MTH211A: Theory of Statistics

Problem set 4

Methods of Point Estimation

1. Let $X_1, \ldots, X_n$ be a random sample from the following distribution. In each case, find the method of moments (MoM) estimator for $g(\theta)$:

   (a) $\texttt{Gamma}(\alpha, \beta)$, and $g(\boldsymbol{\theta}) = (\alpha, \beta)^\top$. $\to \left(\frac{(\bar{X}_n)^2}{S_n^2}, \frac{\bar{X}_n}{S} \right)^\top$

   (b) $\texttt{Beta}(\alpha, \beta)$ and $g(\theta) = \alpha/\beta$.

   (c) $\texttt{Poisson}(\lambda)$ and $g(\theta) = \exp\{-\lambda\}$. $= \exp(-\bar{X}_n)$

   (d) $\texttt{Location} - \texttt{scale Exponential}(\mu, \sigma)$ and and $g(\theta) = (\mu, \sigma)$. $\mu = \bar{X}_n - S_n$; $\sigma = S_n$

2. Let $X_1, \ldots, X_n$ be a random sample from the following distribution. In each case, find the maximum likelihood estimator (MLE) for $g(\theta)$:

   (a) $\texttt{Binomial}(m, \theta)$, and $g(\theta) = \theta$.

   $l(\theta) = \sum x_i \ln\theta + \sum (m - x_i) \ln(1-\theta)$

   (b) $\texttt{Binomial}(\theta, p)$, and $g(\theta) = \theta$ when $n = 1$.

   (c) $\texttt{Binomial}(m, \theta)$, and $g(\theta) = P(X_1 + X_2 = 0)$.

   (d) $\texttt{Hypergeometric}(m, r, \theta)$ with p.m.f.

   $$f_X(x; m, r, \theta) = \frac{\binom{m}{x}\binom{\theta - m}{r - x}}{\binom{\theta}{r}}, \quad \theta = m+1, m+2, \ldots; \quad \max\{0, r + m - \theta\} \leq x \leq \min\{m, r\},$$

   $g(\theta) = \theta$ and $n = 1$.

   (e) Double exponential: pdf $f_X(x; \theta) = 2^{-1} \exp\{-|x - \theta|\}$; with $x \in \mathbb{R}$ and $\theta \in \mathbb{R}$.

   (f) $\texttt{Uniform}(\alpha, \beta)$, and $g(\theta) = \alpha + \beta$. 1 $\alpha = X_{(1)}$; $\beta = X_{(n)}$

   (g) $\texttt{Normal}(\theta, \theta^2)$, and $g(\theta) = \theta$.

   (h) $\texttt{Inverse Gaussian}(\theta_1, \theta_2)$ and $g(\boldsymbol{\theta}) = (\theta_1, \theta_2)$.

   (i) $\texttt{Uniform}(\theta, \theta + |\theta|)$, $\theta \in \mathbb{R}$ and $g(\theta) = \theta$.

   (j) $\texttt{Weibull}(\theta, k)$ distribution with pdf as follows

   $$f_X(x; \theta, k) = \frac{k}{\theta}\left(\frac{x}{\theta}\right)^{k-1} e^{-(x/\theta)^k}, \qquad x \geq 0, \quad \theta, k > 0,$$

   and $g(\theta) = \theta^k$.

3. Suppose that the random variables $Y_1, \ldots, Y_n$ satisfy $Y_i = \beta x_i + \epsilon_i,\ i = 1, \ldots, n$, where $x_1, \ldots, x_n$ are fixed constants, and $\epsilon_1, \ldots, \epsilon_n$ are iid $\mathrm{N}\left(0, \sigma^2\right), \sigma^2$ unknown.

   (a) Find a two-dimensional sufficient statistic for $\left(\beta, \sigma^2\right)$.

   (b) Find the MLE of β, and show that it is an unbiased estimator of β. $\to \hat{\beta}_{MLE} = \frac{\sum Y_i x_i}{\sum x_i^2}$

   (c) Find the distribution of the MLE of β. $N\left(\beta, \frac{\sigma^2}{\sum x_i^2}\right)$

$N\left(\beta, \frac{\sigma^2}{\sum x_i^2}\right.$

$x_{(1)} \quad x_{(m)} \quad 0 \quad x_{(m+1)} \quad \cdots \quad x_{(n)}$

(d) Show that $\sum Y_i / \sum x_i$ is an unbiased estimator of β.

(e) Calculate the exact variance of $\sum Y_i / \sum x_i$ and compare it to the variance of the MLE.

(f) Show that $\left[\sum (Y_i/x_i)\right]/n$ is also an unbiased estimator of β.

(g) Calculate the exact variance of $\left[\sum (Y_i/x_i)\right]/n$ and compare it to the variances of the estimators in the previous two estimates.

4. Suppose n independent observations are taken from a random variable X with distribution $\texttt{normal}(\mu, 1)$, but instead of recording all the observations, one notes only whether or not the observation is less than 0. If m observations are less than 0, then find the MLE of μ.

