Lecture Notes: Feb 07, 2024

In this lecture note we will be discussing one and two dimensional random walk. The random walk is a a very important Markov Chain and it can be described as follows.

Definition A random walk $\{X_n; n \geq 0\}$ is called a one dimensional random walk if it has the following transition probability matrix

$$P(X_{n+1} = j | X_n = i) = \begin{cases} 1-p & \text{if} \quad j = i-1 \\ p & \text{if} \quad j = i+1, \end{cases}$$

here $0 < p < 1$, and $i = 0, \mp 1, \mp 2, \ldots$. Therefore, it is clear that the state space is the set of all integers, and the particle moves forward with the probability p and backward with probability $1-p$. Since $0 < p < 1$, it is obvious that all the states communicate with each other, and it has a period 2. Since it is an irreducible Markov Chain, therefore, all the states will be either recurrent or all the states will be transient. Now let us compute $p_{00}^{(m)}$ for $m \geq 0$. By definition of of $p_{00}^{(m)} = P(X_{n+m} = 0 | X_n = 0)$. It is obvious that if m is an odd integer, $p_{00}^{(m)} = 0$. Suppose m is an even integer $m = 2k$, then

$$p_{00}^{(m)} = p_{00}^{(2k)} = \binom{2k}{k} p^k (1-p)^k = \frac{(2k)!}{k!k!} p^k (1-p)^k. \tag{1}$$

Note that (1) is due to that fact the particle has to move k steps forward and k steps backward to come back to the same stage. Now let us approximate (1) using Stirling's approximation. Let us recall Stirling's approximation of $n!$, and that is

$$n! \approx \sqrt{2\pi} e^{-n} n^{n+1/2}.$$

Therefore,

$$\frac{(2k)!}{k!k!} p^k (1-p)^k \approx \frac{1}{\sqrt{2\pi}} \frac{e^{-2k}(2k)^{2k+1/2}}{e^{-2k} k^{2k+1}} p^k (1-p)^k = \frac{(4p(1-p))^k}{\sqrt{k\pi}}. \tag{2}$$

It is clear from (2) that

$$\sum_{k=0}^{\infty} \frac{(2k)!}{k!k!} p^k (1-p)^k < \infty,$$

if and only if $p \neq \frac{1}{2}$. Hence, the Markov Chain is recurrent if and only if $p = \frac{1}{2}$, for any other $0 < p < 1$, it is transient. It means if $p \neq \frac{1}{2}$, there is a positive probability that starting from state 0, it may not come back to state 0 in finite number of steps. Alternatively, due to Markov property, we can say that if $p = \frac{1}{2}$, then starting form state 0, it comes back to state 0 infinitely often. It may be also mentioned that in this case it is quite difficult to compute $f_{00}^{(k)}$ than $p_{00}^{(k)}$. Hence, we have computed the later. A one dimensional random walk with $p = \frac{1}{2}$ is called the symmetric random walk. Now let us discuss about two dimensional random walk.

Two Dimensional Random Walk A Markov Chain $\{X_n; n \geq 0\}$ is called a two dimensional random walk if it has the state space of the form $\{(i,j); i,j = 0, \mp 1, \mp 2, \ldots\}$, and it has the following transition probability matrix

$$\begin{aligned} P(X_{n+1} = (i+1,j)|X_n = (i,j)) &= p_1, \\ P(X_{n+1} = (i-1,j)|X_n = (i,j)) &= p_2, \\ P(X_{n+1} = (i,j+1)|X_n = (i,j)) &= p_3, \\ P(X_{n+1} = (i,j-1)|X_n = (i,j)) &= p_4, \end{aligned}$$

where $0 \leq p_1, p_2, p_3, p_4 \leq 1$, and $p_1 + p_2 + p_3 + p_4 = 1$. It means from any state, it can go one step forward, or one step backward, or one step upward or one step downward. It is also very clear that if $0 < p_1, p_2, p_3, p_4 < 1$, then all the states communicate with each other and it is of period 2. In this case all the states will be transient or recurrent. Now we will be discussing about the symmetric two dimensional random walk, i.e. when $p_1 = p_2 = p_3 = p_4 = \frac{1}{4}$. In the symmetric two dimensional random walk,

$$\begin{aligned} p_{00}^{(2n+1)} &= 0; \quad n = 0,1,2,\ldots \\ p_{00}^{(2n)} &= \sum_{i=0}^{n} \frac{(2n)!}{i!i!(n-i)!(n-i)!} \left(\frac{1}{4}\right)^{2n}; \quad n = 1,2,3,\ldots. \end{aligned} \tag{3}$$

