MTH211A: Theory of Statistics

Problem set 2

1. Find a sufficient statistic for θ (or $\boldsymbol{\theta} = (\theta_1, \theta_2)^\top$) when $X_1, \cdots, X_n$ are random samples from each of the following distributions:

   (a) Location exponential: pdf $f_X(x;\theta) = \exp\{-(x-\theta)\}$; with $x > \theta$ and $\theta \in \mathbb{R}$.

   (b) Cauchy: pdf $f_X(x;\theta) = 1/\left[\pi\left\{1+(x-\theta)^2\right\}\right]$; with $x \in \mathbb{R}$ and $\theta \in \mathbb{R}$.

   (c) Logistic: pdf $f_X(x;\theta) = \exp\{-(x-\theta)\}/\left[1+\exp\{-(x-\theta)\}\right]^2$; with $x \in \mathbb{R}$ and $\theta \in \mathbb{R}$.

   (d) Double exponential: pdf $f_X(x;\theta) = 2^{-1}\exp\{-|x-\theta|\}$; with $x \in \mathbb{R}$ and $\theta \in \mathbb{R}$.

   (e) Location-scale exponential: pdf $f_X(x;\boldsymbol{\theta}) = \theta_2^{-1}\exp\{-(x-\theta_1)/\theta_2\}$; with $x > \theta_1$, $\theta_1 \in \mathbb{R}$ and $\theta_2 > 0$.

   (f) Gamma: pdf $f_X(x;\boldsymbol{\theta}) = \theta_2^{\theta_1}x^{\theta_1-1}\exp\{-\theta_2 x\}/\Gamma(\theta_1)$; with $x > 0$ and $\theta_i > 0$, $i = 1, 2$.

   (g) Uniform: pdf $f_X(x;\theta) = 1$; with $\theta < x < \theta+1$ and $\theta > 0$.

   (h) Normal distribution with mean θ and θ^2.

   (i) Geometric distribution: pmf $f_X(x;\theta) = \theta(1-\theta)^{x-1}$; with $x = 1, 2, \ldots$ and $0 < \theta < 1$.

   (j) Inverse Gaussian distribution: pdf $f(x;\boldsymbol{\theta}) = \sqrt{\theta_1}\exp\left\{-\theta_1(x-\theta_2)^2/(2\theta_2^2 x)\right\}/\sqrt{2\pi x^3}$, with $x > 0$, $\theta_1 > 0$ and $\theta_2 \in \mathbb{R}$

2. [Independent but not identically distributed] Let $X_1, \ldots, X_n$ be independent but not identically distributed from the following distributions. Find a sufficient statistic for θ.

   (a) $X_i \sim \texttt{uniform}(1-i\theta, 1+i\theta)$.

   (b) X_i has the pdf $f_{X_i}(x;\theta) = \exp\{i\theta - x\}$, with $x \geq i\theta$, $\theta \in \mathbb{R}$.

   (c) $X_i \sim \texttt{Poisson}(a_i\theta)$ for some positive real numbers $a_1, \ldots, a_n$ and $\theta > 0$.

3. Find minimal sufficient statistic in each case in Q.1.

4. Let $\mathbf{Z}_1 = (X_1, Y_1), \cdots, \mathbf{Z}_n = (X_n, Y_n)$ be a random sample from an uniform distribution over a a circle with radius θ. The pdf of $\mathbf{Z}_1$ is

$$f_{\mathbf{Z}|\theta}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{cases} \dfrac{1}{\pi\theta^2} & \text{if } x^2 + y^2 \leq \theta, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Let $W_i = \|\mathbf{Z}_i\|$. Find the distribution of Y_1.

   (b) Show that $\mathbf{T}(\mathbf{Z}_1, \cdots, \mathbf{Z}_n) = (W_1, \cdots, W_n)$ is jointly sufficient for θ.

   (c) Find a minimal sufficient statistic $T^\star$ for θ. Also, find the distribution of $T^\star$.

5. Let $X_1, \cdots, X_n$ be a random sample from `exponential`(θ). Consider the statistics $T_1 = \arg\min_{i\in\{1,\ldots,n\}} X_i$ and $T_2 = X_{(1)}$.

   (a) Derive the pre-images $A_t = \{\mathbf{z} \in [0, \infty)^n : T_1(\mathbf{z}) = t\}$ and $B_t = \{\mathbf{z} \in [0, \infty)^n : T_2(\mathbf{z}) = t\}$.

   (b) Hence derive the distributions of T_1 and T_2.

   (c) Which statistic contains more information about θ. Justify your answer.

6. Let $f((x, y) \mid \boldsymbol{\theta})$, where $\boldsymbol{\theta} = (\alpha_1, \alpha_2, \beta_1, \beta_4)'$, be a bivariate pdf for the uniform distribution on the rectangle with lower left corner (α_1, β_1) and upper-right corner (α_2, β_2) in $\mathbb{R}^2$. The parameters satisfy $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$. Let $(X_1, Y_1), \cdots, (X_n, Y_n)$ be a random sample from this pdf. Find a four-dimensional sufficient statistics for $\boldsymbol{\theta}$.

7. Let $X_1, \cdots, X_n$ be a random sample from a population with pdf of the form $h(x - \theta)$ where $h : \mathbb{R} \to \mathbb{R}$ is a function. Show that the order statistics, $\mathbf{T}(\mathbf{X}) = (X_{(1)}, \cdots, X_{(n)})$ is sufficient for θ, and no further reduction is possible.

8. Let $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_m$ be independently distributed according to `normal`(μ, σ^2) and `normal`(ν, τ^2), respectively. Find a minimal sufficient statistics for the following cases:

   (a) $\mu = \nu \in \mathbb{R}$ and $\sigma, \tau > 0$.

   (b) $\sigma = \tau > 0$ and $\mu, \nu \in R$.