Lecture Notes: Feb. 14, 2024

In this lecture note we will be discussing about the basic limit theorem for Markov Chain and also about the stationary probabilities, whenever they exist. Now first provide the basic limit theorem for a finite or infinite Markov Chain, and it is very helpful in determining the limiting behavior of the transition probability matrix.

Theorem: Consider a recurrent, aperiodic, irreducible Markov Chain. Then we have the following simple result for all i and j:

$$\lim n \to \infty p_{ii}^{(n)} = \frac{1}{\sum_{n=0}^{\infty} n f_{ii}^{(n)}} = \frac{1}{m_i}.$$

$$\lim n \to \infty p_{ji}^{(n)} = \frac{1}{\sum_{n=0}^{\infty} n f_{ii}^{(n)}} = \frac{1}{m_i}.$$

Here $m_i = \sum_{n=1}^{\infty} n f_{ii}^{(n)}$. We have already provided the proof of this theorem in the class, and it mainly follows from the Renewal equation. In my opinion it is a wonderful application on an independent mathematical result in Markov Chain. I would like to make couple of comments in this context.

Comment: Let us recall that if $\{a_k\}$ is a sequence of real numbers, such that $\lim_{n\to\infty} a_k = a$, then $\lim_{n\to\infty} \frac{1}{m}\sum_{k=1}^{m} a_k = a$. Hence, it immediately follows that for a recurrent, aperiodic and irreducible Markov Chain

$$\lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} p_{ii}^{(m)} = \frac{1}{\sum_{n=0}^{\infty} n f_{ii}^{(n)}} = \frac{1}{m_i}.$$

Comment: Let $\mathcal{C}$ be a aperiodic recurrent Markov Chain and not irreducible. Hence, $p_{ij}^{(n)} = 0$, for all $i \in \mathcal{C}$, for all $j \notin \mathcal{C}$ and $n \geq 0$. Therefore, the submatrix $\mathbf{P}_1 = ((p_{ij}))$, for $i, j \in \mathcal{C}$ is a proper transition matrix, of a recurrent aperiodic and irreducible recurrent Markov Chain. Therefore, the theorem can be applied to this submatrix.

Let us consider the following example. Suppose a Markov Chain $\mathcal{C}$ with the state space $\{1,2,3,4\}$ having the following transition probability matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \end{bmatrix}.$$

It is clear that it is an aperiodic recurrent Markov chain, but it is not irreducible. On the other hand there are two recurrent equivalent recurrent classes namely $\{1,3\}$ and $\{2,4\}$. Hence, the limiting results hold on two sub-matrices namely

$$\mathbf{P}_1 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \quad \text{and} \quad \mathbf{P}_2 = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}.$$

Now we will state one result without proof. Suppose we have a recurrent irreducible Markov Chain with period d, then

$$\lim_{n\to\infty} p_{ii}^{nd} = \frac{d}{\sum_{k=1}^{\infty} k f_{ii}^{(k)}} = \frac{d}{m_i}.$$

From now on we will be restricting mainly for aperiodic Markov Chain, unless it is mentioned otherwise. If for an aperiodic, recurrent and irreducible Markov Chain,

$$\lim_{n\to\infty} p_{ii}^{(n)} = \frac{1}{\sum_{k=1}^{\infty} k f_{ii}^{(k)}} = \frac{1}{m_i} = \pi > 0,$$

then it is called a positive recurrent Markov Chain.

We have proved the following result in the video that for a positive recurrent, aperiodic and irreducible Markov Chain, π_i can be obtained as the unique solution of the following set of linear equation:

$$\pi_j = \sum_{j=0}^{\infty} \pi_i p_{ij}, \qquad \sum_{j=0}^{\infty} \pi_i = 1.$$

Here it is assumed that the state space is the set of all non-negative integers, i.e. $\{0,1,2,\ldots\}$.

