Lecture 5: Divide & Conquer 1

2-D Maxima & Closest Pair

CS 341: Algorithms

Tuesday, Jan 22^{nd} 2019

Outline For Today

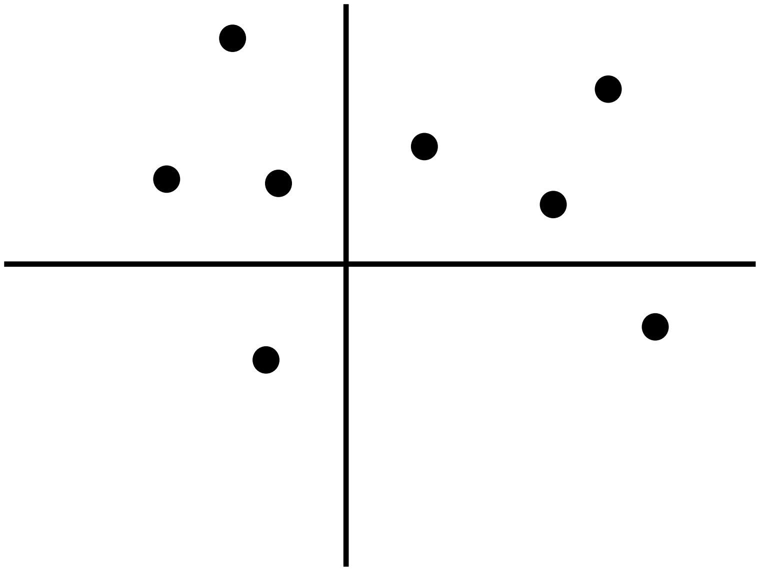
1. 2-D Maxima
2. Closest Pair

Outline For Today

1. 2-D Maxima
2. Closest Pair

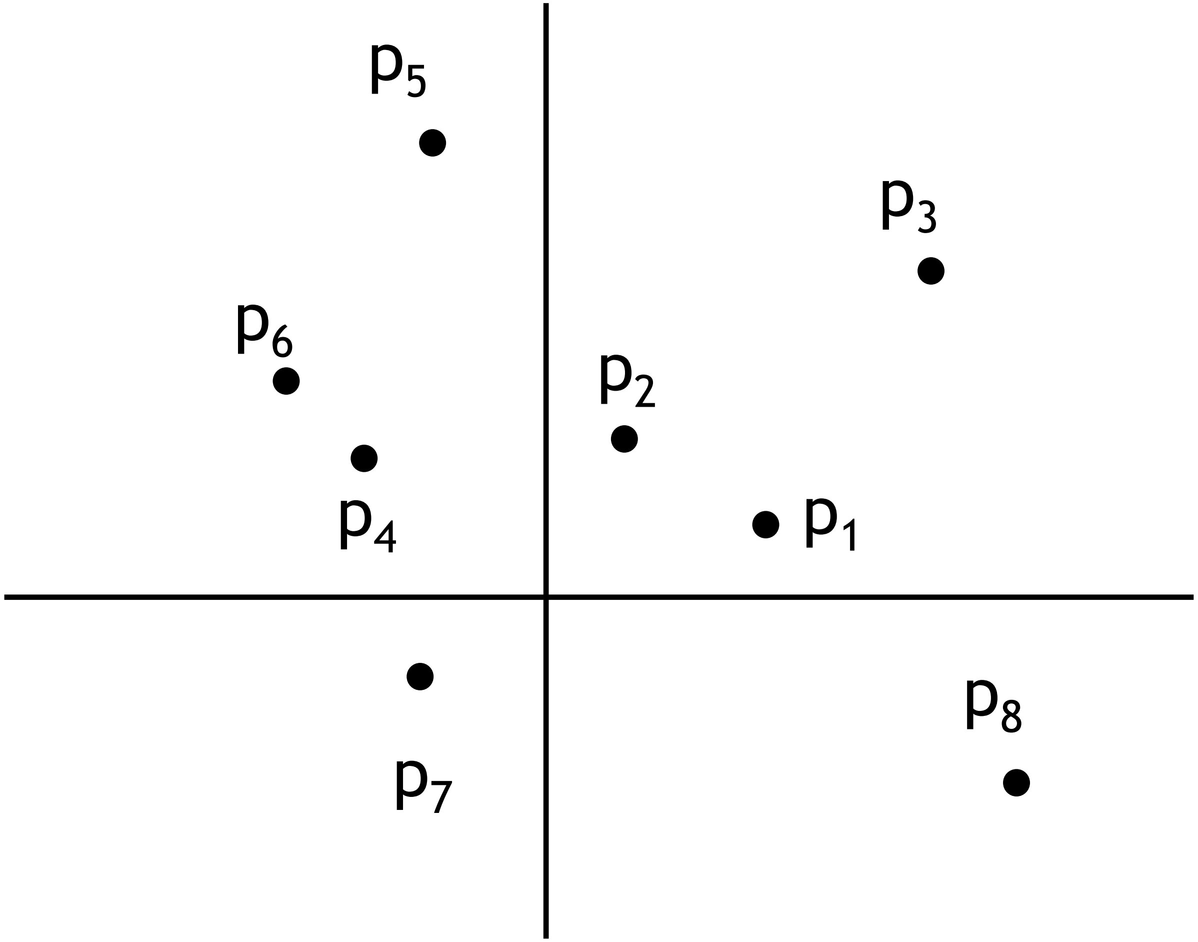
2-D Maxima

- Input: Set P of n 2-D points

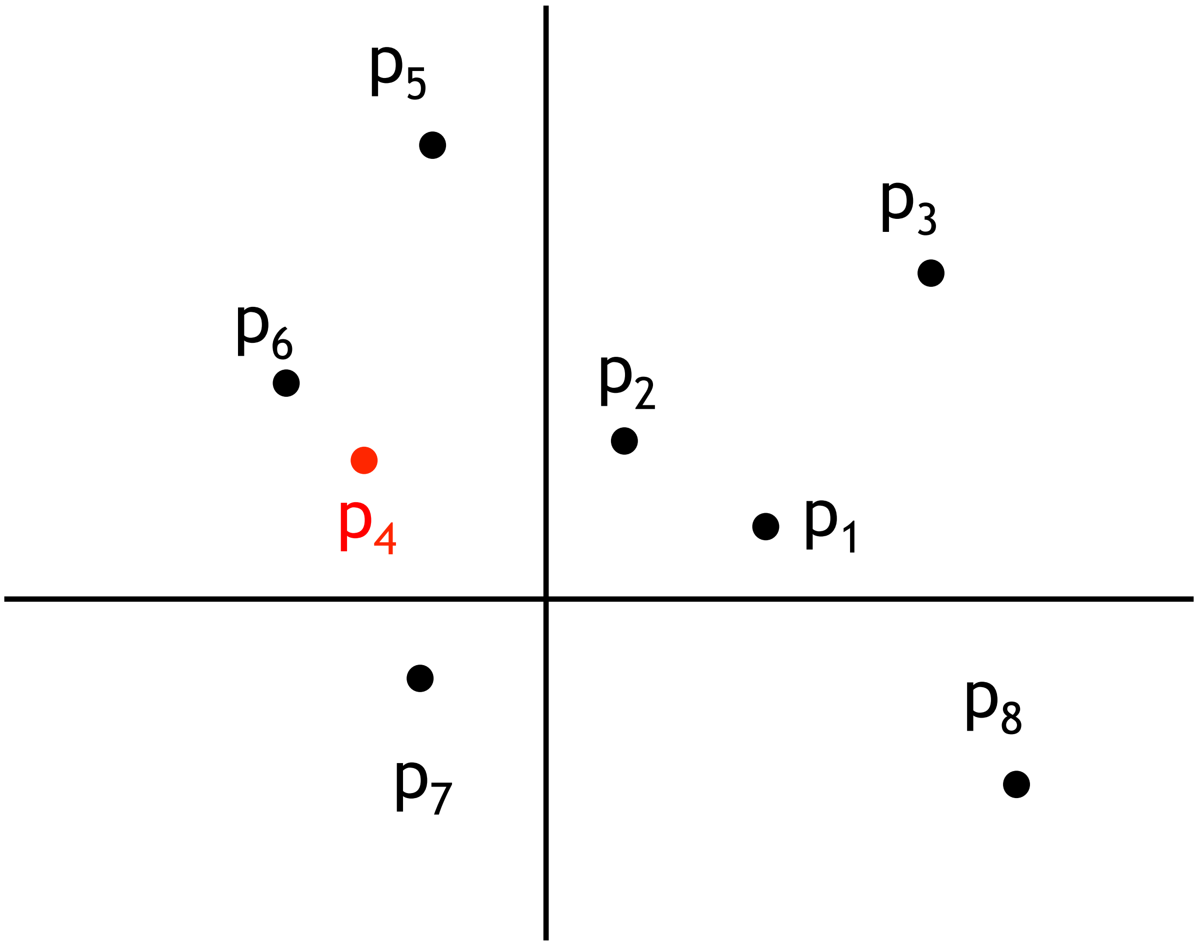


- Output: All *maximal* points
- Dfn 1: Point p *dominates* point q iff
  - p.x > q.x AND p.y > q.y
- Dfn 2: Point p is maximal if no point dominates it

Example

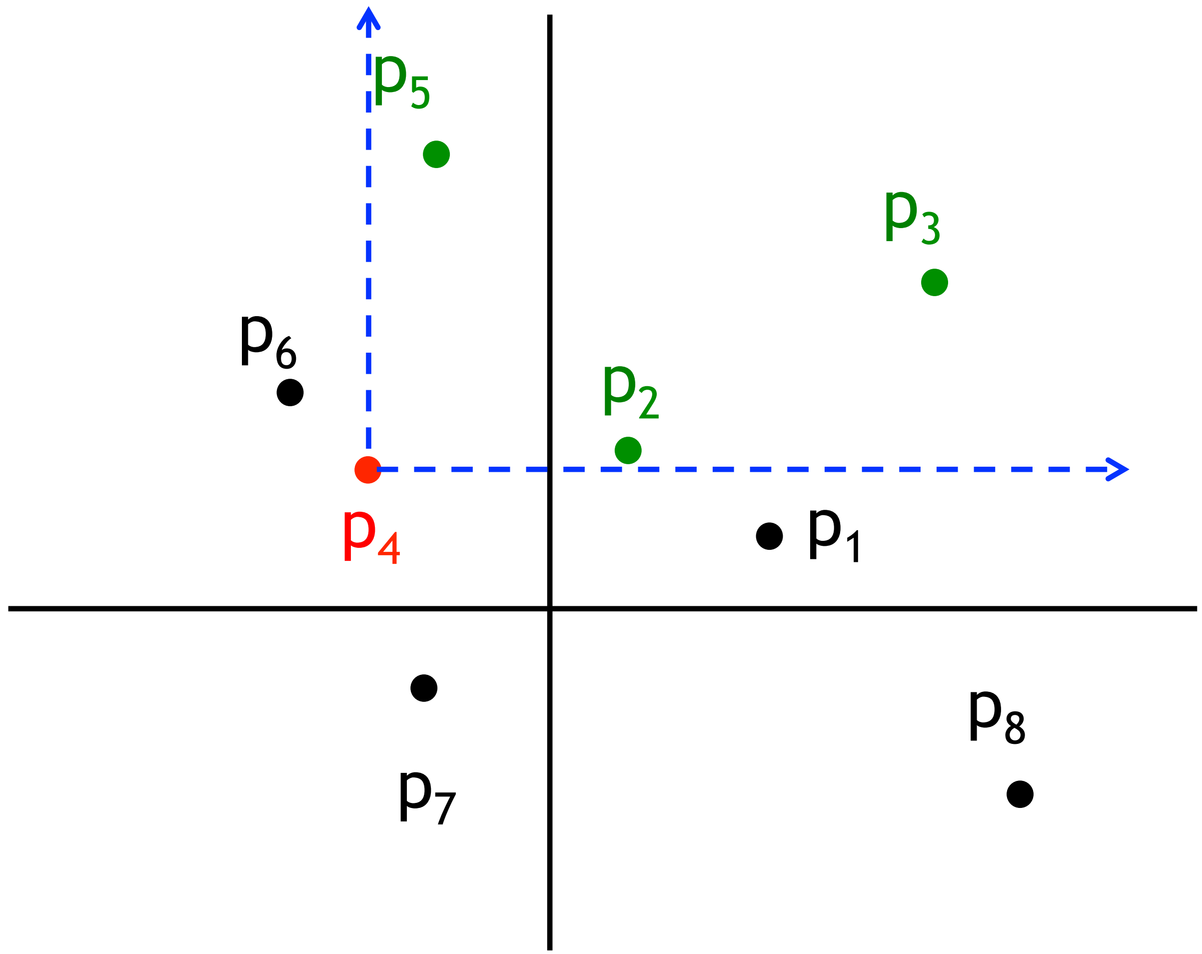


Example



Q: Is p_4 maximal?

