Lecture Notes: Jan 23, 2024

In the last lecture notes I had discussed about one-step transition probability matrix. In this lecture notes, I will be discussing about the n-th step transition probability matrix and also some of the related issues. Let $\{X_n\}$ be a Markov Chain with the state space $S = \{0, 1, 2, \ldots\}$, and the transition probability matrix $\boldsymbol{P}$. Note that $\boldsymbol{P}$ is of infinite dimension. Since it is a transition probability matrix, it means

$$p_{ij} = P(X_{n+1} = j | X_n = i); \qquad i, j = 0, 1, 2, \ldots.$$

Now let us look at the two-step transition probability matrix, i.e. $P(X_{n+2} = j | X_n = i)$, for $i, j = 0, 1, 2, \ldots$. Now

$$\begin{aligned}
P(X_{n+2} &= j | X_n = i) = \sum_{k=0}^{\infty} P(X_{n+2} = j, X_{n+1} = k | X_n = i) \\
&= \sum_{k=0}^{\infty} P(X_{n+2} = j | X_{n+1} = k, X_n = i) P(X_{n+1} = k | X_n = i) \\
&= \sum_{k=0}^{\infty} P(X_{n+2} = j | X_{n+1} = k) P(X_{n+1} = k | X_n = i) \\
&= \sum_{k=0}^{\infty} p_{kj} p_{ik} = \sum_{k=0}^{\infty} p_{ik} p_{kj}.
\end{aligned}$$

Note that the right hand side is the (i, j)-th element of the matrix $\boldsymbol{P}^2$. Along the same line show that $\boldsymbol{P}^n$ is the n-th step transition probability matrix. Show that a $m + n$-step transition probability matrix can be written as a product of m-step and n-step transition probability matrices.

Now we will be discussing one important concept of a Markov Chain. It is called the classification of states of a Markov Chain. In a Markov Chain $\{X_n\}$ a state 'j' is said to be accessible from a state 'i', if there exists an integer $n \geq 0$, such that $P(X_{n+1} = j | X_1 = i) > 0$. Note that since $P(X_1 = i | X_1 = i) = 1$, for all i, hence, any state is always accessible to itself. If the state 'j' is accessible to 'i', it will be denoted by $i \to j$. If the state 'j'

is accessible to the state 'i', and the state 'i' is accessible to the state 'j', then 'i's is said to communicate with 'j', and it will be denoted by $i \leftrightarrow j$. It is clear that $i \leftrightarrow i$, for all i. It has been shown (video lecture) that this communication relation is an 'equivalent' relation, hence it divides the state space into disjoint equivalent classes.

We will try to explain with various examples. Let us remember that a Markov Chain is completely specified with the transition probability matrix and the initial distribution. Moreover the classification of states depends only on the transition probability matrix. Hence we will give this example with respect to the transition probability matrix only, we may not go back to the original Markov Chain most of the times.

Let us consider the example of the two persons zero sum game, where Player A has 'a' amount of money and Player B has 'b' amount of money, and $P(H) = p$, $P(T) = 1 - p$. Hence, in this case the state space $S = \{0, 1, 2, \ldots, a+b\}$. The transition probability matrix $\boldsymbol{P}$ is a $a+b+1 \times a+b+1$ matrix, and

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ 1-p & 0 & p & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1-p & 0 & p \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}.$$

Note that here the state '0' is accessible to the states $1, 2, \ldots, a + b - 1$, similarly, the state '$a + b$' is accessible to the states $1, 2, \ldots, a + b - 1$. The states $1, 2, \ldots, a + b - 1$ is not accessible to either the state '0' or to the state '$a + b$'. The state '0' only communicates with itself, similarly, the state '$a + b$' only communicates with itself. Where as the states $1, 2, \ldots, a + b - 1$ communicate with each other. Hence, we have the following communication relation

$$0 \leftrightarrow 0, \quad a + b \leftrightarrow a + b, \quad i \leftrightarrow j, \quad \text{if } 1 \leq i, j \leq a + b - 1.$$

Hence, in this case we have three equivalent classes, namely E_1, E_2 and E_3, where $S = E_1 \cup E_2 \cup E_3$,

$$E_1 = \{0\}, \quad E_2 = \{a + b\}, \quad E_3 = \{1, 2, \ldots, a + b - 1\}.$$

Within an equivalent class all the states communicate with each other, and the state 'i' and 'j' belong to two different equivalent classes, then they do not communicate with each other, $i \nleftrightarrow j$.

Now consider the two persons zero sum game when Player B has infinite fortune. In this case the state space becomes $S = \{0, 1, 2, \ldots\}$, and the transition probability matrix becomes

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & \ldots \\ 1-p & 0 & p & 0 & \ldots & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 1-p & 0 & p & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

In this case it is clear that there are two equivalent classes, and they are namely E_1 and E_2, where

$$E_1 = \{0\} \quad \text{and} \quad E_2 = \{1, 2, \ldots\}.$$

Suppose a Markov Chain has the state space $S = \{1, 2, \ldots, M\}$, with the following transition probability matrix

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 1-p & 0 & p & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1-p & 0 & p \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}.$$

Here all the states communicate with each other, hence there is only one equivalent class namely $E = S = \{1, 2, \ldots, M\}$. In a Markov Chain if all the states communicate with each other it is called an irreducible Markov Chain. An irreducible Markov Chain plays a very important role in a stochastic process. It has several interesting properties. Moreover, we will see that the states within the same class behave in a very similar manner.

Now do the following problems:

Problem: Suppose $\{X_n; n \geq 0\}$ is a Markov Chain with the following transition probability matrix having state space $S = \{1, 2, 3, 4\}$.

$$\boldsymbol{P} = \begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

What is (are) the equivalent classe(s)? Find $\boldsymbol{P}^n$, for all $n \geq 1$. Hence, find $\lim_{n \to \infty} p_{ij}^n$, for all $1 \leq i, j \leq 4$.

Problem: Suppose $\{X_n; n \geq 0\}$ is a Markov Chain with the following transition probability matrix having state space $S = \{1, 2, \ldots\}$.

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

What is (are) the equivalent classe(s)? Find $\boldsymbol{P}^n$, for all $n \geq 1$. Hence, find $\lim_{n \to \infty} p_{ij}^n$, for all $1 \leq i, j \leq 4$.

Problem: Suppose $\{X_n; n \geq 0\}$ is a Markov Chain with the following transition probability matrix having state space $S = \{0, \mp 1, \mp 2, \ldots\}$,

$$P(X_{n+1} = i + 1 | X_n = i) = P(X_{n+1} = i - 1 | X_n = i) = \frac{1}{2}.$$

What is (are) the equivalent classe(s)? Find p_{ii}^n for all n. Find $\lim_{n \to \infty} p_{ii}^n$.

Problem: Let $\{X_n; n \geq 1\}$ be a sequence of independent and identically distributed random variables with $P(X_1 = 0) = P(X_1 = 1) = 1/2$. Let $Y_n = X_1 + \ldots + X_n$, for $n \geq 1$. Show that $\{Y_n; n \geq 1\}$ is a Markov Chain. Find the transition probability matrix. Find the equivalent classes. Find the period of each state.