MTH210: Lab 7

Likelihood functions

1. Suppose $X_1, X_2 \overset{iid}{\sim} N(\theta, 1)$, and you observed $X_1 = x_1 = 2$ and $X_2 = x_2 = 3$. For various values of θ, draw the likelihood function of θ. Recall that

$$L(\theta|\mathbf{X}) = f(x_1|\theta)f(x_2|\theta)\,,$$

where $f(x|\theta)$ is the density of a $N(\theta, 1)$ distribution evaluated at the value x. So you have to find this product and then choose a grid of values of θ on the x-axis and plot the corresponding value of $L(\theta|x_1 = 2, x_2 = 3)$.

2. Suppose you obtain $X_1, X_2, \dots, X_{100} \overset{iid}{\sim} \text{Gamma}(\alpha, 2)$, with $\alpha = 10$. For various values of α, draw the likelihood function $L(\alpha|\mathbf{X})$ and draw log of the likelihood function: $\log L(\alpha|\mathbf{X})$. "Verify" that both $L(\alpha|\mathbf{X})$ and $\log L(\alpha|\mathbf{X})$ have the same maxima.

3. Our goal in this next problem is to draw the likelihood function as n, the data size increases. Consider the N$(\theta, 1)$ distribution. Suppose we obtain one data-point from this distribution and calculate the log-likelihood:

$$\log L(\theta|X_1) = \log f(X_1|\theta),.$$

Every time we obtain different data points, we will get different log-likelihood functions. Thus, the log-likelihood function is random, where the source of randomness is the data.

Similarly, suppose we get now $n = 100$ data points from $N(\theta, 1)$. Here again, we obtain the log-likelihood

$$\log L(\theta|X_1, \dots, X_n) = \sum_{i=1}^{n} \log f(X_i|\theta),.$$

For both situations, in file `demonstration.R` we will plot

$$\frac{\text{Log-likelihood}}{n}\,.$$

Notice that

$$\hat{\theta}_{MLE} = \arg\max\{\text{Log-likelihood}\} = \arg\max \frac{\text{Log-likelihood}}{n}$$

In the demonstration, we can see what happens to the MLE estimator as n increases. Now, change the code in the demonstration to increase n to 500 to see how this affects the MLE.