Lecture 14: BFS/DFS Applications

CS 341: Algorithms

Tuesday, March 5th 2019

Outline For Today

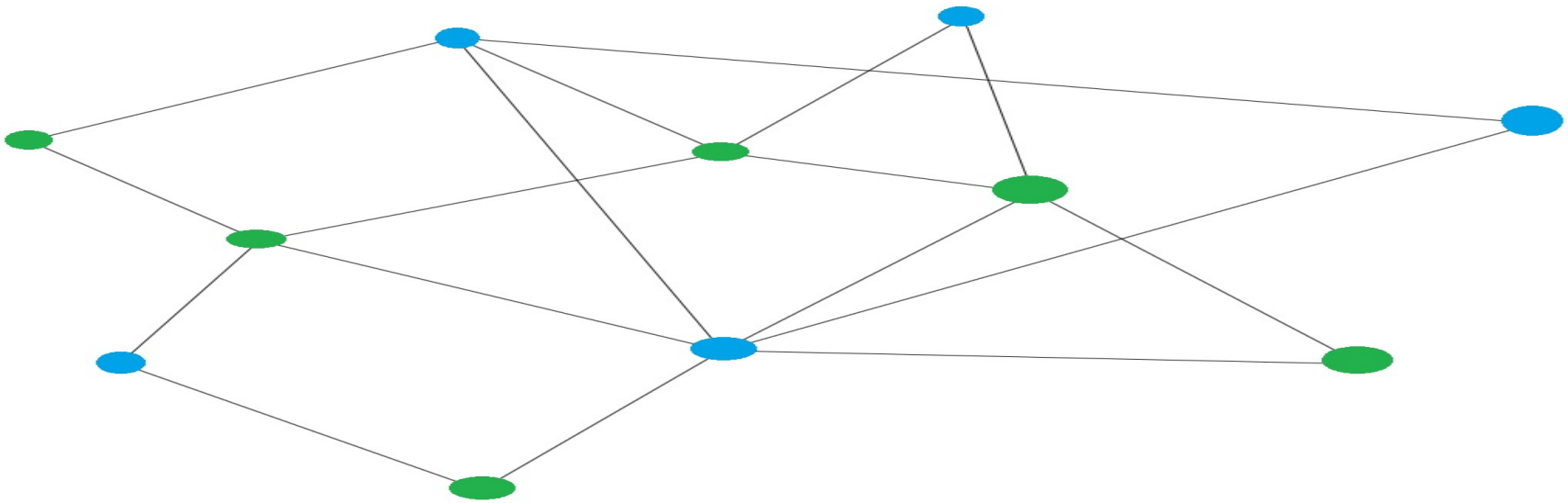
1. Graph Terminology
2. BFS/DFS & BFS/DFS Tree
3. Application 1: Unweighted Single Source Shortest Paths
4. Application 2: Bipartiteness/2-coloring
5. Application 3: Connected Comp. in Undir. Graphs
6. Application 4: Topological Sort

Outline For Today

1. Graph Terminology
2. BFS/DFS & BFS/DFS Tree
3. Application 1: Unweighted Single Source Shortest Paths
4. Application 2: Bipartiteness/2-coloring
5. Application 3: Connected Comp. in Undir. Graphs
6. Application 4: Topological Sort

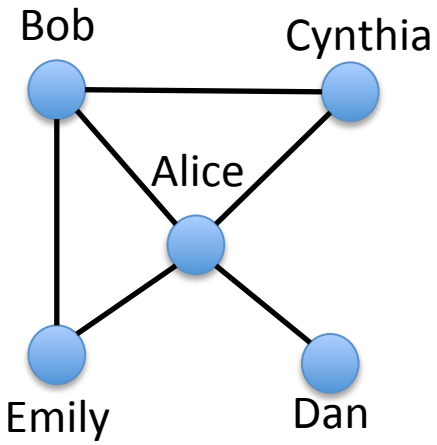
Graphs

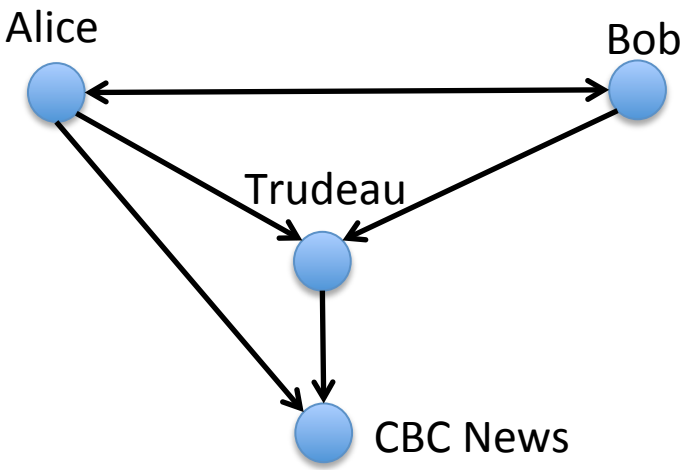
- A graph G(V, E) is a pair of sets:
  - V is a set of nodes/vertices
  - E is a set of edges (u,v) s.t u,v ∈ V



Examples

- Social Networks:
  - FB: V: people; E: (u, v) → u and v are friends
  - Twitter: V: people/organizations; E: (u, v) → u follows v





- World Wide Web: V: web pages; E: (u,v) → page u links to v
- Molecular Networks: V: atoms; E: (u, v) → bond btw u and v
- Many more …

Some Graph Terminology (1)

- Directed vs Undirected
  - Directed: edges not (necessarily) symmetric
    - E.g. Twitter, WWW
  - Undirected: edges are symmetric $(u, v) \in E \Rightarrow (v, u) \in E$
    - FB friendship graph, Road-maps (V: cities, E: roads)
- Simple vs Multigraphs
  - Simple: no parallel edges can exist between (u, v)
  - Multigraphs: parallel edges are allowed

Some Graph Terminology (2)

- Cyclic vs Acyclic
  - Cyclic: can start from v; follow a path; and come back to v
  - Acyclic: no such *cycles* exist in the graph
- Connected vs Unconnected
  - Connected: graph in "one piece" (will elaborate more)
  - Unconnected: graph can be in "multiple pieces"

More Terminology & Conventions (1)

- Convention 1
  - $|V| = n$
  - $|E| = m$
- Note: m is not a free parameter!

Q: In an undirected G w/ n vertices, what's the max # edges?

A: $m \leq$ (n choose 2)$=n(n-1)/2$

Q: In a directed G w/ n vertices, what's the max # edges?

A: $m \leq 2n(n-1)/2=n(n-1)$

Q: In an (undirected) connected G w/ n vertices, the min # edges?

A: $m \geq n-1$ (Exercise: Proof by induction on n)

More Terminology & Conventions (2)

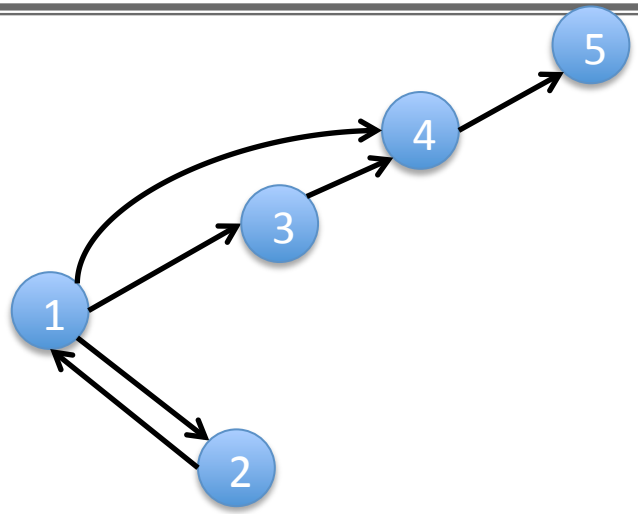
- If m varies between O(n) and $O(n^2)$

  $\log(m)=\Theta(\log(n))$

- Convention 2: We'll always use log(n) in analysis (not log(m))
- E.g: Dijkstra's Alg is O(nlog(n)) instead of O(nlog(m)).
- Degree of a vertex:
  - Out Degree: out-deg(v): # outgoing edges of v
    - i.e # $(v, w) \in E$
  - In Degree: in-deg(v): # incoming edges of v
    - i.e # $(w, v) \in E$
  - Note: deg(v) usually means out-deg not in-deg

2 Common Graph Storage Formats (1)

1. Adjacency Matrix: an n x n matrix A
   - A[i, j] = 1 if (i, j) ∈ E
   - A[i, j] = 0 otherwise
   - $O(n^2)$ storage.
- Operations:
  - Lookup (u,v) exists: O(1)
  - Iterate over u's out/in edges: O(n)
- Usually good for very "dense" graphs, i.e. when most edges exist



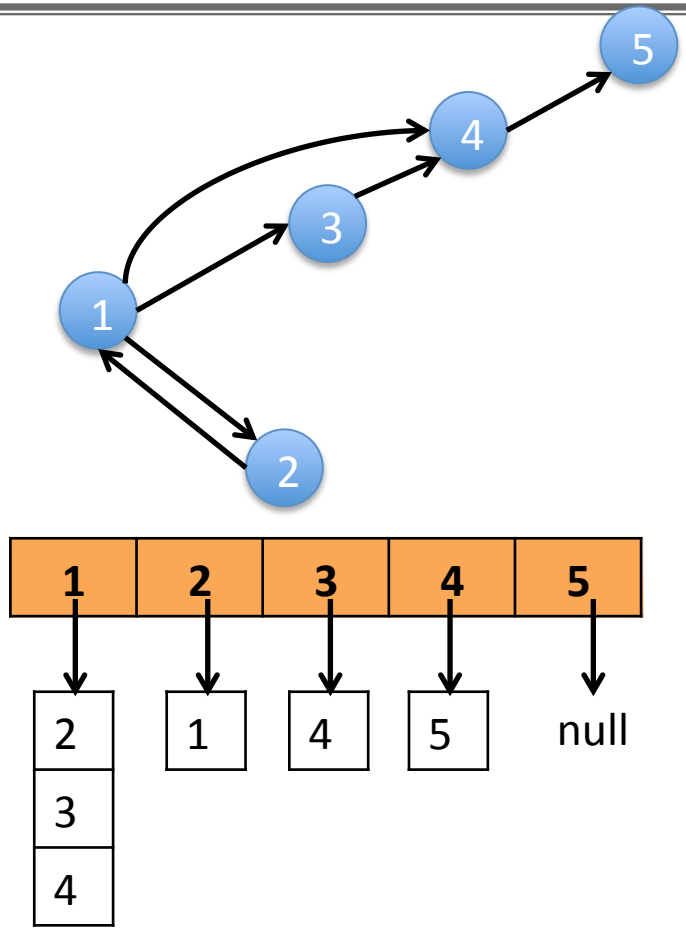
	1	2	3	4	5
1	0	1	1	1	0
2	1	0	0	0	0
3	0	0	0	1	0
4	0	0	0	0	1
5	0	0	0	0	0

2 Common Graph Storage Formats (2)

2. Adjacency List

- O(m + n) storage.
- Operations:
  - Lookup (u,v) exists: deg(u)
  - Iterate over u's out/in edges: deg(u)
- Usually good for "sparse" graphs

Mostly, we'll be using Adj. List Format



If needed, can also store incoming edges in separate arrays

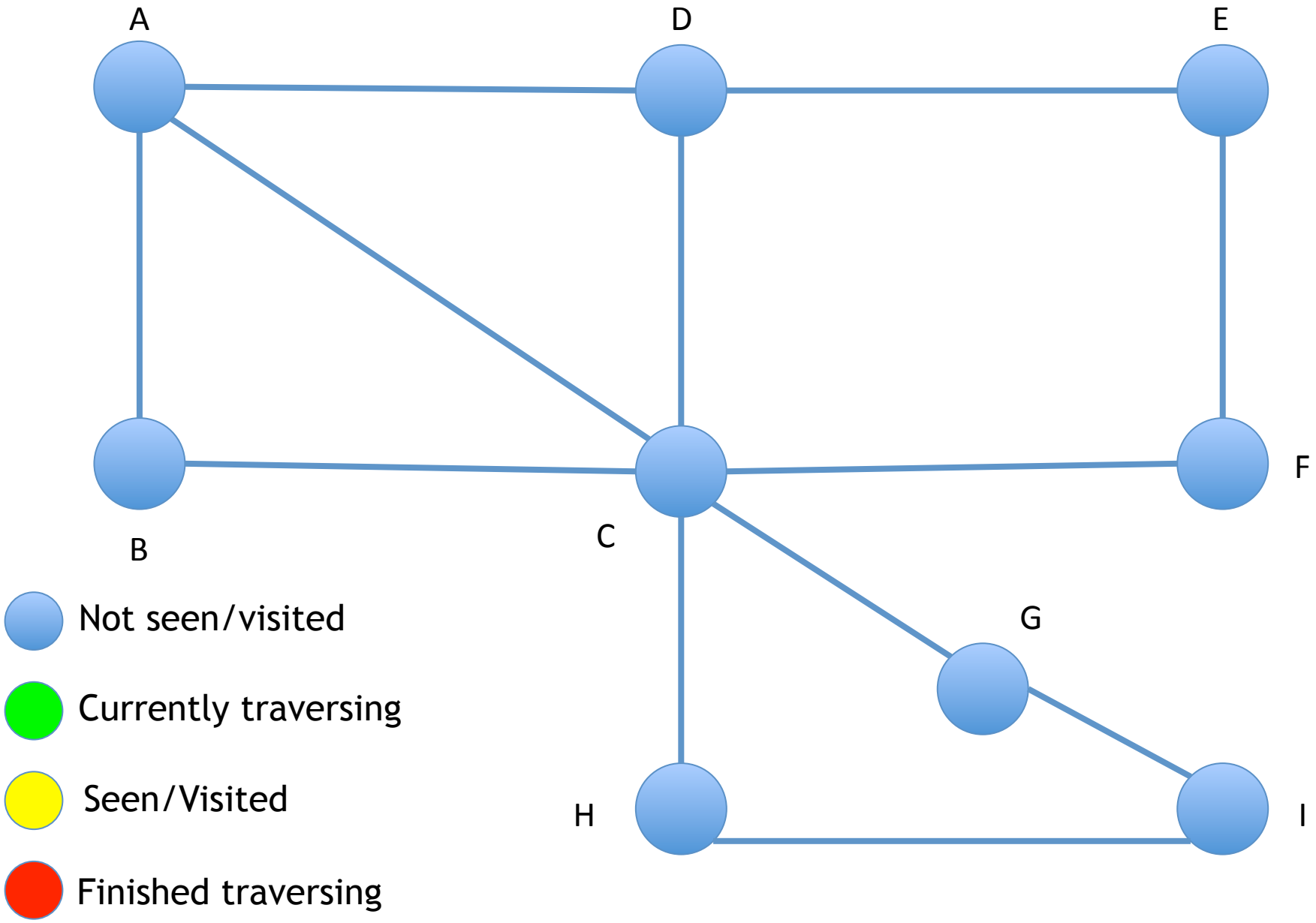
Outline For Today

1. Graph Terminology
2. BFS/DFS & BFS/DFS Tree
3. Application 1: Unweighted Single Source Shortest Paths
4. Application 2: Bipartiteness/2-coloring
5. Application 3: Connected Comp. in Undir. Graphs
6. Application 4: Topological Sort

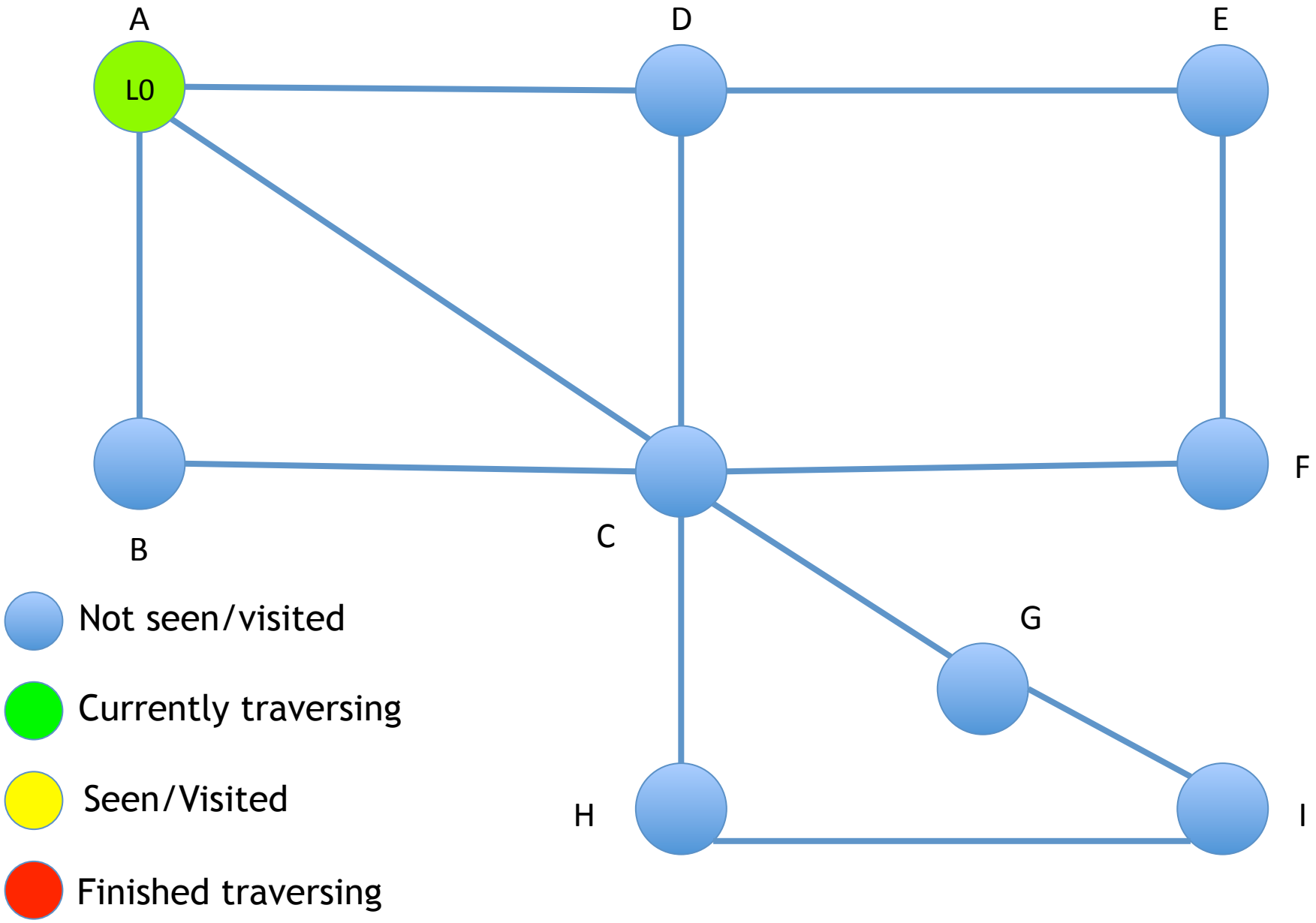
2 Basic Graph Traversal Algorithms

- BFS: Breadth-First Search Traversal
  - Starts from s and traverses the graph *in waves*
  - s->s's first degree nbrs->s's 2nd degree nbrs -> …
- DFS: Depth-First Search Traversal
  - From s tries to go *"as far as"* it can *"as fast as"* it can
  - Backtracking when stuck

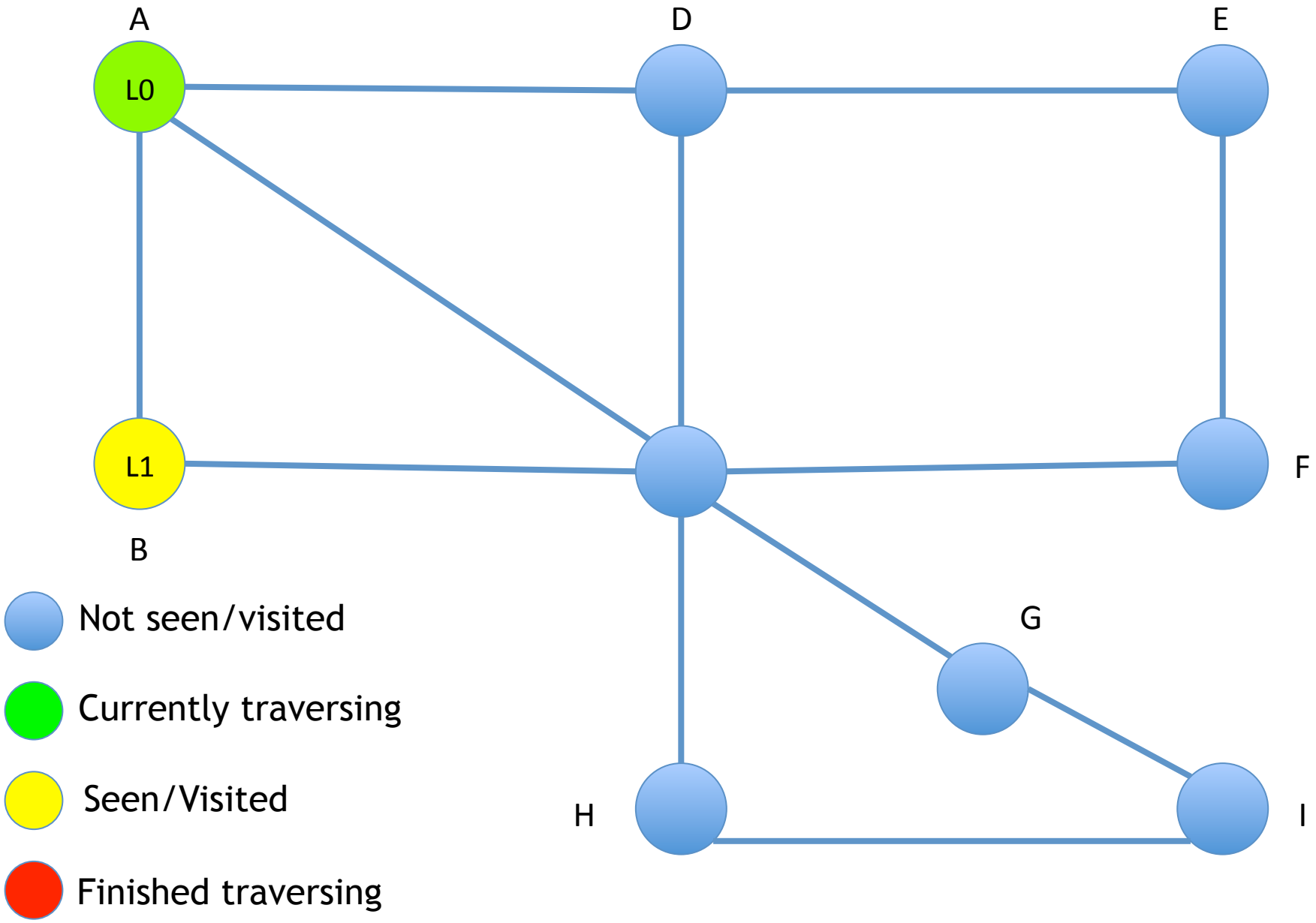
BFS


A
D
E
B
C
F
G
H
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

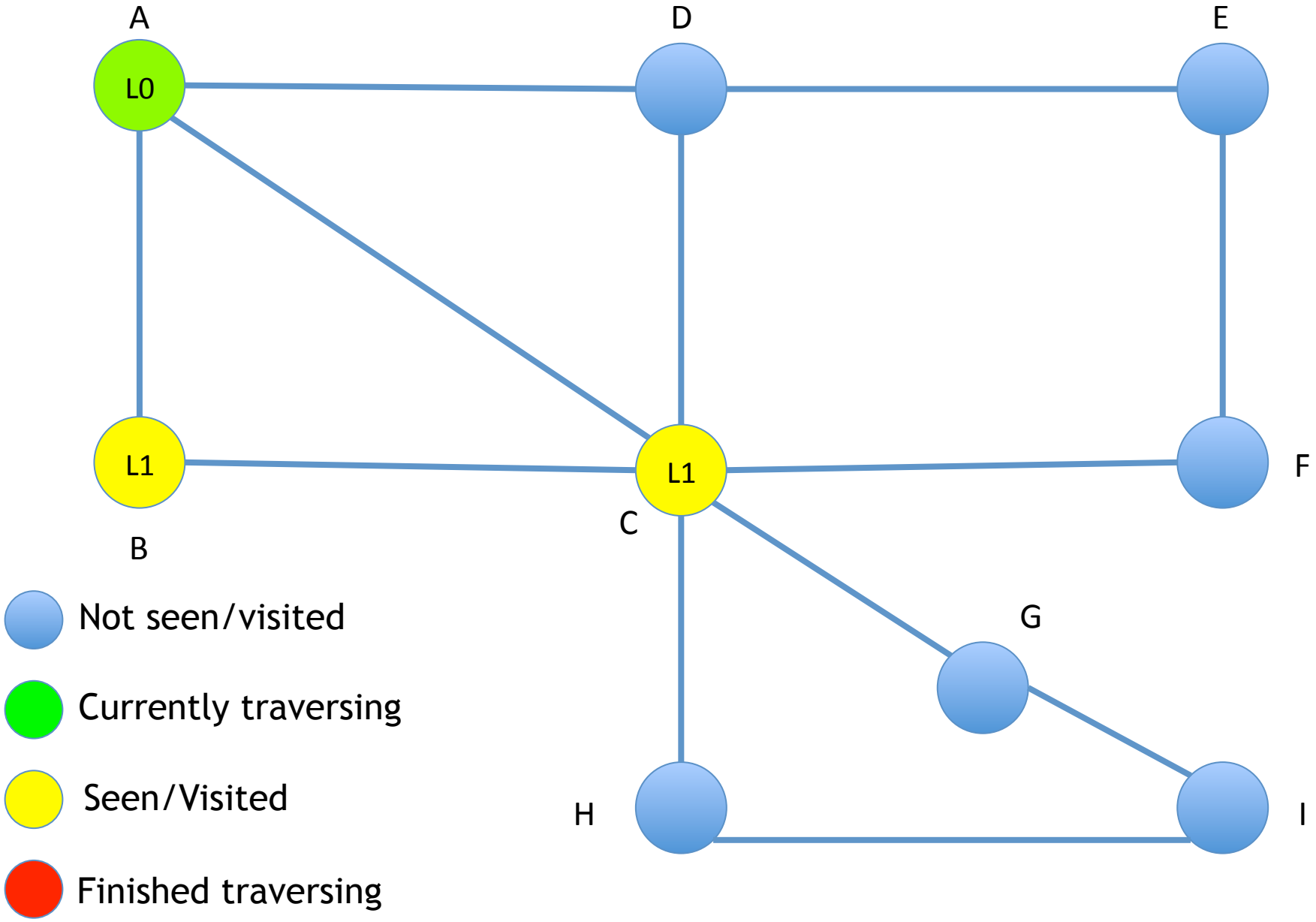
BFS


A
L0
D
E
F
B
C
G
H
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

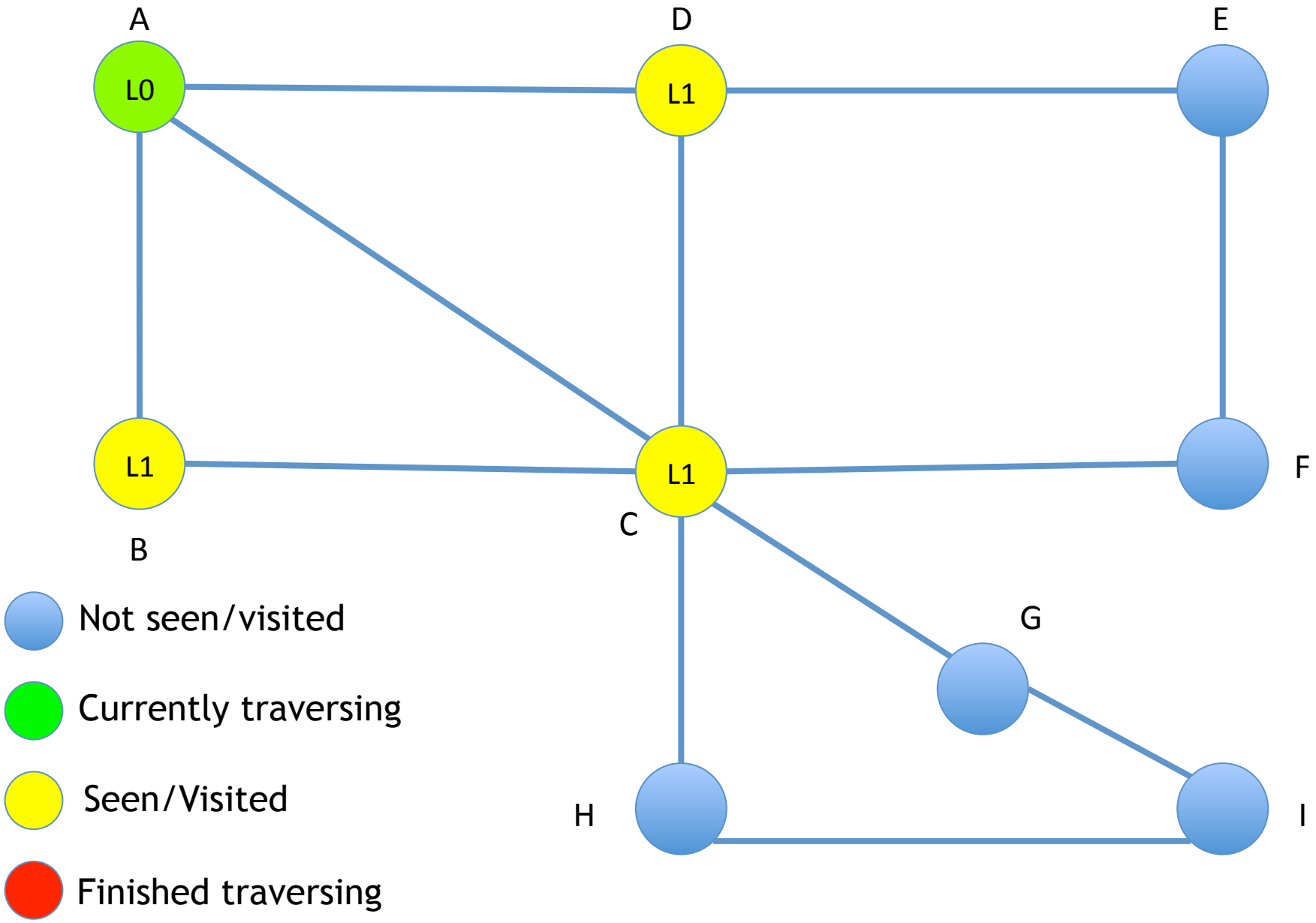
BFS


A
D
E
L0
L1
B
F
G
H
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

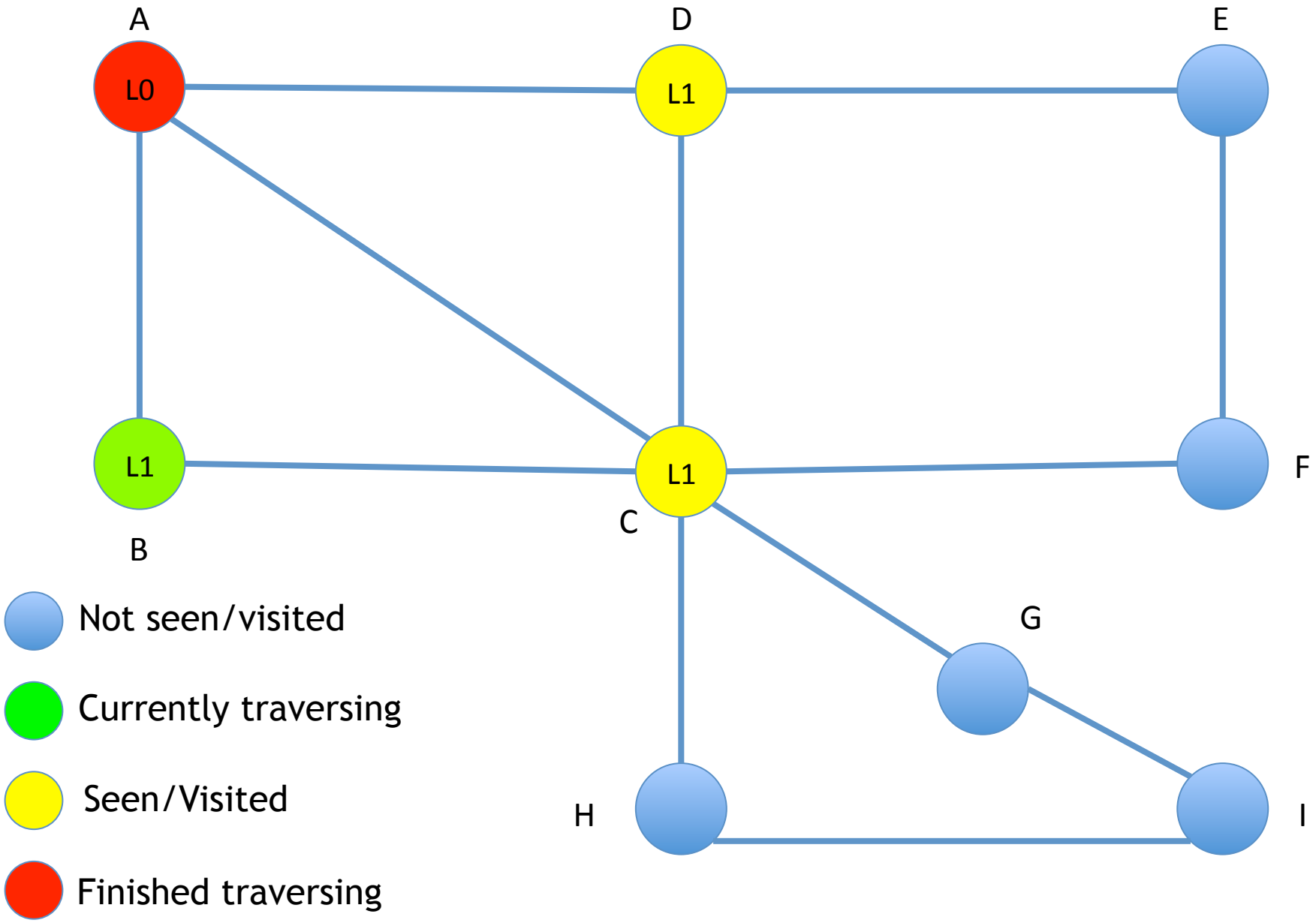
BFS


A
D
E
L0
L1
L1
F
C
B
G
H
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

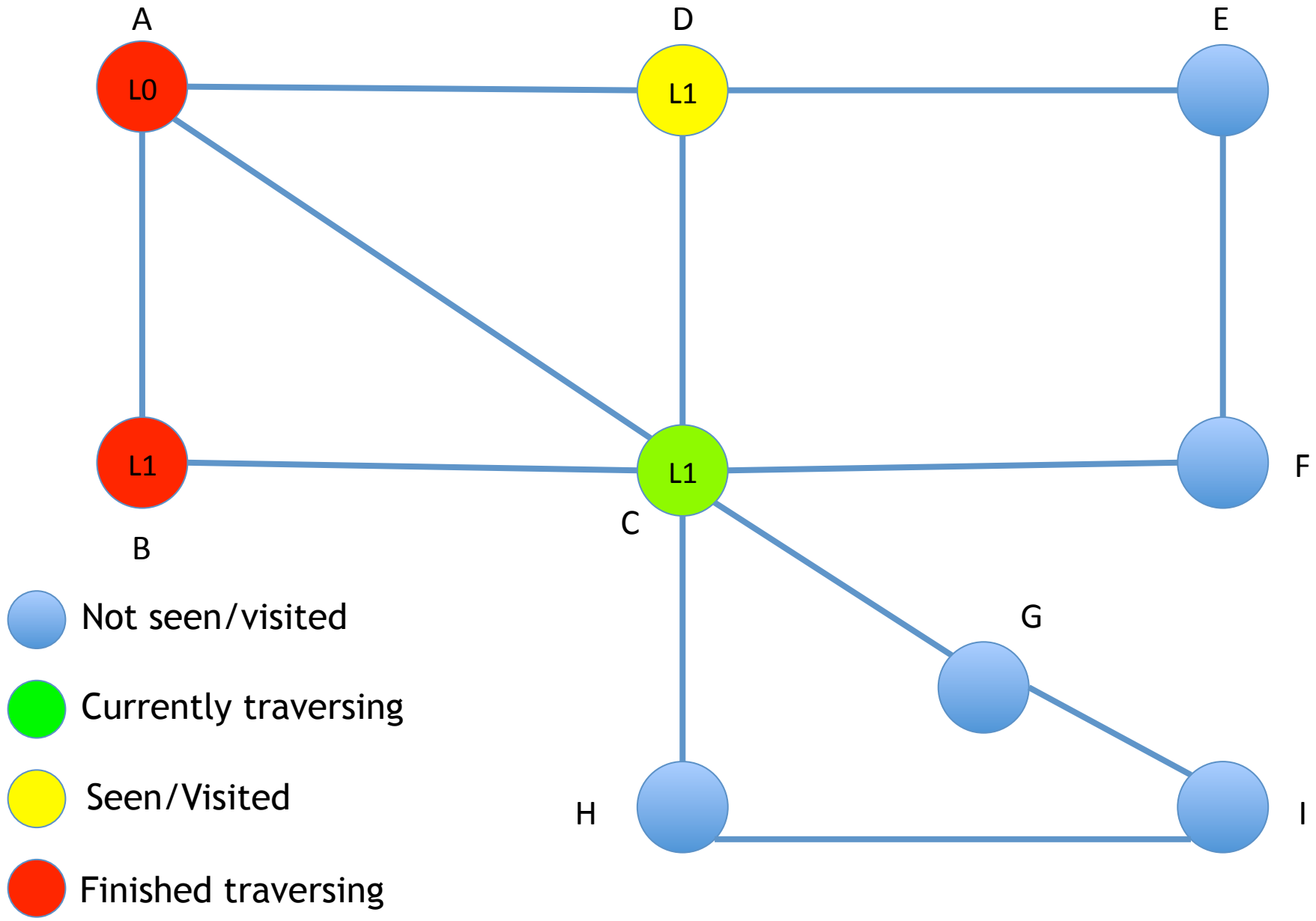
BFS


A
D
E
L0
L1
L1
L1
F
C
B
G
H
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

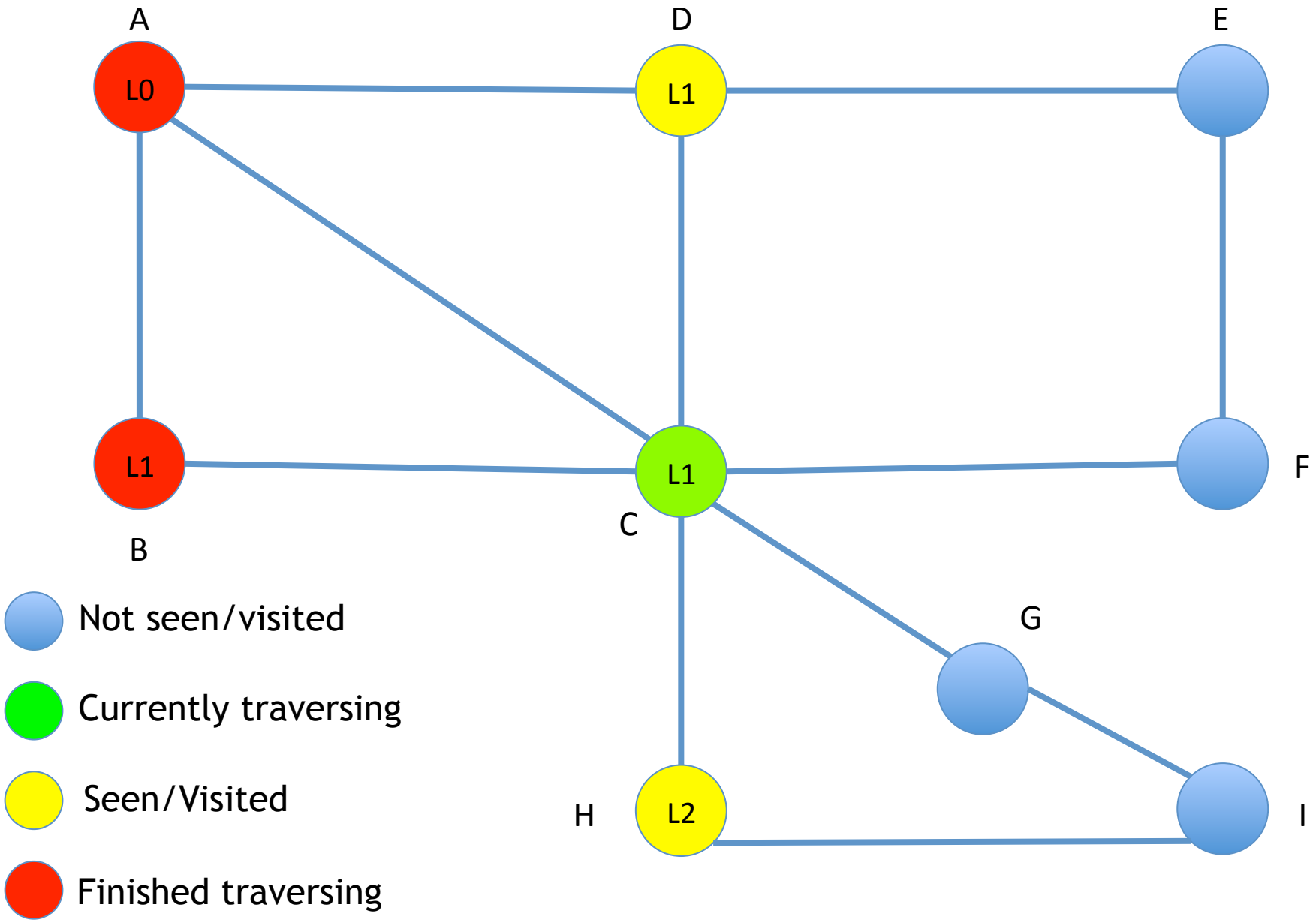
BFS


A
L0
D
L1
E
L1
B
L1
C
F
G
H
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

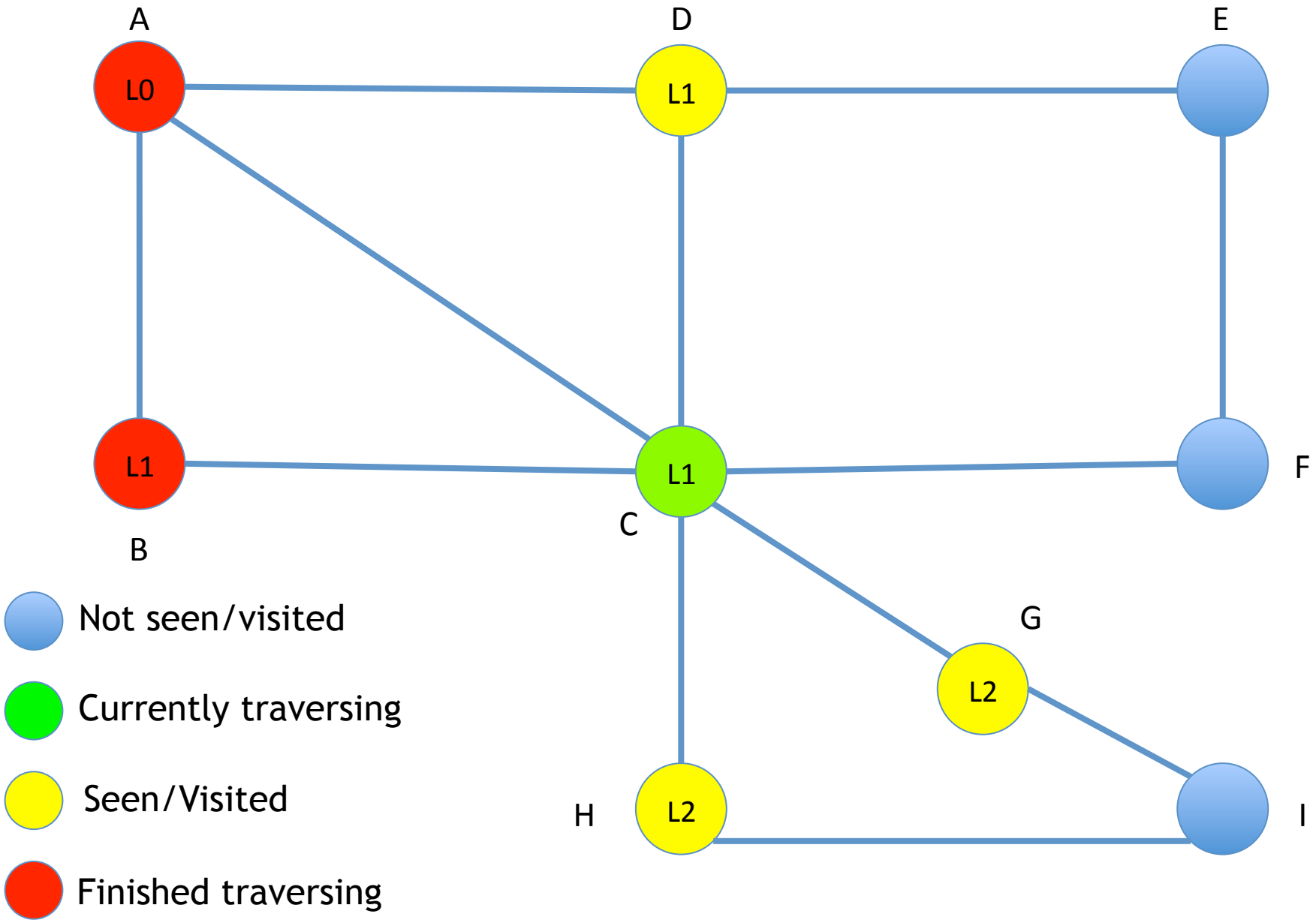
BFS


A
L0
D
L1
E
L1
B
L1
C
F
G
H
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

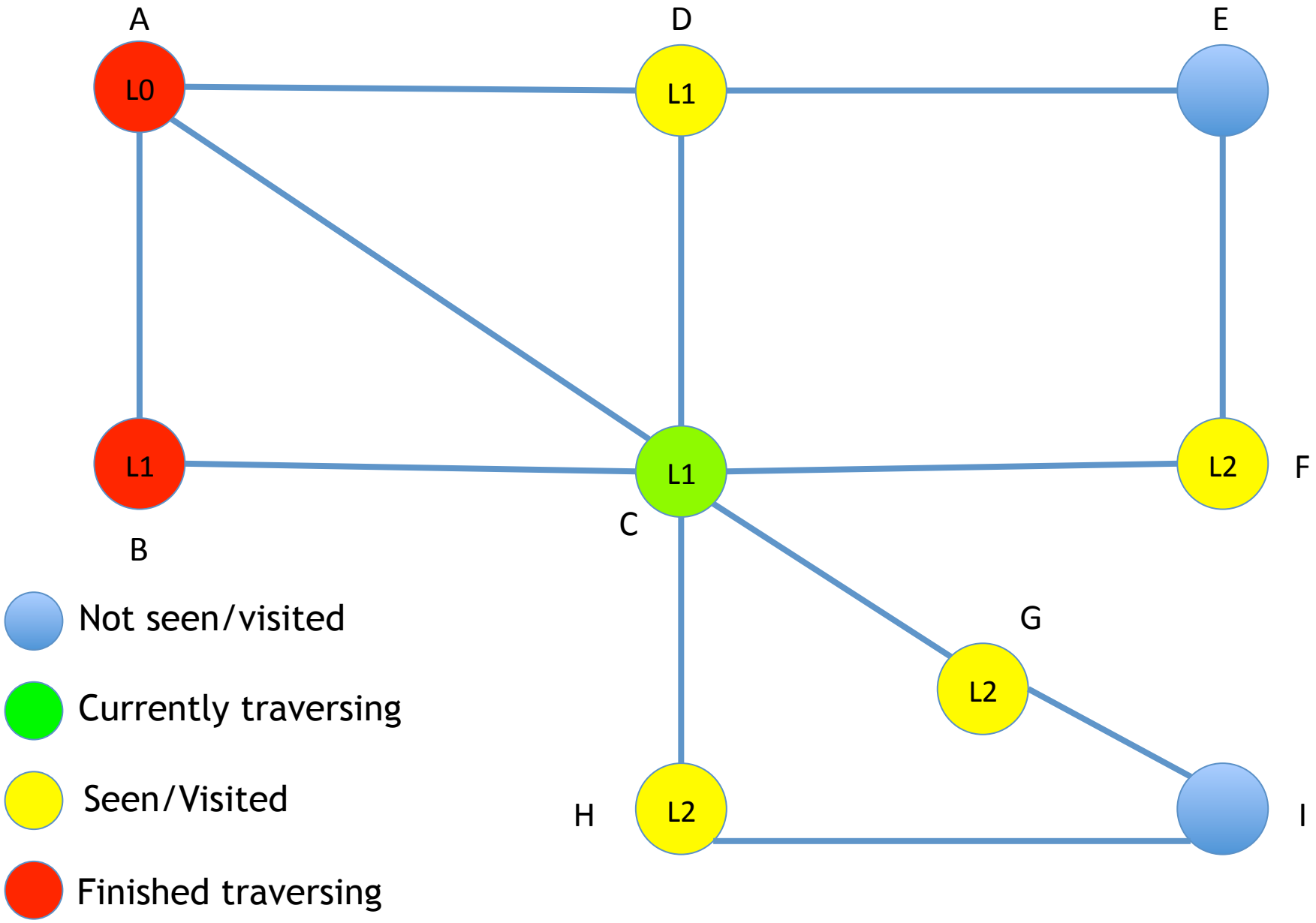
BFS


A
L0
D
L1
E
L1
B
L1
C
F
G
H
L2
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

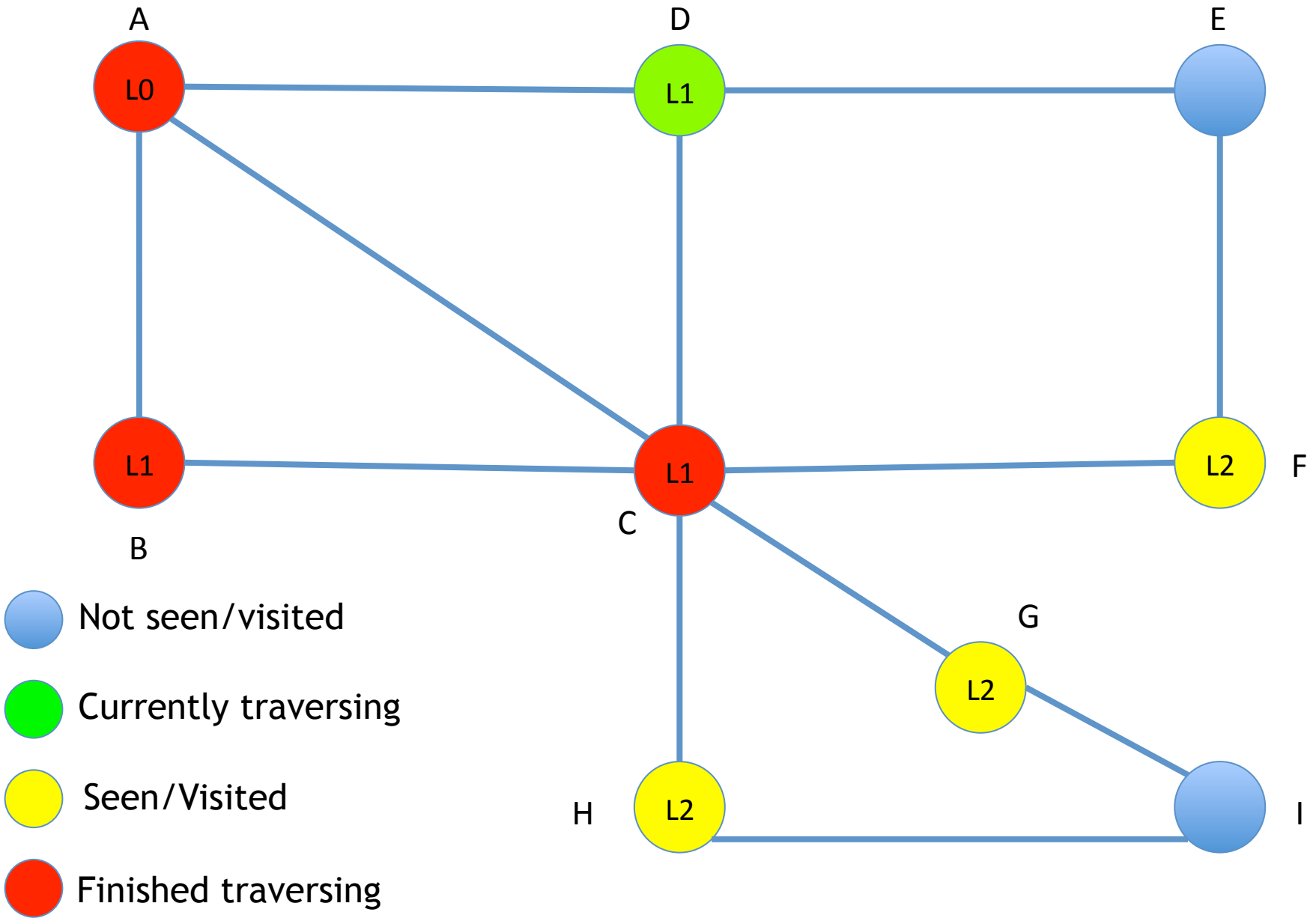
BFS


A
L0
D
L1
E
L1
L1
F
B
C
G
L2
H
L2
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

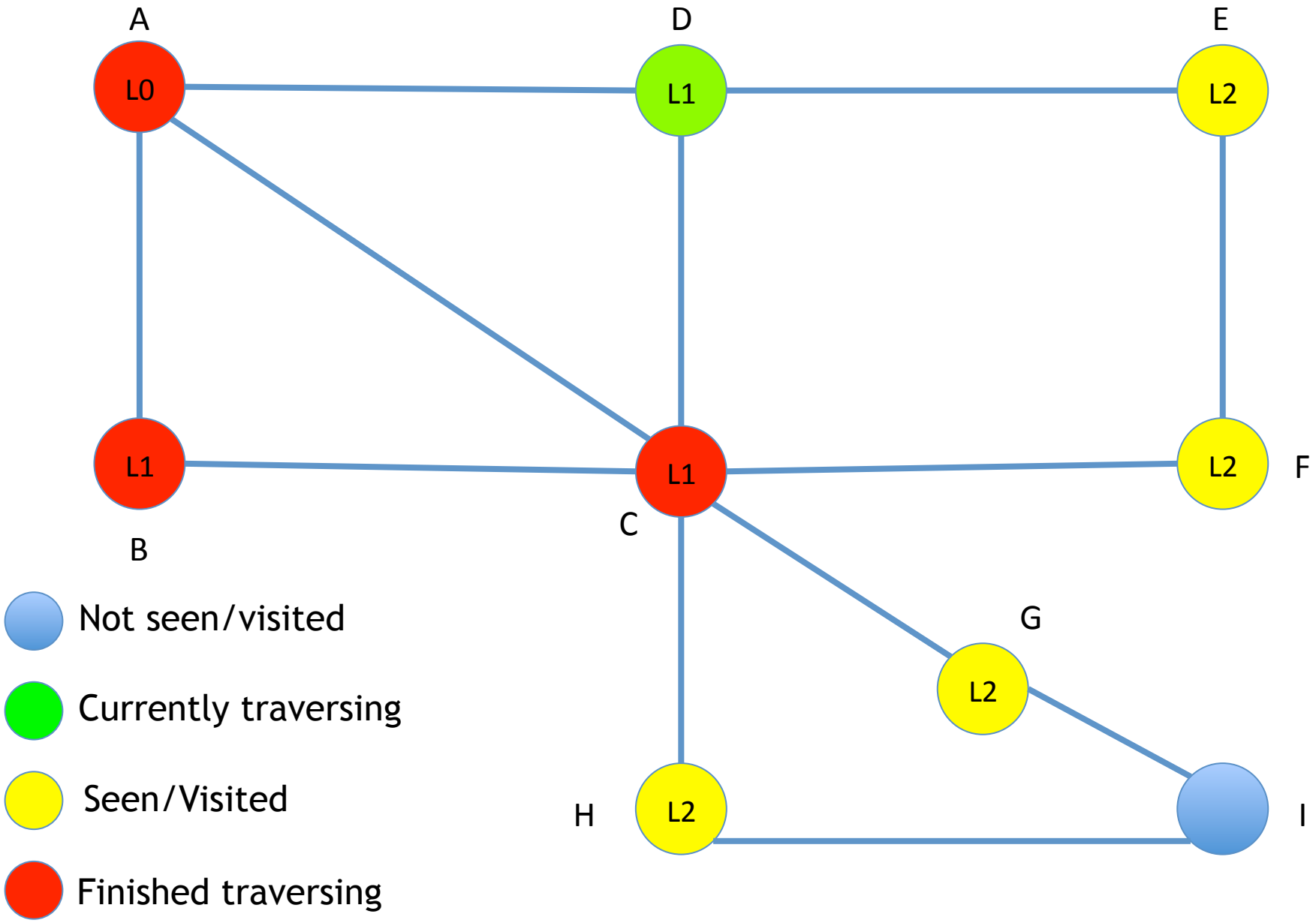
BFS


A
L0
D
L1
E
L1
B
L1
C
L2
F
G
L2
H
L2
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

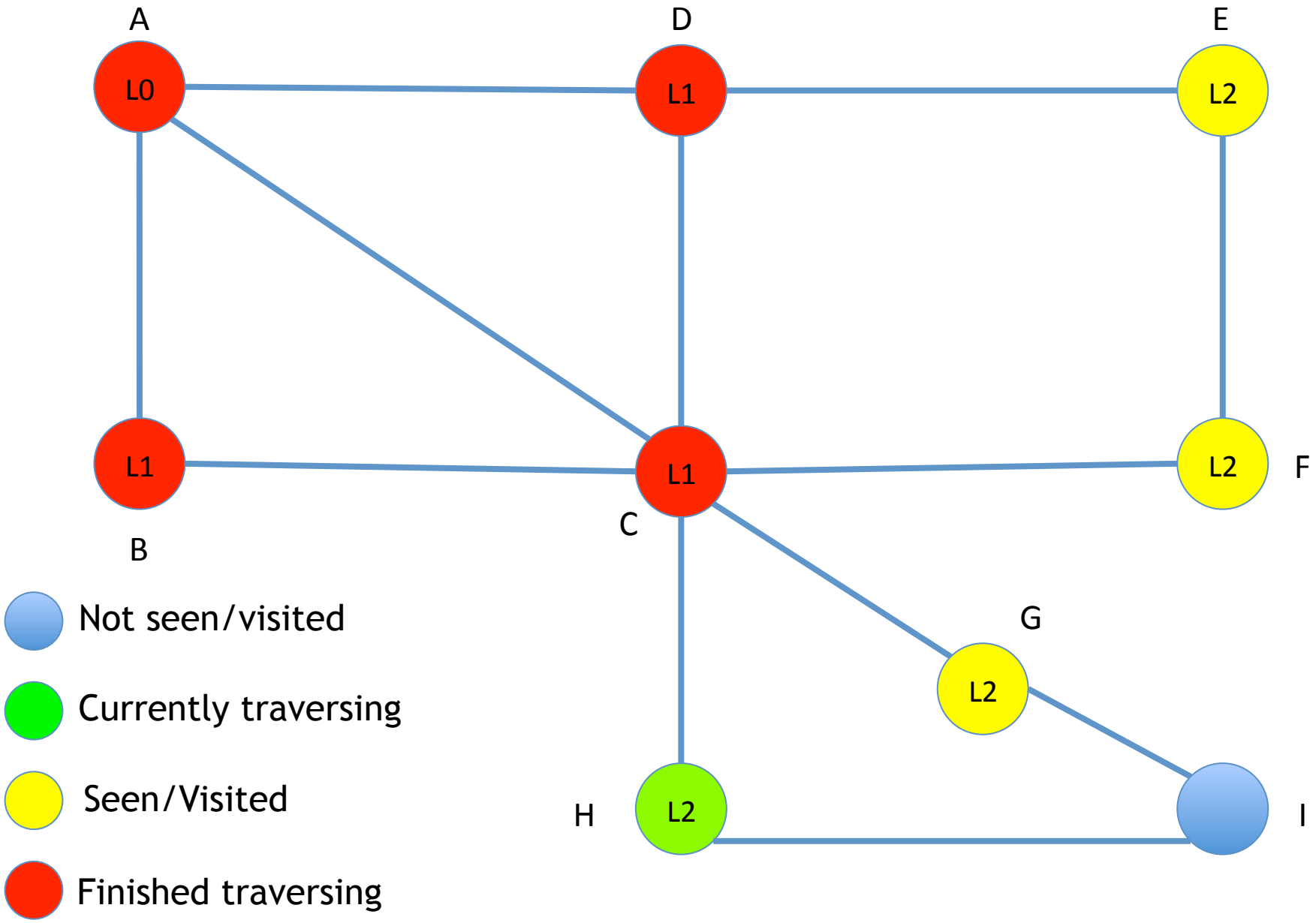
BFS


A
D
E
L0
L1
L1
L1
L2
F
B
C
G
L2
H
L2
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

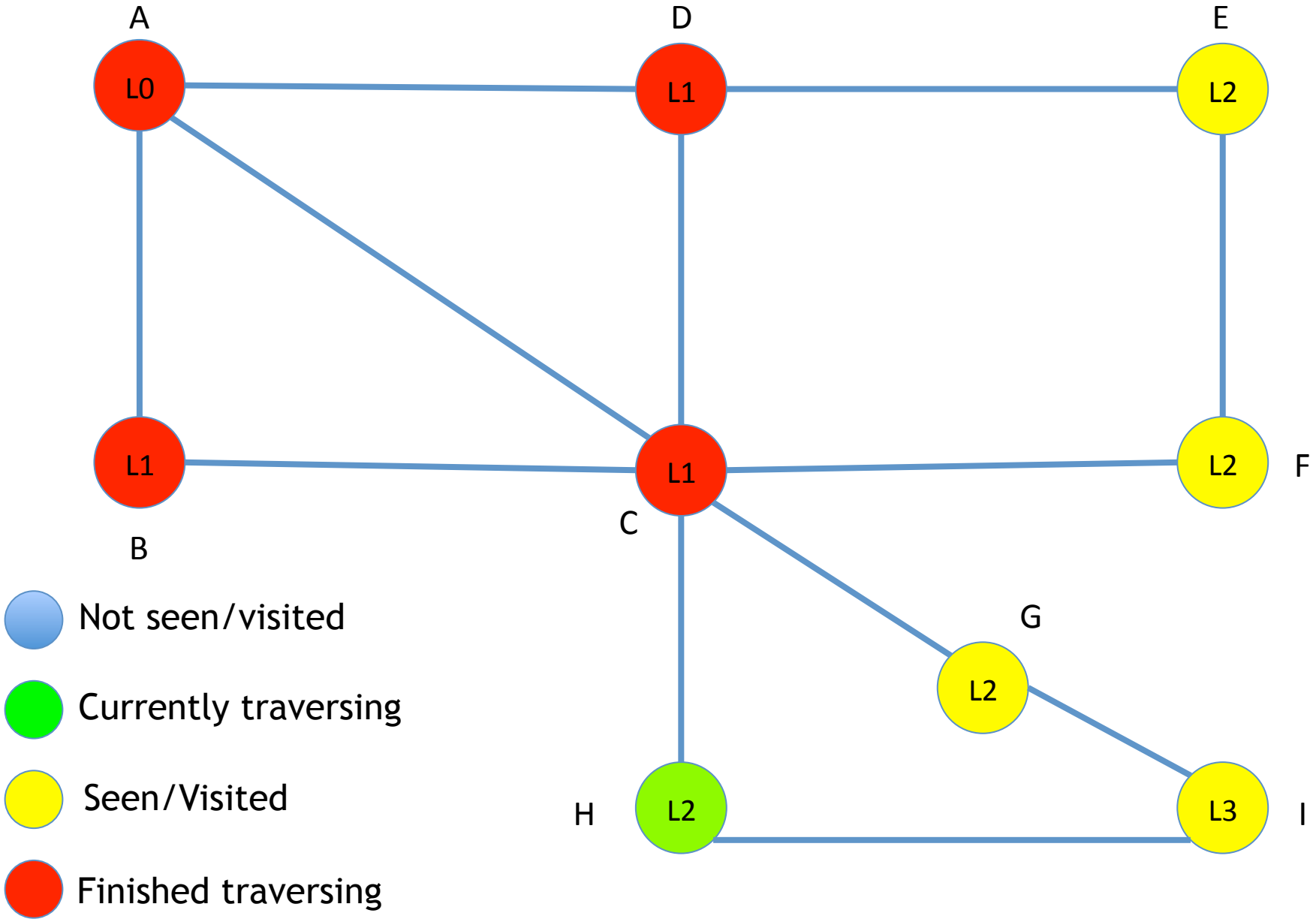
BFS


A
L0
D
L1
E
L2
L1
B
L1
C
L2
F
G
L2
H
L2
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

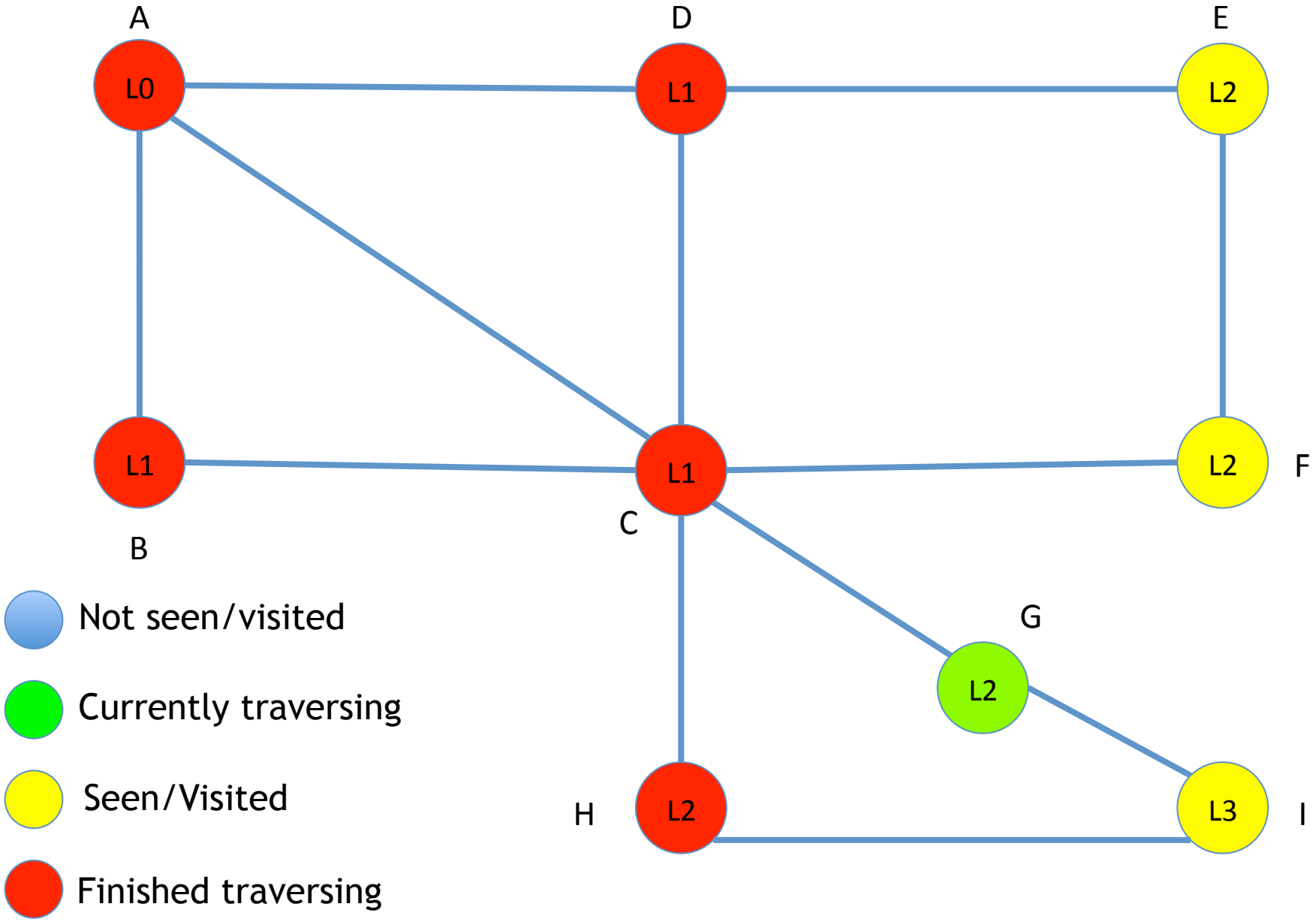
BFS


A
L0
D
L1
E
L2
L1
B
L1
C
L2
F
G
L2
H
L2
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

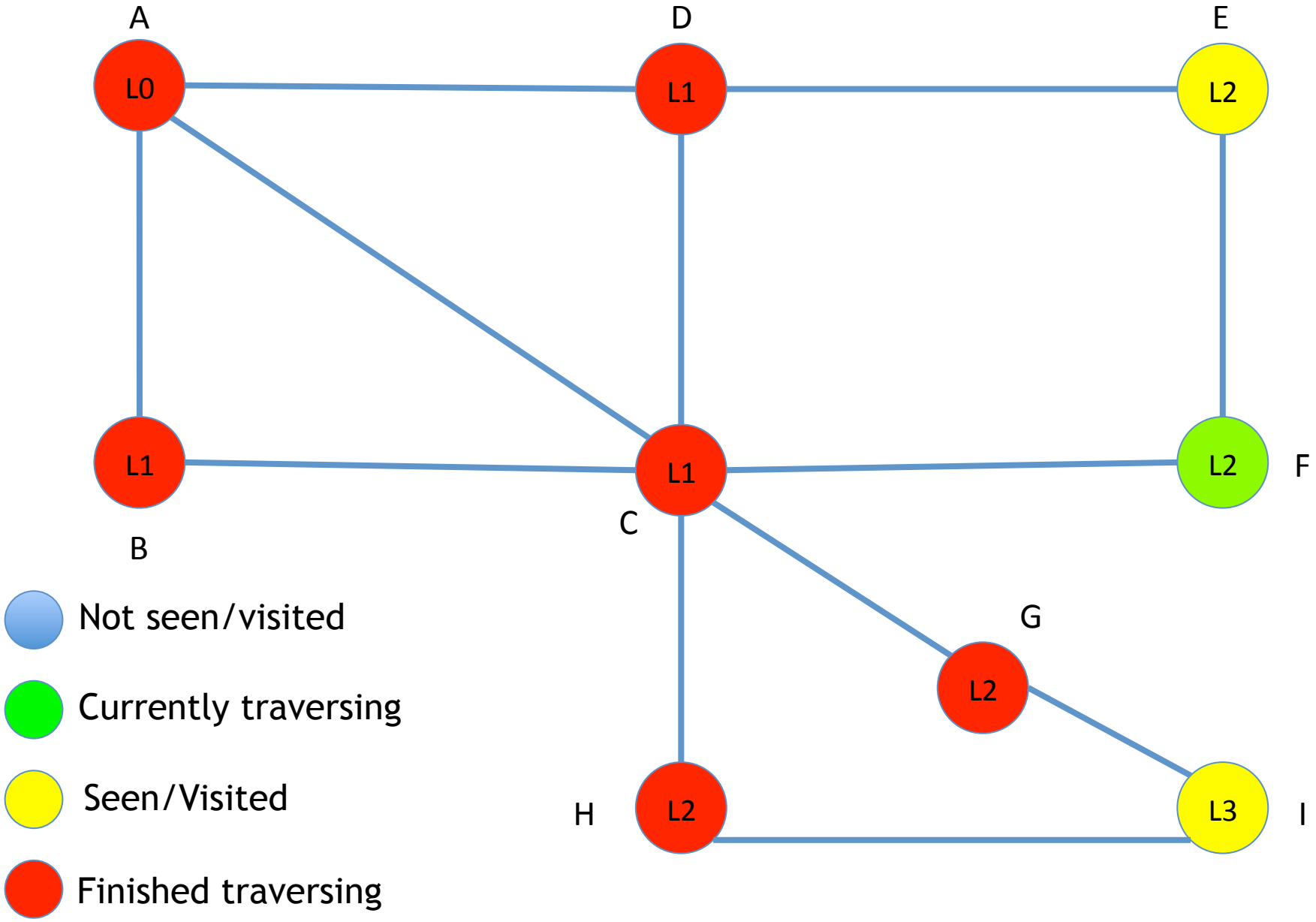
BFS


A
L0
D
L1
E
L2
L1
B
L1
C
L2
F
G
L2
H
L2
L3
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

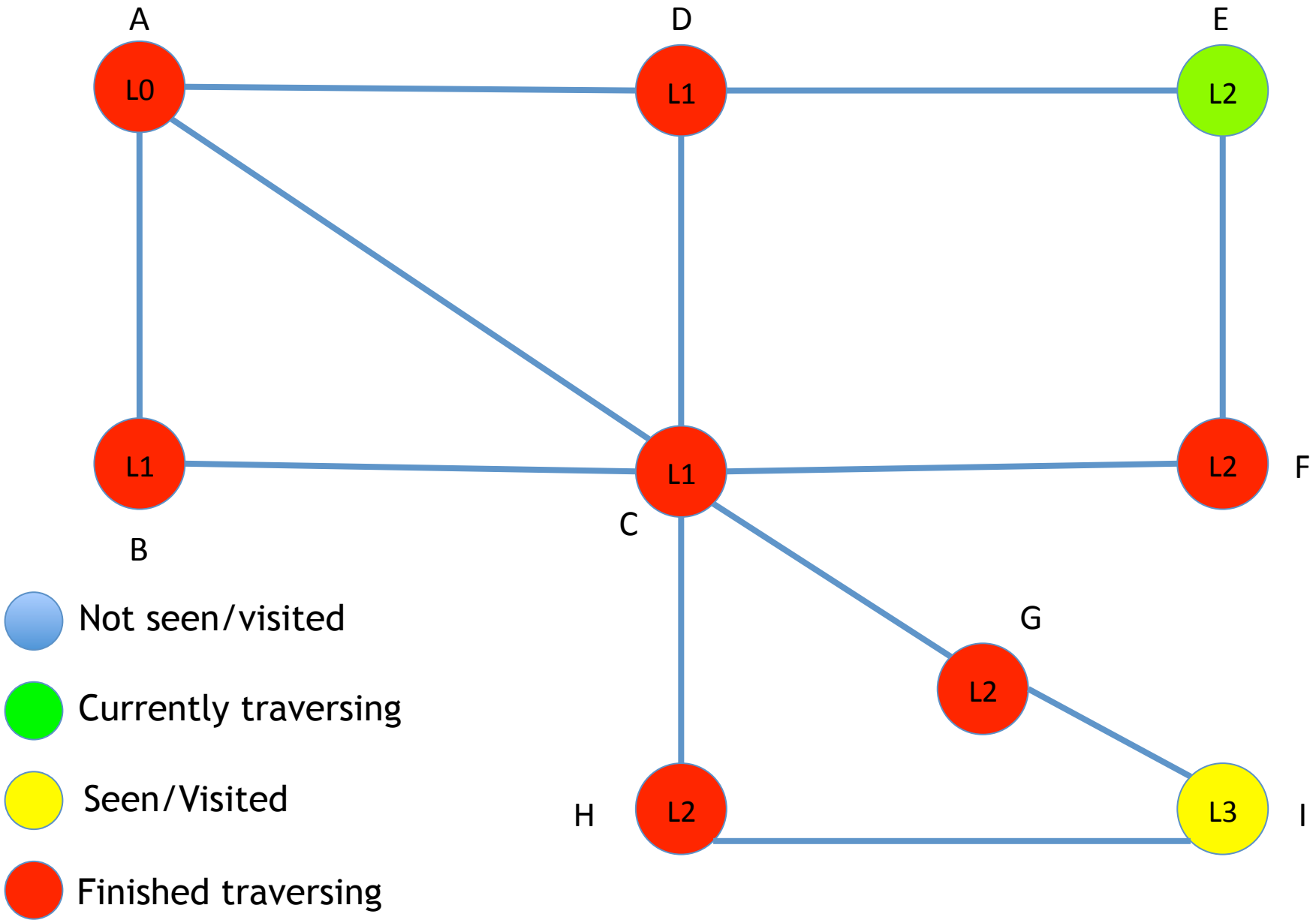
BFS


A
L0
D
L1
E
L2
L1
B
L1
C
L2
F
G
L2
H
L2
L3
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

