MTH211A: Theory of Statistics

Problem set 6

Testing of Hypothesis

1. Consider the problem of testing $H_0 : X \sim f_0$ against $H_1 : X \sim f_1$. Let β denote the power of the most powerful test at level-α. Then show that $\alpha < \beta$ unless $f_0 = f_1$.

2. Consider the problem of testing $H_0 : X \sim f_0$ against $H_1 : X \sim f_1$, and let
$$\phi(\mathbf{x}) = \begin{cases} 1 & \text{if } T(\mathbf{x}) \geq C_\alpha, \\ 0 & \text{if } T(\mathbf{x}) < C_\alpha, \end{cases}$$
be the most powerful size-α test. Suppose further that for any $\alpha' > \alpha$, $C_{\alpha'} < C_\alpha$. Then find the distribution of p-values under H_0.

3. Let $\phi^\star$ be a UMP size α test for $H_0 : \mathbf{X} \sim f_0(\mathbf{x})$ against $H_1 : \mathbf{X} \sim f_1(\mathbf{x})$, and $k(\alpha)$ denote the boundary of the critical region as given in Neyman-Pearson's lemma. Show that if $\alpha_1 > \alpha_2$, $k(\alpha_1) \leq k(\alpha_2)$ when the test is nonrandomized.

4. Let $\phi^\star$ be an UMP size α test for $H_0 : \mathbf{X} \sim f_0(\mathbf{x})$ against $H_1 : \mathbf{X} \sim f_1(\mathbf{x})$, and let $\beta^\star$ be the power of $\phi^\star$, $0 < \beta^\star < 1$. Show that $(1 - \phi^\star)$ is an UMP test for testing $H_0 : \mathbf{X} \sim f_1(\mathbf{x})$ against $H_1 : \mathbf{X} \sim f_0(\mathbf{x})$ at level $(1 - \beta^\star)$.

5. If ϕ_1 and ϕ_2 are 2 size-α tests for testing $H_0 : \theta \in \Theta_0$ against $H_A : \theta \in \Theta_A$, then prove that any convex combination of ϕ_1 and ϕ_2 is a level-α test.

6. In a city it is assumed that the number of automobile accidents in a given year follows a Poisson distribution. In past years, the average number of accidents per year was 15, and this year it is 10. Is it justified to claim that the accident rate has dropped?

7. The lifetime of equipment is normally distributed with mean θ and standard distribution 5. For testing the null hypothesis $H_0 : \theta \leq 30$ against the alternative hypothesis $H_A : \theta > 30$, a random sample of size n is chosen. Determine n and the cutoff c such that the test
$$\phi(\mathbf{x}) = 1, \text{ if } \bar{x} > c, \qquad \phi(\mathbf{x}) = 0, \text{ if } \bar{x} < c$$
has power function values 0.1 and 0.9 at the points $\theta = 30$ and $\theta = 35$ respectively.

8. Let X be distributed as $U(0, \theta)$ and $X_{(n)}$ denote the largest order statistic based on a random sample of size n from this distribution. We reject $H_0 : \theta = 1$ and accept $H_1 : \theta \neq 1$ if either $x_{(n)} \leq 1/2$ or $x_{(n)} \geq 1$. Find the power function of the test.

9. Based on a random sample $X_1, \ldots, X_n$, derive the MP size- α test for testing $H_0 : \theta = \theta_0$ against $H_A : \theta = \theta_1 \, (> \theta_0)$ for each of the following populations:

   (a) $f_X(x; \theta) = 2\theta x + 2(1 - \theta)(1 - x)$, $x \in [0, 1], \theta \in [0, 1]$, and $f_X(x; \theta) = 0$, otherwise, with $n = 1$.

   (b) $f_X(x; \theta) = (\sqrt{2\pi}\theta)^{-1} e^{-x^2/2\theta^2}; -\infty < x < \infty;\ \theta > 0$, and $f_X(x; \theta) = 0$, otherwise.

10. Let X be distributed as $f_X(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1, \theta > 0$, and $f_X(x;\theta) = 0$, otherwise, $\theta > 0$. To test $H_0 : \theta = \theta_1$ against $H_A : \theta > \theta_1$ based on a sample of size n , the following critical region is proposed $\mathbb{C} = \{\mathbf{x} : \prod_{i=1}^{n} x_i \geq 0.5\}$. Find the power function of the above test.

11. Let X be an observation in $(0,1)$. Find an UMP level-α test of $H_0 : X \sim f_0$ against $H_A : X \sim$ `Uniform`$(0,1)$, where, $f_0(x) = 4x$ if $0 < x < \dfrac{1}{2}$, or $f_0(x) = 4(1-x)$ if $\dfrac{1}{2} \leq x < 1$.

