y_{l,i} - \bar{Y}_l)$.
   (c) $E(\bar{\mathbf{X}}_n) = \boldsymbol{\mu}$ and $\text{var}(\bar{\mathbf{X}}_n) = E\left\{(\bar{\mathbf{X}}_n - \boldsymbol{\mu})(\bar{\mathbf{X}}_n - \boldsymbol{\mu})^\top\right\} = n^{-1}\Sigma$.
   (d) $E(nS_n) = (n-1)\Sigma$.

13. Let $X_1, \cdots, X_n$ be iid with pdf f_X and CDF F_X, and let $r < s$. Then find the conditional distribution of $X_{(r)}$ given $X_{(s)} = y$ in terms of f_X and F_X. In particular, if $X_1, \cdots, X_n$ is a random sample from $\mathtt{uniform}(0,1)$ distribution, then can you identify the conditional distribution of $X_{(r)}$ given $X_{(s)} = y$?

14. Let $X_1, \cdots, X_n$ be iid with pdf f_X and CDF F_X. Find the CDF of r-th order statistics $X_{(r)}$. Hence derive the pdf of $X_{(r)}$.

15. Let Y have a $\mathtt{Cauchy}(0,1)$ distribution.
   (a) Find the CDF of Y.
   (b) Hence provide a method of simulating random samples from $\mathtt{Cauchy}(0,1)$ distribution, starting from $\mathtt{uniform}(0,1)$ random variables.

(10) $\{X_i\}_n \overset{iid}{\sim} X : F_X$ & $\mathbb{E}(X_1) = \mu$

$$Y_i = \begin{cases} 1 & \text{if } X_i > \mu \\ 0, & \text{o.w.} \end{cases}$$

(a) $\mathbb{E}(Y_1) = 1 \cdot \mathbb{P}(X_1 > \mu) = 1 - \mathbb{P}(X_1 \leq \mu)$
$= 1 - F_X(\mu)$

(b) distribution of $\sum Y_i = Z$

$\because$ Y_i's are independent $\Rightarrow$ $MgF_Z = \mathbb{E}e^{t(Y_1 + Y_2 + \cdots}$

$\Rightarrow M_Z(t) = [M_Y(t)]^n = ((1-p)e^t + p)^n \Rightarrow Z \sim$ $p =$

$M_Y(t) = \mathbb{E}e^{tY} = e^t \cdot (1 - F_X(\mu)) + F_X($

#way 2 : $Z = \sum Y_i$: discrete $\Rightarrow S_Z = \{0, 1, \ldots, n\}$

$f_Z(z) = \mathbb{P}[$ z of Y_i's are 1, else are zero $] = {}^nC_z [1 - F_X(\mu)]^z [F_X(\mu)]^z$

$Z \sim \text{Binomial}[n, 1-p]$

(11) $g(S_n^2)$ s.t. $\mathbb{E}[g(S_n^2)] = \sigma$

$S_n^2 = \frac{1}{n}\sum (X_i - \bar{X}_n)^2$ $\Rightarrow$ we know, $\mathbb{E}\left(\frac{nS_n^2}{n-1}\right) =$

#M1 $(+\mu, -\mu) \Leftarrow$ No benefit → all corresponding contrib

$= \frac{1}{n-1}\mathbb{E}\left(\sum_{i=1}^{n}(X_i - \bar{X}_n)^2\right) = \frac{1}{n-1}\sum_{i=1}^{n}\left[\mathbb{E}(\cancel{X_i - \mu})^2 + \mathbb{E}(\cancel{\mu - \bar{X}_n})^2 + 2\mathbb{E}(X_i - \mu)\ldots\right.$

$= \frac{1}{n-1}\left[n\sigma^2 + n\mu^2 + n\mathbb{E}\bar{X}_n^2 - 2n\mu^2\right] = \frac{1}{n-1}[(n$
$+ 2n\mathbb{E}\left[\cancel{\mu X_1 - \mu^2} - X_i\bar{X}_n + \mu\bar{X}_n\right]$

$\mathbb{E}[\bar{X}_n^2] = \text{Var}(\bar{X}_n) + [\mathbb{E}\bar{X}_n]^2 = \mu^2$
$= \frac{1}{n^2}\sum \text{Var}(X_1) = \frac{\sigma^2}{n}$ ✓

#M2 $\mathbb{E}\left(\frac{nS_n^2}{n-1}\right) = \mathbb{E}\left[\frac{1}{n-1}\left(\sum X_i^2 - n\bar{X}_n^2\right)\right] = \frac{n}{n-1}\mathbb{E}(X_1^2) - \frac{n}{n-1}\mathbb{E}(\bar{X}_n^2) = \frac{n}{n-1}$ $\sigma^2 + \mu^2$ $\frac{\sigma^2}{n} + \mu^2$

7. (a) Let (X, Y) is jointly distributed as $\mathtt{N}_2(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$. Suppose (X, Y) are uncorrelated in the sense that $\text{cov}(X, Y) = E[\{X - E(X)\}\{Y - E(Y)\}] = 0$. Then show that X and Y are independently distributed.
   (b) Let $\mathbf{X} = (X_1, \cdots, X_k)^\top$ is distributed as a k-variate normal distribution with parameters $\boldsymbol{\mu}$ and $\Sigma = \text{Diag}(\sigma_1^2, \cdots, \sigma_k^2)$, $\sigma_j > 0$ for all $j = 1, \ldots, k$. Show that $X_1, \cdots, X_k$ are mutually independent.