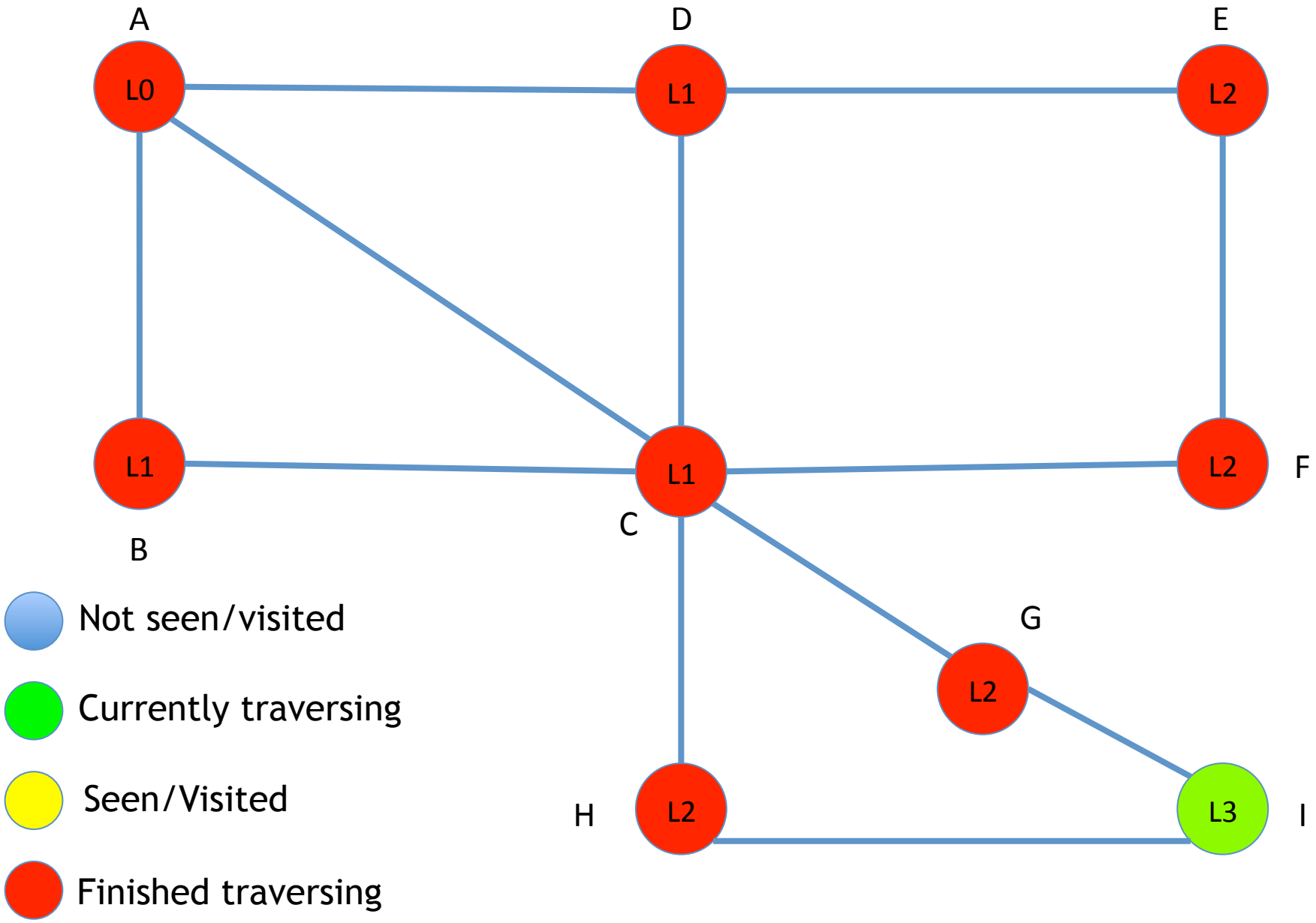
BFS


A
L0
D
L1
E
L2
L1
B
L1
C
L2
F
G
L2
H
L2
L3
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

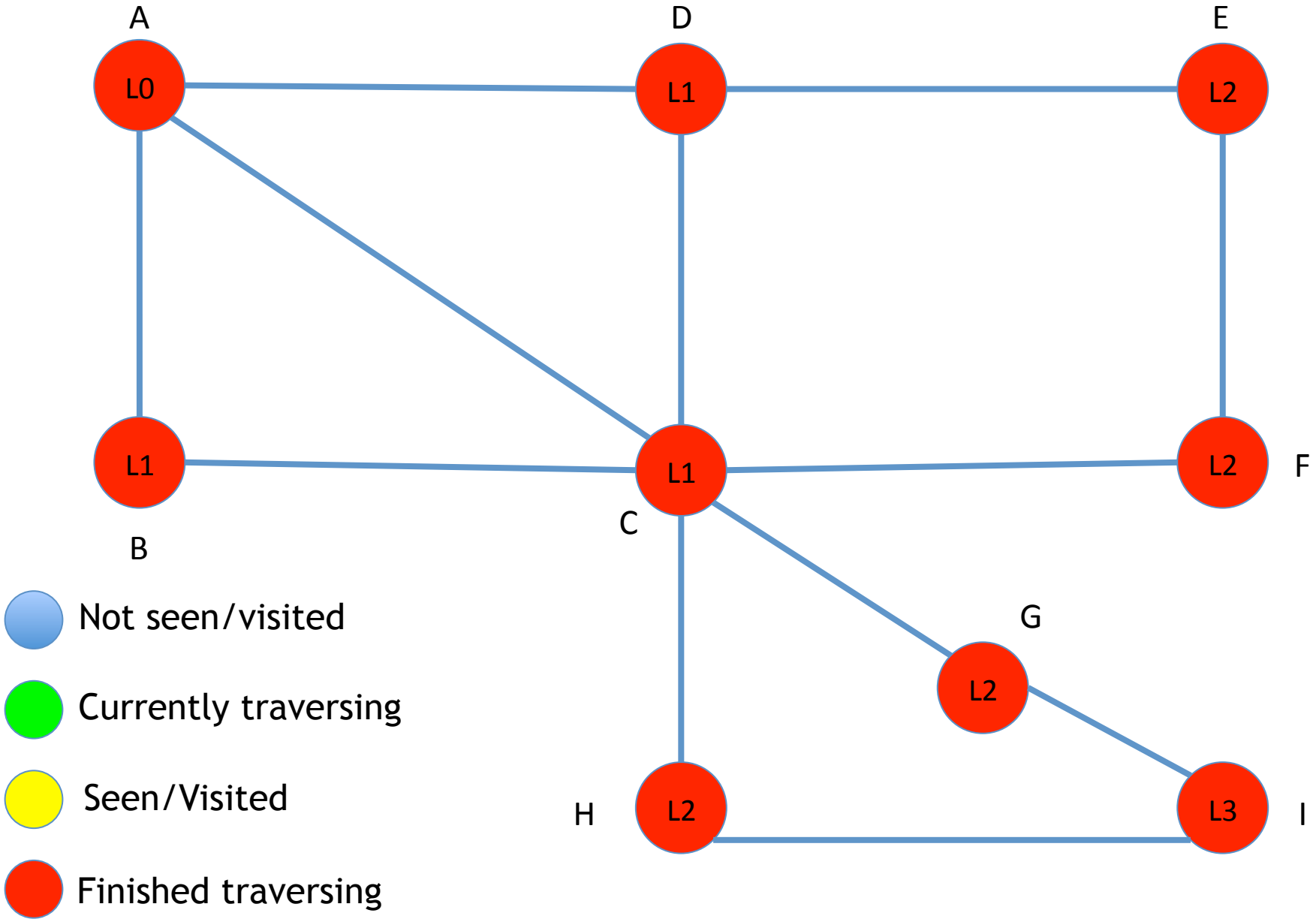
BFS


A
D
E
L0
L1
L2
L1
L1
L2
F
B
C
G
L2
H
L2
L3
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

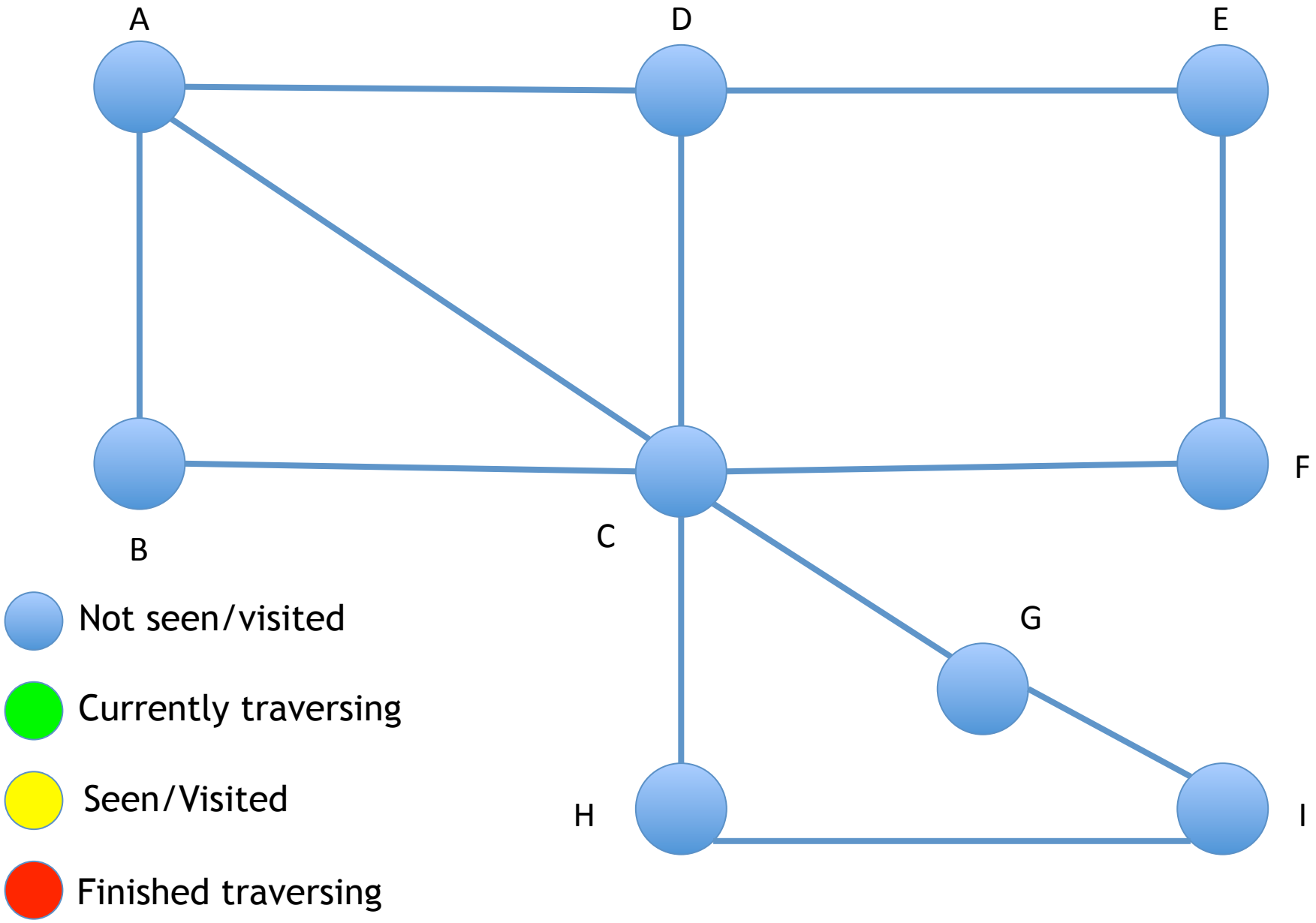
BFS


A
D
E
L0
L1
L2
L1
L1
L2
F
B
C
G
L2
H
L2
L3
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

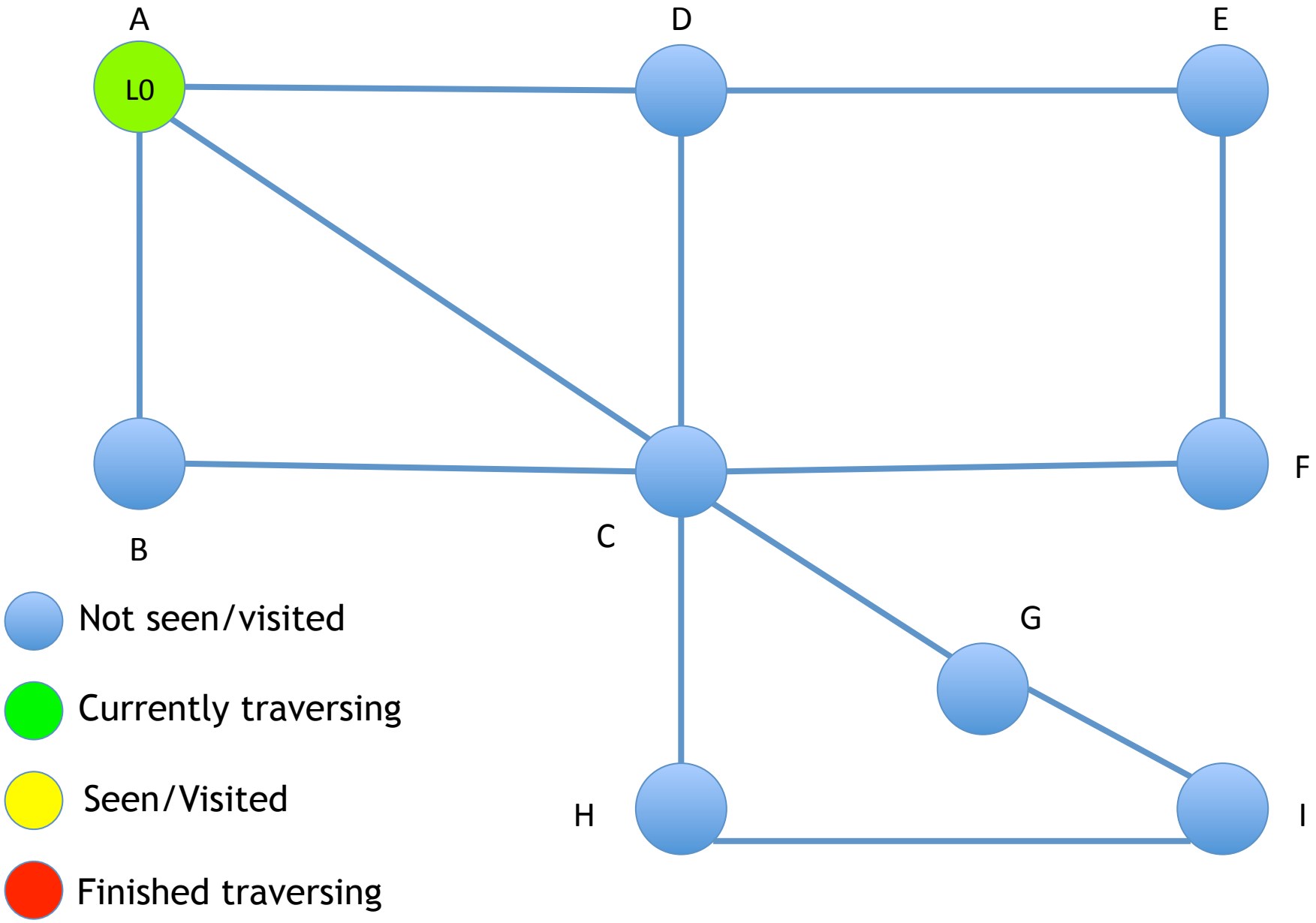
BFS


A
D
E
L0
L1
L2
L1
L1
L2
F
B
C
G
L2
H
L2
L3
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

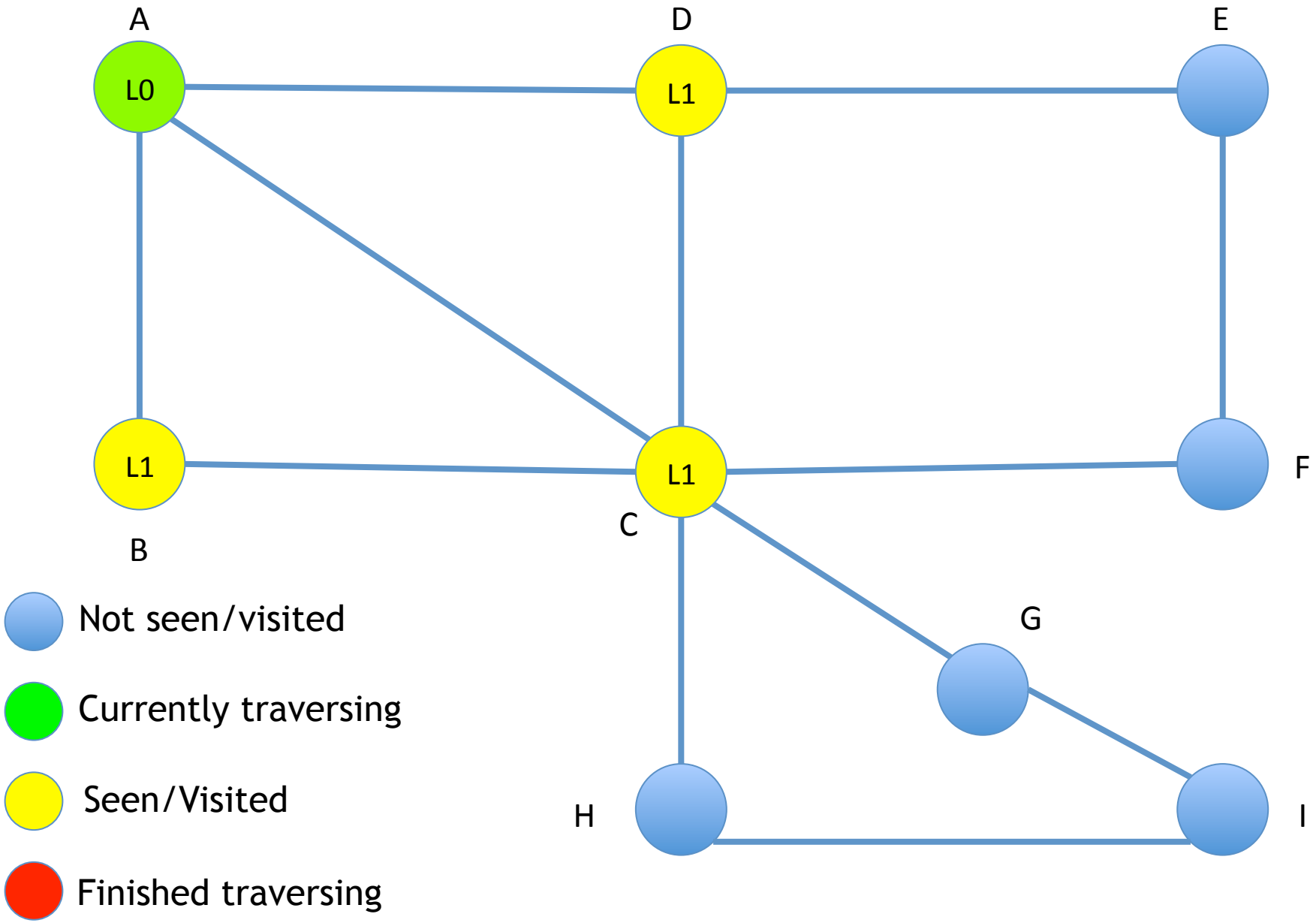
BFS At a Higher Level



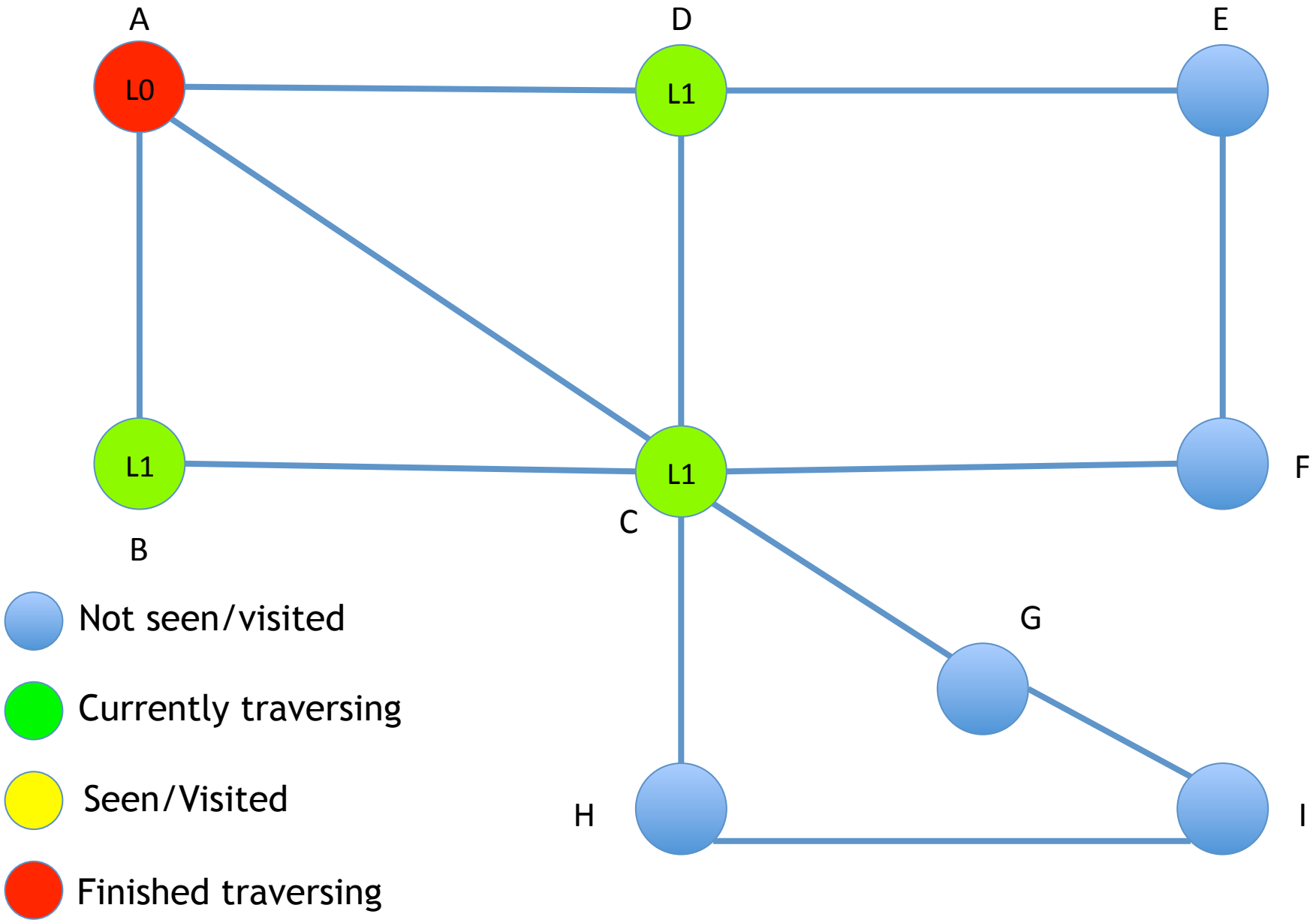
BFS At a Higher Level



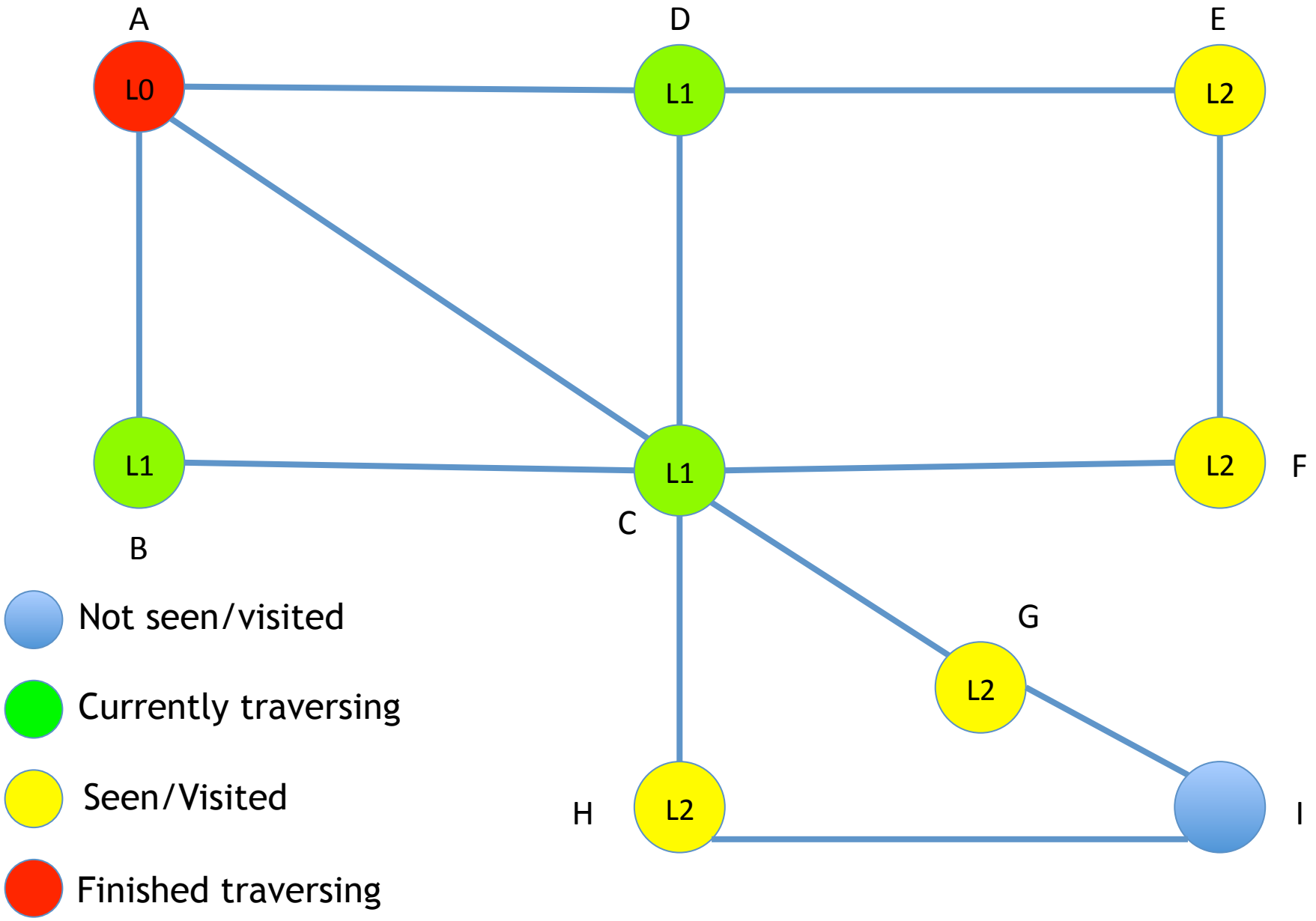
BFS At a Higher Level



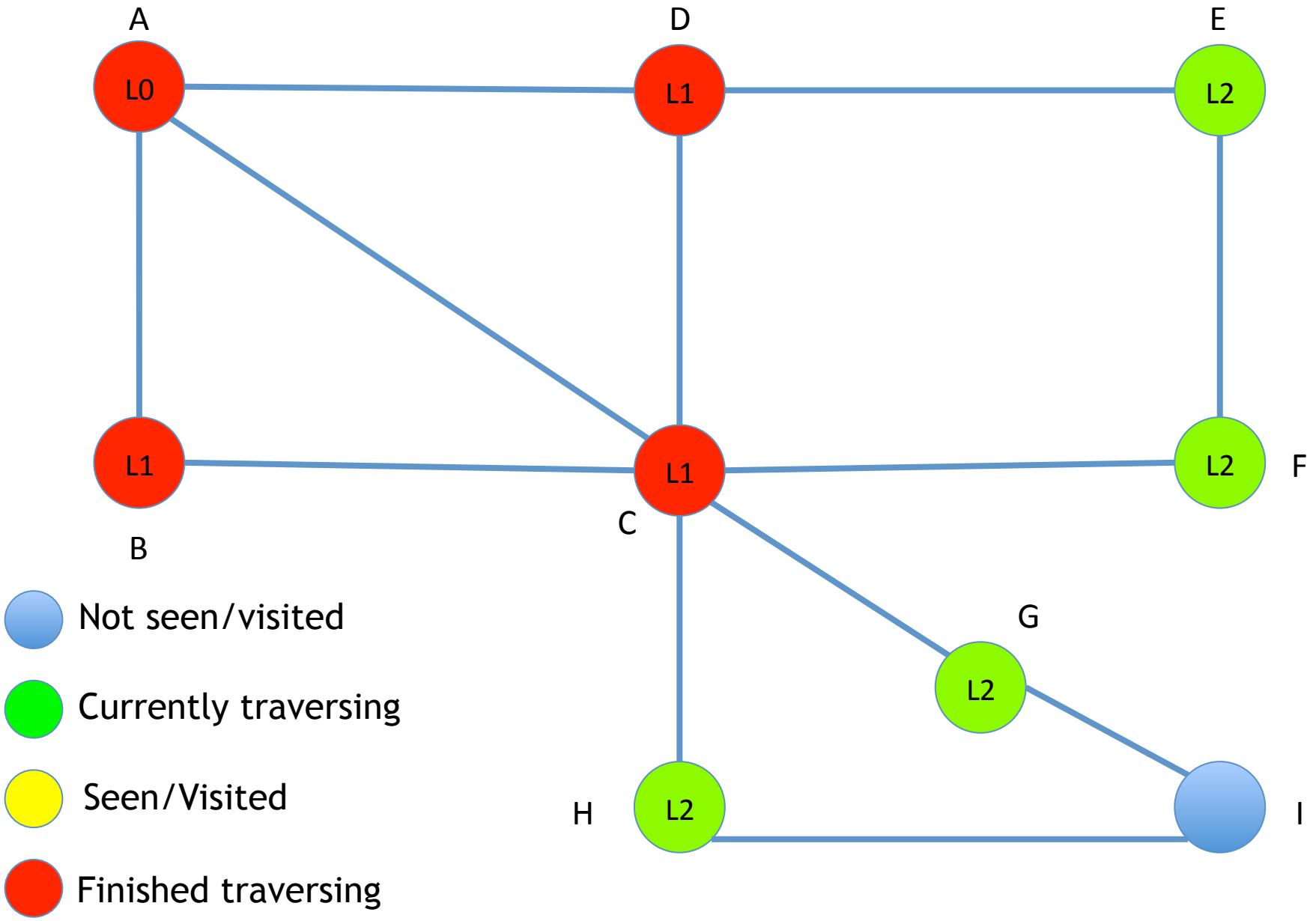
BFS At a Higher Level



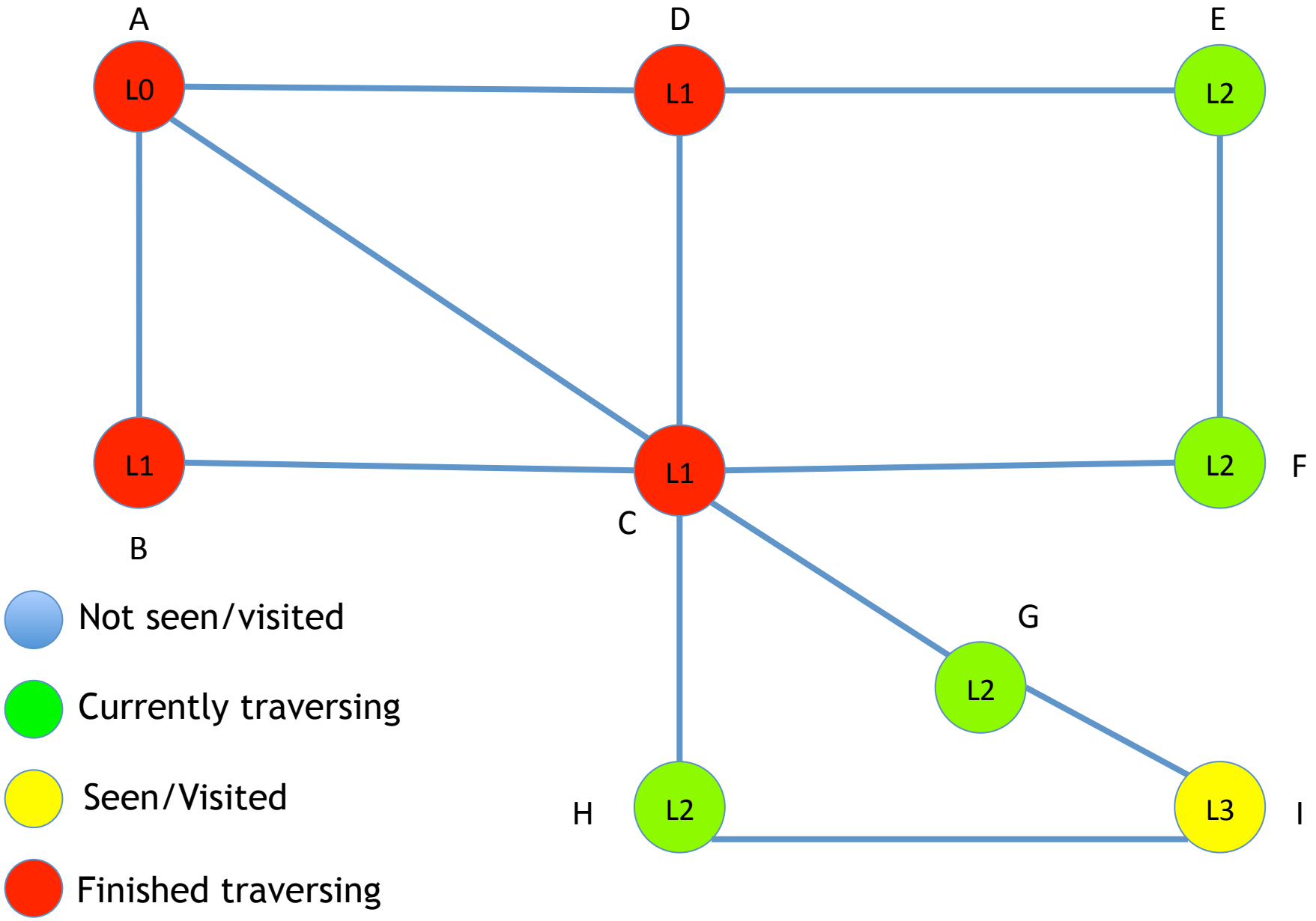
BFS At a Higher Level



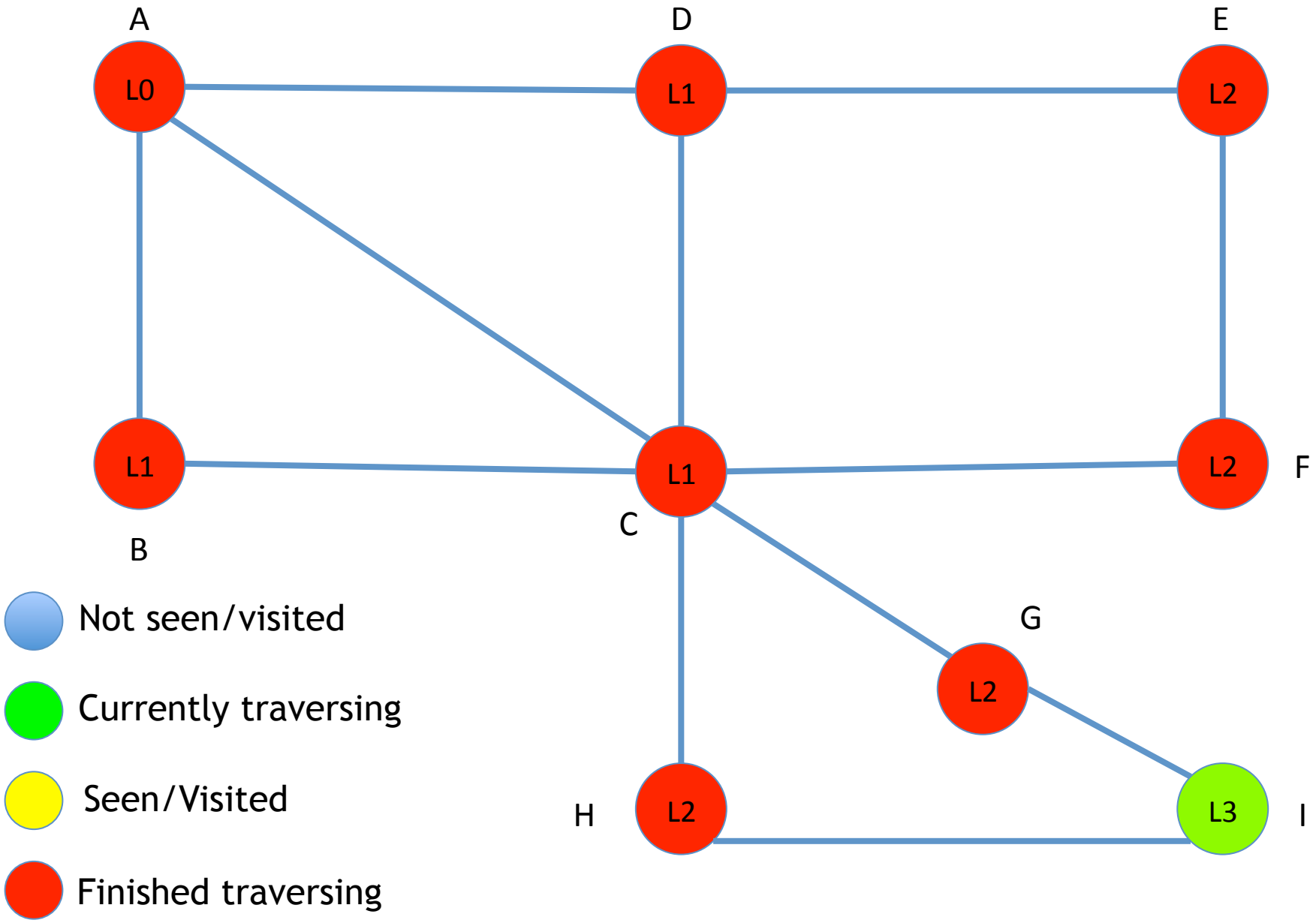
BFS At a Higher Level



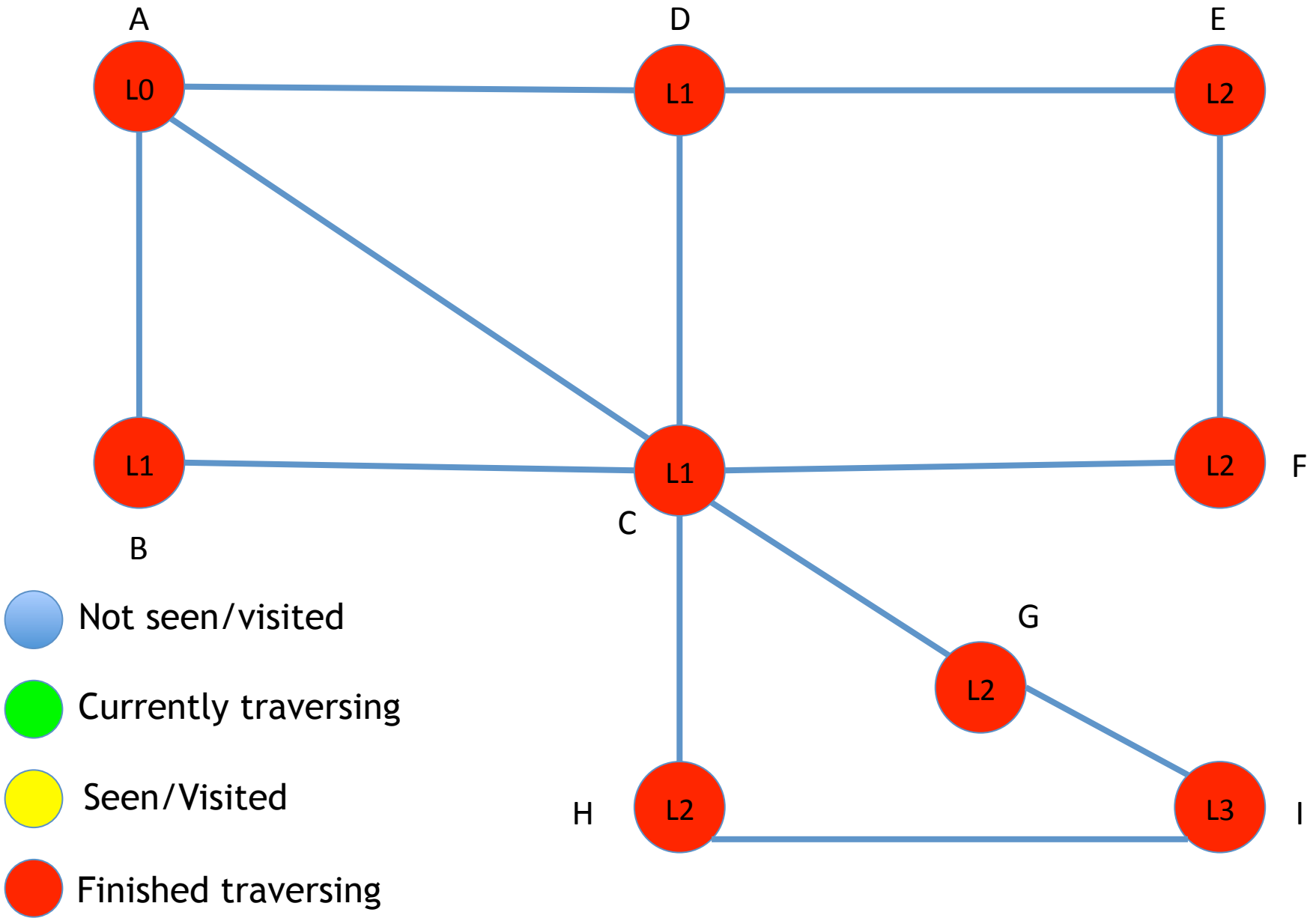
BFS At a Higher Level



BFS At a Higher Level



BFS At a Higher Level



BFS Pseudocode

```
1.  procedure BFS(G(V, E), s)
2.      let Q be a new queue;
3.      mark s visited
4.      enqueue(s, Q)
5.      while (Q not empty):
6.          let v = dequeue(Q)
7.          for each neighbor u of v:
8.              if (u is not-visited):
9.                  mark u as visited;
10.                 enqueu(u, Q)
11.         mark v as finished
```

Total Runtime: O(n+m)

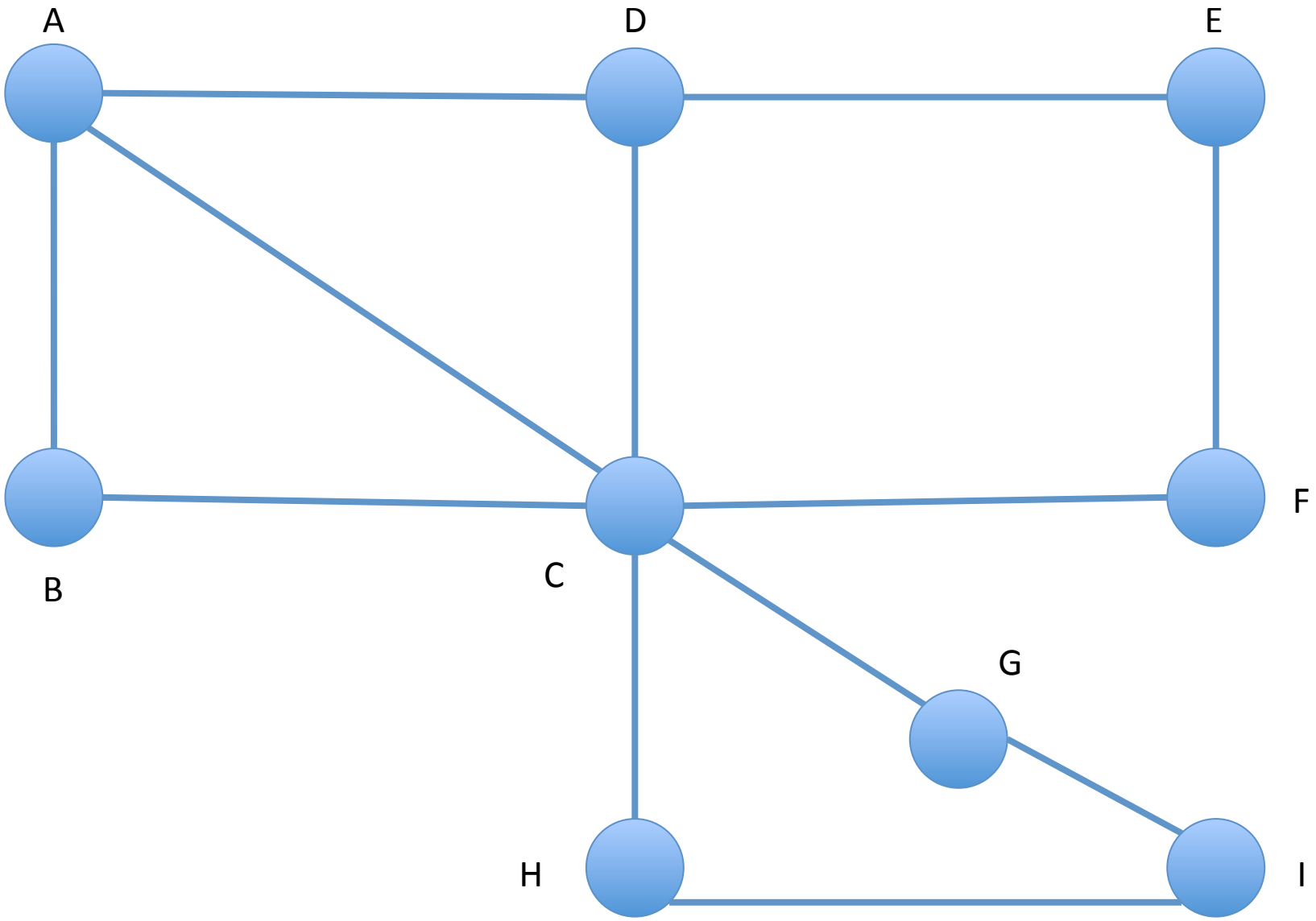
(with adj list)

$$\sum_{v=1}^{n} out - \deg(v) = m$$

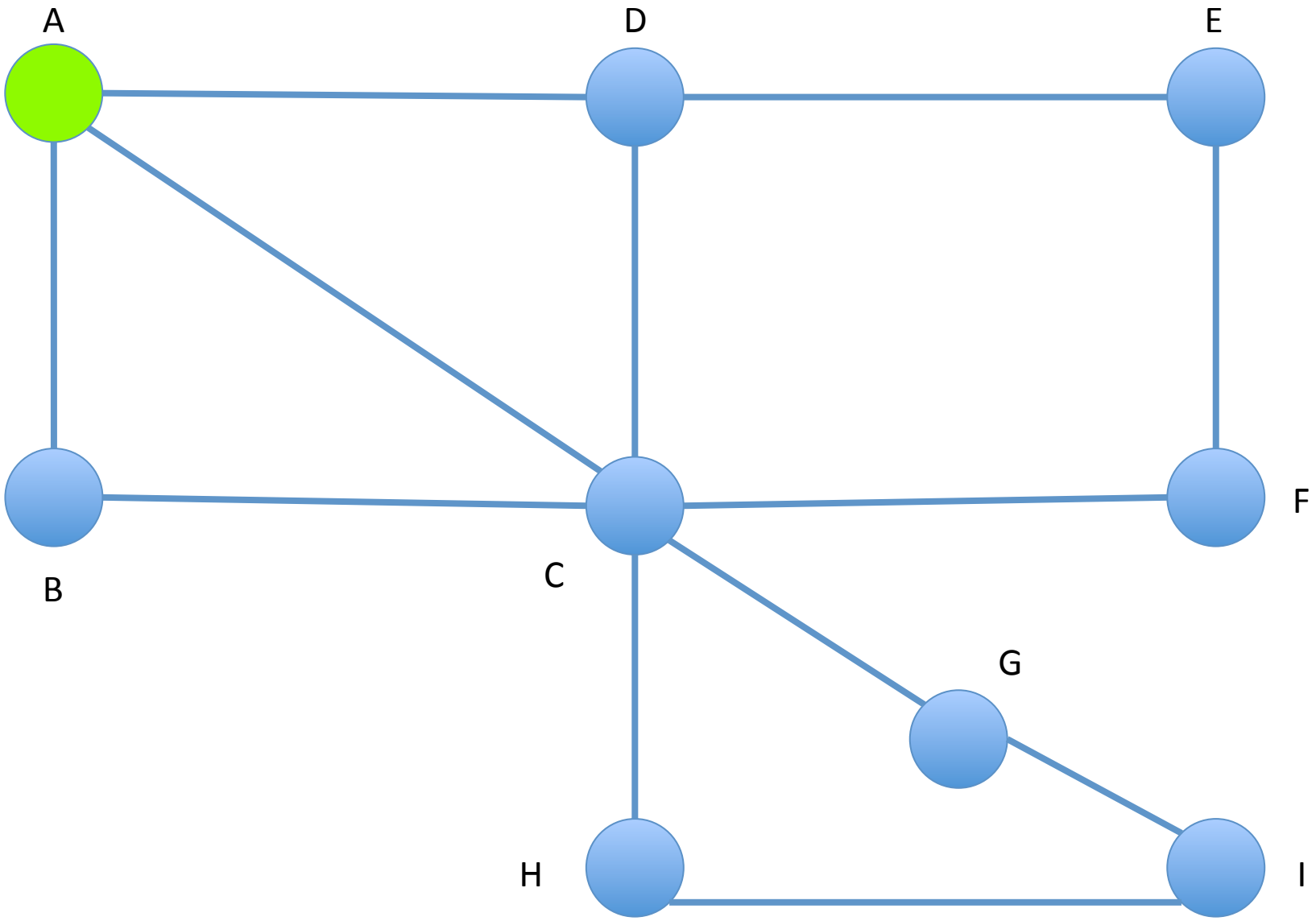
O(n)

```
1.  procedure BFS(G(V, E))
2.      mark all vertices as not-visited
3.      for each (v ∈ V, if v is not-visited do BFS(G, v)
```

