MTH211A: Theory of Statistics

Quiz 3

Name: Solution Set Roll number:

Time: 15 minutes Total marks: 7 + 5 = 12

Q.1 Let $X_1, \cdots, X_n$ be a random sample from the discrete uniform distribution supported on the set $\{1, \ldots, N\}$, $N \in \mathbb{N}$. Find complete sufficient statistics for $\psi(N)$, where ψ is a function of natural numbers. ~~[2.5+4.5]~~ [9]

(A) We first find ~~the~~ a minimal sufficient statistic for N and then verify if the ~~solution~~ statistic is complete.

Ratio of the joint pmf of $\underset{\sim}{X}$ based on 2 realizations $\underset{\sim}{x}$ and $\underset{\sim}{y}$:

$$\lambda_N(\underset{\sim}{x}, \underset{\sim}{y}) = \frac{\mathbb{I}(x_{(n)} \leq N)}{\mathbb{I}(y_{(n)} \leq N)} \text{ is const. w.r.t. } N \text{ iff } x_{(n)} = y_{(n)}.$$

$\Rightarrow X_{(n)}$ is minimal sufficient stat. ~~[2]~~ [2]

(B) Next we find the pmf of $X_{(n)}$.

$$P(X_{(n)} \leq t) = [P(X_1 \leq t)]^n = \begin{cases} 0 & \text{for } t < 1 \\ \left(\frac{[t]}{N}\right)^n & \text{for } t \in [1, N] \\ 1 & \text{for } t > N \end{cases}$$

$$\therefore P(X_{(n)} = t) = P(X_{(n)} \leq t) - P(X_{(n)} \leq t^-) =$$

$$= \begin{cases} 0 & \text{if } t < 1 \text{ or } t > N \text{ or } t \text{ is not an integer} \\ \frac{t^n - (t-1)^n}{N^n} & \text{if } t \in \{1, 2, \ldots, N\} \end{cases}$$

[~~2.5~~ 3]

(C) [Completeness] $\underset{N}{E}[g(T)] = 0 \quad \forall N \in \mathbb{N}$ where $T = X_{(n)}$

$\Rightarrow$ For $N = 1$, $g(1) \cdot 1 = 0 \Rightarrow g(1) = 0$

[3] $N = 2$, $\frac{g(1)}{2^n} + \frac{g(2)}{2^n} \cdot (2^n - 1) = 0 \Rightarrow g(2) = 0$

Proceeding similarly, we get: $g(t) = 0$ for each $t = 1, 2, \ldots$

$\Rightarrow P[g(T) = 0] = 1$.

(D) Note that, $f_{\underset{\sim}{X} \mid X_{(n)}}$ is free of N, which implies $f_{\underset{\sim}{X} \mid N}$ is free of $\psi(N)$. So, $X_{(n)}$ is CSS for N. [1]

Q.2 Using Q.1 or otherwise, find the UMVUE of N.

It is easy to see that $E(x_1) = \frac{1+2+\cdots+N}{N} = \frac{N(N+1)}{2N} = \frac{N}{2}+\frac{1}{2}$

$$\Rightarrow E[\underbrace{2X_1 - 1}_{T_1}] = N$$

As $T = X_{(n)}$ is the minimal sufficient statistic, we have:

$$\phi(T) = E[X_1 | T] \text{ to be the UMVUE.}$$ [2]

Finding the distribution of $f_{X_1|T}$.

$$f_{X_1|T=t}(u) = \frac{P(X_1 = u, X_{(n)} = t)}{P(X_{(n)} = t)} = \begin{cases} \frac{P(X_1=u, \max\{X_2,\dots,X_n\}=t)}{P(X_n=t)} & u<t \text{ i.e. } u=1,\dots,t-1 \\ \frac{P(X_1=t, X_2\le t,\dots,X_n\le t)}{P(X_n=t)} & u=t \\ 0 & u>t \end{cases}$$

$$= \begin{cases} \frac{\frac{1}{N}\frac{t^{n-1}-(t-1)^{n-1}}{N^{n-1}}}{\frac{t^n-(t-1)^n}{N^n}} = \frac{t^{n-1}-(t-1)^{n-1}}{t^n-(t-1)^n} & \text{for } u<t \\ \frac{\frac{1}{N}\cdot\left(\frac{t}{N}\right)^{n-1}}{\{t^n-(t-1)^n\}/N^n} = \frac{t^{n-1}}{t^n-(t-1)^n} & \text{for } u=t \\ 0 & \text{o/w} \end{cases}$$ [3]

$$E[X_1|T] = \frac{(t-1)t^{n-1} - (t-1)^n + t^{n-1}}{t^n-(t-1)^n} = \frac{2t^n - t^{n-1} - (t-1)^n}{t^n-(t-1)^n}.$$ [2]

$$\phi(T) = \frac{2T^n - T^{n-1} - (T-1)^n}{T^n - (T-1)^n} \text{ is the UMVUE}$$