HOME WORK 1
MTH 212M/412A (2024)
APPLIED STOCHASTIC PROCESS - I

$X^{-1}((-\infty,a]) = \begin{cases} \phi & a<0 \\ [0,a] & 0\le a<1 \\ \Omega & a\ge 1 \end{cases}$

1. Suppose $\Omega = [0,1]$, and $\mathcal{F}$ is the class of all subsets of Ω. Let X be a real valued function as follows $X(\omega) = \omega$. Is X a random variable? Yes

$\Omega = [0,1]$

Doubt

2. Suppose $\Omega = [0,1]$, and $\mathcal{F}$ is the class of all subsets of Ω. Find a real valued function X defined on Ω which is NOT a random variable. Not possible.

$X^{-1}((-\infty,a]) \subseteq \Omega$ and $\mathcal{F}$ contains all subsets of Ω hence it isn't possible.

3. Suppose $\Omega = \{1,2,3,4\}$, and $\mathcal{F} = \{\Omega, \{1,3\}, \{2,4\}, \phi\}$. Suppose X is a real valued function defined on Ω as follows $X(1) = X(2) = X(3) = X(4) = 1$. Show that X is a random variable and find $F(x) = P(X \le x)$, for all $-\infty < x < \infty$, if $P(\Omega) = 1$, $P(\{1,3\}) = 1/4$, $P(\{2,4\}) = 3/4$ and $P(\phi) = 0$).

4. Suppose $\Omega = \{1,2,3,4\}$, and $\mathcal{F} = \{\Omega, \{1,3\}, \{2,4\}, \phi\}$. Suppose X is a real valued function defined on Ω as follows $X(1) = X(3) = 1$ and $X(2) = X(4) = 10$. Show that X is a random variable and find $F(x) = P(X \le x)$, for all $-\infty < x < \infty$, if $P(\Omega) = 1$, $P(\{1,3\}) = 1/4$, $P(\{2,4\}) = 3/4$ and $P(\phi) = 0$).

5. Suppose $\Omega = \{1,2,3,4\}$, and $\mathcal{F} = \{\Omega, \{1,3\}, \{2,4\}, \phi\}$. Find a real valued function X defined on Ω, so that it is NOT a random variable. $\longrightarrow$ $X^{-1}((-\infty,a]) \notin \mathcal{F}$

6. Suppose $\Omega = \{1,2,3,4\}$, and X is a real valued function defined on Ω as follows $X(1) = 1$, $X(2) = 4$, $X(3) = -1$ and $X(4) = 5^6$. Find a σ-field $\mathcal{F}$, so that X becomes a random variable. $\{1,2\}$

$X(1) = X(2) = 1$
$X(3) = X(4) = 5$

7. Suppose $\Omega = \{1,2,3,4\}$, and $\mathcal{F}$ is the class of all subsets of Ω. Let $P(1) = 0.1$, $P(2) = 0.2$, $P(3) = 0.3$ and $P(4) = 0.4$, and for any other subsets of Ω it is defined in such a manner so that it satisfies the properties of a probability function. Let X and Y be two real valued functions defined on Ω as follows: $X(1) = X(2) = 1$, $X(3) = X(4) = 0$ and $Y(1) = Y(3) = 1$ and $Y(2) = Y(4) = 0$. So that both X and Y are random variables. Find their distribution functions. Are they independent?

8. Let $\{r_1, r_2, \ldots\}$ be a particular enumeration of rational numbers in $(-\infty, \infty\}$. Suppose $\Omega = (-\infty, \infty)$ and $\mathcal{F}$ is the class of all subsets of Ω. Let X be a real valued function in as follows $X(\omega) = \omega$. So that X is a random variable. Suppose the probability $P(\cdot)$ defined on any subset of Ω as follows

$$P(A) = \sum_{i:r_i \in A} \frac{1}{2^i}.$$

Find the distribution function of X.

4. $X^{-1}((-\infty,a]) = \begin{cases} \phi & a<1 \\ \{1,3\} & 10>a\geq 1 \\ \{1,2,3,4\} & a\geq 10 \end{cases}$

$\phi \in \mathcal{F}$, $\{1,3\} \in \mathcal{F}$ & $\{1,2,3,4\} = \Omega \in \mathcal{F}$

$F(x) = P(X\leq x) = P(X \in (-\infty, x])$

$= \begin{cases} P(\phi) = 0 & x<1 \\ \frac{1}{4} & 10>x\geq 1 \\ 1 & x\geq 10 \end{cases}$

3. $\Omega = \{1,2,3,4\}$

$X^{-1}((-\infty,a]) = \begin{cases} \phi & a<1 \\ \Omega & a\geq 1 \end{cases}$

$F(x) = P(X\leq x) = P(X^{-1}((-\infty,a])) = \begin{cases} 0 & a<1 \\ 1 & a\geq 1 \end{cases}$