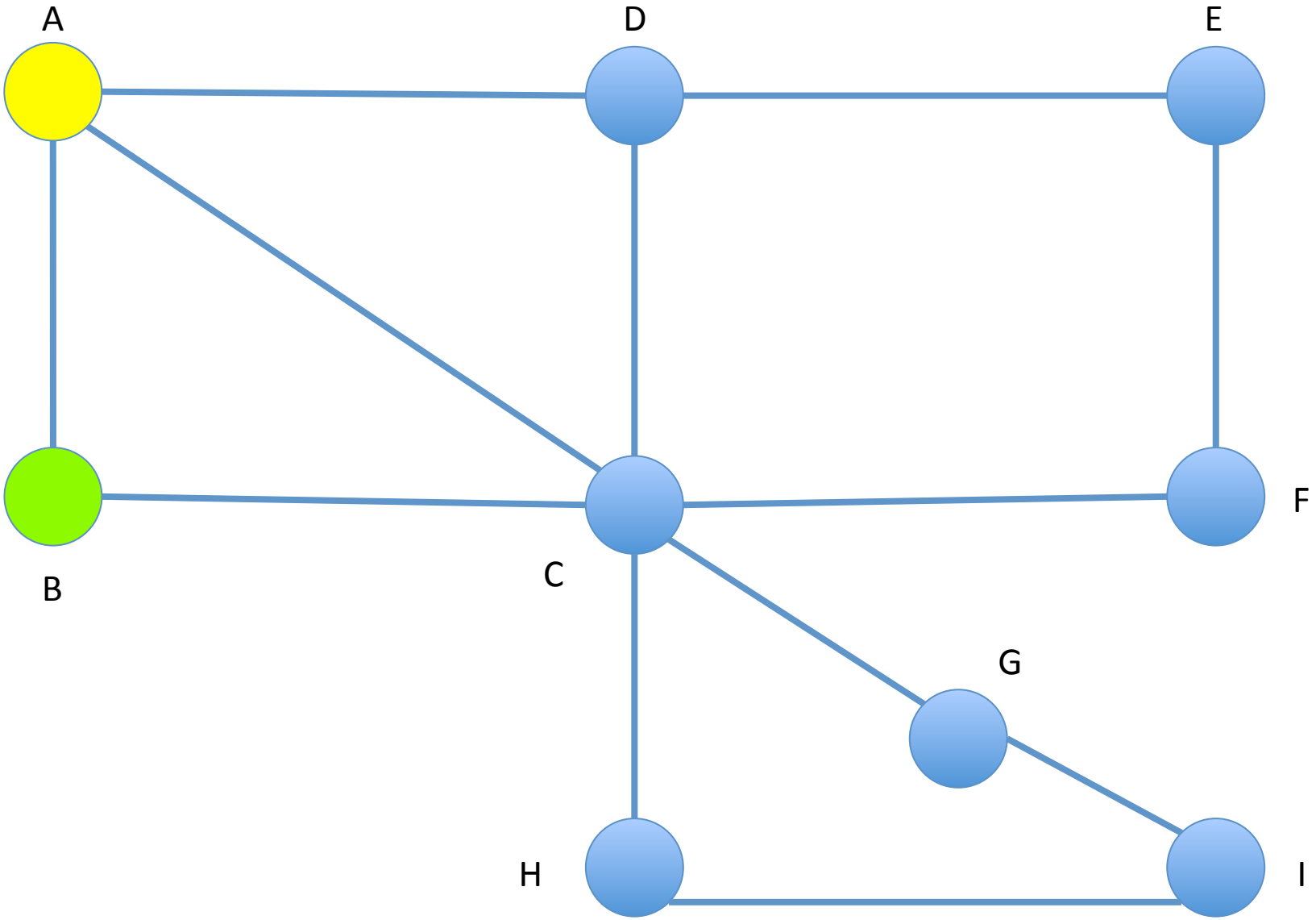
DFS



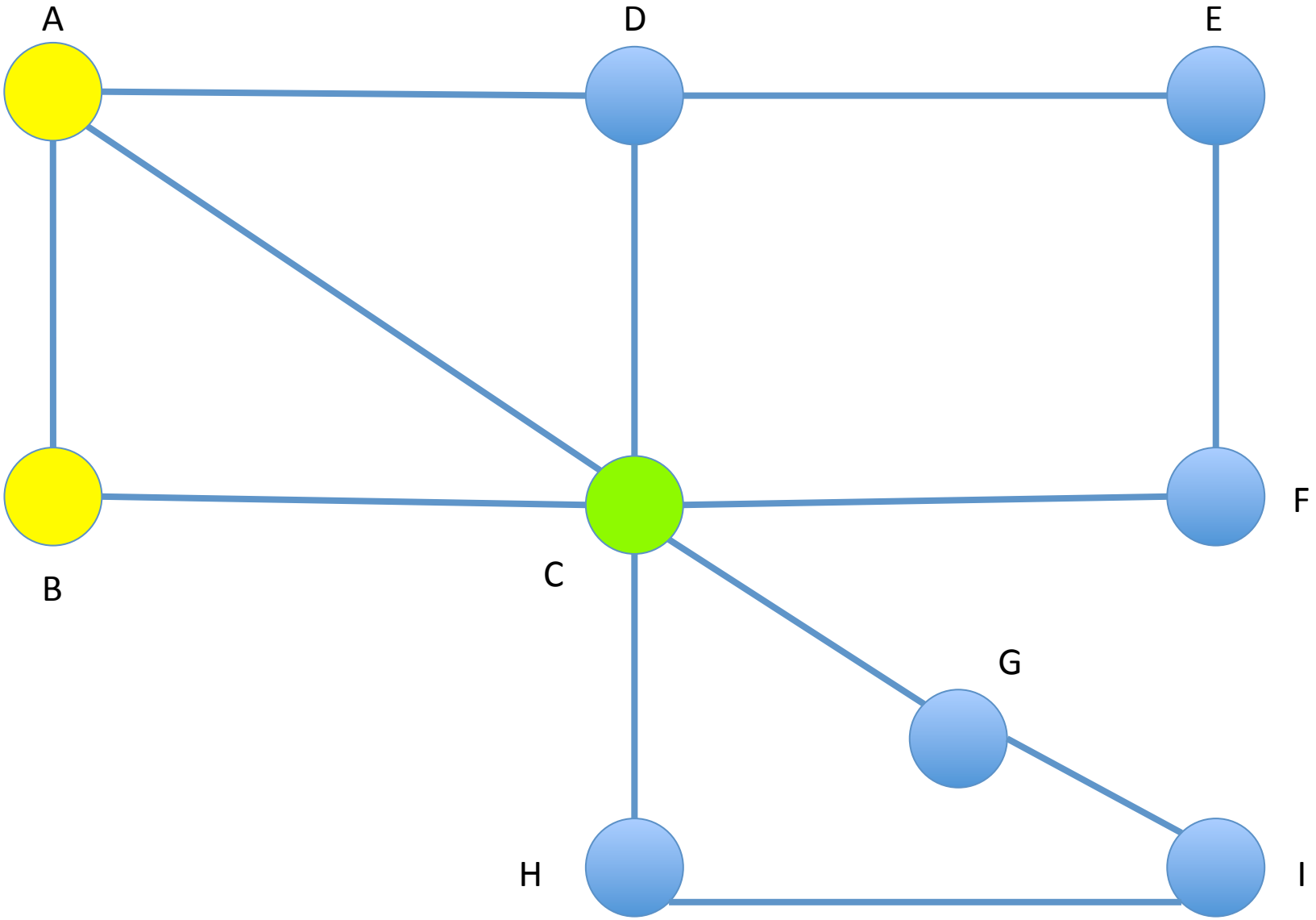
DFS



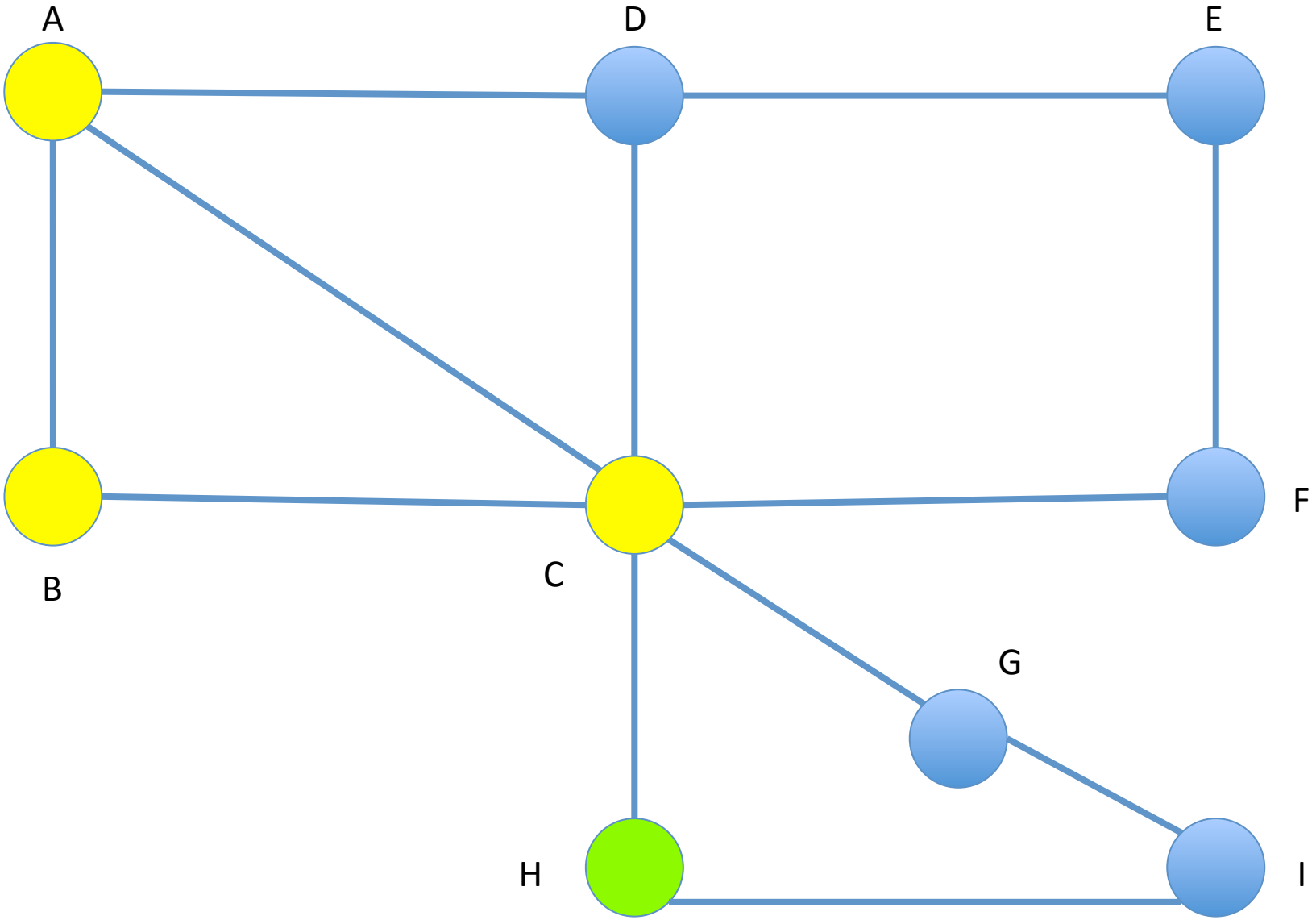
DFS



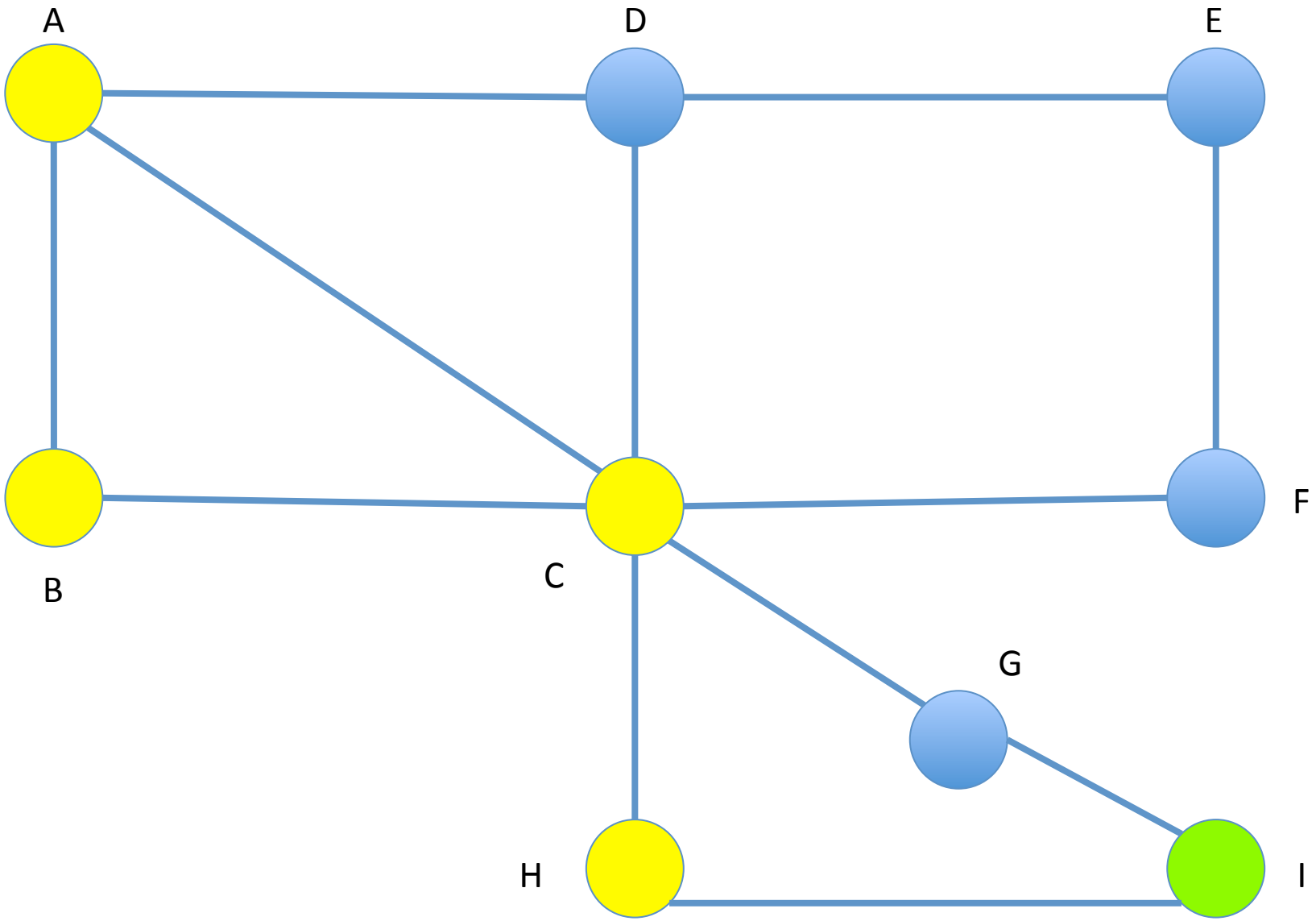
DFS



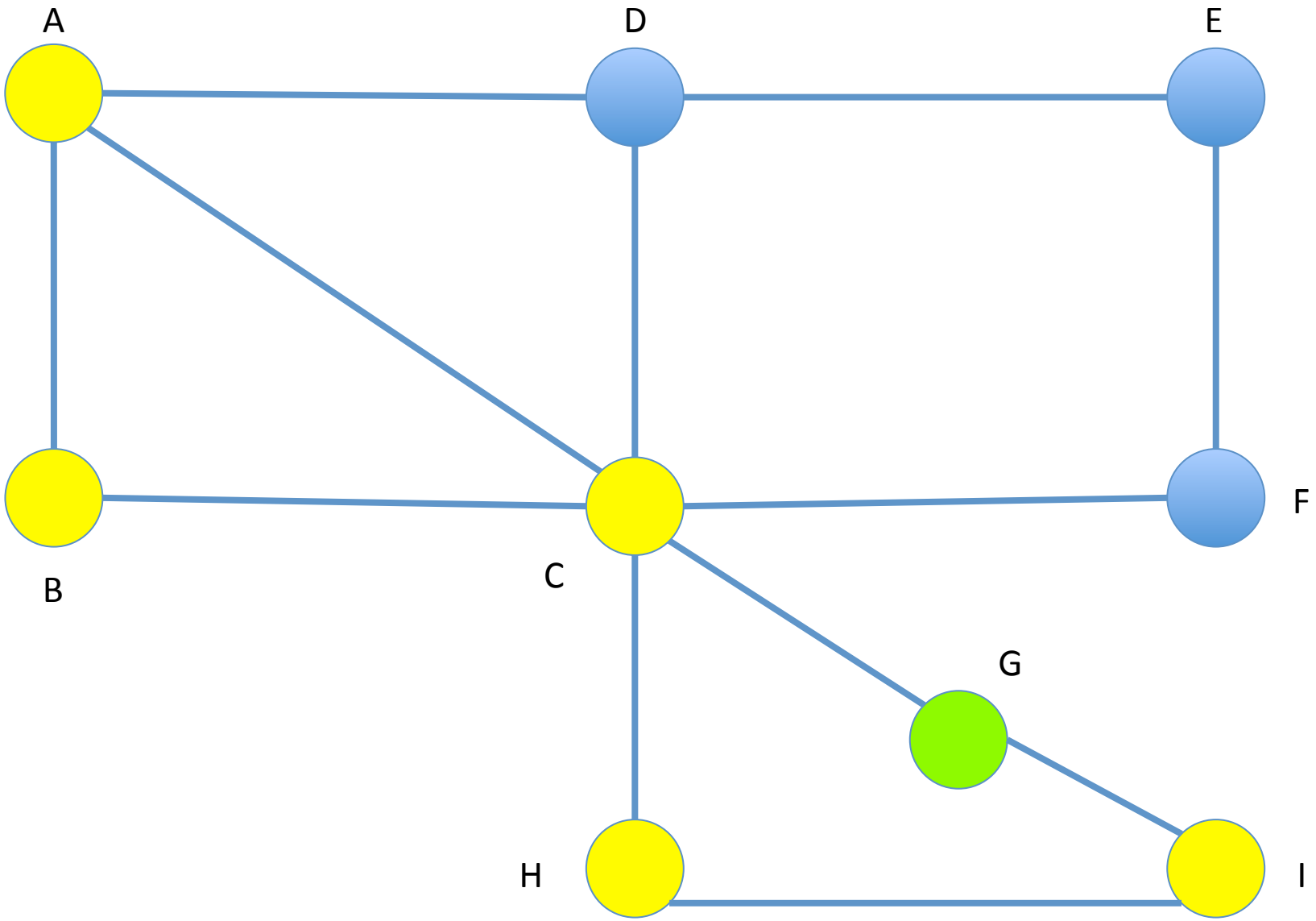
DFS



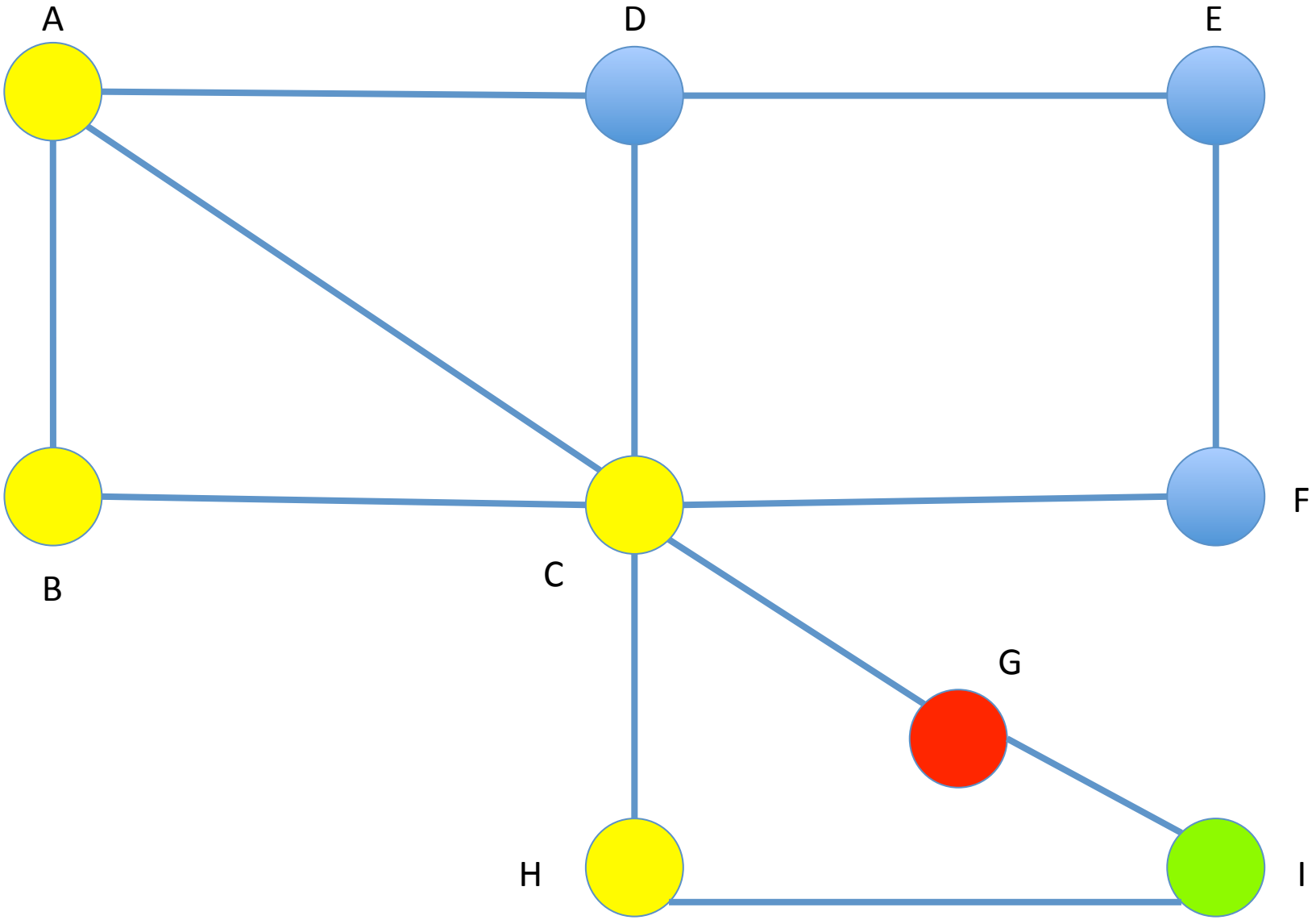
DFS



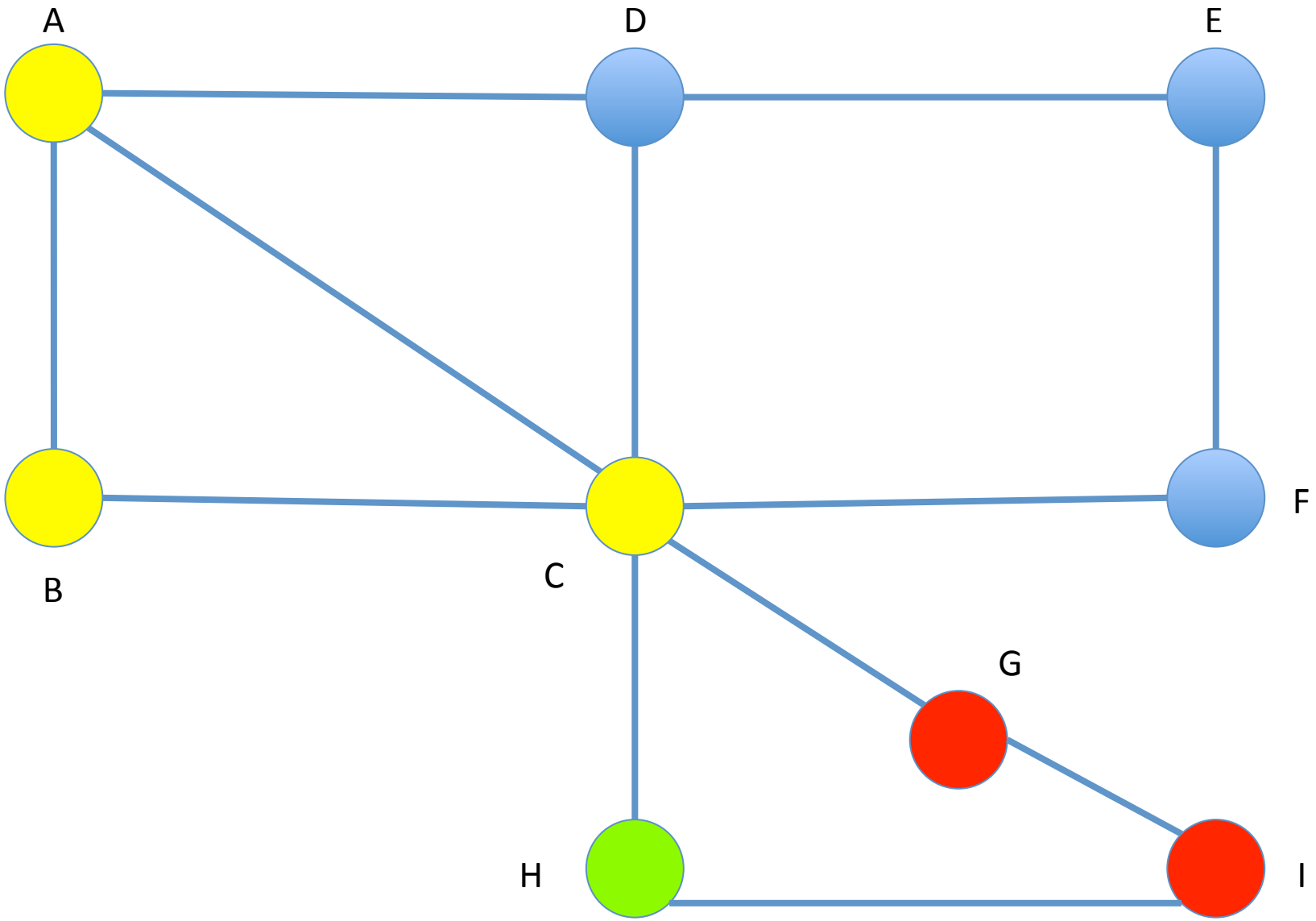
DFS



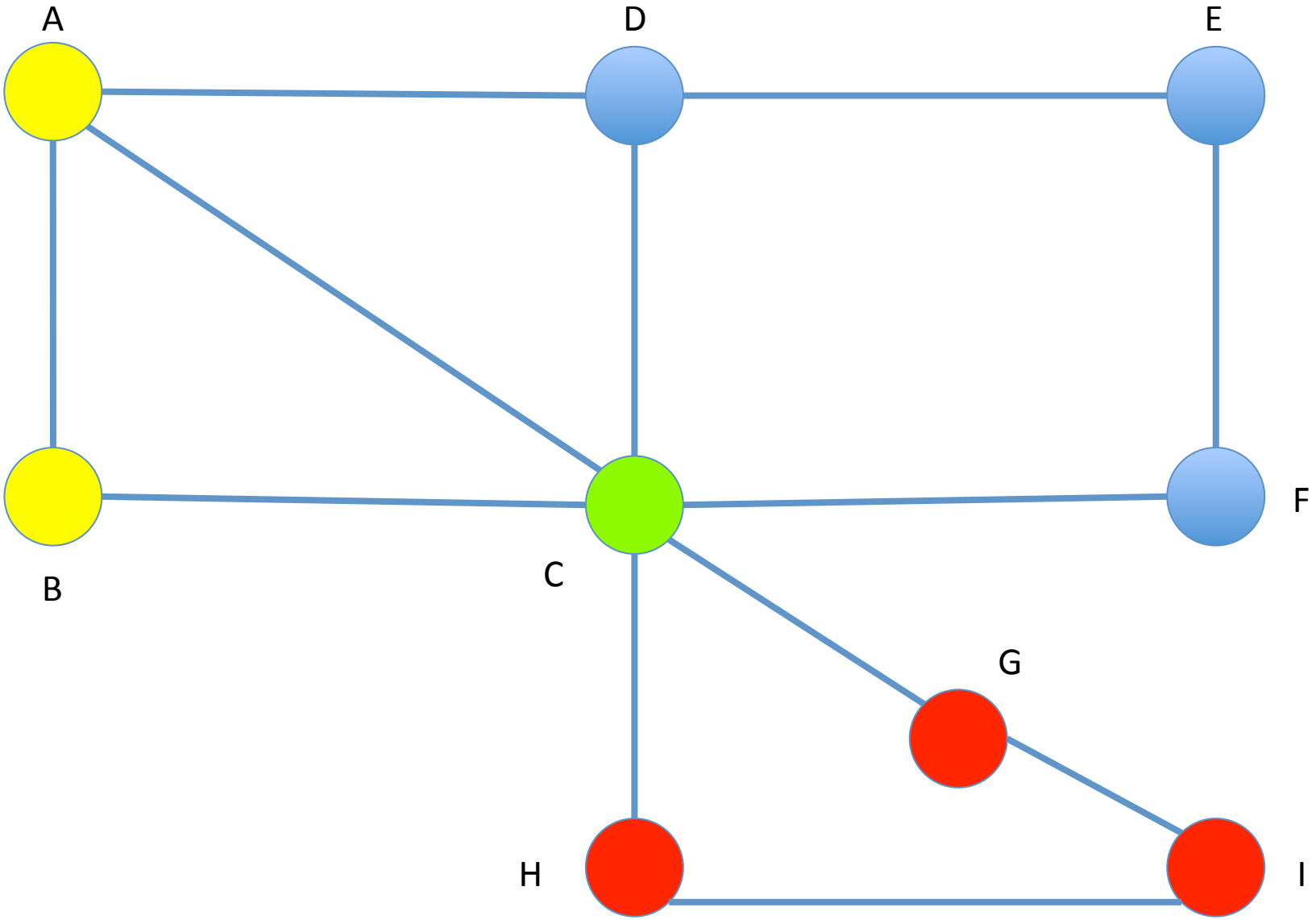
DFS



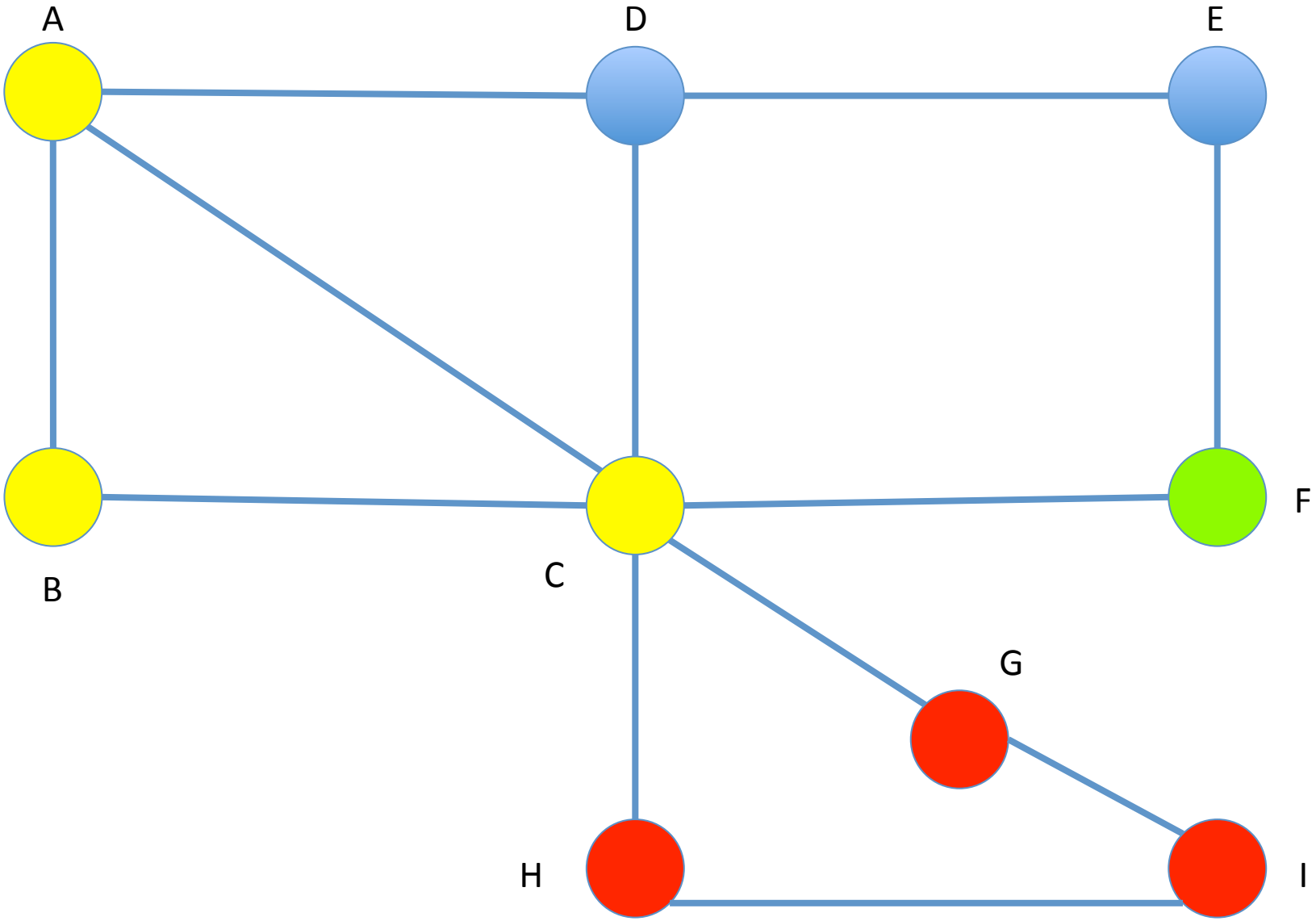
DFS



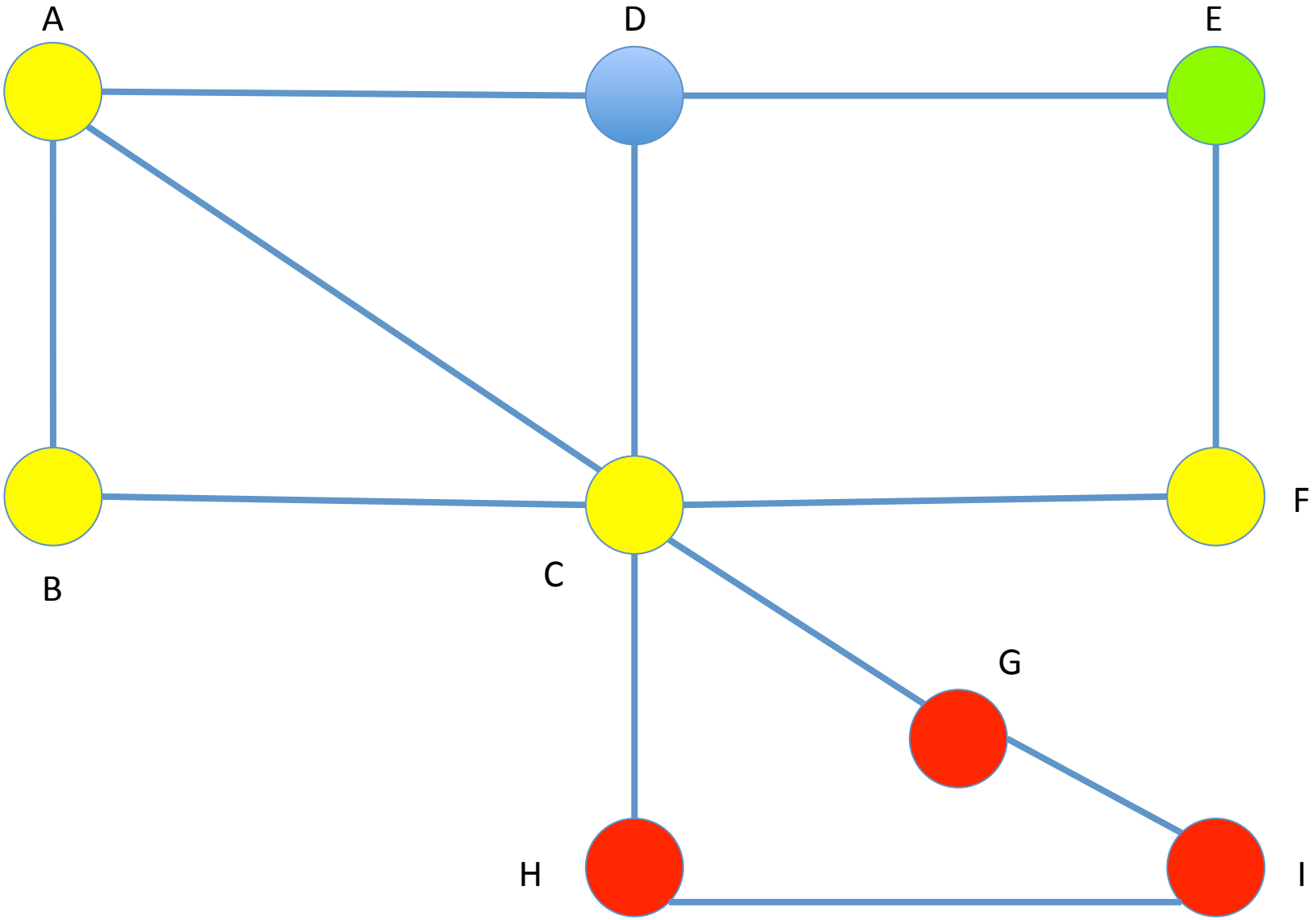
DFS



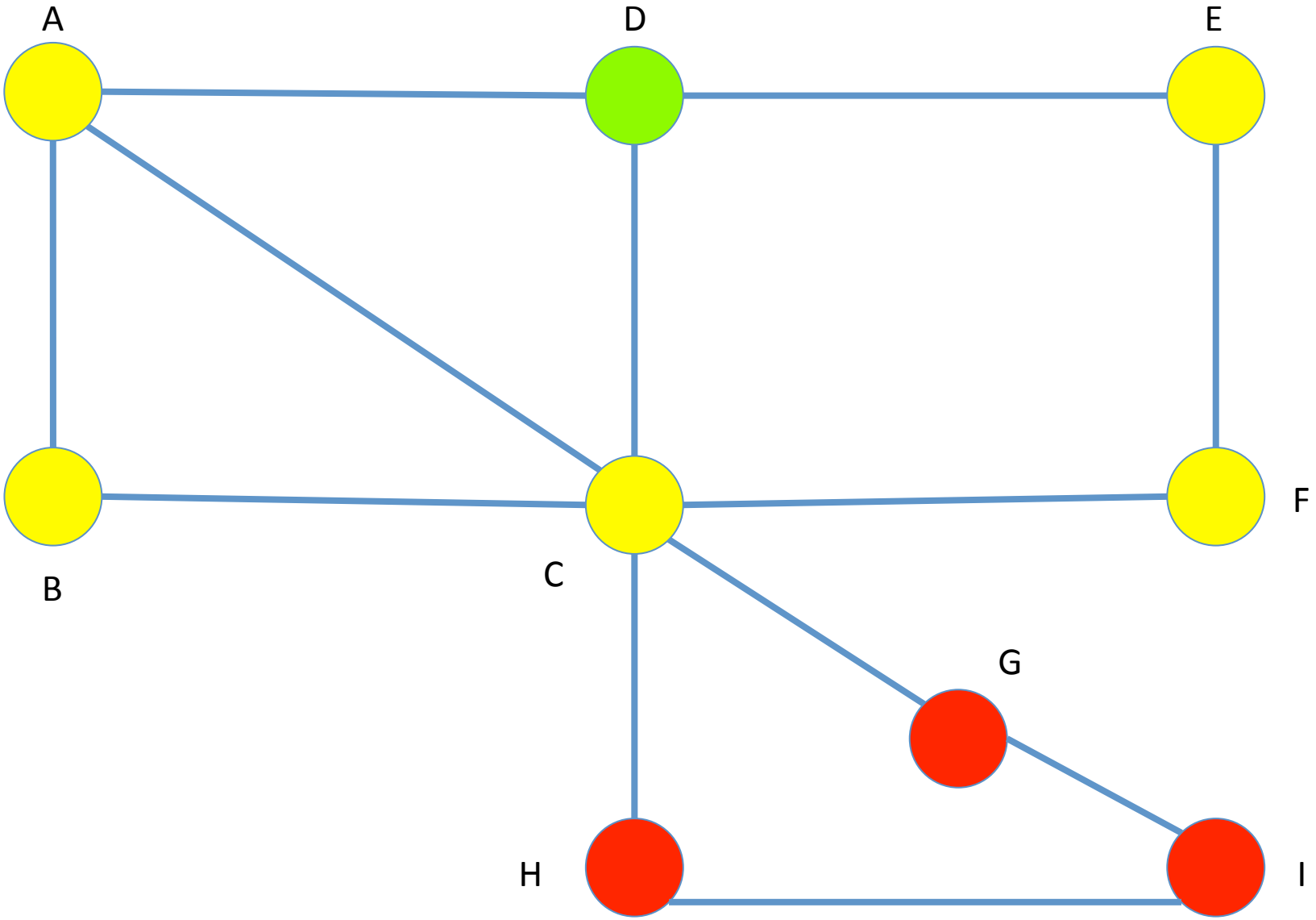
DFS



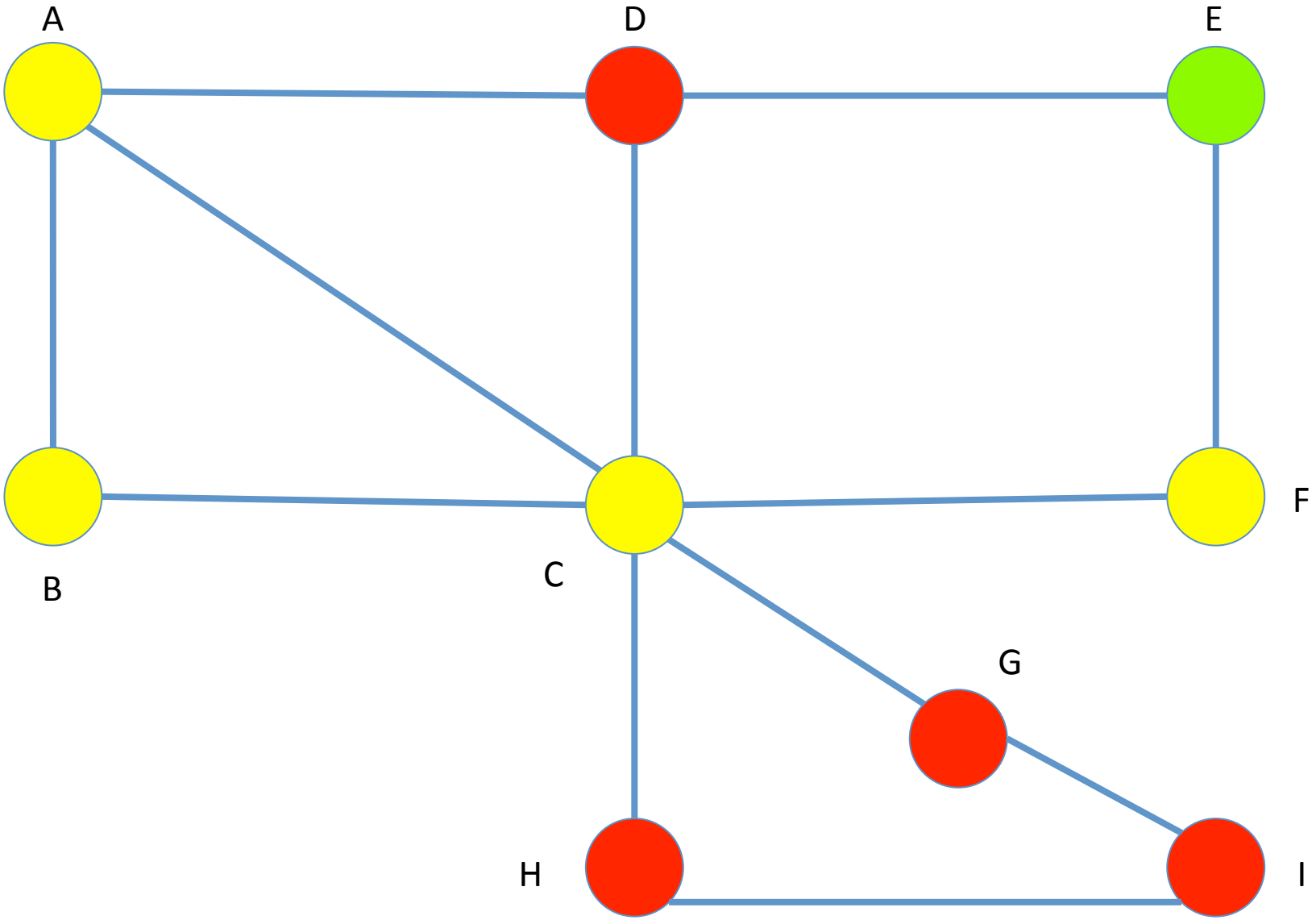
DFS



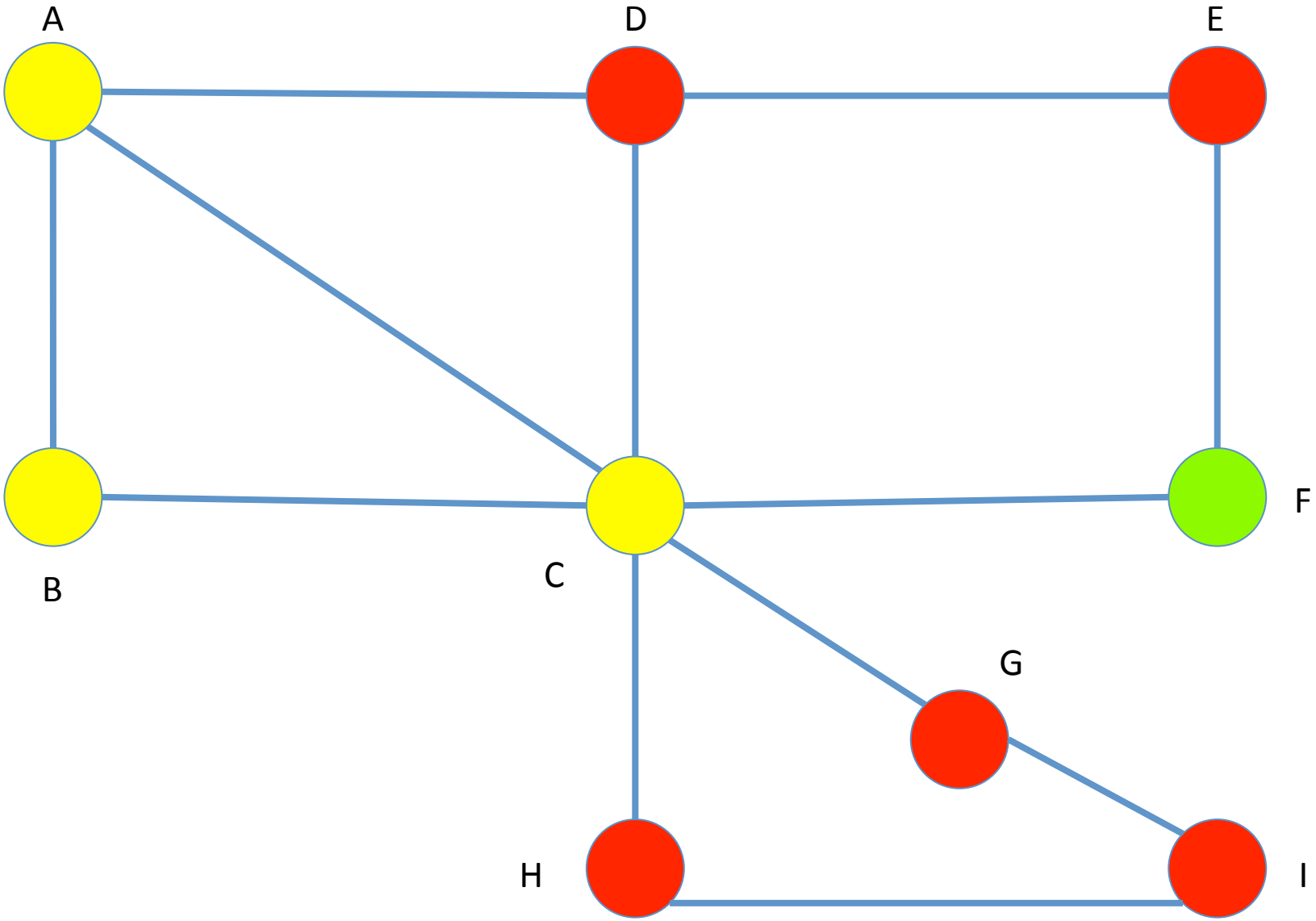
DFS



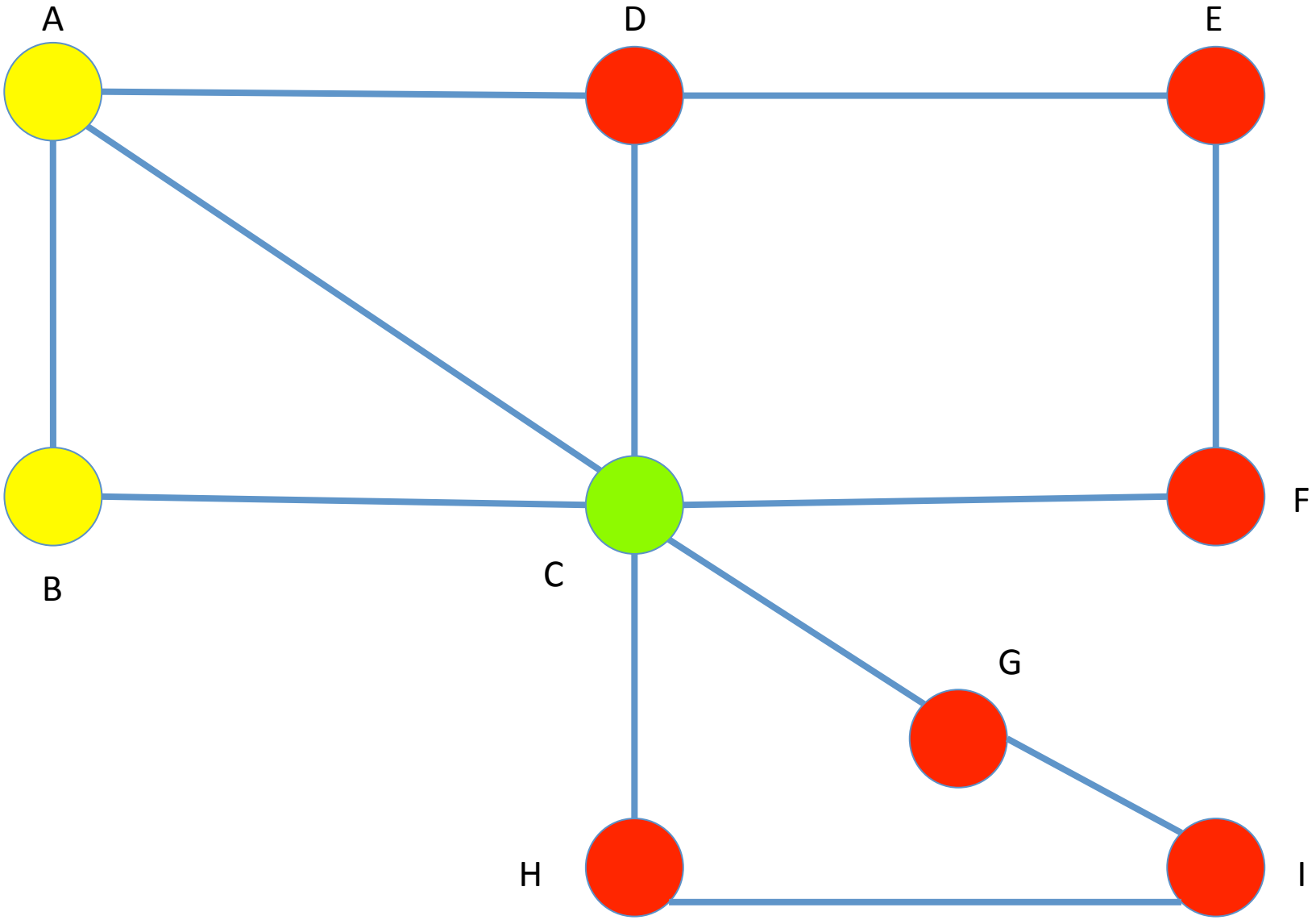
DFS



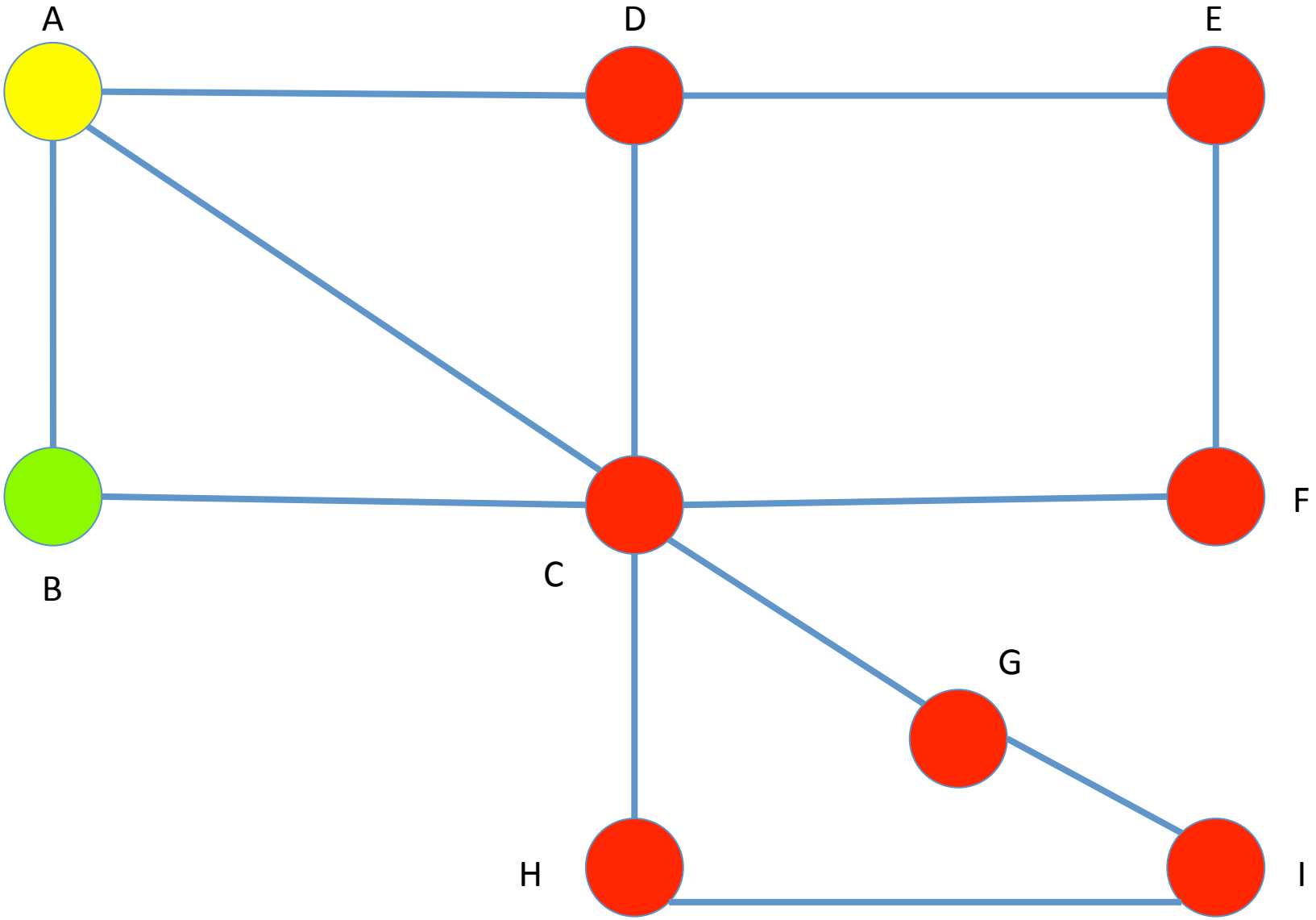
DFS



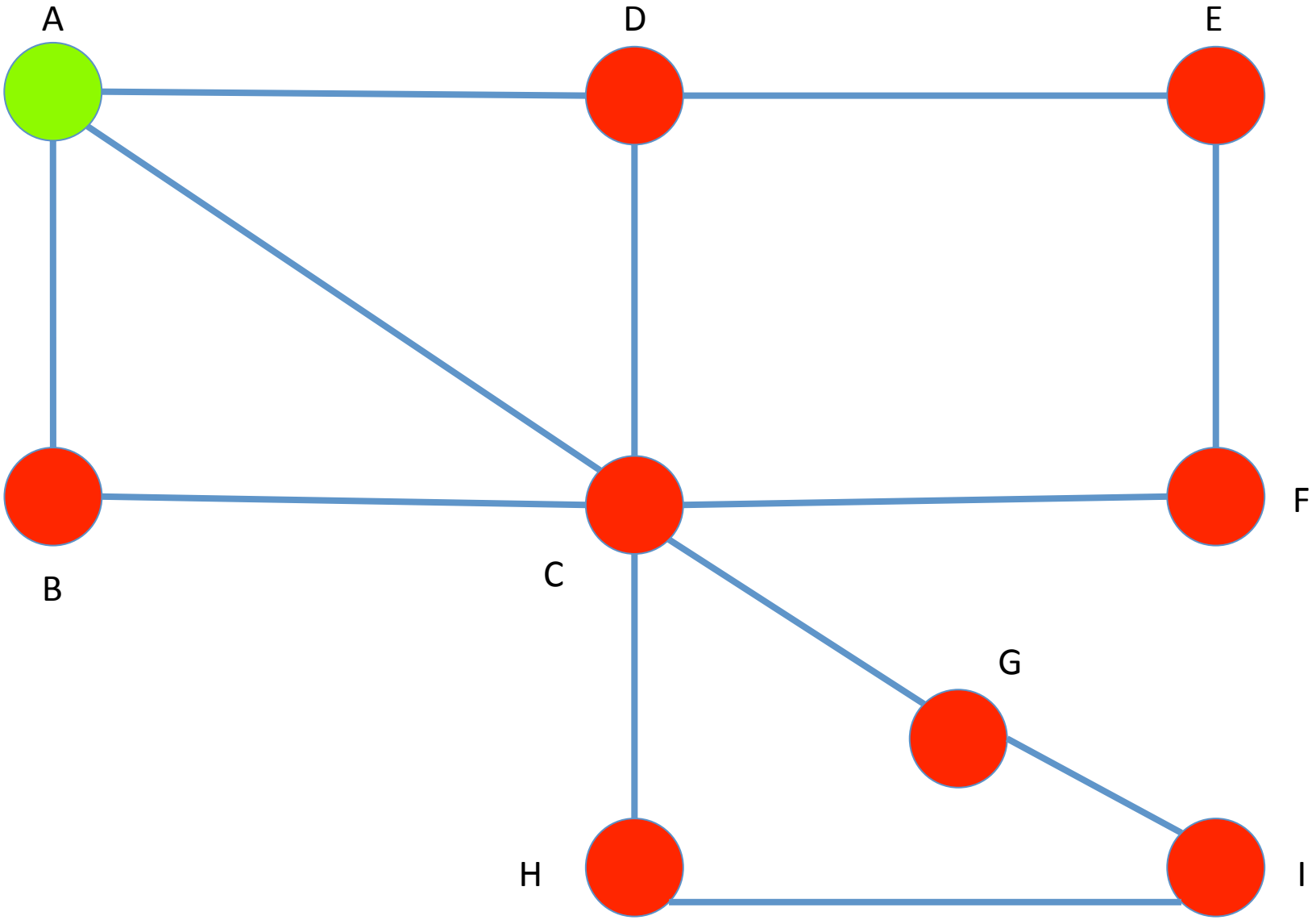
DFS



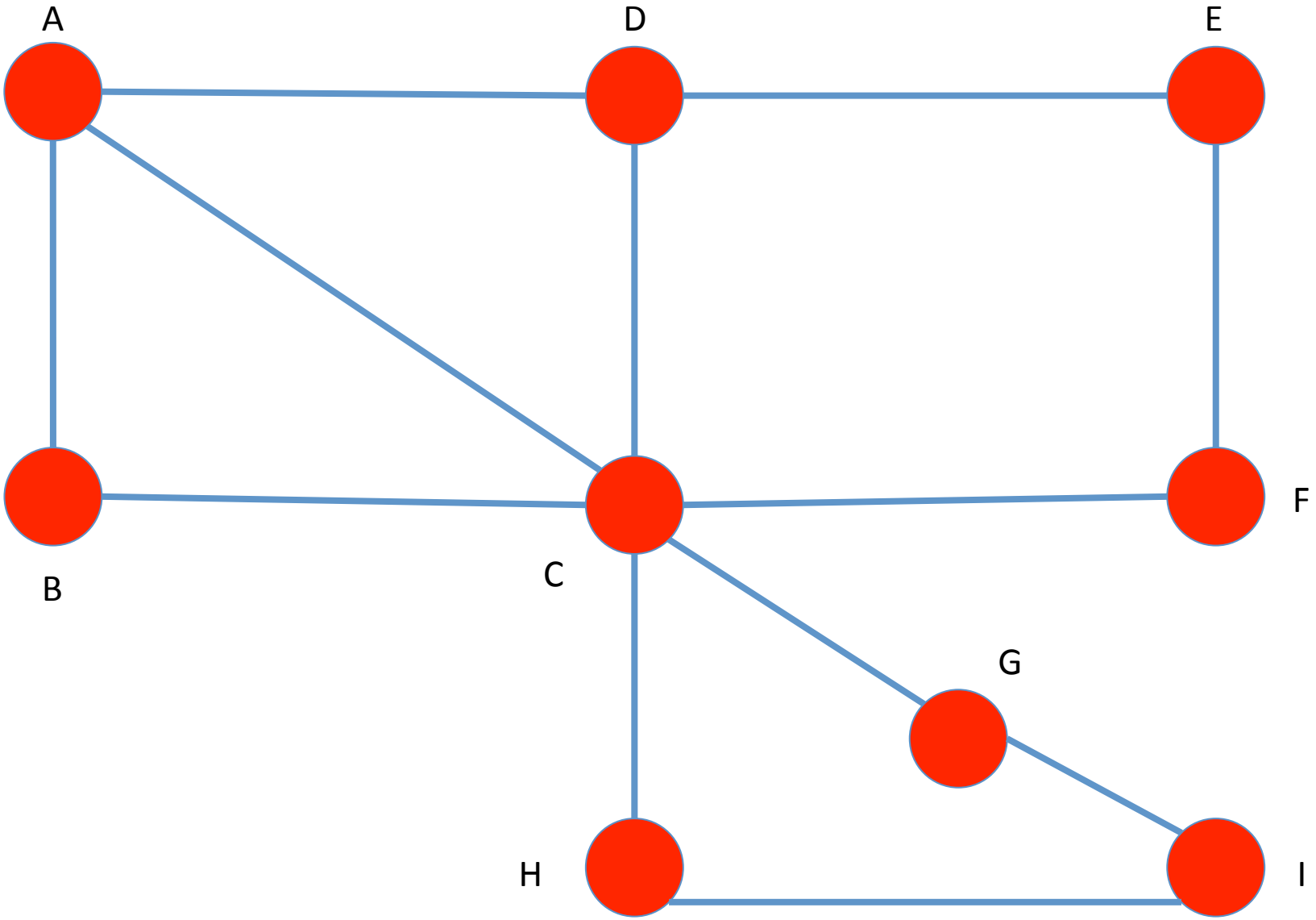
DFS



DFS



DFS



DFS Pseudocode (Recursive Version)

```
1.  procedure DFS(G(V, E), s)
2.      mark s visited
3.      for each (neighbor v of s):
4.          if (v is not-visited):
5.              DFS(G, v);
6.      mark s finished
```

```
1.  procedure DFS(G(V, E))
2.      mark all vertices as not-visited
3.      for each (v ∈ V, if v is not-visited do DFS(G, v)
```

Runtime: O(n + m) (with adj. list)

Visually:

- each v traversed once
- each (u, v) will be "traversed" at most twice:
  1. when attempting to visit v
  2. and possibly once when backtracking)

Properties of BFS/DFS

- Key Observation 1:

BFS/DFS are both linear time

(when implemented by the right data structures)

- Key Observation 2:

A BFS/DFS starting from s will reach all vertices t such that s has a path to t.

Exercise: Prove this claim by induction (on the length of the path that s has to t).

BFS Tree of a "Connected Graph"

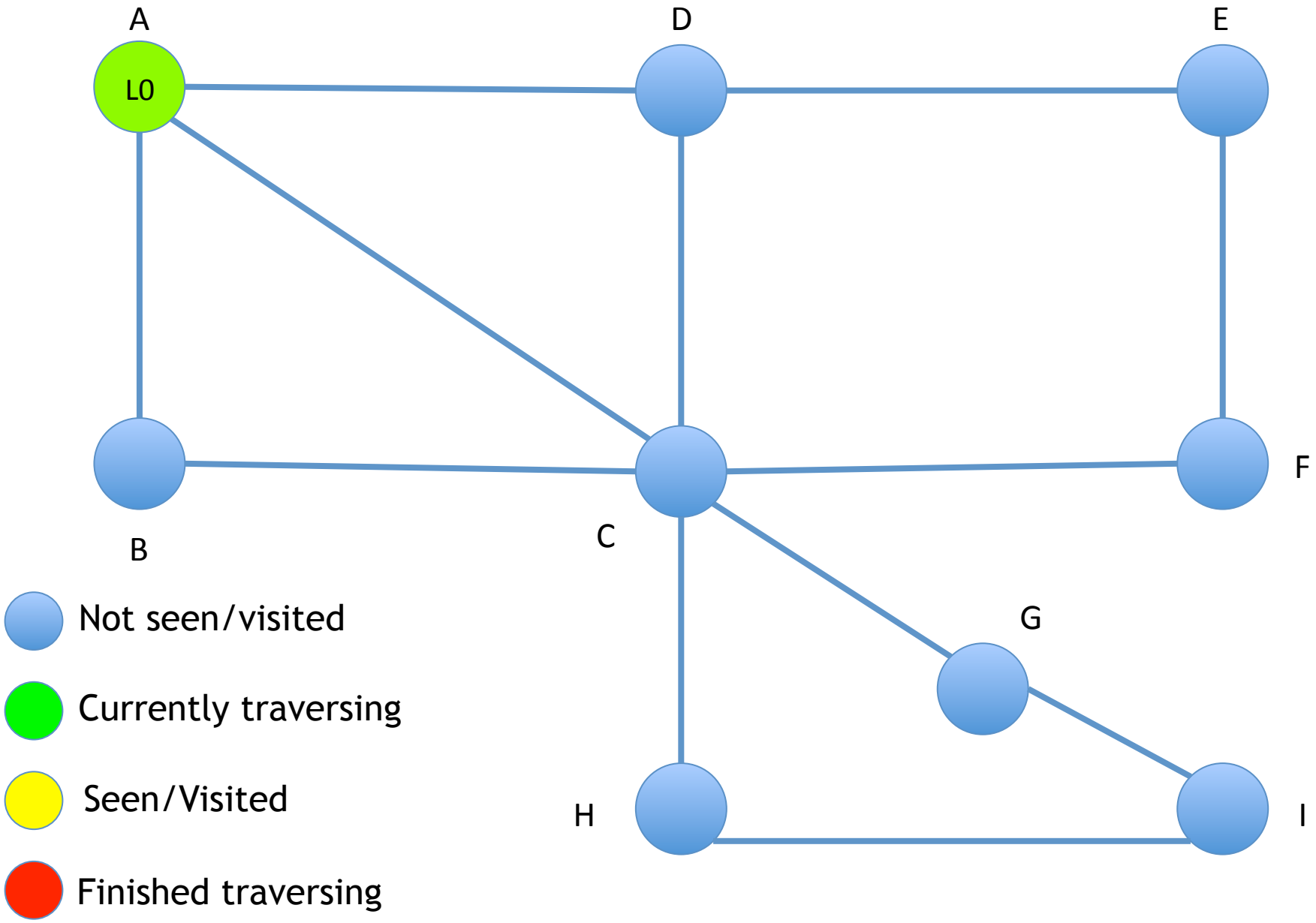
- Dfn "parent of v" p(v):

  Vertex u that was being traversed *when v was first visited*

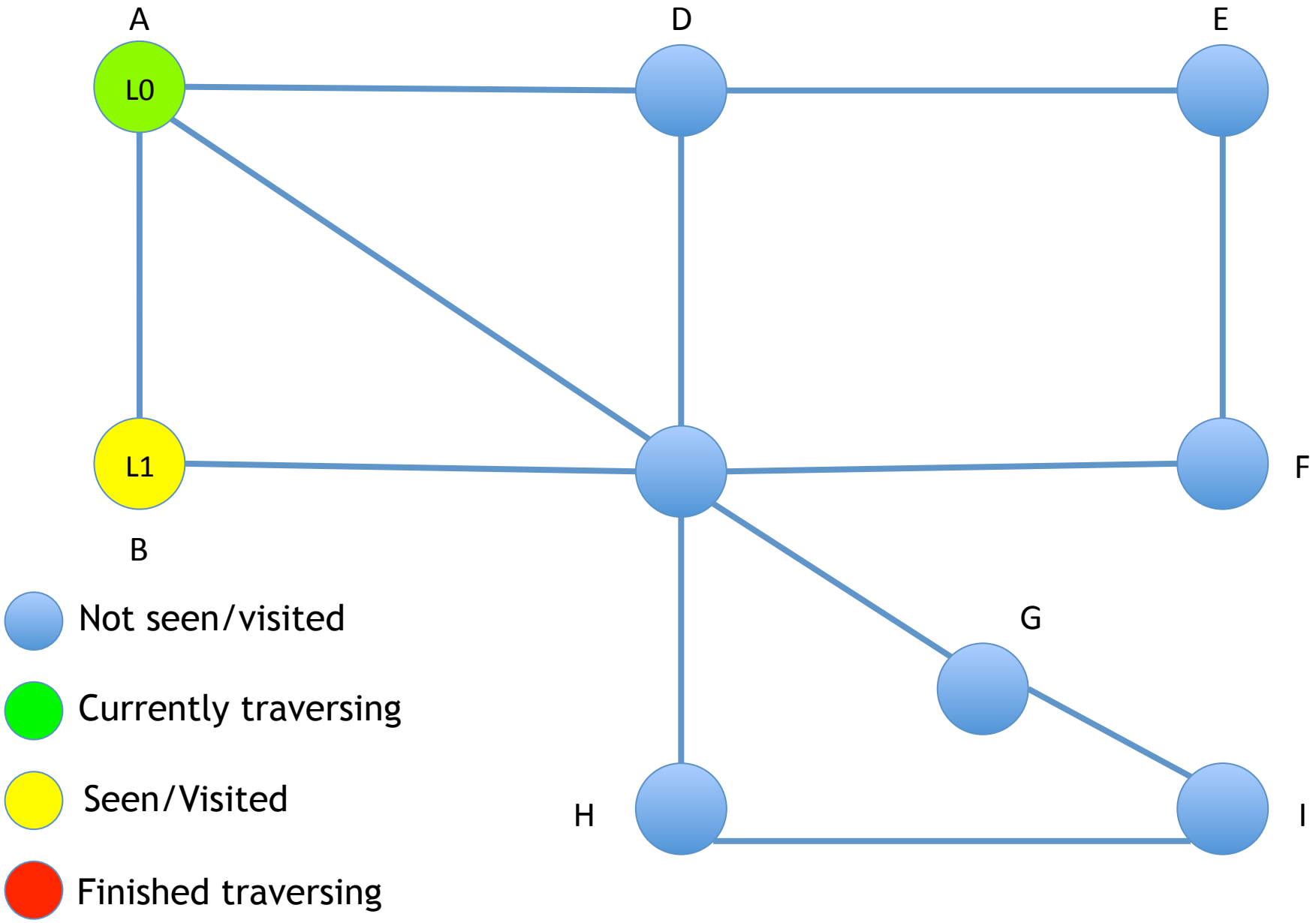
  In simulation: the green vertex u when v became yellow

- BFS Tree(V_T, E_T) is the graph s.t.
  - $V_T = V$
  - $E_T = \{\forall v: (p(v), v)\}$ (can define as (v, p(v)) as well, depending on a application or as undirected)

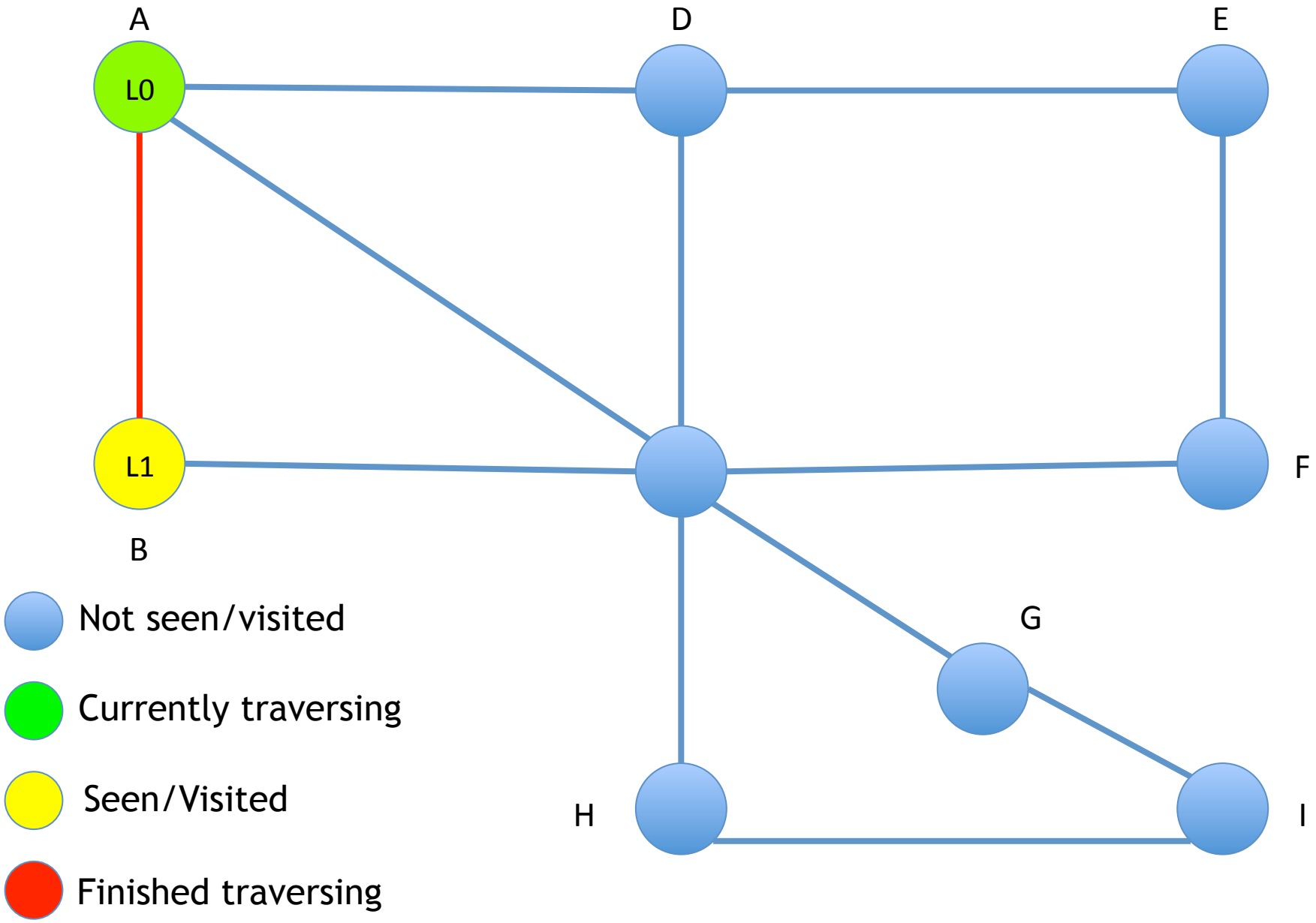
BFS Tree


A
L0
D
E
F
B
C
G
H
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

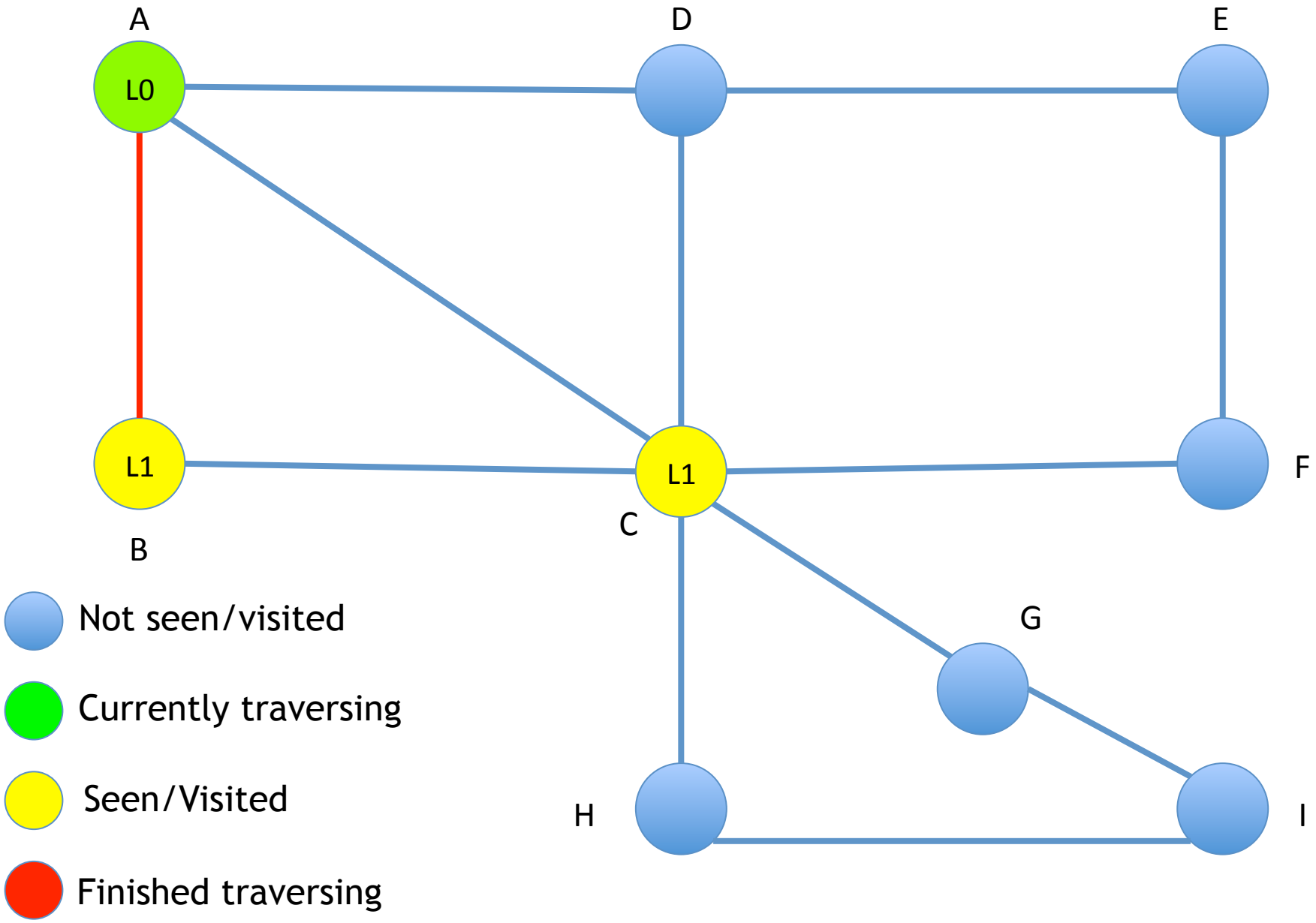
BFS Tree


A
D
E
L0
L1
F
B
G
H
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

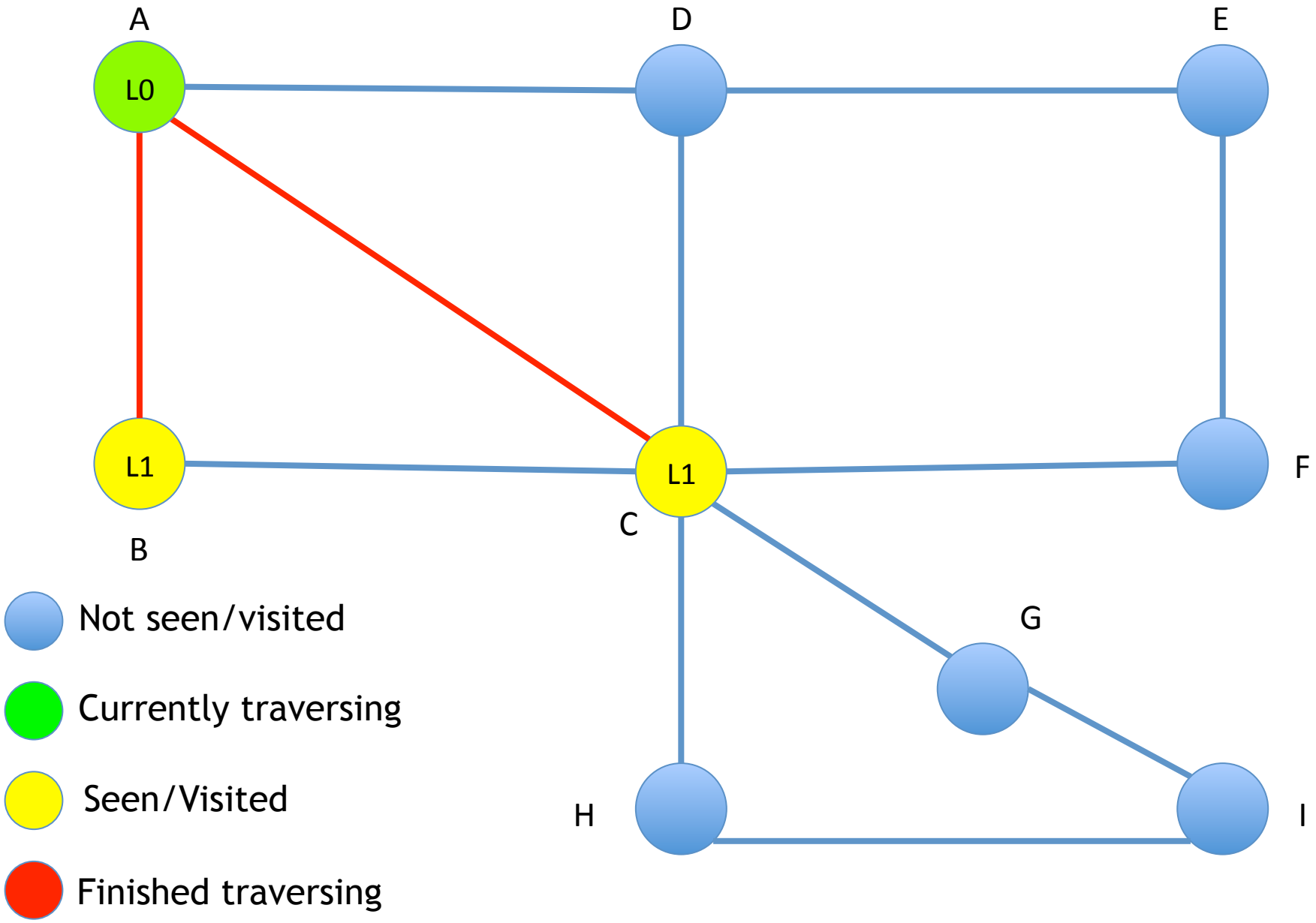
BFS Tree


A
D
E
L0
L1
F
B
G
H
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

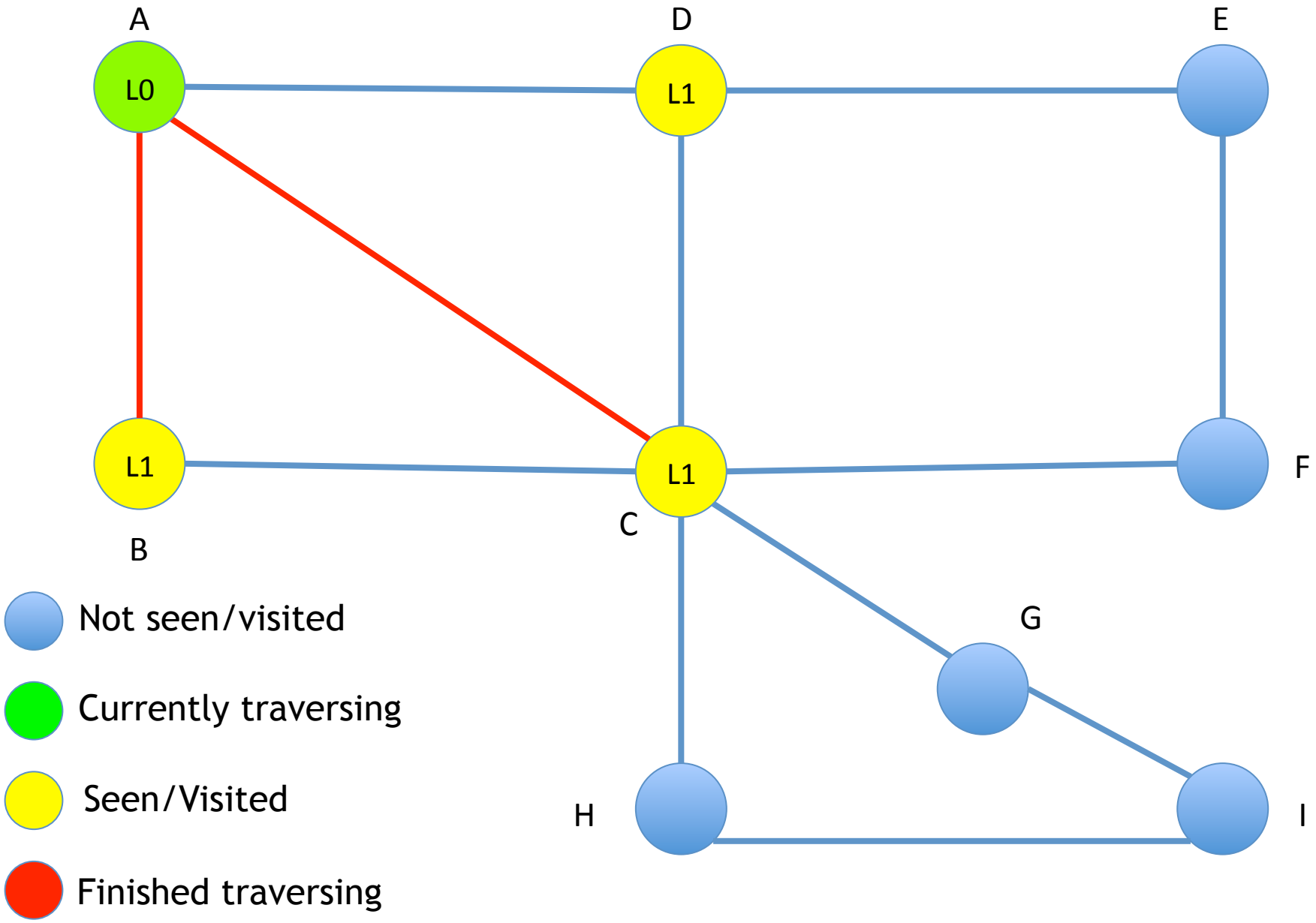
BFS Tree


A
D
E
L0
L1
L1
F
B
C
G
H
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

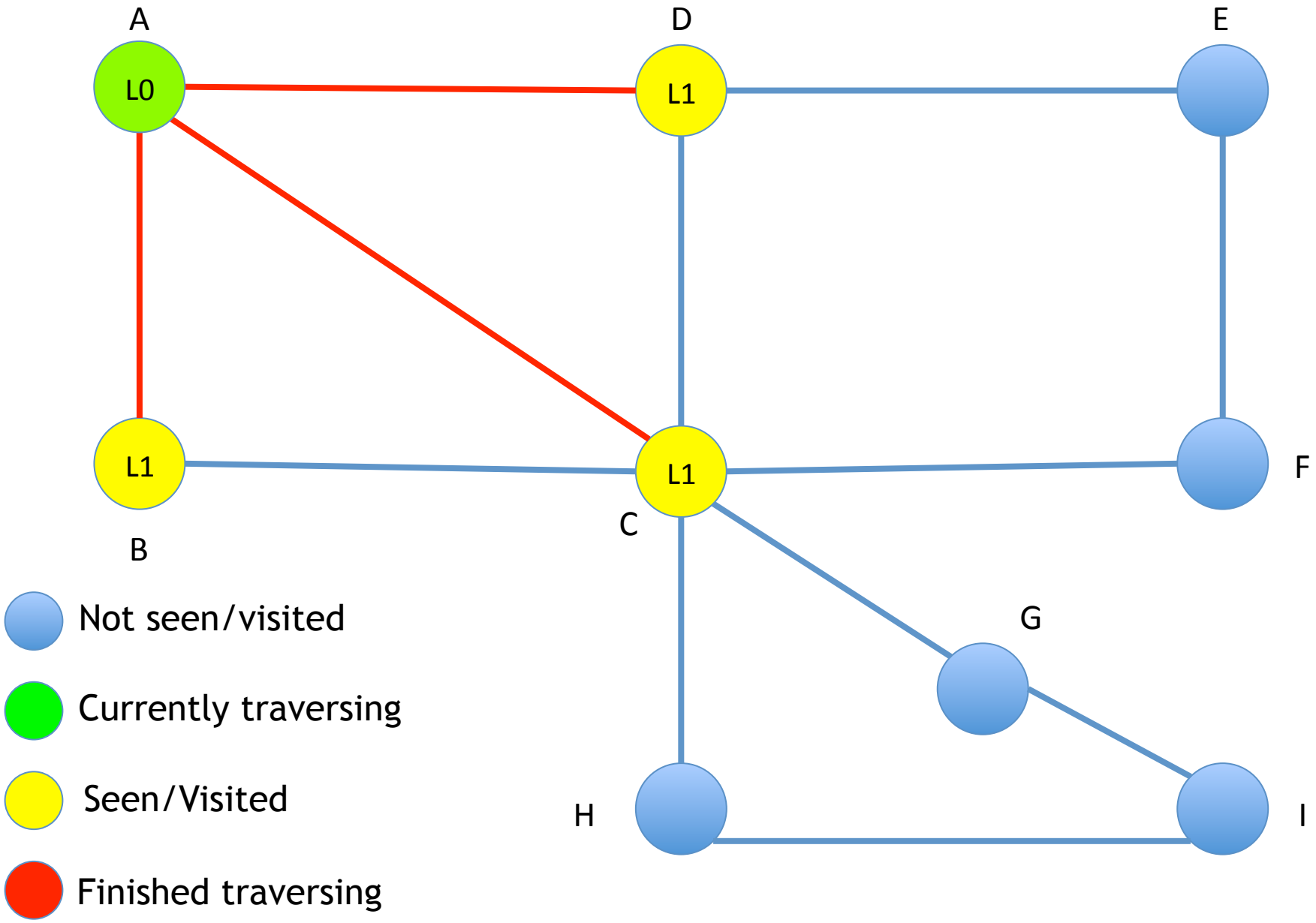
BFS Tree


A
D
E
L0
L1
L1
F
C
B
G
H
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

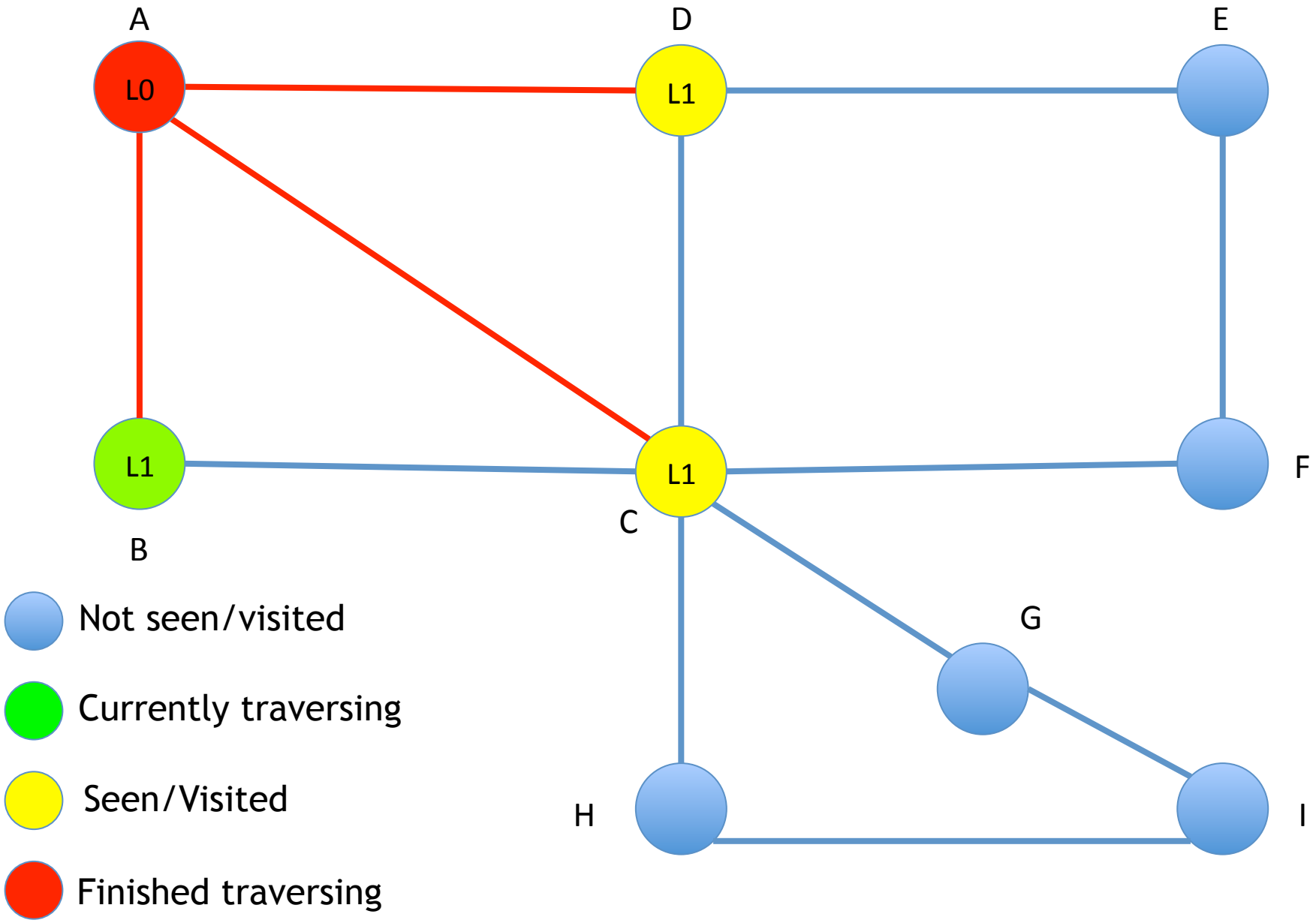
BFS Tree


A
D
E
L0
L1
L1
L1
F
C
B
G
H
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

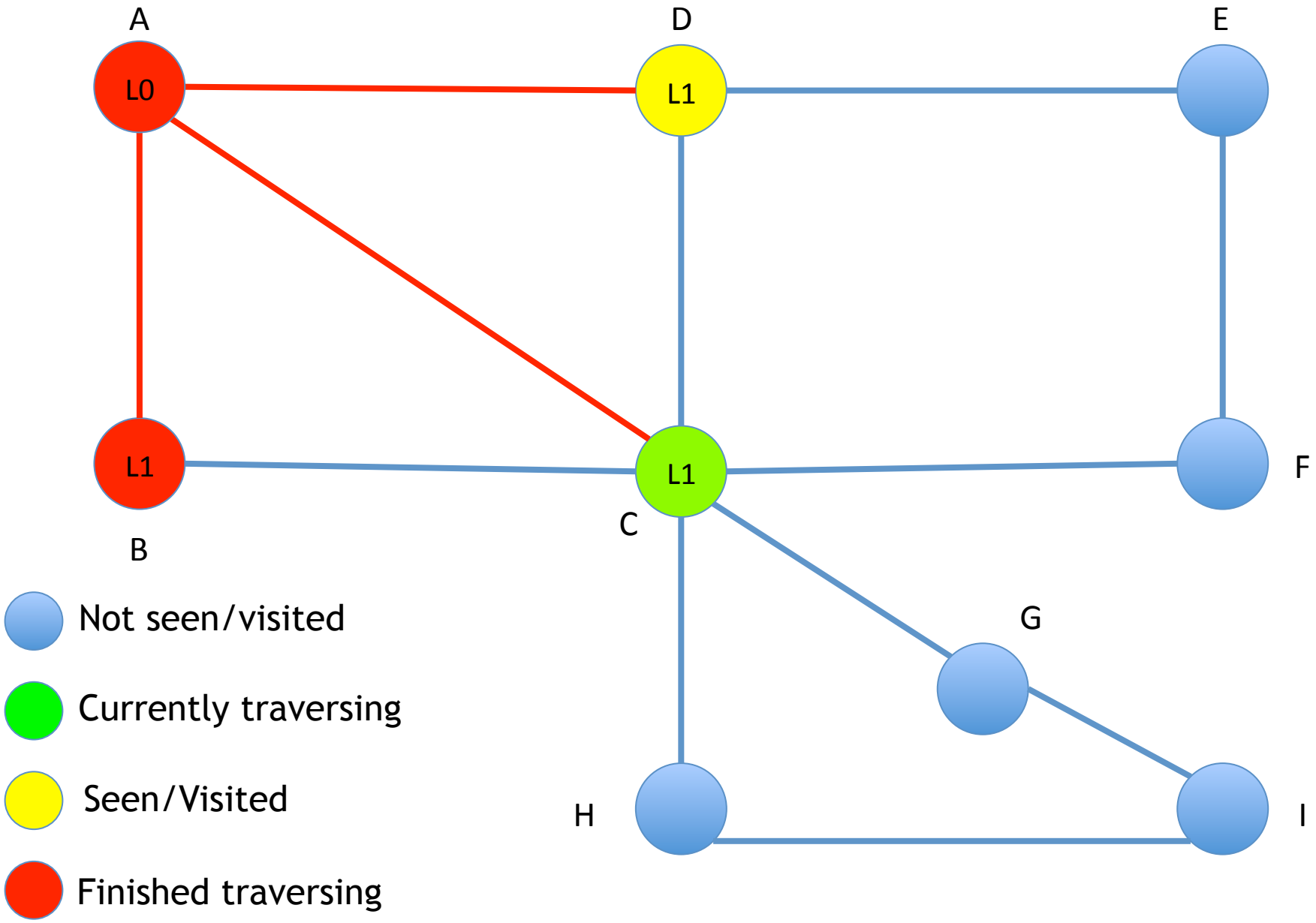
BFS Tree


A
D
E
L0
L1
L1
L1
F
C
B
G
H
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

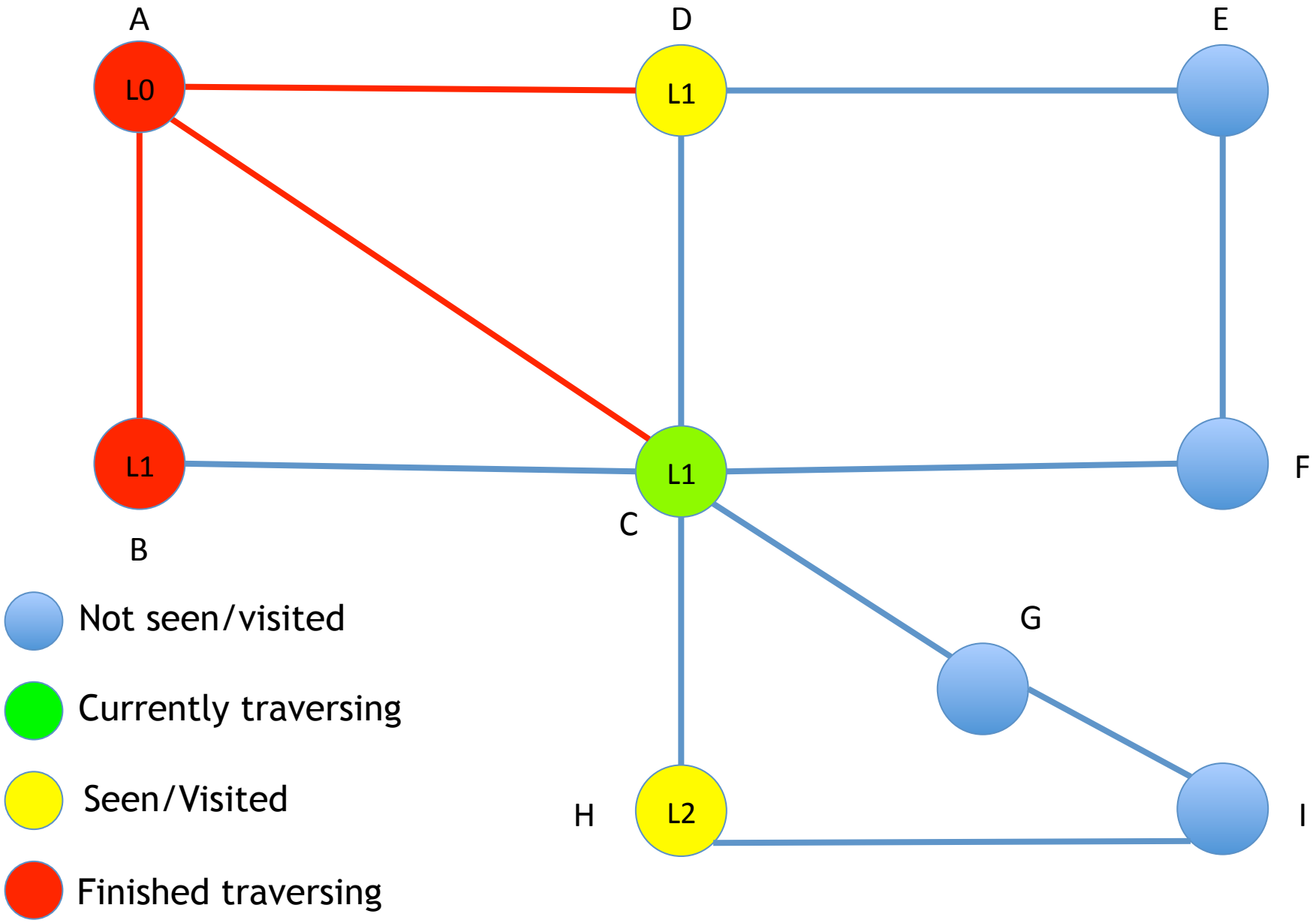
BFS Tree


A
L0
D
L1
E
L1
B
L1
C
F
G
H
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

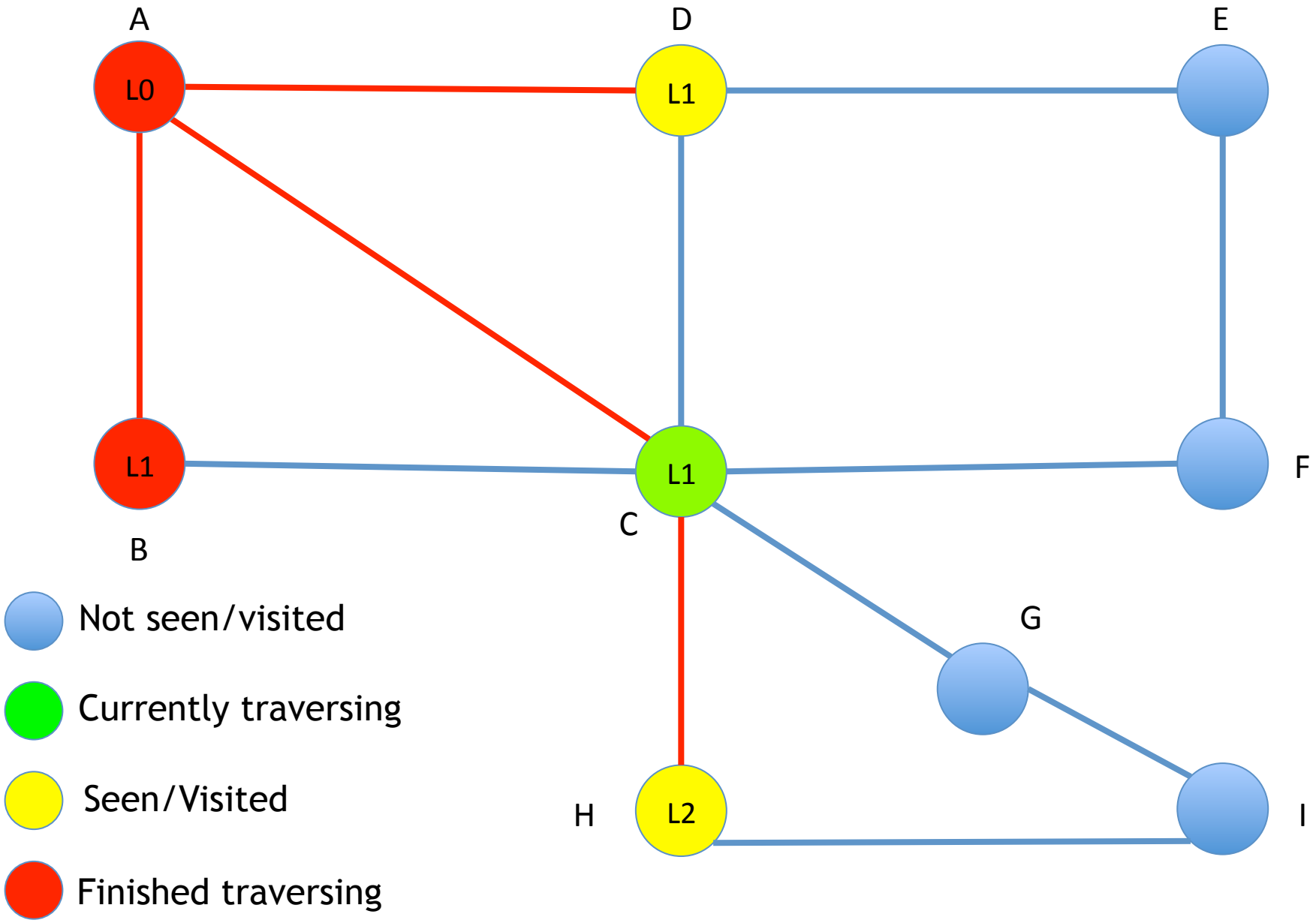
BFS Tree


A
L0
D
L1
E
L1
B
L1
C
F
G
H
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

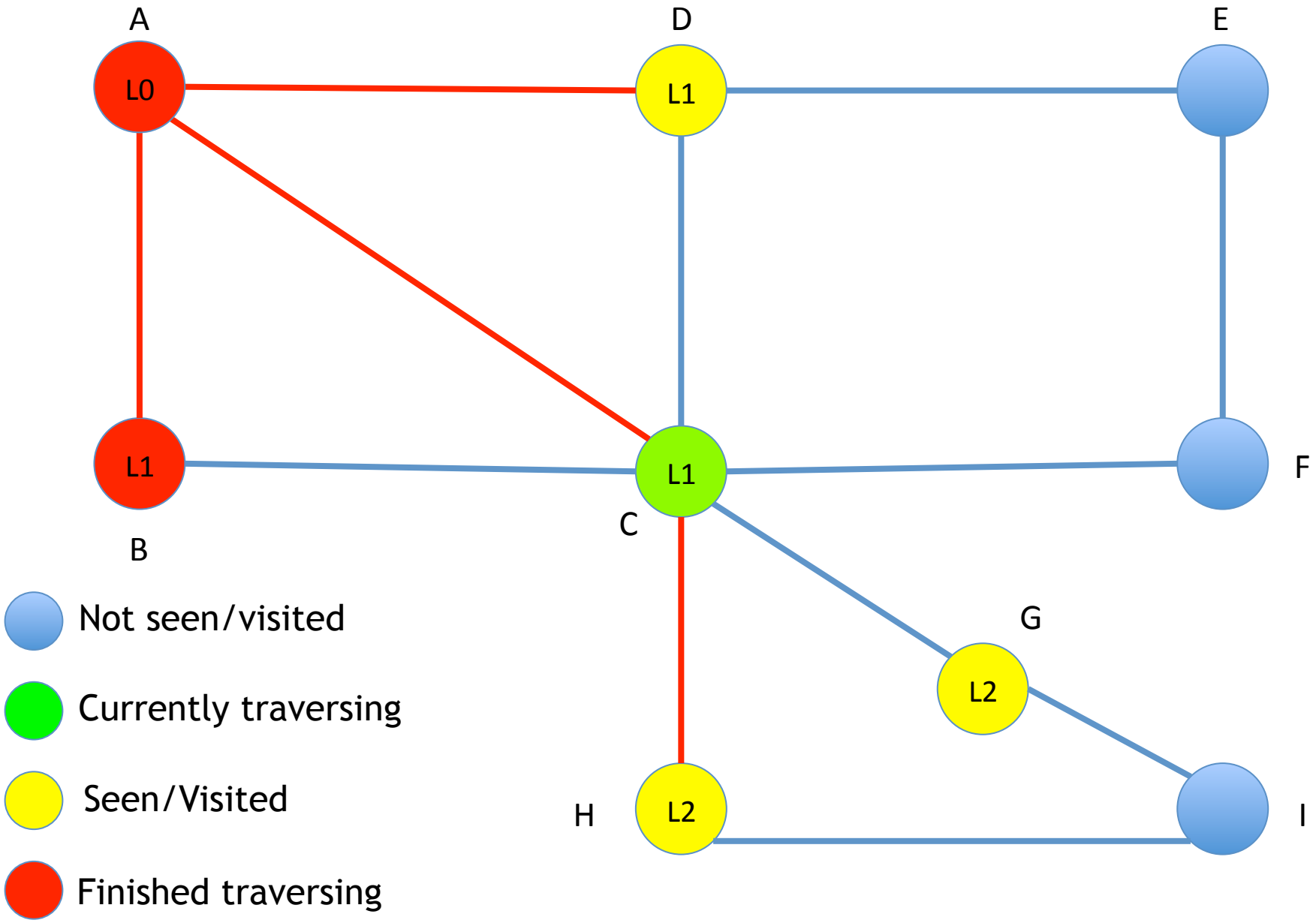
BFS Tree


A
L0
D
L1
E
L1
B
L1
C
F
G
H
L2
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

