MTH210a (2023): Quiz 2

Name .... Roll. No. ....

Instructions:

(a) Show **all** mathematical details. Marks will not be given unless all work is shown.

Suppose $X \sim N(0, 1)$. Use importance sampling to estimate $\Pr(-1 < X < 1)$. Write down all the steps clearly, and make sure to simplify all possible calculations.

solution

(1) $\Pr(-1<X<1)$; $X \sim N(0,1)$

$= E[\mathbb{I}(-1<X<1)]$

$= \int_{-\infty}^{\infty} \mathbb{I}(-1<x<1)\, f(x)\, dx$; $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$; $x \in \mathbb{R}$

$g(x)$ is importance density : $N(0, \sigma^2)$ ($\sigma^2 \neq 1$) (-3 if $N(0,1)$ chosen) / Cauchy (-5 if proposal here doesn't have $\mathbb{R}$ as support.)

$g(x) = \frac{1}{\pi} \frac{1}{1+x^2}$; $x \in \mathbb{R}$.

$$\boxed{\hat{\theta}_g = \frac{1}{N} \sum_{i=1}^{N} \mathbb{I}(-1<Z_t<1) \cdot \frac{f(Z_t)}{g(Z_t)}}$$; $Z_1, \ldots, Z_N \overset{iid}{\sim}$ Cauchy

(-1) if not iid

(-2) if x written instead of Z_t

solution.

(2) $\Pr(-1<X<1) = E[\mathbb{I}(-1<X<1)] = \int_{-1}^{1} f(x)\,dx = 2\int_{-1}^{1} \underbrace{f(x)}_{\approx h(x)} \cdot \underbrace{\frac{1}{2}}_{\text{density of } U(-1,1) \approx \tilde{f}(x)} dx$

$= 2\int_{-1}^{1} f(x)\,\tilde{f}(x)\,dx$; $\tilde{f}(x) = \frac{1}{2}\mathbb{I}(-1<x<1)$ (-3) if doing simple Monte Carlo.

Consider another density $g(x)$ on $[-1,1]$, say (-3) if chosen g is not correct

$g(x) = \frac{3}{2}x^2$; $x \in [-1,1]$.

$= 2\int_{-1}^{1} f(x) \frac{\tilde{f}(x)}{g(x)} \cdot g(x)\,dx = 2\, E_G\left[\frac{f(X)\,\tilde{f}(X)}{g(X)}\right]$

$Z_1, \ldots, Z_N \overset{iid}{\sim} G$

(-1) if not iid

$$\boxed{\hat{\theta}_g = \frac{2}{N} \sum_{t=1}^{N} \frac{\frac{1}{\sqrt{2\pi}} e^{-Z_t^2/2} \cdot \frac{1}{2}}{\frac{3}{2} Z_t^2}}$$

(-2) if x written instead of Z_t

if doing Simple Monte Carlo

$Z_1, \ldots, Z_N \overset{iid}{\sim}$: Unif $[-1, 1]$.

$$\hat{\theta} = \frac{2}{N} \sum_{t=1}^{N} f(Z_t) = \frac{2}{N} \sum_{t=1}^{N} \frac{1}{\sqrt{2\pi}} e^{-Z_t^2/2}$$

-2

This estimates θ, but is Simple Monte Carlo and not importance sampling.