12. Suppose that $X_1, \dots, X_n$ are iid with a common pdf $f(x)$, which one of the following two forms:
    (a)
$$f_0(x) = \begin{cases} 3x^2/64 & 0 < x < 4 \\ 0 & \text{otherwise,} \end{cases} \qquad f_1(x) = \begin{cases} 3\sqrt{x}/16 & 0 < x < 4 \\ 0 & \text{otherwise.} \end{cases}$$
    (b) f_0 is the pdf of `Normal`$(0,1)$ and f_1 is the pdf of `Cauchy`$(0,1)$, when $n = 1$.

    Find the UMP level-α test for testing $H_0 : X \sim f_0(x)$ against $H_A : X \sim f_1(x)$.

13. Based on a random sample $X_1, \dots, X_n$, derive the UMP size- α test for testing $H_0 : \theta \leq \theta_0$ against $H_A : \theta > \theta_0$ for each of the following populations
    (a) Location exponential distribution with pdf $f_X(x;\theta) = \sigma^{-1}\exp\{-(x-\theta)/\sigma\}$; with $x > \theta$, $\theta \in \mathbb{R}$ and $\sigma > 0$, when σ is known.
    (b) Location exponential distribution with pdf $f_X(x;\theta) = \theta^{-1}\exp\{-(x-\mu)/\theta\}$; with $x > \mu$, $\mu \in \mathbb{R}$ and $\theta > 0$, when μ is known.
    (c) `Normal` $\left(0, \theta^2\right)$.

14. Let X be an observation from `Poisson`(θ). Find an UMP level-α test for testing $H_0 : \theta \leq \theta_0$ against $H_A : \theta > \theta_0$.

15. Suppose $X_1, \cdots, X_m$ be a random sample of size m from $B(n,p)$, find the UMP level- α test for testing $H_0 : p \leq p_0$ against $H_A : p > p_0$.

16. Do the following families have MLR property in some statistic $T(\mathbf{X})$? If yes, then identify $T(\mathbf{X})$.
    (a) `Uniform`$(\theta, \theta+1)$ based on one sample.
    (b) `Cauchy`(θ), where θ is the location parameter, based on one sample.
    (c) `Exponential`(θ).
    (d) Laplace family of distributions with pdf $f_X(x;\theta) = 2^{-1}e^{-|x-\theta|}$, $x \in \mathbb{R}, \theta \in \mathbb{R}$, based on one sample.
    (e) `Normal`(θ, θ).

17. Let $X_1, \dots, X_n$ is a random sample from discrete uniform distribution supported on $\{1, \dots, N\}$. Find an UMP test for testing $H_0 : N \leq N_0$ against $H_1 : N > N_0$.

18. Consider the distributions `Normal` $\left(\mu_1, 20^2\right)$ and `Normal` $\left(\mu_2, 15^2\right)$, and define $\theta = \mu_1 - \mu_2$. Let there be n samples, $\mathbf{X} = \{X_1, \cdots, X_n\}$ and $\mathbf{Y} = \{Y_1, \cdots, Y_n\}$ from each of these distributions. Suppose $\mathbf{X}$ and $\mathbf{Y}$ are mutually independent. Consider the test ϕ for testing $H_0 : \theta = 0$ against $H_1 : \theta \neq 0$, such that
$$\phi(\mathbf{x}, \mathbf{y}) = \begin{cases} 1 & \text{if } \ |\bar{x} - \bar{y}| > c_0 \\ 0 & \text{otherwise} \end{cases}.$$
    Find n and c_0 such that $\beta_\phi(0) = 0.05$ and $\beta_\phi(10) = 0.9$.

19. Let X be an observation from Cauchy distribution with pdf

$$f_X(x;\theta) = \frac{\theta}{\pi}\frac{1}{\theta^2 + x^2}, \quad x \in \mathbb{R},\ \theta > 0.$$

(a) Show that f_X does not have an MLR.

(b) Find an UMP test for testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$.

20. Let $X_1, \cdots, X_n$ be a random sample from $\texttt{Uniform}(\theta, \theta + 1)$ distribution. Consider the test ϕ for testing $H_0 : \theta = 0$ against $H_1 : \theta > 0$, such that

$$\phi(\mathbf{x}, \mathbf{y}) = \begin{cases} 1 & \text{if} \quad x_{(n)} \geq 1,\ \textbf{or}\ x_{(1)} \geq k \\ 0 & \text{otherwise} \end{cases}.$$

(a) Determine k so that the test will have size α.

(b) Find an expression for $\beta_\phi(\theta)$ for $\theta \in \mathbb{R}$.

(c) Find values of n and k such that the above test has size 0.1 and power at least 0.8 if $\theta > 1$.