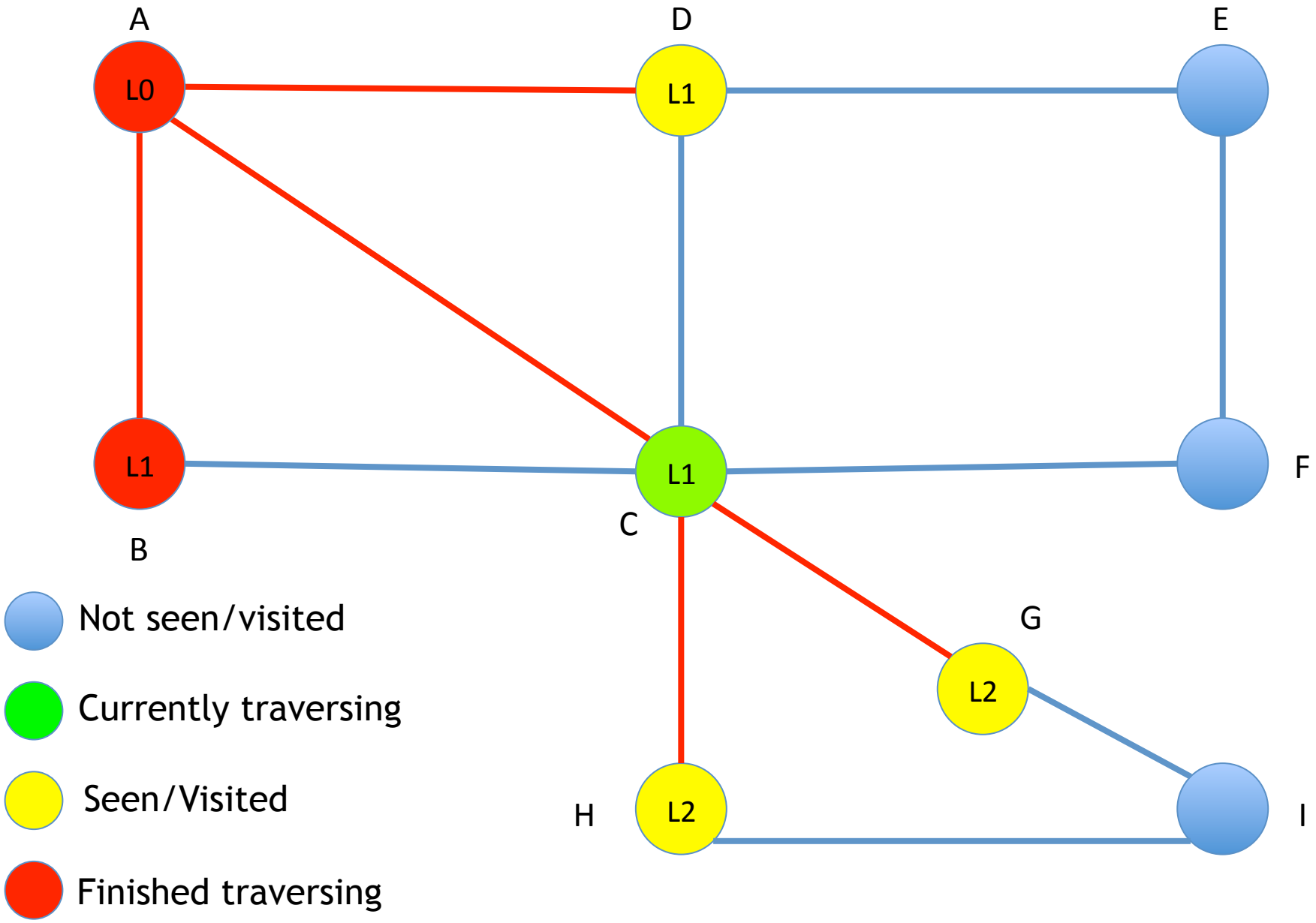
BFS Tree


A
L0
D
L1
E
L1
B
L1
C
F
G
H
L2
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

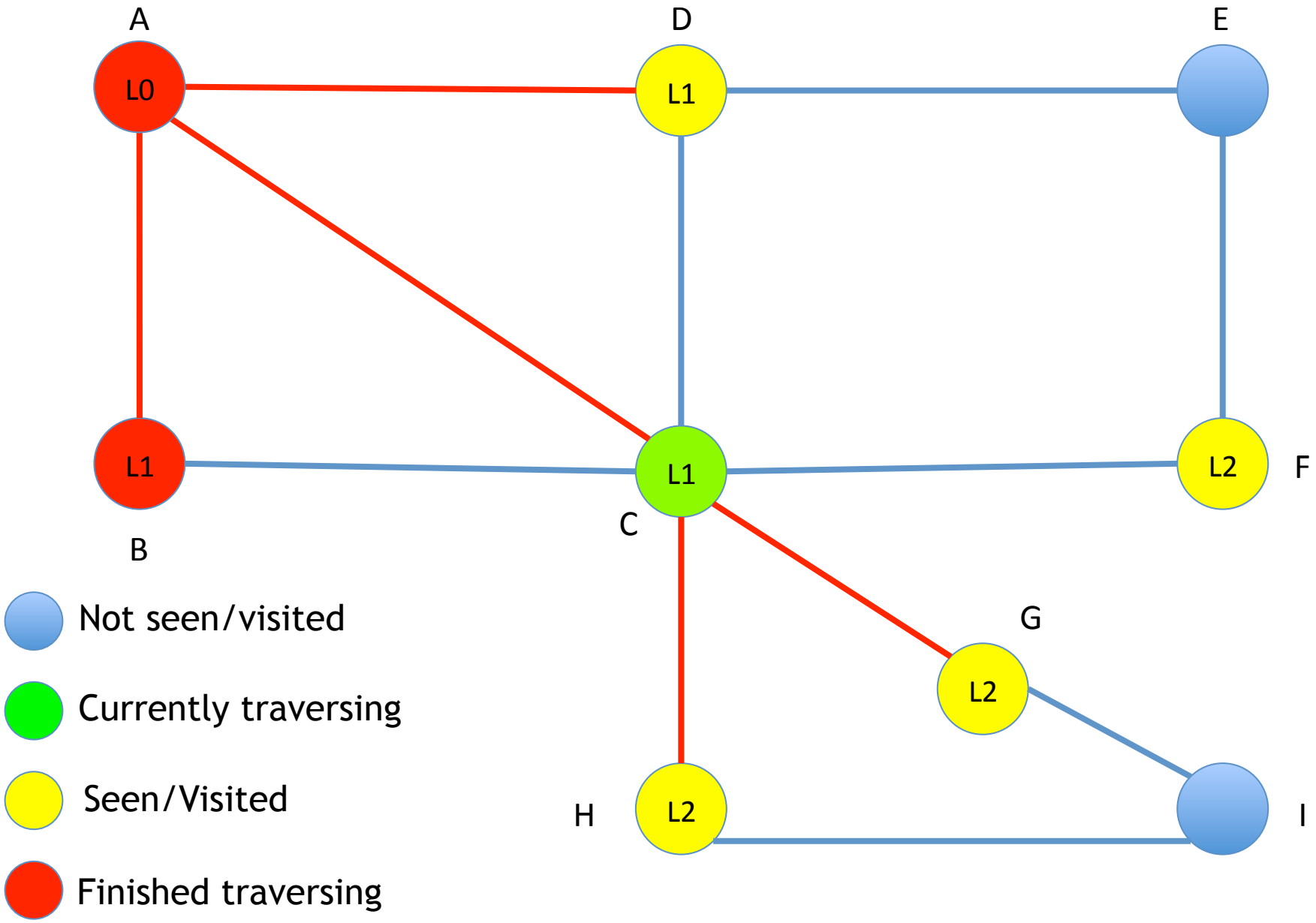
BFS Tree


A
L0
D
L1
E
L1
B
L1
C
F
G
L2
H
L2
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

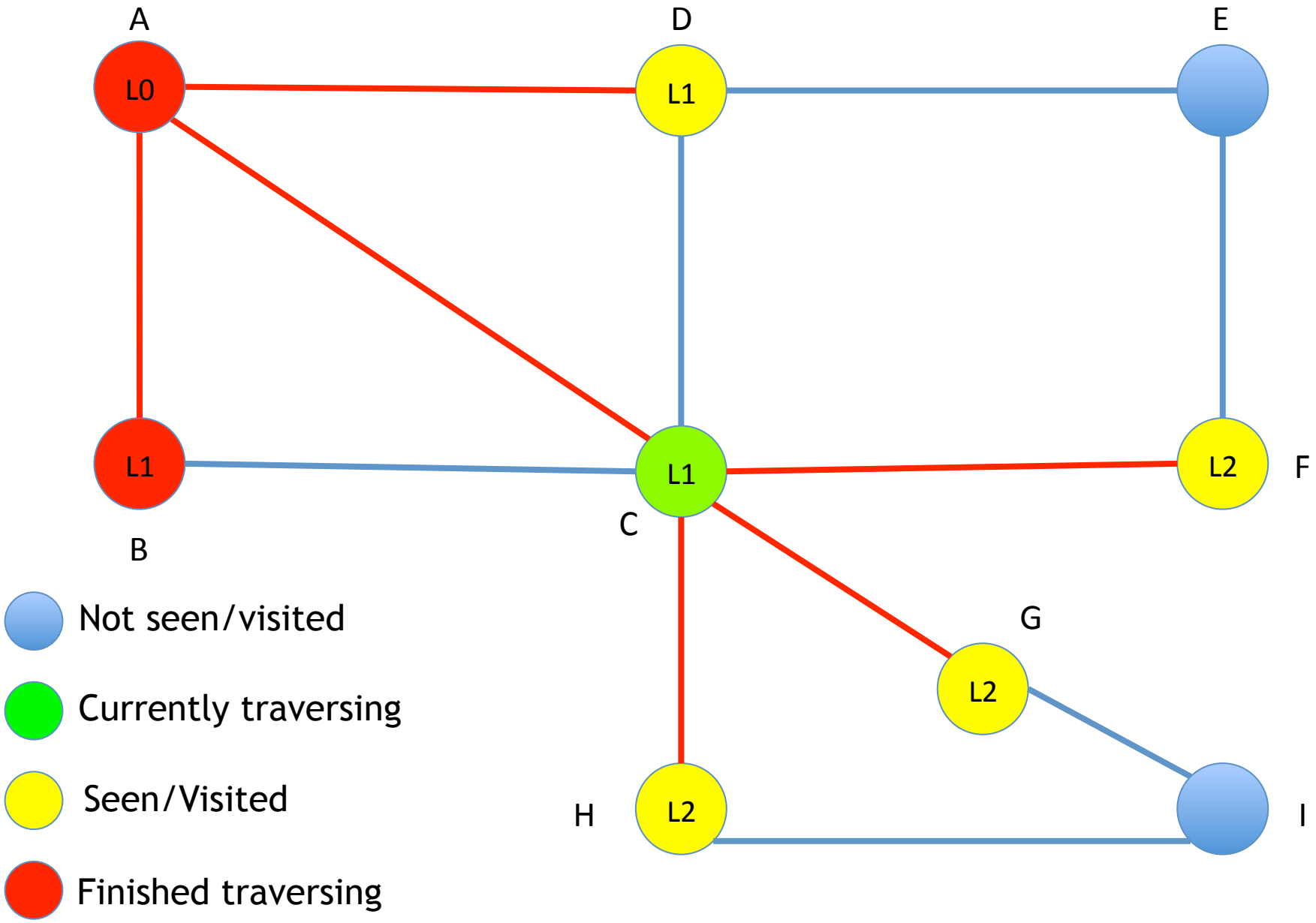
BFS Tree


A
L0
D
L1
E
L1
B
L1
C
F
G
L2
H
L2
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

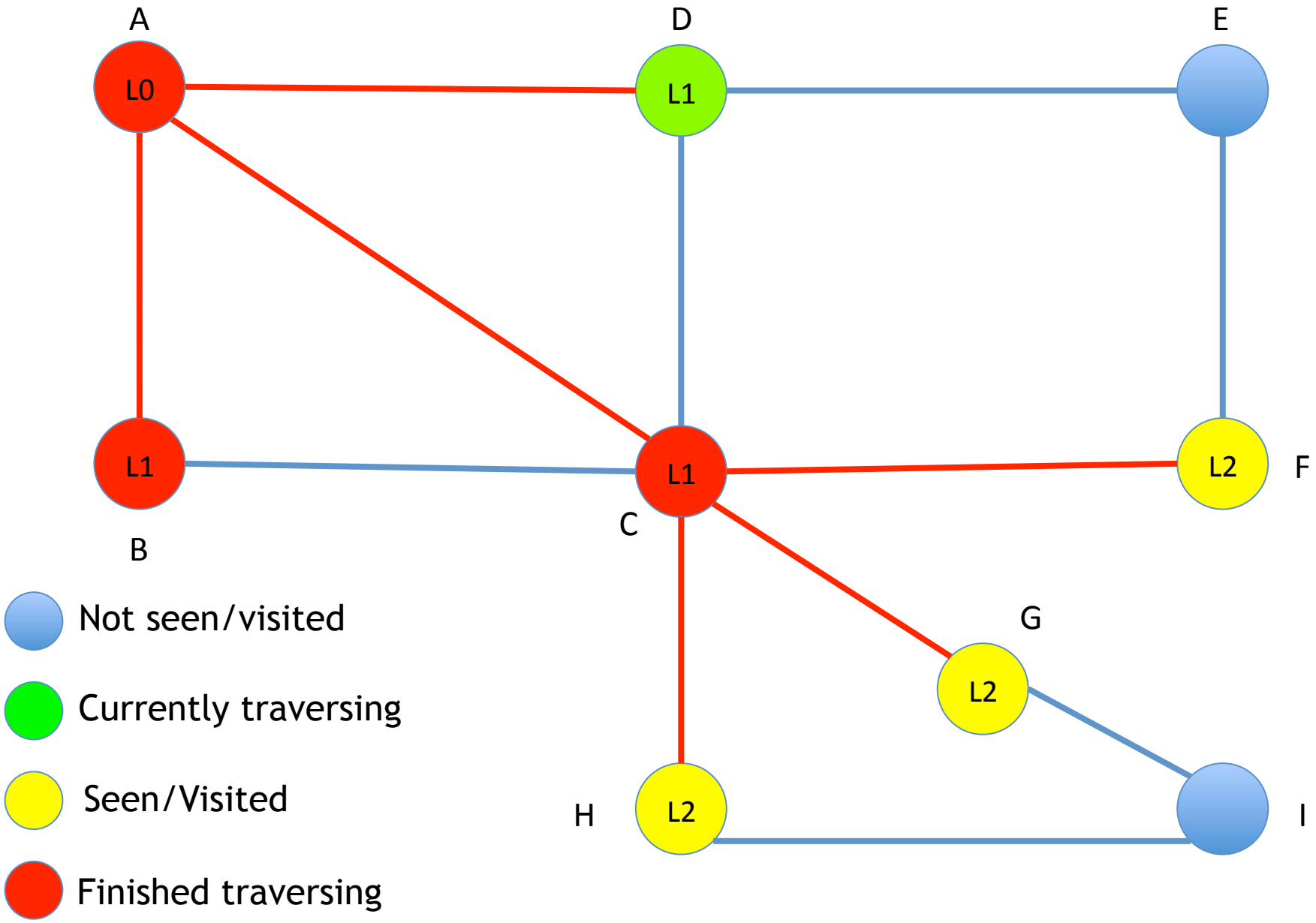
BFS Tree


A
L0
D
L1
E
L1
B
L1
C
L2
F
G
L2
H
L2
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

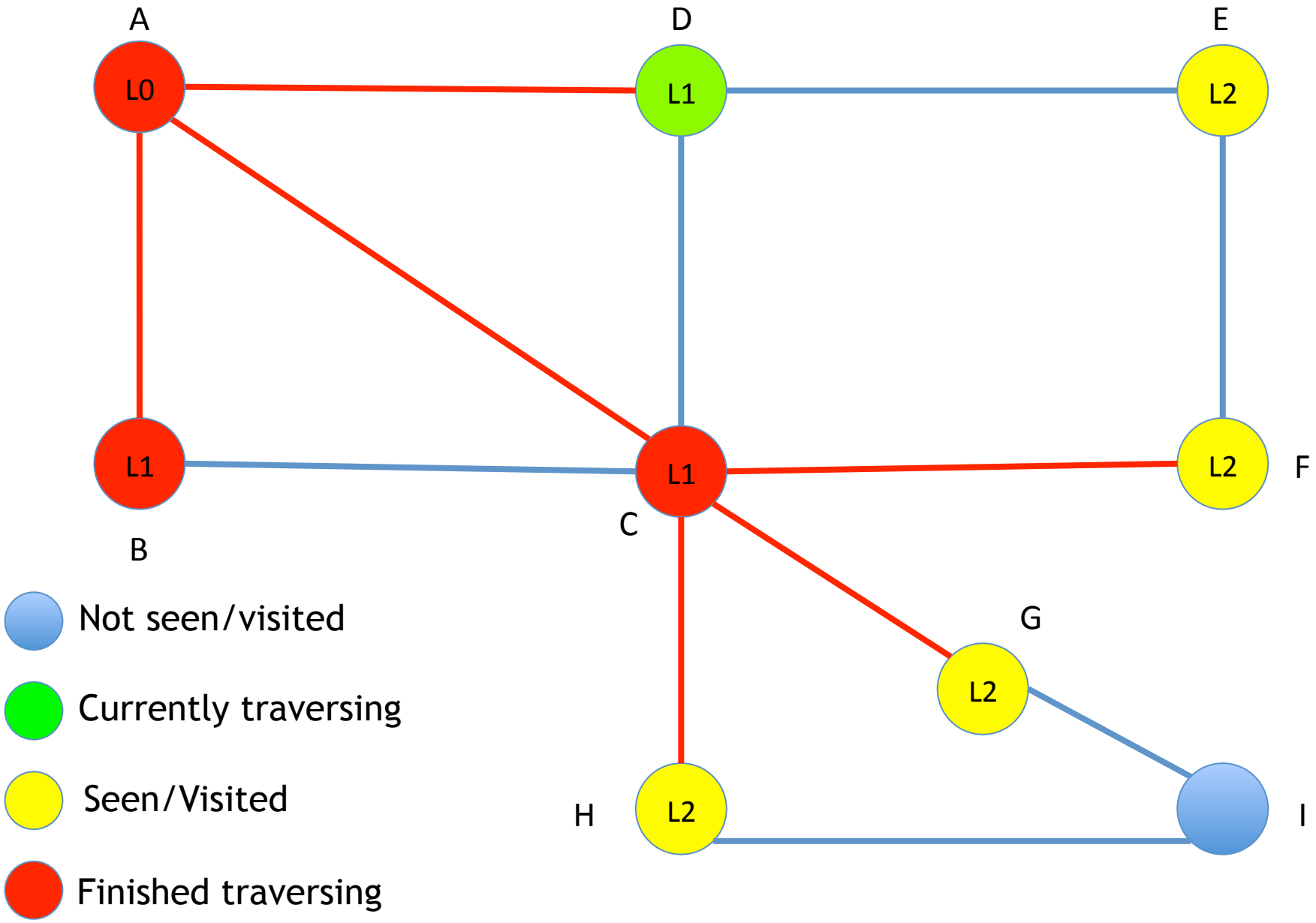
BFS Tree


A
L0
D
L1
E
L1
B
L1
C
L2
F
G
L2
H
L2
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

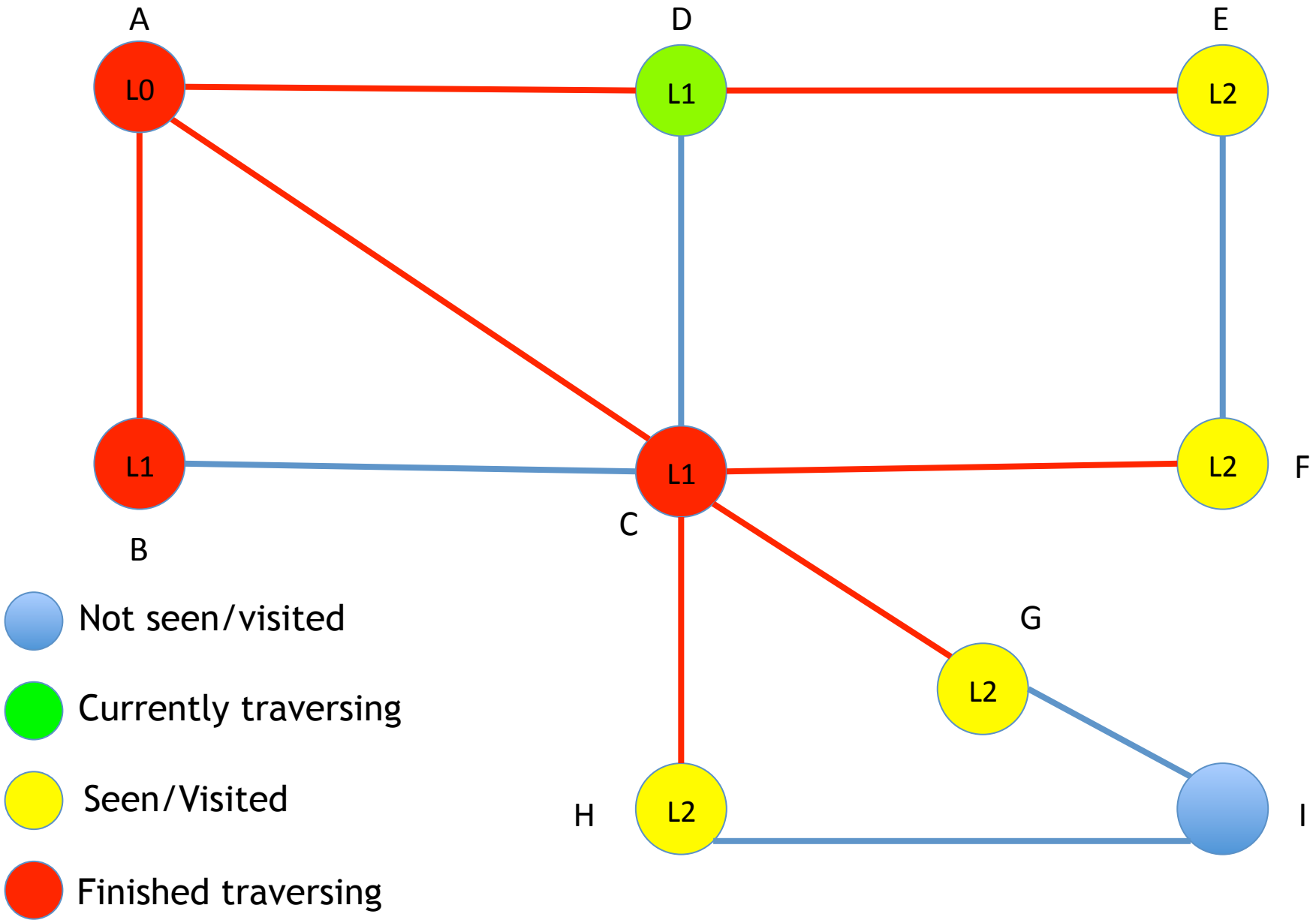
BFS Tree


A
D
E
L0
L1
L1
L1
L2
F
B
C
G
L2
H
L2
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

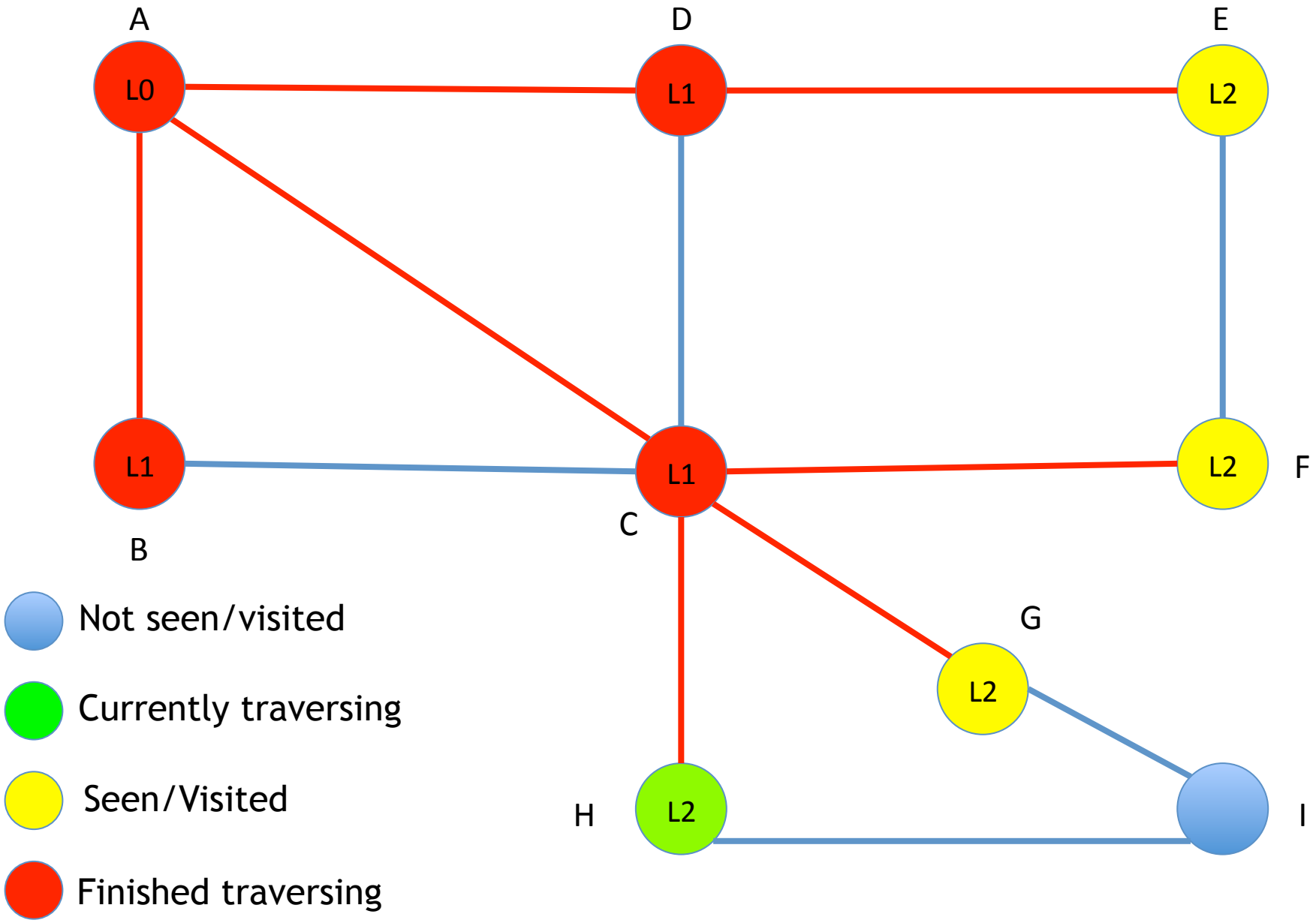
BFS Tree


A
L0
D
L1
E
L2
L1
B
L1
C
L2
F
G
L2
H
L2
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

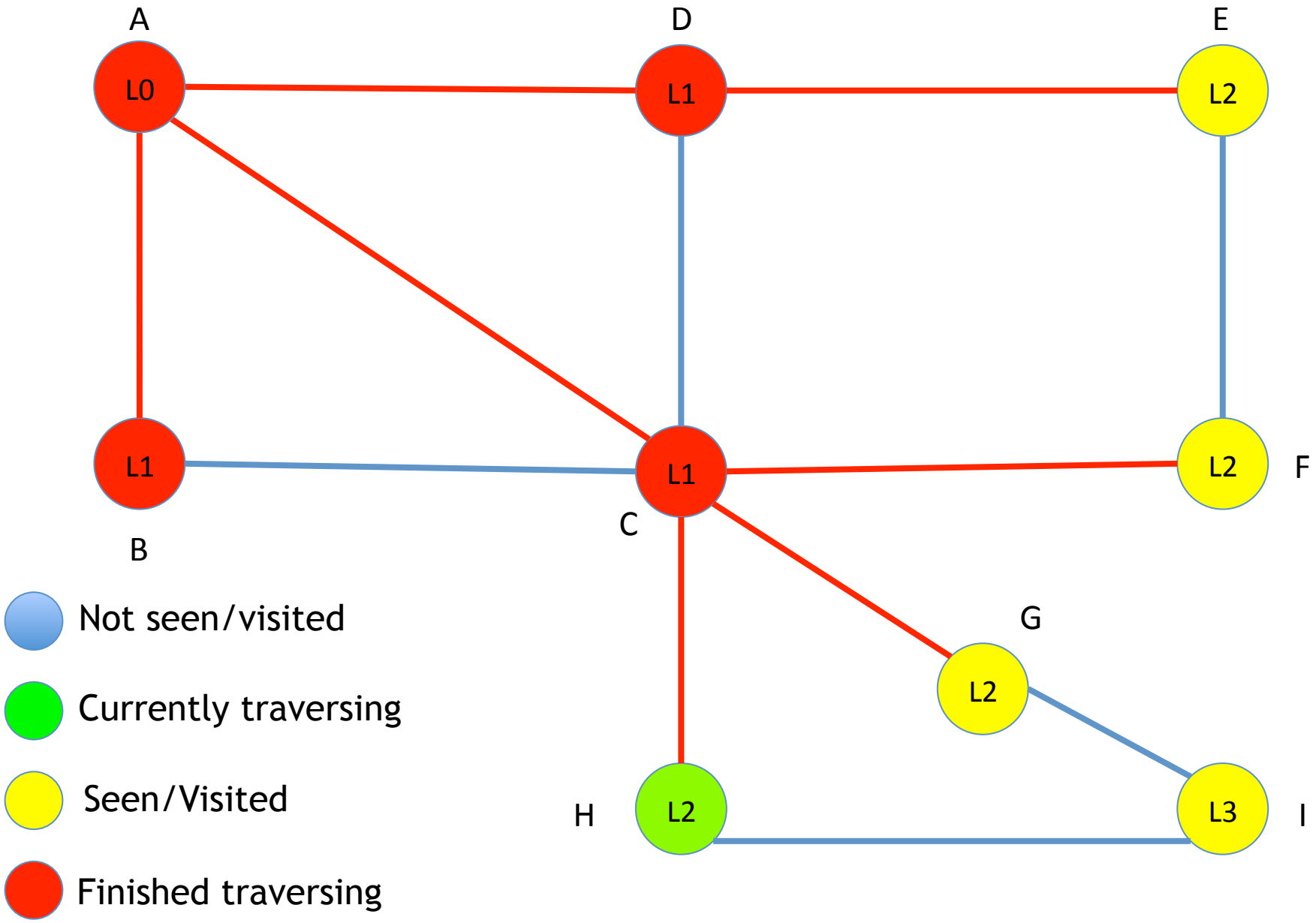
BFS Tree


A
L0
D
L1
E
L2
L1
B
L1
C
L2
F
G
L2
H
L2
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

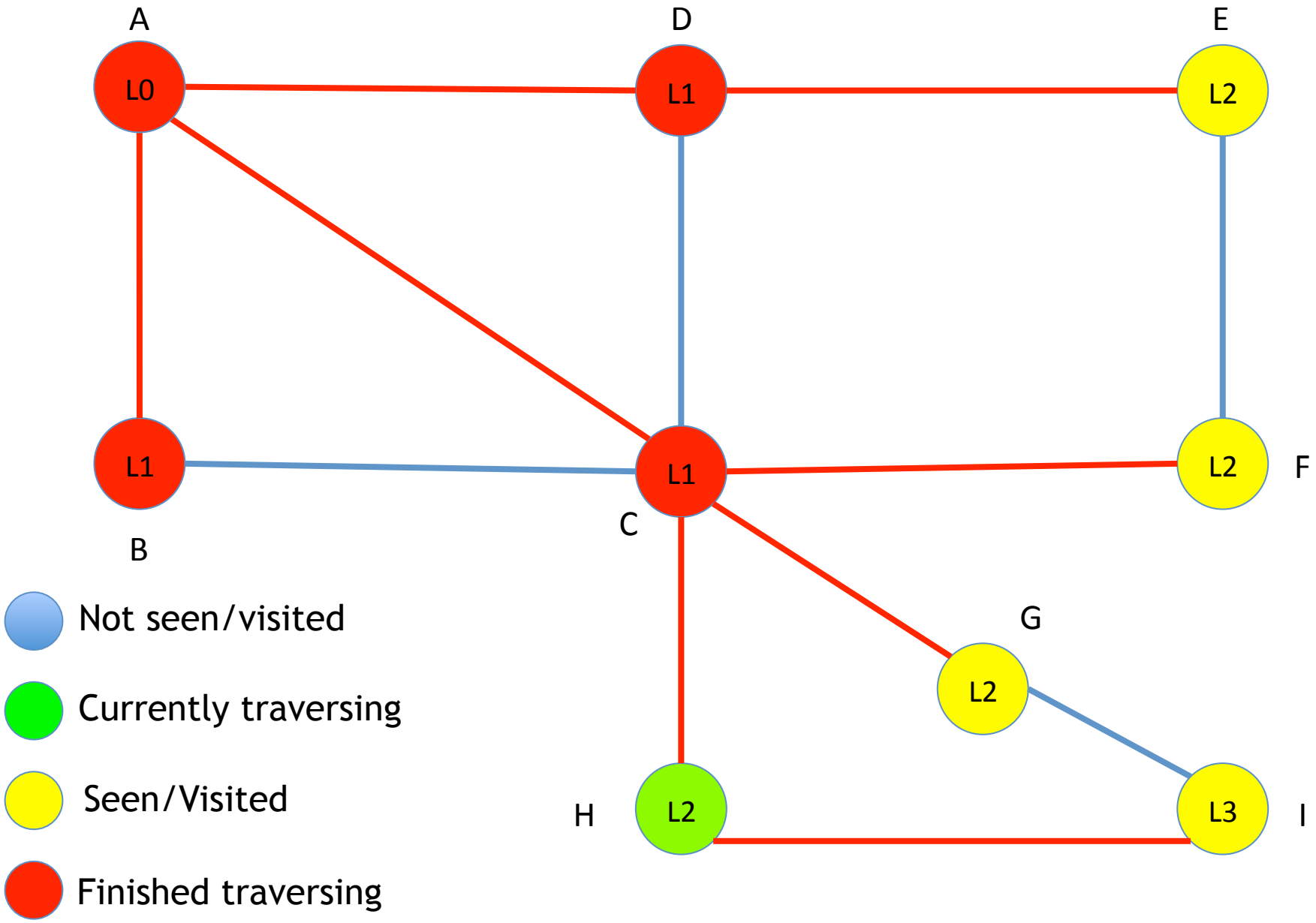
BFS Tree


A
D
E
L0
L1
L2
L1
L1
L2
F
B
C
G
L2
H
L2
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

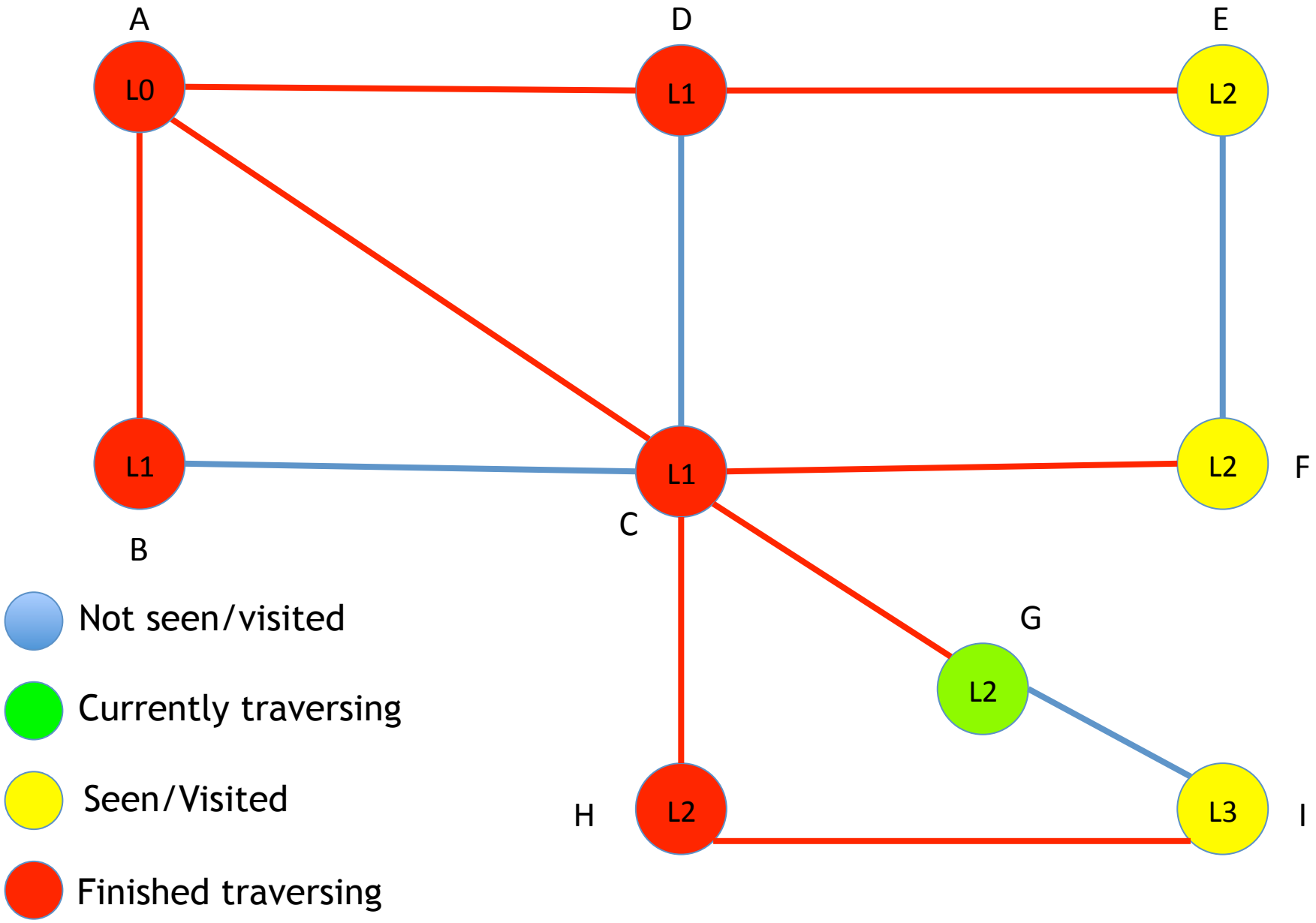
BFS Tree


A
L0
D
L1
E
L2
L1
B
L1
C
L2
F
G
L2
H
L2
L3
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

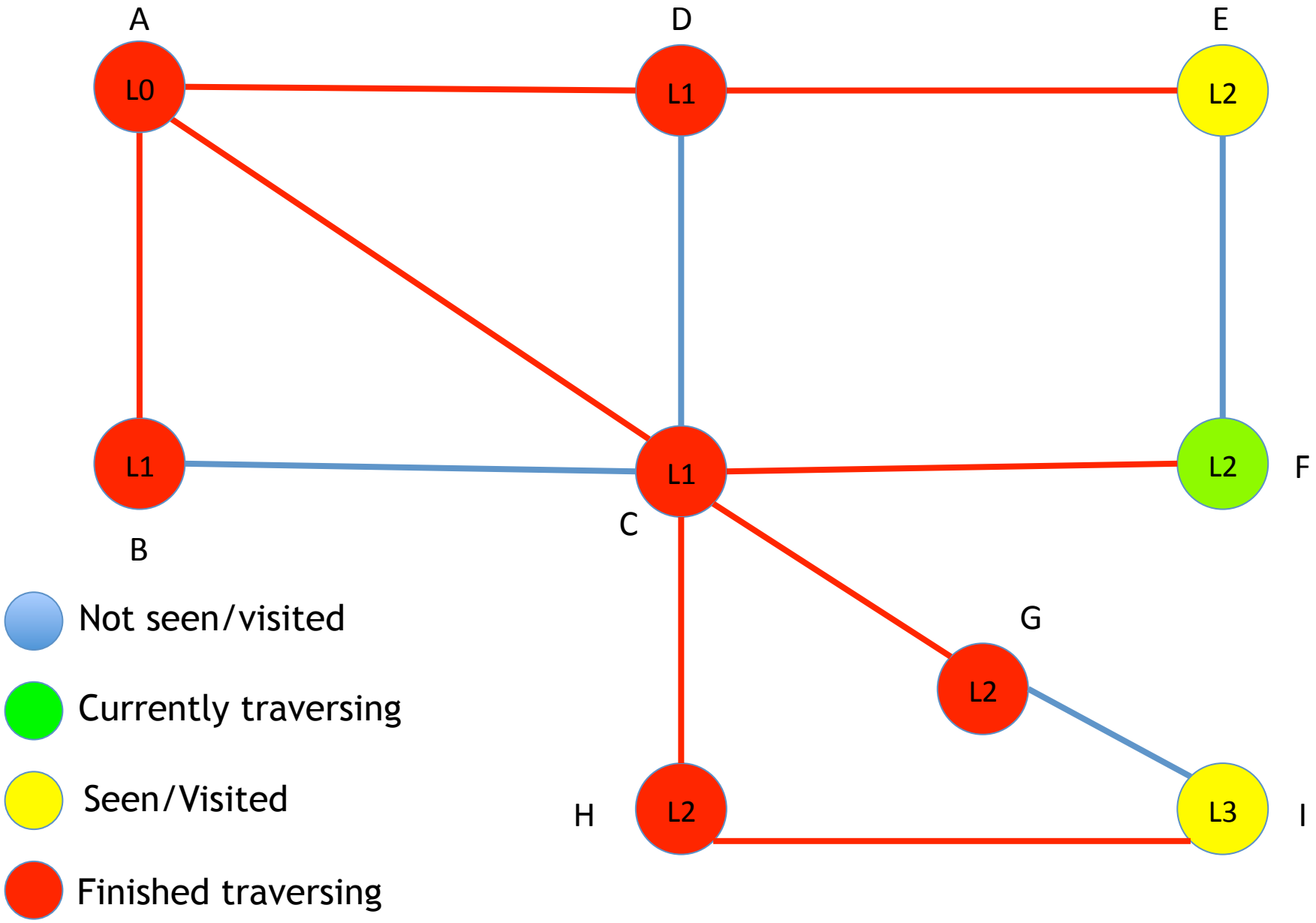
BFS Tree


A
L0
D
L1
E
L2
L1
B
L1
C
L2
F
G
L2
H
L2
L3
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

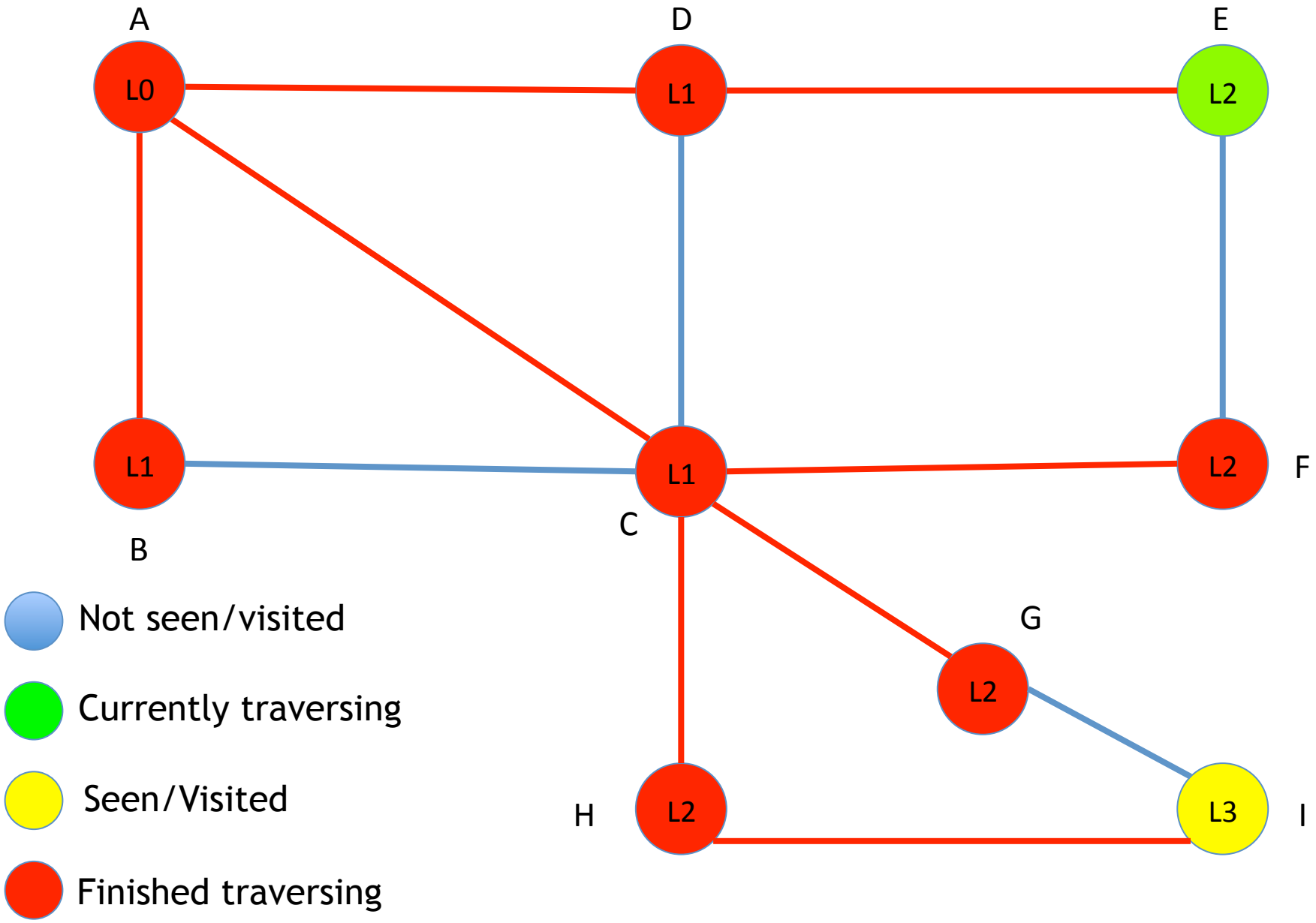
BFS Tree


A
D
E
L0
L1
L2
L1
L1
L2
F
B
C
G
L2
H
L2
L3
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

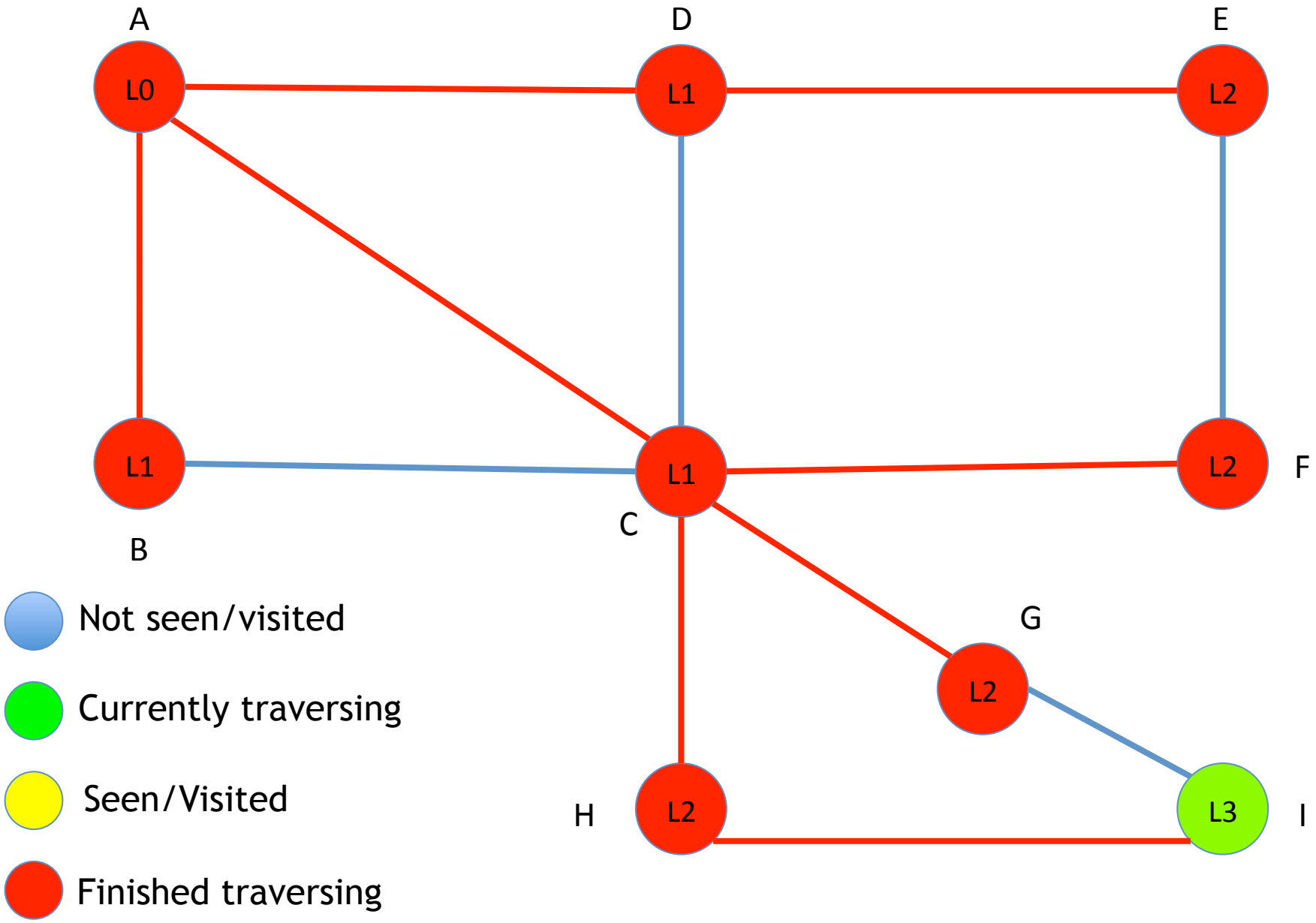
BFS Tree


A
L0
D
L1
E
L2
L1
B
L1
C
L2
F
G
L2
H
L2
L3
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

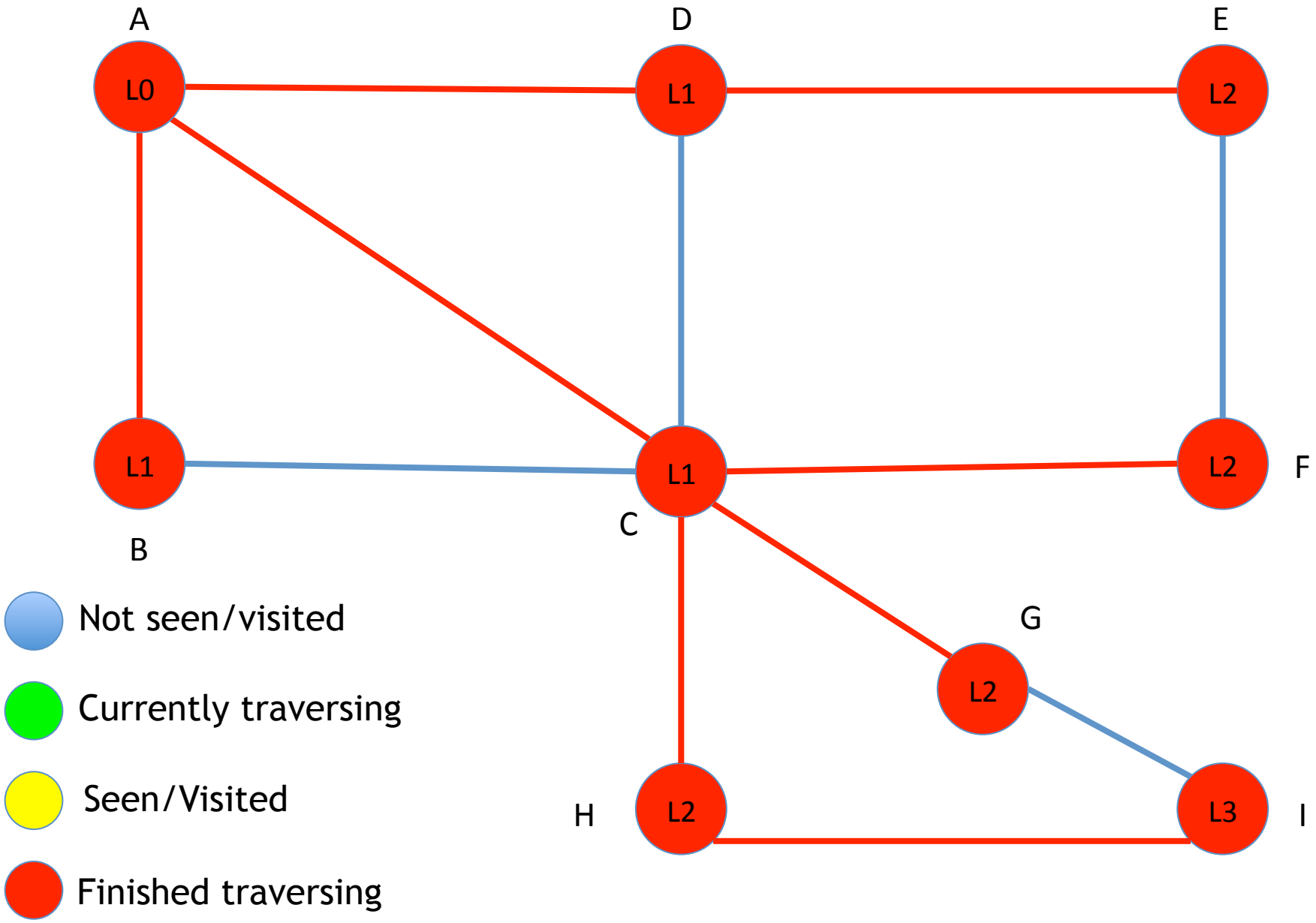
BFS Tree


A
D
E
L0
L1
L2
L1
L1
L2
F
B
C
G
L2
H
L2
L3
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

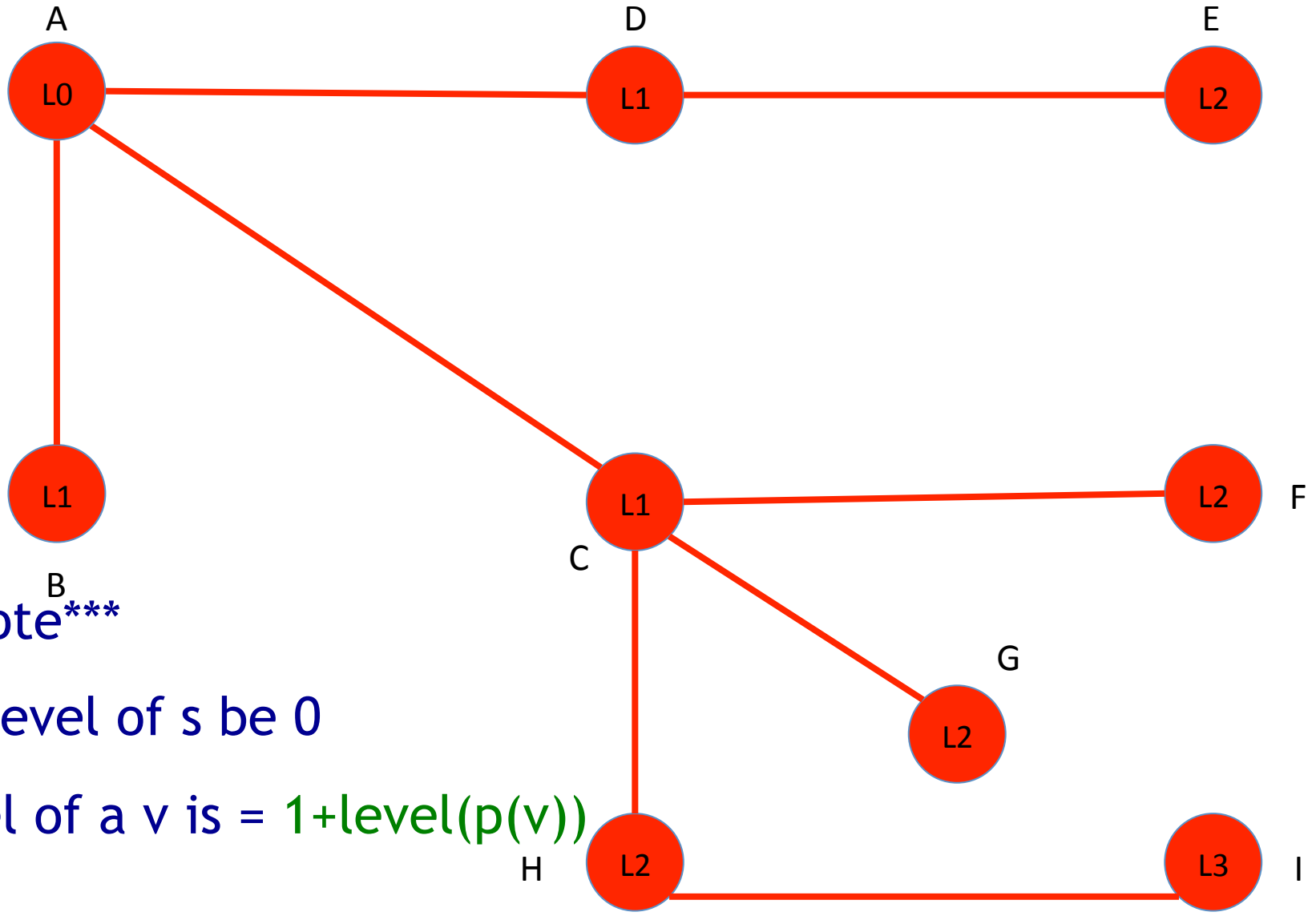
BFS Tree


A
D
E
L0
L1
L2
L1
L1
L2
F
B
C
G
L2
H
L2
L3
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

BFS Tree


A
D
E
L0
L1
L2
L1
L1
L2
F
B
C
G
L2
H
L2
L3
I
Not seen/visited
Currently traversing
Seen/Visited
Finished traversing

BFS Tree



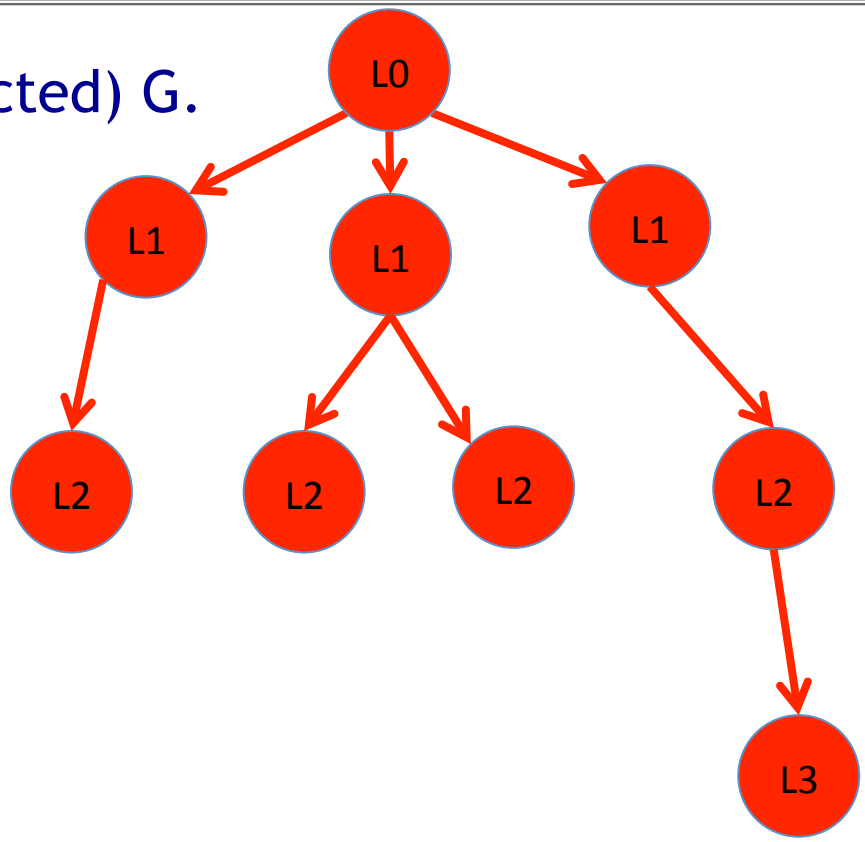
Note

Let level of s be 0

Level of a v is = 1+level(p(v))

BFS Tree

Suppose this is the BFS-Tree of a (directed) G.



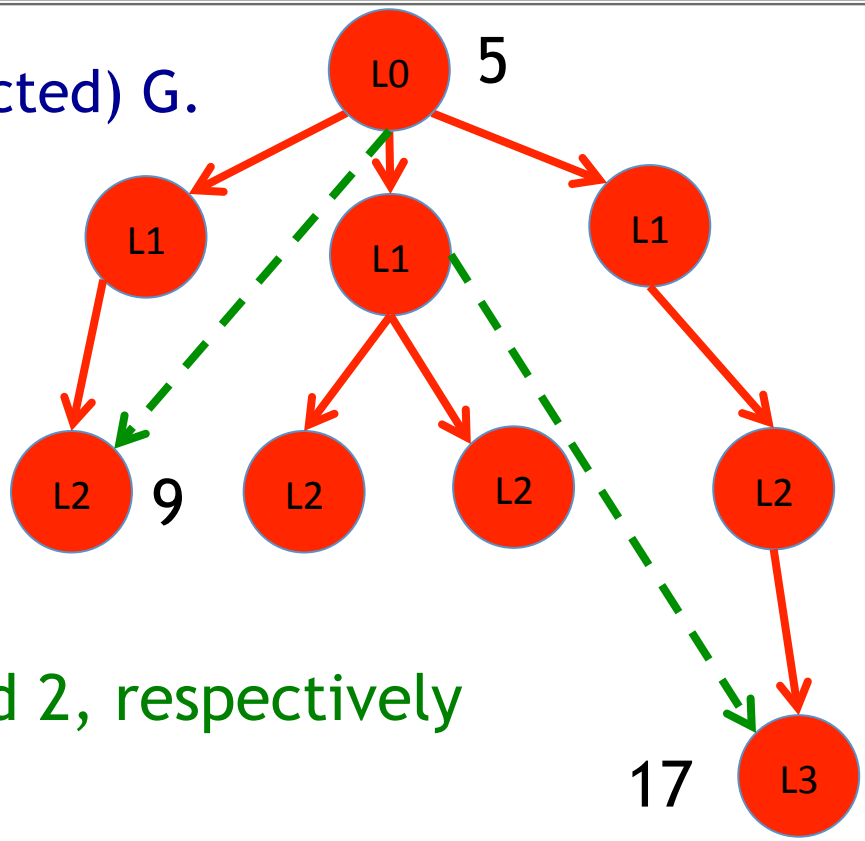
BFS Tree

Suppose this is the BFS-Tree of a (directed) G.

Q: Can "forward" edge, i.e. btw non-consecutive levels, exist in G?

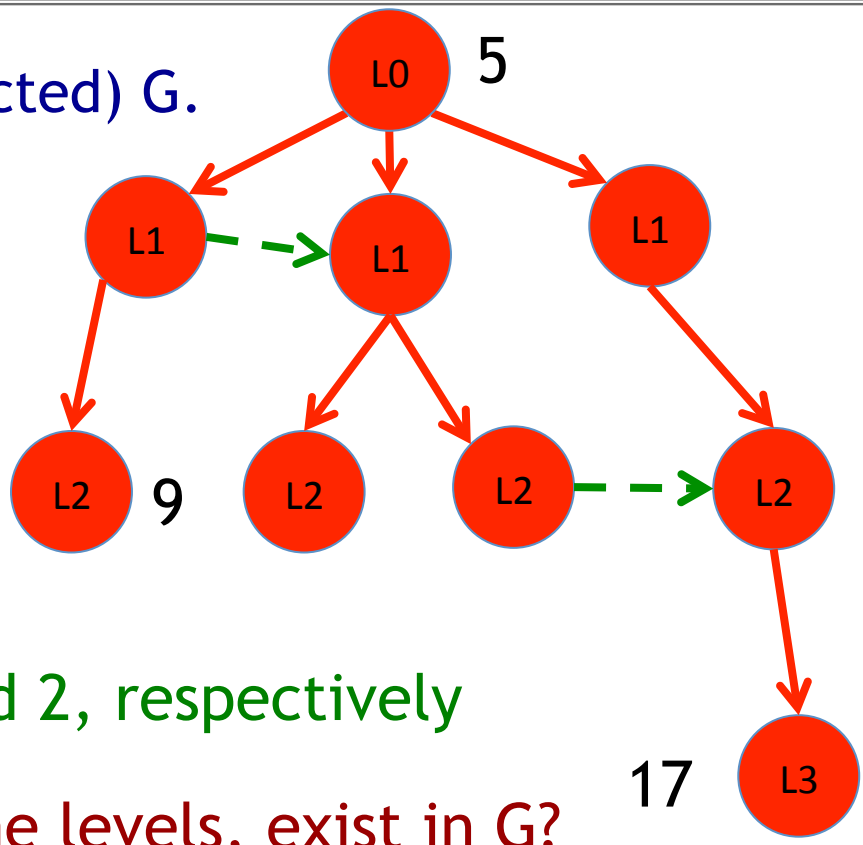
A: No

B/c 9 and 17 would be levels 1 and 2, respectively



BFS Tree

Suppose this is the BFS-Tree of a (directed) G.



Q: Can "forward" edge, i.e. btw non-consecutive levels, exist in G?

A: No

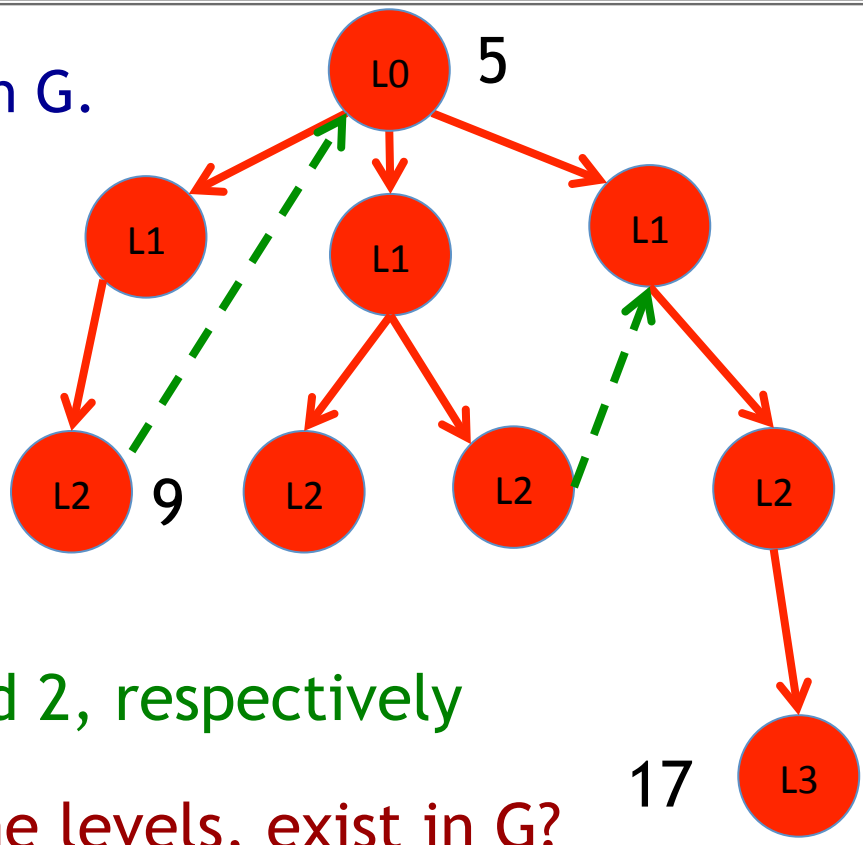
B/c 9 and 17 would be levels 1 and 2, respectively

Q: Can "cross" edge, i.e. btw same levels, exist in G?

A: Yes

BFS Tree

Suppose this is the BFS-Tree of a Graph G.



Q: Can "forward" edge, i.e. btw non-consecutive levels, exist in G?

A: No

B/c 9 and 17 would be levels 1 and 2, respectively

Q: Can "cross" edge, i.e. btw same levels, exist in G?

A: Yes

Q: Can "back" edge, i.e. from larger to smaller level, exist in G?

A: Yes in a directed graph. No in an undirected graph.

DFS Tree of a “Connected Graph”

- Can be defined in the exact same way as BFS-tree
- Exercise: Do the analysis of what types of edges can exist given a DFS-Tree of a graph G.

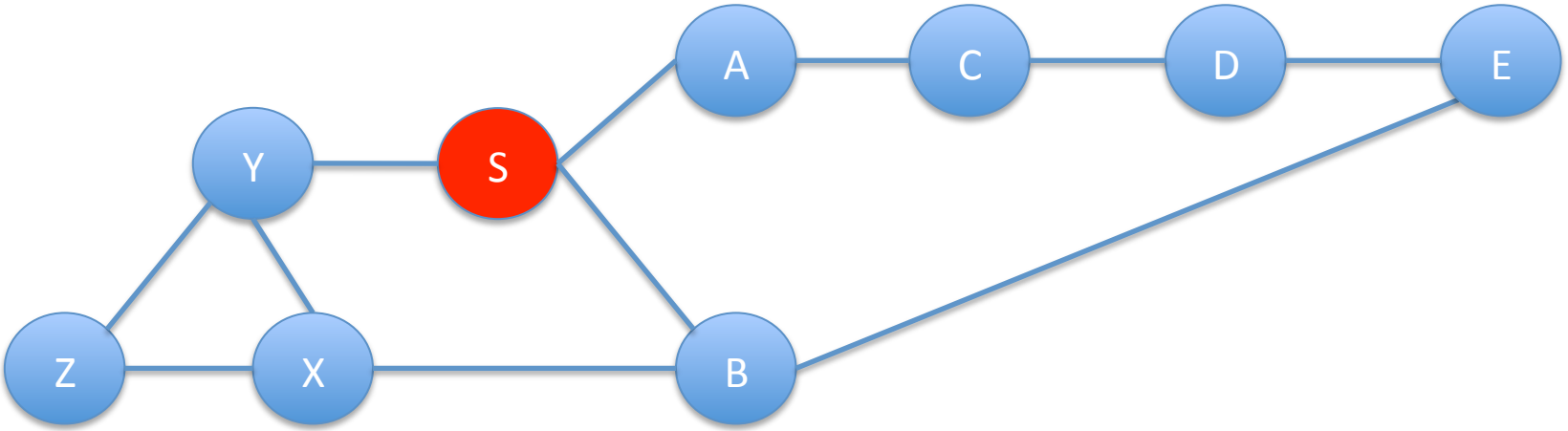
Outline For Today

1. Graph Terminology
2. BFS/DFS & BFS/DFS Tree
3. Application 1: Unweighted Single Source Shortest Paths
4. Application 2: Bipartiteness/2-coloring
5. Application 3: Connected Comp. in Undir. Graphs
6. Application 4: Topological Sort

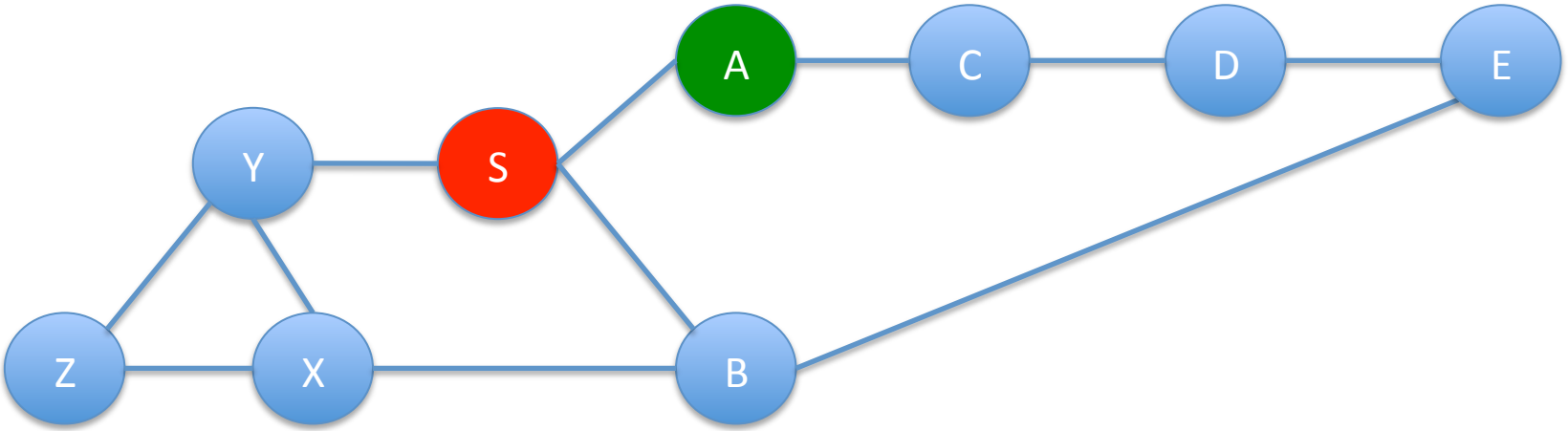
Unweighted Single Source Shortest Paths

- Input: G(V, E) (directed or undirected), source vertex s
- Output: $\forall v \in V$, shortest path and shortest dist(s, v)

Shortest Path Example



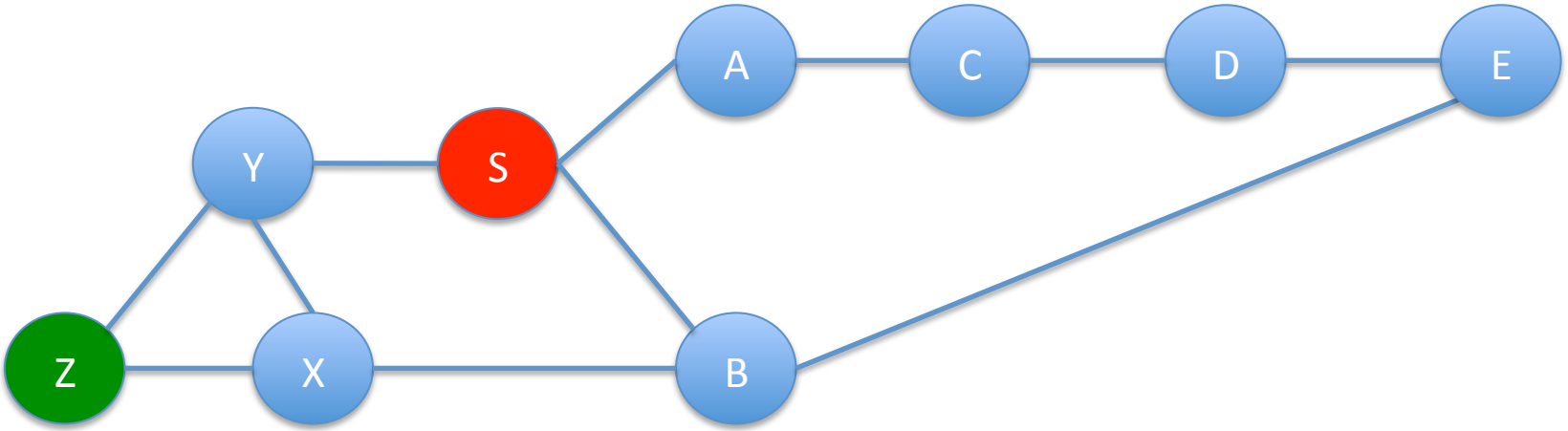
Shortest Path Example



Q: What's dist(S, A)?

A: 1 (Path: S->A)

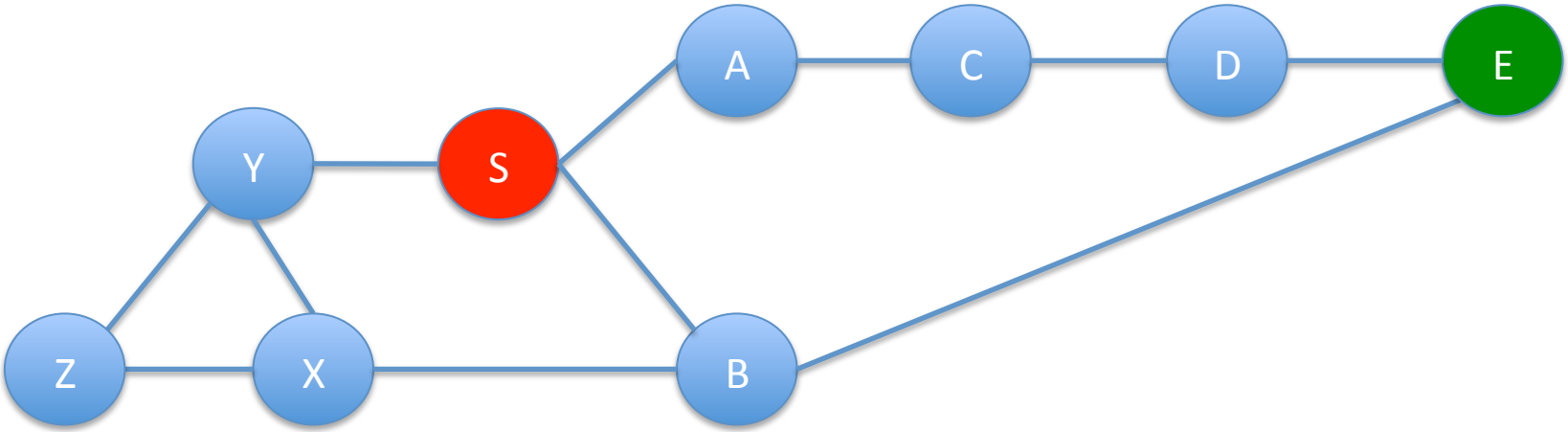
Shortest Path Example



Q: What's dist(S, Z)?