Note that (3) is due to the fact that from the state (0,0) to come back to the state (0,0) in $2n$ steps it has to go i step forward, i step backward, $(n-i)$

step upward, and $(n-i)$ step downward, where $0 \le i \le n$. Now note that

$$\sum_{i=0}^{n} \frac{(2n)!}{i!i!(n-i)!(n-i)!} \left(\frac{1}{4}\right)^{2n} = \left(\frac{1}{4}\right)^{2n} \binom{2n}{n} \sum_{i=0}^{n} \binom{n}{i} \binom{n}{n-i}.$$

Since,

$$\sum_{i=0}^{n} \binom{n}{i} \binom{n}{n-i} = \binom{2n}{n},$$

hence,

$$p_{00}^{(2n)} = \left(\frac{1}{4}\right)^{2n} \left(\binom{2n}{n}\right)^2.$$

Again using Stirling's approximation, it follows that

$$p_{00}^{(2n)} \approx \frac{1}{\pi n} \qquad \Rightarrow \qquad \sum_{n=0}^{\infty} p_{00}^{(2n)} = \infty.$$

Hence, a symmetric two-dimensional random walk is a recurrent Markov Chain.

Example: Suppose $\{X_n\}$ is a Markov Chain with the following transition probability matrix

$$P(X_{n+1} = j | X_n = i) = \begin{cases} \frac{1}{4} & \text{if} \quad j = i+1 \\ \\ \frac{1}{4} & \text{if} \quad j = i-1 \\ \\ \frac{1}{2} & \text{if} \quad j = i. \end{cases}$$

It is a modified one dimensional random walk, and instead of moving only forward and backward, it can stay back in the same position also with a positive probability. Here all the states communicate with each other, and the period of the Markov Chain is 1. Now the question is whether it is a recurrent or transient Markov Chain? Intuitively, it seems that it should be a recurrent Markov Chain, but how do we prove it. One point is clear that $p_{00}^{(n)} > 0$, for both odd and even n, for example $p_{00}^{(1)} = \dfrac{1}{2}$, unlike the symmetric one dimensional Markov Chain.

Now to prove that it is a recurrent Markov Chain, we compare it with the two dimensional symmetric Markov Chain $\{Y_n\}$, and suppose the associate transition probabilities are defined as q_{ij}. Now we compare the two Markov Chains as follows: in $\{Y_n\}$ whenever we move up or down, in $\{X_n\}$ we stay back at the same position. Now observe that staying back in the same position $2k$ times in Markov Chain $\{X_n\}$ is $\left(\frac{1}{2}\right)^{2k}$, where as in Markov Chain $\{Y_n\}$, coming back to the same state only moving up down after $2k$ steps is $\binom{2k}{k}\left(\frac{1}{4}\right)^{2k}$. Since

$$\left(\frac{1}{2}\right)^{2k} = \left(\frac{1}{4}+\frac{1}{4}\right)^{2k} = \sum_{i=1}^{2k}\binom{2k}{i}\left(\frac{1}{4}\right)^{2k} > \binom{2k}{k}\left(\frac{1}{4}\right)^{2k},$$

we have $p_{00}^{(n)} \geq q_{00}^{(n)}$, for all $n \geq 0$. Since

$$\sum_{n=0}^{\infty} q_{00}^{(n)} = \infty \qquad \Rightarrow \qquad \sum_{n=0}^{\infty} p_{00}^{(n)} = \infty.$$