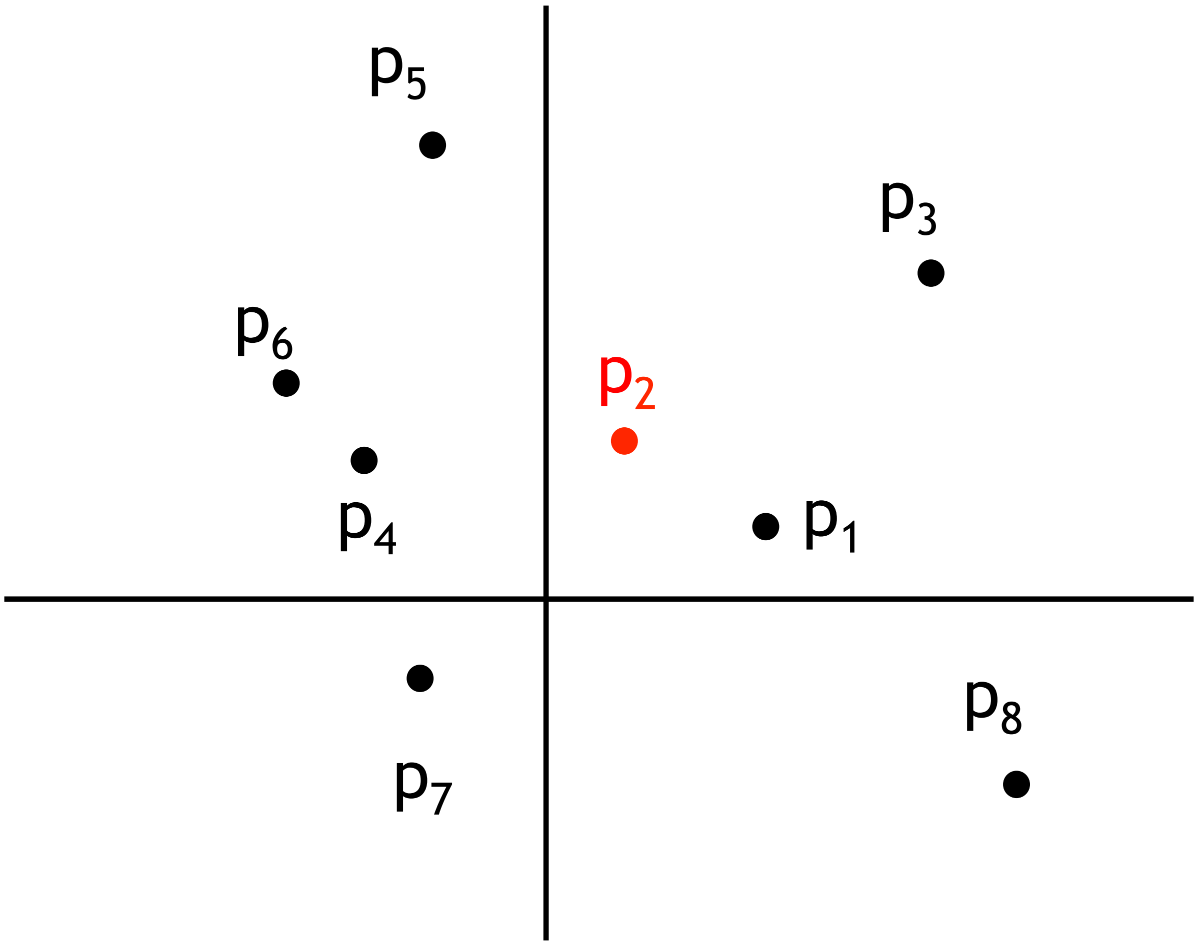
Example



Q: Is p_4 maximal?

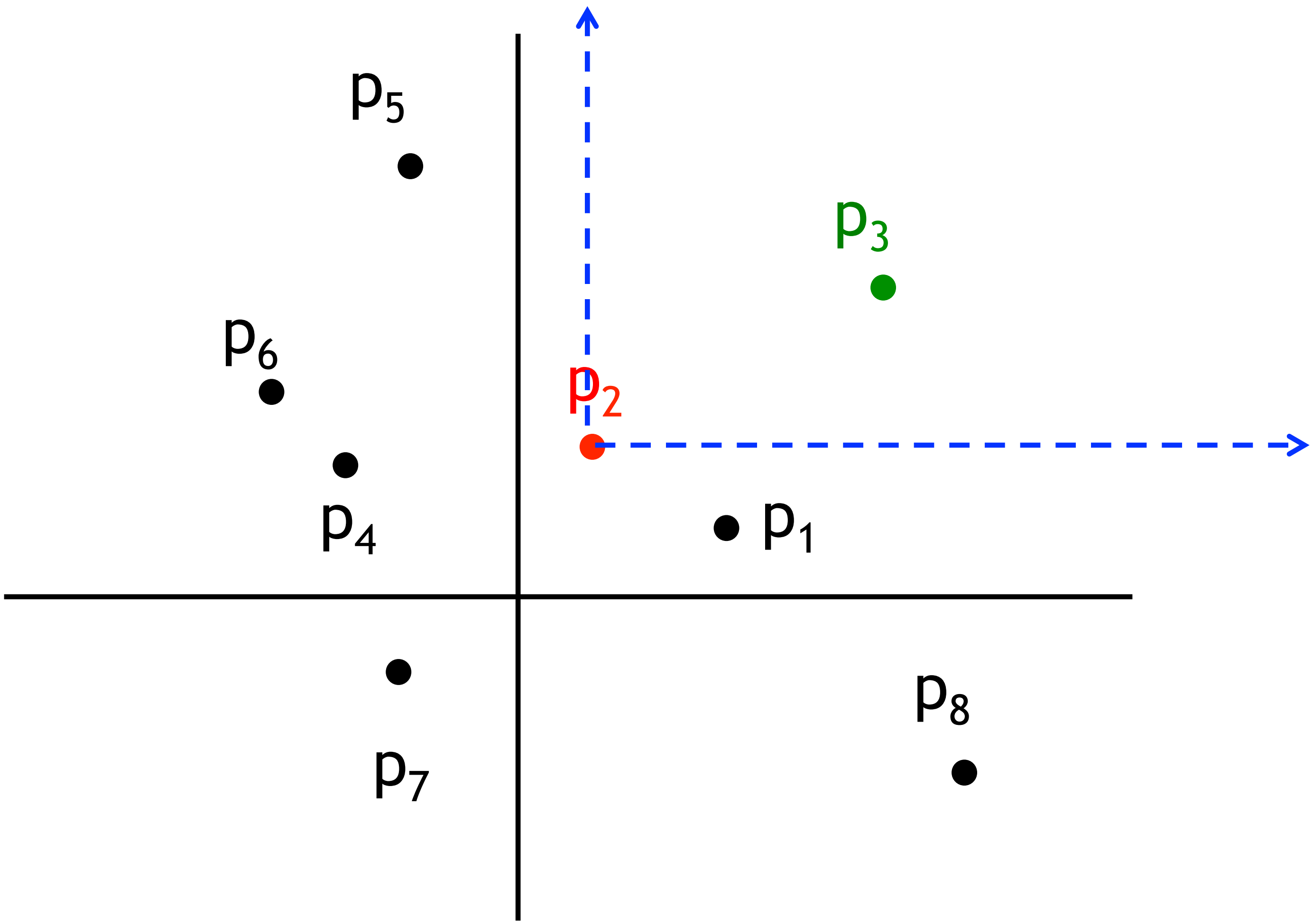
A: No. p_2, p_3, and p_5 dominate p_4.

Example



Q: Is p_2 maximal?

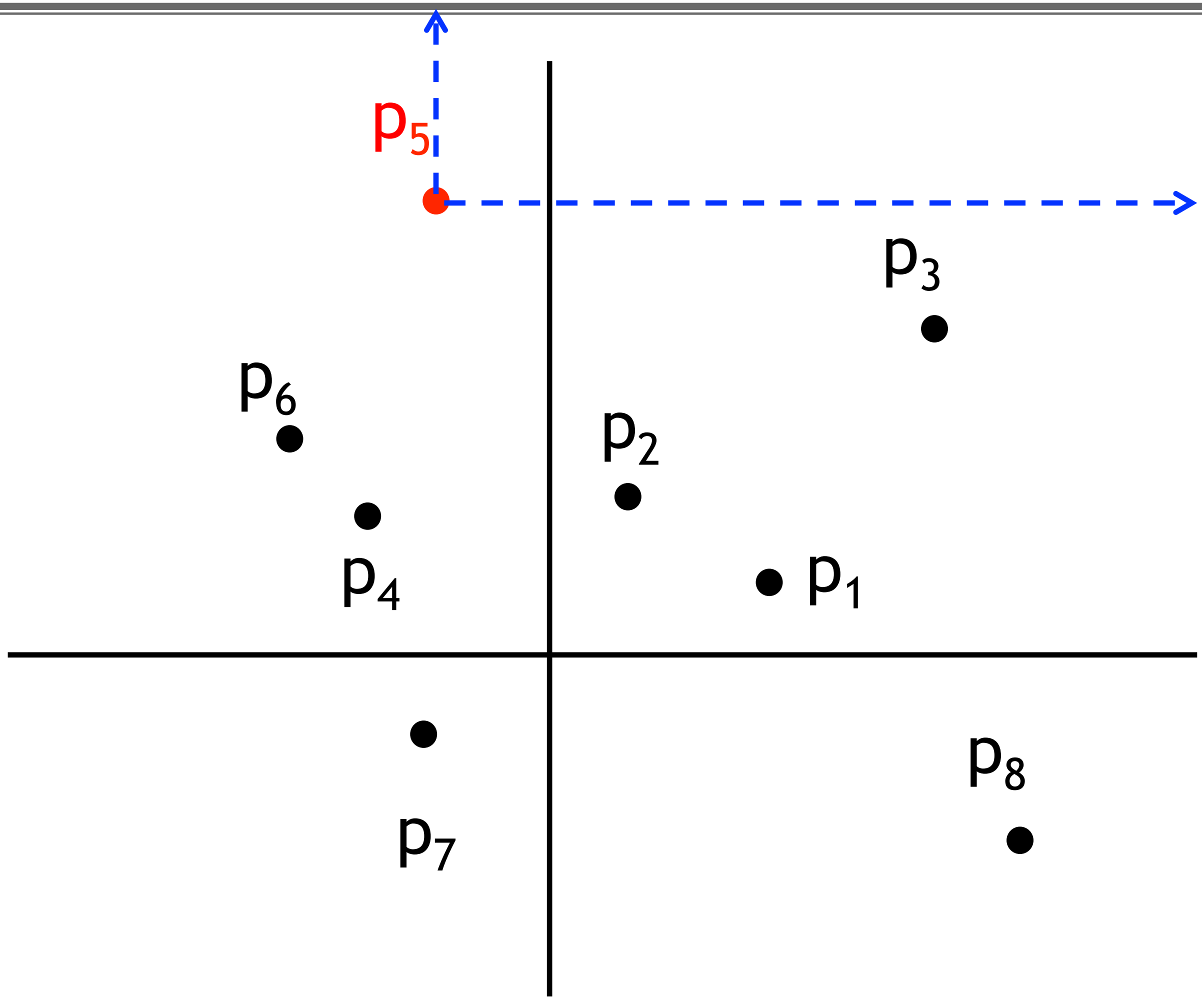
Example



Q: Is p_2 maximal?

A: No. p_3 dominates p_4.