A: 2 (Path: S->Y->Z)

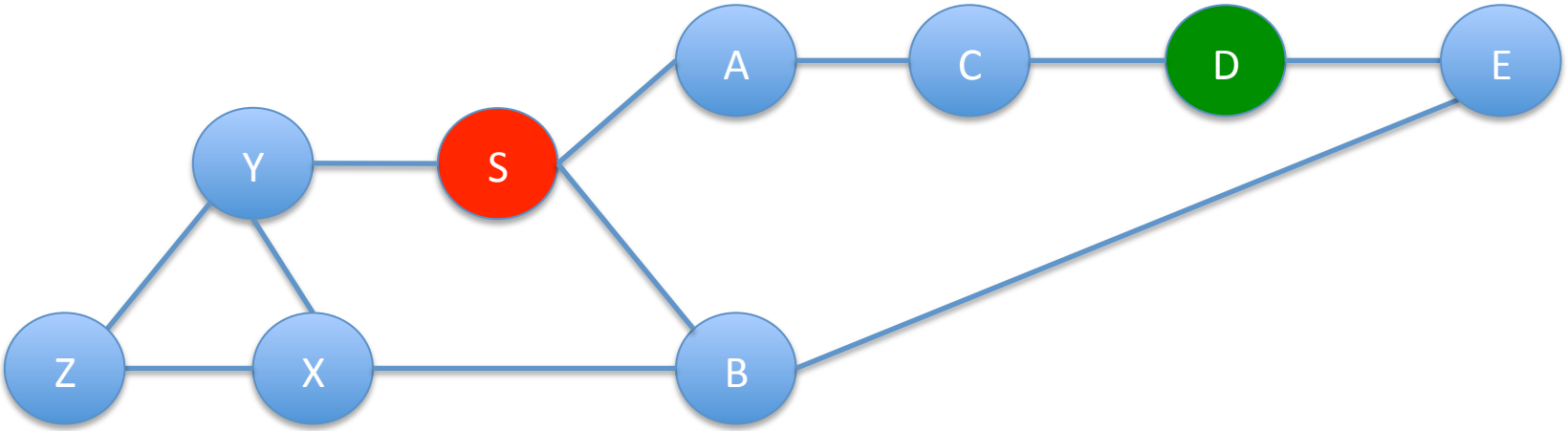
Shortest Path Example



Q: What's dist(S, E)?

A: 2 (Path: S->B->Z)

Shortest Path Example



Q: What's dist(S, D)?

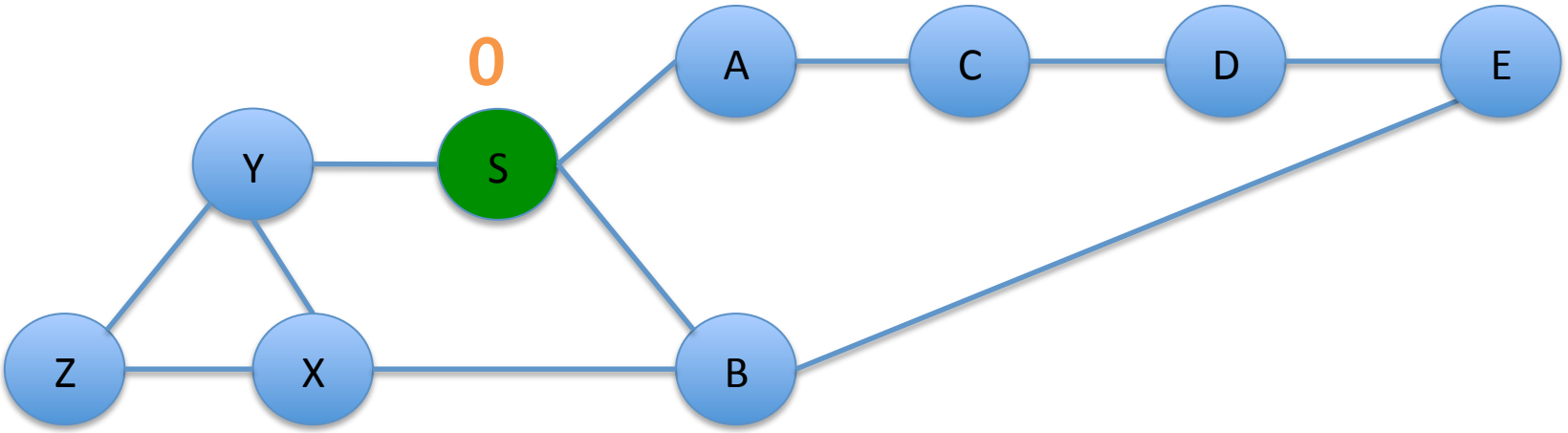
A: 3

Paths: S->B->E->D or S->A->C->D

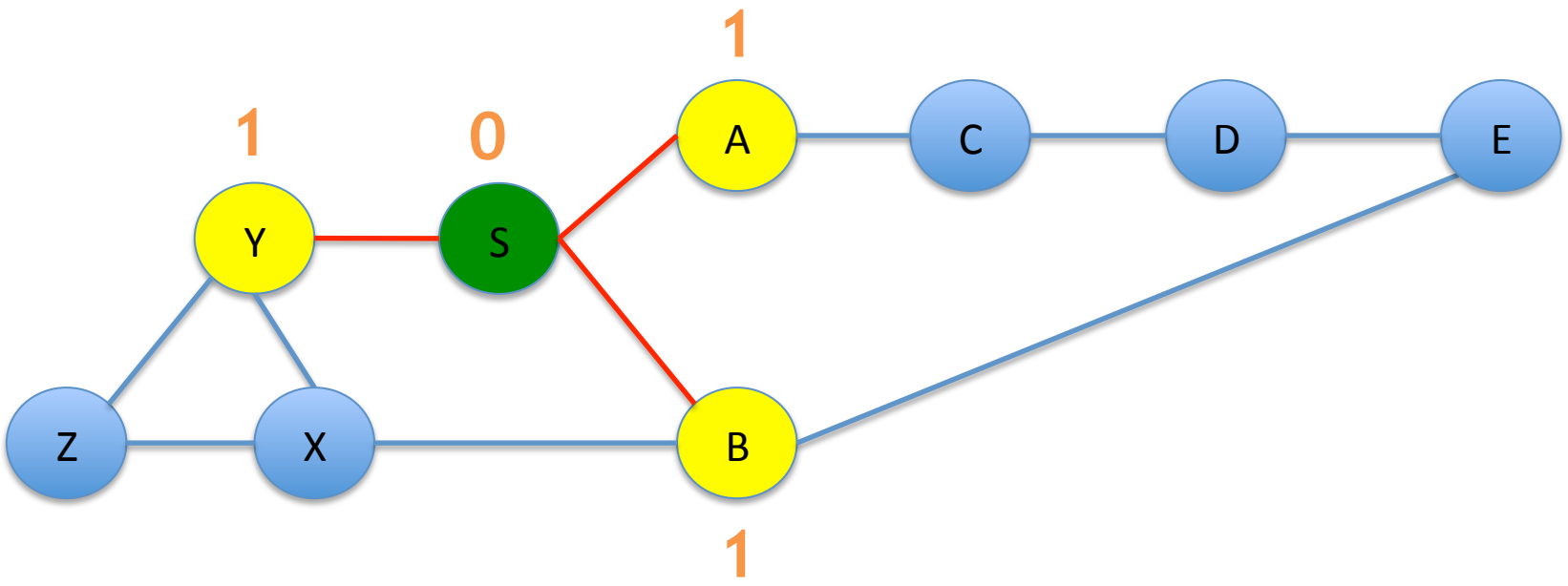
Just Run BFS!

The level numbers will exactly be the distances!

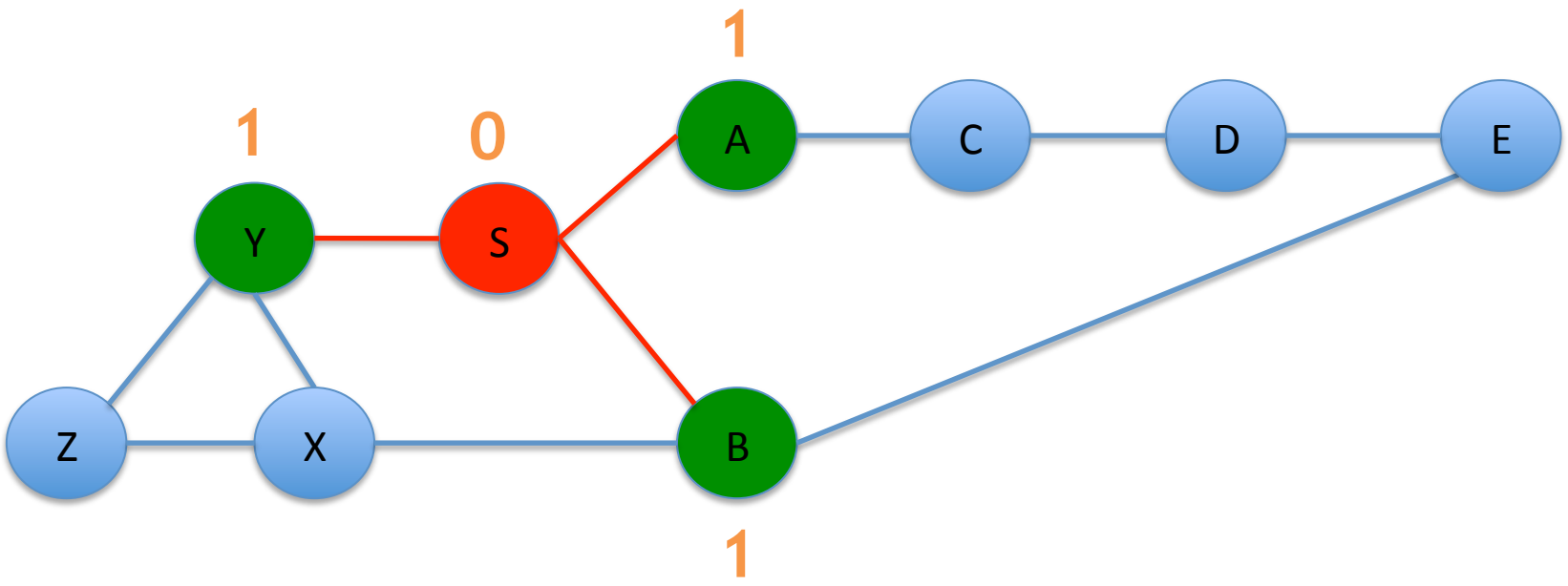
Shortest Path Example



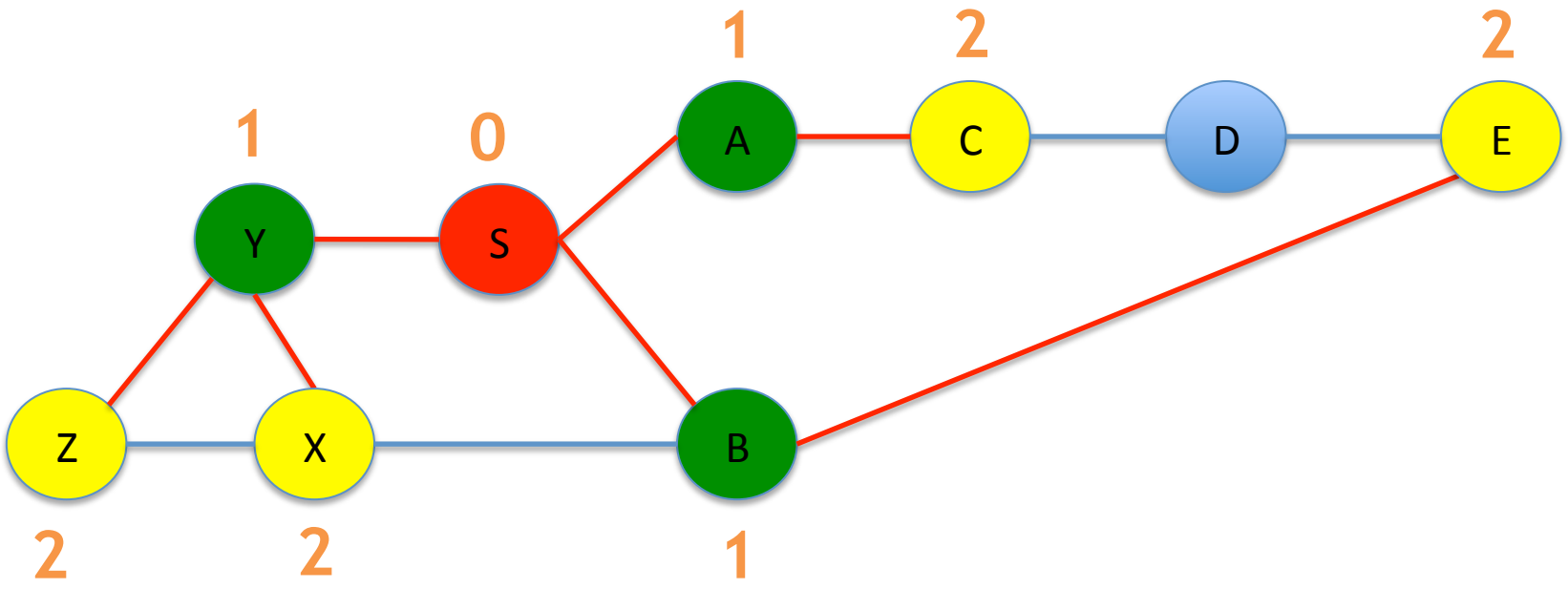
Shortest Path Example



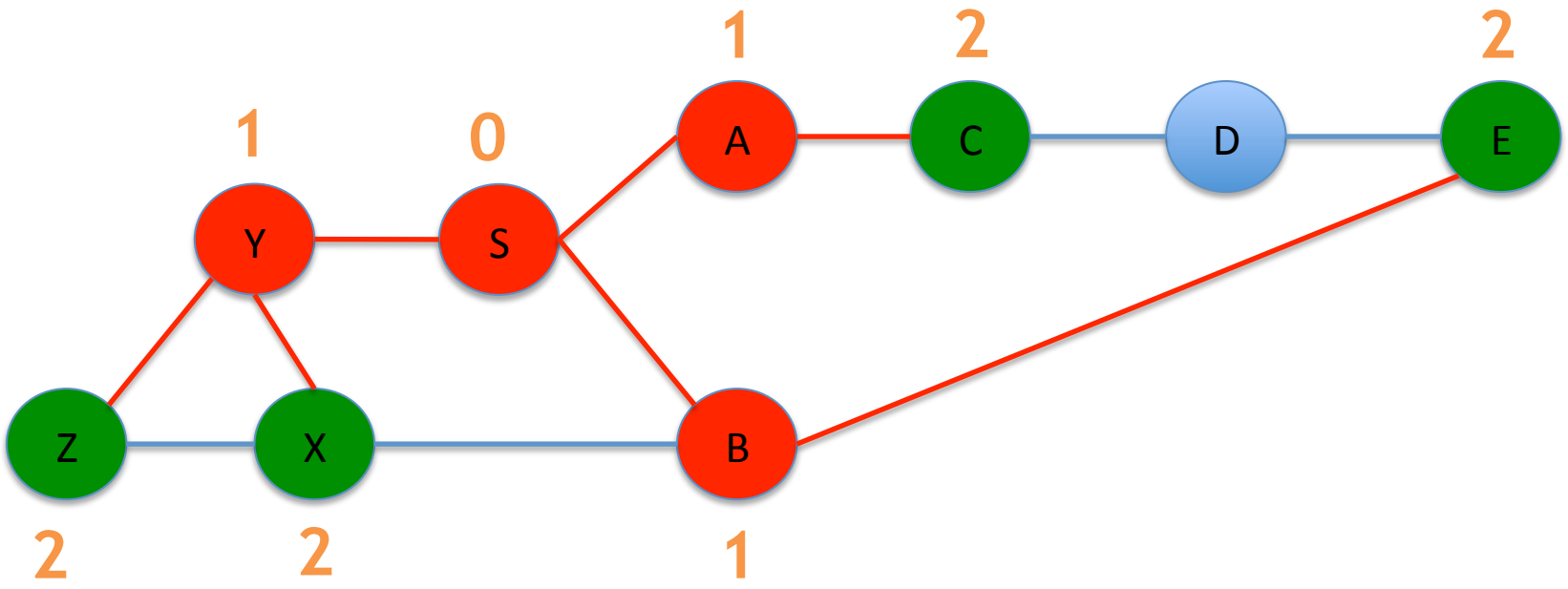
Shortest Path Example



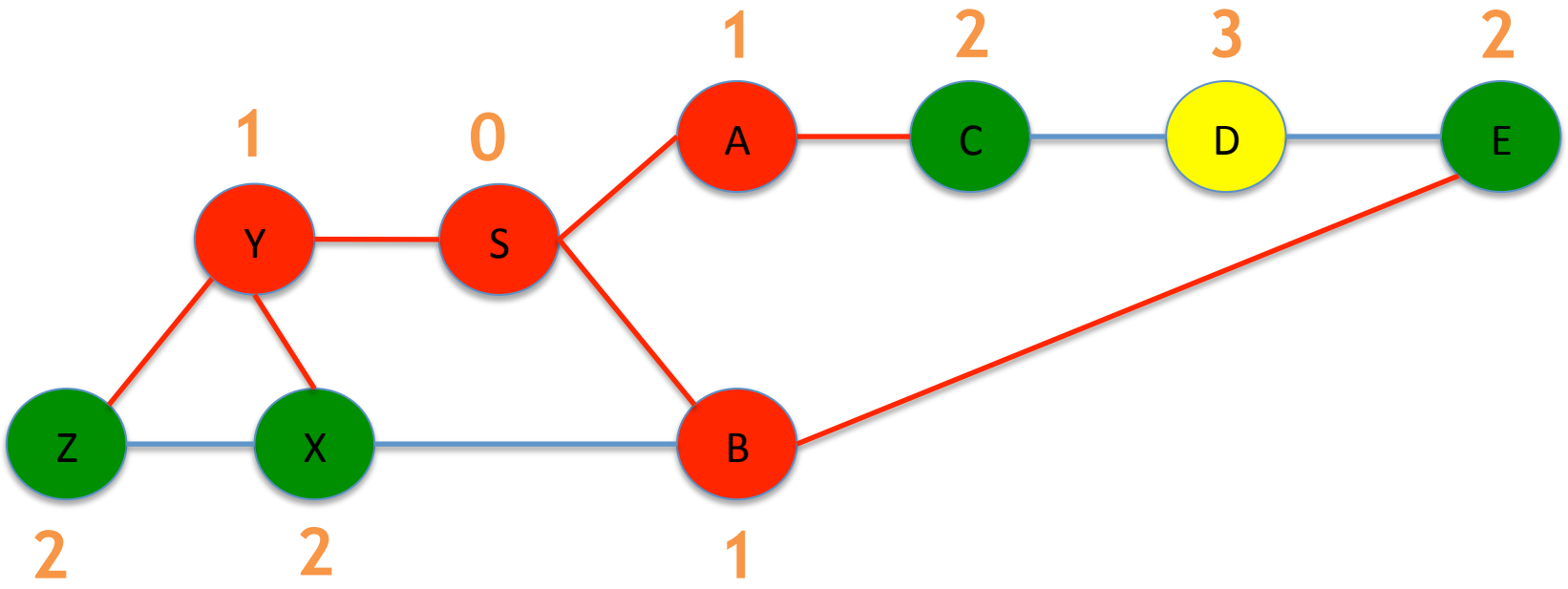
Shortest Path Example



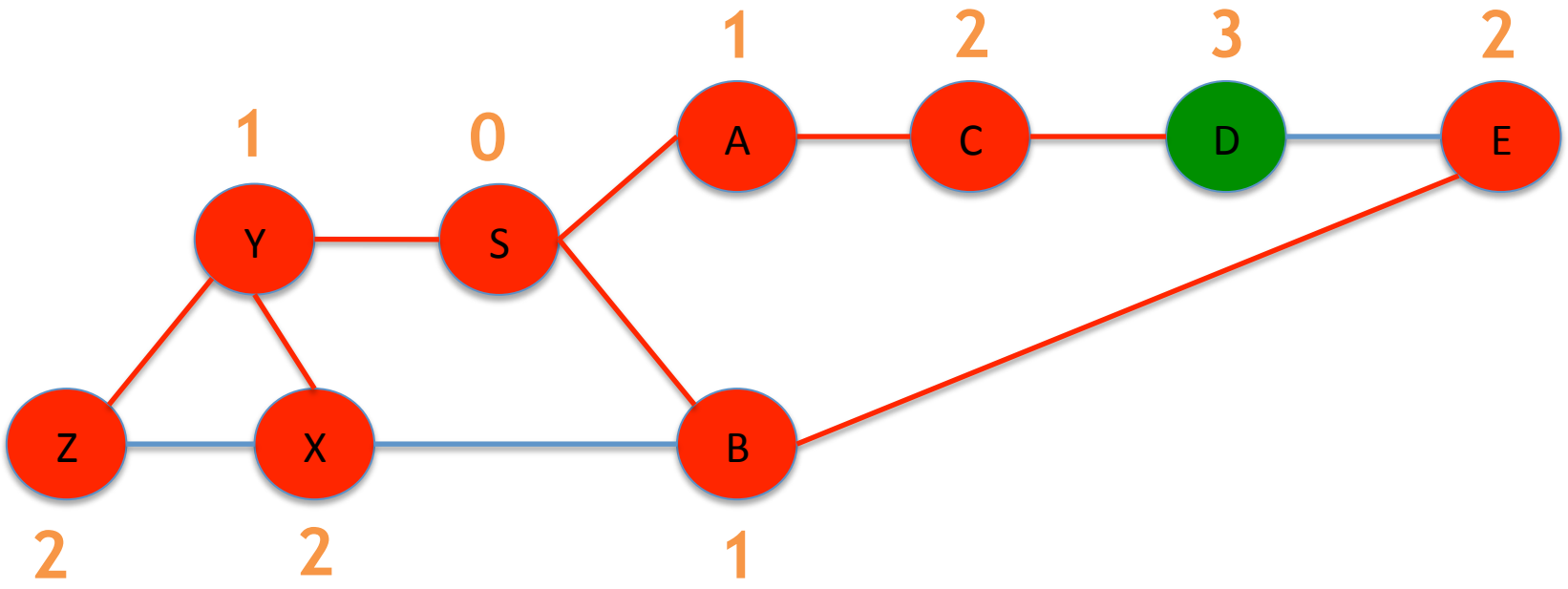
Shortest Path Example



Shortest Path Example

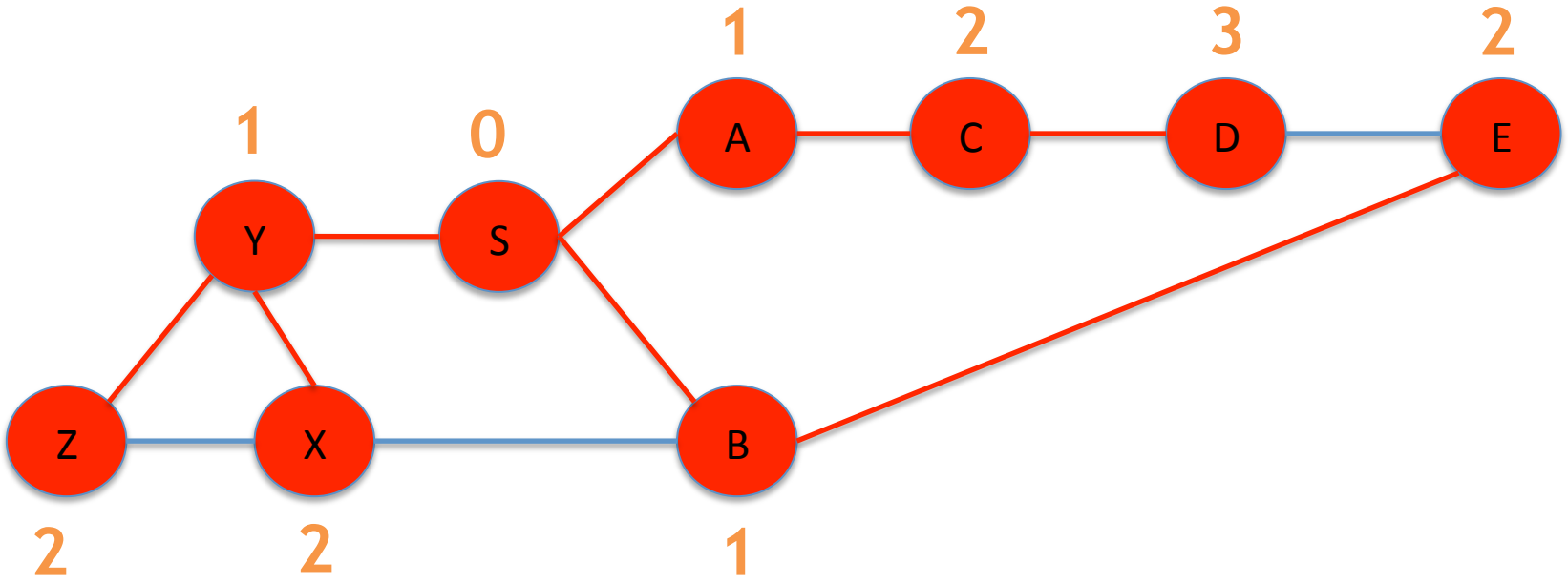


Shortest Path Example



Shortest Path Example

Formal correctness proof is by induction on the length of the (s, t) paths. (Check)

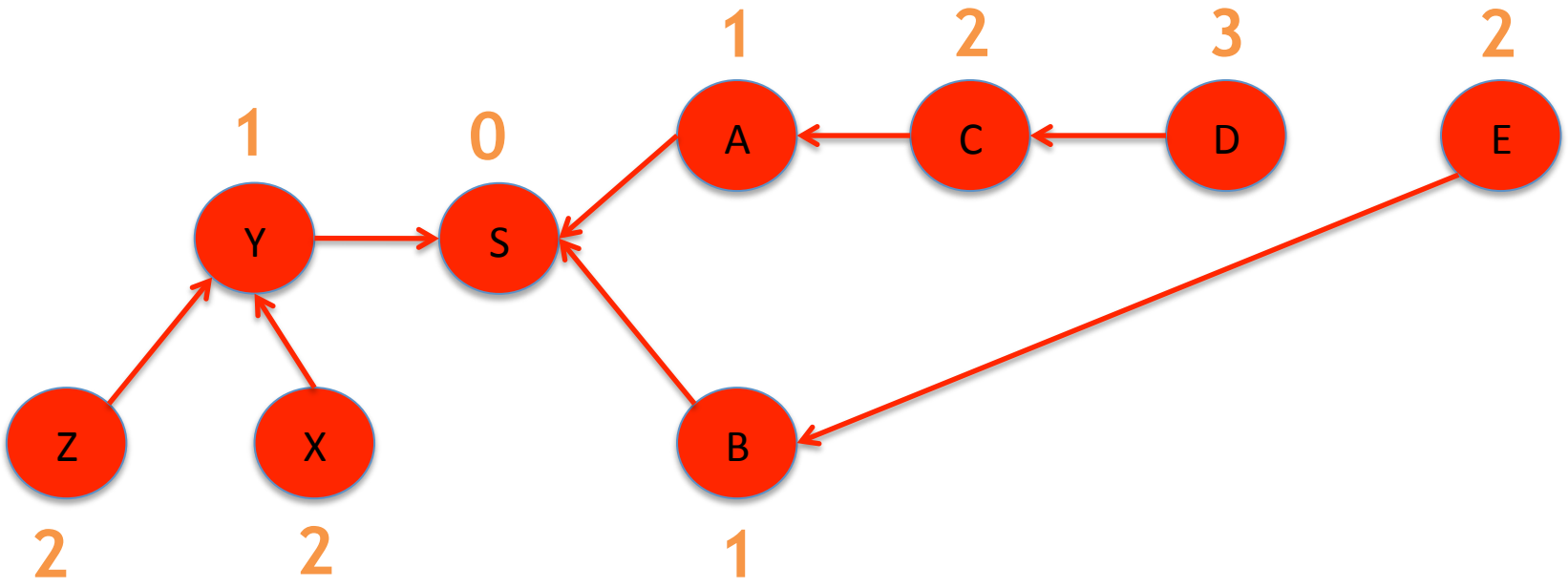


Runtime: O(n + m)

In linear time, all shortest paths from s to every other vertex in V!

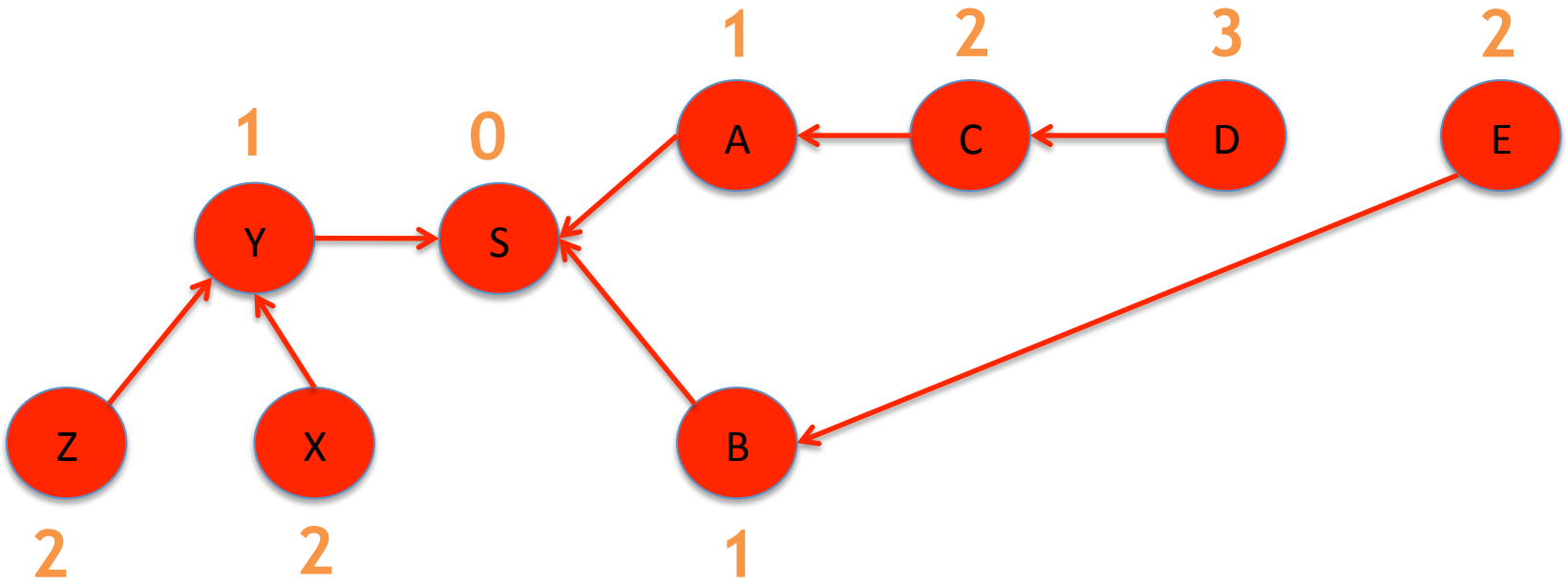
Using The BFS Tree To Construct Paths

- The BFS tree can be used to construct actual paths
- Just follow your parent! (suppose we stored (v,p(v)) direction)



Using The BFS Tree To Construct Paths

- The BFS tree can be used to construct actual paths
- Just follow your parent! (suppose we stored (v,p(v)) direction)

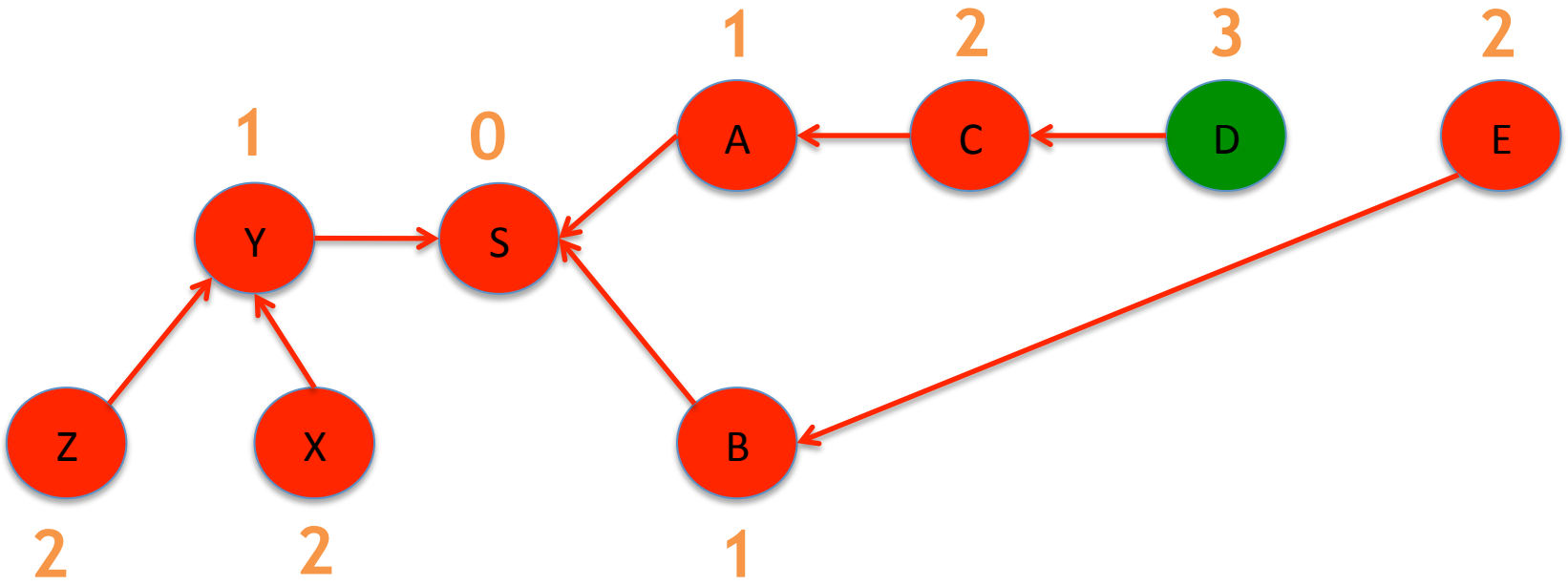


Ex: What's the path from S to D?

Just follow back from D on BFS Tree.

Using The BFS Tree To Construct Paths

- The BFS tree can be used to construct actual paths
- Just follow your parent! (suppose we stored (v,p(v)) direction)



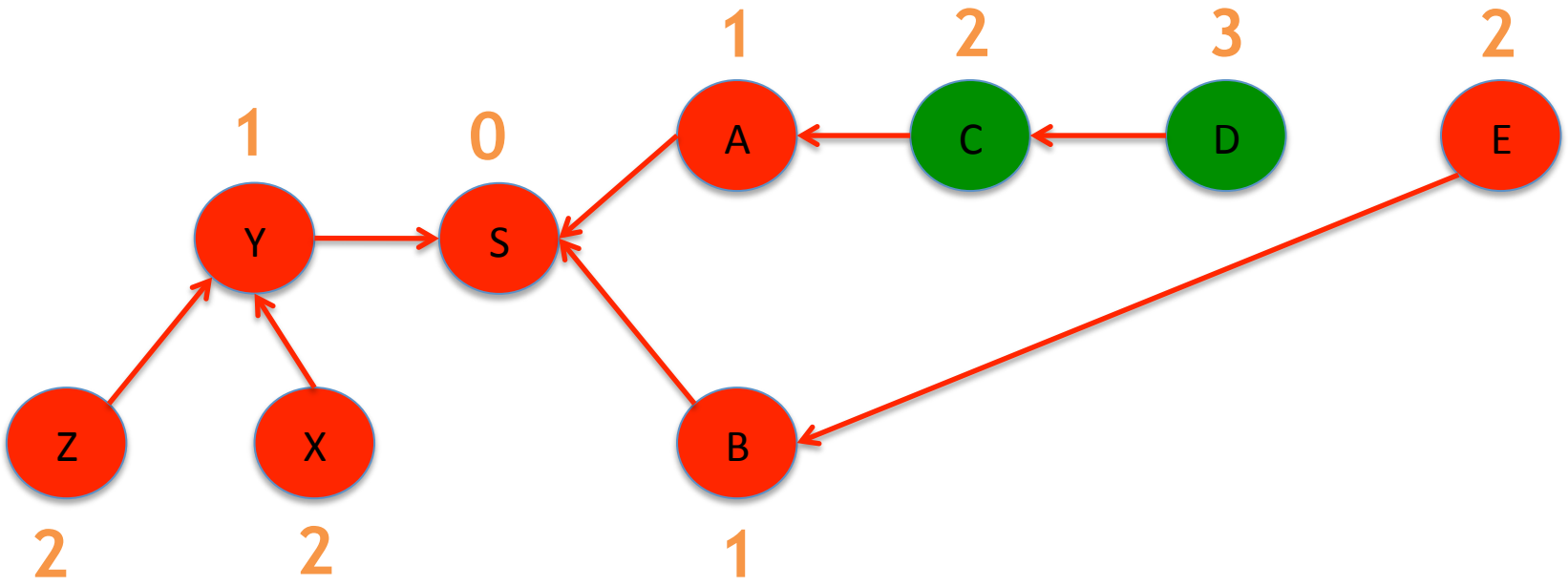
Ex: What's the path from S to D?

Just follow back from D on BFS Tree.

D

Using The BFS Tree To Construct Paths

- The BFS tree can be used to construct actual paths
- Just follow your parent! (suppose we stored (v,p(v)) direction)



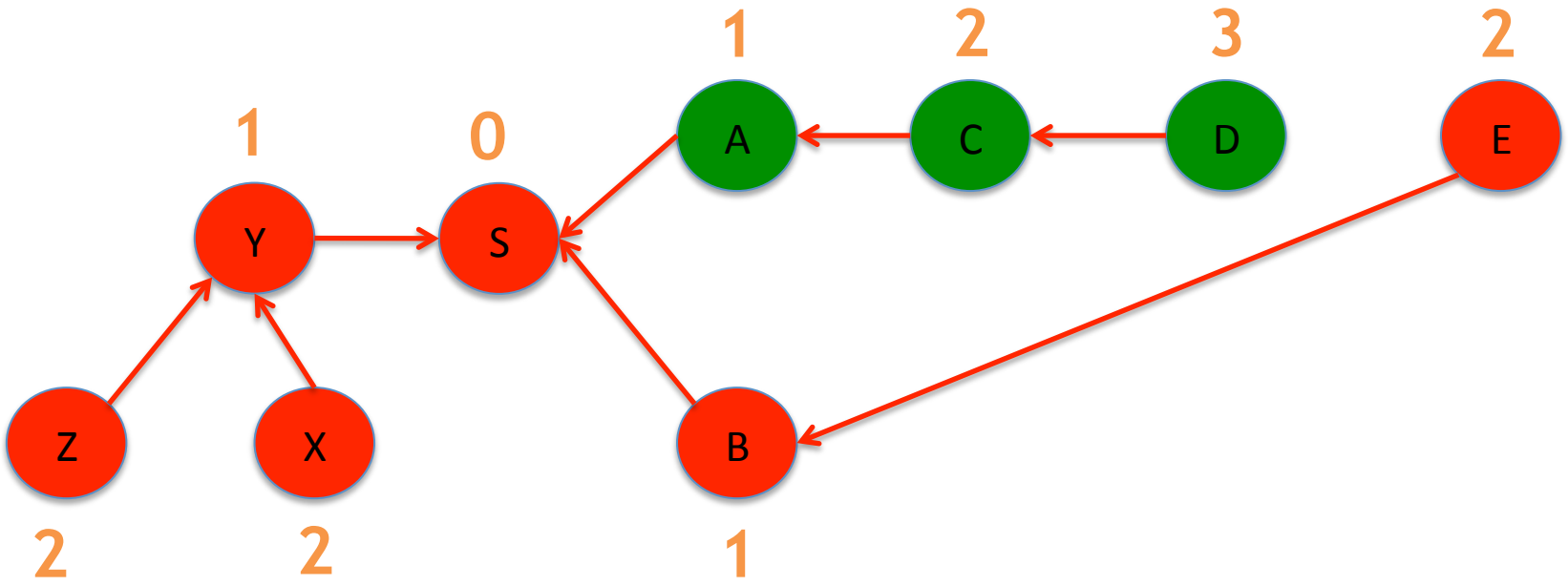
Ex: What's the path from S to D?

Just follow back from D on BFS Tree.

C->D

Using The BFS Tree To Construct Paths

- The BFS tree can be used to construct actual paths
- Just follow your parent! (suppose we stored (v,p(v)) direction)



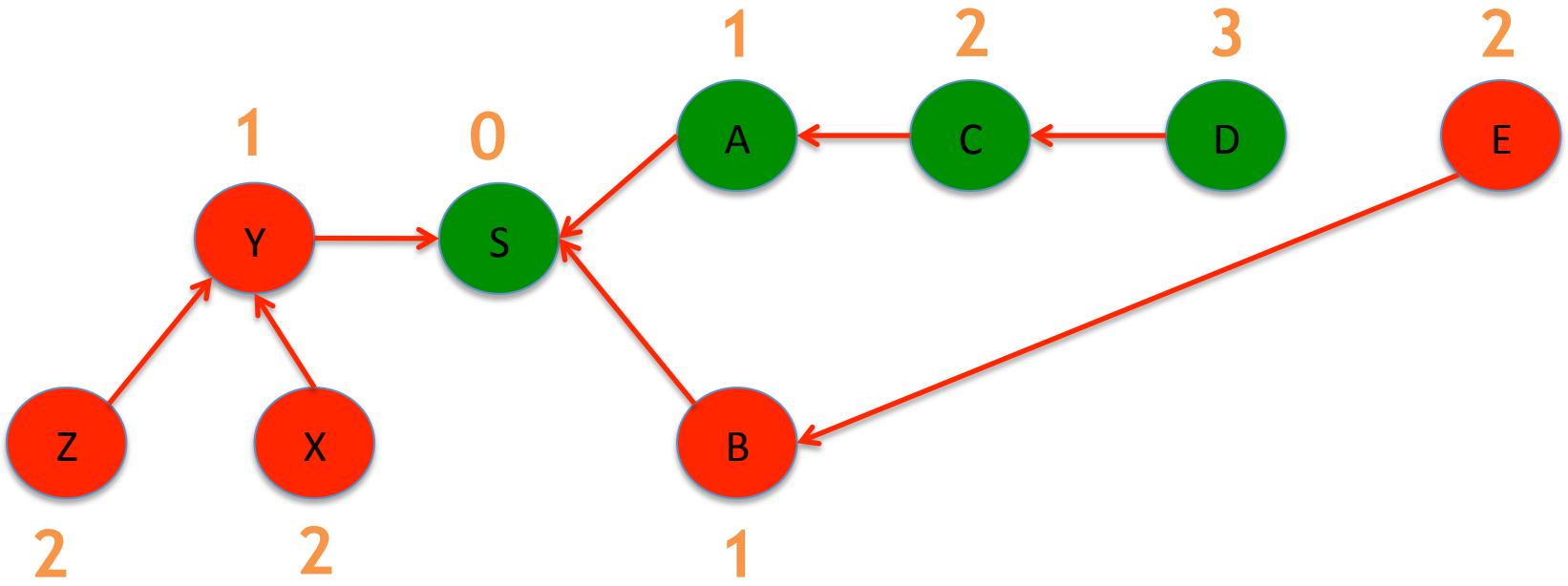
Ex: What's the path from S to D?

Just follow back from D on BFS Tree.

A->C->D

Using The BFS Tree To Construct Paths

- The BFS tree can be used to construct actual paths
- Just follow your parent! (suppose we stored (v,p(v)) direction)



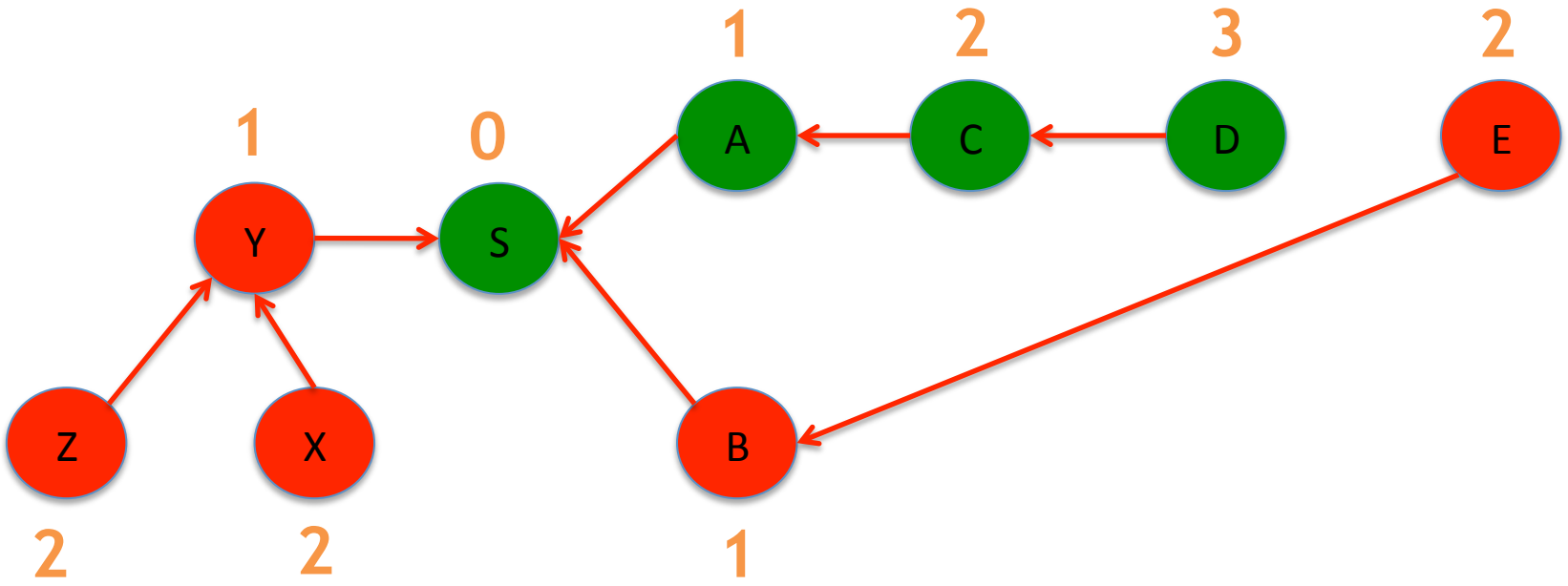
Ex: What's the path from S to D?

Just follow back from D on BFS Tree.

S->A->C->D

Using The BFS Tree To Construct Paths

- The BFS tree can be used to construct actual paths
- Just follow your parent! (suppose we stored (v,p(v)) direction)



Constructs a path (s, t) in d(s,t) time!

Outline For Today

1. Graph Terminology
2. BFS/DFS & BFS/DFS Tree
3. Application 1: Unweighted Single Source Shortest Paths
4. Application 2: Bipartiteness/2-coloring
5. Application 3: Connected Comp. in Undir. Graphs
6. Application 4: Topological Sort

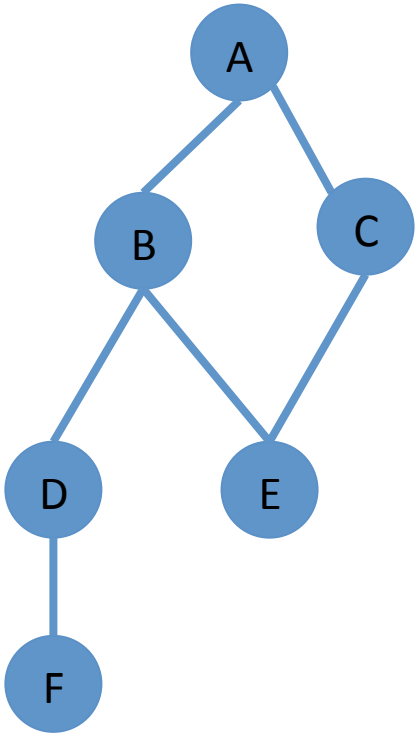
Bipartite Checking/2-coloring

- Input: undirected G(V, E)
- Output: True/False to the following question:

  Can V be partitioned as V1 (red) and V2 (green)

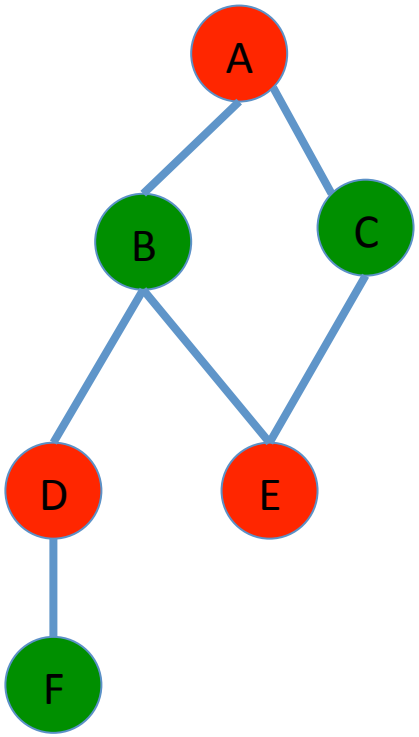
  s.t. $\forall (u,v) \in E$, u is red and v is green

Bipartite Checking/2-coloring



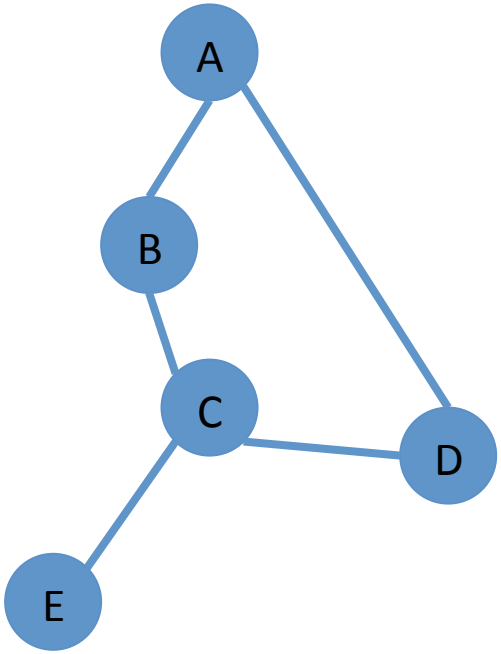
Is this graph 2-colorable?

Bipartite Checking/2-coloring



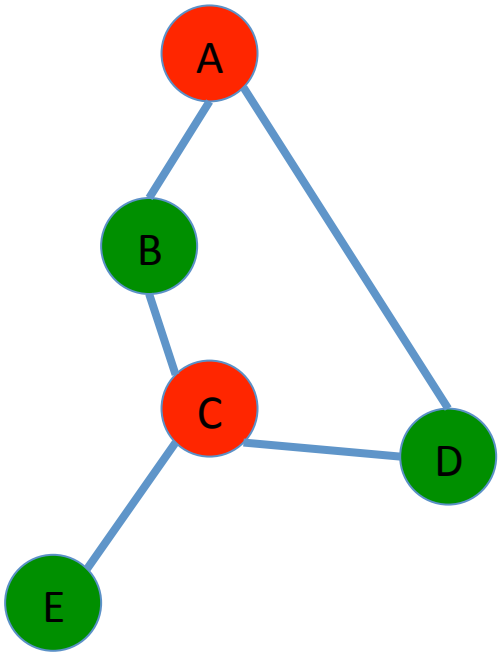
Yes

Bipartite Checking/2-coloring



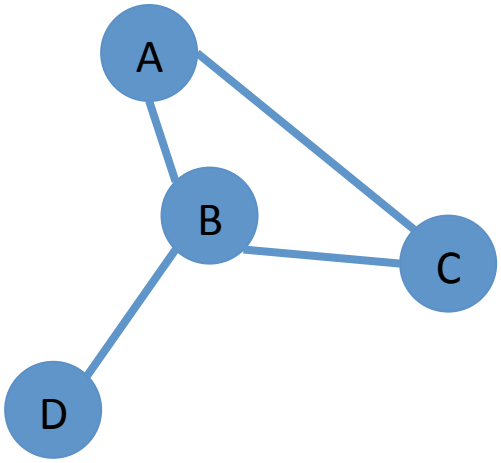
Is this graph 2-colorable?

Bipartite Checking/2-coloring



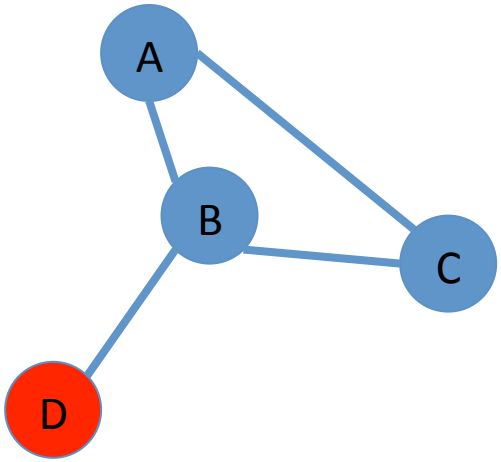
Yes

Bipartite Checking/2-coloring



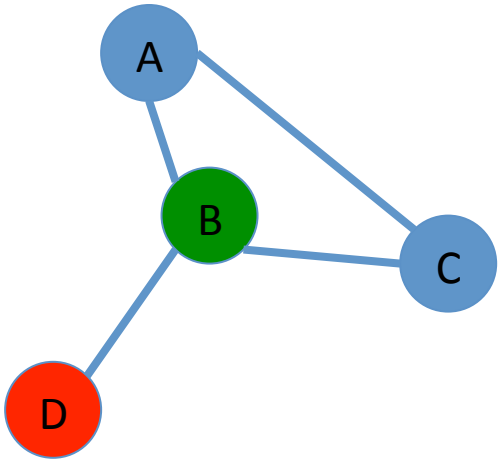
Is this graph 2-colorable?

Bipartite Checking/2-coloring



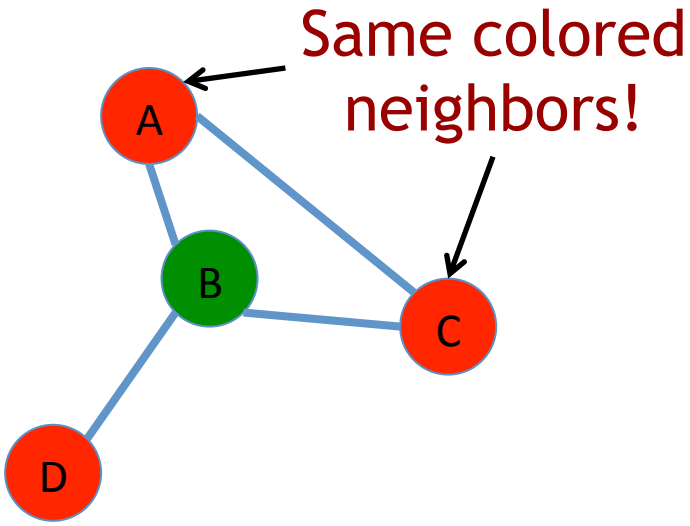
Is this graph 2-colorable?

Bipartite Checking/2-coloring



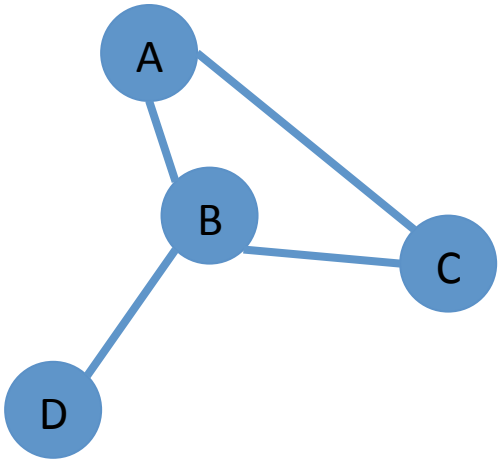
Is this graph 2-colorable?

Bipartite Checking/2-coloring



Is this graph 2-colorable?

Bipartite Checking/2-coloring



Not colorable.

Observation

Suppose we do BFS from s and assign s (w.l.o.g) the color red.

Then if G is 2-colorable:

s is level 0 and colored red.

Then all of level 1 vertices has to be green.

Then all of level 2 has to be red.

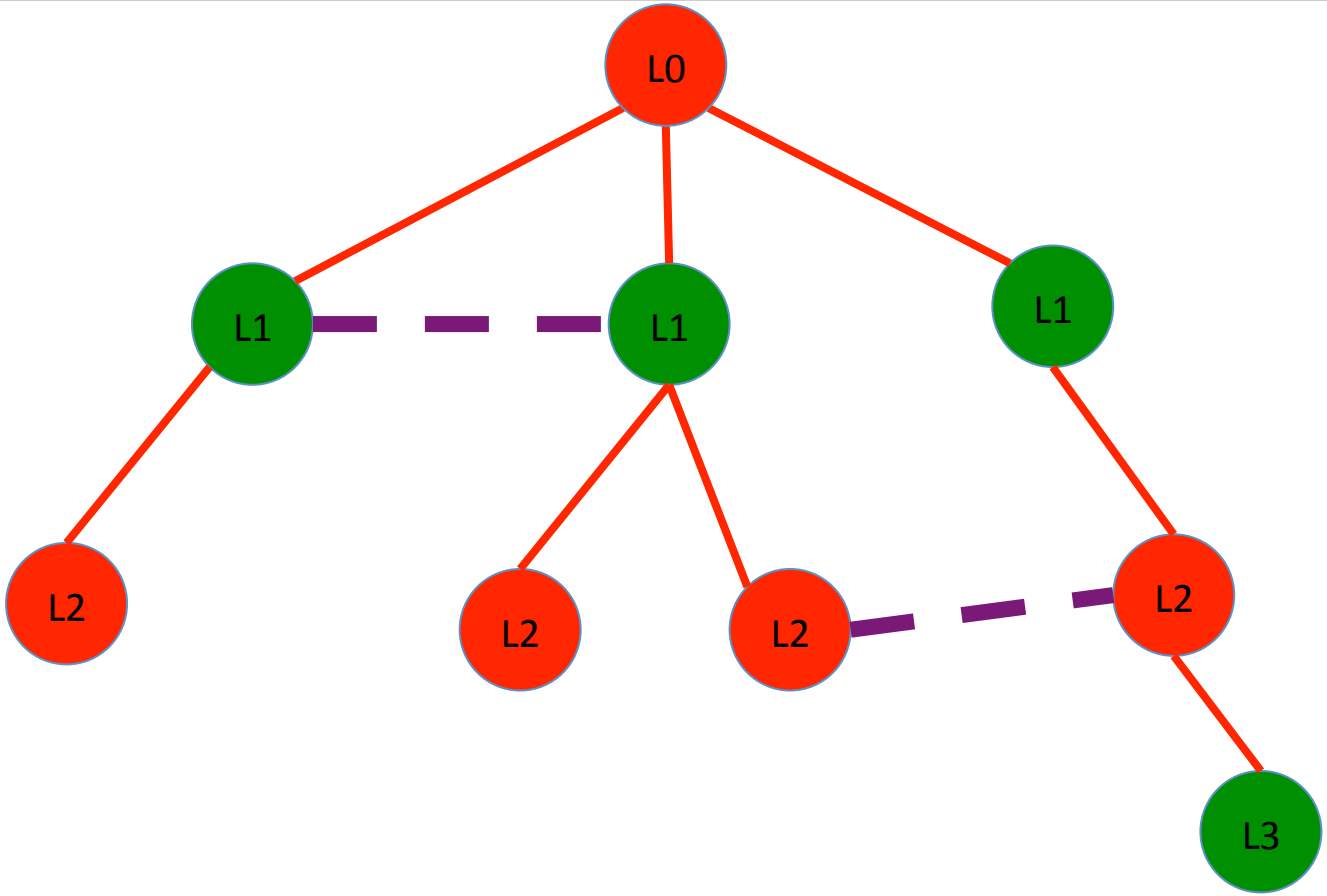
Then all of level 3 has to be green.

Then all of level 4 has to be red.

…

So let's "tentatively" color vertices according to the levels of the vertices in the BFS-tree.

What if G is not colorable?



Then there must be a “cross edge” in the graph.

So let’s just check if a cross edge breaks our coloring.

BFS-based Bipartite Checking Algorithm

```
1. procedure Bipartite-Checking(undir. G(V, E))
2.    run BFS(G(V,E))
3.    give even levels red; odd levels green
4.    if ∃(u, v) ∈ E s.t. u & v are the same color
5.       return false
6.    else
7.       return true
```

Runtime: O(n + m)

Outline For Today

1. Graph Terminology
2. BFS/DFS & BFS/DFS Tree
3. Application 1: Unweighted Single Source Shortest Paths
4. Application 2: Bipartiteness/2-coloring
5. Application 3: Connected Comp. in Undir. Graphs
6. Application 4: Topological Sort

Undirected (Weakly) Connected Components

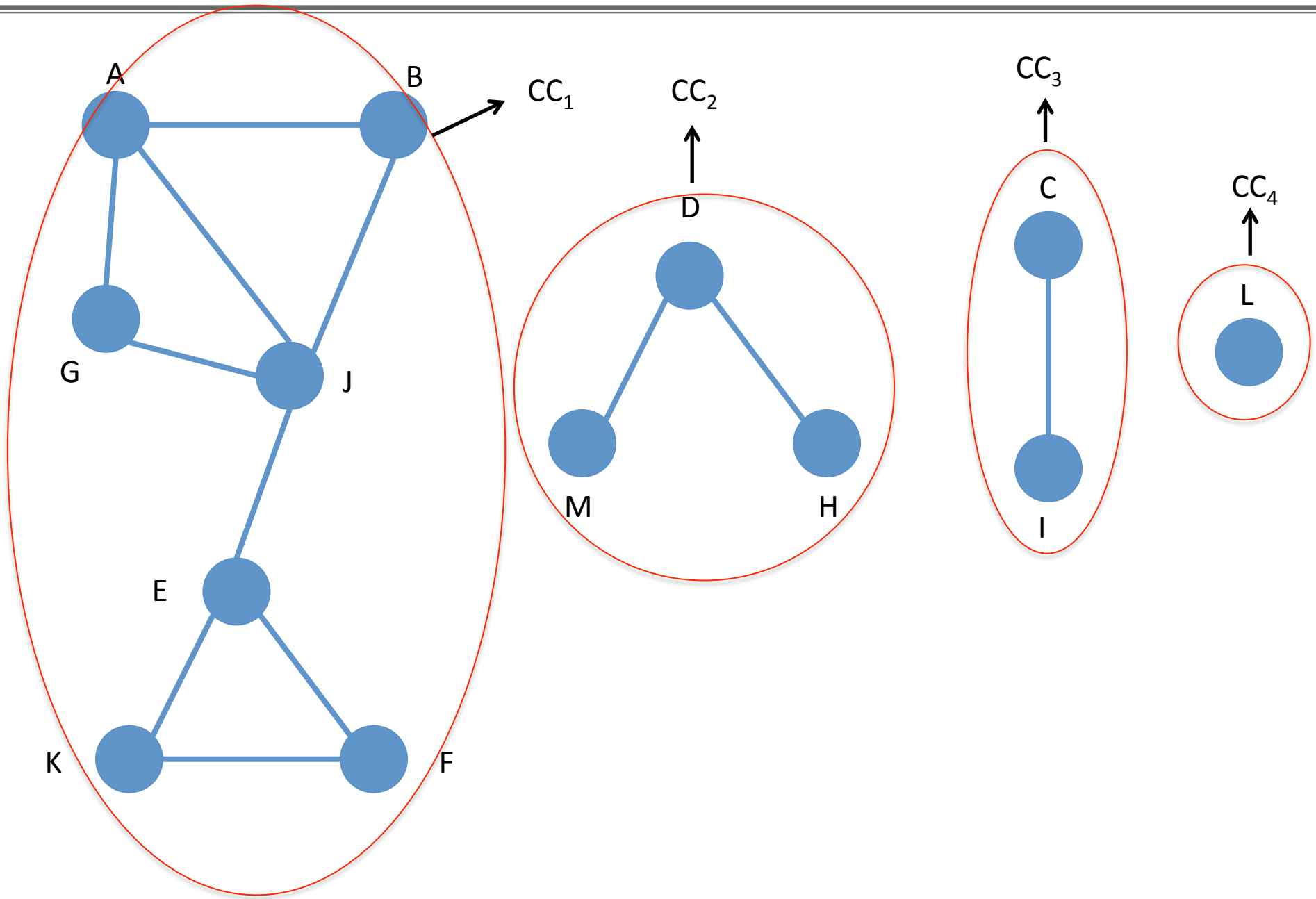
- Input: undirected G(V, E)
- Output: CC_1, CC_2, …, CC_k

  where each CC_i is a maximal subset of V, s.t.

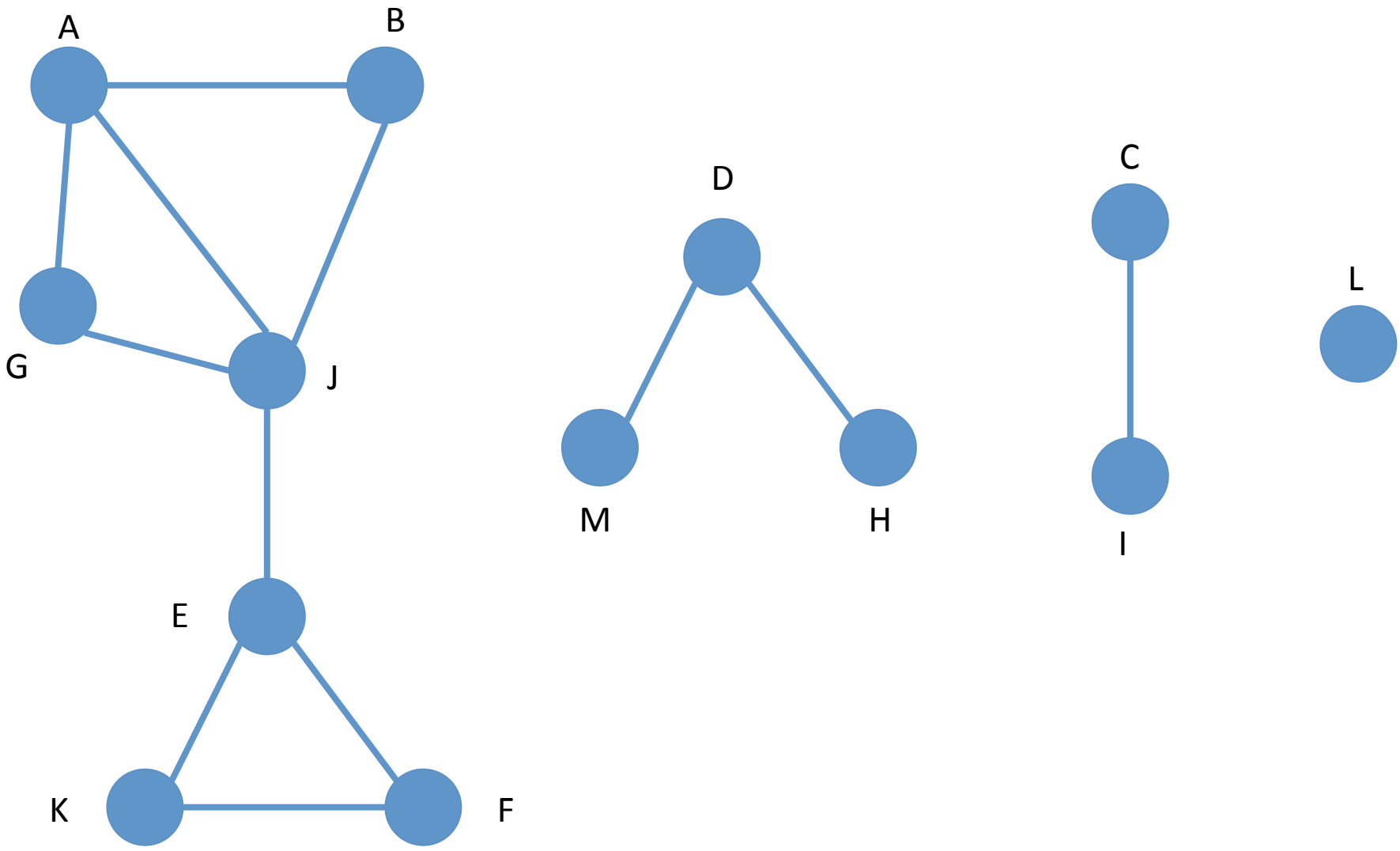
  each s, t in CC_i has a path to each other AND

  each v ∈ V is part of one CC_i.

