Home Work 2
MTH 212M/ MTH 412A (2024)
Applied Stochastic Process - I

1. Suppose $\{X_n; n \geq 1\}$ is a sequence of i.i.d. Bernoulli random variables, such that

$$P(X_1 = 0) = p \quad \text{and} \quad P(X_1 = 1) = 1 - p; \qquad 0 < p < 1.$$

Suppose $Y_n = \min\{M, X_1 + \ldots + X_n\}$, for $n = 1, 2, \ldots$. Show that $\{Y_n; n \geq 1\}$ is a Markov Chain, and find the transition probability P.

2. A urn contains B black balls and R red balls at the beginning. A ball is drawn at random, and it is replaced by a ball with opposite color. If X_n denotes the number of Black balls after n th draw, show that $\{X_n; n \geq 1\}$ is a Markov Chain. Find the transition probability matrix P.

3. Suppose A and B are two players playing a game with the initial amount of fortune as Rs. 5 and Rs. 10, respectively. At each time they toss a fair coin. If Head appears, A wins and gets Rs. 1 from B, otherwise A has to give Rs. 1 to B. The game stops whenever either A or B gets all the money. Let X_n denote the amount of money A has after the n-th toss. Show that $\{X_n; n \geq 1\}$ is a Markov Chain. Find the transition probability matrix. Based on computer simulation (a) Find the probability that A has all the money when the game stops, (b) Find the expected duration of the game.

4. Customers arrive for service and take their place in a waiting line. During each period of time a single customer is served, provided that at least one customer is present. If no customer awaits service then during this period no service is performed. During a service period new customers may arrive. It is assumed that the actual number of arrivals in the n-th period is a random variable Z_n, whose distribution function is independent of the period and it is given by

$$P(Z_n = k) = p_k; \quad k = 0, 1, 2, \ldots,$$

where $0 \leq p_k \leq 1$, and $\sum_{k=0}^{\infty} = 1$. It is further assumed that $Z_1, Z_2, \ldots$ are independently distributed. If X_n denotes the number customers waiting in the line for service, show that $\{X_n; n \geq 1\}$ is a Markov Chain. Find the transition probability matrix P.

5. Suppose a coin is tossed indefinitely, and $P(H) = p$, where $0 < p < 1$. If Head appears we call it a success. We define a random variable X_n after n-th toss as follows: $X_n = k$, if there is a run of k successes from the last failure, where $k = 0, 1, 2, \ldots$. Show that X_n is a Markov Chain, and find its transition probability matrix P.

6. Suppose an organism at the end of its lifetime produces a random number Y of offspring with probability distribution

$$P(Y = k) = p_k; \quad k = 0, 1, 2, \ldots, \quad p_k \geq 0, \quad \sum_{k=0}^{\infty} p_k = 1.$$

It is assumed that all offspring act independently of each other, and at the end of their lifetime (for simplicity, the life span of all organisms are assumed to be the same) individually have progeny in accordance with the above probability distribution., thus propagating their species. Let X_n denote the population size at the n-th generation. Show that $\{X_n; n \geq 1\}$ is a Markov Chain. If Y follow Binomial(10,1/2), find the transition probability matrix P.

7. Show that if P is stochastic matrix, then P^2 is also a stochastic matrix. In fact P^m, for any positive integer m is also a stochastic matrix.

8. If Q is a stochastic matrix,is it always possible to find a stochastic matrix P, such that $P^2 = Q$?

1. $Y_n = \min \{ M, \sum_{i=1}^{n} X_i \}$

$$Y_n = \begin{cases} M & \sum_{i=1}^{n} X_i > M \\ \sum_{i=1}^{n} X_i & \sum_{i=1}^{n} X_i \le M \end{cases}$$