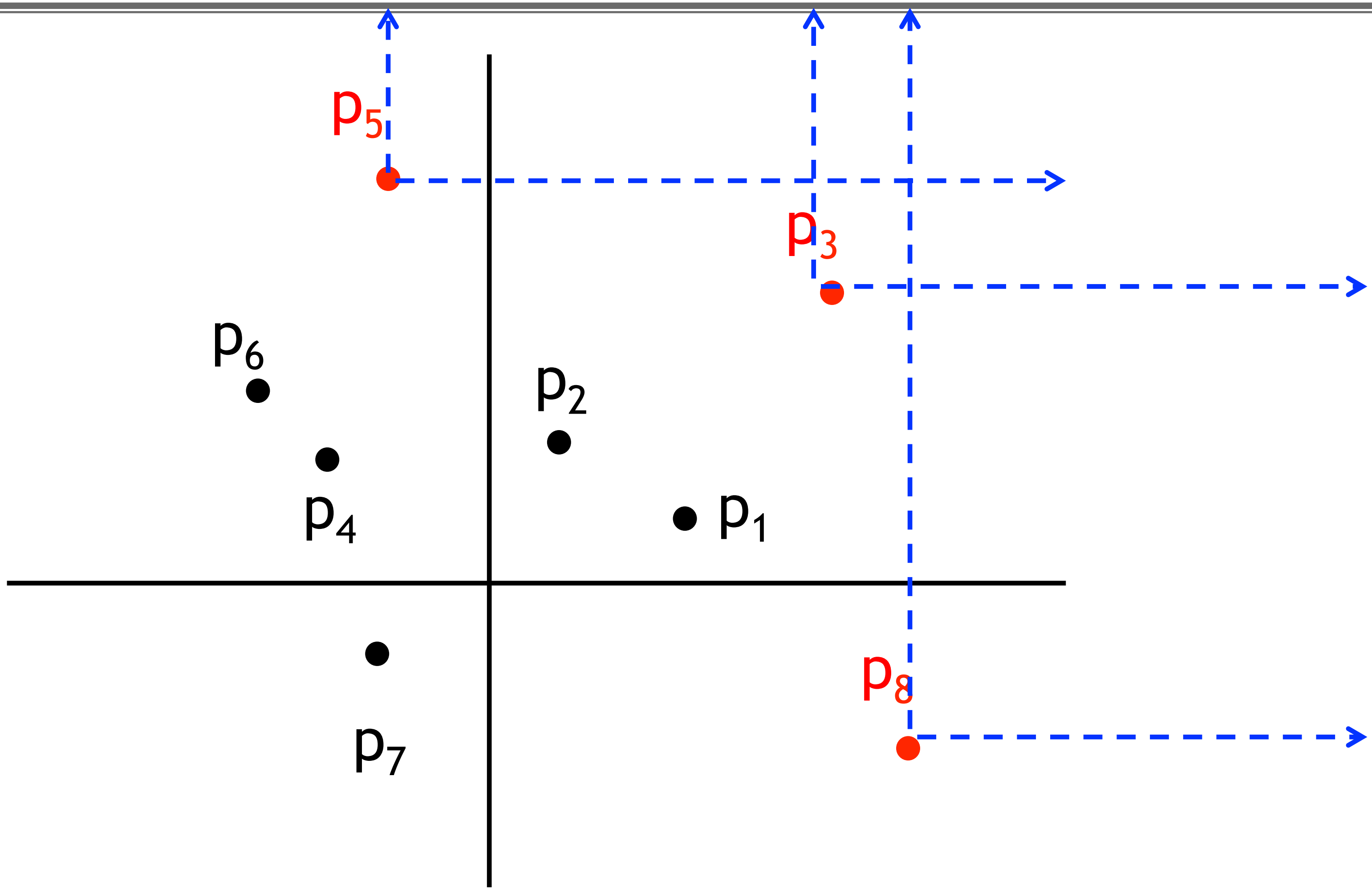
Example



Q: Is p_5 maximal?

A: Yes

Example

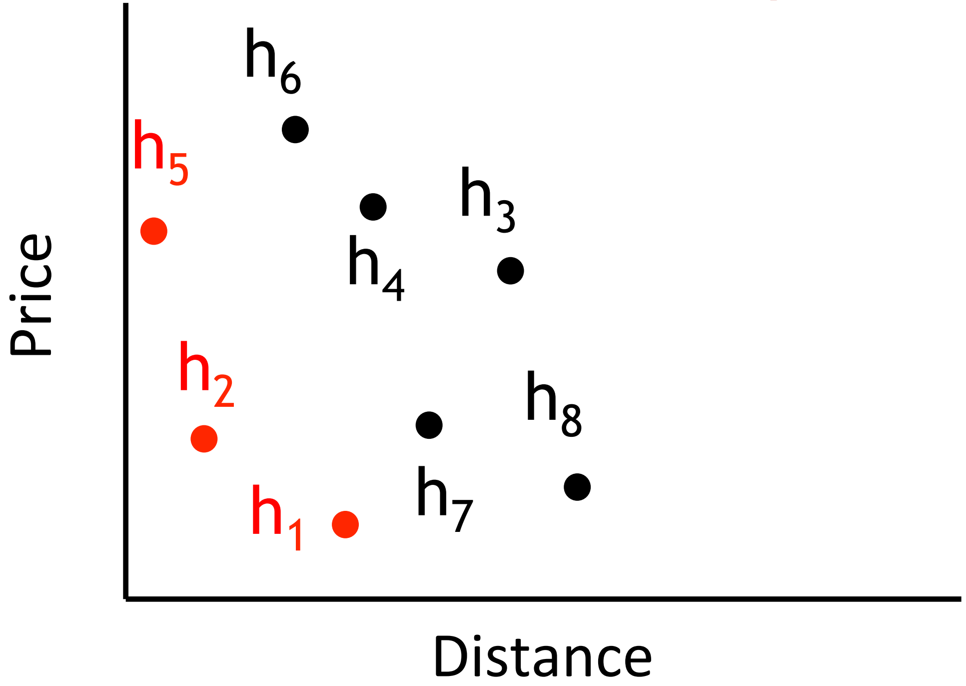


p_3, p_5, and p_7 are maximal

other points are not

Applications

- Databases: Skyline queries
  - Example "Skyline query": *minima*
  - Find "minimal" hotels in a price & distance graph



- Economics: Finding "Pareto Optimal" points

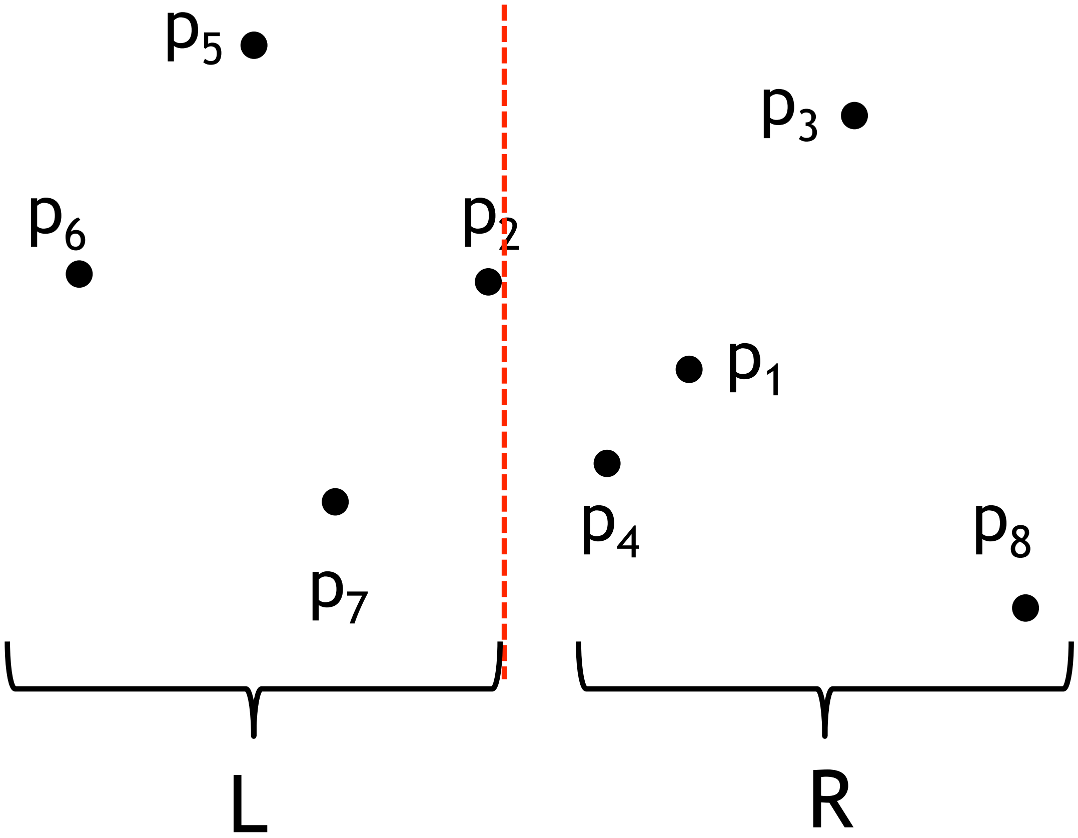
Alg 1: Brute Force

```
procedure bruteForceMaxima(Set P of n points):
   M = {}
   for each p in P:
      for each q in P:
         check if p is dominated by q
      if p is not dominated:
         M.insert(p);
   return M
```

Runtime: $O(n^2)$

Can we Divide & Conquer?

- Idea: Let's divide P vertically (on x-axis)



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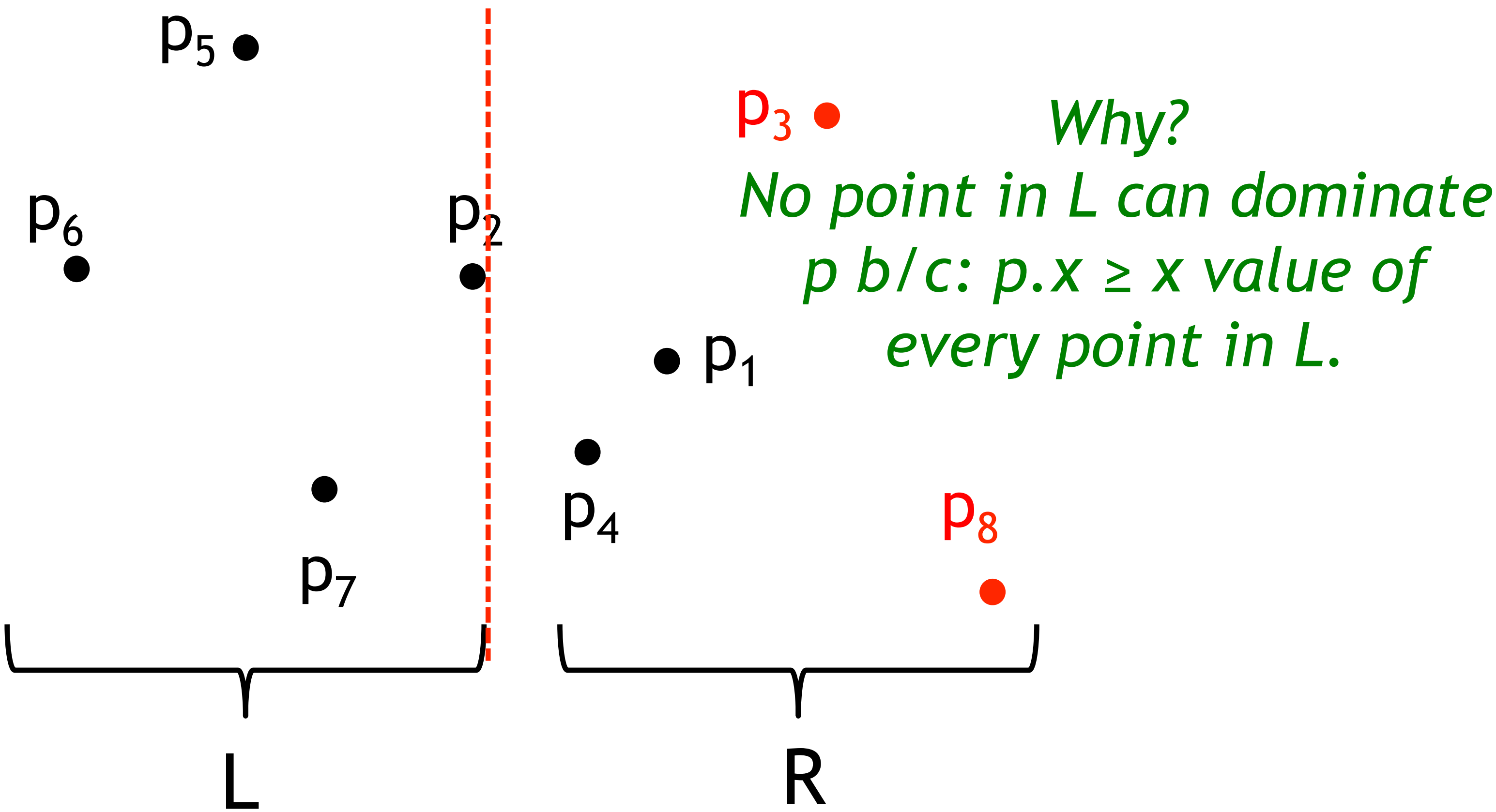
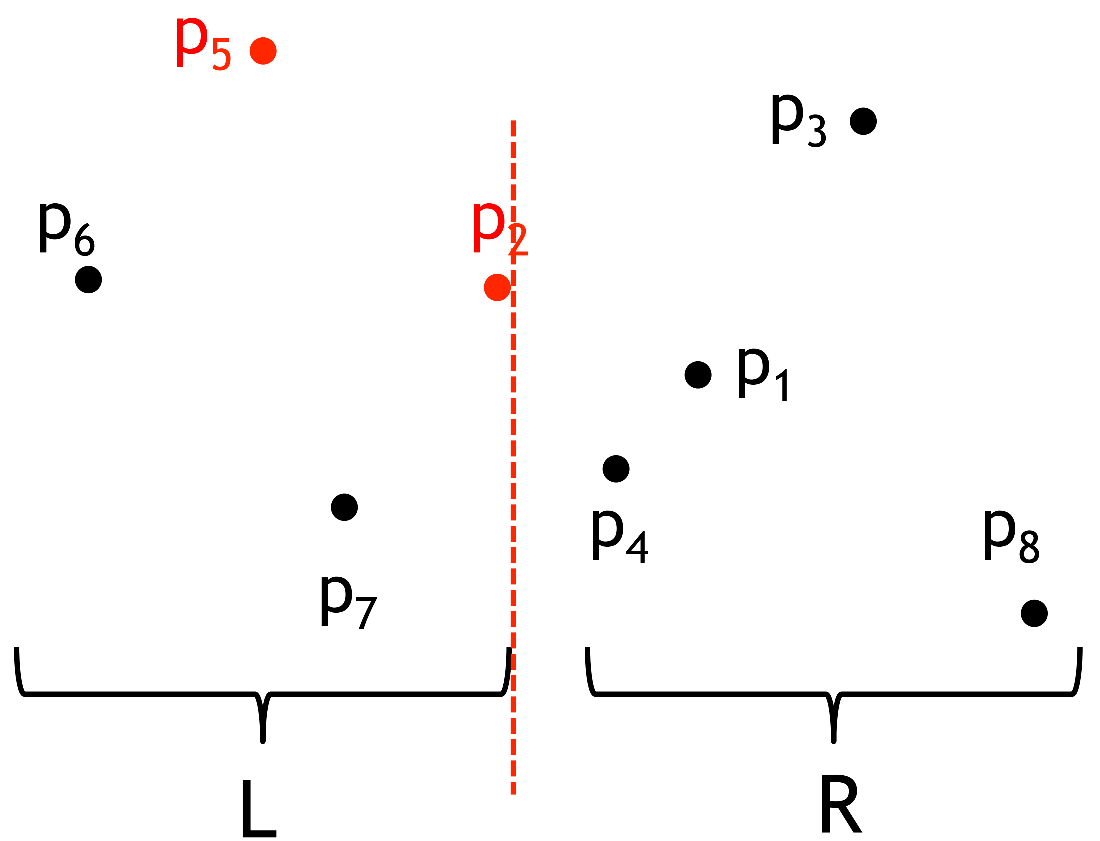
- Idea: Let's divide P vertically (on x-axis)



Q1: What can you say about a maximal point p in R?