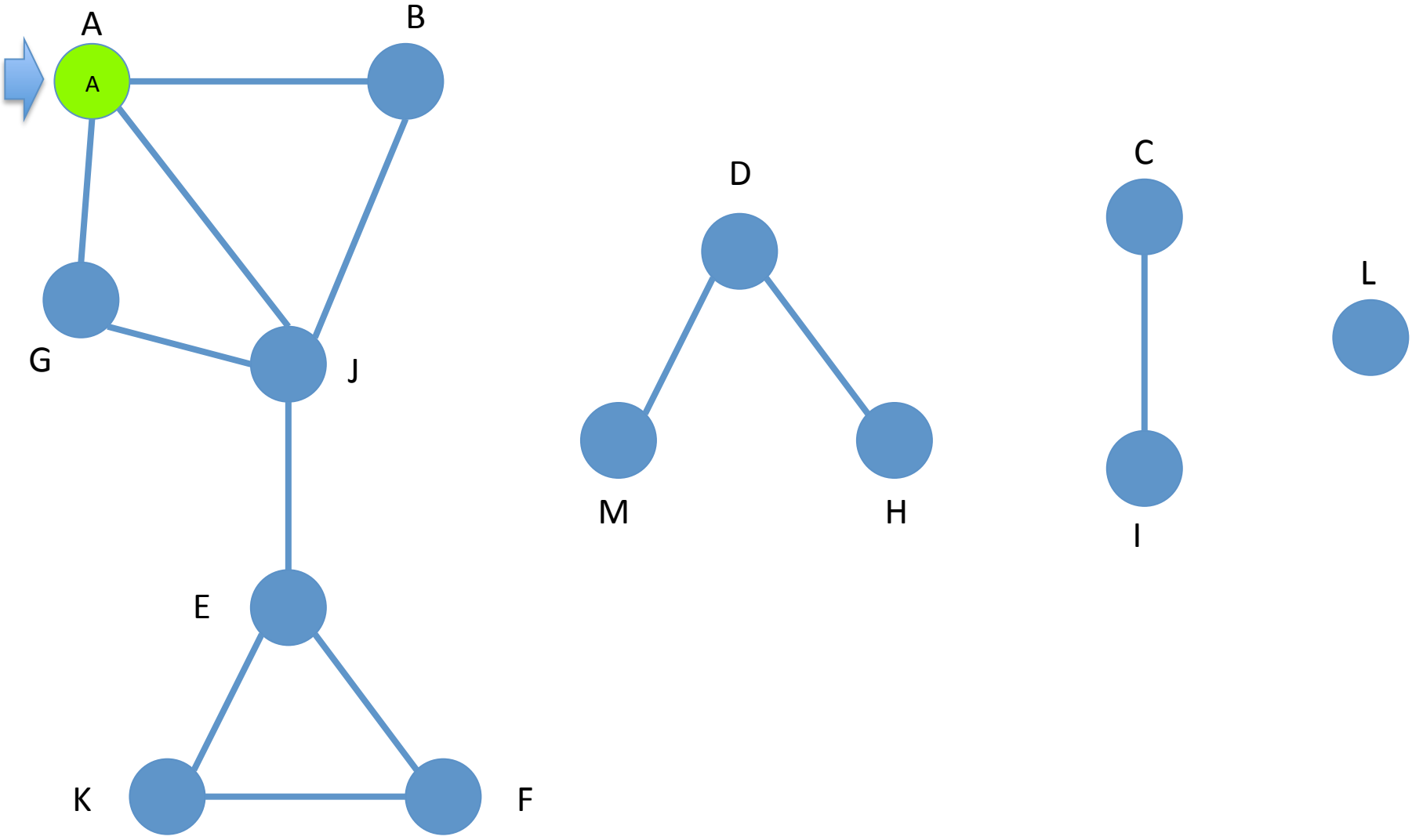
Undirected Connected Components



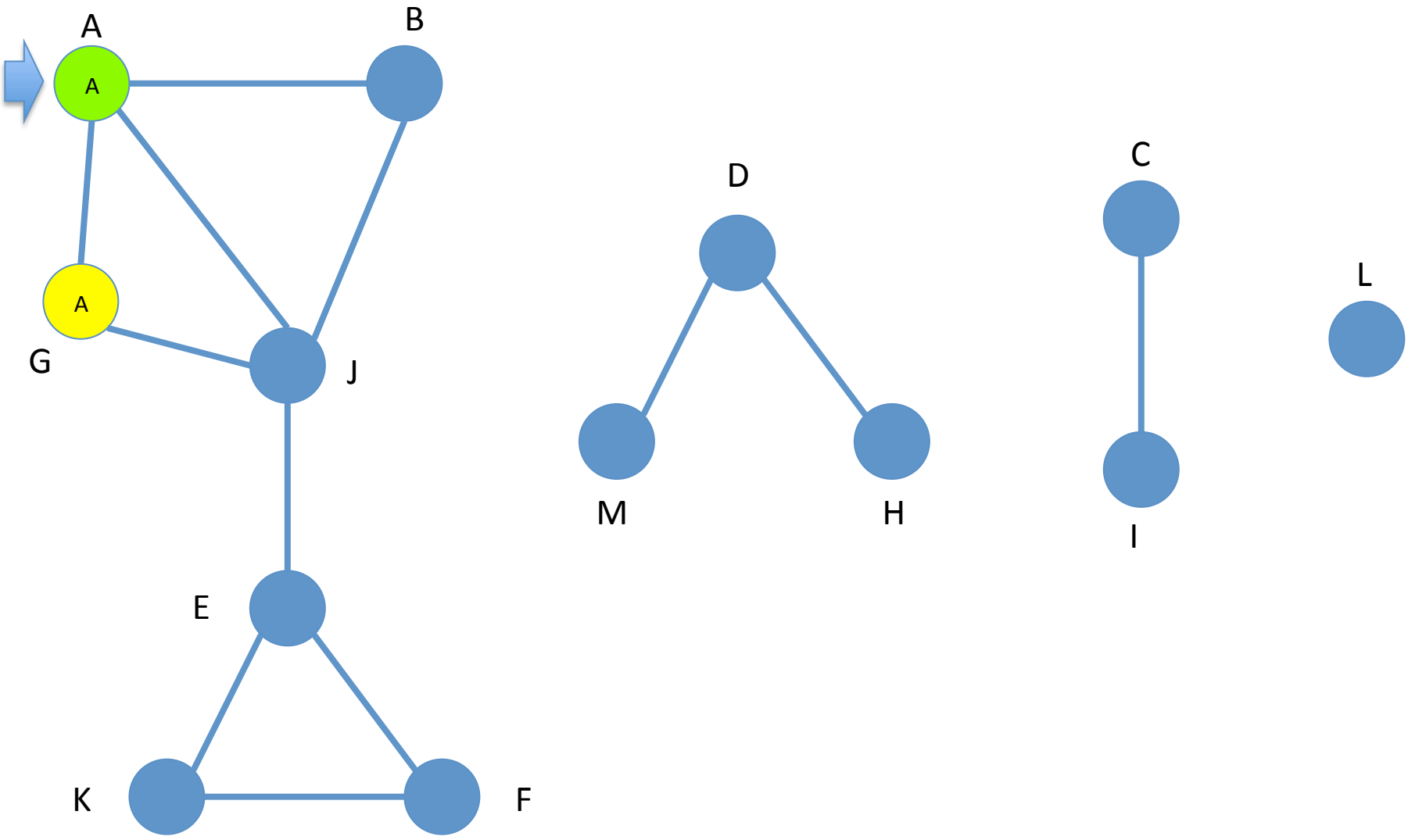
BFS - Con. Comp. w/ Label Propagation



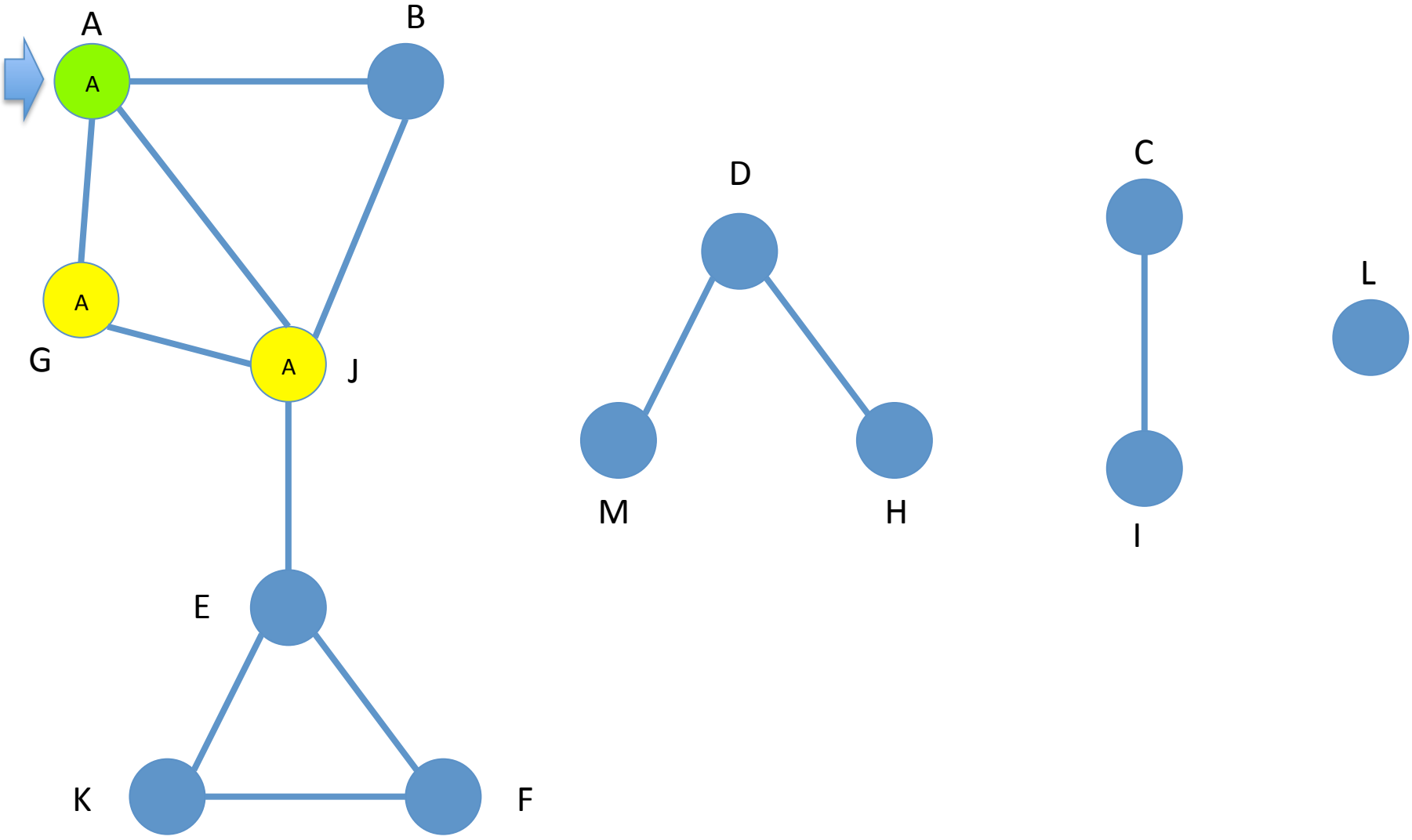
BFS - Con. Comp. w/ Label Propagation



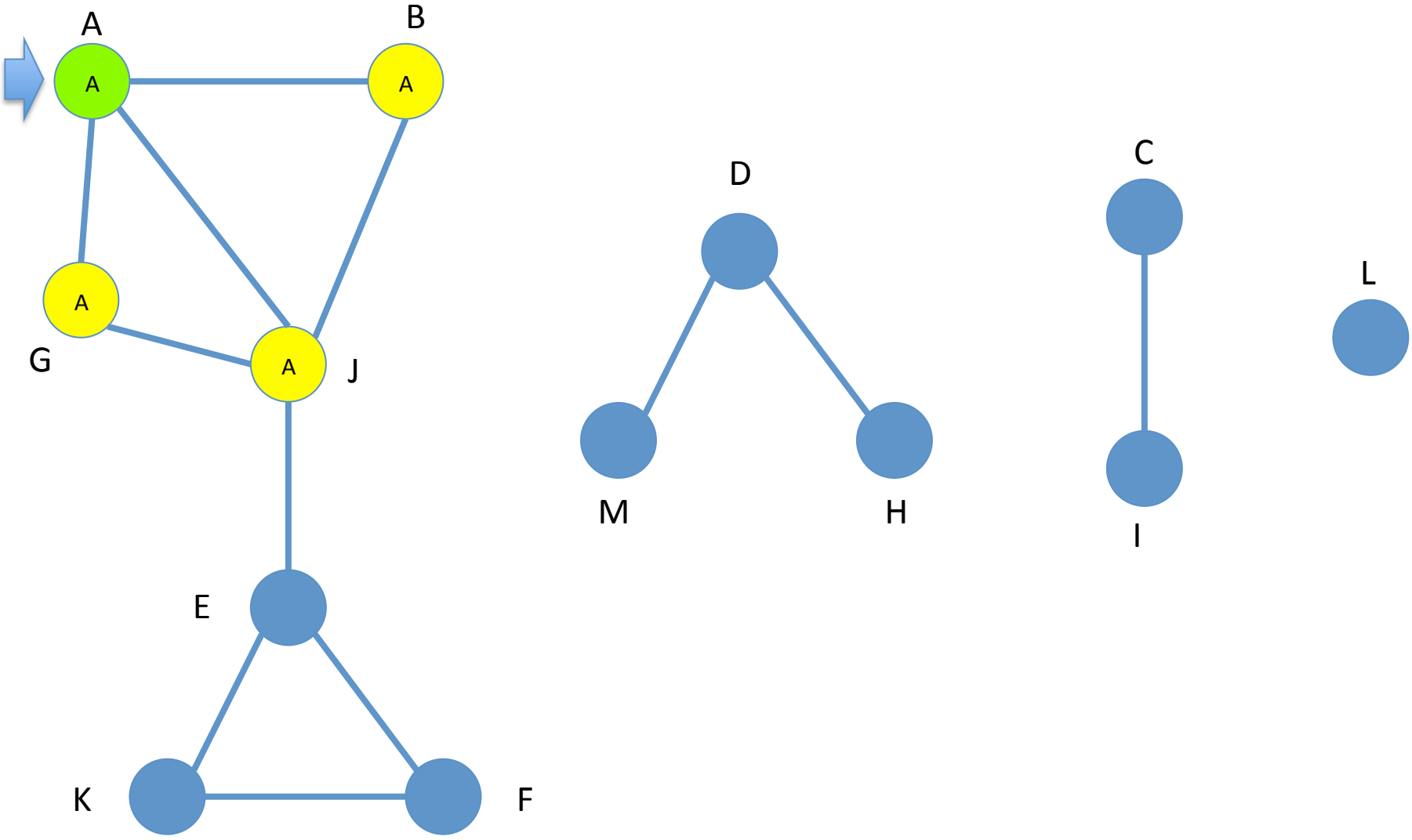
BFS - Con. Comp. w/ Label Propagation



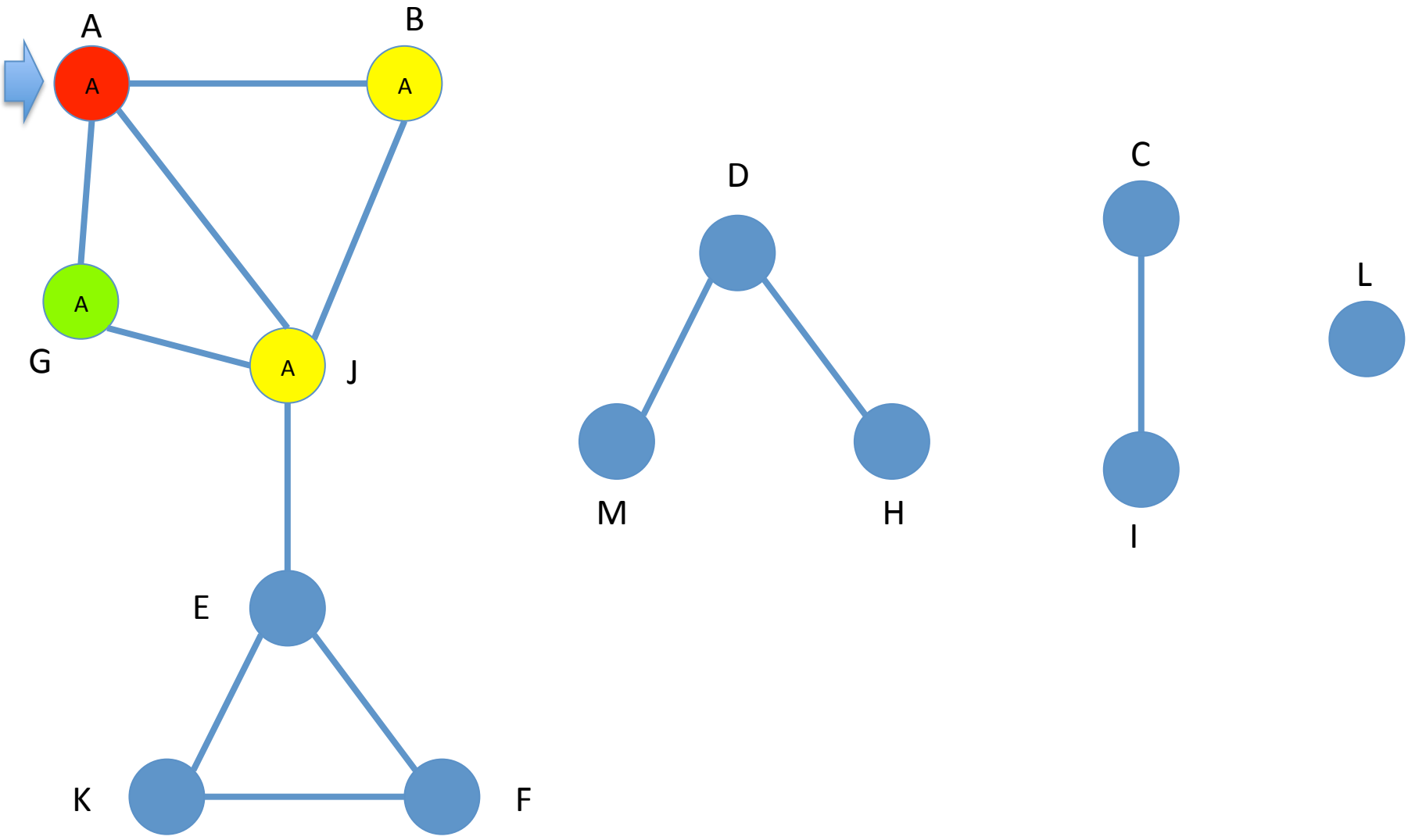
BFS - Con. Comp. w/ Label Propagation



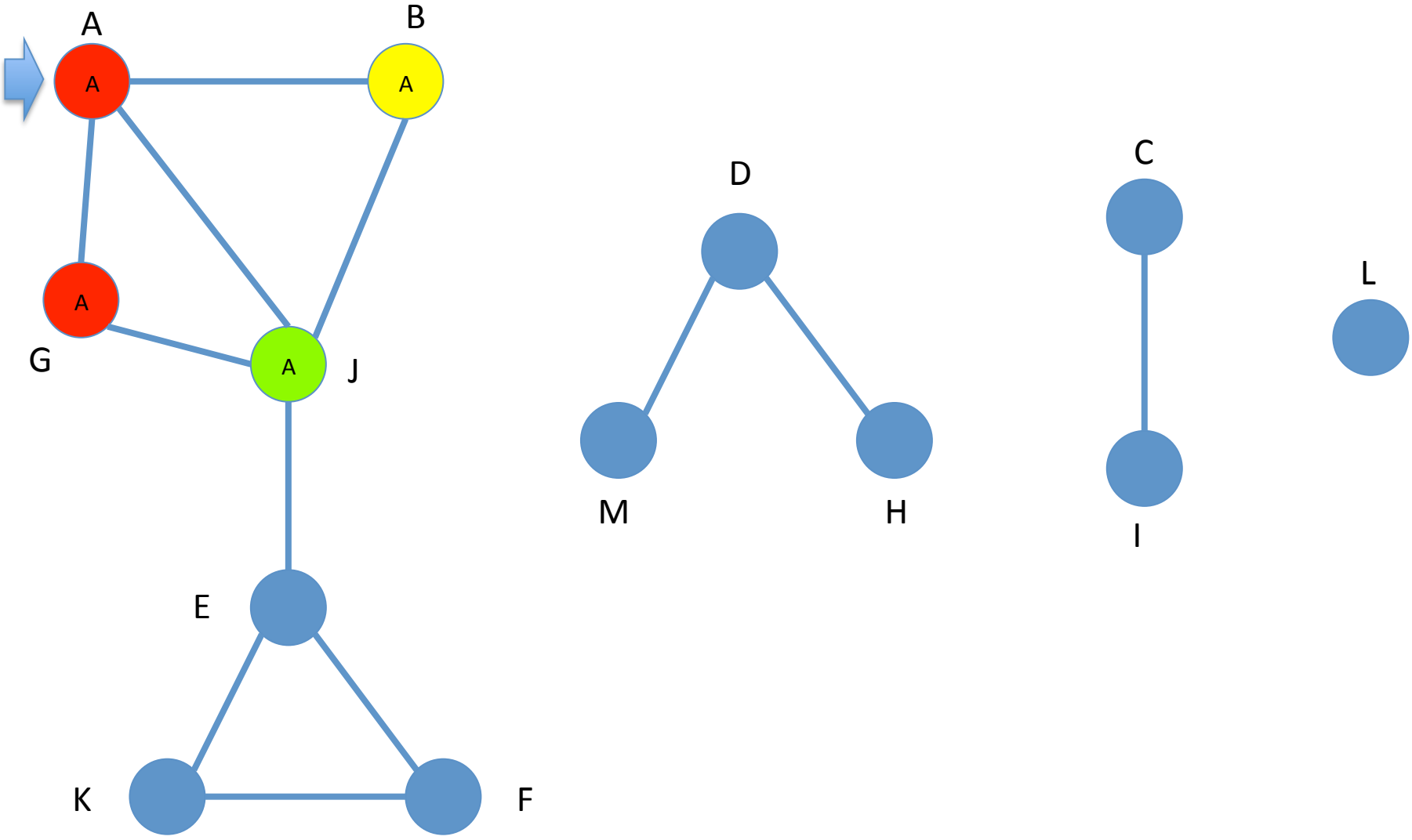
BFS - Con. Comp. w/ Label Propagation



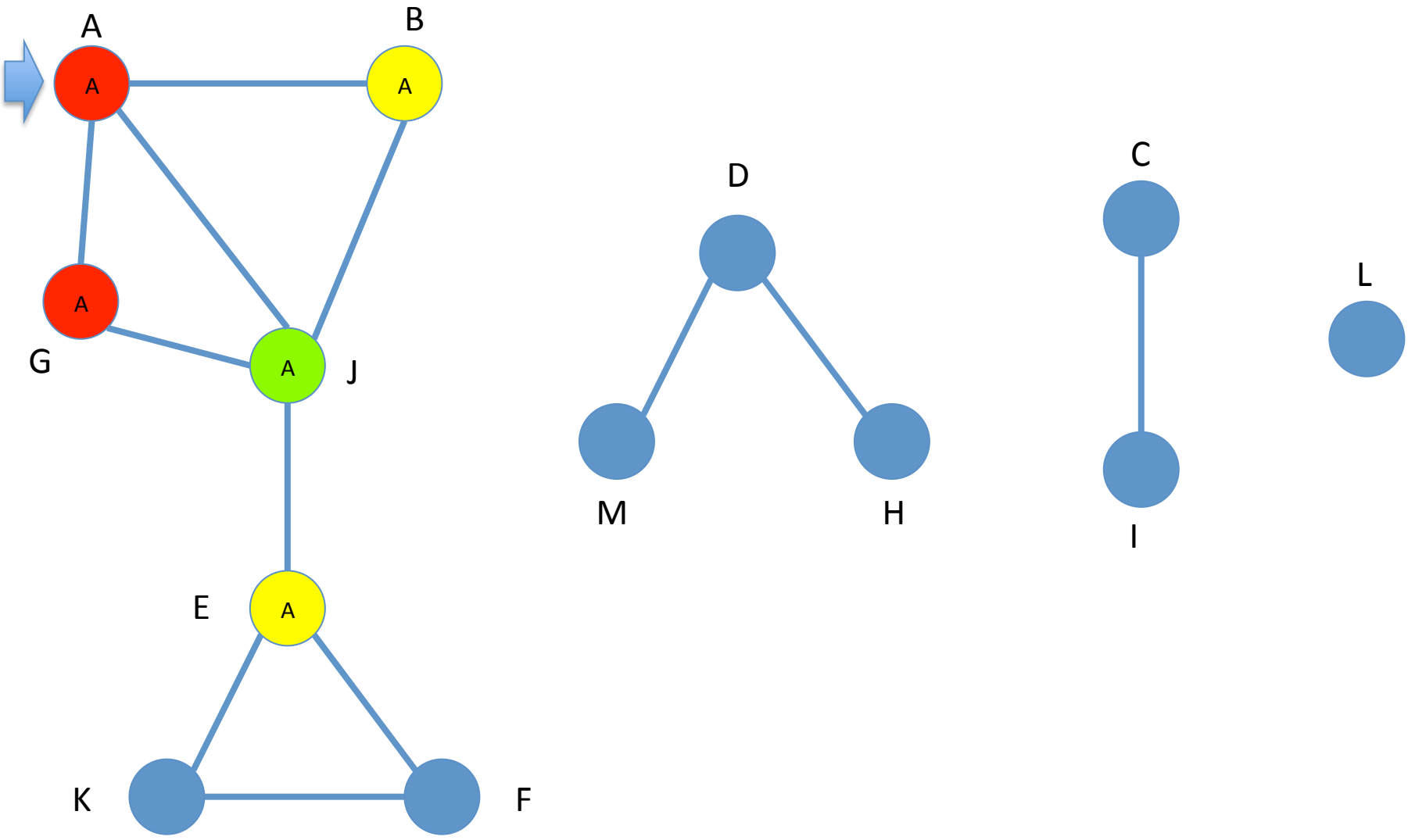
BFS - Con. Comp. w/ Label Propagation



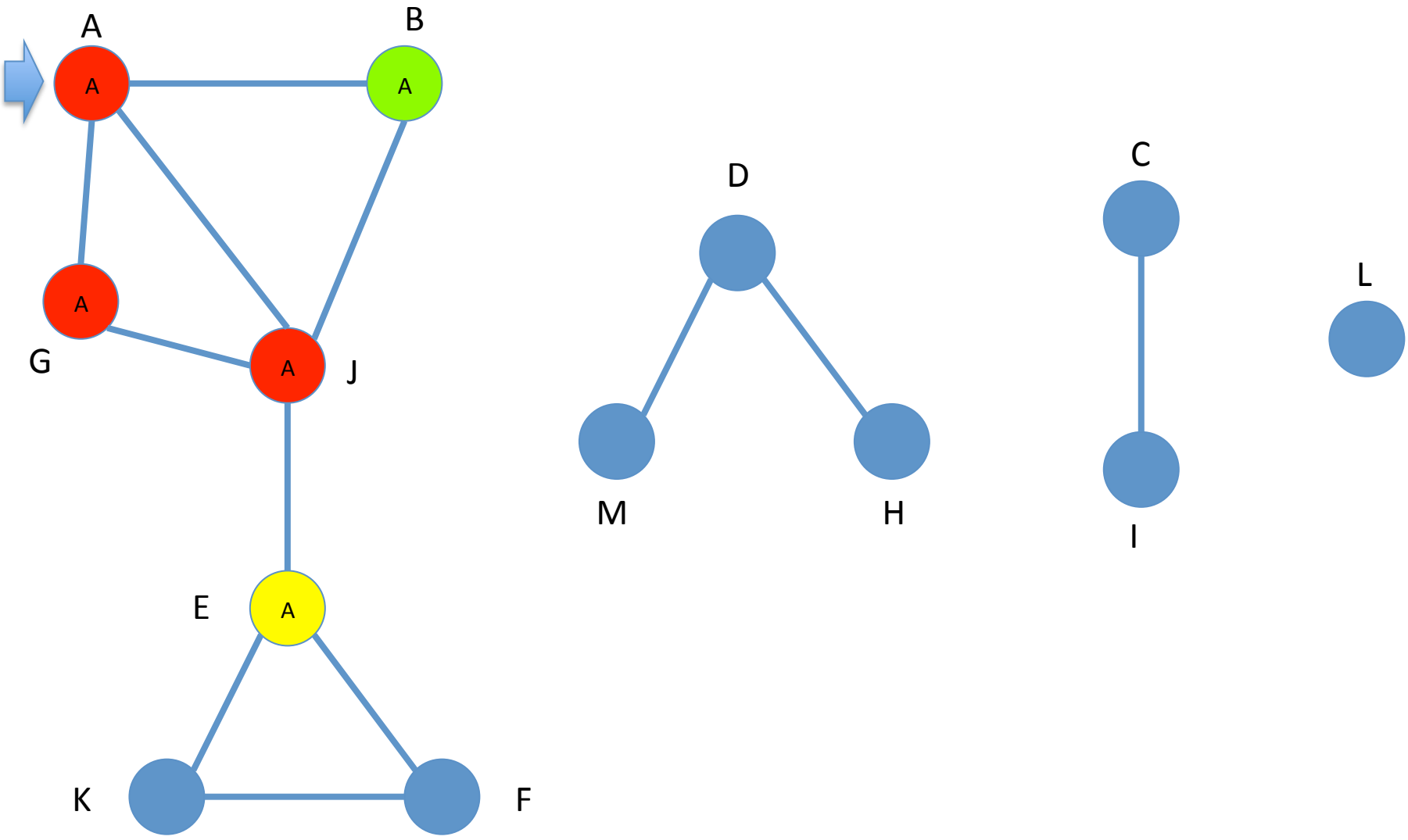
BFS - Con. Comp. w/ Label Propagation



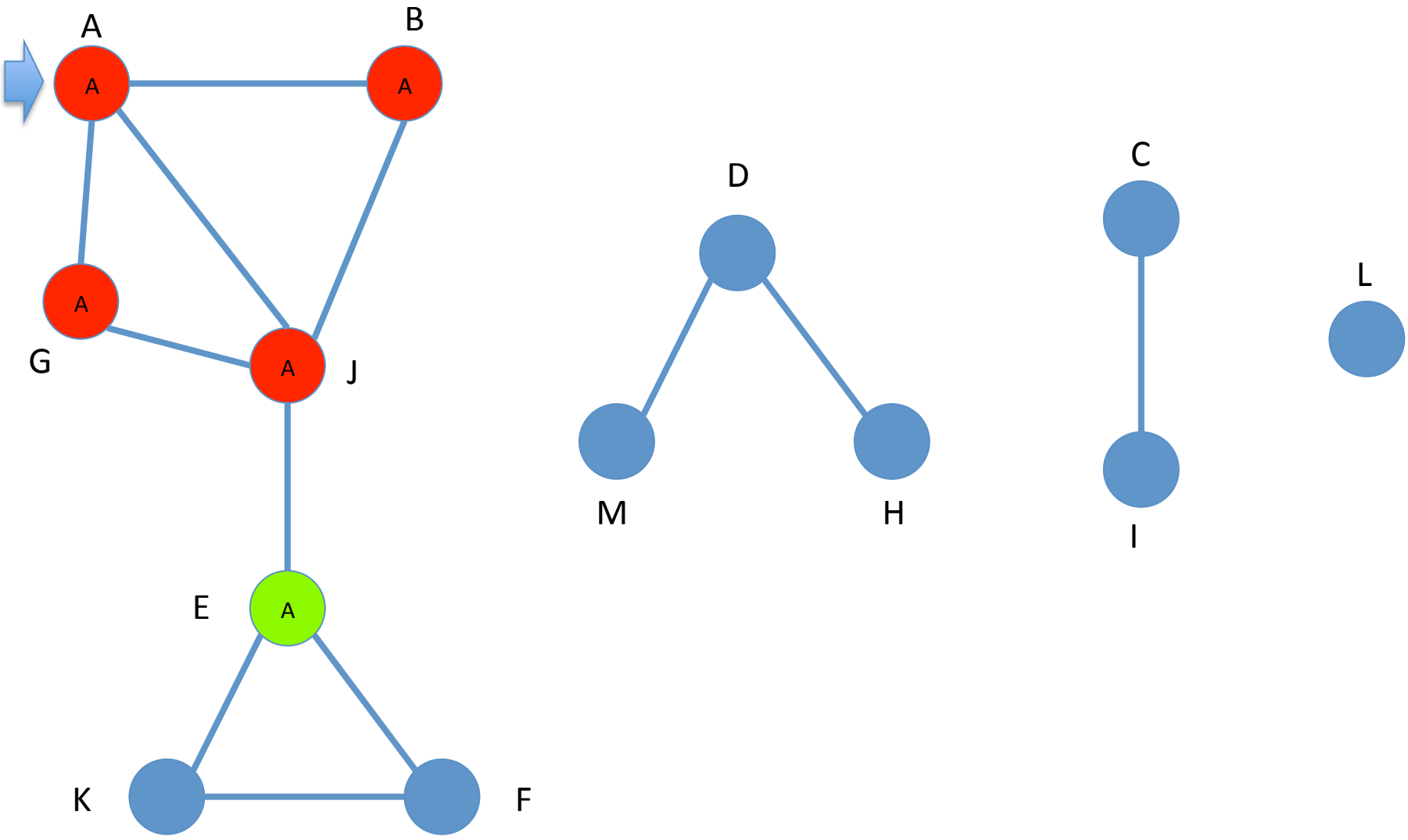
BFS - Con. Comp. w/ Label Propagation



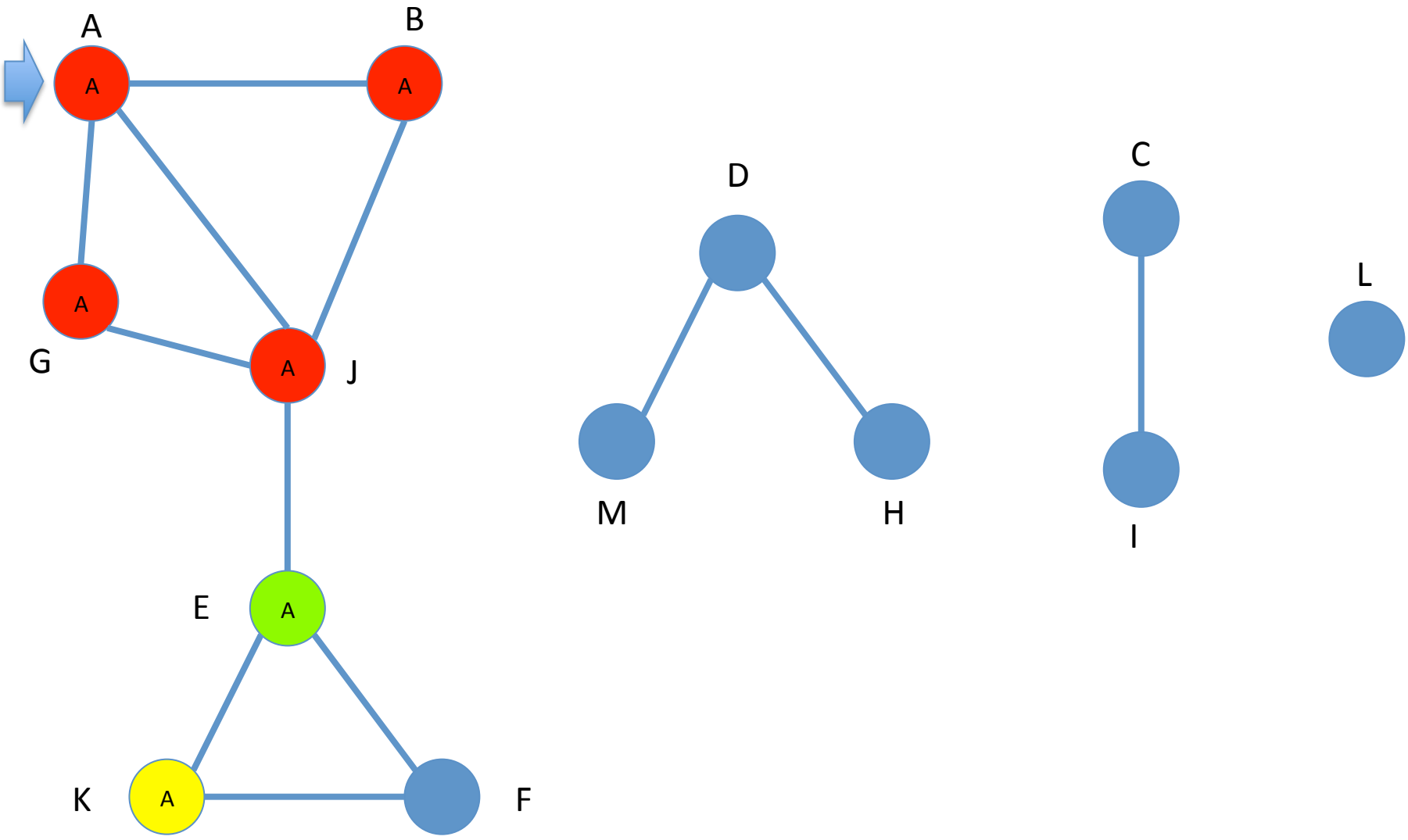
BFS - Con. Comp. w/ Label Propagation



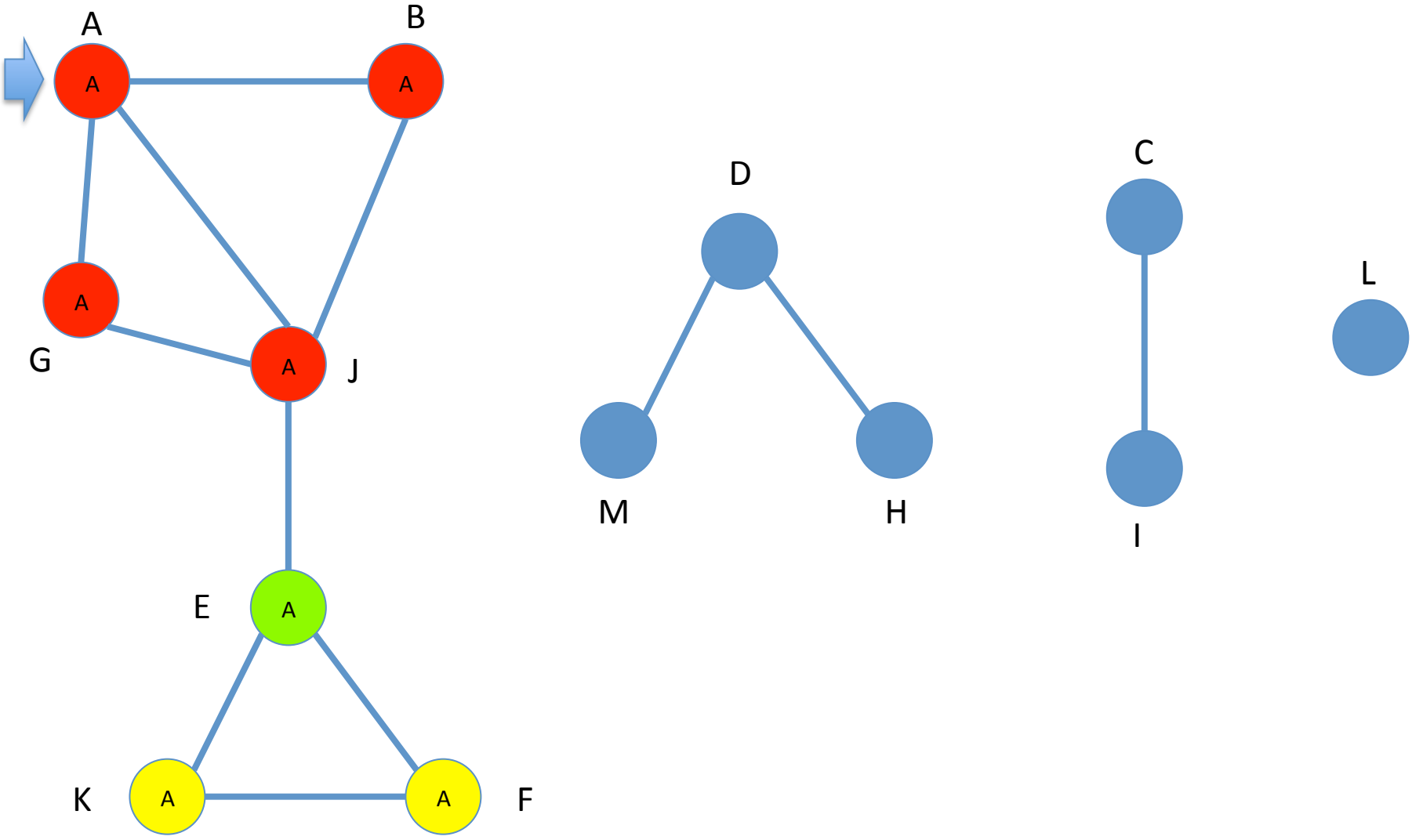
BFS - Con. Comp. w/ Label Propagation



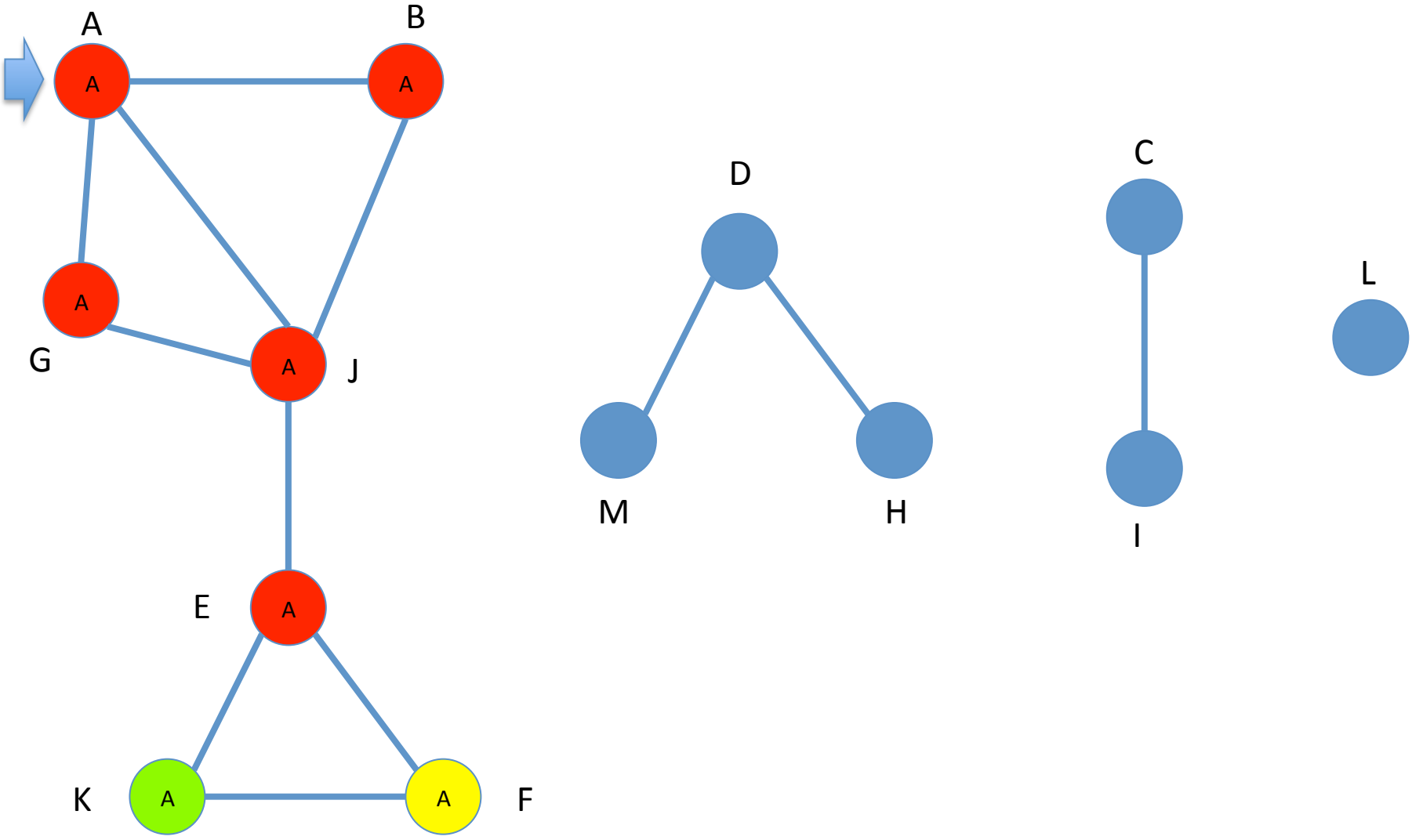
BFS - Con. Comp. w/ Label Propagation



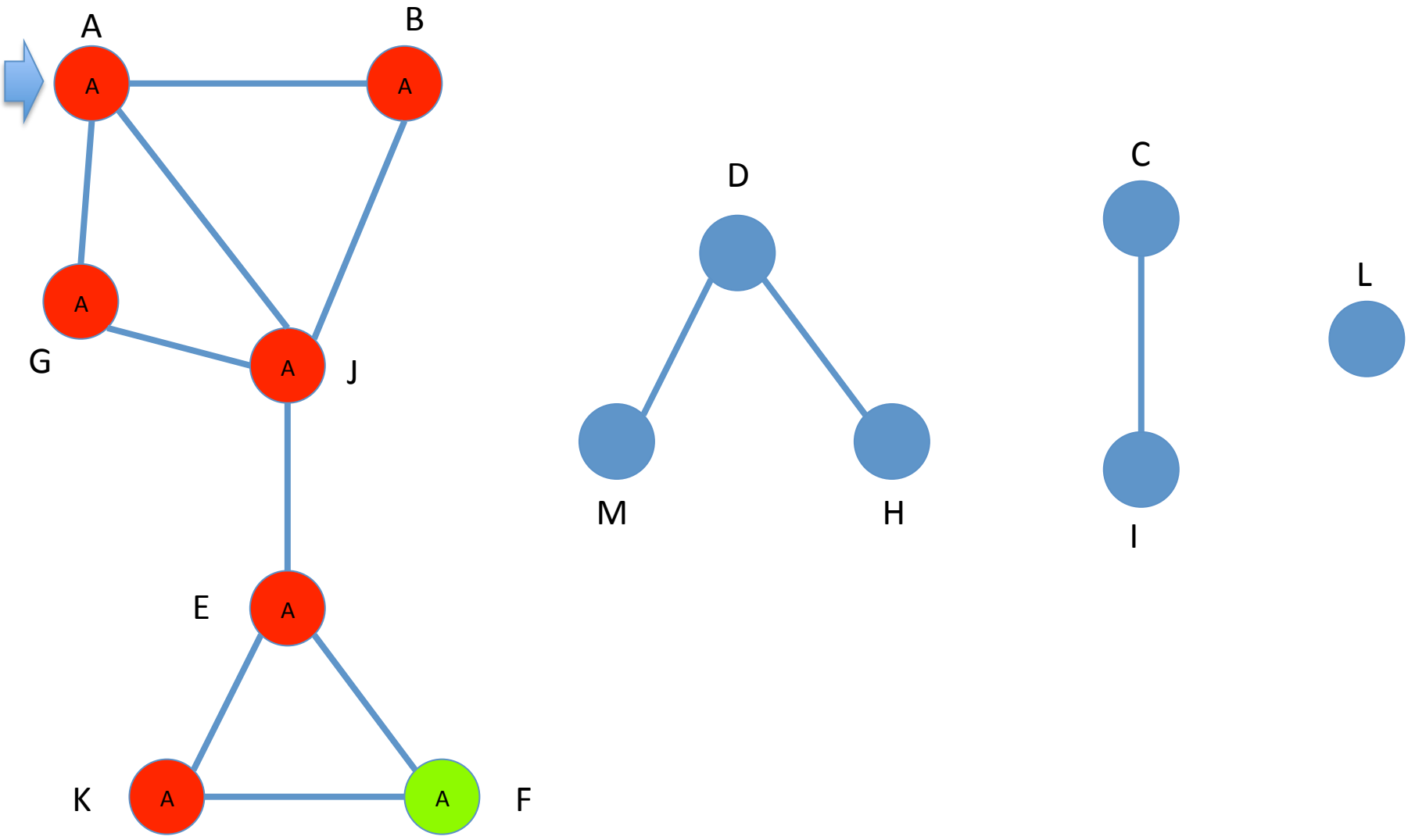
BFS - Con. Comp. w/ Label Propagation



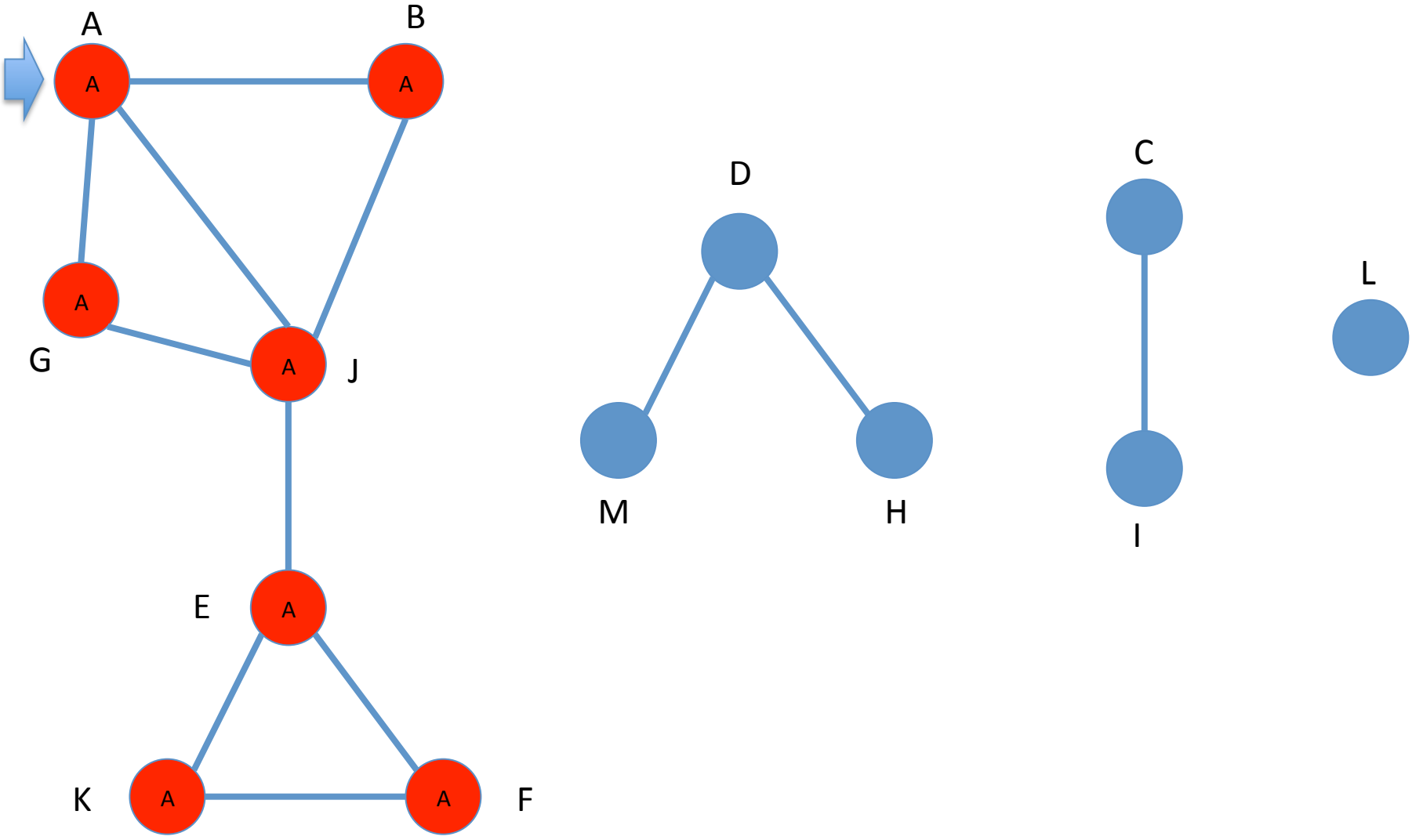
BFS - Con. Comp. w/ Label Propagation



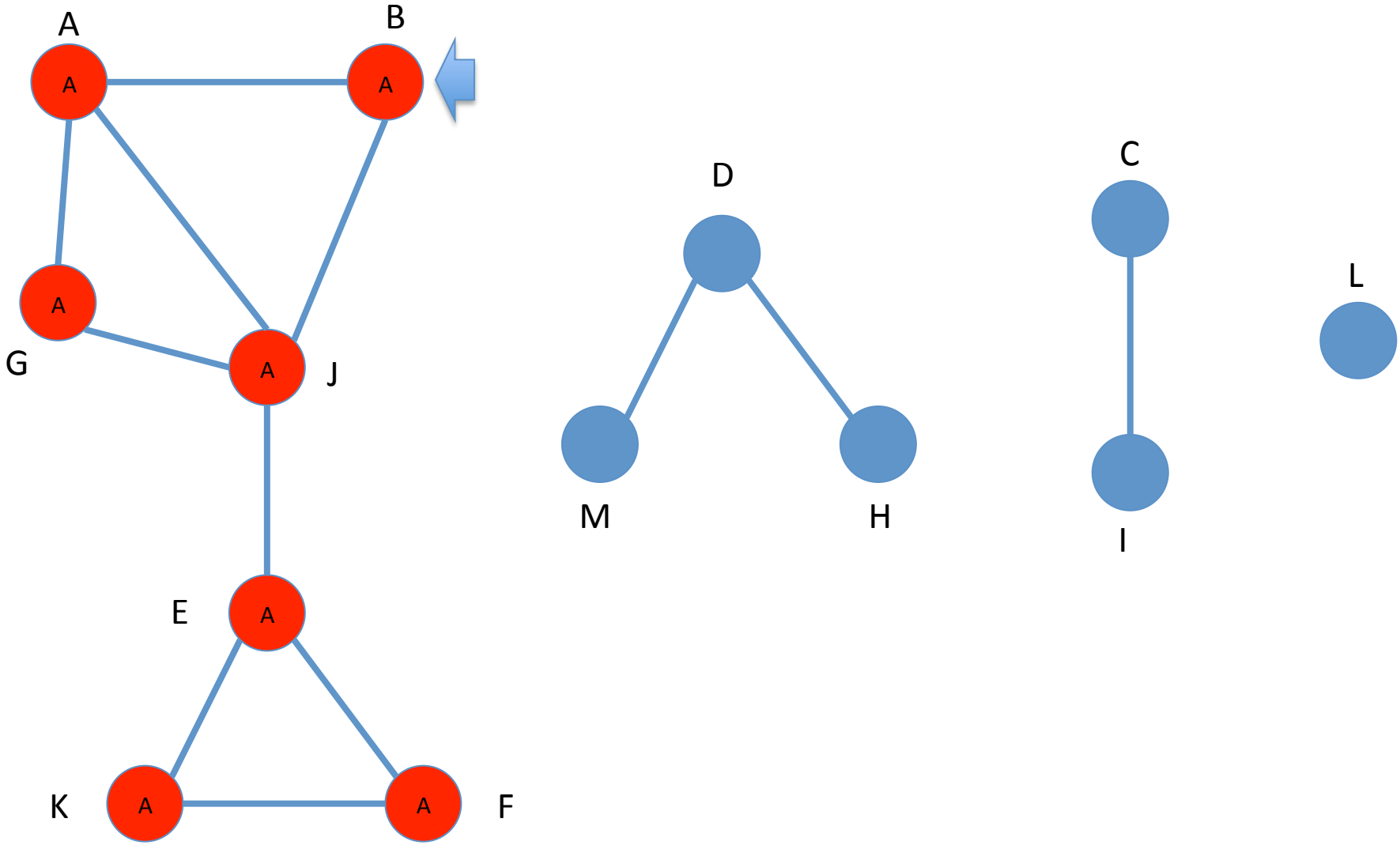
BFS - Con. Comp. w/ Label Propagation



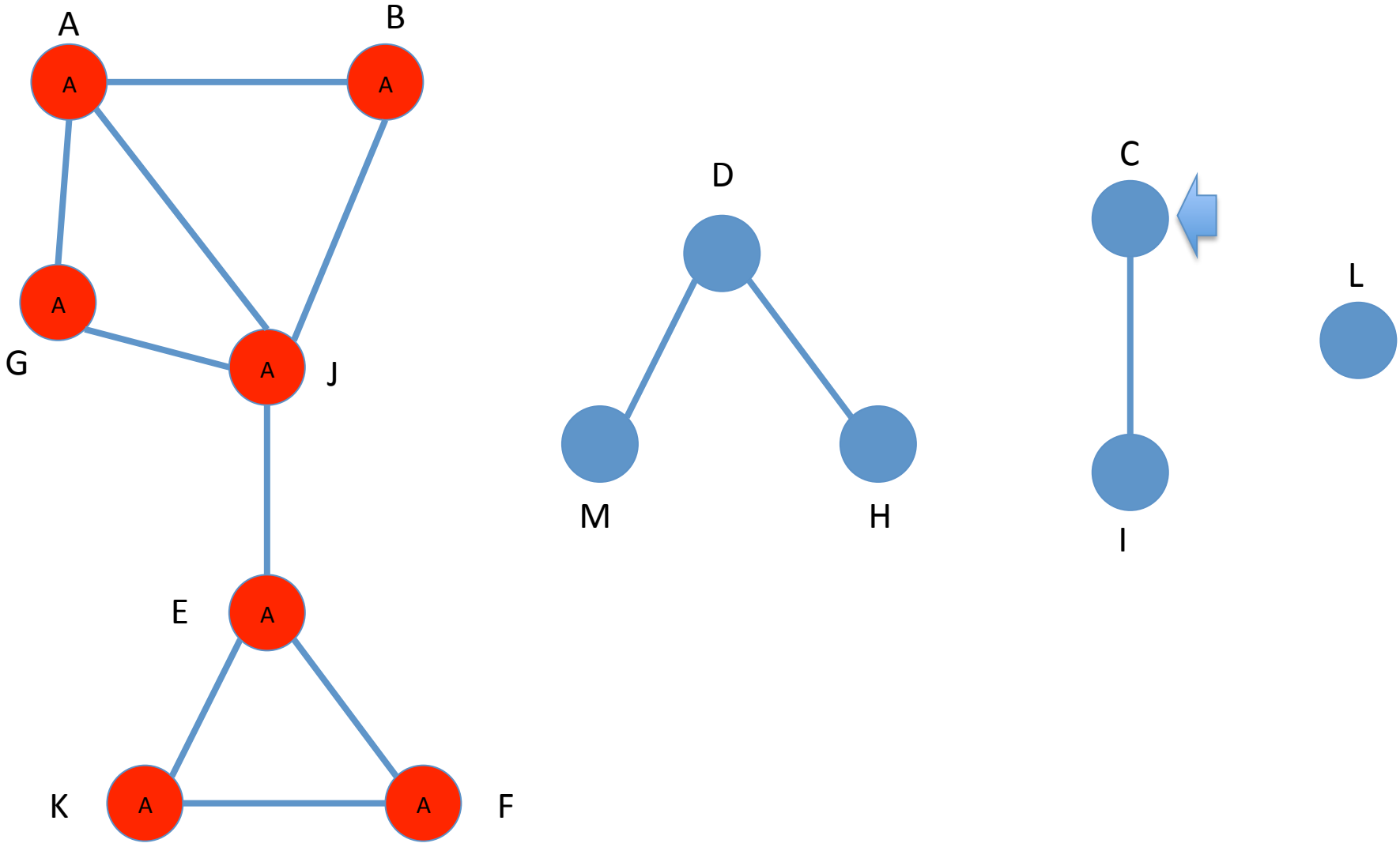
BFS - Con. Comp. w/ Label Propagation



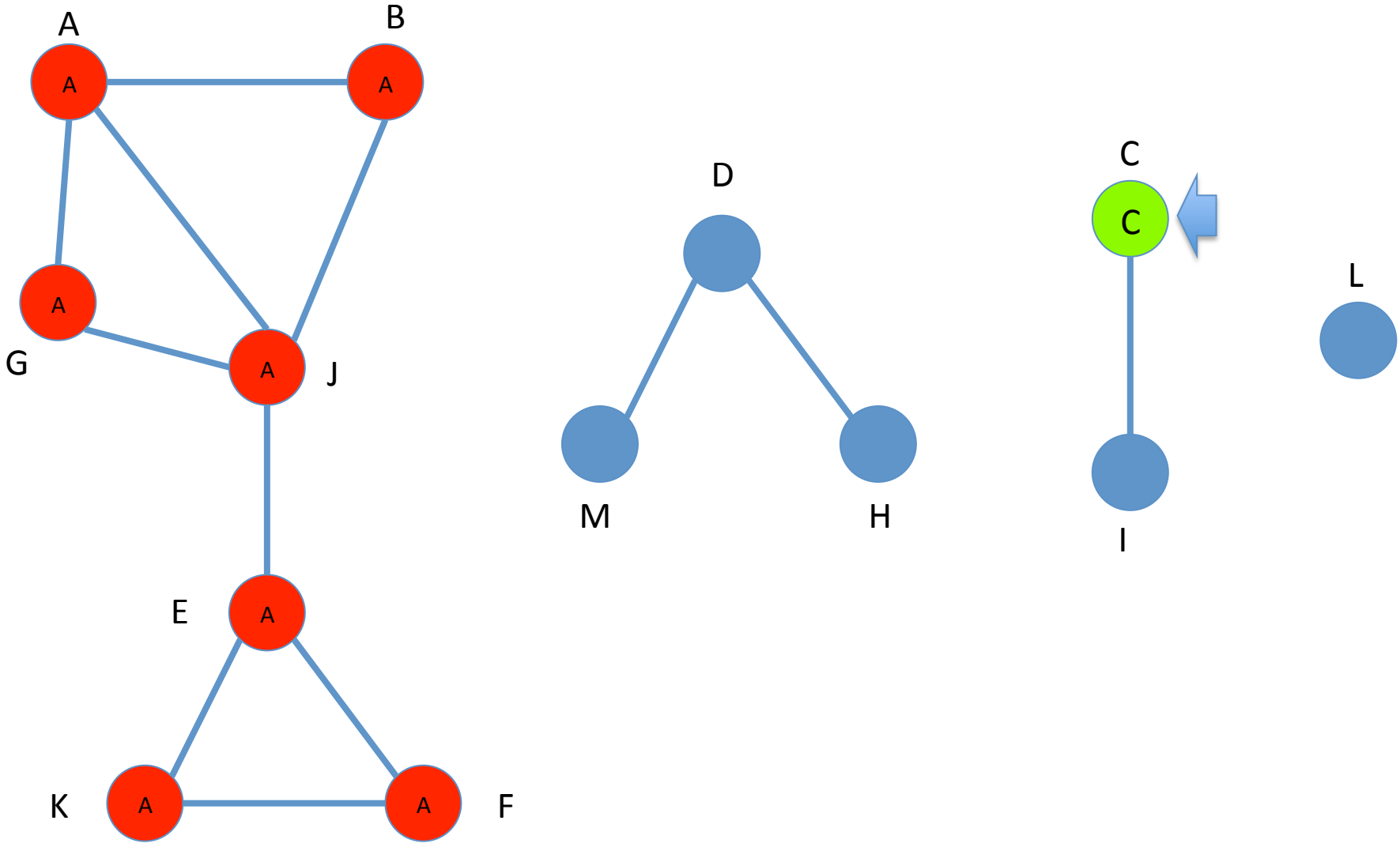
BFS - Con. Comp. w/ Label Propagation



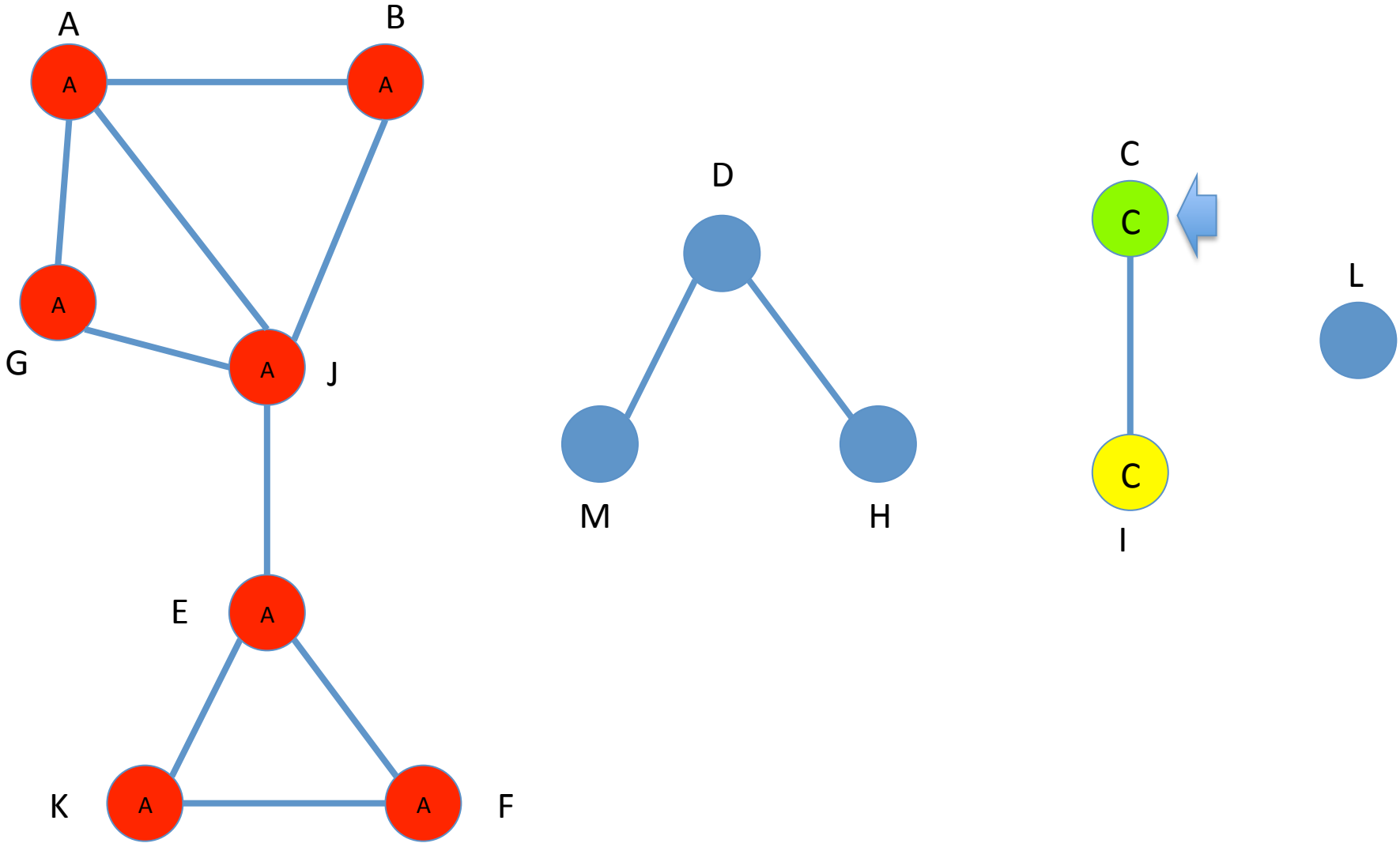
BFS - Con. Comp. w/ Label Propagation



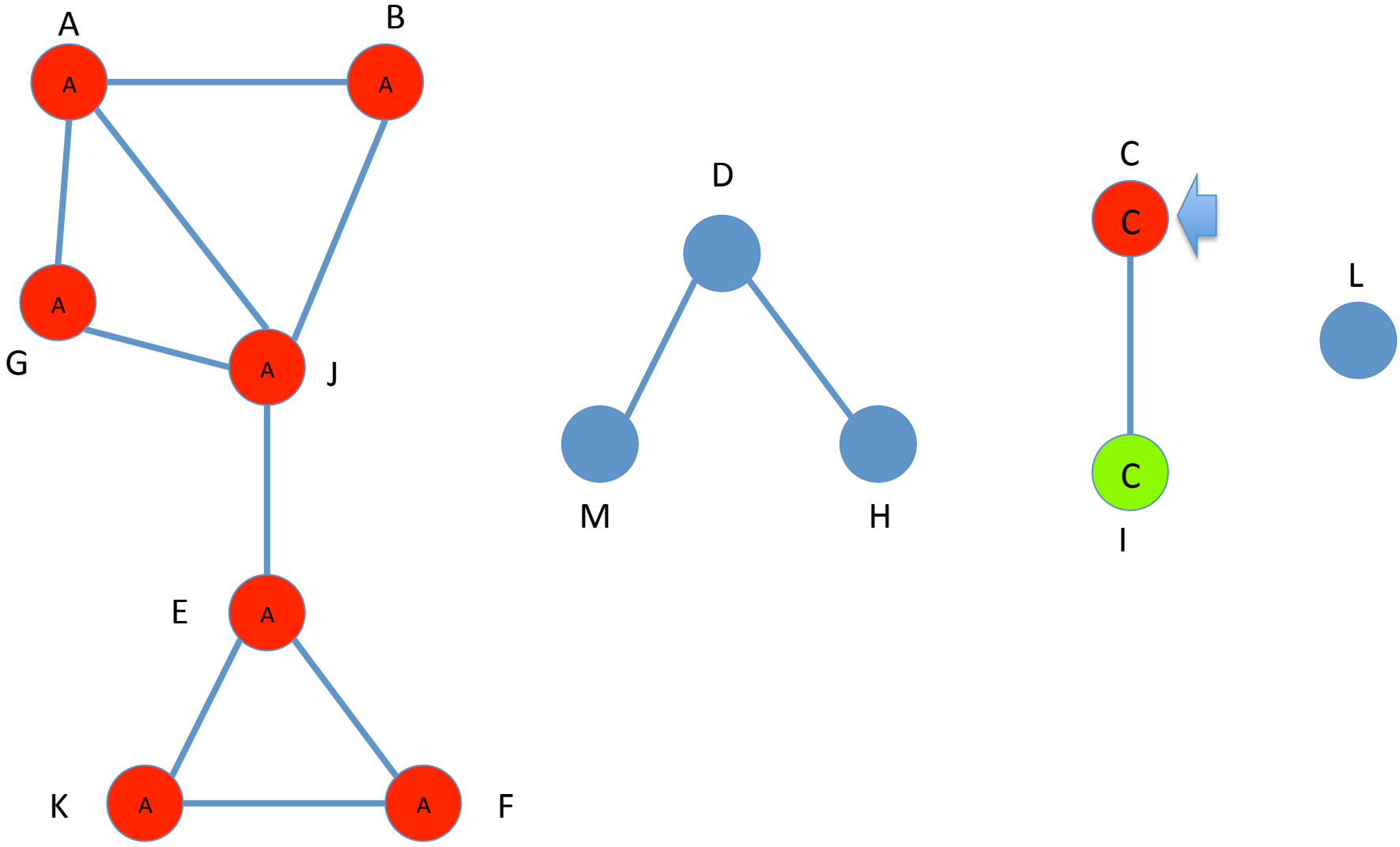
BFS - Con. Comp. w/ Label Propagation



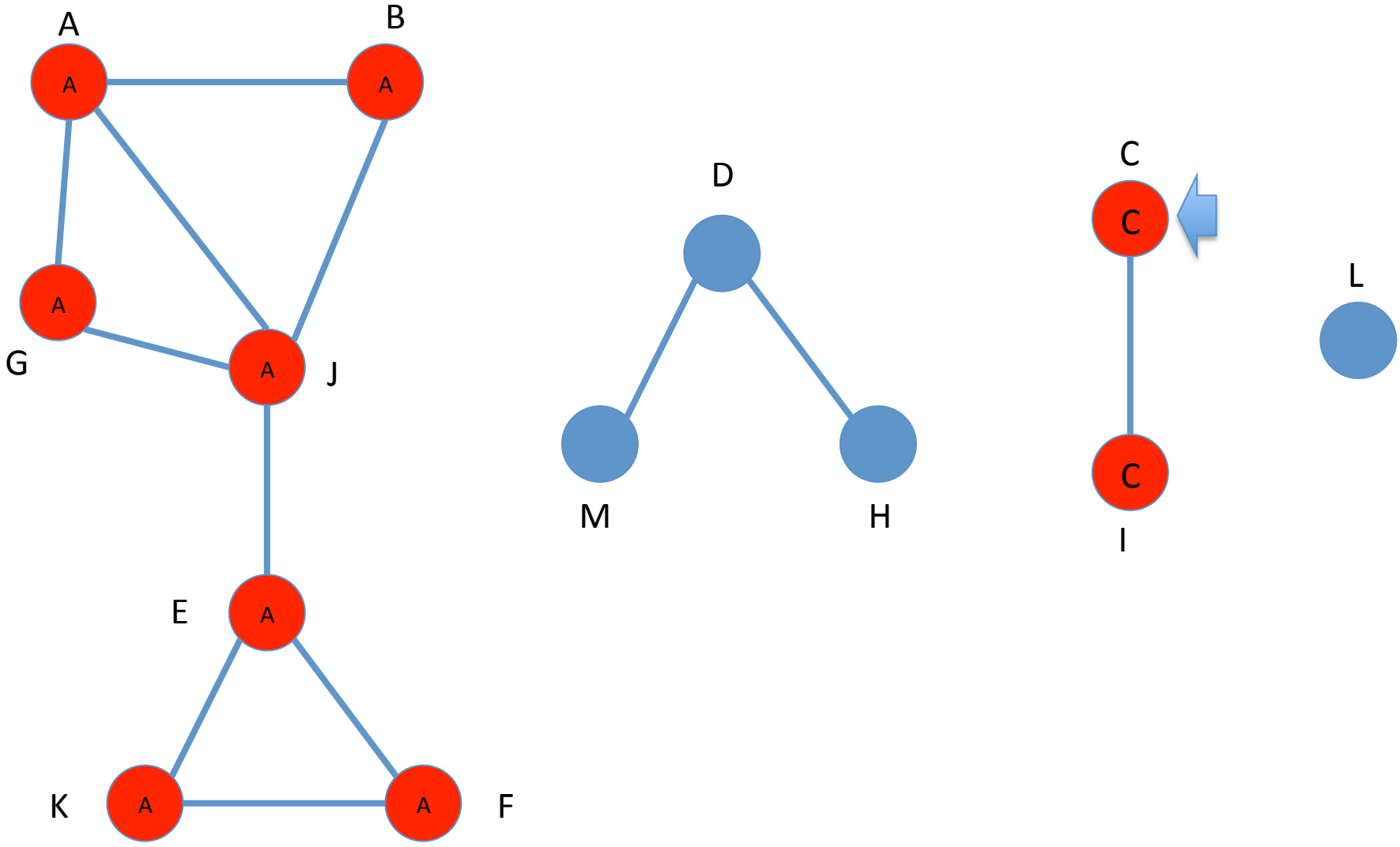
BFS - Con. Comp. w/ Label Propagation



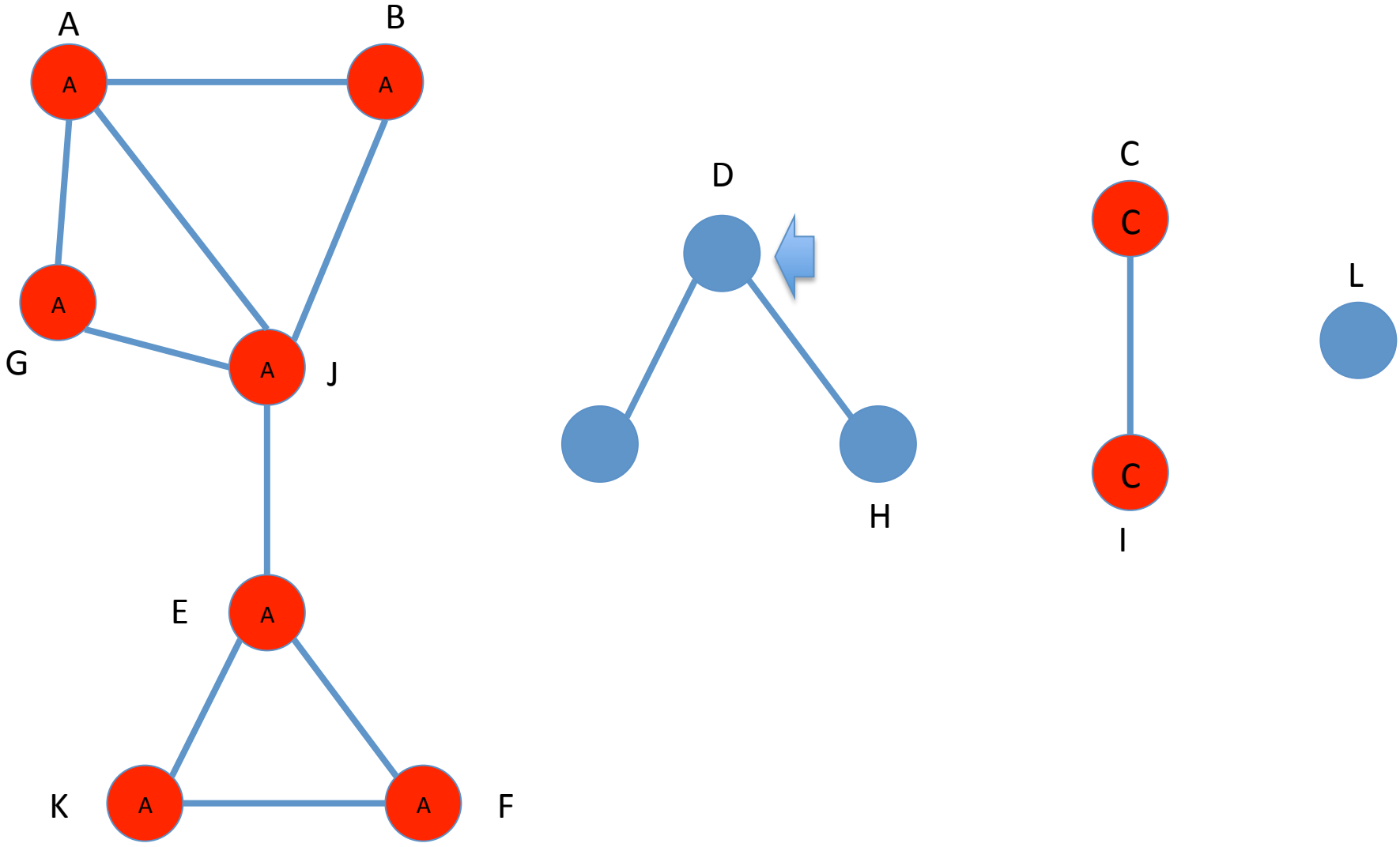
BFS - Con. Comp. w/ Label Propagation



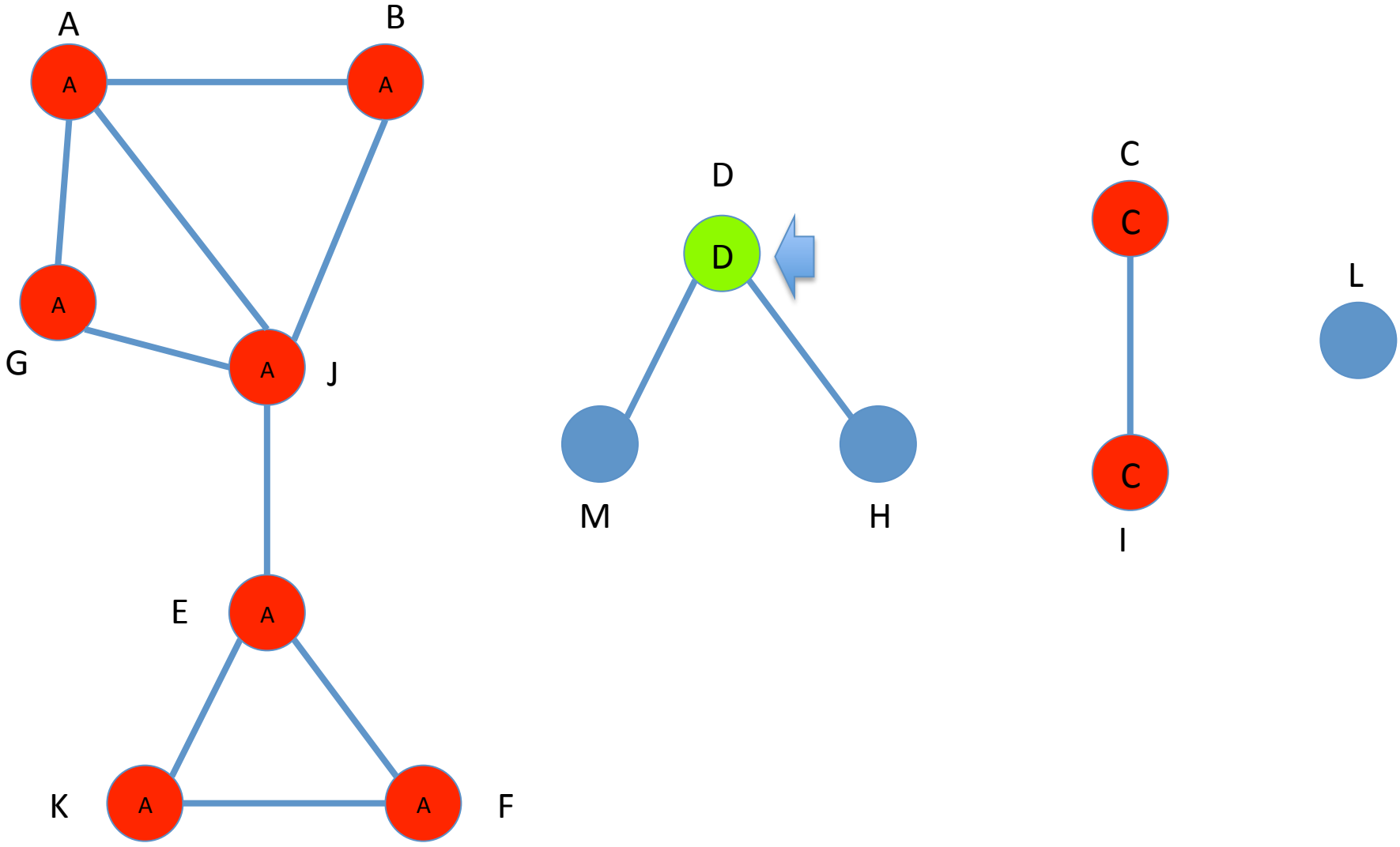
BFS - Con. Comp. w/ Label Propagation



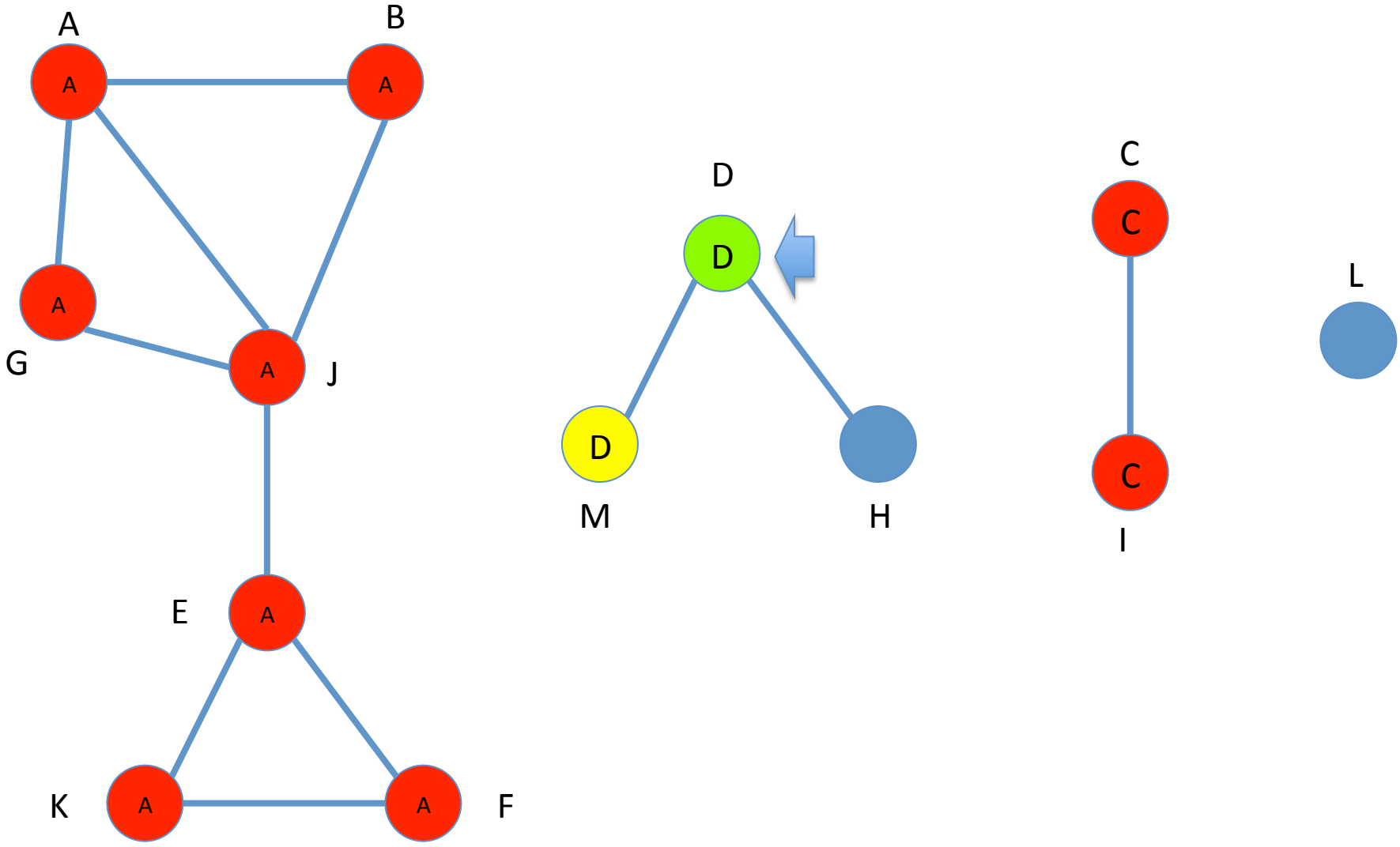
BFS - Con. Comp. w/ Label Propagation



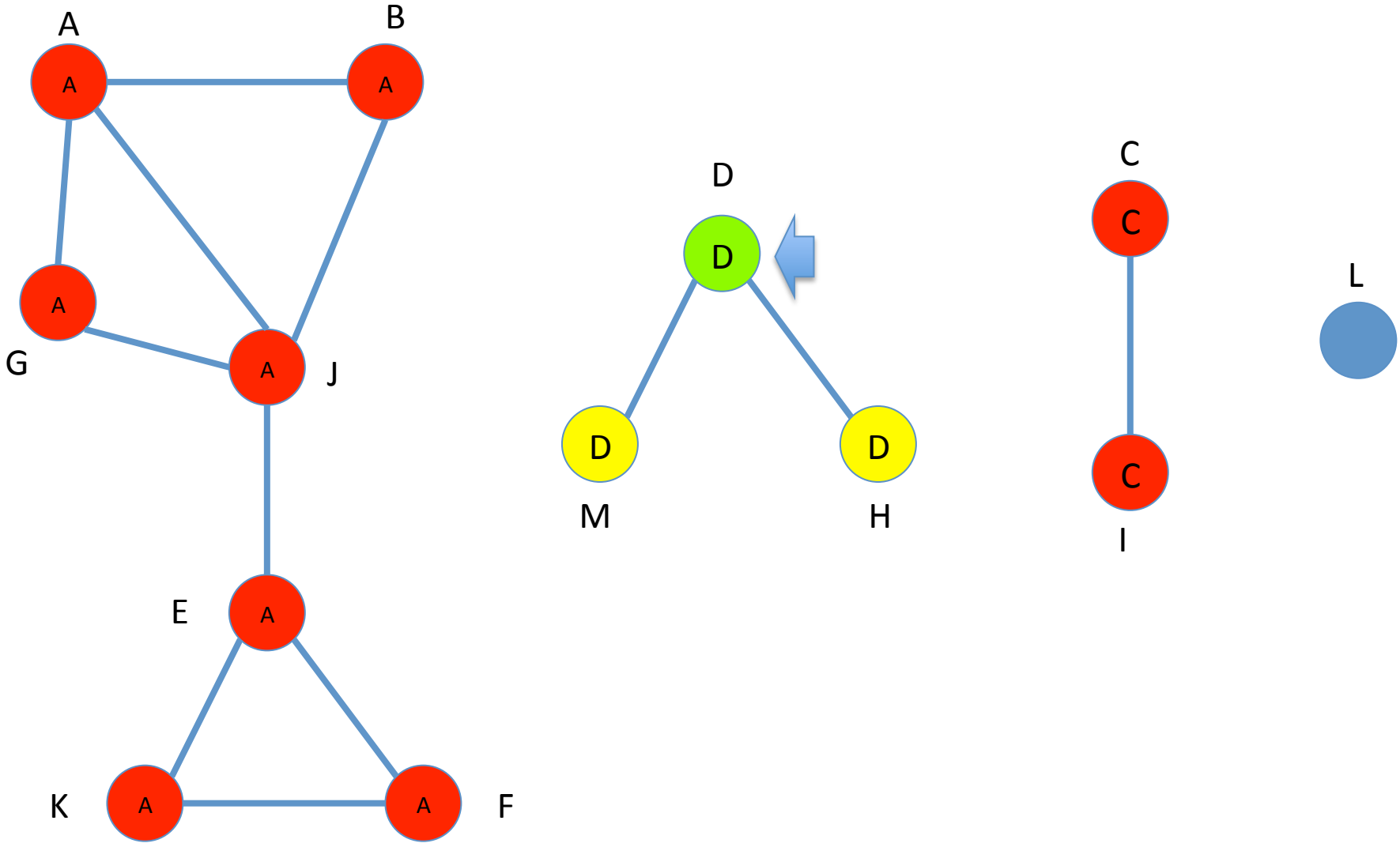
BFS - Con. Comp. w/ Label Propagation



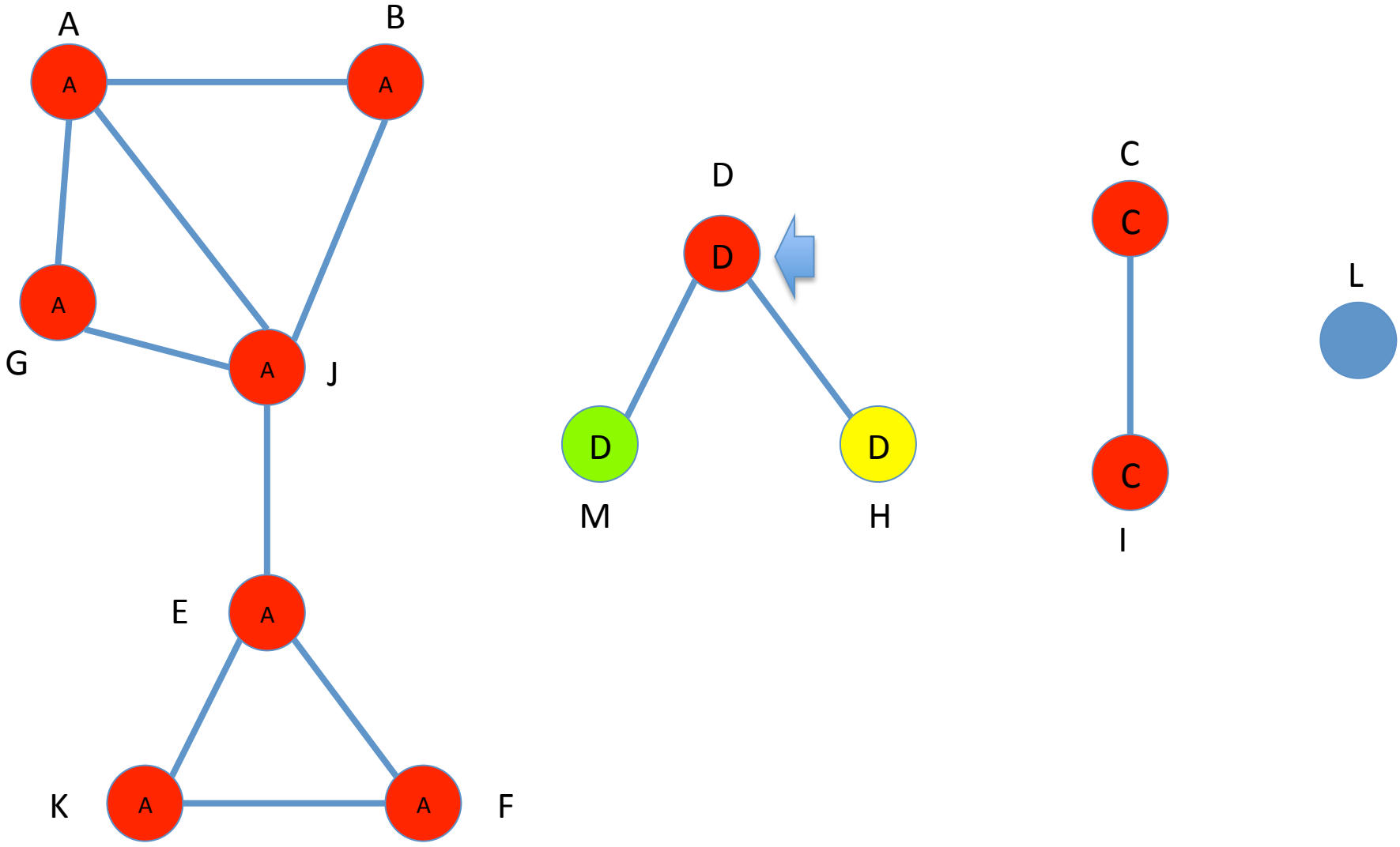
BFS - Con. Comp. w/ Label Propagation



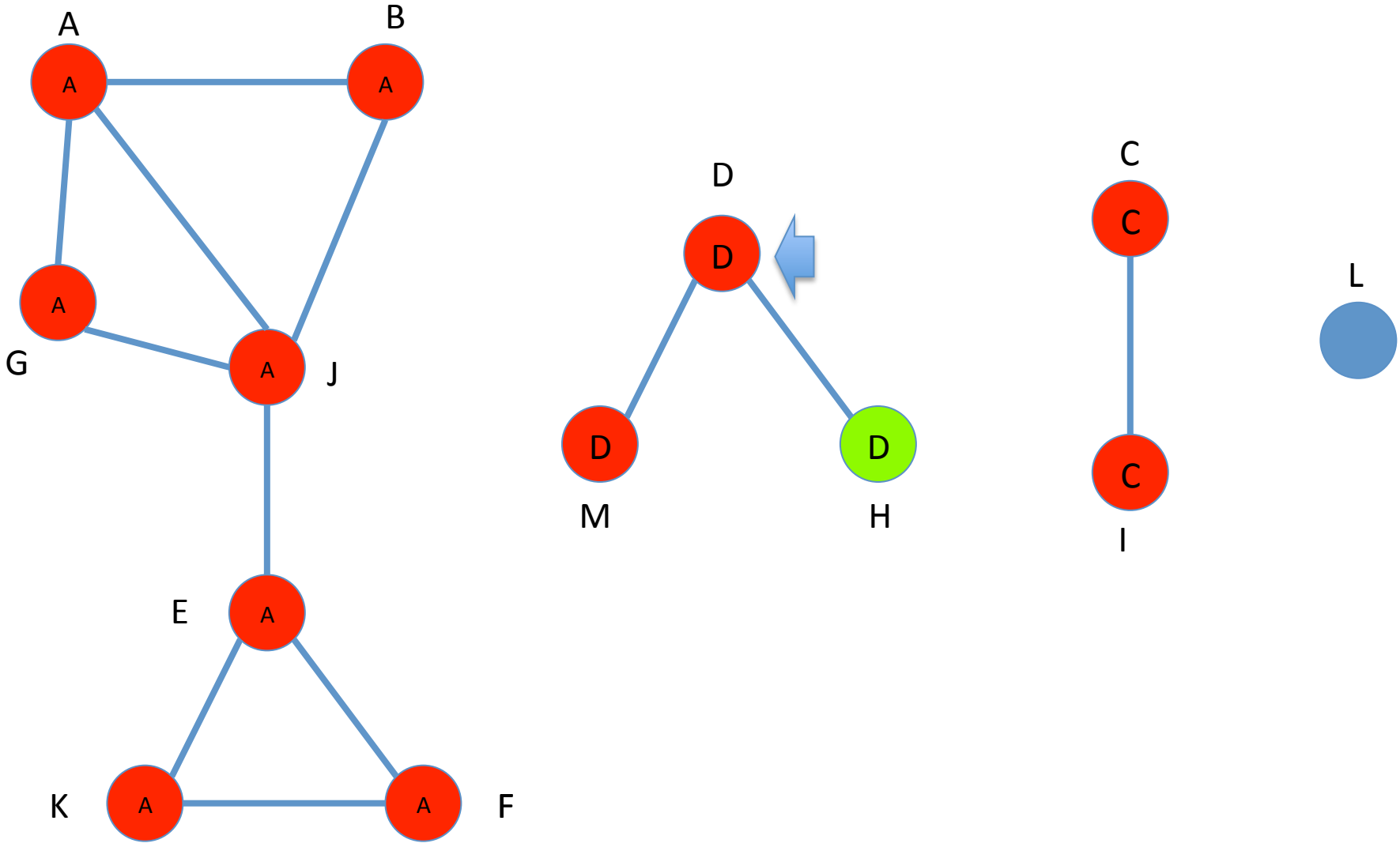
BFS - Con. Comp. w/ Label Propagation



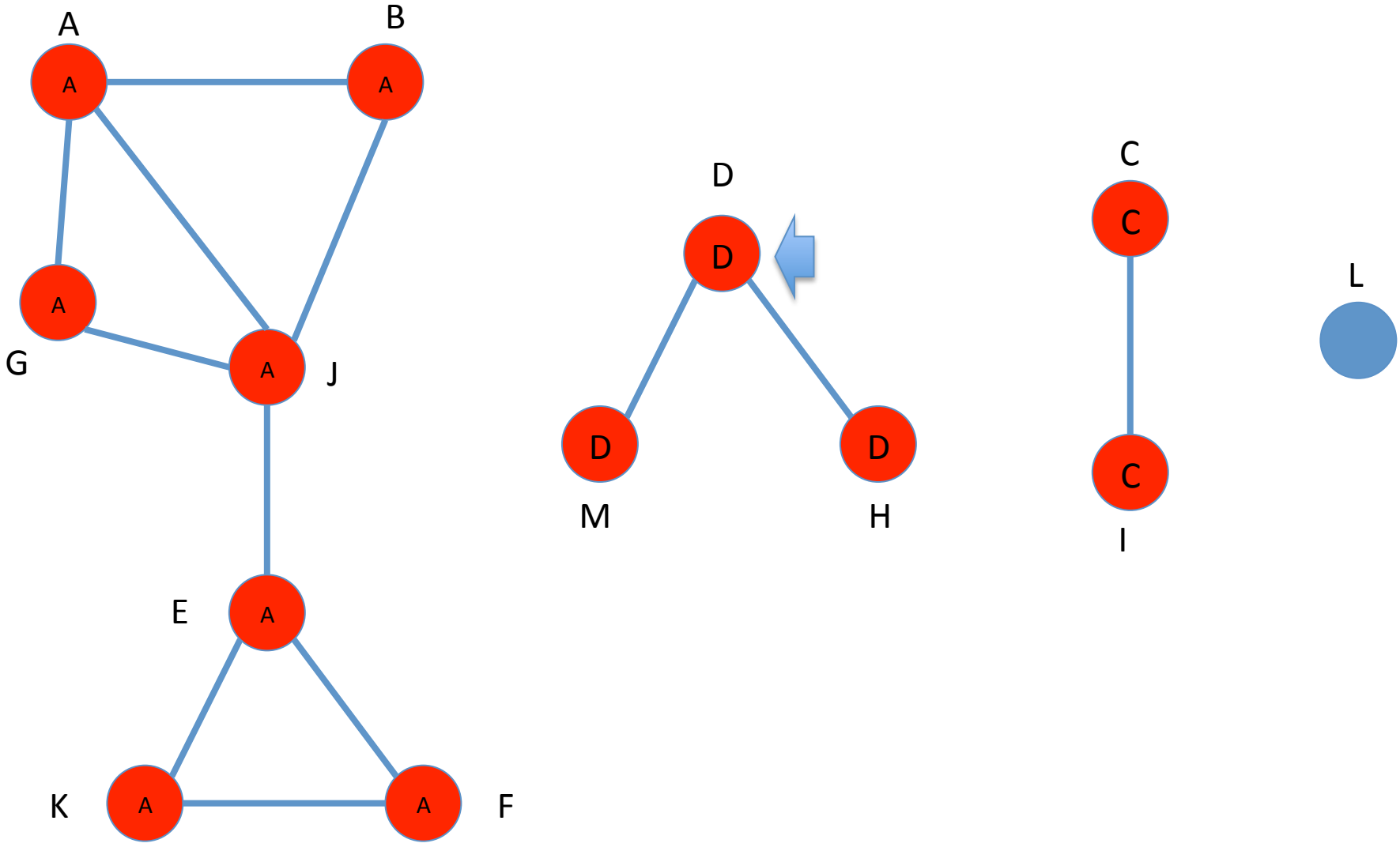
BFS - Con. Comp. w/ Label Propagation



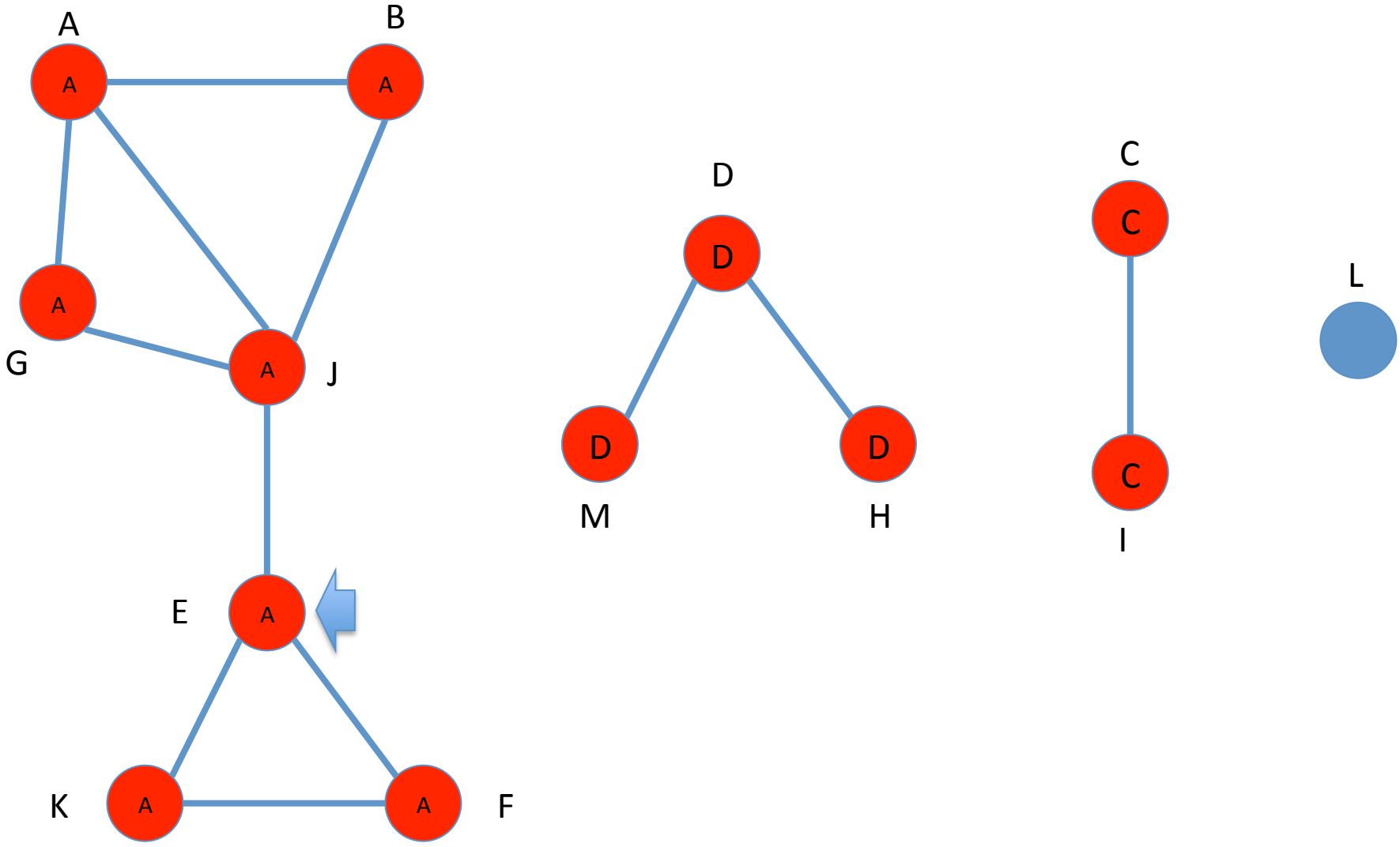
BFS - Con. Comp. w/ Label Propagation



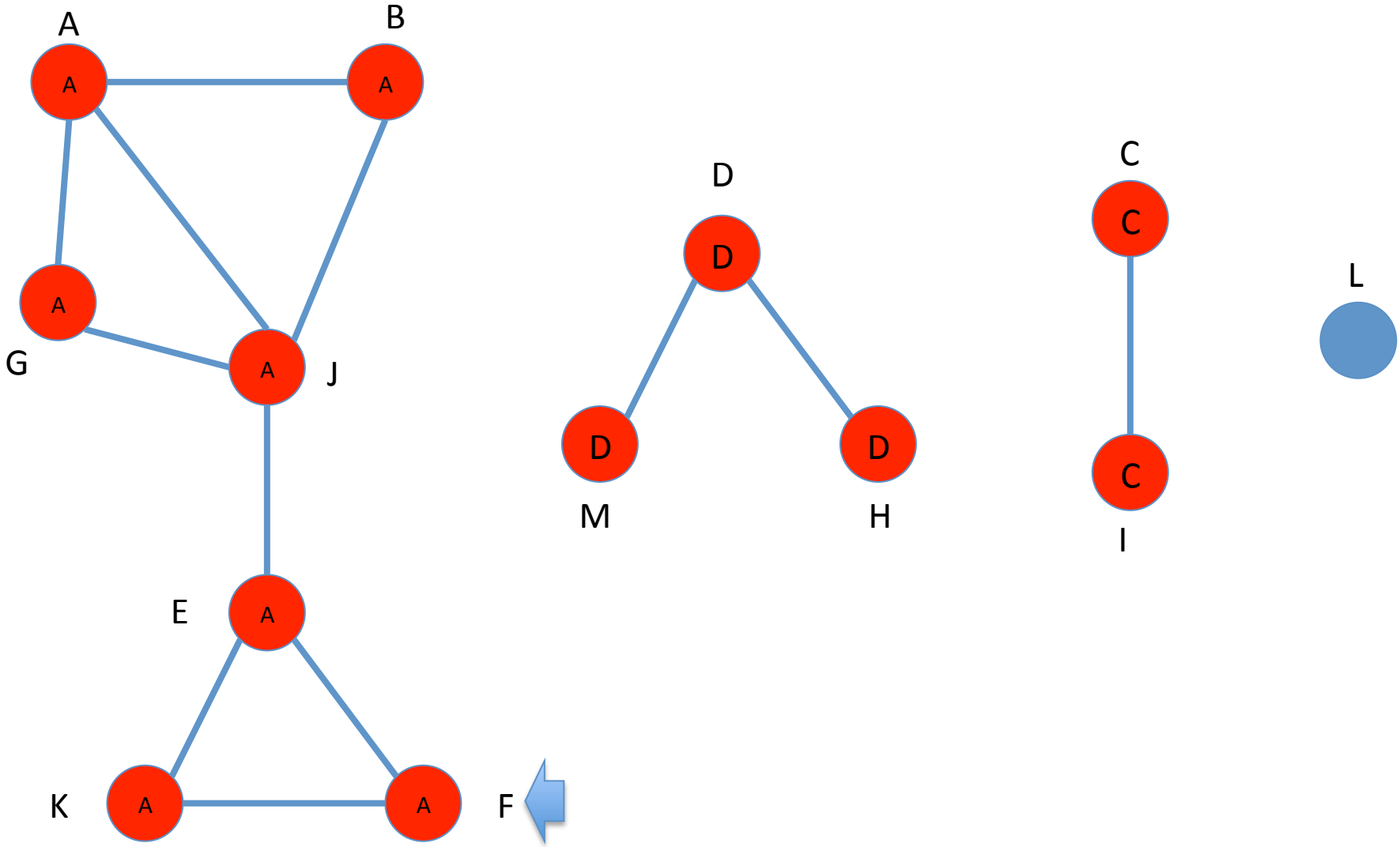
BFS - Con. Comp. w/ Label Propagation



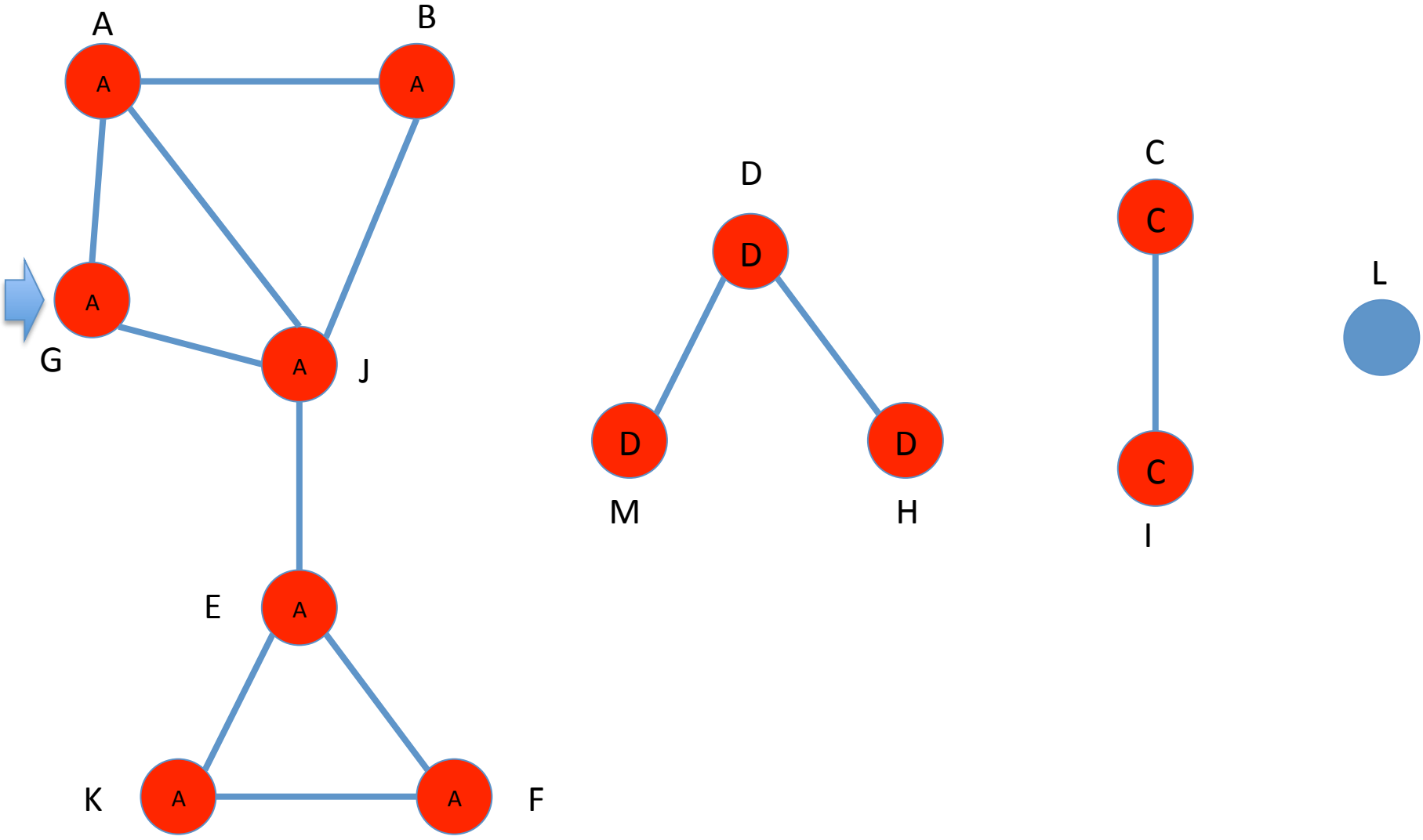
BFS - Con. Comp. w/ Label Propagation



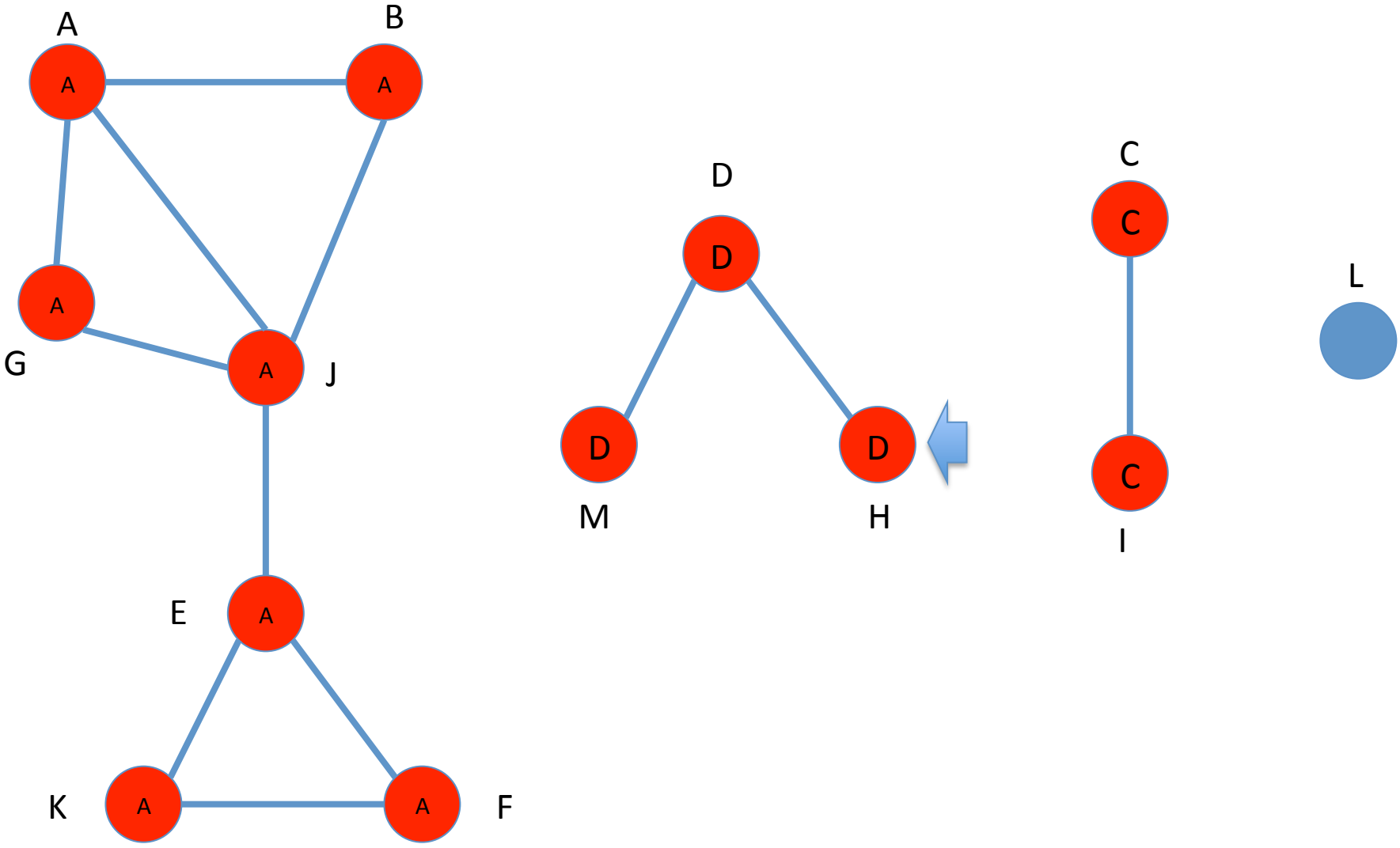
BFS - Con. Comp. w/ Label Propagation



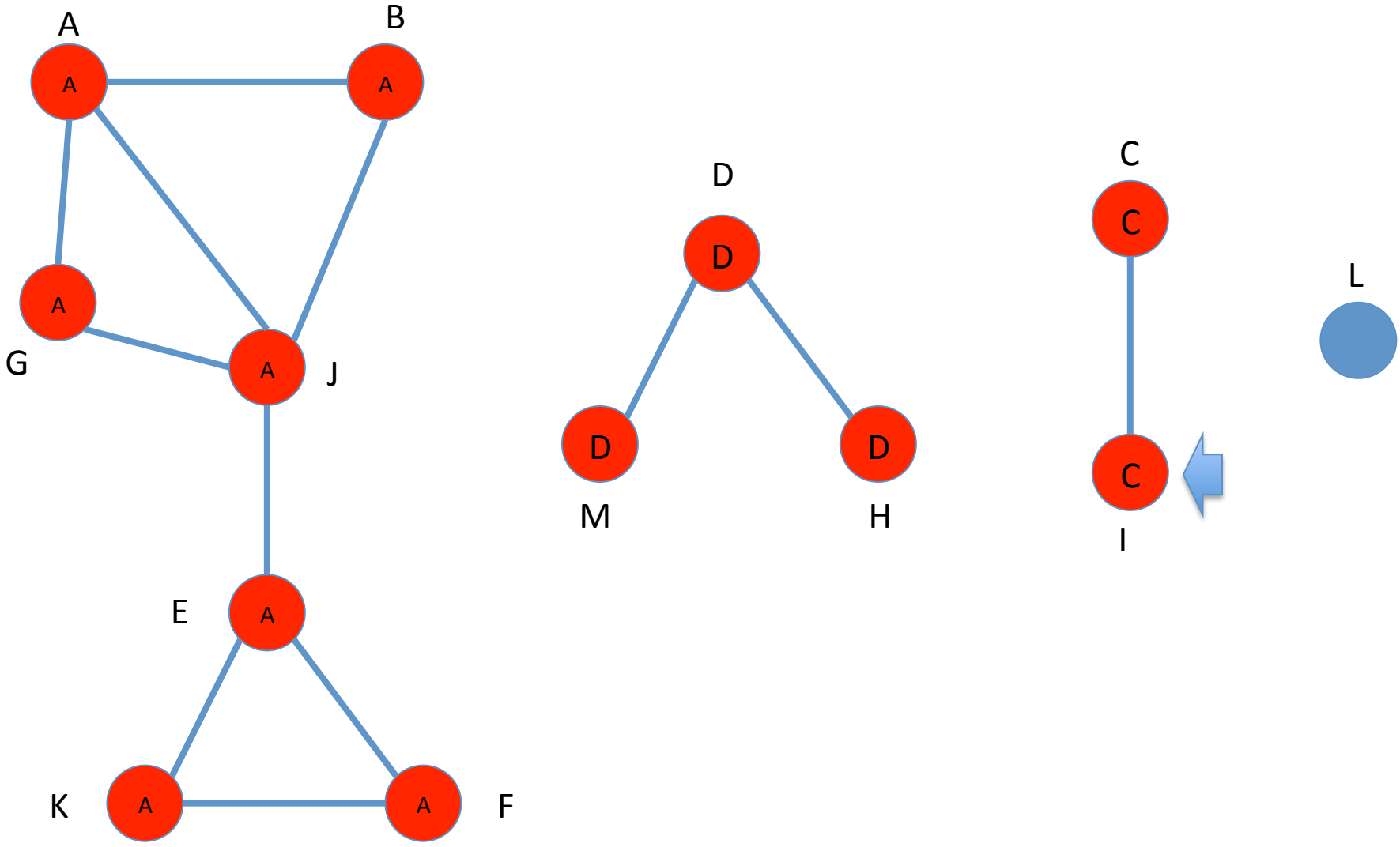
BFS - Con. Comp. w/ Label Propagation



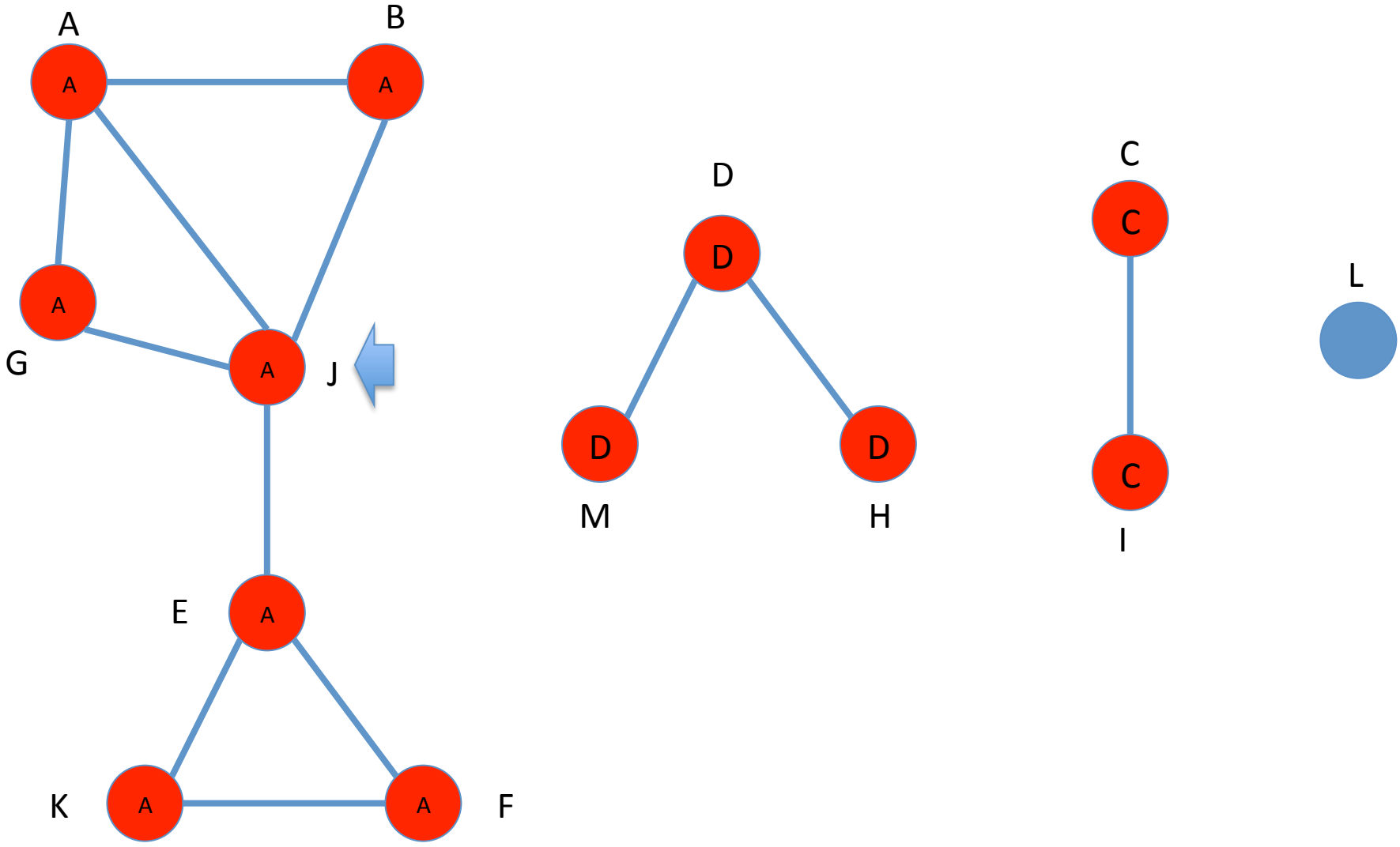
BFS - Con. Comp. w/ Label Propagation



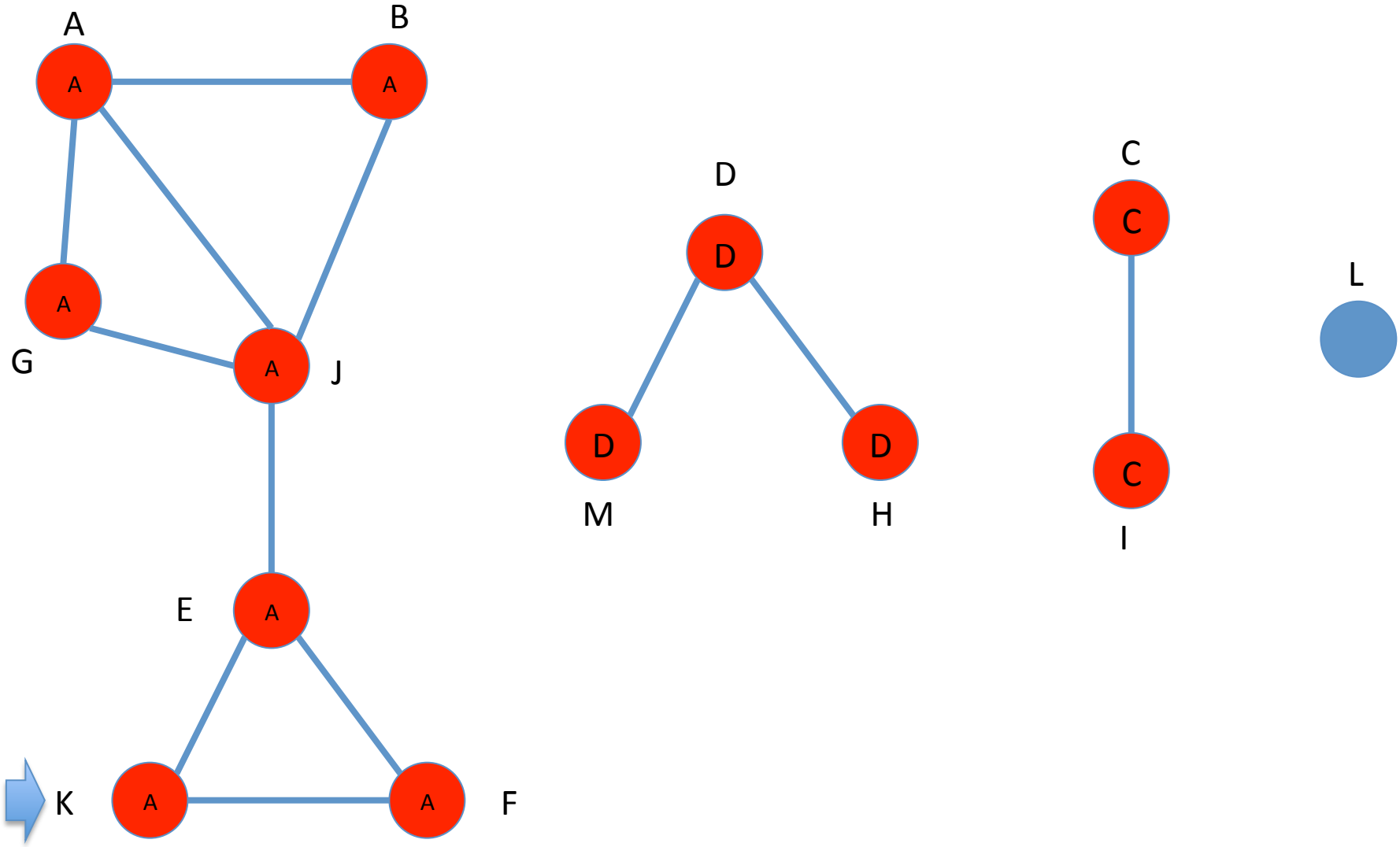
BFS - Con. Comp. w/ Label Propagation



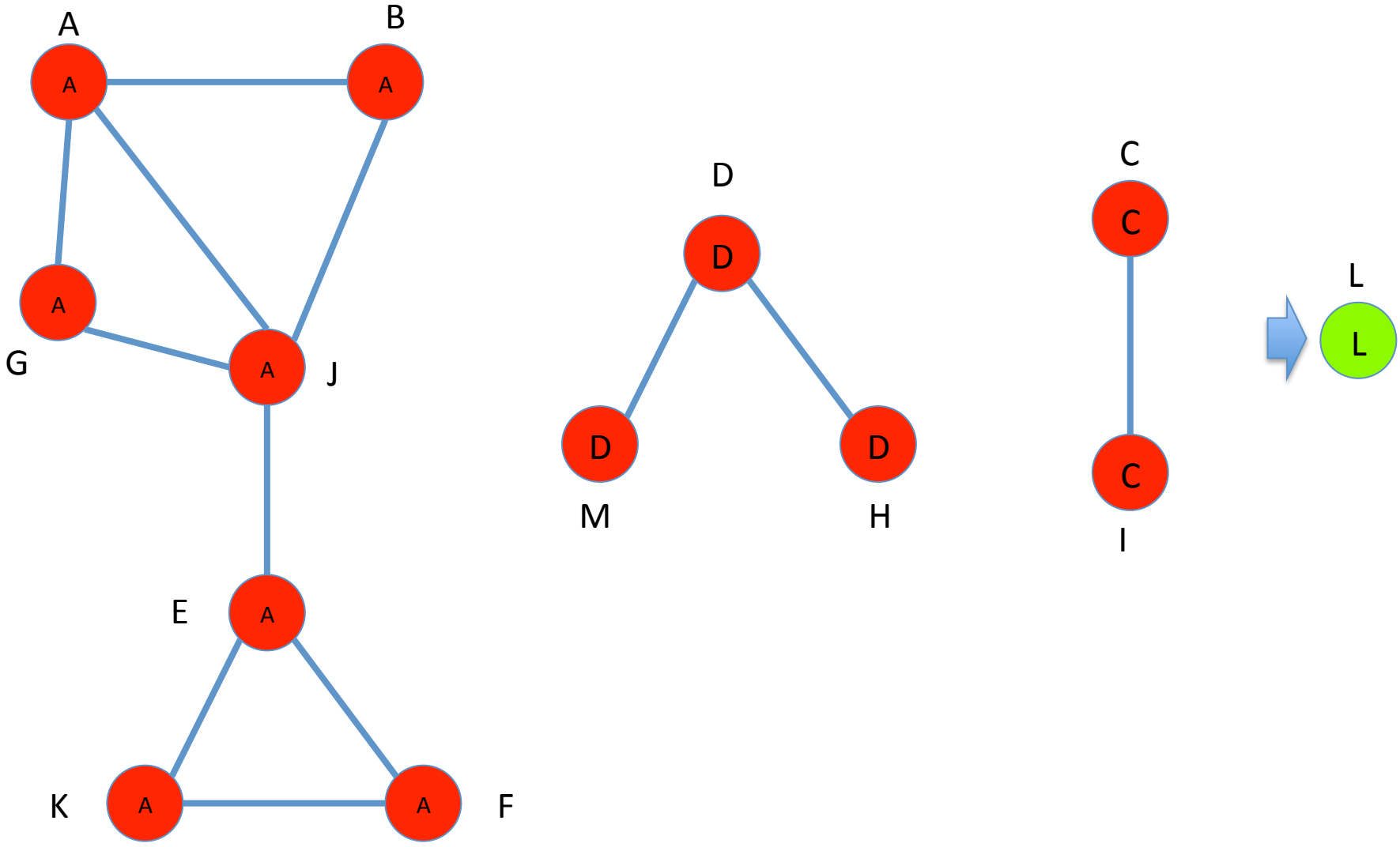
