Exercise 4.7:

1) $X \sim Exp(\lambda)$: $f_X(x) = \lambda e^{-\lambda x}$ | Using Inverse transform: $F(t) = \int^t f_X(x)\partial x$

$\Rightarrow F(t) = 1 - e^{-\lambda t}$: $F^{-1}(u) = t \Rightarrow t = \log(1-u) \times \left(-\frac{1}{\lambda}\right)$
where $u \sim Uniform(0,1)$

Thus, draw $u \sim Uniform(0,1)$
return $x = -\frac{1}{\lambda}\log u$:

2) Weibull (α, λ): $f(x) = \alpha\lambda x^{\alpha-1}e^{-\lambda x^\alpha}$, $\boxed{x > 0}$. : Inverse transform

$F(x) = \int_0^x \alpha\lambda t^{\alpha-1}e^{-\lambda t^\alpha}\partial t$: Let $\lambda t^\alpha = k$: $\alpha\lambda t^{\alpha-1}\partial t = \partial k$

$\Rightarrow$ Take $\alpha \geq 1$: so that $\lim_{t\to 0} t^{\alpha-1} < \infty$ $\Rightarrow \int_0^{\lambda x^\alpha} e^{-k}\partial k \Rightarrow F(x) = 1 - e^{-\lambda x^\alpha}$

$F^{-1}(u) = x \Rightarrow \left(\log(1-u) \times \left(\frac{-1}{\lambda}\right)\right)^{\frac{1}{\alpha}} = x$.

3) $f(x) = \frac{e^x}{e-1}$; $x \in [0,1]$??

(I) By Inverse transform: $F(x) = \int_0^x \frac{e^t}{e-1}\partial t = \frac{e^x - 1}{e-1}$

$F_X(x) = F^{-1}(u) = \ln(u(e-1)+1)$ ✓

(II) By Accept reject: let $g(x) = 1$: $x \in [0,1]$: $g \sim U(0,1)$

then, $c = \max_{x \in [0,1]} \frac{f(x)}{g(x)} = \max_{x\in[0,1]} \frac{e^x}{e-1} = \frac{e}{e-1}$ | $\frac{f(x)}{c\,g(x)} = \frac{e^x}{e}$.

Thus Draw proposal $Y \sim U(0,1)$ & $U \sim U(0,1)$ if $U \leq \frac{e^Y}{e}$ then return $X = Y$ else.

(4) $f(x) = \begin{cases} x-2/2 & x \in [2,3] \\ (2-x/3)/2 & x \in [3,6] \end{cases}$ | $F(x) = \begin{cases} \int_2^x (t-2)/2\,\partial t & x \in [2,3] \\ \int_2^3 \frac{(t-2)}{2}\partial t + \int_3^x \frac{(2-t/3)}{2}\partial t & x \in [3,6] \end{cases}$

$\Rightarrow F(x) = \begin{cases} \frac{(x-2)^2}{4}, & x \in [2,3] \\ \frac{1}{4} + x - 3 - \frac{1}{12}[x^2 - 9] = \frac{-x^2}{12} + x - 2, & 3 \leq x \leq 6 \end{cases}$

$\Rightarrow F^{-1}(u) = \begin{cases} 2\sqrt{u} + 2, & 0 \leq u \leq 1/4 \\ \text{solve quadratic}, & \frac{1}{4} \leq u \leq 1 \end{cases}$ ✓ Inverse transform

(5) CDF: $F(x) = \frac{x^2 + x}{2}$ $0 \leq x \leq 1$ $\Rightarrow x^2 + x - 2U = 0$: $x = \frac{-1 \pm \sqrt{1+8U}}{2} \Rightarrow (+)$ ✓

Thus: draw $U \sim Uniform(0,1)$
set $X = \frac{-1 + \sqrt{1+8U}}{2}$ ✓

(6) $f(x) = \begin{cases} \frac{3}{4}(1-x^2) & x \in (-1,1) \\ 0 \end{cases}$

(7) $X \sim Exp(1)$ $Y \sim X \mid X < 0.05$: $f_Y(x) = \frac{e^{-x}}{1-e^{-0.05}}$: $0 < x < 0.05$