A1: It is maximal in P as well.

Can we Divide & Conquer?



Q2: What can you say about a maximal point q in L?

A2: It's maximal iff q.y ≥ y value of every point in R.
In particular, let p be the point in R with max y*
Then q is maximal iff q.y ≥ p.y,*

DC-Maxima

```
Procedure Algorithm(Set P of n points):
   Sort P by x values;          → O(nlog(n)) work
   return DCMaxima(P)

Procedure DCMaxima (P sorted by x values):
   if (P.size == 1) return P;
   L = DCMaxima(P[1…n/2]);
   R = DCMaxima(P[n/2+1…n]);
   let p* be max y valued point in R; → O(n) work
   let M = R;
   for each q in L:
      if (q.y ≥ p*.y)                  → O(n) work
         M.insert(q);
   return M;
```

Total: O(n) work outside recursive calls

Runtime Analysis

Recursive part: $T(n) = 2T(n/2) + O(n)$

By Master Thm: $O(n\log(n))$

Total Work:

1. Initial Sorting: $O(n\log(n))$
2. Recursive part: $O(n\log(n))$

Total: $O(n\log(n))$

Exercise

After initial sorting by x-axis, design an $O(n)$ time algorithm (not DC) that gives all of the maximal points.

Note: Total time still $O(n\log(n))$ b/c of sorting. But post-sorting work is linear instead o f $O(n\log(n))$ of DCMaxima.

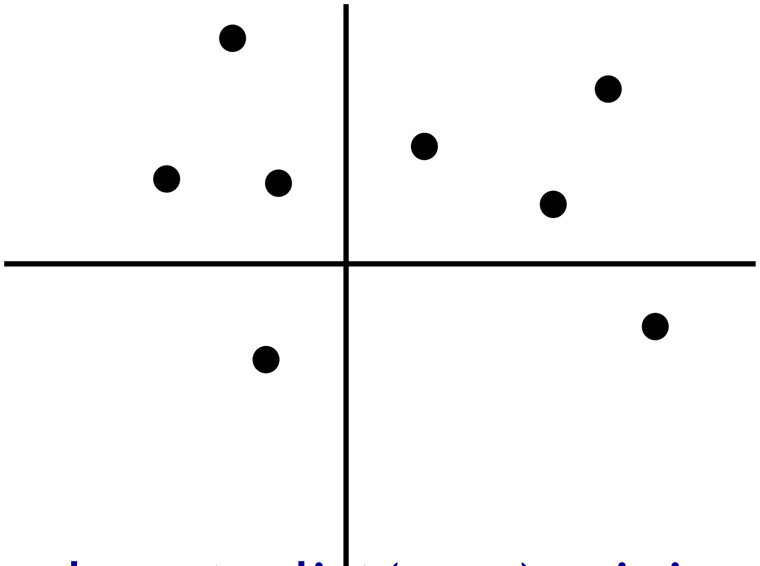
Fact: Ω $(n\log(n))$ lower bound for comparison based algs.

Outline For Today

1. 2-D Maxima
2. Closest Pair

Closest Pair Problem

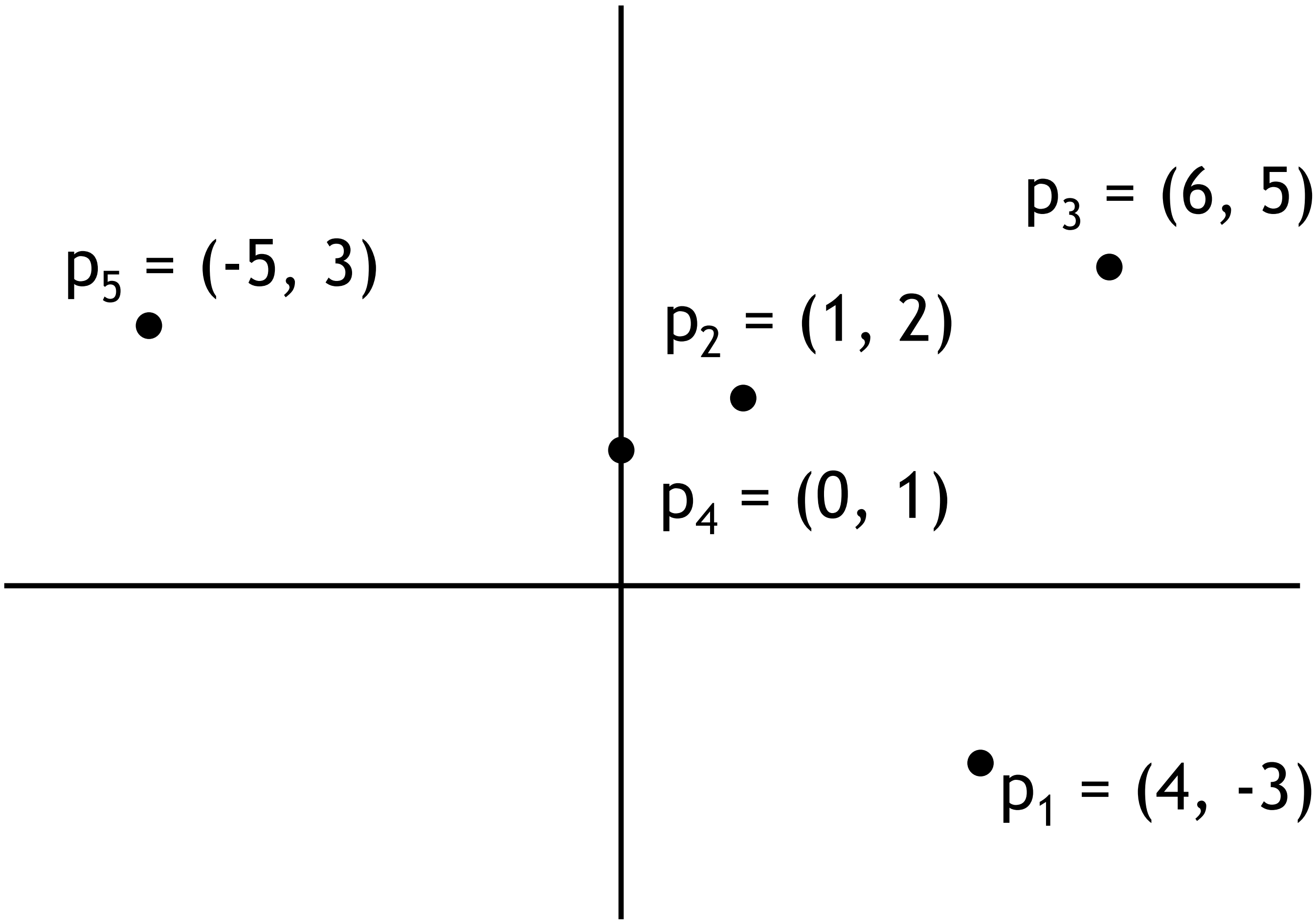
- Input: Set P of n 2-D points



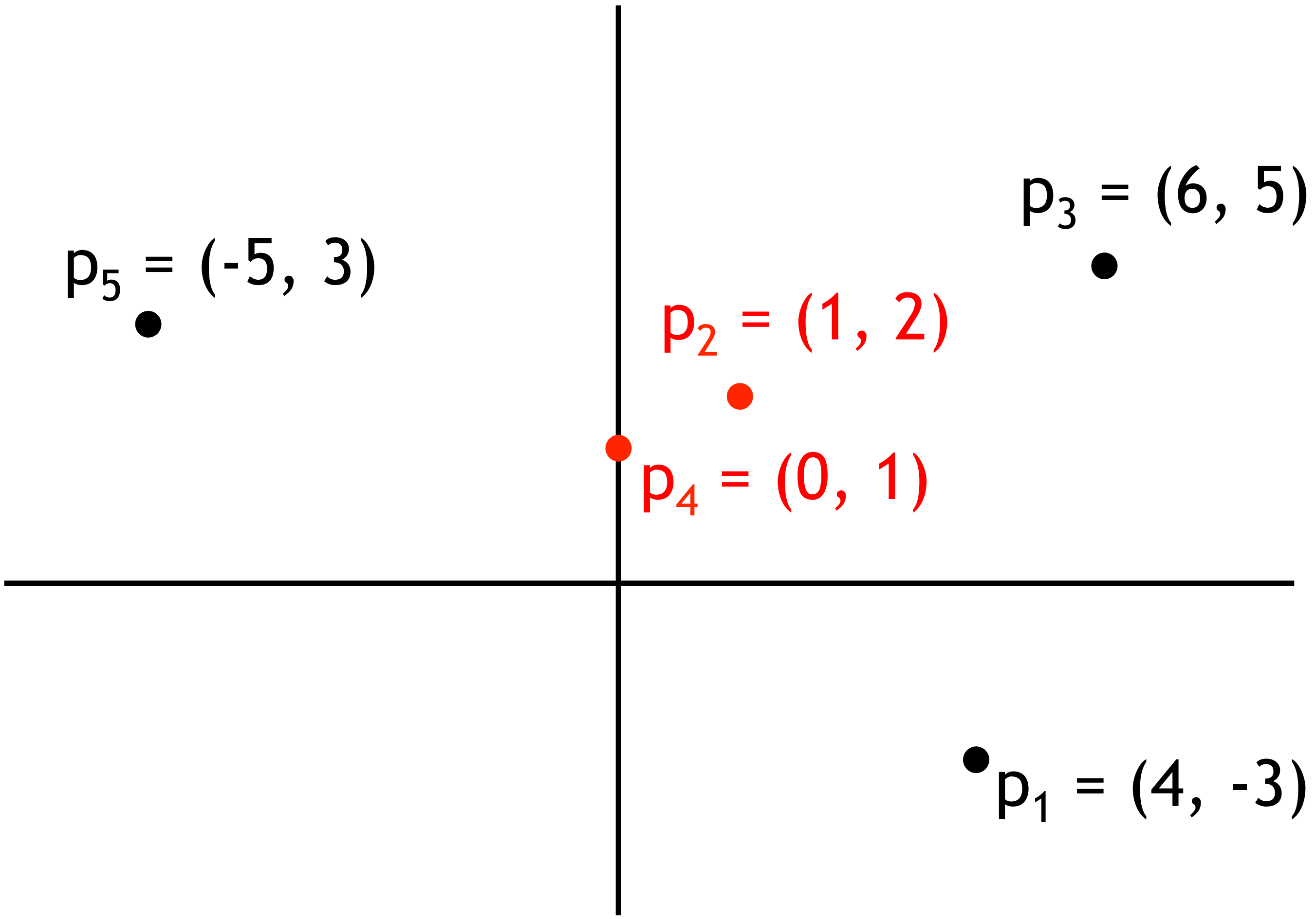
- Output: pair p and q s.t. dist(p, q) minimum over all pairs
- Break ties arbitrarily
- dist(p, q): Euclidean distance

$$dist(p,q) = \sqrt{(p.x - q.x)^2 + (p.y - q.y)^2}$$

Example



Example



Closest pair: (p_2, p_4); dist(p_2, p_4): sqrt(1^2+1^2) = sqrt(2)

Applications

- Very fundamental computational problem
  - Databases
  - Machine Learning
  - Image Processing
  - Computational Geometry

1-D Version

- $(x_1, x_2, \ldots, x_n) = (2.2, 5.8, 1.1, -3.0, 1.2, \ldots)$
- Just sort and scan:
  - compare each point with the next point in the sorted order
  - b/c closest pair has to be a consecutive pair
- Sort: O(nlog(n)) time
- Scan: O(n) time
- Total: O(nlog(n)) time

Question: Can we do (nlog(n)) in 2-D?