BFS - Con. Comp. w/ Label Propagation



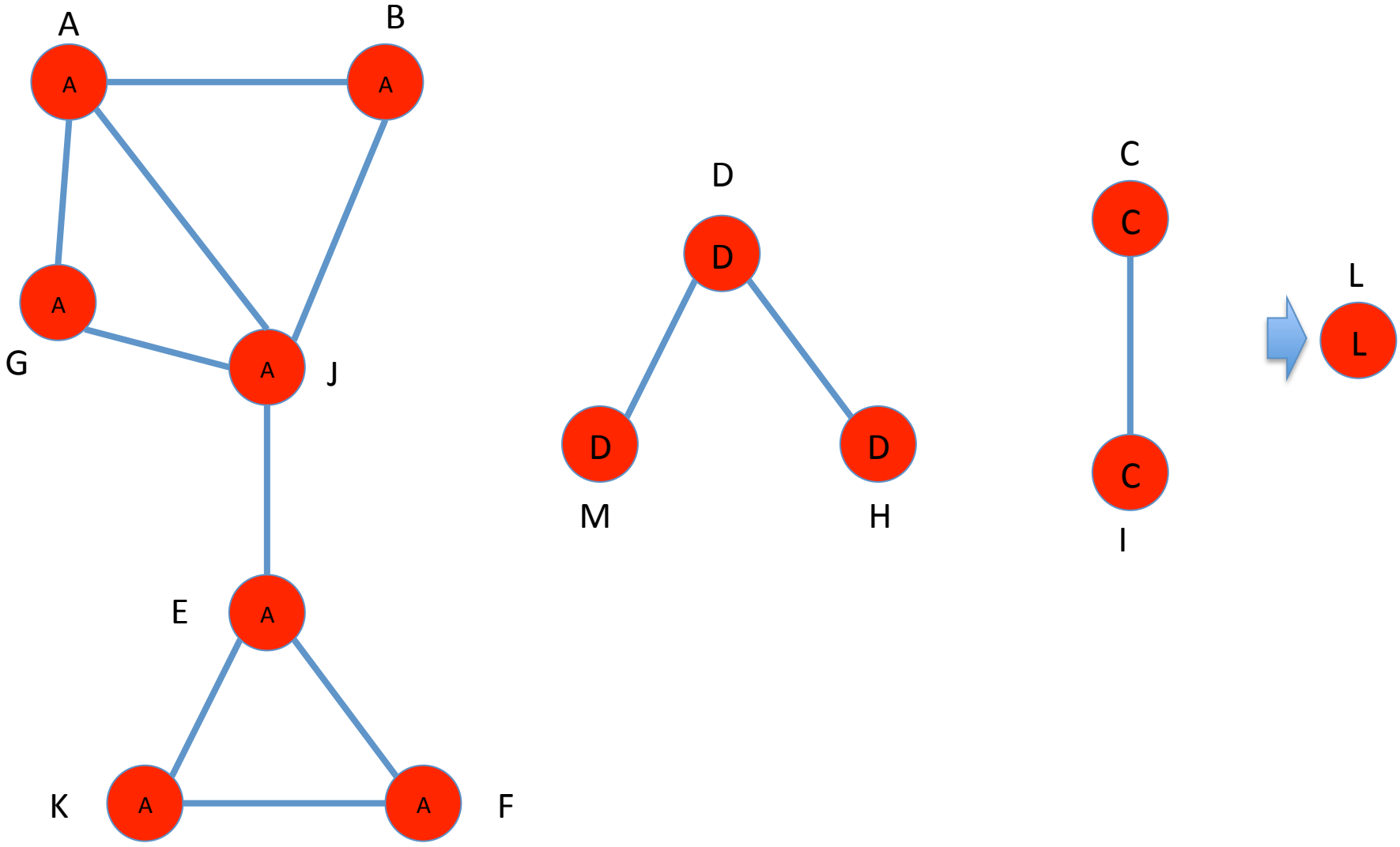
BFS - Con. Comp. w/ Label Propagation



BFS - Con. Comp. w/ Label Propagation



BFS - Con. Comp. w/ Label Propagation



Run BFS Just With Label Propagation

```
1.  procedure BFS-LP(G(V, E), s)
2.     let Q be a new queue;
3.     mark s visited
4.     label s with s
5.     enqueue(s, Q)
6.     while (Q not empty):
7.        let v = dequeue(Q)
8.        for each neighbor u of v:
9.           if (u is not-visited):
10.             mark u as visited;
11.             label u with s
12.             enqueu(u, Q)
13.       mark v as finished
```

Runtime: O(n+m)

(with adj list)

```
1.  procedure BFS-LP(G(V, E))
2.     mark all vertices as not-visited
3.     for each v ∈ V, if v is not-visited do BFS-LP(G, v)
```

Undirected Connected Components

Note that using BFS was not critical.

We could have easily propagated the labels

using DFS.

Outline For Today

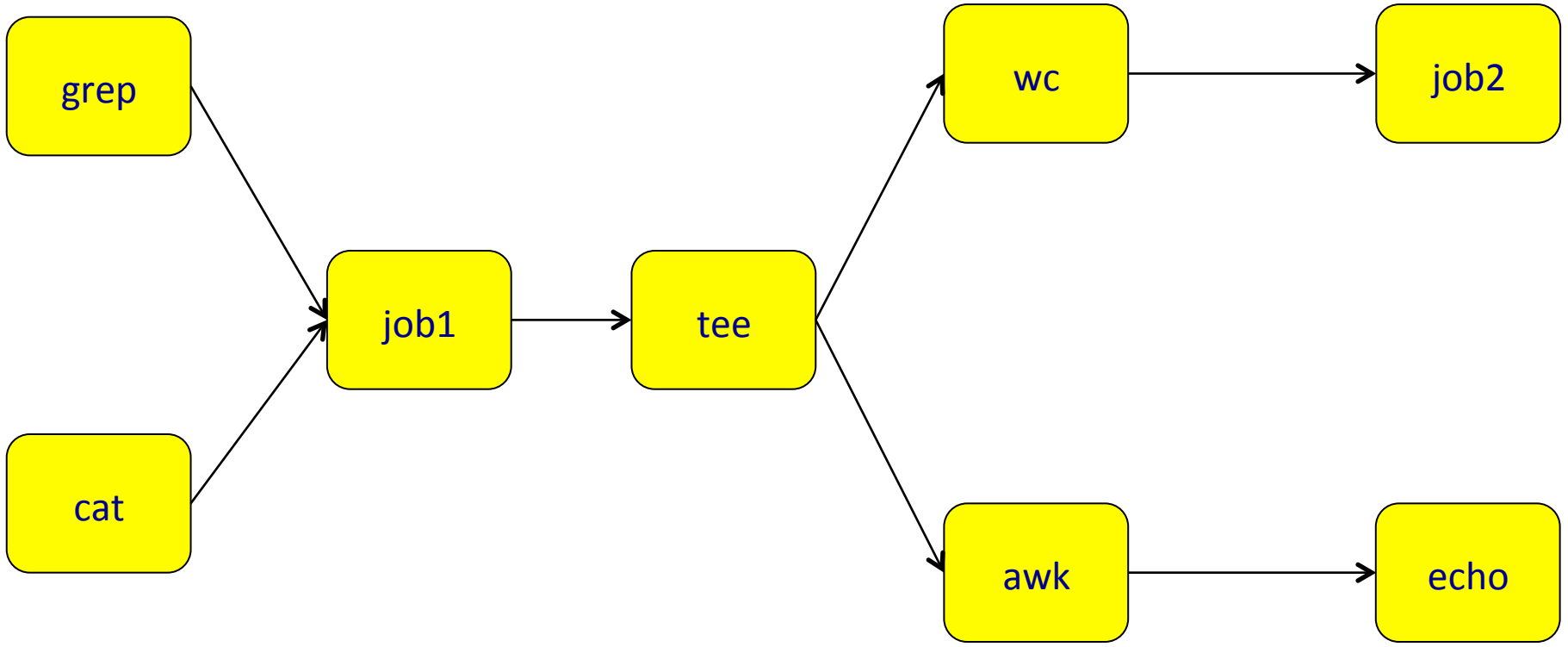
1. Graph Terminology
2. BFS/DFS & BFS/DFS Tree
3. Application 1: Unweighted Single Source Shortest Paths
4. Application 2: Bipartiteness/2-coloring
5. Application 3: Connected Comp. in Undir. Graphs
6. Application 4: Topological Sort

Directed Acyclic Graphs (DAGs)

- Graphs to model “dependency”/“prerequisite” relationships
  - directed & acyclic

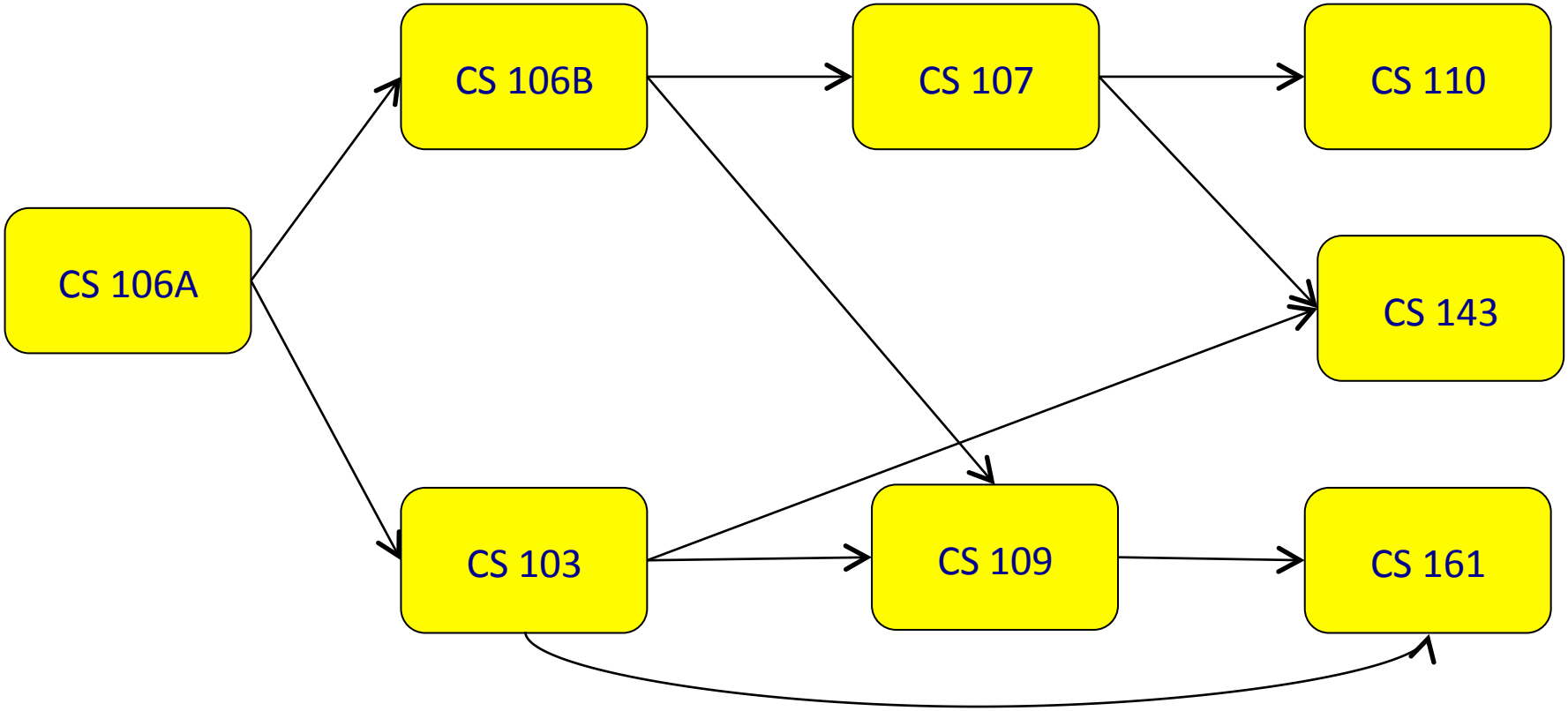
Example #1

- Job Scheduling in Operating Systems



Example #2

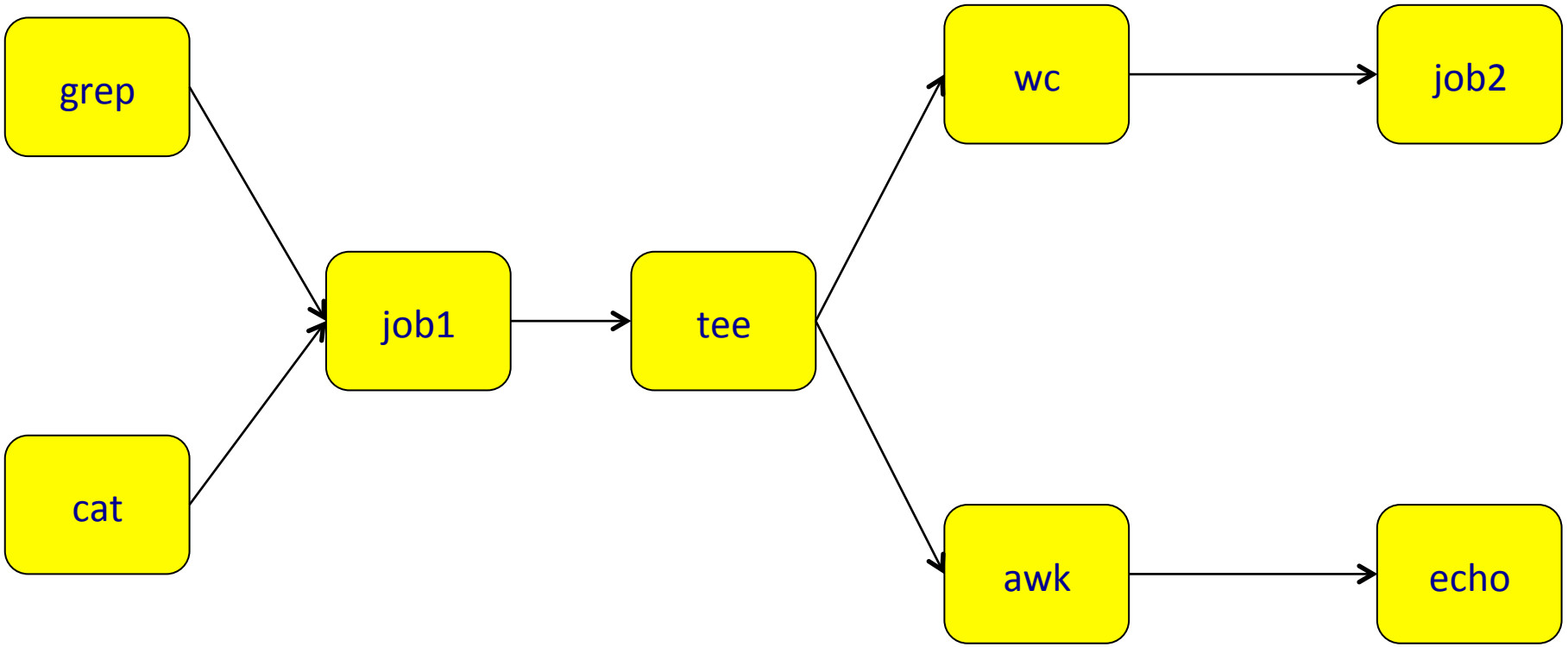
- CS Course Dependencies



Why Acyclic?

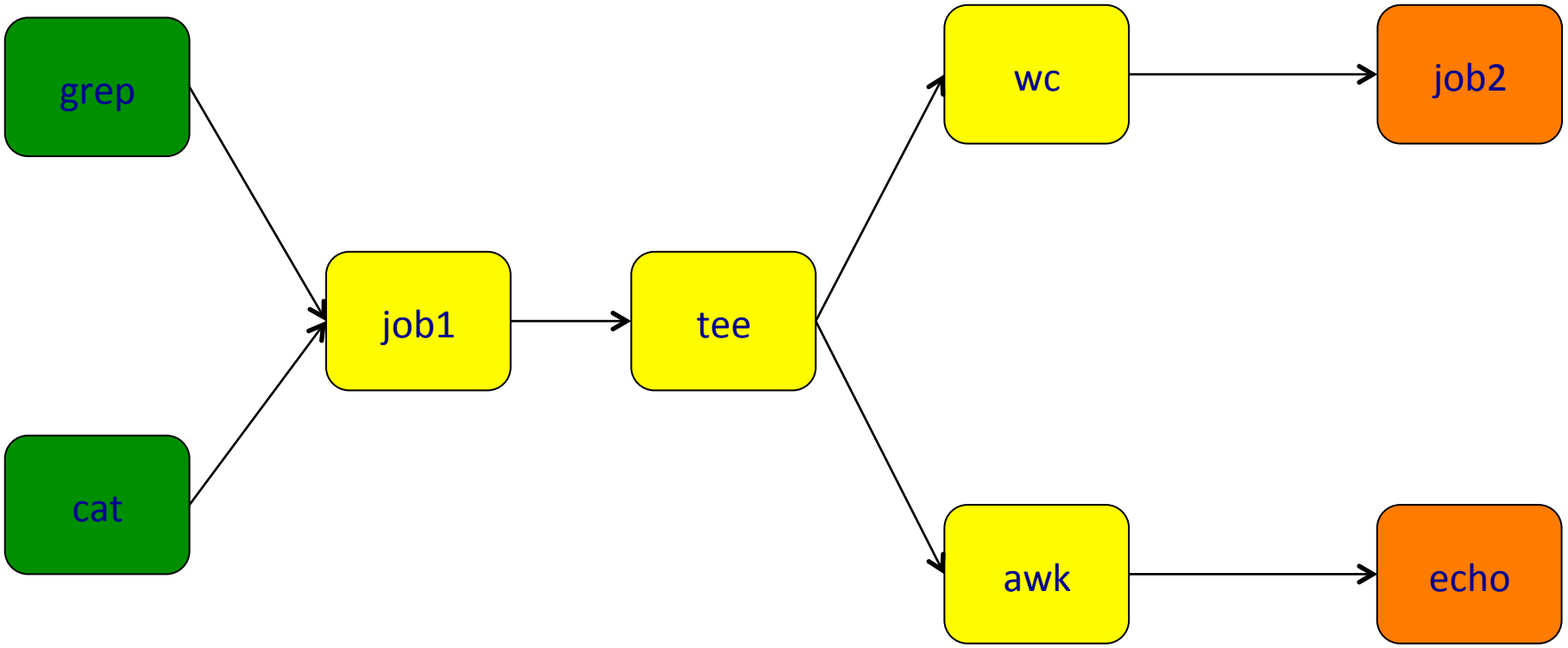
DAG Terminology

- source: no incoming edges
- sink: no outgoing edges



DAG Terminology

- source: no incoming edges
- sink: no outgoing edges

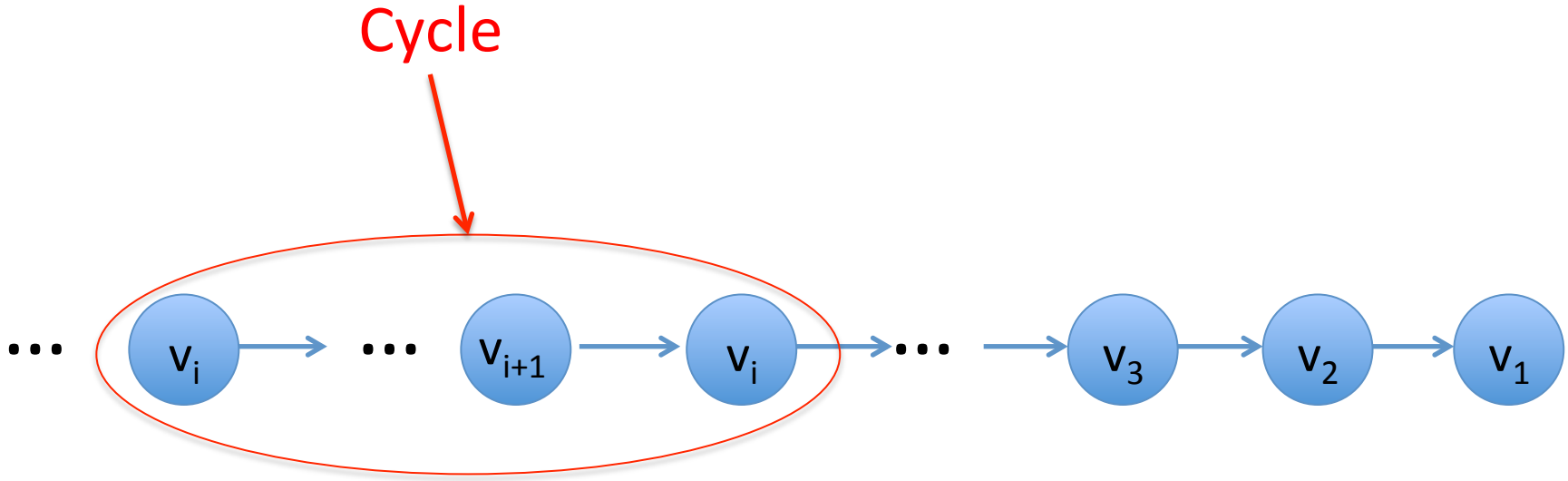


DAG Properties

- Property 1: Every DAG has at least 1 source
- Proof:

  Assume contrary: then every v has at least 1 incoming edge

  Start at v_1 and follow, repeatedly, one of v_1's in-edges



- Property 2: Every DAG has at least 1 sink: (similar proof)

Topological Sorting of a DAG

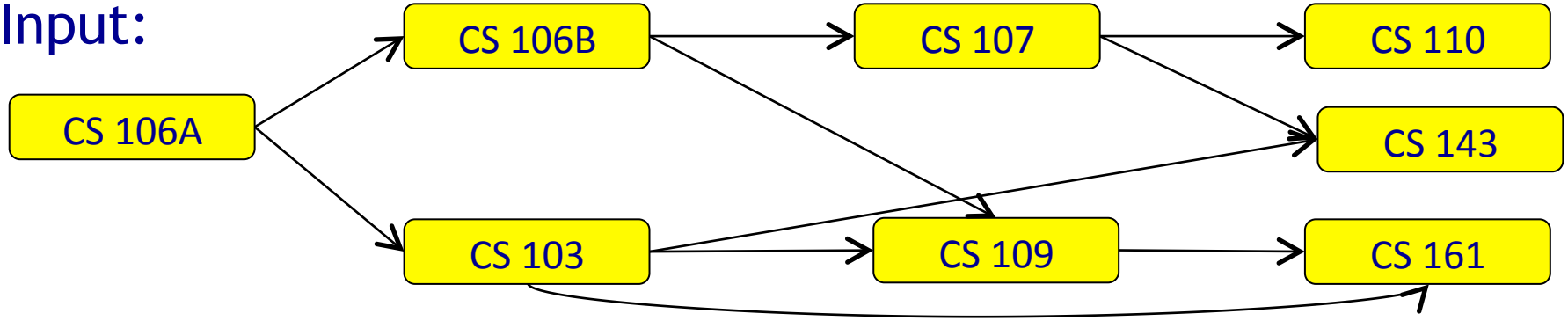
- Input: A G(V, E) which is a DAG
- Output: vertices in sorted order s.t
  - each vertex v appears after its dependencies
- A More Formal Definition:

Given an input DAG G(V, E) order the nodes such that

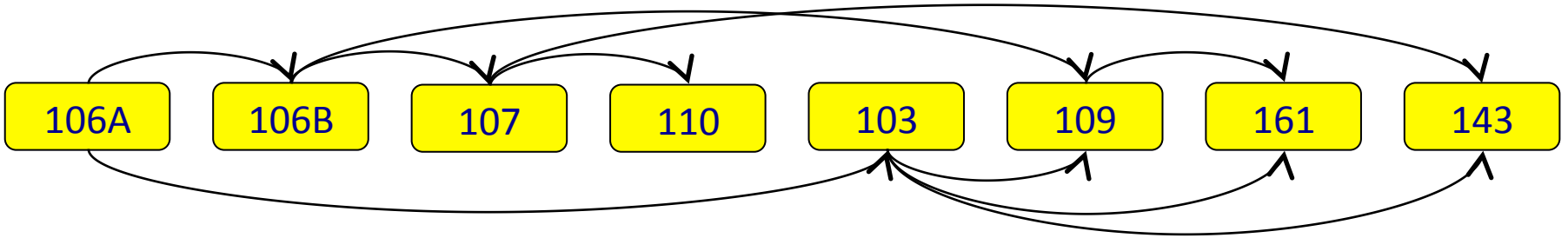
if (u, v) $\in$ E, then u appears before v

Example

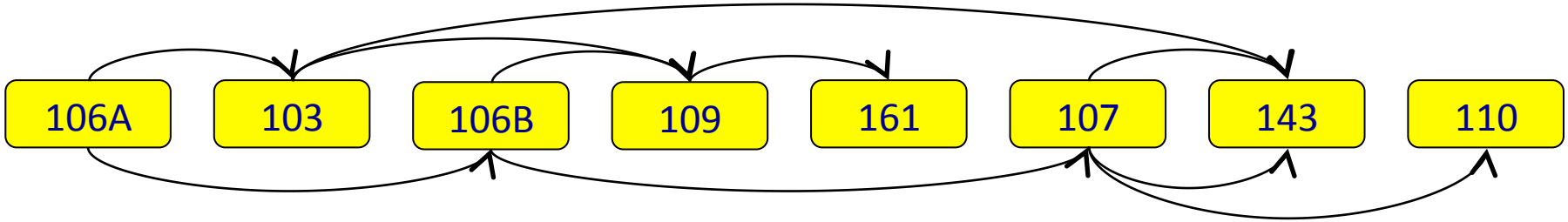
Input:



Output:



OR



OR…

Output is not unique

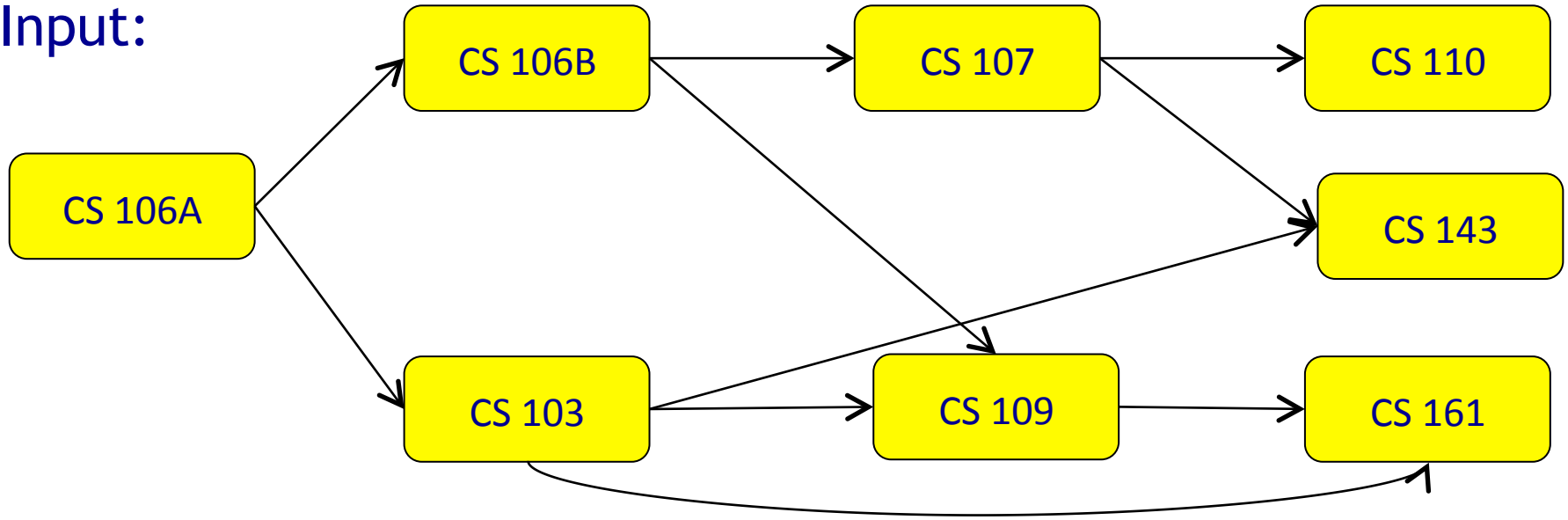
Topological Sorting Algorithm #1

```
procedure topologicalSort1(DAG G):
    let result empty list
    while G is not empty:
        1. let v be a source node in G
        2. add v to result
        3. remove v from G
    return result/reverse
```

Note: You can also pick a sink iteratively!