Now there are some important points. First of all it has been shown that π's are unique. Moreover, if and $\pi_i > 0$, then $\pi_j > 0$, for all j. These $\{\pi_i; i = 0, 1, 2, \ldots\}$.

Example: Suppose we have a Markov Chain with the following transition probability matrix having the state space $\{0, 1, 2, \ldots\}$.

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \ldots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \ldots \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & \ldots \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Clearly, it is an aperiodic, recurrent and irreducible Markov Chain. We have already shown it before. Now let us see whether there exists the stationary probabilities or not. Suppose it exist, then it has to satisfy the following set of linear equations:

$$\pi_0 = \frac{1}{2}\sum_{i=0}^{\infty} \pi_i \quad \text{and} \quad \pi_i = \frac{1}{2}\pi_{i-1}; \quad i \geq 1.$$

It implies that it has the unique solution

$$\pi_i = \frac{1}{2^{i+1}} = \frac{1}{m_i} > 0; \quad i \geq 0.$$

Therefore, it is a positive recurrent Markov Chain and starting from the state i, the expected time to return to state i for the first time is 2^{i+1}.

Example: Now let us consider the one dimensional modified random walk, with the state space $\{0, \mp 1, \mp 2, \ldots\}$, and having the transition probability matrix $\mathbf{P} = ((p_{ij}))$, where

$$p_{ij} = \begin{cases} \frac{1}{4} & \text{if } \; j = i - 1 \\ \frac{1}{4} & \text{if } \; j = i + 1 \\ \frac{1}{2} & \text{if } \quad j = i \end{cases}$$

Let us recall that in the last lecture note, we have shown that it is an aperiodic, irreducible and recurrent Markov Chain. Now let us try to find out π.

It is known that π should satisfy the following set of linear equations:

$$\frac{1}{2}\pi_i = \frac{1}{4}\pi_{i-1} + \frac{1}{4}\pi_{i+1}; \quad \text{for} \quad i = 0, \mp 1, \mp 2, \ldots.$$

It is a homogeneous difference equation of the form:

$$x_{i+1} - 2x_i + x_{i-1} = 0.$$

The associated auxiliary equation associated with the difference equation is

$$x^2 - 2x + 1 = 0.$$

Hence, 1 is the repeated root. Therefore, the general solution will be of the form $x_i = a + bi$. Hence, in this case the constant solution is the only solution. Therefore, $x_i = 0$. Hence, the stationary probabilities do not exist in this case.

Example: Suppose we have a Markov Chain with the following transition probability matrix with the state space $\{0, 1, 2, \ldots\}$.

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \ldots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \ldots \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \ldots \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

We want to explore whether the stationary probabilities exist or not. Suppose it exist, then it has to satisfy the following set of linear equations:

$$x_0 = \frac{1}{2}x_0 + \frac{1}{2}x_1 \quad \text{and} \quad x_i = \frac{1}{2}x_{i-1} + \frac{1}{2}x_{i+1}; \quad i \geq 1.$$

It means all the x_i's have to be equal. It is not possible, hence the stationary probabilities do not exist.

Exercise: Consider the one dimensional modified random walk, with the state space $\{0, \mp 1, \mp 2, \ldots\}$, and having the transition probability matrix $\mathbf{P} = ((p_{ij}))$, where

$$p_{ij} = \begin{cases} \frac{1}{6} & \text{if} \quad j = i - 1 \\ \frac{1}{6} & \text{if} \quad j = i + 1 \\ \frac{2}{3} & \text{if} \quad j = i \end{cases}$$

Verify, whether it has the stationary probabilities (positive recurrent) or not?

Exercise: Suppose we have a Markov Chain with the following transition probability matrix with the state space $\{0, 1, 2, \ldots\}$.

$$\mathbf{P} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & \ldots \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 & \ldots \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & \ldots \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Verify, whether it has the stationary probabilities (positive recurrent) or not?

Exercise: Prove that for a finite, irreducible, aperiodic and recurrent Markov Chain all the states are positive recurrent.