MTH211A: Theory of Statistics

Quiz 2

SOLUTION SET

Name: ______________________ **Roll number:** __________

Time: 15 minutes **Total marks:** $4 + 6 = 10$

Q.1 Let $X_1, \cdots, X_n$ be a random sample from uniform$(\theta - 0.5, \theta + 0.5)$ distribution.

Show that $\mathbf{T}(\mathbf{X}) = \begin{bmatrix} X_{(n)} + X_{(1)} \\ X_{(n)} - X_{(1)} \end{bmatrix}$ is jointly minimal sufficient. [4]

First we will show that $\underset{\sim}{T}^{*}(\underset{\sim}{x}) = \begin{bmatrix} X_{(1)} \\ X_{(n)} \end{bmatrix}$ is a minimal sufficient statistic for θ.

To see this consider the ratio of PDFs of $X_1, \cdots, X_n$ for 2 different realizations $\underset{\sim}{x}$ and $\underset{\sim}{y}$ as:

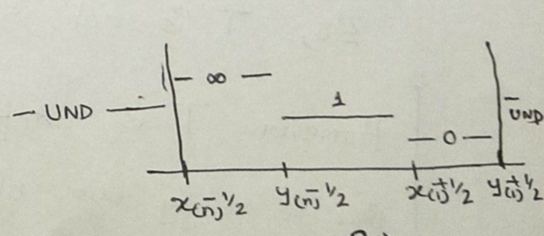
$$\lambda_\theta(\underset{\sim}{x}, \underset{\sim}{y}) = \frac{\mathbb{I}\left(x_{(1)} > \theta - \frac{1}{2},\ x_{(n)} < \theta + \frac{1}{2}\right)}{\mathbb{I}\left(y_{(1)} > \theta - \frac{1}{2},\ y_{(n)} < \theta + \frac{1}{2}\right)}$$

Observe that $\lambda_\theta(\underset{\sim}{x}, \underset{\sim}{y})$ will be a constant function of θ iff $x_{(1)} = y_{(1)}$ and $x_{(n)} = y_{(n)}$.

$x_{(n)} - \frac{1}{2} < \theta < x_{(1)} + \frac{1}{2}$

$y_{(n)} - \frac{1}{2} < \theta < y_{(1)} + \frac{1}{2}$.

So $\underset{\sim}{T}^{*}(\underset{\sim}{x}) = \begin{bmatrix} X_{(1)} \\ X_{(n)} \end{bmatrix}$ is minimal sufficient.



Now, $\underset{\sim}{T}(\underset{\sim}{x}) = h(T^{*}(\underset{\sim}{x}))$ where.

$h\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y-x \end{pmatrix}$. Clearly h is a bijective function. So, $\underset{\sim}{T}(\underset{\sim}{x})$ is also minimal sufficient.

X———————X

Q.2 Let $X_1, \cdots, X_n$ be a random sample from $\texttt{Bernoulli}(p)$, $n > 2$. Let $T = T(\mathbf{X}) = 3X_1 + 3X_2 + X_3$. Find the p.m.f. of T. Hence, or otherwise show that T is not sufficient for p. [3 + 3]

PMF of T : As $X_i \sim \text{Bern}(p)$, each X_i can take 2 values 0 and 1.

So, $T(\underset{\sim}{x})$ can take 6 values 0, 1, 3, 4, 6, 7

$$P(T=0) = P(X_1=0, X_2=0, X_3=0) = (1-p)^3$$

$$P(T=1) = P(X_1=0, X_2=0, X_3=1) = p(1-p)^2$$

$$P(T=3) = P(X_1=1, X_2=0, X_3=0) + P(X_1=0, X_2=1, X_3=0) = 2p(1-p)^2$$

$$P(T=4) = P(X_1=1, X_2=0, X_3=1) + P(X_1=0, X_2=1, X_3=1) = 2p^2(1-p)$$

$$P(T=6) = P(X_1=1, X_2=1, X_3=0) = p^2(1-p)$$

$$P(T=7) = P(X_1=1, X_2=1, X_3=1) = p^3.$$

Sufficiency of T : Let $n > 3$ ~~then~~ and $\underset{\sim}{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_4 \\ \vdots \\ x_n \end{bmatrix}$ and $T=0$,

then $$P(\underset{\sim}{X} = \underset{\sim}{x} \mid T=0) = p^{\sum_{i=4}^{n} x_i} (1-p)^{n - \sum_{i=4}^{n} x_i}$$

$$= \left(\frac{p}{1-p}\right)^{\sum_{i=4}^{n} x_i} (1-p)^n$$ which is not free of p.

So, T is not sufficient for p for $n > 3$.

However T is sufficient for $n = 3$.

extra → To verify this, observe that whenever $T(\underset{\sim}{x}) = T(\underset{\sim}{y})$ then $S(\underset{\sim}{x}) = S(\underset{\sim}{y})$ where $S = (X_1 + X_2 + X_3)$: a sufficient statistic.

$T = 0 \Rightarrow S = 0$; $T \in \{1, 3\} \Rightarrow S = 1$;

$T \in \{4, 6\} \Rightarrow S = 2$ and $T = 7 \Rightarrow S = 3$.

Thus, S is a function of T. Hence T is sufficient.