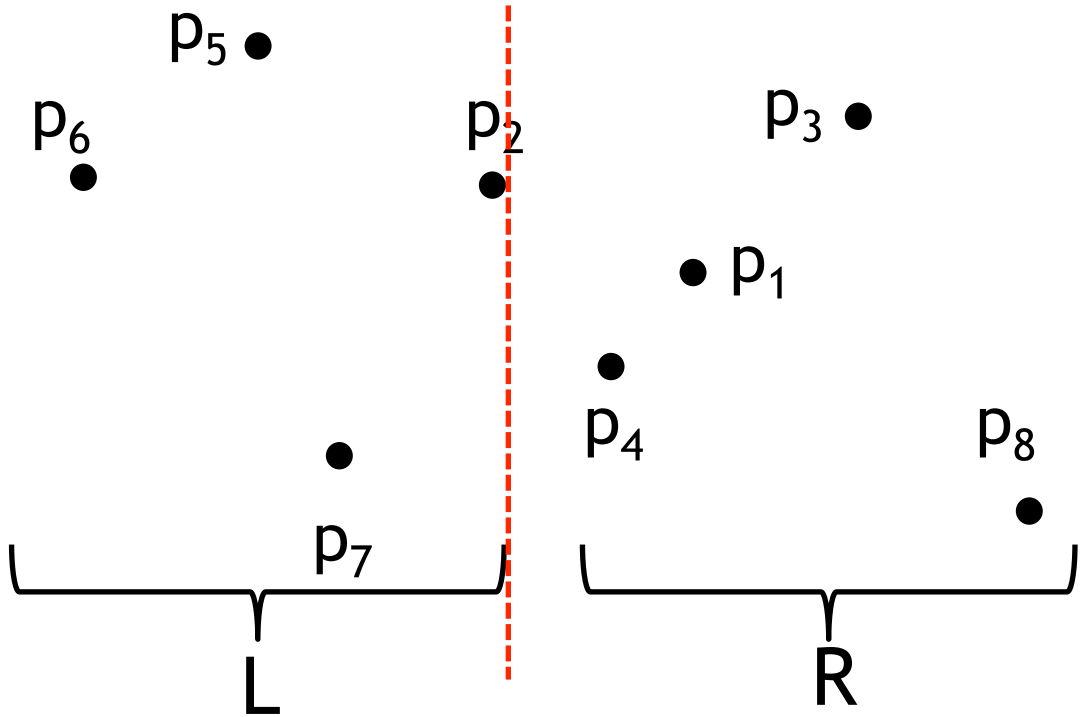
Alg 1: Brute Force

```
procedure bruteForceCP(Set P of n points):
    minPair = {}
    minDist = +∞
   for each p in P:
       for each q in P:
          if (dist(p, q) < minDist)
             minPair.insert(p, q);
             minDist = dist(p, q)
    return minPair
```

Runtime: $O(n^2)$

Can we Divide & Conquer?

- Same idea as maxima: Divide P on x-axis



Claim that doesn't require a proof: closest pair (p, q):

1. *(p, q) both in L;*
2. *(p, q) both in R; or*
3. *p is in L and q is in R*

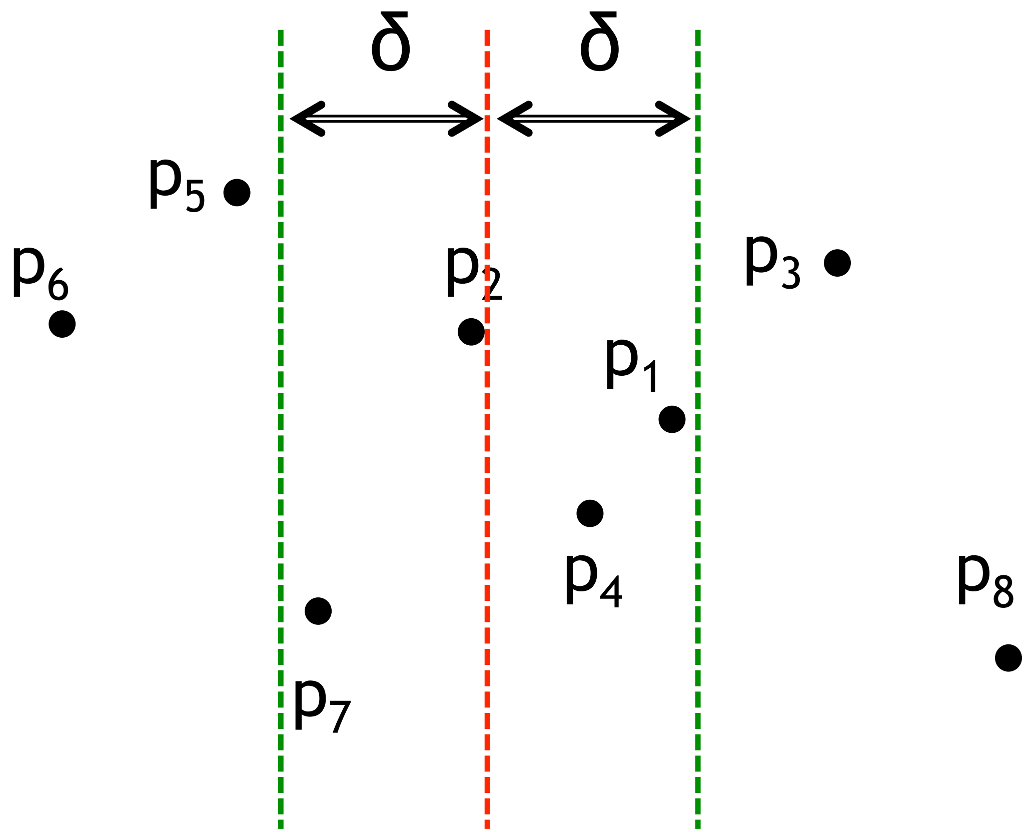
DC Algorithm Template:

```
procedure Algorithm(P of n points):
   sort P by x values
   DC-CP(P)
procedure DC-CP(P sorted by x values):
   if (P.size ≤ 3) compare all & return closest;
   pair_L = DC-CP(P[1,…,n/2])
   pair_R = DC-CP(P[n/2+1,…,n])
   pair_s = findMinSpanningPair(P)
   return min(pair_L, pair_R, pair_S)
```

Q: How can we find the spanning pair quickly?

Observation 1

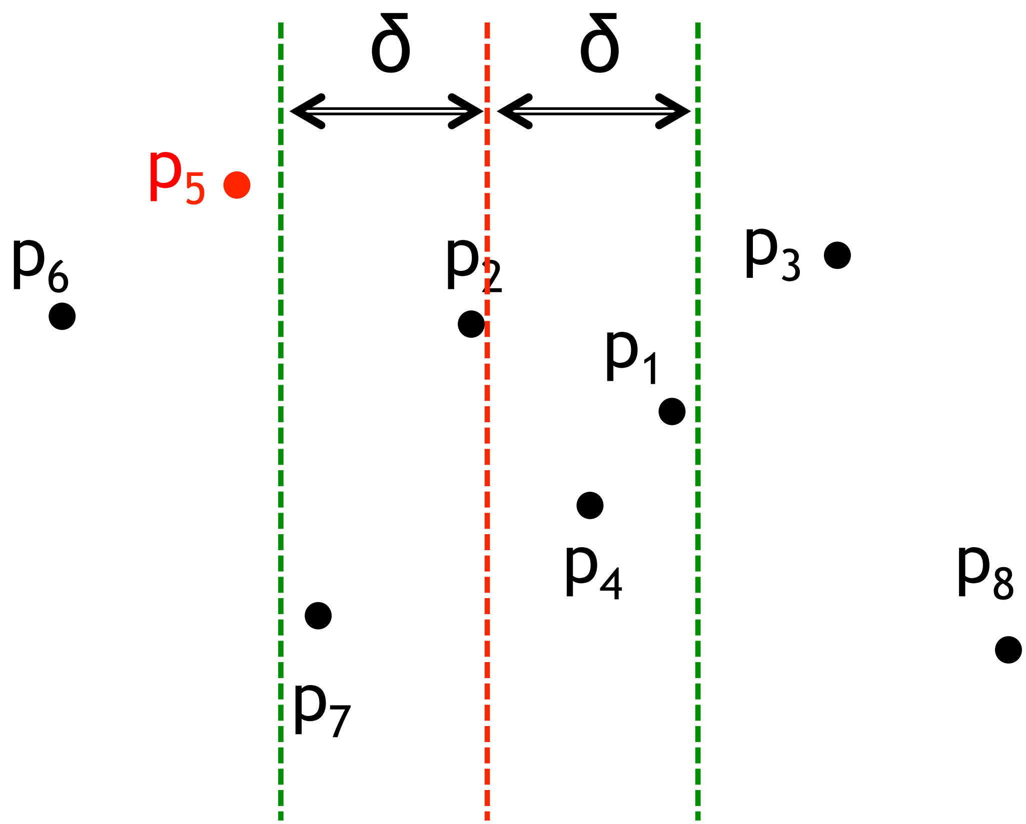
- Let $\delta = \min(\mathrm{dist}(pair_L), \mathrm{dist}(pair_R))$



- Then $pair_s$ (if closest globally) lies in the above 2δ-wide green strip

Q: Why?

Example for Observation 1

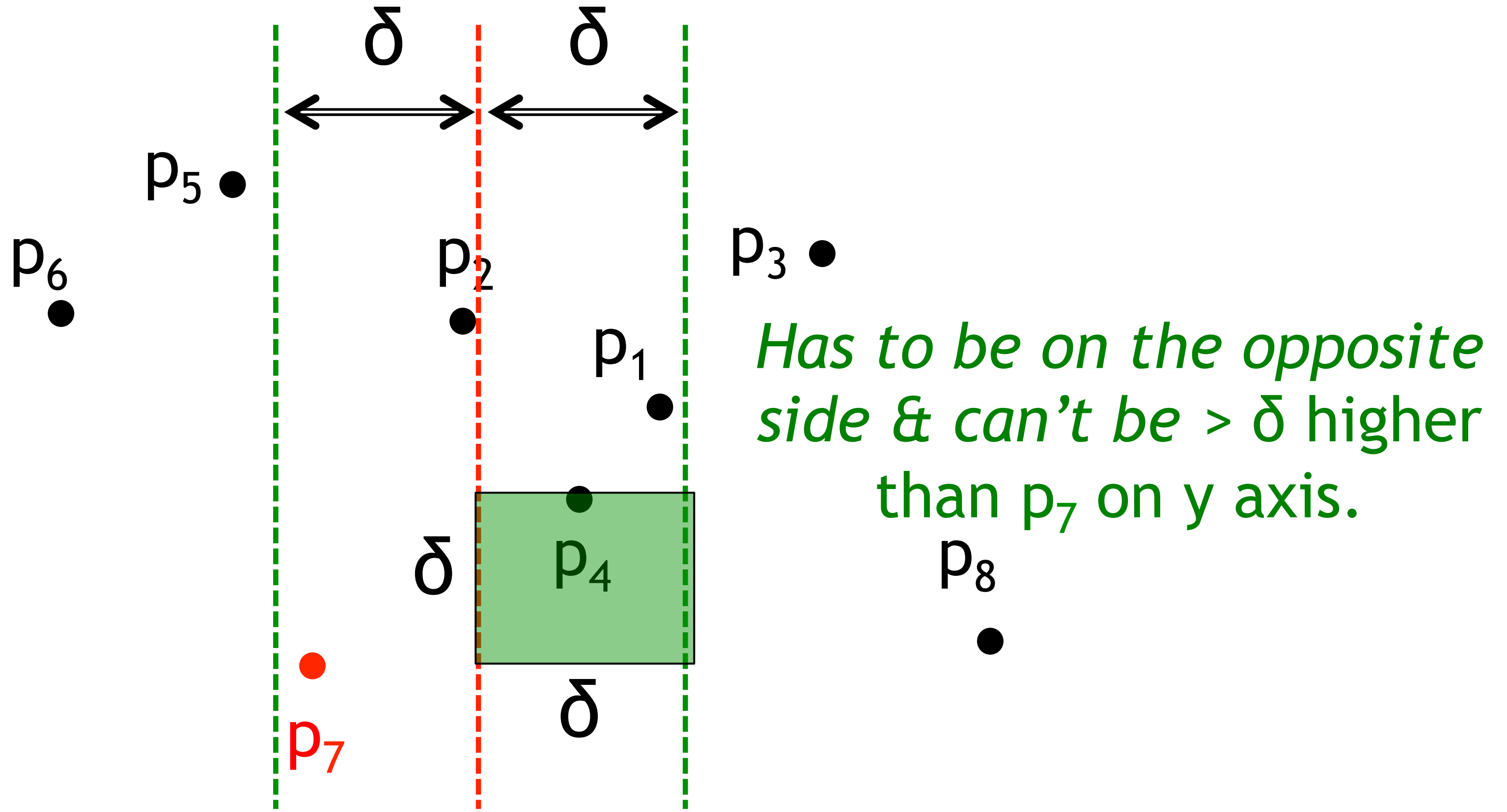


Q: Can p_5 be part of a globally closest pair$_s$?