6. $X^{-1}((-\infty,a]) = \begin{cases} \phi & a<-1 \\ \{3\} = A & -1\leq a<1 \\ \{1,3\} = B & 1\leq a<4 \\ \{1,2,3\} = C & 4\leq a<56 \\ \{1,2,3,4\} = D & a\geq 56 \end{cases}$

Minimalistic set out of A, B, C, D are

$\mathcal{C} = \{\{3\}, \{1\}, \{2\}, \{4\}\}$

$\sigma(\mathcal{C})$ is the required σ-field

7. $X^{-1}((-\infty, a]) = \begin{cases} \phi & a < 0 \\ \{3,4\} & 0 \leq a < 1 \\ \Omega & 1 \leq a \end{cases}$ $\quad Y^{-1}((-\infty, a]) = \begin{cases} \phi & a < 0 \\ \{2,4\} & 0 \leq a < 1 \\ \Omega & 1 \leq a \end{cases}$

$\{3,4\} \subseteq \Omega \in \mathcal{F}$ & $\phi, \Omega \in \mathcal{F}$ $\therefore$ X & Y both are RV.

$F_X(x) = \begin{cases} 0 & a < 0 \\ 0.7 & 1 > a \geq 0 \\ 1 & a \geq 1 \end{cases}$ $\quad F_Y(y) = \begin{cases} 0 & a \\ 0.6 & 1 > a \geq 0 \\ 1 & a \geq 1 \end{cases}$

$F_{X,Y}(x,y) = P[X \leq x, Y \leq y] = P[X^{-1}((-\infty, x]) \,\&\, Y^{-1}(-\infty, y]]$

$= P[X^{-1}((-\infty, x]) \cap Y^{-1}((-\infty, y])]$

Case 1: atleast one of x or y is less than zero.

WLOG, consider $y < 0$ then $Y^{-1}((-\infty, 0]) = \phi$

$F_{X,Y}(x,y) = P[X^{-1}((-\infty, x]) \cap \phi] = 0$

Case 2: If both $1 > x \geq 0$ & $1 > y \geq 0$

$F_{X,Y}(x,y) = P[X^{-1}((-\infty, x]) \cap Y^{-1}((-\infty, y])]$

$= P[\{3,4\} \cap \{2,4\}] = P[\{4\}] = 0.4$

Case 3: If $x \geq 1$ & $1 > y \geq 0$

So, $X^{-1}((-\infty, x]) = \Omega$

$F_{X,Y}(x,y) = P[X^{-1}((-\infty, x]) \cap Y^{-1}((-\infty, y])]$

$= P[\Omega \cap \{2,4\}] = P[\{2,4\}] = 0.6$

Case 4: If $1 > x \geq 0$ & $y \geq 1$

$$Y^{-1}((-\infty, y]) = \Omega$$

$$F_{X,Y}(x,y) = P[X^{-1}((-\infty,x]) \cap Y^{-1}((-\infty,y])]$$
$$= P[\{3,4\} \cap \Omega] = P[\{3,4\}] = 0.7$$

Case 5: If $x \geq 1$ & $y \geq 0$

$$F_{X,Y}(x,y) = P(\Omega \cap \Omega) = P(\Omega) = 1$$

Hence

$$F_{X,Y}(x,y) = \begin{cases} 0 & \text{atleast one of } x,y < 0 \\ 0.4 & 0 \leq x < 1 \ \& \ 0 \leq y < 1 \\ 0.6 & 1 \leq x \ \& \ 0 \leq y < 1 \\ 0.7 & 0 \leq x < 1 \ \& \ 1 \leq y \\ 1 & 1 \leq x \ \& \ 1 \leq y \end{cases}$$

Consider $x = 0.5$ & $y = 0.5$ so $0 \leq x < 1$ & $0 \leq y < 1$

$$F_{X,Y}(0.5, 0.5) = 0.4 \neq 0.42 = 0.6 * 0.7 = F_X(0.5)\, F_Y(0.5)$$

Hence X & Y are NOT independent RV.

$$X^{-1}((-\infty,\omega]) = \{ r_i \;:\; r_i \le \omega \quad r \in \mathbb{Q} \}$$

$\because$ all rational numbers $\subseteq$ by $= \{r_1, r_2, \dots\}$

enumeration of $\mathbb{Q}$

$\therefore$ $$X^{-1}((-\infty, \omega]) \subseteq \mathcal{F}$$

$$F(x) = P(X \le x) = \sum_{r_i \le x} \frac{1}{2^i}$$