$P(Y_{n+1} = y_{n+1} \mid Y_n = y_n, Y_2,$

$$Y_{n+1} = \begin{cases} M & \sum_{i=1}^{n} X_i > M \\ \sum_{i=1}^{n+1} X_i & \sum_{i=1}^{n+1} X_i \le M \\ M & \sum_{i=1}^{n} X_i < M \ \& \ \sum_{i=1}^{n+1} X_i \ge M \end{cases}$$

$$Y_{n+1} = \begin{cases} M & Y_n > M \\ Y_n + X_{n+1} & Y_n + X_{n+1} \le M \\ M & Y_n < M \ \& \ Y_n + X_{n+1} \ge M \end{cases}$$

$$P(Y_{n+1} = i \mid Y_n = j) = \begin{cases} 1 & j = m, \ i = m \\ {}^{n}C_j \, p^j (1-p)^{n-j} (1-p) & i < M, \ \cancel{if} \ i = j \\ {}^{n}C_j \, p^j (1-p)^{n-j} p & j + x_{n+1} = i, \ i \ne j \end{cases}$$

j

$$P(Y_{n+1} = i \mid Y_n = j) = \begin{cases} 1 & i = j \ \& \ j = M \\ (1-p) & i = j \ \& \ \underline{j < M} \\ p & i \ne j \quad i = j+1 \\ 0 & \text{o.w.} \end{cases}$$

$$P = \begin{array}{c|c} X_n \backslash X_{n+1} & \begin{matrix} 0 & 1 & 2 & \cdots & M \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \\ M \end{matrix} & \begin{bmatrix} (1-p) & p & 0 & \cdots & 0 \\ 0 & (1-p) & p & \cdots & 0 \\ & \vdots & & & 0 \\ & \vdots & & & 0 \\ & \vdots & & (1-p) & p \\ 0 & 0 & 0 & 0 \; 0 \; 0 & 1 \end{bmatrix} \end{array}$$

$\underline{M \times M}$

2.

B
R

X_n = no. of Balls after n^{th} draw

$$X_{n+1} = \begin{cases} X_n + 1 & \text{if red is draw} \\ X_n - 1 & \text{if black is drawn} \end{cases}$$

$$P(X_{n+1} = i \mid X_n = j) = \begin{cases} \left(\frac{B+R-j}{B+R}\right) & i = j+1 \\ \frac{j}{B+R} & i = j-1 \\ 0 & \text{otherwise} \end{cases}$$

Red   Black
& $(B+R)-j$ j

$X_n = \{0, 1, 2, \ldots B+R\}$

$$P = \begin{array}{c|c} X_n \backslash X_{n+1} & \begin{matrix} 0 & 1 & 2 & \cdots & \cdots & B+R \end{matrix} \\ \hline \begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \\ B+R \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ \frac{1}{B+R} & 0 & \frac{B+R-1}{B+R} & 0 & \cdots & 0 \\ & & \vdots & & & \\ & & & & & \\ 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$(B+R+1) \times (B+R+1)$

4.

$$X_{n+1} = \begin{cases} X_n + Z_n - 1 & \text{if } X_n > 0 \\ Z_n & \text{if } X_n = 0 \end{cases}$$

$$P(X_{n+1} = i \mid X_n = j) = \begin{cases} p_k & i = k \quad i = j+k-1 \quad \underline{j > 0} \\ p_k & i = k \quad j = 0 \\ 0 & \text{o.w.} \end{cases}$$

$$P = \begin{array}{c|l} {}_{X_n} \backslash {}^{X_{n+1}} & \begin{matrix} 0 & 1 & 2 & \cdots \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \end{matrix} & \begin{bmatrix} p_0 & p_1 & p_2 & \cdots & & \\ p_0 & p_1 & p_2 & p_3 & \cdots & \\ 0 & p_0 & p_1 & p_2 & \cdots & \\ 0 & 0 & p_0 & p_1 & \cdots & \\ 0 & 0 & 0 & p_0 & p_1 & \cdots \\ 0 & 0 & 0 & & & \\ \vdots & \vdots & \vdots & & & \end{bmatrix} \end{array}$$

5.