A: No. Any p in R is already has dist > δ to p_5.

Observation 2

- Say, p_7 (the lowest y valued point in strip) is in $pair_s$

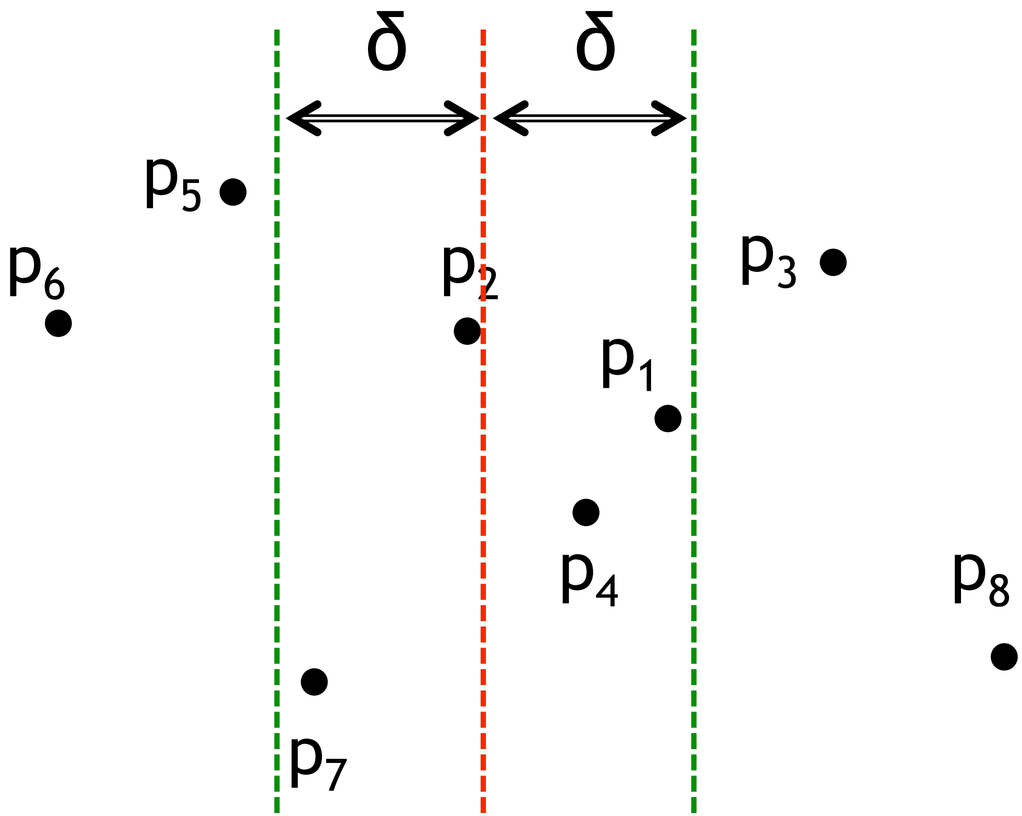


- Then the other point can only lie in this δxδ square.

Q: Why?

Core Idea For Finding Spanning Pair

1. Start from lowest y valued point in the strip
2. Search the δxδ square points on the opposite side
3. Repeat 1 & 2this for the next lowest y-valued point
4. So on and so forth…



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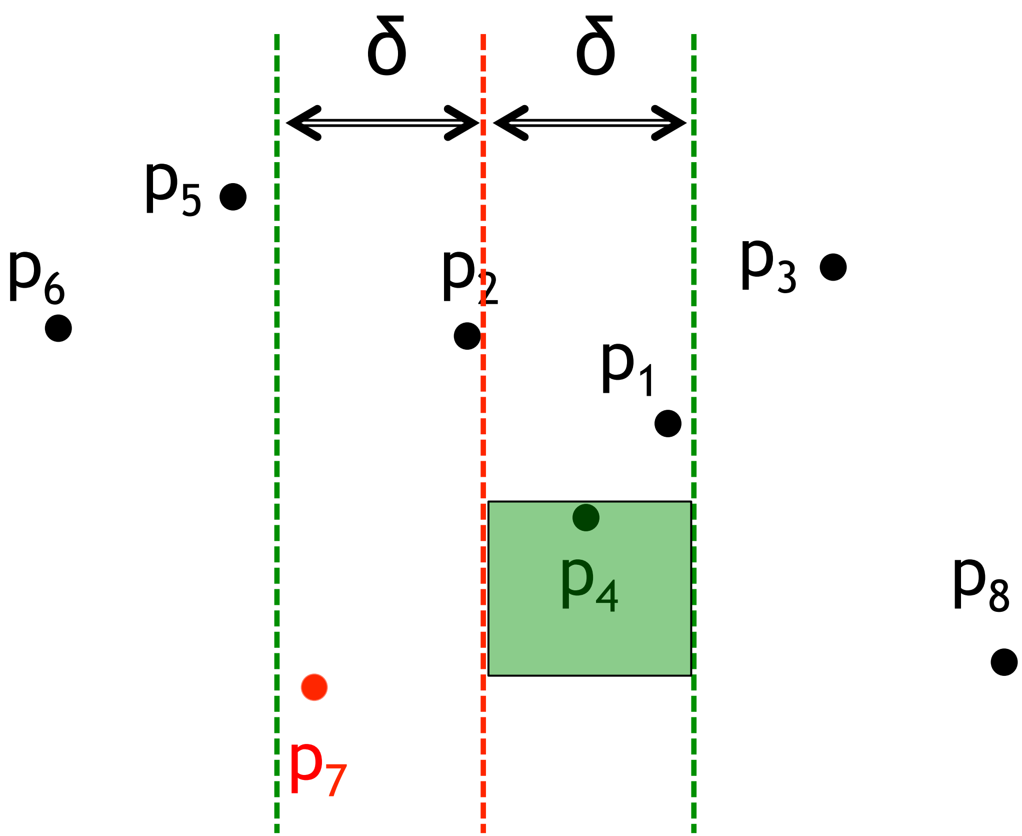

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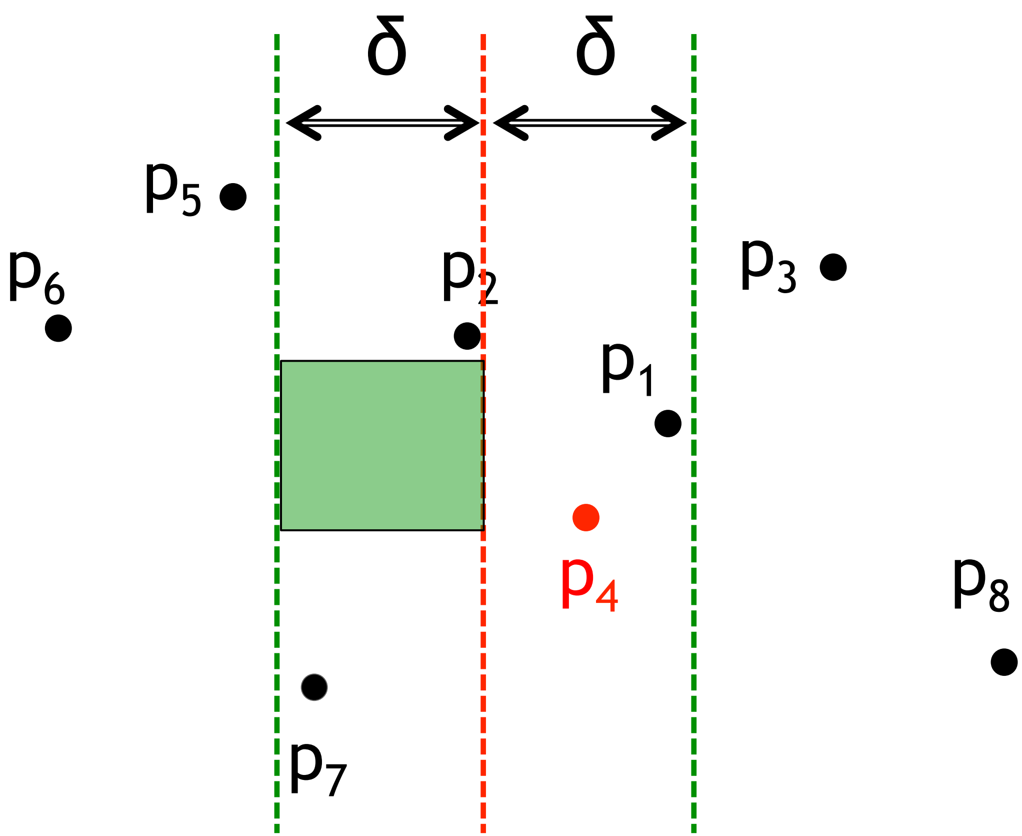

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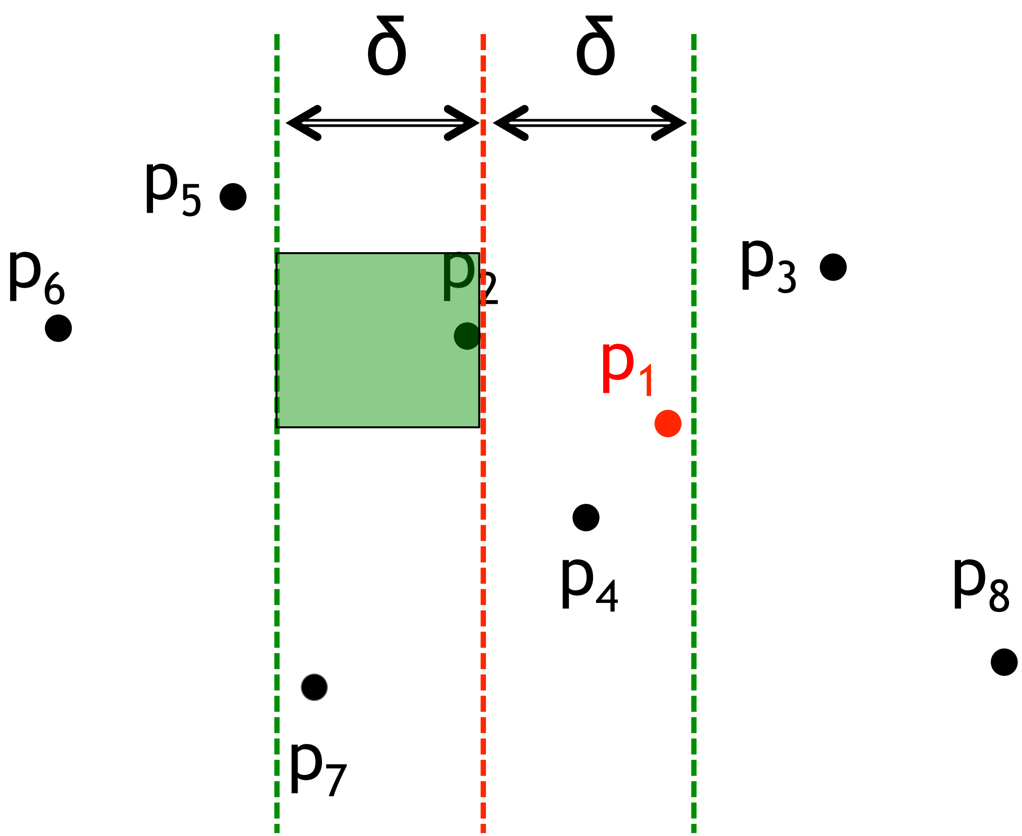

