# Lecture Notes: Jan 09, 2024

Welcome to this course MTH412A/MTH212M on *Applied Stochastic Process*. I am teaching this course for quite some time and I really enjoy teaching this course. I hope you will also enjoy and will learn some new interesting topics. I will try my best, and if you have any suggestions or comments to improve please let me know, I will try my best to incorporate them if it is possible. Do not hesitate to send me e-mail: kundu@iitk.ac.in.

I have already circulated my first course hand out, I have already suggested the books I am planning to follow and also the broad topics. Now let us talk about the course. It is expected that the students of this course has some basic knowledge of probability. The students should know what is a random variable. Remember probability (function), from now on it will be denoted by $P(\cdot)$, is a set function define on the class of subsets of the sample space, say $\Omega$. It has to satisfy the following properties:

1. If $A \subset \Omega$, $P(A) \in [0,1]$.
2. $P(\Omega) = 1$.
3. If $A_1, A_2, \ldots$ are disjoint subsets of $\Omega$, then
$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

It would have been nice if we could define $P(\cdot)$ on the class of all subsets of $\Omega$, but it is not always possible. Hence, we restrict the class of subsets of $\Omega$ on which we define $P(\cdot)$ so that it satisfies the above properties. It is known as $\sigma$-field. A $\sigma$-field $\mathcal{F}$ is a class of subsets of $\Omega$ which satisfies the following properties:

1. $\Omega \in \mathcal{F}$.
2. If $A \subset \Omega$, and $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.

3. If $A_1, A_2, \ldots$ are subsets of $\Omega$ and $A_i \in \mathcal{F}$, for all $i = 1, 2, \ldots,$, then

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}.$$

We can choose an appropriate $\mathcal{F}$ to define $P(\cdot)$. Now let us define a random variable $X$. We need a random variable because often the set $\Omega$ has arbitrary elements, hence, it may not be very convenient to deal with them. To define a random variable we need the sample space $\Omega$ and an associated $\sigma$-field defined of the subsets of $\Omega$. We also need a probability function $P(\cdot)$ defined on the elements of $\mathcal{F}$, eventually. Although just to define a random variable we do not need $P(\cdot)$. A random variable $X$ is a real valued function defined on $\Omega$. Hence, the domain of $X$ is $\Omega$ and the range is the set of real numbers. A real valued function $X$ is called a random variable if $X^{-1}(-\infty, a] \in \mathcal{F}$ for all $a \in (-\infty, \infty)$. Here

$$X^{-1}(-\infty, a] = \{y; y \in \Omega, X(y) \leq a\}.$$

Further, we define the distribution function $F(\cdot)$ associate with a random variable is as follows:

$$F(a) = Prob(X \leq a) = P\left(X^{-1}(-\infty, a]\right).$$

It is clear from the above definition of a distribution function is that for a given random variable it has a unique distribution function. If the distribution function is a step function, i.e. it is of the form

$$F(x) = \begin{cases} 0 & \text{if} \quad -\infty < x < a_1 \\ p_1 & \text{if} \quad a_1 \leq x < a_2 \\ p_1 + p_2 & \text{if} \quad a_2 \leq x < a_3 \\ \vdots & \vdots \qquad \vdots \\ \sum_{i=1}^{k} p_k & \text{if} \quad a_k \leq x < a_{k+1} \\ \vdots & \vdots \qquad \vdots \end{cases}$$

where $\infty < a_1 < a_2 < a_3 < \ldots < \infty$ are set of real numbers and $p_i \geq 0$, for $i = 1, 2, \ldots$ and $\sum_{i=1}^{\infty} p_i = 1$, then $X$ is called a discrete random variable.

In this case $Prob(X = a_i) = p_i$, for $i = 1, 2, \ldots$. On the other hand if there exists a $f(x) \geq 0$, such that $F(a) = \int_0^a f(x)dx$, for all $a \in (-\infty, \infty)$, then $X$ is called a continuous (absolute) random variable.

**Exercise:** suppose $\Omega = [0, 1]$, $\mathcal{F} = \{\Omega, [0, .1], (.1, 1], \phi\}$ and we have $P(\Omega) = 1$, $P([0, .1]) = 0.8$, $P((0.1, 1]) = 0.2$ and $P(\phi) = 0$. Suppose $X(\omega) = 1$, for all $\omega \in [0, 1]$. Is $X$ a random variable? If so find the distribution function of $X$.

