Quiz - 5 (Makeup)
Elementary Stochastic Process (MTH-212a)

Name (Roll Number)
Time: 20 mins.
Maximum Marks: 10
We are using the same notation as we have used in the class.

Let a Markov Chain have the state space $S = \{1, 2, 3, 4, 5, 6\}$ and the transition probability matrix

$\{1, 4\}$ $\{3, 6\}$ $\{2, 5\}$

$\tau_1 = \frac{1}{3}\tau_1 + \frac{2}{3}\tau_4$

$2\tau_1 = 2\tau_4$

$\tau_4 = \frac{2}{3}\tau_1 + \frac{1}{3}\tau_4 \Rightarrow \left(\frac{1}{2} = \tau_1 = \tau_4\right)$

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} \end{array}$$

(i) Find $\lim_{n\to\infty} p_{11}^{(n)}$ (ii) Find $\lim_{n\to\infty} p_{14}^{(n)}$ (iii) Find $\lim_{n\to\infty} p_{21}^{(n)}$ (iv) If $i \in T$ find $\pi_i(C_k)$ for all $i \in T$ and for all k. (v) Suppose for $i \in T$, the discrete random variable $N_i = j$ if $X_j \in T$, $X_{j+1} \notin T$, given that $X_0 = i$. Find $E(N_i)$ for all $i \in T$. (2+2+2+2+2=10)

Solution: If we rewrite the state space as $S = \{1, 4, 3, 6, 2, 5\}$, then the transition probability matrix becomes:

$$\mathbf{P} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

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Here, there are three equivalent classes $C_1 = \{1, 4\}$, $C_2 = \{3, 6\}$ and $T = \{2, 5\}$. Hence, here $(\mathbf{I} - \mathbf{Q})^{-1} = \frac{3}{2}\mathbf{I}$. All the required elements can be obtained from this. $\lim_{n\to\infty} p_{11}^{(n)} = \dfrac{1}{2} = \lim_{n\to\infty} p_{14}^{(n)}$. $\pi_2(C_1) = \pi_2(C_2) = \pi_5(C_1) = \pi_5(C_2) = \dfrac{1}{2}$. Hence, $\lim_{n\to\infty} p_{21}^{(n)} = \dfrac{1}{4}$. $P(N_2 = k) = (1/3)^k(2/3)$, for $k = 0, 1, \ldots$.

$\lim_{n\to\infty} p_{21}^{(n)} = \underline{\pi_2(C_1)}\, \tau_1 = \frac{1}{2} \times \frac{1}{2}$

$N_i = \{ j$ if $X_j \in T$ & $X_{j+1} \notin T \mid X_0 = i$, $i \in T$

$P(N_2 = k) = \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right) \leftarrow$ Geometric RV