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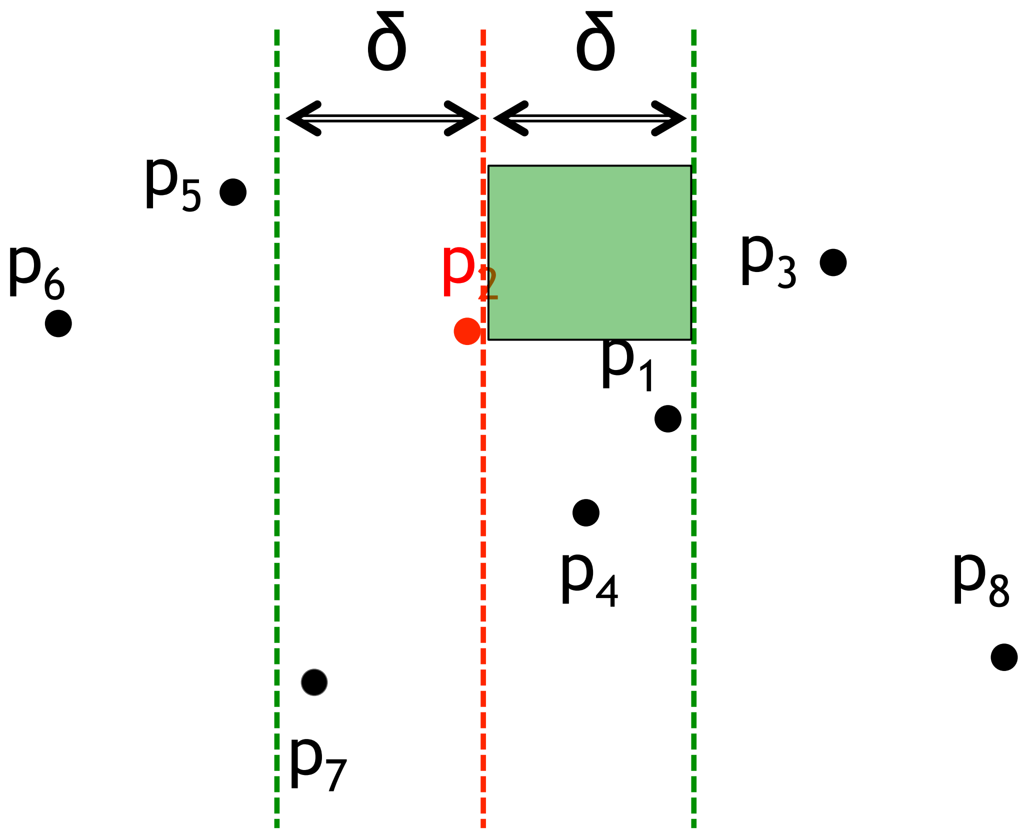
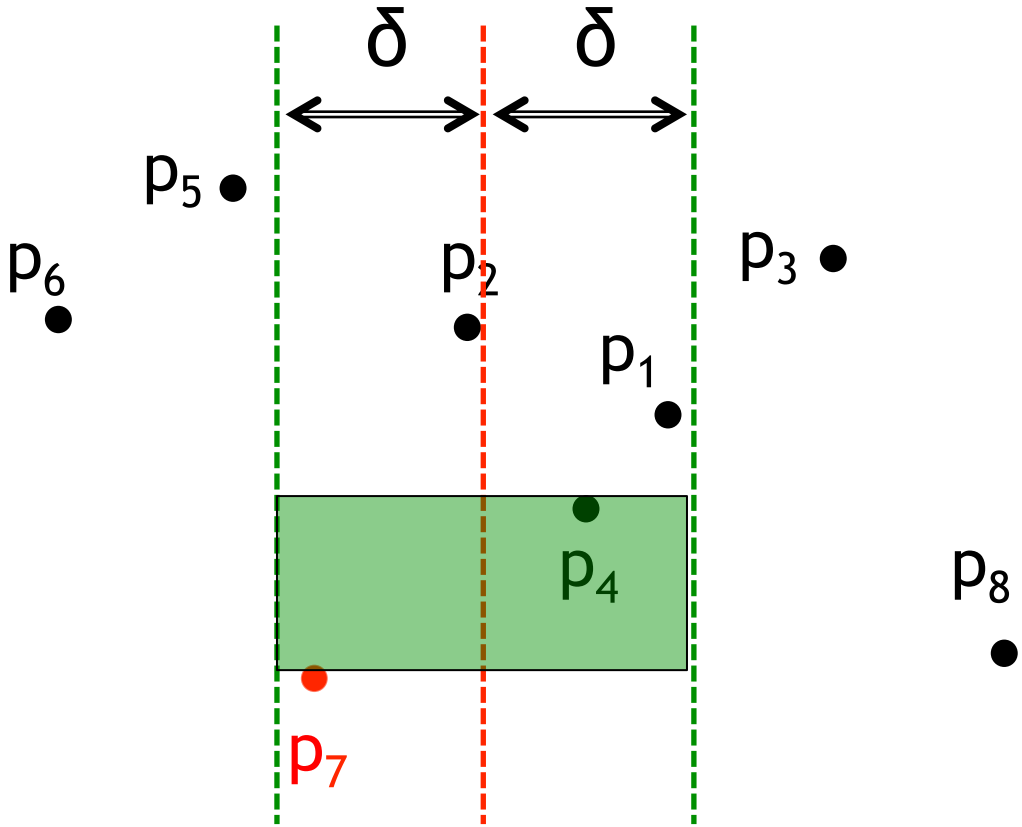


A More Practical Idea

- Don't differentiate between same and opposite side
- Just search the 2δxδ above rectangle each time



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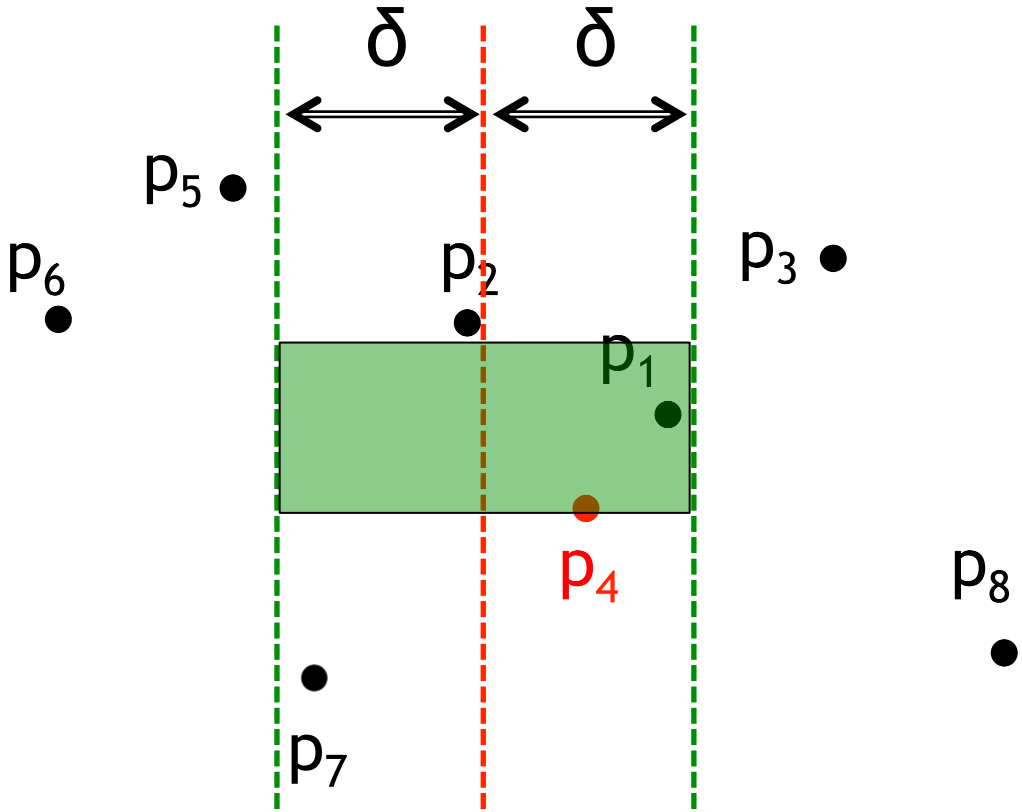

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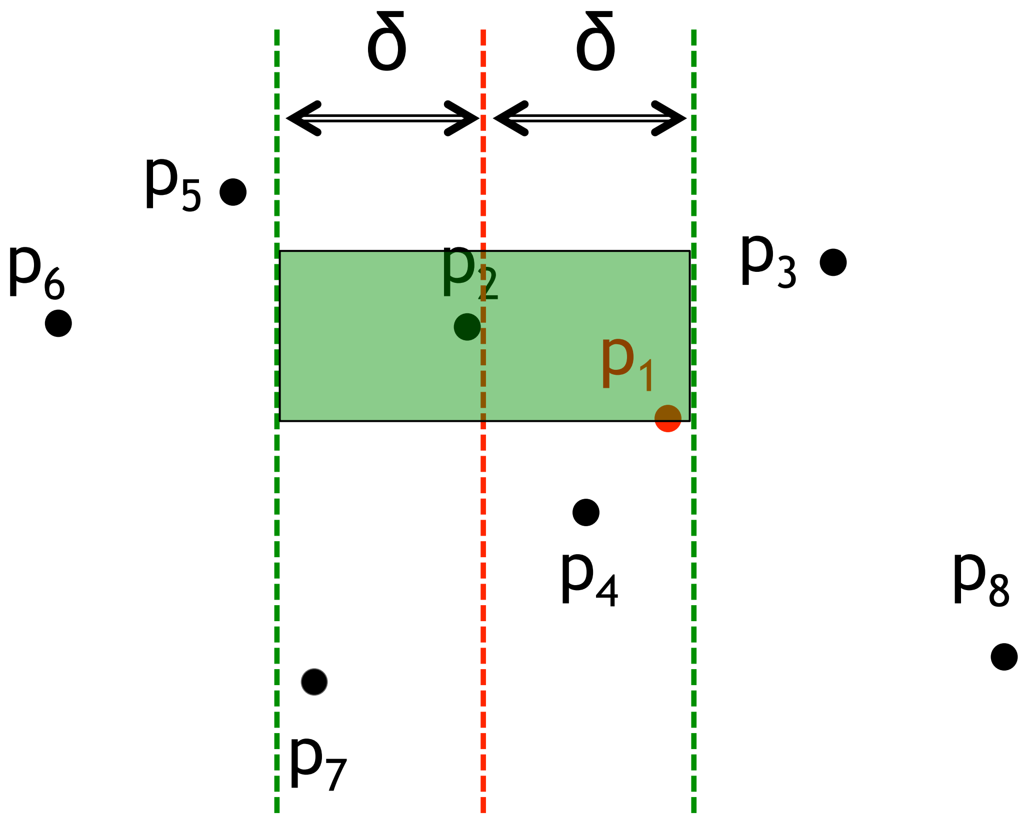

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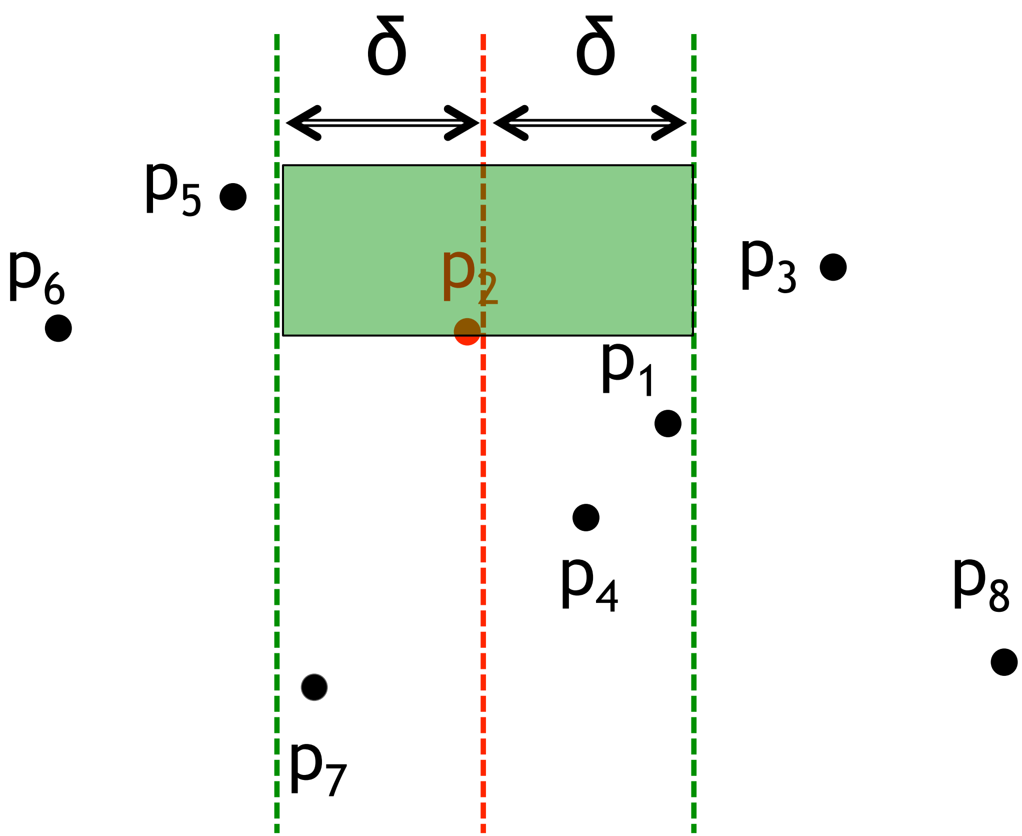


DC-CP 1 (1)

```
procedure Algorithm(P of n points):
   sort P by x values
   DC-CP1(P)
procedure DC-CP1(P sorted by x values):
   if (P.size ≤ 3) compare all & return closest;
   pair_L = DC-CP1(P[1,…,n/2])
   pair_R = DC-CP1(P[n/2+1,…,n])
   δ = min(dist(pair_L, pair_R))
   pair_s = findMinSpanningPair(δ, P)
   return min(pair_L, pair_R, pair_S)
```

DC-CP 1 (2)

```
procedure findMinSpanningPair (δ, P):
   S = select each p in P s.t |p_{n/2}.x-p.x| ≤ δ  ⟶ O(n)
   sort(S by increasing y values)  ⟶ O(nlog(n))
   minDist = +∞
   minPair = null;
   for i = 1 to S.length:  ⟶  O(n)
      j = i+1 (compare S[i] to only the points above S[i])
      while (|S[j].y – S[i].y| ≤ δ):
         if (dist(S[i], S[j]) < minDist):
            minPair = (S[i], S[j]);        ⟶ ?
            minDist = dist(S[i], S[j])
         j++;
   return minPair
```

Q: How many times does the while loop execute?

