MTH210: Lab 5

Ratio-of-Uniforms and Multivariate Normal

1. Complete the code in the file `RoU.R` that has the code for RoU for the Exponential(1) distribution. This generates 10^4 samples from Exp(1) distribution (in a way that's different from before).

2. Consider using RoU method to sample from $N(\theta, \sigma^2)$, where the pdf is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\theta)^2/(2\sigma^2)}$$

   Note that the set D is

$$D = \left\{ (u, v) : 0 \leq u \leq \left(\frac{1}{2\pi\sigma^2}\right)^{1/4} e^{-(v-u\theta)^2/4\sigma^2 u^2} \right\}$$

   Go through the example in the notes to find a, b, c, and then draw 10^4 samples using RoU method.

3. Run the code in `RoURegion.R` to visualize the RoU region for Exp(1) and $N(\theta, \sigma^2)$. The code is complete, but carefully understand all the steps.

   a. Change the values of θ and σ^2 to see what the shape turns out to be.

   b. For a fixed value of σ^2, what do you think will happen to the efficiency of the RoU algorithm as θ increases?

4. Implement the RoU algorithm for $f(x) = \mathbb{I}(2 < x < 3)$.

5. **Multivariate Normal:** In class we have learned about sampling from the multivariate normal. Suppose $\mu \in \mathbb{R}^p$ and $\Sigma \in \mathbb{R}^{p \times p}$ be a positive-definite matrix, so that we want to draw

$$X \sim N_p(\mu, \Sigma) \,.$$

   We will consider $p = 2$ and $\mu = (-5 \;\, 10)^T$ and for $|\rho| < 1$

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} .$$

```r
# defining mean and variance matrix
mu <- c(-5, 10)
Sigma <- matrix(c(1, .5, .5, 1), nrow = 2, ncol = 2)
```

Recall that drawing from this distribution involves first finding the eigenvalue decomposition of Σ

$$\Sigma = Q\Lambda Q^{-1}\,,$$

where Q is the matrix of eigenvectors and Λ is the diagonal matrix of eigenvalues $(\lambda_1, \lambda_2, \dots, \lambda_p)$. An eigenvalue decomposition for a matrix can be done using `eigen()` function:

```r
# eigen value decomposition
decomp <- eigen(Sigma)

# eigenvectors
decomp$vectors
```

```
          [,1]       [,2]
[1,] 0.7071068 -0.7071068
[2,] 0.7071068  0.7071068
```

```r
# eigenvalues
decomp$values
```

```
[1] 1.5 0.5
```

```r
# Let's check if the decomposition is right
# yes it is!!
decomp$vectors %*% diag(decomp$values) %*% solve(decomp$vectors)
```

```
     [,1] [,2]
[1,]  1.0  0.5
[2,]  0.5  1.0
```

From this, we can calculate the "square-root" of the matrix

$$\Sigma^{1/2} = Q\Lambda^{1/2}Q^{-1}\,.$$

```r
# Finding matrix square-root
Sig.sq <- decomp$vectors %*% diag(decomp$values^(1/2)) %*% solve(decomp$vectors)
```

Now, in order to generate observations from $N_p(\mu, \Sigma)$, we obtain $Z = (Z_1, Z_2, \dots, Z_p)^T$ and set

$$X = \mu + \Sigma^{1/2} Z\,.$$

```
Z <- rnorm(2)   # Z
X = mu + Sig.sq %*% Z
X  # one draw from N(mu, Sigma)
```

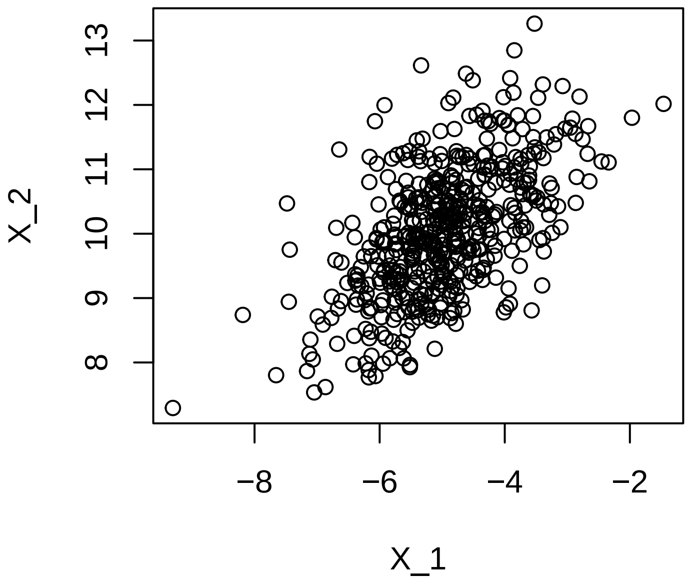
```
          [,1]
[1,] -3.044857
[2,] 10.443596
```

Using all of this information, we can now write a function `multinorm(mu, rho, N)` which takes arguments μ, ρ, and number of samples N, and returns the $N \times 2$ matrix of sampled values.

```
multinorm <- function(mu, rho, N = 5e2)
{
  Sigma <- matrix(c(1, rho, rho, 1), nrow = 2, ncol = 2)
  ...
  ...
  samples <- matrix(0, nrow = N, ncol = 2)
  for(i in 1:N)
  {
    .....
    samples[i, ] <-
  }
}
```

Use your to draw 500 samples from the bivariate normal with $\mu = (-5, 10)^T$ and $\rho = .5$.

```
samples <- multinorm(mu = c(-5, 10), rho = .5)
plot(samples, xlab = "X_1", ylab = "X_2")
```



a. Make a similar plot for $\rho = -.9, -.5, 0, .5, .99$.

b. Repeat the same where marginal variances are 10 and 1 and the off-diagonal elements are 2.