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   (a) $\bar{\mathbf{X}}_n = (\bar{Y}_1, \cdots, \bar{Y}_k)^\top$
   (b) For all $j, l = 1, \ldots, k$, the (j,l)-th component of Σ is $\sigma_{j,l} = E[\{Y_j - E(Y_j)\}\{Y_l - E(Y_l)\}]$. Also, for all $j, l = 1, \ldots, k$, the (j,l)-th component of S_n is $S_{j,l} = n^{-1}\sum_{i=1}^n (Y_{j,i} - \bar{Y}_j)(Y_{l,i} - \bar{Y}_l)$.
   (c) $E(\bar{\mathbf{X}}_n) = \boldsymbol{\mu}$ and $\text{var}(\bar{\mathbf{X}}_n) = E\left\{(\bar{\mathbf{X}}_n - \boldsymbol{\mu})(\bar{\mathbf{X}}_n - \boldsymbol{\mu})^\top\right\} = n^{-1}\Sigma$.
   (d) $E(nS_n) = (n-1)\Sigma$.

13. Let $X_1, \cdots, X_n$ be iid with pdf f_X and CDF F_X, and let $r < s$. Then find the conditional distribution of $X_{(r)}$ given $X_{(s)} = y$ in terms of f_X and F_X. In particular, if $X_1, \cdots, X_n$ is a random sample from $\mathtt{uniform}(0,1)$ distribution, then can you identify the conditional distribution of $X_{(r)}$ given $X_{(s)} = y$?

14. Let $X_1, \cdots, X_n$ be iid with pdf f_X and CDF F_X. Find the CDF of r-th order statistics $X_{(r)}$. Hence derive the pdf of $X_{(r)}$.

15. Let Y have a $\mathtt{Cauchy}(0,1)$ distribution.
   (a) Find the CDF of Y.
   (b) Hence provide a method of simulating random samples from $\mathtt{Cauchy}(0,1)$ distribution, starting

$k \times 1$ $1 \times k$

(b) $j, l = 1, \ldots, k$.

(12) $\underset{\sim}{X}_i = \begin{pmatrix} Y_{1,i} \\ \vdots \\ Y_{k,i} \end{pmatrix}$ $E(\underset{\sim}{X}_1) = \underset{\sim}{\mu}$; X

$\Sigma = \mathbb{E}\{(\underset{\sim}{X}_1 - \underset{\sim}{\mu}) \ldots$

$S_n = \frac{1}{n}\sum(\underset{\sim}{X}_i - \bar{X}_n)(\underset{\sim}{X}_i - \bar{X}_n)$ ⟵

$\bar{X}_n = \frac{1}{n}\sum \underset{\sim}{X}_i$.

(a) $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n}(Y_{1,i}, \ldots, Y_{k,i})^T = (\bar{Y}_1, \ldots, \bar{Y}_k)$ ✓

$\Sigma_{(j,l)} = \mathbb{E}[\{Y_j - \mathbb{E}(Y_j)\}\{Y_l - \mathbb{E}(Y_l)\}]$

$\Sigma = \mathbb{E}\left[\begin{pmatrix} Y_1 - \mu_1 \\ \vdots \\ Y_k - \mu_k \end{pmatrix}(Y_1 - \mu_1 \quad \ldots \quad Y_k - \mu_k)\right]$

$\Rightarrow \Sigma_{(j,l)} = \mathbb{E}[(Y_j - \underbrace{\mu_j}_{\mathbb{E}(Y_j)})(Y_l - \underbrace{\mu_l}_{\mathbb{E}(Y_l)})]$