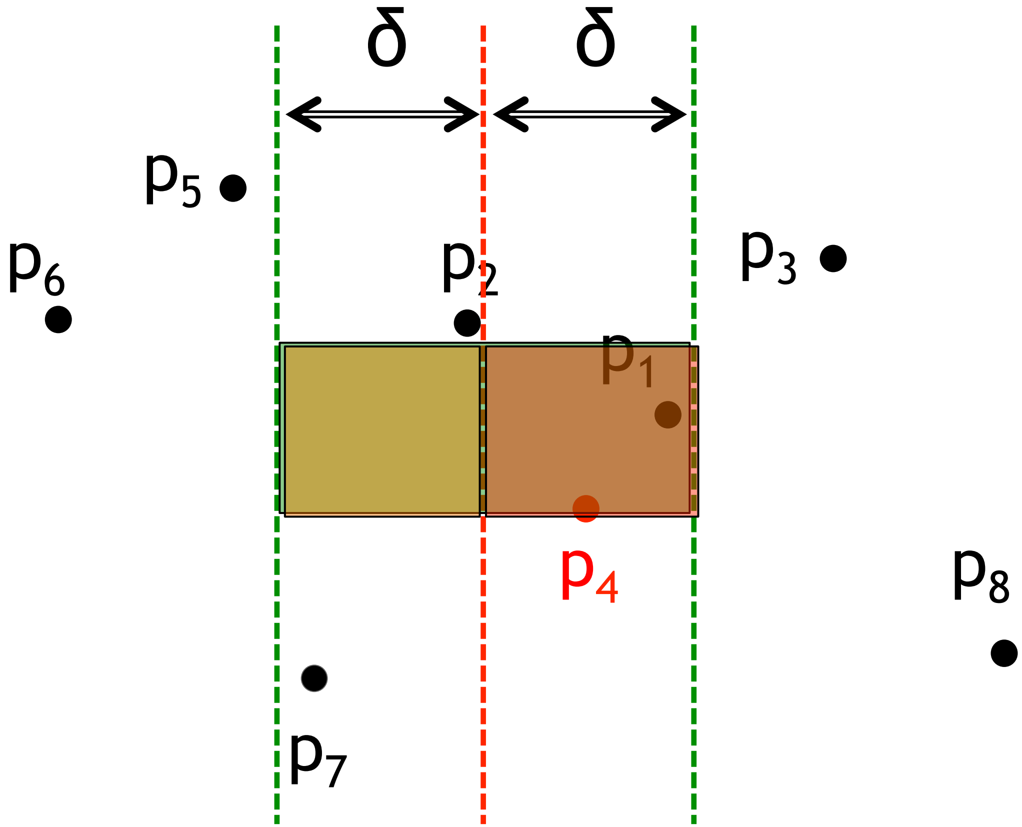
Claim: O(1) times

For a point p, how many times does while loop execute?

Obs: as many times as there are points in the 2δ x δ rectangle.

Q: How many points can be in a 2δ x δ rectangle?

A: As many as in the left δ x δ square + right δ x δ square.



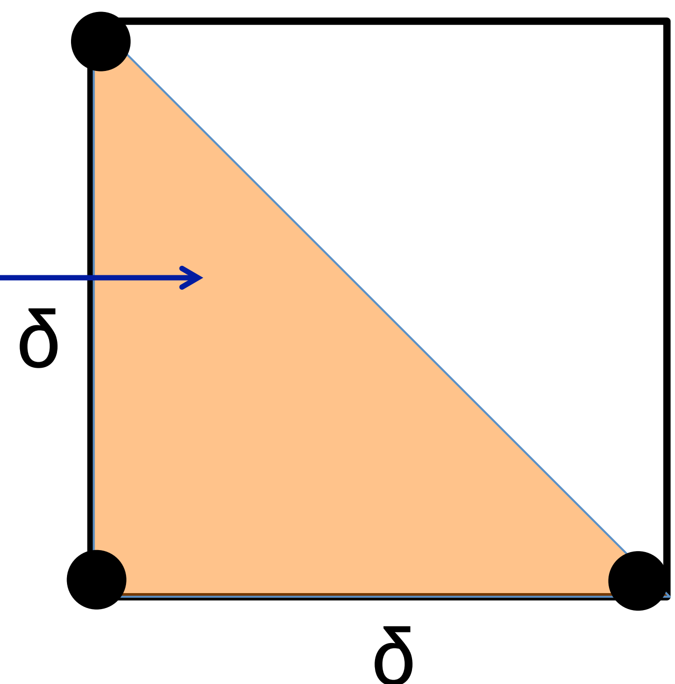
Points in a δ x δ Square

Recall: Each point in the square is at least at distance δ.

Q1: How many can fit the lower triangle?

A: 3

no other point can be inside the triangle except the other two corners



Points in a δ x δ Square

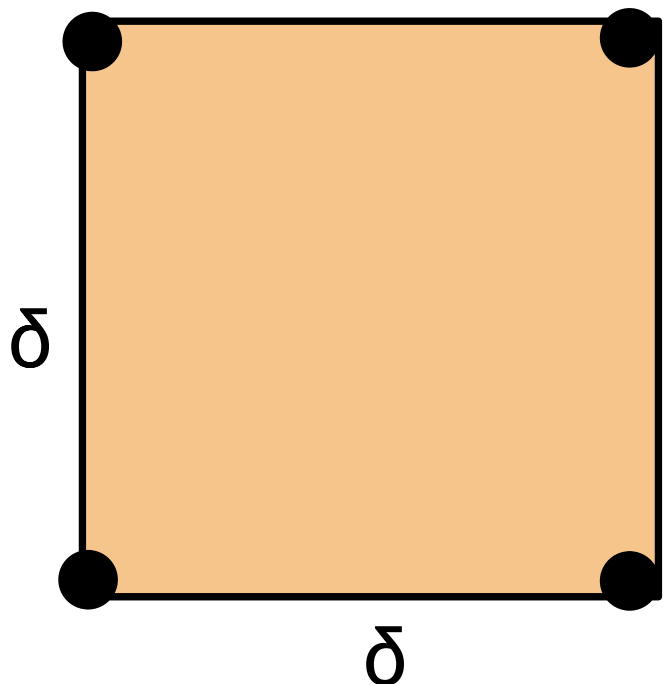
Recall: Each point in the square is at least at distance δ.

Q1: How many can fit the lower triangle?

A: 3

Q2: How many can fit the square?

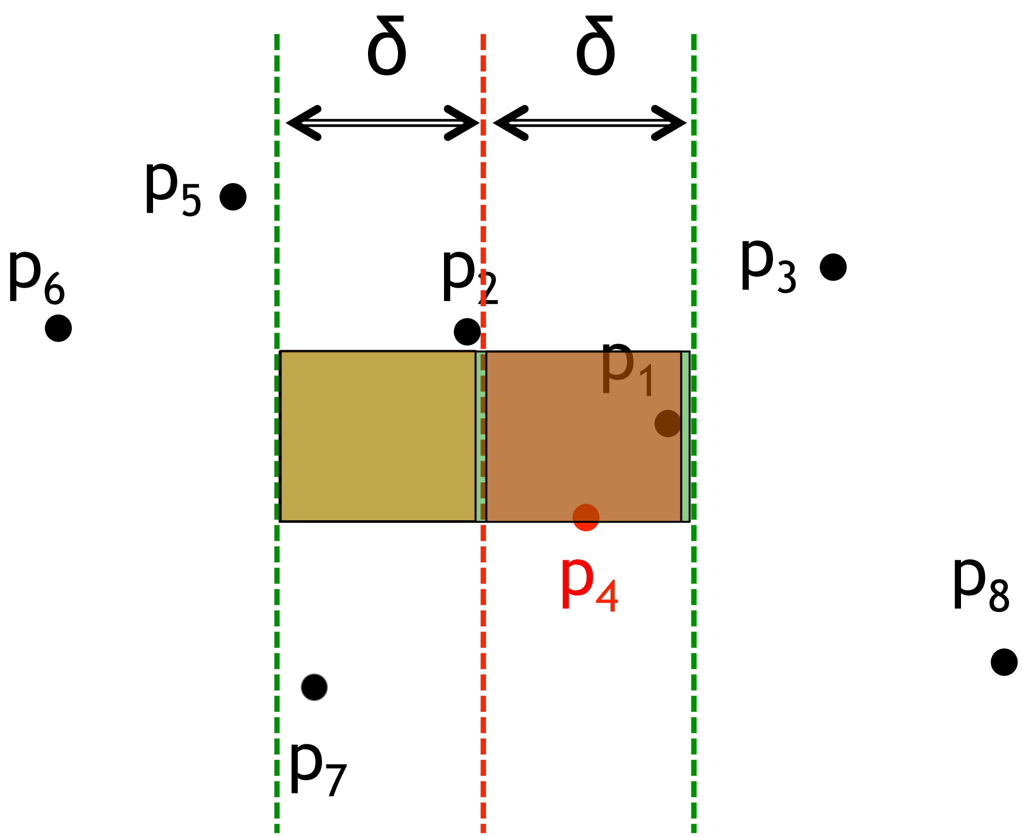
A: 4



For a point p, how many times does while loop execute?

Obs: as many times as there are points in the 2δ x δ rectangle.

points in the the 2δ x δ rectangle ≤ 4 + 4 = 8



DC-CP 1 (2)

```
procedure findMinSpanningPair (δ, P):
   S = select each p in P s.t |P[n/2].x-p.x| ≤ δ
   sort(S by increasing y values)
   minDist = +∞
   minPair = null;
   for i = 1 to S.length:
      j = i+1
      while (|S[j].y – S[i].y| ≤ δ):
         if (dist(S[i], S[j]) < minDist)
            minPair = (S[i], S[j])
         j++;
   return minPair
```

O(n)

O(nlog(n))

O(n)

O(1)

Total:O(nlog(n))

DC-CP 1: Runtime Analysis (1)

```
procedure DC-CP1(P sorted by x values):
   if (P.size ≤ 3) compare all & return closest;
   pair_L = DC-CP1(P[1,…,n/2])
   pair_R = DC-CP1(P[n/2+1,…,n])
   δ = min(dist(pair_L, pair_R))
   pair_s = findMinSpanningPair(δ, P)
   return min(pair_L, pair_R, pair_S)
```

Recursive part: Outside Recursive Calls: nlog(n) work.

$$T(n) = 2T(n/2) + n\log(n)$$

Exercise: Show by induction or recursion tree that total work of recursive part is $O(n\log^2(n))$.

Total Alg Work: $O(n\log(n)) + O(n\log^2(n)) = O(n\log^2(n))$.

Can improve to $O(n\log(n))$ by pre-sorting P also by y.

Shamos' DC Algorithm (1975) (1)

```
procedure Algorithm(P of n points):
   P_x=sort P by x values in increasing order
   P_y=sort P by y values in increasing order
   DC-Shamos(P_x, P_y)
procedure DC-Shamos(P_x, P_y):
   if (P_x.size ≤ 3) …;
   P_yL = select from P_y points with x ≤ P_x[n/2].x
   P_yR = select from P_y points with x > P_x[n/2].x
   pair_L = DC-Shamos(P_x[1,…,n/2], P_yL)
   pair_R = DC-Shamos(P_x[n/2+1,…,n] P_yR)
   δ = min(dist(pair_L, pair_R))
   pair_s = findMinSpanningPairShamos(δ, P_x, P_y)
   return min(pair_L, pair_R, pair_S)
```

Sorted by y already!

Shamos' DC Algorithm (1975) (2)

Don't need to sort by y!

Sorted by y already!

```
procedure findMinSpanningPairShamos(δ, Px, Py):
   S = select each p in Py s.t |Px[n/2].x-p.x| ≤ δ
   minDist = +∞
   minPair = null;
   for i = 1 to S.length:
      j = i+1
      while (|S[j].y – S[i].y| ≤ δ):
         if (dist(S[i], S[j]) < minDist):
            minPair = (S[i], S[j])
         j++;
   return minPair
```

O(n)

O(n)

O(1)

Total: O(n)

Runtime Analysis of Shamos' Algorithm

Key Idea of Shamos: Avoid sorting by y values in each recursive call by pre-sorting P by y values.

Recursive part: Outside Recursive Calls: O(n) work.

$$T(n) = 2T(n/2) + O(n)$$

By Master Thm, total: O(nlog(n))

Total Work for Shamos

$$O(n\log(n)) + O(n\log(n)) = O(n\log(n)).$$