Quiz-4a (Solutions)
Elementary Stochastic Process (MTH-212A)

Name (Roll Number)
Time: 20 mins.
Maximum Marks: 15
Minimum Marks: 0

Instructions: Both the questions are of multiple choice. More than one answers might be correct in both the questions. Each correct answer will give you one point and if you tick a wrong answer it will be negative 1. In Question 1, if you can identify all the correct answers without ticking any wrong answer, you will get eight points, and similarly in Question 2, you will get 7 points. No extra sheet will be provided. T and R deote the class of all transient and recurrent states, respectively.

1. Suppose a Markov Chain with the following state space $\{1, 2, 3, 4, 5\}$ has the following transition probability matrix

$1 \longleftrightarrow 3$

$\underbrace{\{1,3\}}_{C_1}$ $\underbrace{\{2,4\}}_{C_2}$ $\underbrace{\{5\}}_{T}$

Equivalent class = ✓

$$
\boldsymbol{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{bmatrix}
\frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\
0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
\frac{1}{10} & \frac{3}{10} & \frac{1}{20} & \frac{7}{20} & \frac{1}{5}
\end{bmatrix}
\end{array}
$$

Statements:

$i = 5$

$P(X_n \in C_1 \text{ for some } n \mid X_0 = 5) = \frac{3}{20} + \frac{1}{5} \times \frac{3}{20} + \frac{1}{5^2} \times \frac{3}{20} = \frac{3}{20} \times \frac{5}{4} = \frac{3}{16}$

✗ 1. If $i \in T$, and $X_0 = i$, then the probability to get absorbed in a recurrent class is 0.5.

✗ 2. There are two equivalent classes.

✓ 3. There are two recurrent classes.

$P(X_n \in C_2 \text{ for some } n \mid X_0 = 5) = \frac{5}{4} \times \frac{13}{20} = \frac{13}{16}$

✗ 4. If $i \in T$, and $X_0 = i$, then the probability to get absorbed in a specific recurrent class is 0.5.

$P_i(X_{10} = 5 \mid X_0 = 5) = p_{55}^{10} = \left(\frac{1}{5}\right)^{10}$

✗ 5. If $i \in T$, and then $P(X_{10} \in T | X_0 = i) < (1/5)^{10}$.

6. If $i \in T$, and $X_0 = i$, then the expected time to get absorbed in a recurrent class is 1.5.

The correct statements are: [3].

Solutions: The equivalent classes are $C_1 = \{1, 3\}$, $C_2 = \{2, 4\}$ and $T = \{5\}$ C_1 and C_2 are recurrent classes and T is the transient class. $P(X_n \in C_1$ for large n $|X_0 = 5) = \frac{5}{4} \times \frac{3}{20} = \frac{3}{16}$ and $P(X_n \in C_2$ for large n $|X_0 = 5) = \frac{13}{16}$. Clealrly $P(X_{10} \in T | X_0 = i) = (1/5)^{10}$. $E(\tau) = \frac{5}{4}$.

2. Consider the usual two persons zero sum game as we have discussed in the class, where each player has two units of money, and $P(H) = \frac{1}{3}$ and $P(T) = \frac{2}{3}$. If H appears then Playes 1, wins and gets one unit of money from Player 2. If T appears it is the other way. Here X_n denotes the money of Player 1, after n-th game.

Statements:

$$\begin{array}{ccc} & A & B \\ \# & 2 & 2 \\ H: & +1 & -1 \\ T: & -1 & +1 \end{array}$$

$P(H) = 1/3$

$P(T) = 2/3$

$S = \{0, 1, 2, 3, 4\}$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 & 0 \\ & & \vdots & & \\ & & & & 1 \end{bmatrix}$$

1. The state $\{0\}$ is a recurrent state.
2. The state $\{4\}$ is a recurrent state.
3. All the states are recurrent.
4. $f_{22}^2 < \frac{1}{2}$. $\quad 2 \to 3 \to 2$, $2 \to 1 \to 2$ $= 2/9 + 2/9 = 4/9 < \frac{1}{2}$ $\quad \{1, 2, 3\}$
5. $\sum_{k=1}^{\infty} f_{22}^k = 1$.
6. $\sum_{k=1}^{\infty} k f_{22}^k \geq 1$.

The correct statements are: [1],[2],[4],[6].

Solution: The state space in this case is $S = \{0, 1, 2, 3, 4\}$. The transition probability matrix is

$$\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

The equivalent classes are $\{0\}, \{4\}, \{1, 2, 3\}$. The state $\{0\}$ and the state $\{4\}$ are both recurrent classes. The states $\{1, 2, 3\}$ are transient classes. $f_{22}^{(k)} = \frac{1}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{3} = \frac{4}{9}$, for $k = 2$, and zero, otherwise. Here $\sum_{k=1}^{\infty} k f_{22}^k = \infty$, since $\{2\}$ is a transiecnt state.