$\Rightarrow F_Y(x) = \int_0^x \frac{e^{-t}}{1-e^{-0.05}}\partial t = \frac{1-e^{-x}}{1-e^{-0.05}}$ $x \in [0, 0.05]$

$\Rightarrow F_Y(x) = F^{-1}(U) = -\ln\left(1 - (1-e^{-0.05})U\right)$

(8) F_i ✓ To simulate from $F(x) = \sum_{i=1}^n p_i F_i(x)$: $p_i \geq 0$ & $\sum p_i = 1$

① $U \sim Unif(0,1)$ ② Apply discrete case Inverse transform ; choose F_i wp p_i

(9)(a) $F(x) = \frac{x + x^3 + x^5}{3}$, $0 \leq x \leq 1$ | consider $F(x) = Cx^\alpha$; $0 \leq x \leq 1$; $\alpha > 0$
$F(1) = 1 \Rightarrow C = 1$
$F(0) = 0 \Rightarrow$ ✓

To simulate from $Cx^\alpha \mid C=1$

$\frac{\quad}{3}$

$\Rightarrow F(x) = \frac{1}{3}\underbrace{(x)}_{F_1(x)} + \frac{1}{3}\underbrace{(x^3)}_{F_2(x)} + \frac{1}{3}\underbrace{(x^5)}_{F_3(x)}$

$F(1) = 1 \Rightarrow c = 1$
$F(0) = 0 \Rightarrow$ ✓

To simulate from $cx^{\alpha}|_{c=1}$
$x = F^{-1}(U) = U^{1/\alpha}$

Algo:
Draw $U[0,1] \Rightarrow U$
→ If $U \le \frac{1}{3}$: Draw from $F_1(x) \Rightarrow$ Draw $U_2(0,1)$; $x = U_2$, stop
→ If $\frac{1}{3} < U \le \frac{2}{3}$: Draw from $F_2(x)$: Draw U_2 ; $x = U_2^{\frac{1}{3}}$
→ else : Draw from $F_3(x)$: $x = U_2^{1/5}$

(b) $F(x) = \begin{cases} \frac{1 - e^{-2x} + 2x}{3} & \text{if } x \in (0,1) \\ \frac{\cancel{3}^{2+1} - e^{-2x}}{3} & \text{if } x \in [1, \infty) \end{cases}$

$F(x) = \frac{2}{3}\left[x\mathbb{I}_{(0,1)}(x) + 1\mathbb{I}_{(1,\infty)}(x)\right] + \frac{1}{3}\left[1 - e^{-2x}\right]$ $F_1(x)$

$x \in [0, \infty)$

$F_1(x) = \begin{cases} x & 0 \le x \le 1 \\ 1 & 0 \le x \le \infty \end{cases}$ wp $2/3$ → uniform distribution

$F_2(x) = \frac{1}{3}(1 - e^{-2x})$ ↓ using inverse transform

(10) RV : $f(x) = \int_0^\infty x^y e^{-y}\, dy$; $0 \le x \le 1$ | $(xe^{-1}) \le 1$

Let us solve $\int_0^\infty c^t\, dt$ $(0 \le c \le 1)$ let $c^t = k$ $t\log c = \log k \Rightarrow dt = \frac{dk}{k \log c}$

$\Rightarrow \int_1^0 \cancel{k} \frac{dk}{\cancel{k}\log c} \Rightarrow \frac{-1}{\log c}$

Thus, $F(x) = \frac{-1}{\log(xe^{-1})} = \frac{-1}{\log x - 1}$; $0 \le x \le 1$

By Inverse transform approach: $U = F(x) \Rightarrow x = F^{-1}(U) = e^{-\frac{1}{U} + 1}$ ✓

Algo: ① Draw $U \sim \text{Unif}[0,1]$ ② set $x = e^{\left(-\frac{1}{U} + 1\right)}$

(11) $f(x) = xe^{-x}$, $0 \le x < \infty$ ← $X \sim \text{Gamma}(2, 1)$

#M1 : $F(x) = \int_0^x te^{-t}\, dt = -xe^{-x} - e^{-x} + 1$ | $0 \le x \le \infty$