**Exercise:** suppose $\Omega = [0, 1]$, $\mathcal{F} = \{\Omega, [0, .1], (.1, 1], \phi\}$ and we have $P(\Omega) = 1$, $P([0, .1]) = 0.8$, $P((0.1, 1]) = 0.2$ and $P(\phi) = 0$. Suppose $X(\omega) = 1$ if $\omega \in [0, 0.1]$ and $X(\omega) = 2$ if $\omega \in (0.1, 1]$. Is $X$ a random variable? If so find the distribution function of $X$.

**Exercise:** suppose $\Omega = [0, 1]$, $\mathcal{F} = \{\Omega, [0, .1], (.1, 1], \phi\}$ and we have $P(\Omega) = 1$, $P([0, .1]) = 0.8$, $P((0.1, 1]) = 0.2$ and $P(\phi) = 0$. Suppose $X(\omega) = \omega$, for all $\omega \in [0, 1]$. Is $X$ a random variable? If so find the distribution function of $X$.

Whenever, we talk about a random variable, we mainly mean either a discrete or a continuous random variable. Although, there is a concept of mixed random variable, we do not need in this course. Moreover, whenever we talk about a continuous random variable, we mainly mean an absolute continuous random variable, i.e. it has a probability density function. Let us recall that a random variable $X$ is called a discrete random variable, if there exists $\{a_1, a_2, \ldots\}$, and $\{p_1, p_2, \ldots,\}$, such that $p_i \geq 0$ for all $i = 1, 2, \ldots$, and $\sum_{i=1}^{\infty} p_i = 1$, then

$$P(X = a_i) = p_i; \quad i = 1, 2, \ldots.$$

Similarly, a random $X$ is called an absolute continuous random variable, if there exists a function $f(x)$, known as the probability density function (PDF), such that $f(x) \geq 0$, for all $-\infty < x < \infty$, and $\int_{-\infty}^{\infty} f(x)dx = 1$, then for any $-\infty < a < b < \infty$,

$$P(-a \leq X \leq b) = \int_a^b f(x)dx.$$

The cumulative distribution function (CDF) associate with a random variable (either discrete or continuous) $X$ is

$$F(x) = P(X \leq x); \qquad -\infty < x < \infty.$$

It can be easily verified that for a discrete random variable the CDF is a right continuous function, and in case of a continuous random variable it is a continuous function for all $-\infty < x < \infty$. The CDF is always a non-decreasing function, and

$$\lim_{x \to -\infty} F(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} F(x) = 1.$$

For an absolute continuous distribution function the PDF can be obtained from the CDF as

$$f(x) = \frac{d}{dx} F(x).$$

Some of the well known discrete random variables are Bernoulli, binomial, Poisson, geometric etc. The probability mass function (PMF) of a Bernoulli random variable with parameter $0 \leq p \leq 1$, Bernoulli($p$), is defined as

$$P(X = 1) = p \quad \text{and} \quad P(X = 0) = 1 - p.$$

**Exercise:** Find the mean, variance, the moment generating function of a Bernoulli($p$) random variable.

The PMF of a binomial random variable with parameters $n$, a positive integer, and $0 \leq p \leq 1$, Binomial($n, p$), is defined as

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}; \qquad k = 0, 1, \ldots, n.$$

**Exercise:** Find the mean, variance, the moment generating function of a Binomial($n, p$) random variable.

**Exercise:** Show that if $X_1, \ldots, X_n$ are $n$ independent identically distributed Bernoulli($p$) random variables, then $X = X_1 + \ldots + X_n$ is a Binomial($n, p$) random variable.

**Exercise:** Suppose $X$ and $Y$ are independent Binomial($n, p$) and Binomial($m, p$) random variables, respectively, then find the distribution of $X + Y$.

The PMF of a Poisson random variable with parameters $\lambda > 0$, Poisson($\lambda$), is defined as

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}; \qquad k = 0, 1, \ldots.$$

**Exercise:** Find the mean, variance, the moment generating function of a Poisson($\lambda$) random variable.

**Exercise:** Suppose $X_1, \ldots, X_n$ are $n$ independent Poisson random variables with parameters $\lambda_1, \ldots, \lambda_n$, respectively, then find the distribution of $Y$, where $Y = X_1 + \ldots + X_n$.

Recall that a random variable $Y$ is called a truncated (at zero) Poisson random variable if

$$Y = X | X > 0,$$

where $X$ is a Poisson($\lambda$) random variable. It will be denoted by TPoisson($\lambda$).

**Exercise:** Find the PMF of a TPoisson($\lambda$).

**Exercise:** Find the mean, variance, the moment generating function of a TPoisson($\lambda$) random variable.