MTH211A: Theory of Statistics

Quiz 4

Name: ____________________ Roll number: ____________

Time: 25 minutes Total marks: ~~6 + 5 = 11~~ 4+6 = 10

Q.1 Based on a random sample X of size one from Cauchy$(\theta, 1)$ distribution, find the shortest length 5% confidence interval for θ. [4]

Let $X \sim$ Cauchy $(\theta, 1)$ distribution. Then $f_X(x) = \dfrac{1}{\pi(1+(x-\theta)^2)}$; $x \in \mathbb{R}$.

Define $T(x,\theta) := x - \theta$ as the pivot.

Clearly, $T(x,\theta) \sim$ Cauchy $(0,1)$ distribution, i.e., $f_T(t) = \dfrac{1}{\pi(1+t^2)}$; $t \in \mathbb{R}$.

Let c_1, c_2 be such that $\displaystyle\int_{c_1}^{c_2} \frac{1}{\pi(1+t^2)}\,dt = 0.05.$ — (*)

Then $P\left(c_1 < T(x;\theta) < c_2\right) = 0.05$

$\Rightarrow P\left(c_1 < x - \theta < c_2\right) = P\left(x - c_2 < \theta < x - c_1\right) = 0.05.$

Thus, $(\underbrace{x - c_2}_{L(x)}, \underbrace{x - c_1}_{U(x)})$ is a 5% confidence interval.

Length of the interval, $U(x) - L(x) = c_2 - c_1$ is lowest if

(i) $\dfrac{1}{\pi(1+c_1^2)} = \dfrac{1}{\pi(1+c_2^2)} \Leftrightarrow c_1^2 = c_2^2$

and (ii) $c_1 <$ mode of $f_T(t) < c_2$; and mode $f_T(t)$ is 0.

Thus, $c_1 = -c_2$.

Further, from (*) we get: $\displaystyle\int_{-c_2}^{c_2} \frac{dt}{\pi(1+t^2)} = \left.\frac{\tan^{-1} t}{\pi}\right|_{-c_2}^{c_2} = 0.05$

$\Rightarrow 2\tan^{-1} c_2 = \pi \times 0.05 \Rightarrow \tan^{-1} c_2 = \frac{\pi}{2} \times 0.05$

$\Rightarrow c_2 = \tan\left(\frac{\pi}{2} \times 0.05\right)$

Thus, $\left(x - \tan\left(\frac{\pi}{2} \times 0.05\right), x + \tan\left(\frac{\pi}{2} \times 0.05\right)\right)$ is the shortest length C.I.

Q.2 The lifetime of an electric bulb, denoted as X, is assumed to follow an exponential distribution with parameter λ. However, in practice, the lifetime is rounded to the nearest number of hours. Given a data point indicating that a bulb's lifetime is y_0 hours (where y_0 is a positive integer greater than 0), find the maximum likelihood estimate for λ. 6

Let Y denote the random variable indicating integer valued lifetime (in nearest hours).

PMF of Y :

$$P(Y=0) = \int_0^{0.5} \lambda e^{-\lambda x}\,dx = \left[-e^{-\lambda x}\right]_0^{1/2} = 1 - e^{-\lambda/2}$$

$$P(Y=y) = \int_{y-1/2}^{y+1/2} \lambda e^{-\lambda x}\,dx = \left[-e^{-\lambda x}\right]_{y-1/2}^{y+1/2} = e^{-\lambda(y-1/2)} - e^{-\lambda(y+1/2)}$$

$$= e^{-\lambda y}\left[e^{\lambda/2} - e^{-\lambda/2}\right]$$

$$= e^{-\lambda y}\left(\frac{e^{\lambda}-1}{e^{\lambda/2}}\right)$$

for $y = 1, 2, \cdots$

Log Likelihood of λ given $y_0 > 0$:

$$\ell(\lambda) = -\lambda y_0 - \frac{\lambda}{2} + \log\left(e^{\lambda}-1\right)$$

$$\frac{\partial \ell(\lambda)}{\partial \lambda} = -y_0 - \frac{1}{2} + \frac{e^{\lambda}}{(e^{\lambda}-1)} \Rightarrow$$

First order condition (FOC) gives

$$\frac{e^{\lambda}}{e^{\lambda}-1} = \underbrace{(y_0+1/2)}_{c_0\text{, say}}.$$

$$\Rightarrow e^{\lambda} = c_0 e^{\lambda} - c_0$$

$$\Rightarrow (c_0-1)e^{\lambda} = c_0$$

$$\Rightarrow e^{\lambda} = \frac{c_0}{c_0-1}$$

$$\Rightarrow \lambda = \log c_0 - \log(c_0-1).$$

$\therefore \hat{\lambda} = \left[\log c_0 - \log(c_0-1)\right]$ is a critical point.

Second order condition:

$$\frac{\partial^2 \ell(\lambda)}{\partial \lambda^2} = \frac{(e^{\lambda}-1)e^{\lambda} - e^{\lambda}e^{\lambda}}{(e^{\lambda}-1)^2}$$

$$= -\frac{e^{\lambda}}{(e^{\lambda}-1)^2} < 0 \text{ for any } \lambda > 0$$

$$\therefore \hat{\lambda}_{MLE} = \log\frac{c_0}{c_0-1}$$

$$= \log\left(\frac{y_0+1/2}{y_0-1/2}\right).$$