Inv Transform X

Accept Reject : proposal domain : $\supseteq [0, \infty)$
↳ clearly exponential(λ) where $\lambda_* = \frac{\beta}{\alpha} = \frac{1}{2}$ ✓
with various Exponentials ✓
↓ optimal

#M2 :
Lets try rou : $D = \left\{(U, V) : 0 \le U \le f^{1/2}\left(\frac{V}{U}\right)\right\}$ $\left(\frac{V}{U}\right)^{\frac{1}{2}} e^{-\frac{V}{2U}}$

$a = \sup_{x \in [0,\infty)} x^{\frac{1}{2}} e^{-\frac{x}{2}}$ $\Rightarrow$ differentiate : $\frac{1}{2x^{\frac{1}{2}}} e^{-x/2} = \frac{\cancel{x}}{2}^{\frac{1}{2}} \cancel{e^{-x/2}}$

$a = e^{-1/2}$

$a = \sup_{x\in[0,\infty)} (x^{\frac{1}{2}} e^{-\frac{x}{2}}) \Rightarrow \frac{1}{2x^{\frac{1}{2}}} e^{-x/2} = \frac{x^{\frac{1}{2}}}{2} e^{-x/2}$ (v)

$\hookrightarrow x = 1$: $\boxed{a = e^{-1/2}}$

$b = \inf_{x\leq 0 \,\&\, x\in[0,\infty)} \left(x^{\frac{3}{2}} e^{-x/2}\right) = \underline{0}$ $\therefore \boxed{b = 0}$

$c = \sup_{x\geq 0 \,\&\, x\in[0,\infty)} (x^{3/2} e^{-x/2}) \Rightarrow \frac{3}{2} x^{\frac{1}{2}} = \frac{x^{\frac{3}{2}}}{2} \Rightarrow x = 3 \Rightarrow \underline{c = 3^{3/2} e^{-3/2}}$

Algorithm:

Draw $X \sim$ Uniform $[0, e^{-1/2}]$, $Y \sim$ Uniform $[0, 3^{3/2} e^{-3/2}]$

if $0 \leq X \leq \left(\frac{Y}{X}\right)^{\frac{1}{2}} e^{-\frac{Y}{2X}}$; then return $\frac{X}{Y}$ else

(12) $f(x) = 2x e^{-x^2}$, $\underline{x > 0}$

(I) Inverse Transform: $F(x) = \int_0^x 2t e^{-t^2} dt = \left(-e^{-t^2}\right)_0^x$

$= \underline{1 - e^{-x^2}}$

$U = F(x)$. $(-\log(1-U))^{1/2} = x$

(13) $F(x) = \frac{x + x^2}{2}$ $[0 \leq x \leq 1]$

(I) Inverse transform: $x^2 + x - 2U = 0$

$x = \frac{-1 \pm \sqrt{1+8U}}{2}$ $\quad \because 0 \leq U \leq 1$

$\therefore 1 \leq \sqrt{1+8U} \leq 3$

$(+)$

(II) Accept Reject: $f(x) = \frac{1}{2} + x$; $0 \leq x \leq 1$; Take proposal to be uniform

$c = \sup_{0 \leq x \leq 1} \left(\frac{1}{2} + x\right) = 3/2$

$\frac{2}{3}\left(\frac{1}{2} + x\right) = \frac{1}{3} + \frac{2x}{3}$

Algorithm: Draw $U \sim U[0,1]$, $Y \sim U[0,1]$ $\Rightarrow$ if $U \leq \frac{2f(Y)}{3}$ then return $\underline{Y}$

(III) Composition method: $F_1(x) = x$; $F_2(x) = x^2$ $\quad | \quad 0 \leq x \leq 1$

Draw $U \sim U(0,1)$: If $U \leq \frac{1}{2} \Rightarrow$ return $x = t \sim$ unif $(0,1)$

else $\Rightarrow$ Draw $t \sim$ Unif$[0,1]$, return $x = \underline{\sqrt{t}}$

(14) $f(x) = \frac{(1+x) e^{-x}}{2}$; $0 < x < \infty$

Let ~~prop~~ exponential (λ) ; $g(x) = \lambda e^{-\lambda x}$

$\boxed{\lambda < 1}$ ← then only convergent

Now, $c = \sup_{x\in(0,\infty)} \left| \frac{(1+x)}{2\lambda} e^{(-\lambda+1)x} \right|$