$$X_{n+1} = \begin{cases} X_n + 1 & \text{if head appers in } (n+1)^{th} \text{ trial} \\ 0 & \text{if tail appears in } n^{th} \text{ trial} \end{cases}$$

$$P(X_{n+1} = i \mid X_n = j) = \begin{cases} p & i = j+1 \quad j >= 0 \\ (1-p) & i = 0 \quad j >= 0 \end{cases}$$

$$p = \begin{array}{c|l} {}_{X_n} \backslash {}^{X_{n+1}} & \begin{matrix} 0 & 1 & 2 & 3 & \cdots \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \end{matrix} & \begin{bmatrix} (1-p) & p & 0 & \cdots & \\ (1-p) & \cdots & p & \cdots & \\ (1-p) & \cdots & & p & \\ \vdots & & & & \ddots \end{bmatrix} \end{array}$$

6.

$$X_{n+1} = \begin{cases} \sum_{i=1}^{X_n} Y_i \\ \end{cases}$$

$$P(X_{n+1} = i \mid X_n = j) = \begin{cases} P_{Y_1} \cdot P_{Y_2} \cdots P_{Y_j} & i = \sum_{k=1}^{j} Y_k \\ \end{cases}$$

$$P = \begin{array}{c|ccccc} X_n \backslash X_{n+1} & 0 & 1 & 2 & 3 & \cdots \\ \hline 0 & P_0 & 0 & 0 & P_2 & \cdots \\ 1 & P_0 & P_1 & \cdots & & \\ 2 & P_0P_0 & 2P_0P_1 & & & \\ 3 & & & & & \\ \vdots & & & & & \end{array}$$

$2 \to 2$

$2P_0P_2 + P_1P_1$

$$P(X_{n+1} = i \mid X_n = j) = \begin{cases} \\ \end{cases}$$

$i \geq j$

$x_1 + x_2 + \cdots x_j = i$

$2 \to 1 \quad \to$

6. X_n = population at n-th generation

$$X_{n+1} = \begin{cases} Y_1 + Y_2 + \cdots Y_{X_n} & \text{if } X_n > 0 \\ 0 & \text{if } X_n = 0 \end{cases}$$

We can clearly see that X_{n+1} depends on just X_n and no. of offsprings produced Y i.e. not on any other previous state.

Hence it's a Markov chain.

Given $Y \sim \text{Binomial}(10, \frac{1}{2})$

hence. $M_Y(t) = \left(1 - \frac{1}{2} + \frac{1}{2}e^t\right)^n = \left(\frac{1}{2}\right)^{10}(1+e^t)^{10} = E(e^{tY})$

$$M_{X_{n+1}}(t) = M_{\sum_{i=1}^{X_n} Y_i}(t) = E\left(e^{t\sum_{i=1}^{X_n} Y_i}\right) = E\left(\prod_{i=1}^{X_n} e^{tY_i}\right)$$
$$= \prod_{i=1}^{X_n} E\left(e^{tY_i}\right) = \left(\frac{1}{2}\right)^{10X_n}(1+e^t)^{10X_n}$$

So $X_{n+1} \sim \text{binomial}\left(10X_n, \frac{1}{2}\right)$

Also, $X_n = \{0, 1, 2, \ldots\}$

$$P(X_{n+1} = i \mid X_n = j) = \begin{cases} \binom{10j}{i}\left(\frac{1}{2}\right)^i\left(\frac{1}{2}\right)^{10j-i} = \binom{10j}{i}\left(\frac{1}{2}\right)^{10j} & j > 0 \\ 1 & i = j = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$P = \begin{array}{c|c} X_n \backslash X_{n+1} & \begin{matrix} 0 & 1 & 2 & \cdots \end{matrix} \\ \hline \begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & \cdots \\ \left(\frac{1}{2}\right)^{10} & \binom{10}{1}\left(\frac{1}{2}\right)^{10} & \binom{10}{2}\left(\frac{1}{2}\right)^{10} & \cdots \\ \left(\frac{1}{2}\right)^{20} & \binom{20}{1}\left(\frac{1}{2}\right)^{20} & \cdots & \\ \vdots & \vdots & & \end{bmatrix} \end{array}$$