$S_{j,l} = \frac{1}{n}\sum_{i=1}^{n} T_{j,l,i}$ where $T_{i,j,l} =$

$T_{i,j,l} = (Y_{j,i} - \bar{Y}_j)(Y_{l,i} - \bar{Y}_l)$

$S_{j,l} = n^{-1}\sum_{i=1}^{n} T_{i,j,l}$ ✓

$a_i b_j \mathbb{E} X_i)(\mathbb{E} X_j)$

$^2 + \sigma^2) + \sum_{i \neq j} a_i b_j \mu^2$

μ

$Y_n)$

$^{1 - F_X(n)}$

Binom $(n, 1-p)$

$\mu)$

ing 3 terms ute smtg w the other

$\mu^2 + X_n^2 - 2\mu X_n)$

^{2}

$(\mu - \overline{X}_n)]$

$+1)\sigma^2$

$- 2\mathbb{E}(X_1^2 + X_1X_2 + \dots + X_1X_n)$

$+ 2n\mu^2]$

$-1[n+1 - 2] = \sigma^2$

$\sigma^2 - \frac{\sigma^2}{n}]$

σ^2

$\overset{iid}{\sim} F_{\underline{k}}$

$(\underset{\sim}{X}_1 - \underset{\sim}{\mu})^T\}.$

$< \infty$ $\forall$ elements.

sample covar matrix

$\frac{1}{n}(\sum Y_{1,i}, \sum Y_{2,i}, \dots, \sum Y_{k,i})^T$

$\mu_k)$

$(\underset{\sim}{X}_i - \overline{\underset{\sim}{X}}_n) \leftarrow j^{th}$

$\times (\underset{\sim}{X}_i - \overline{\underset{\sim}{X}}_n)^T$ l^{th}

derive the pdf of $X_{(r)}$.

15. Let Y have a Cauchy(0, 1) distribution.

(a) Find the CDF of Y.

(b) Hence provide a method of simulating random samples from Cauchy(0, 1) distribution, starting from uniform(0, 1) random variables.

2

$S_{j,l} = n^{-1}\sum_{i=1}^{n} T_{i,j,l}$ ✓.

(c) $\mathbb{E}(\bar{X}_n) = \mu$ & $Var(\bar{X}_n) = \mathbb{E}\{(\bar{X}_n -$

$\mathbb{E}(\bar{X}_n) = \mathbb{E}(X_1) = \mu$ ✓

$Var(\bar{X}_n) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right) \overset{\text{by independence}}{=} \frac{1}{n} Var(X_1) = \frac{1}{n}\mathbb{E}[(X_1 - \mu)(X_1-\mu)^T] = \frac{1}{n}\Sigma$

(d) $\mathbb{E}(nS_n) = (n-1)\Sigma$ ✓ $\Rightarrow n \cdot \mathbb{E}\left[n^{-1}\sum(X_i - \bar{X}_n)(X_i - \bar{X}_n)^T\right] = \cancel{n}\frac{1}{\cancel{n}}\mathbb{E}\left[\sum \begin{pmatrix} Y_{1,i} - \bar{Y}_1 \\ \vdots \\ Y_{k,i} - \bar{Y}_k \end{pmatrix}(\quad)\right]$

$\left(n\bar{Y}_1 = \sum_{i=1}^{n} Y_{1,i}\right)$ $= \cancel{n}\frac{1}{\cancel{n}}\mathbb{E}\left[\begin{pmatrix} \sum_{i=1}^{n}(Y_{1,i}-\bar{Y}_1)^2 & \cdots \\ \vdots & \\ & \sum_{i=1}^{n}(Y_{k,i}-\bar{Y}_k)^2 \end{pmatrix}\right] = \begin{bmatrix} \mathbb{E}Y_{1,i}^2 - n\mathbb{E}(\bar{Y}_1^2) & \\ & \ddots \end{bmatrix} = \begin{bmatrix} n\,\sigma_1^2 \end{bmatrix}$

(13) $f_{X_{(r)}|X_{(s)}=y}(x) = \dfrac{f_{X_{(r)},X_{(s)}}(x,y)}{f_{X_{(s)}}(x)}$ $= \dfrac{\dfrac{n!}{(r-1)!(s-r-1)!\cancel{(n-s)!}} F(x)^{r-1} f(x) \cancel{f(y)} (F(y)-F(x))^{s-r-1} \cancel{[1-F(y)]^{n-s}}}{\dfrac{n!}{(s-1)!\cancel{(n-s)!}} F(y)^{s-1} \cancel{f(y)} (1-F(y))}$

$r<s$

$= \dfrac{(s-1)!}{(r-1)!(s-r-1)!} F(x)^{r-1} f(x) (F(y) - F(x))^{s-r-1} \cdot F(y)^{-s+1}$

(14) Notes easy $\therefore \mathbb{P}(\cup A_n) = \sum \mathbb{P}(A_n)$

pw disjoint

$\mathbb{P}(X_{(r)} \le x)$

$A_n =$ exactly n elements are less

$A_{n+1} =$ exactly $n+1$ elements are l

X_i's ✓

(15) MATH210: Inverse transform ✓ $F^{-1}(U) \sim X$ ✓

$-\mu)(\overline{X}_n - \mu)^T]$

$= n^{-1}\Sigma$

$-n(\mu^2 + \frac{\sigma^2}{n})$

Σ

$\cancel{-s}$

$m_s \quad \tilde{n} = s$

$s - r - 1$

$n - s$

than x

ers than $\underline{x}$.