$(1 + x_0)(\lambda - 1) = 1$

$\hookrightarrow x_0 = \frac{-\lambda + 2}{\lambda - 1} < 0$ out of domain

$c = \max\{c_0, c_{x_0}, c_\infty\} \Rightarrow c = 1/2\lambda$

$c_0 = \frac{1}{2\lambda}$, $c_\infty = \infty$

Thus draw for $Y \sim$ exponential (λ) : $U \sim$ Unif$(0,1)$

Thus ✓ draw from exponential(λ) $Y\sim$: $U\sim \text{Unif}(0,1)$

if $U \le \frac{2\lambda f(y)}{\le g(y)}$ $\Rightarrow$ return $X=Y$; else

(15) Target ; Trunc Gamma(α, 1), $\boxed{\alpha<1}$ | proposal: $\exp(\lambda)$: $g(x) = \lambda e^{-\lambda x}$ $x \in (a,\infty)$

$f(x) = \frac{e^{-x}x^{\alpha-1}}{\Gamma(\alpha)}$ $\quad \int_a^\infty \frac{e^{-x}x^{\alpha-1}}{\Gamma(\alpha)} \to \beta$ $\quad \int_a^\infty \lambda e^{-\lambda x} = e^{-\lambda a}$

$c = \sup_{x\in(a,\infty)} \frac{f(x)}{g(x)} = \frac{x^{\alpha-1}}{k\, e^{(1-\lambda)x}}$ Differentiating: $x^{\alpha-1}e^{-(1-\lambda)x}$ $\} \to (\alpha-1) = (1-\lambda)x^2$ ####

(18) $X\sim$ Gamma(α,1) for $\alpha>1$ $\leftarrow$ Exponential(λ): $\lambda = 1/\alpha$

($\lambda = \beta/\alpha$)

(20) $f(x) = \frac{e^{-x}}{1-e^{-a}}$ | $0<x<a$ | $D = \{(u,v) : 0<u \le f^{1/2}(\frac{v}{u})\}$

Trunc Exp   Let $(1-e^{-a})^{1/2} = \lambda$ For such a method $D \to$ bounded

$a = \sup_{0<x<a} \frac{e^{-x/2}}{\lambda} = \frac{1}{\lambda}$

Let us find the constants a, b, c |

$b = \inf_{x\le a} x f^{\frac{1}{2}}(x) = 0$; $c = \sup_{x\ge 0} \frac{xe^{-x/2}}{\lambda} \Rightarrow$ at $x(+\frac{1}{2}) = 1 \Rightarrow x = +2X$

If $a>2$: $c = \frac{2e^{-1}}{\lambda}$; else $c = \frac{ae^{-a/2}}{\lambda}$

(21) RoU : $f(x) = \frac{1}{x^2}$: $x\ge 1$ | Algorithm:

$a = \sup_{x\ge 1} \frac{1}{x^2} = 1$ Draw (U,V) from Unif(0,1) $\times$ ~~Unif(0,1)~~ $\to 1$

$b = 0$ ✗ $\boxed{b=1}$ If $\boxed{U \le U}$ return $\left(x = \frac{1}{U}\right)$ ✓

$c = \sup_{x\ge 1} 1 = 1$ always true

M2 : $F(x) = \int^x \frac{1}{t^2}\,dt \Rightarrow 1 - \frac{1}{x}$ $x \ge 1$

$\left(x = \frac{1}{1-U}\right)$ $\to$ Draw $U\sim$ Uniform(0,1) return $\frac{1}{U}$

(22) As given in notes if $X\sim N(0,1)$ & $Y\sim \chi^2_k$

then $Z = \frac{X}{\sqrt{\frac{Y}{K}}} \sim t_k$ | Thus the algorithm. Sum of k iid normal

(23) (ZIG) : (1) Draw U from Uniform[0,1]

(2) if $U<p$ return $X=0$ else draw Gamma($\cdot$, β)

(23) (ZIEG) : (1) Draw U from Uniform $(0, 1)$

(2) if $U \leq p$ return $X = 0$ else draw Gamma $\underline{(\alpha, \beta)}$