$L(\mu \mid x_{(m)} < 0 \ \& \ x_{(m+1)} > 0)$

$f_{x_{(1)}, \dots, x_{(n)}}(x_{(m)} < 0 \ \& \ x_{(m+1)} > 0)$

5. Let $X_1, \cdots, X_n$ be a random sample from $\texttt{normal}(0, \sigma^2)$ distribution, where $\sigma^2 > 2$. Find the MLE of σ^2. Is it a consistent estimator of σ^2?

6. Let $X_{i,j}$, $i = 1, \dots, s$ and $j = 1, \dots, n$ be independent random variables, where $X_{i,j}$ is distributed as $\texttt{normal}(\mu_i, \sigma^2)$, $i = 1, \dots, s$.

(a) Find MLEs for $\mu_1, \dots, \mu_s$, and σ^2.

(b) Show that the MLE for σ^2 is not consistent as $s \to \infty$, if n is fixed.

7. Let $X_1, \cdots, X_n$ be a sample from the PDF

$$f_\theta(x) = \begin{cases} 2x/(\alpha\theta), & \text{if } \ 0 \le x \le \theta, \\ 2(\alpha - x)/\{\alpha(\alpha - \theta)\}, & \text{if } \ \theta \le x \le \alpha, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha > 0$ is known. Show that the MLE of θ has to be one of the observations.

8. Let $\mathbf{X}_1, \cdots, \mathbf{X}_n$ be a random sample from a multivariate normal distribution $N_k(\boldsymbol{\mu}, \sigma^2 I)$ where I is the identity matrix of order k. Find MLEs of $\boldsymbol{\mu}$ and σ^2.

9. Let $X_1, \cdots, X_n$ be iid with one of the two pdfs. If $\theta = 0$, then

$$f(x \mid \theta) = \begin{cases} 1 & \text{if } \ 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

while if $\theta = 1$, then

$$f(x \mid \theta) = \begin{cases} 1/(2\sqrt{x}) & \text{if } \ 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find a MLE of θ.

10. Let X and Y be independent random variables having exponential distributions with expectations λ and μ, respectively. Define

$$Z = \min\{X, Y\}, \qquad \text{and} \qquad W = \begin{cases} 1 & \text{if } \ Z = X, \\ 0 & \text{if } \ Z = Y. \end{cases}$$

Let $(Z_1, W_1), \cdots, (Z_n, W_n)$ be n iid observations on (Z, W). Based on these samples, find MLEs of λ and μ.

11. Given a random sample $X_1, \cdots, X_n$ from a population with pdf $f(x \mid \theta)$, show that maximizing the likelihood function, $L(\theta \mid \mathbf{x})$, as a function of θ is equivalent to maximizing $l(\theta \mid \mathbf{x}) = \log L(\theta \mid \mathbf{x})$.

12. A density function f_x is called *unimodal* or *log-concave* if $\log f_x$ is a concave function.

(a) Let $X_1, \cdots, X_n$ be iid with density $f(x-\theta)$. Show that the likelihood function has a unique root if $f'(x)/f(x)$ is monotone, and the root is a maxima if $f'(x)/f(x)$ is decreasing. Hence, the densities that are yield unique MLEs.

(b) Let $X_1, \cdots, X_n$ be iid positive random variables (or, symmetrically distributed about zero) with common pdf $f_X(x) = af(ax)$, $a > 0$. Show that the likelihood equation has a unique maxima if $xf'(x)/f(x)$ is strictly decreasing for $x > 0$.

(c) If $X_1, \cdots, X_n$ are iid with density $f(x_i - \theta)$ where f is unimodal and the likelihood equation has a unique root. Show that the likelihood equation also has a unique root if the density of each X_i is $af[a(x_i - \theta)]$, with $a > 0$ known.

13. If $X_1, \cdots, X_n$ are iid with density $f(x_i - \theta)$ or $\sigma f(\sigma x_i)$, and f is the logistic density as follows

$$f(u) = \frac{\exp\{-x\}}{[1+\exp\{-x\}]^2}, \qquad x \in \mathbb{R}. \quad = \frac{e^{-x}}{(1+e^{-x})^2}$$

Find the MLEs of $\widehat{\theta}_{\mathrm{ML}}$ and $\widehat{\sigma}_{\mathrm{ML}}$ of θ and σ. Find the limiting distributions of $\sqrt{n}\left(\widehat{\theta}_{\mathrm{ML}} - \theta\right)$ and $\sqrt{n}\left(\widehat{\sigma}_{\mathrm{ML}} - \sigma\right)$.

$$\frac{f'(v)}{f(v)} = \frac{\dfrac{-e^{-x}}{(1+e^{-x})^2} + \dfrac{2 \cdot e^{-x} \cdot e^{-x}}{(1+e^{-x})^3}}{\dfrac{e^{-x}}{(1+e^{-x})^2}}$$

$$g(v) = \frac{f'(v)}{f(v)} = -1 + \frac{2e^{-x}}{(1+e^{-x})} = -1 + \frac{2}{1+e^{x}}$$

$$g'(v) = \cancel{2[1} \; \frac{-2 \cdot e^{x}}{(1+e^{x})^2} < 0 \checkmark$$