Algorithm #1 Simulation

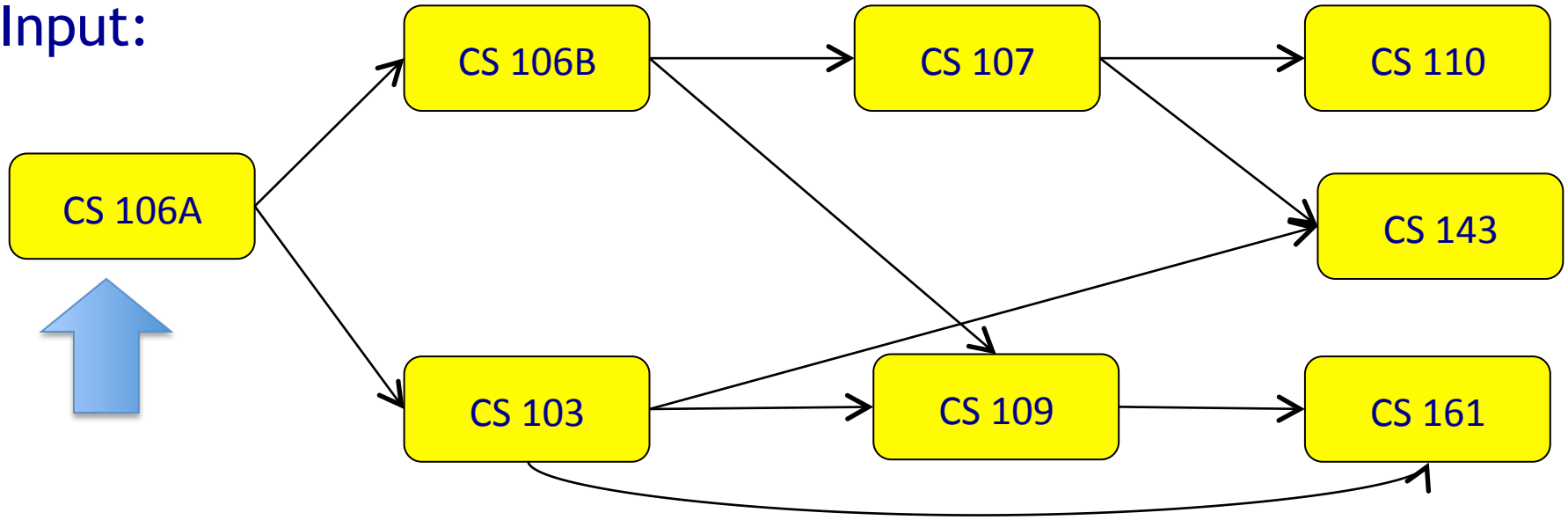
Input:



Output:

Algorithm #1 Simulation

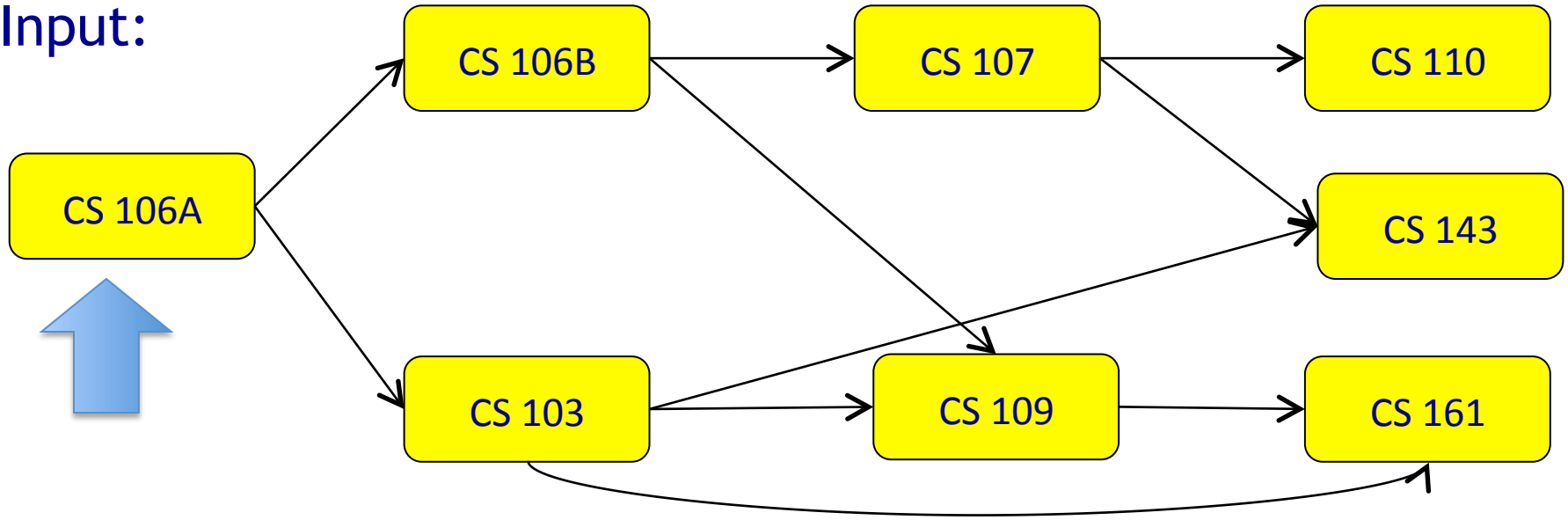
Input:



Output:

Algorithm #1 Simulation

Input:

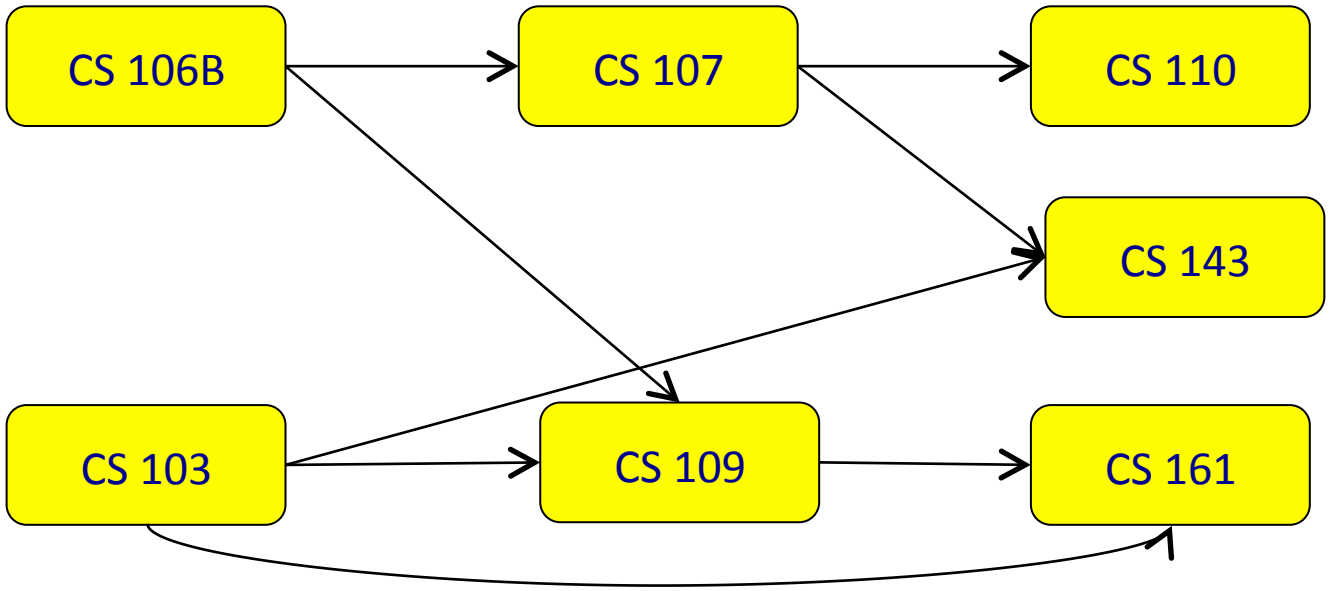


Output:

106A

Algorithm #1 Simulation

Input:

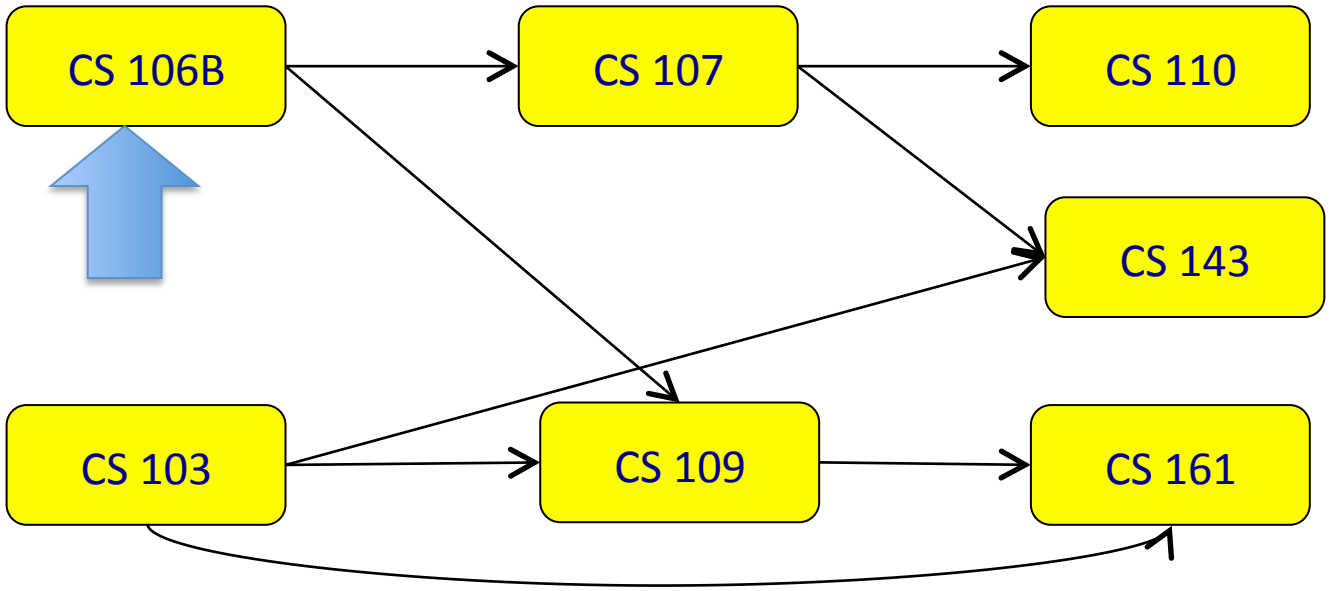


Output:

106A

Algorithm #1 Simulation

Input:

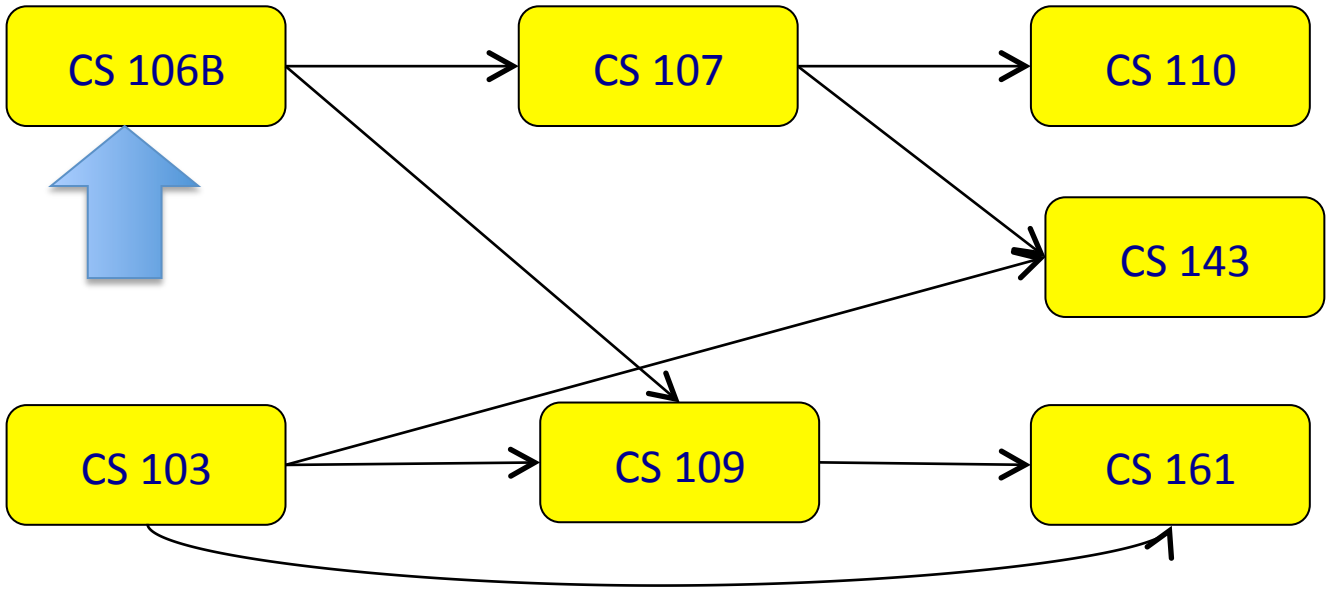


Output:

106A

Algorithm #1 Simulation

Input:



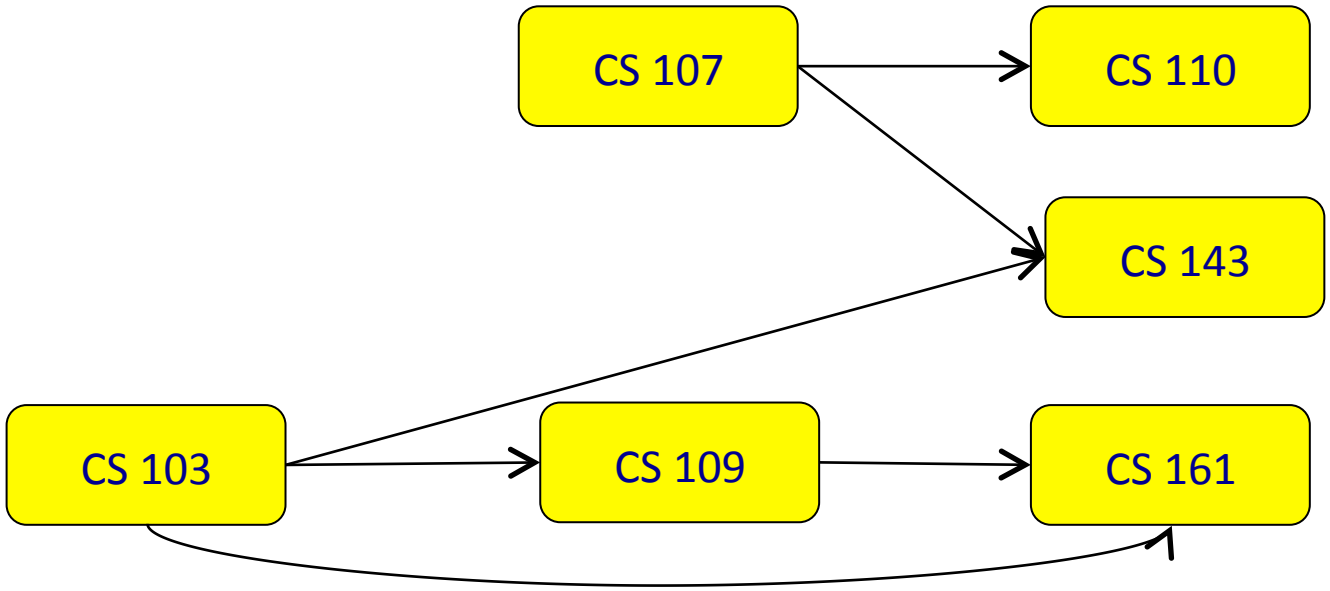
Output:

106A

106B

Algorithm #1 Simulation

Input:



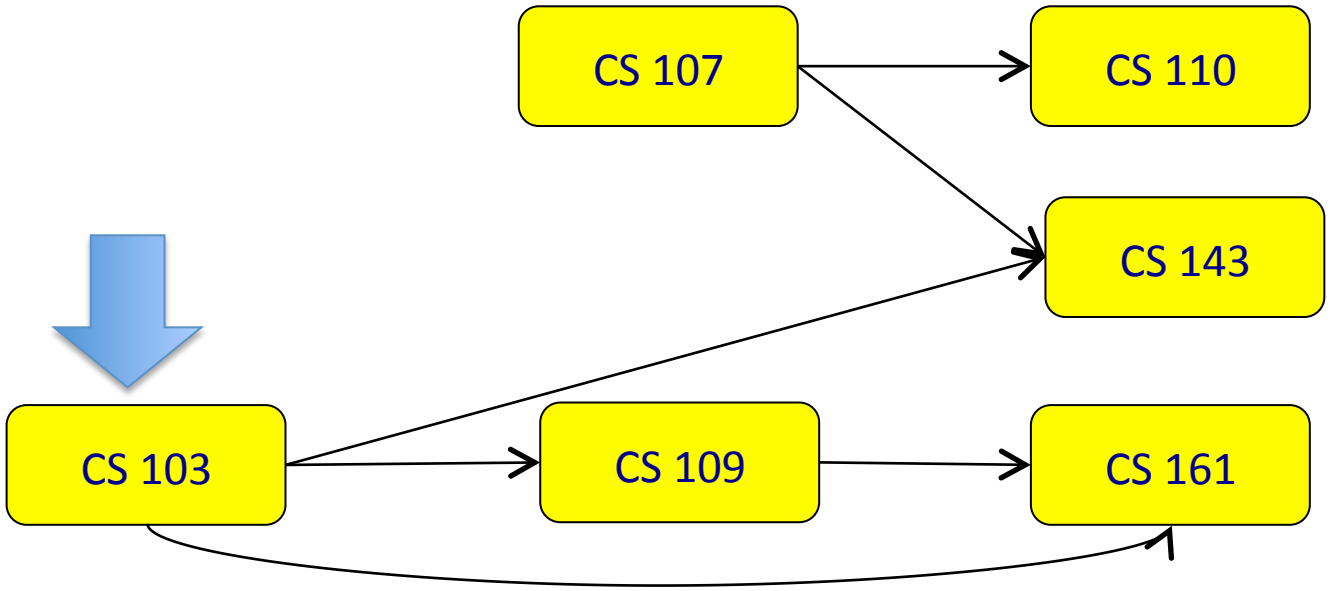
Output:

106A

106B

Algorithm #1 Simulation

Input:



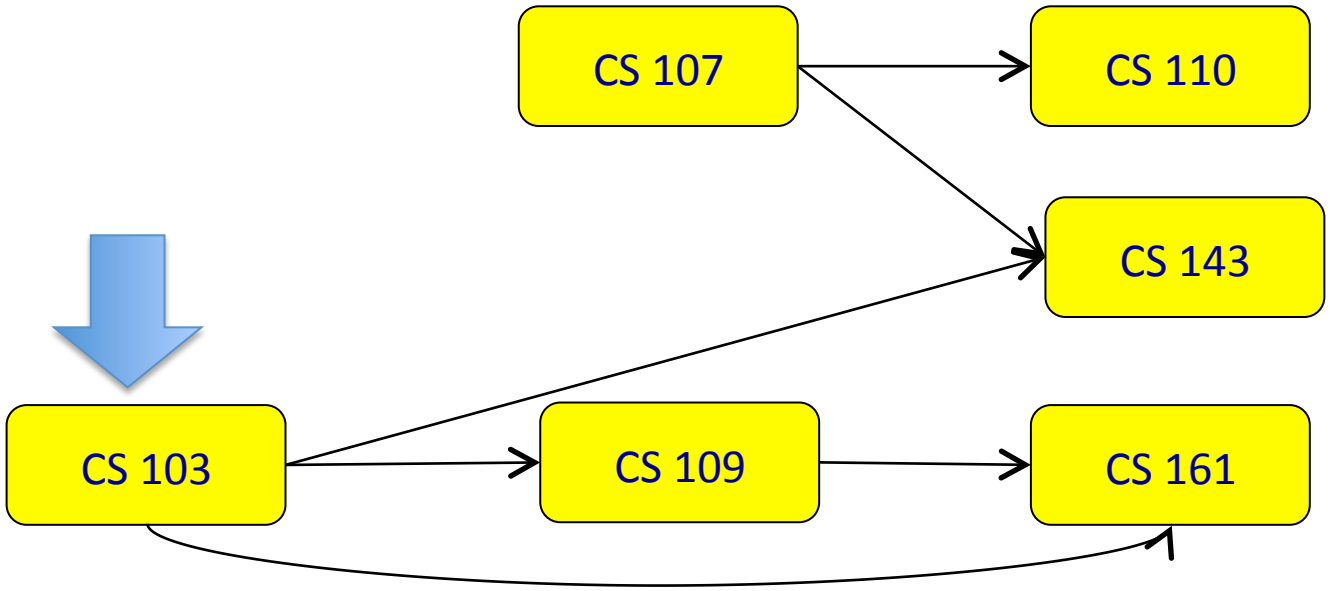
Output:

106A

106B

Algorithm #1 Simulation

Input:

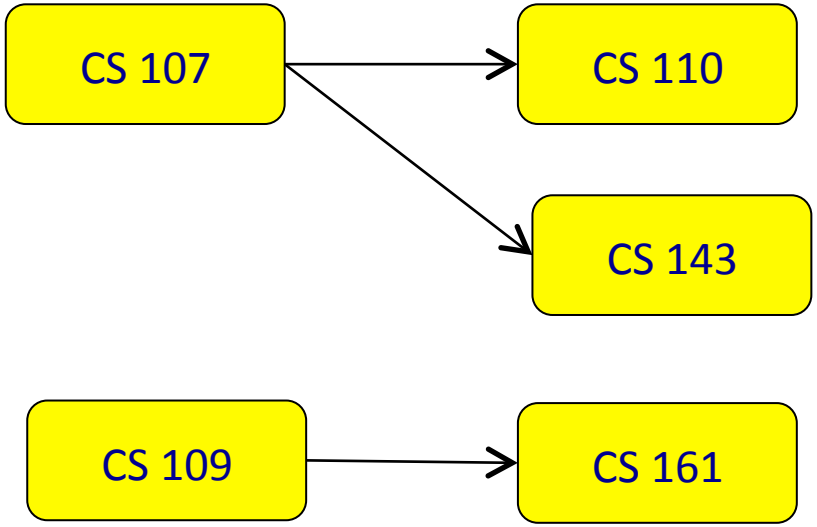


Output:

106A 106B 103

Algorithm #1 Simulation

Input:

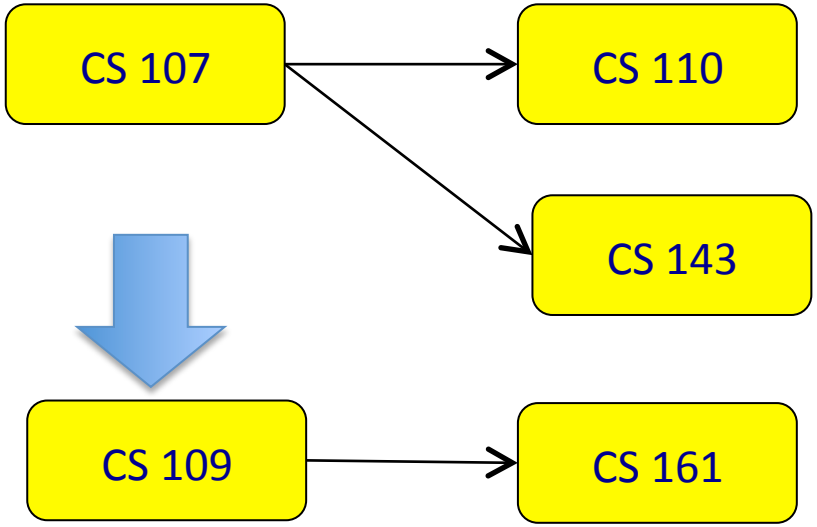


Output:

106A 106B 103

Algorithm #1 Simulation

Input:

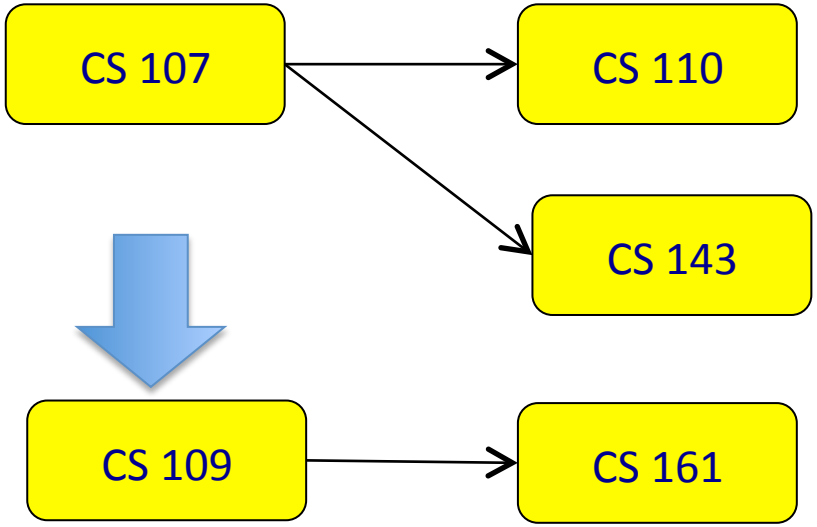


Output:

106A 106B 103

Algorithm #1 Simulation

Input:

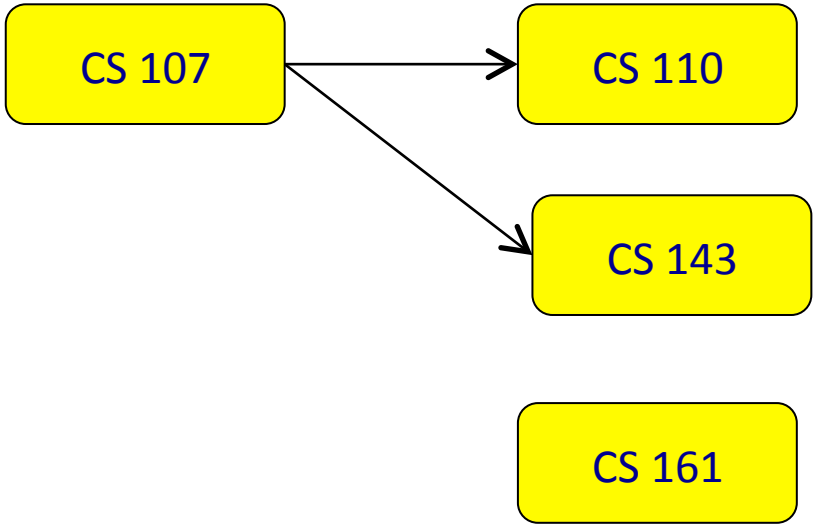


Output:

106A 106B 103 109

Algorithm #1 Simulation

Input:

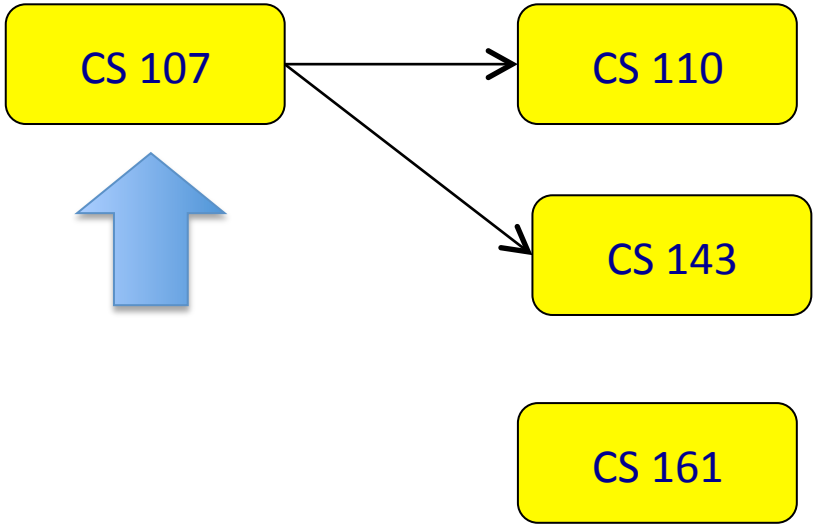


Output:

106A 106B 103 109

Algorithm #1 Simulation

Input:

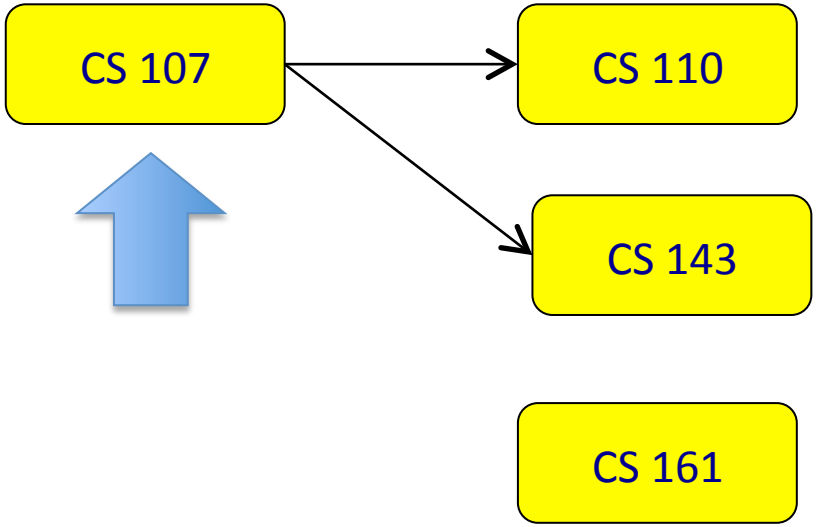


Output:

106A 106B 103 109

Algorithm #1 Simulation

Input:



Output:

106A 106B 103 109 107

Algorithm #1 Simulation

Input:

CS 110

CS 143

CS 161

Output:

106A

106B

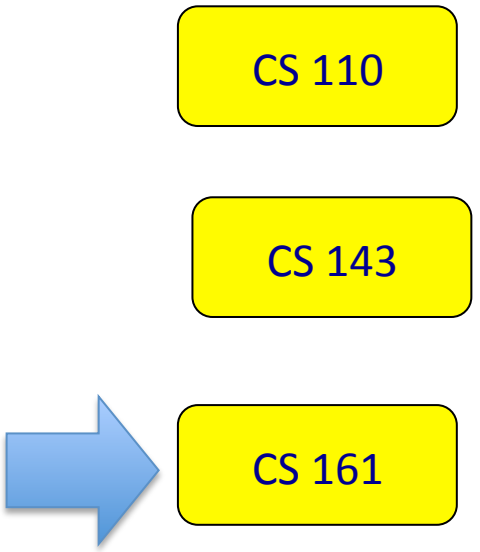
103

109

107

Algorithm #1 Simulation

Input:

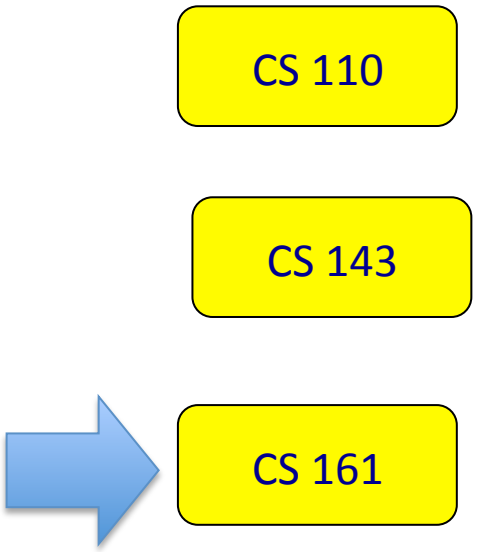


Output:

106A 106B 103 109 107

Algorithm #1 Simulation

Input:



Output:

106A 106B 103 109 107 161

Algorithm #1 Simulation

Input:

CS 110

CS 143

Output:

106A

106B

103

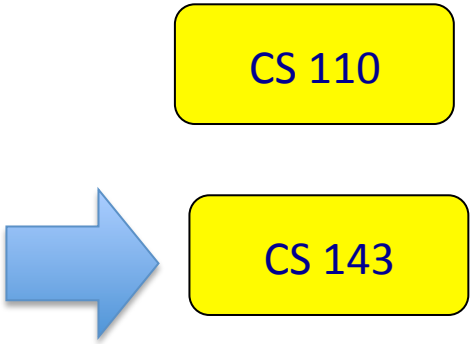
109

107

161

Algorithm #1 Simulation

Input:

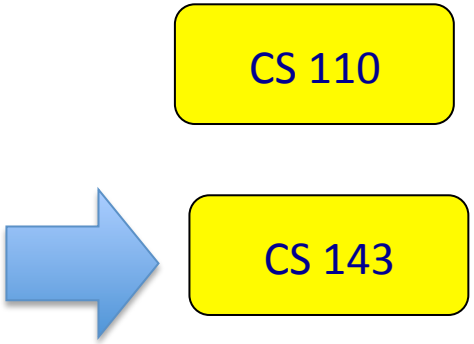


Output:

106A 106B 103 109 107 161

Algorithm #1 Simulation

Input:



Output:

106A 106B 103 109 107 161 143

Algorithm #1 Simulation

Input:

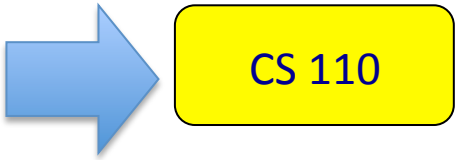
CS 110

Output:

106A 106B 103 109 107 161 143

Algorithm #1 Simulation

Input:

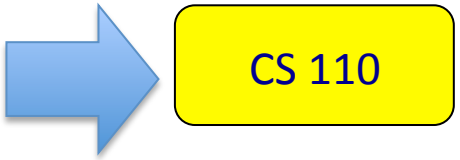


Output:

106A 106B 103 109 107 161 143

Algorithm #1 Simulation

Input:



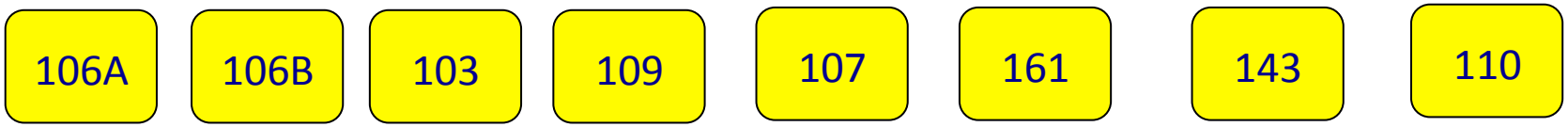
Output:

106A 106B 103 109 107 161 143 110

Algorithm #1 Simulation

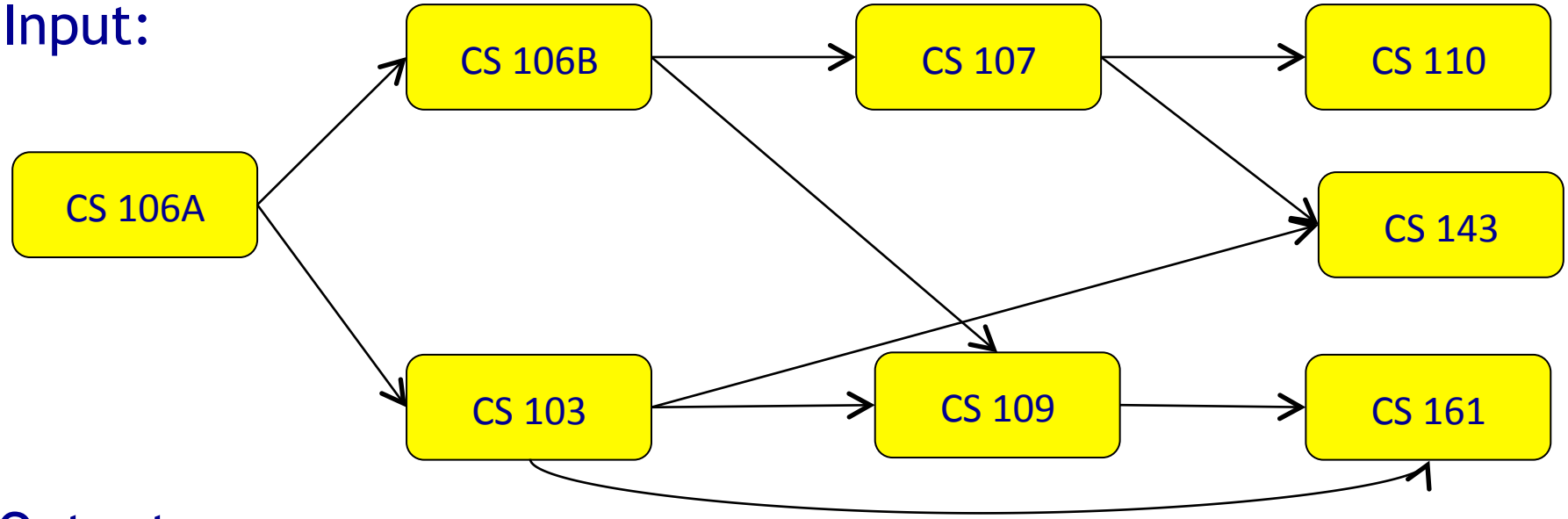
Input:

Output:

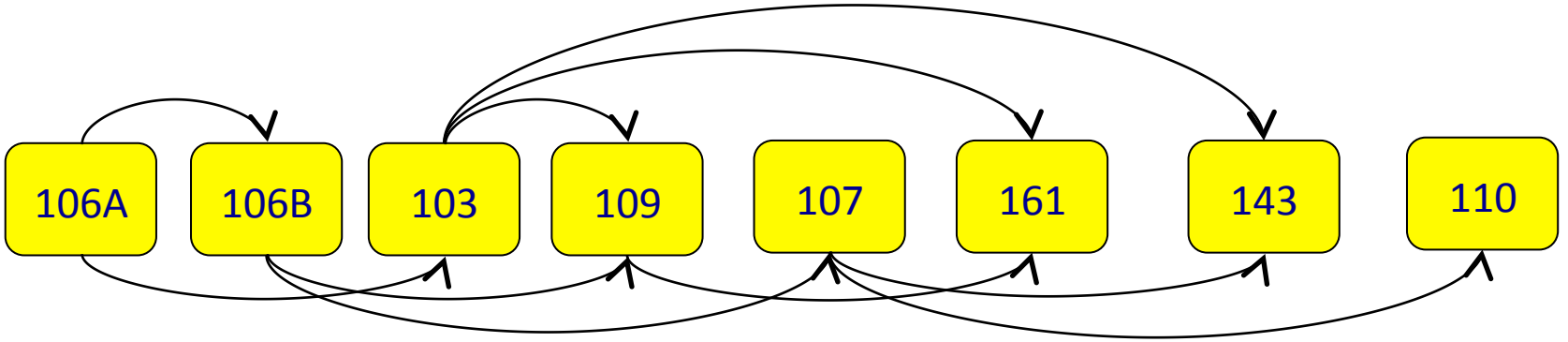


Algorithm #1 Simulation

Input:



Output:



Runtime of TS Algorithm #1

- Depends on implementation

```
procedure topologicalSort1(DAG G):
    let result empty list.
    while G is not empty:
        1. let v be a source node in G
        2. add v to result
        3. remove v from G
    return result
```

Naïve Implementation: find source by looping over V
Runtime: $O(n^2)$

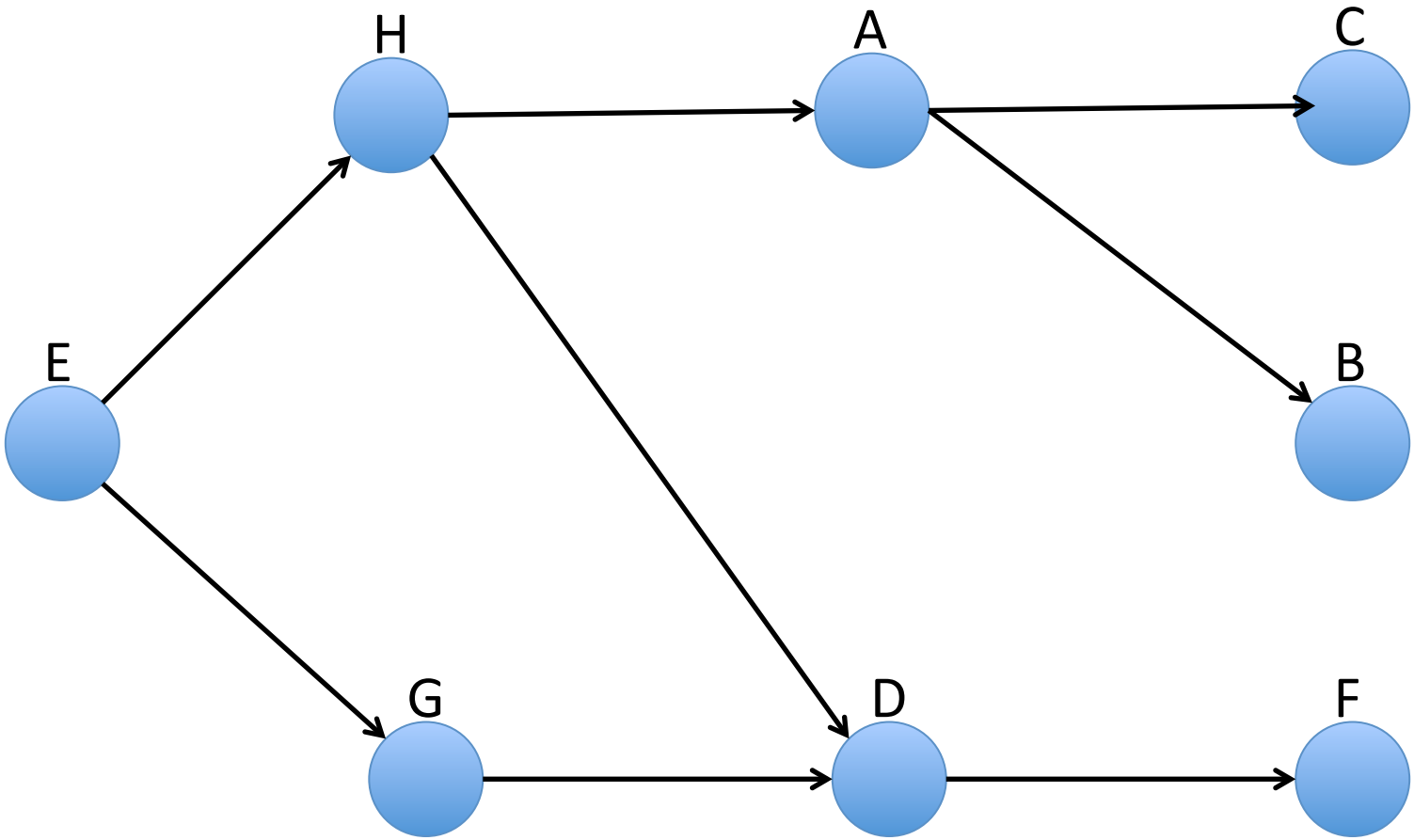
Exercise: Can do it in: O(n + m).

TS Algorithm #2 (Using DFS)

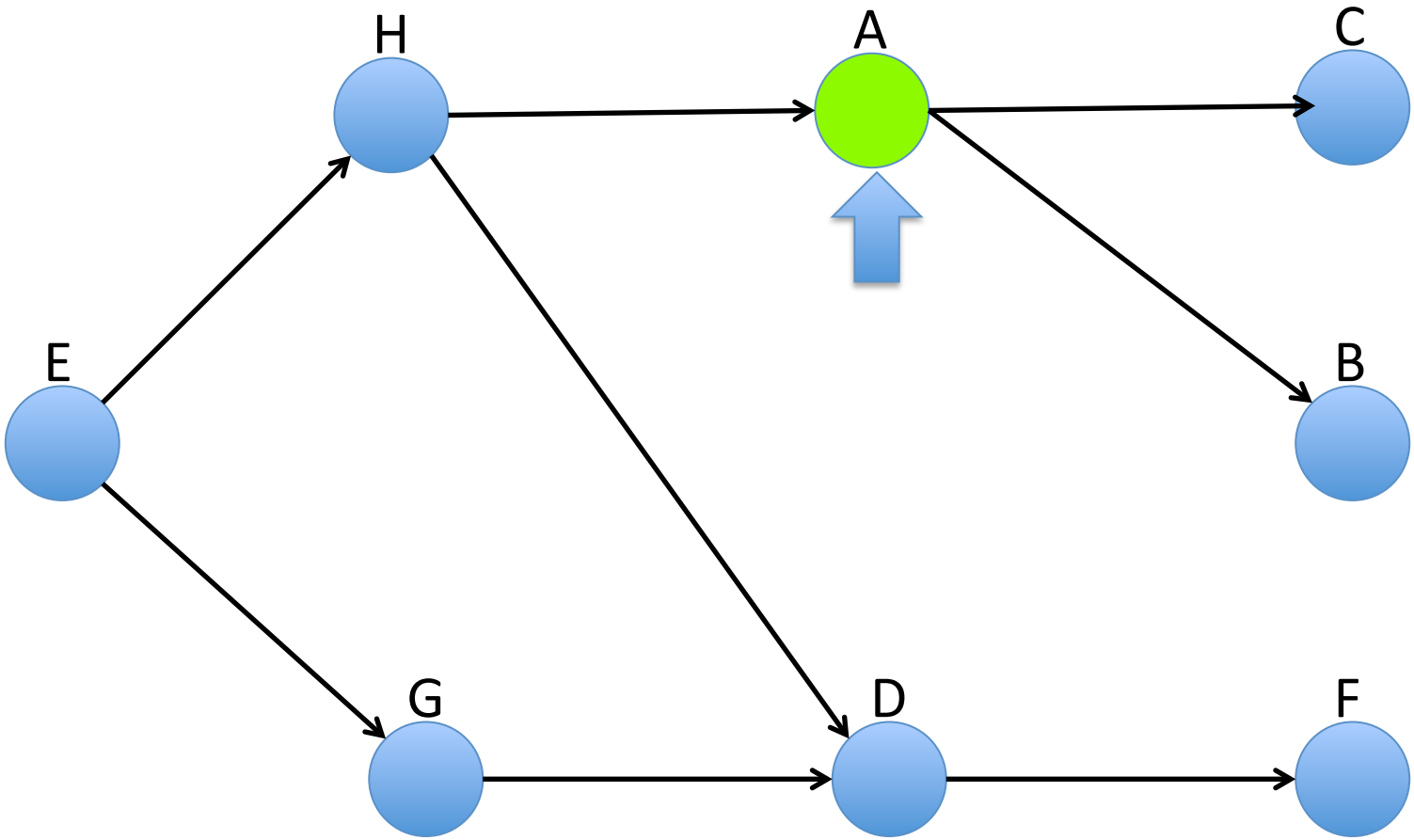
- Run DFS
- Keep track of the “finishing times” of the vertices

Finishing Time f[v]: the time when v turns red

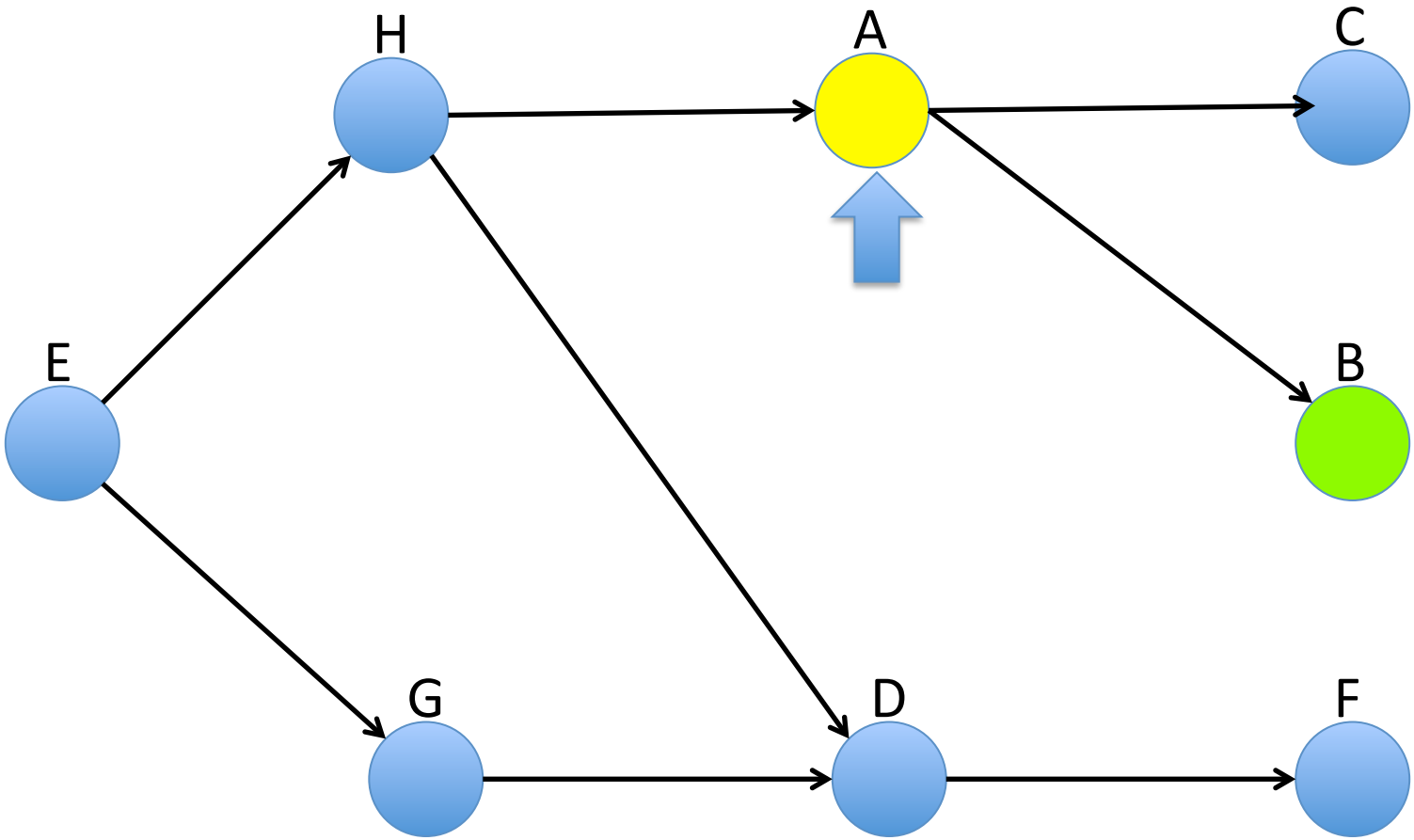
TS Algorithm #2 (Using DFS) Simulation 1



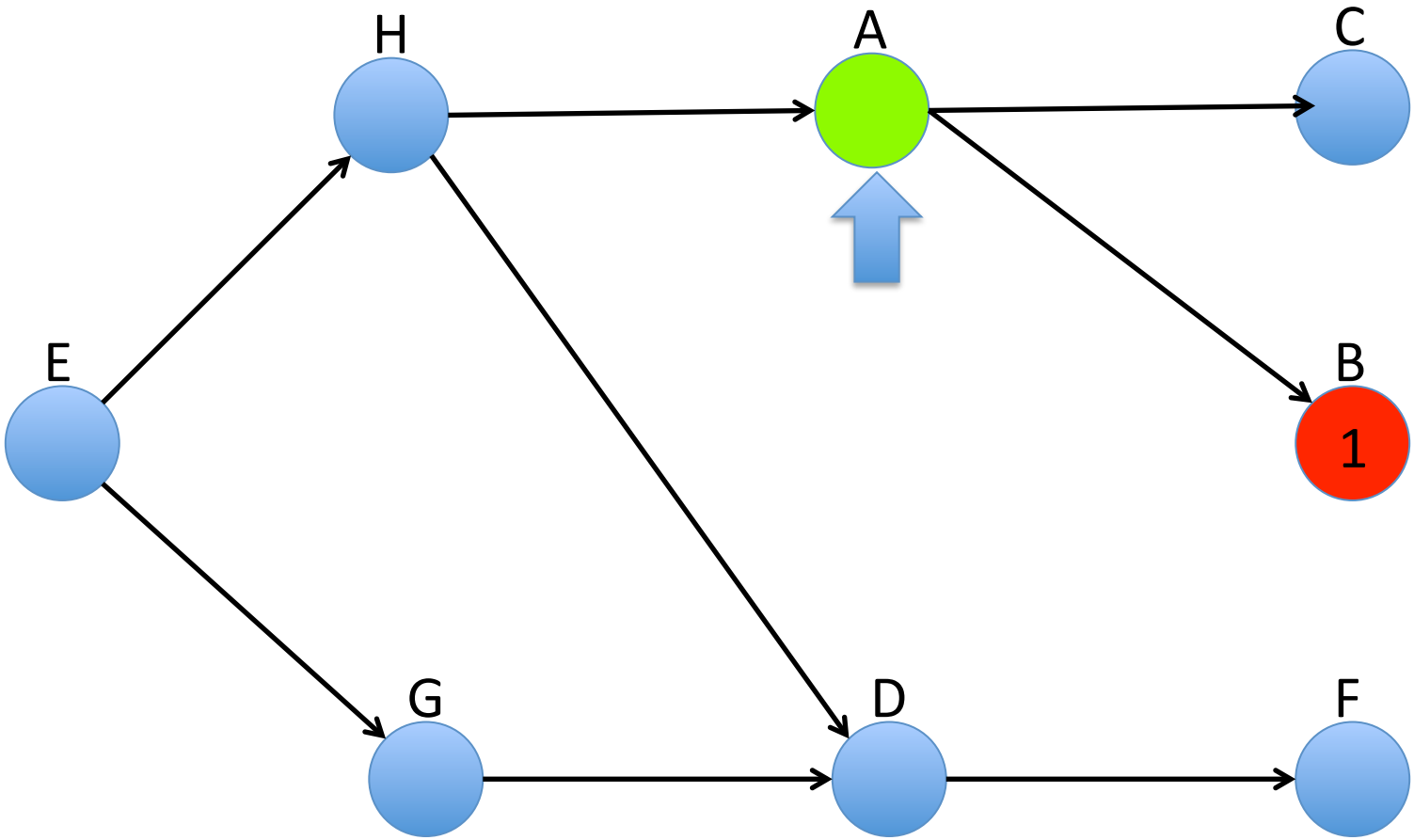
TS Algorithm #2 (Using DFS) Simulation 1



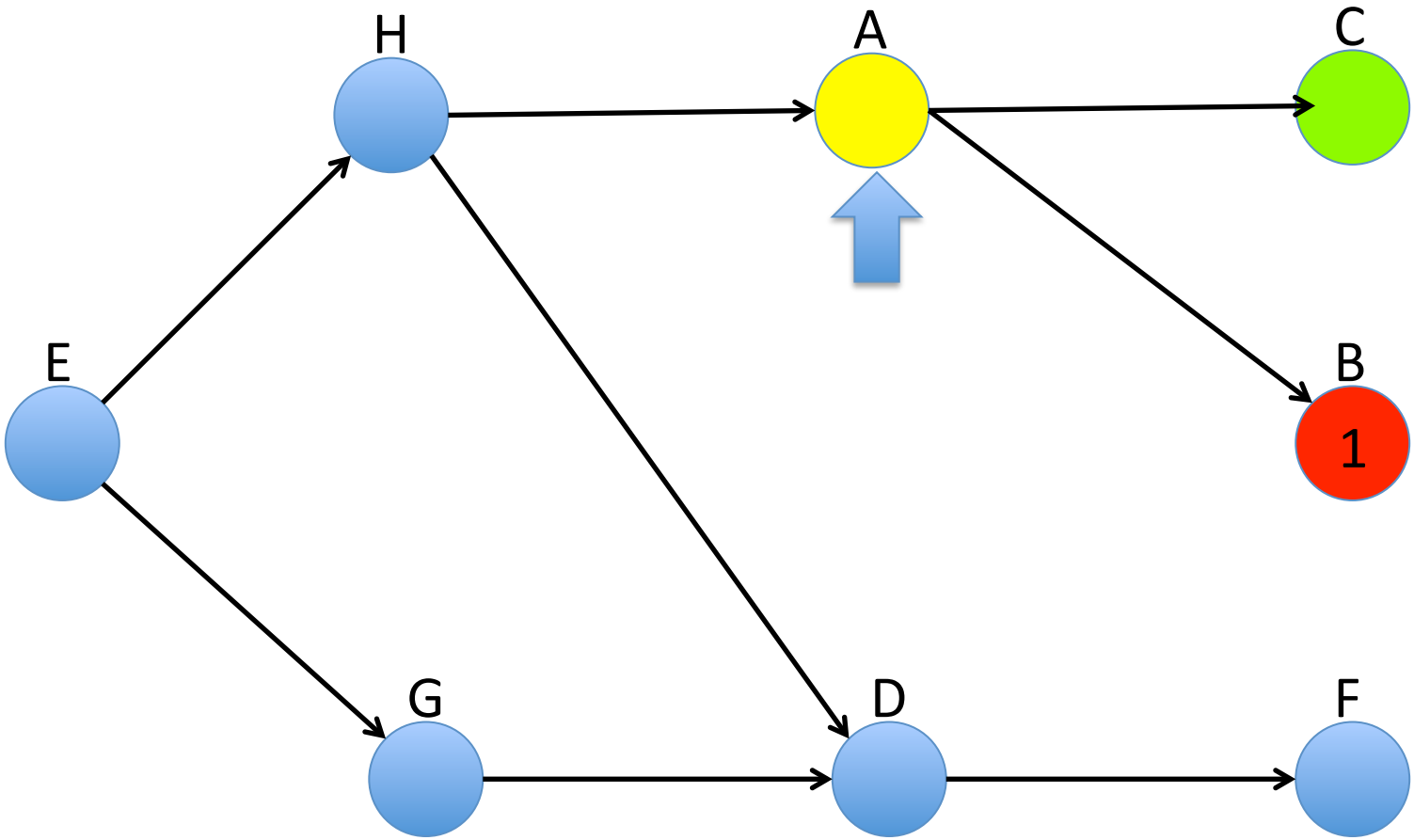
TS Algorithm #2 (Using DFS) Simulation 1



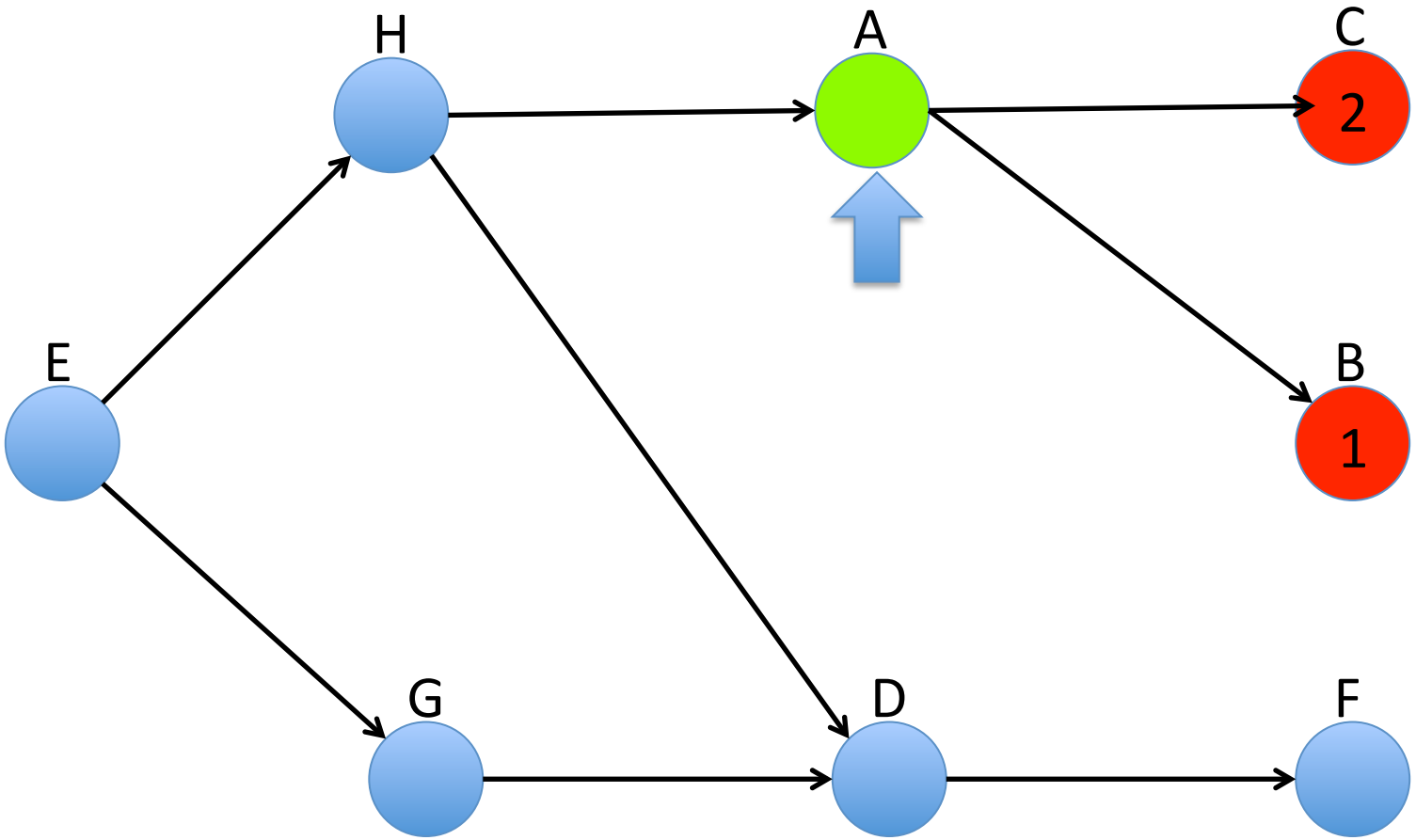
TS Algorithm #2 (Using DFS) Simulation 1



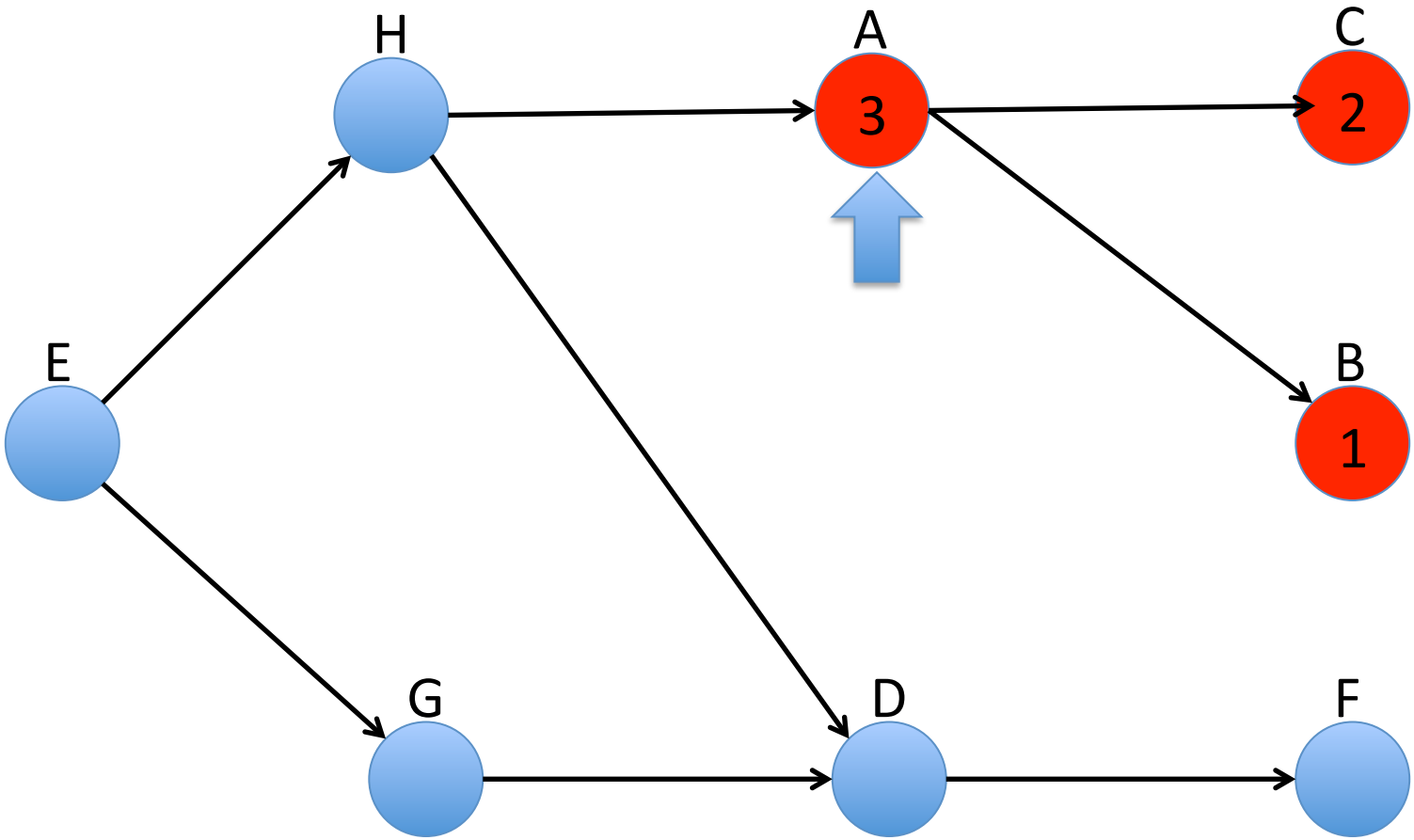
TS Algorithm #2 (Using DFS) Simulation 1



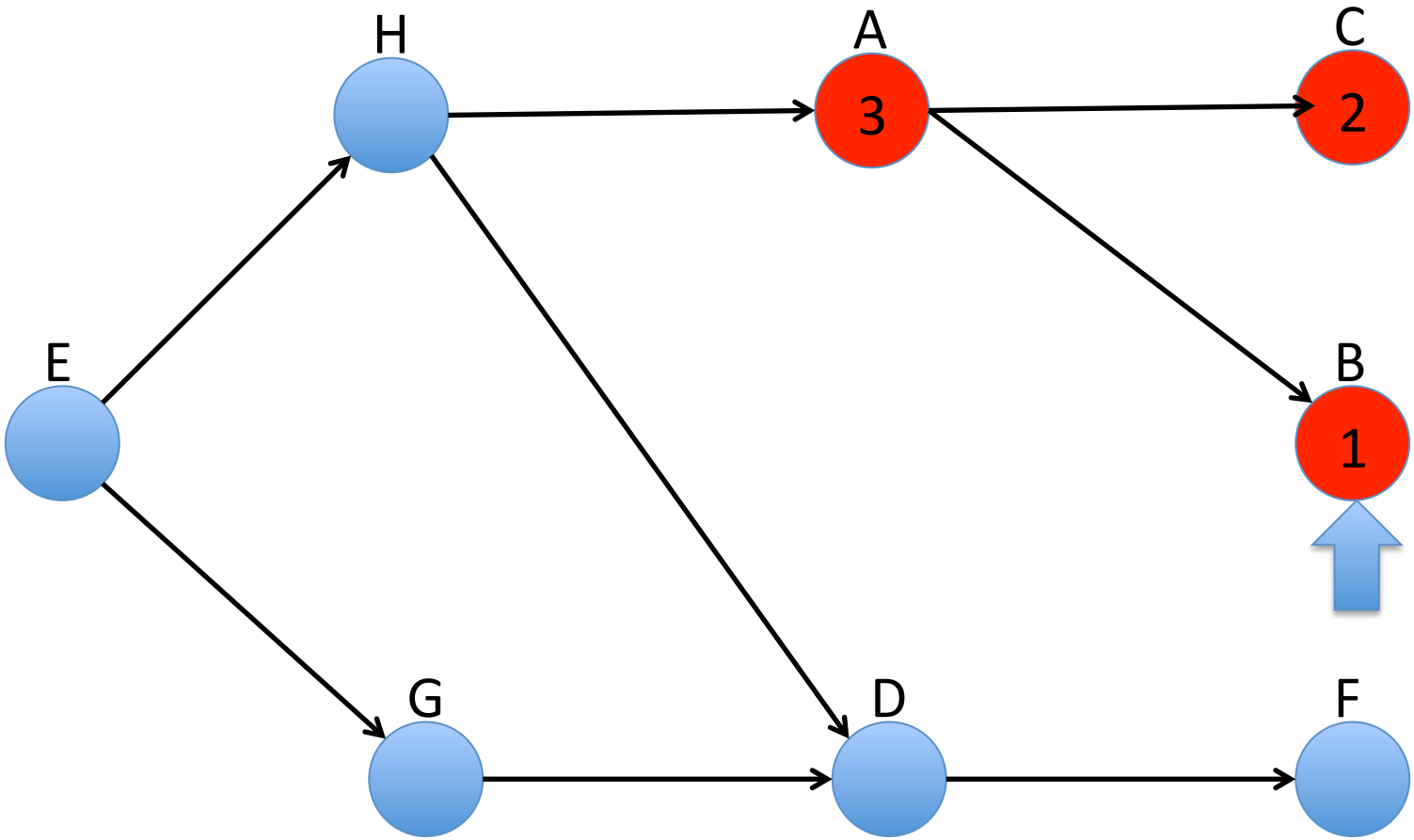
TS Algorithm #2 (Using DFS) Simulation 1



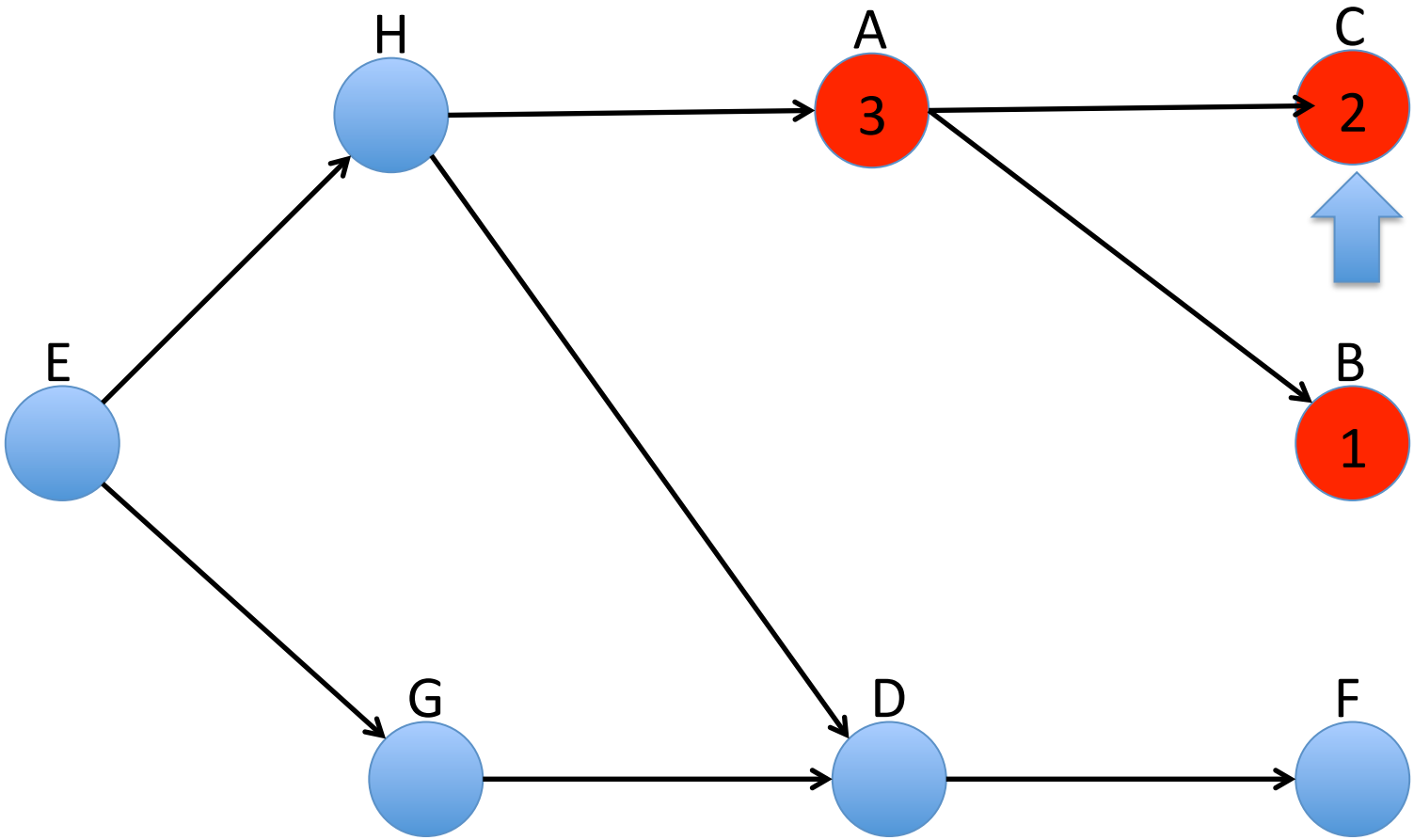
TS Algorithm #2 (Using DFS) Simulation 1



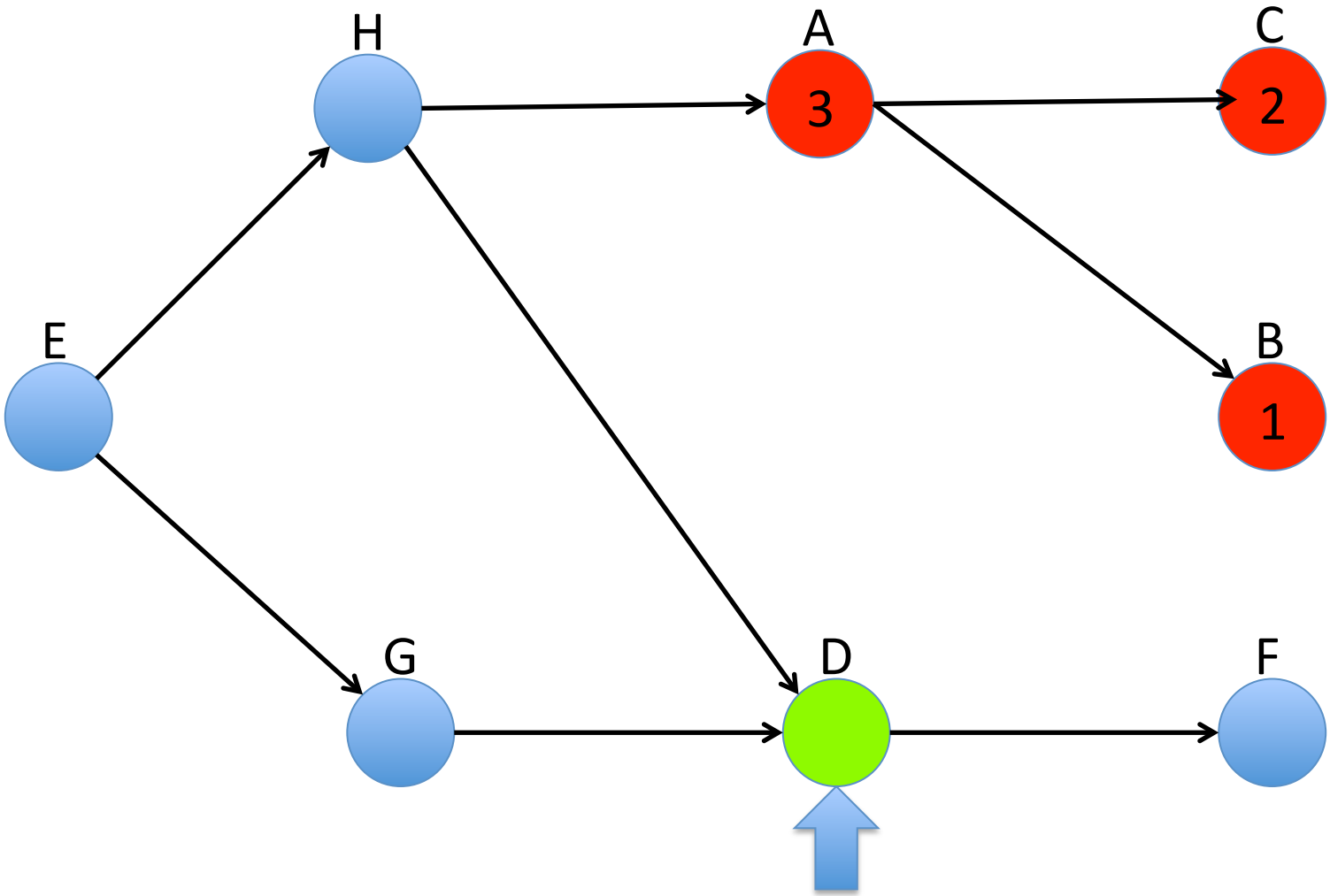
TS Algorithm #2 (Using DFS) Simulation 1



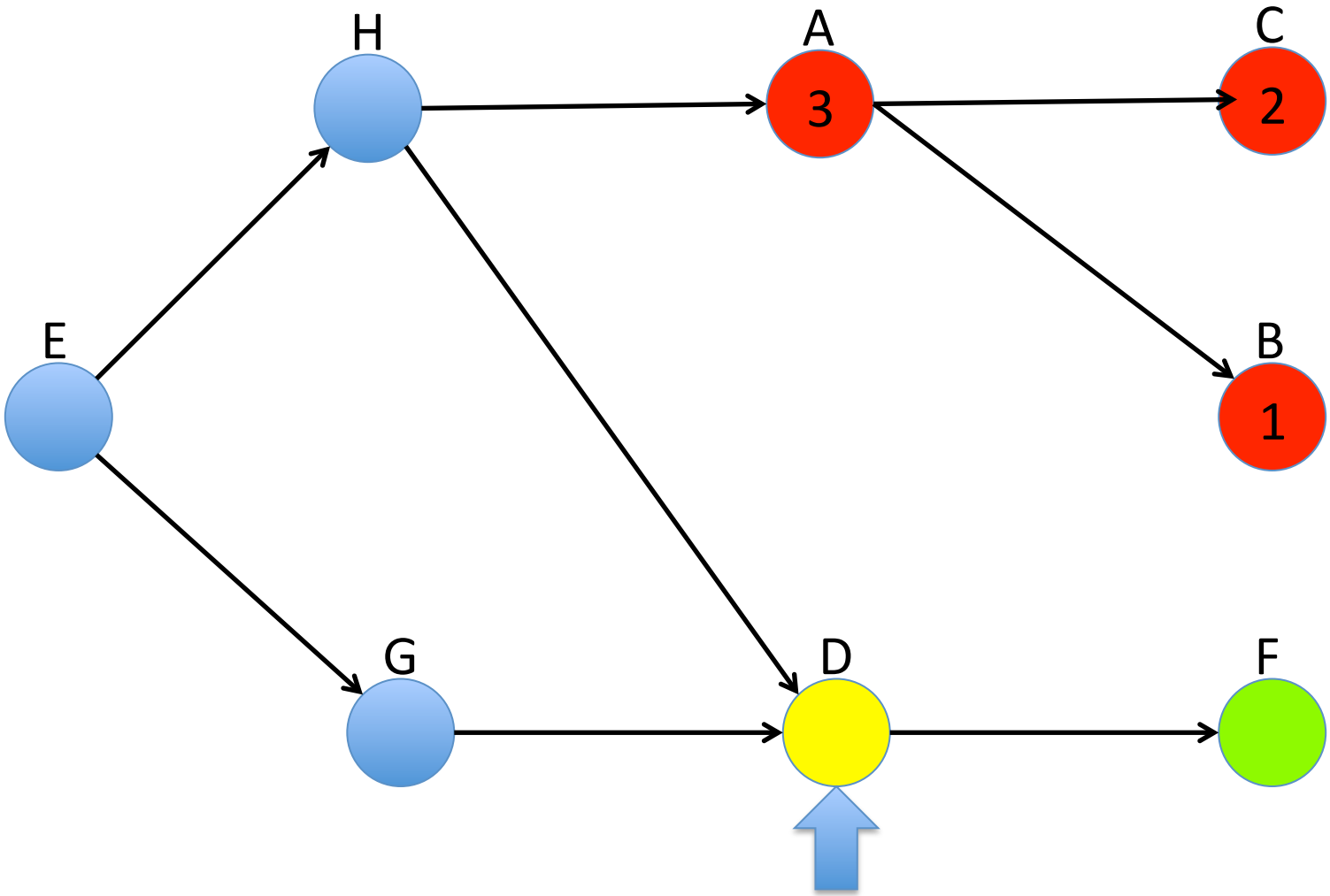
TS Algorithm #2 (Using DFS) Simulation 1



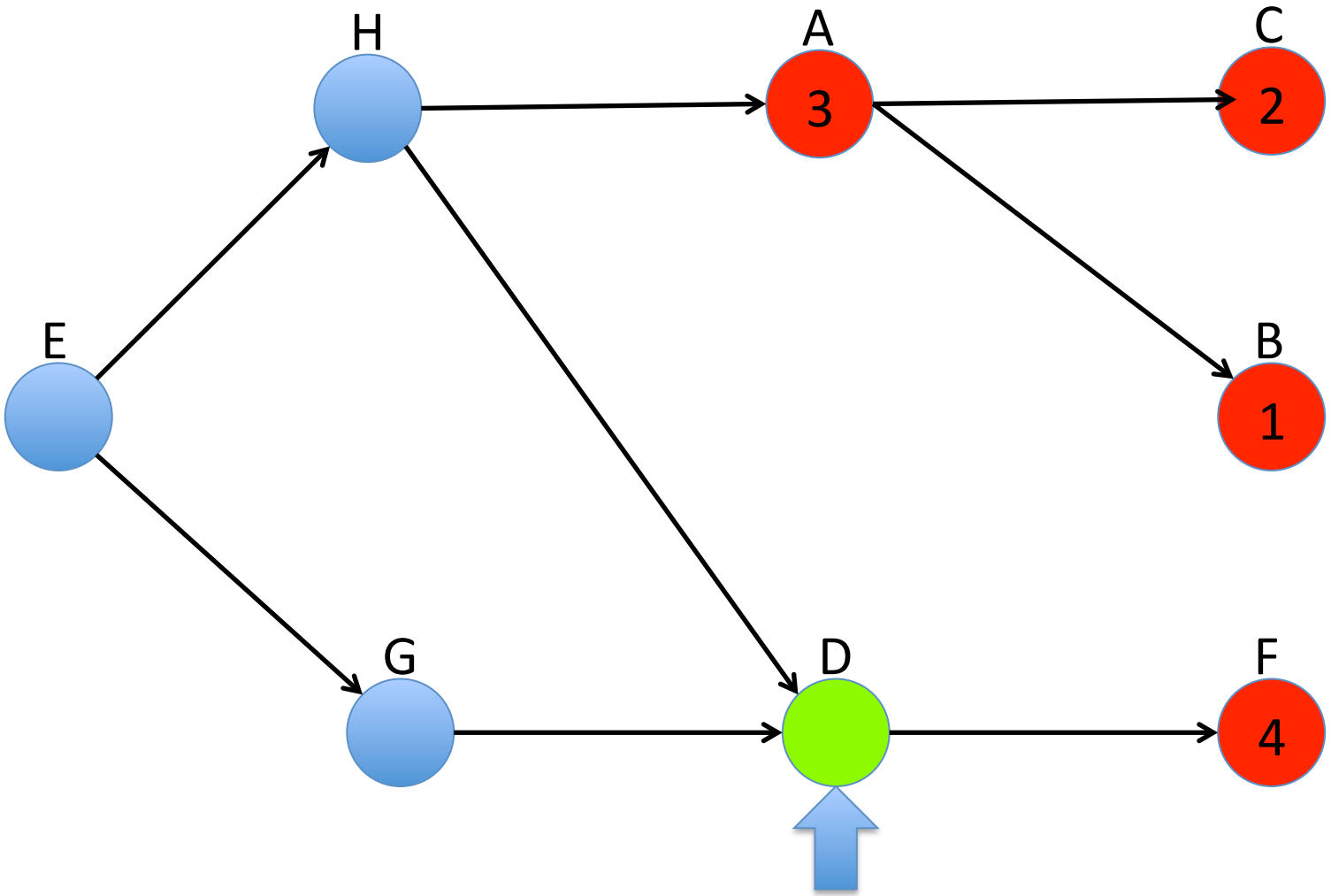
TS Algorithm #2 (Using DFS) Simulation 1



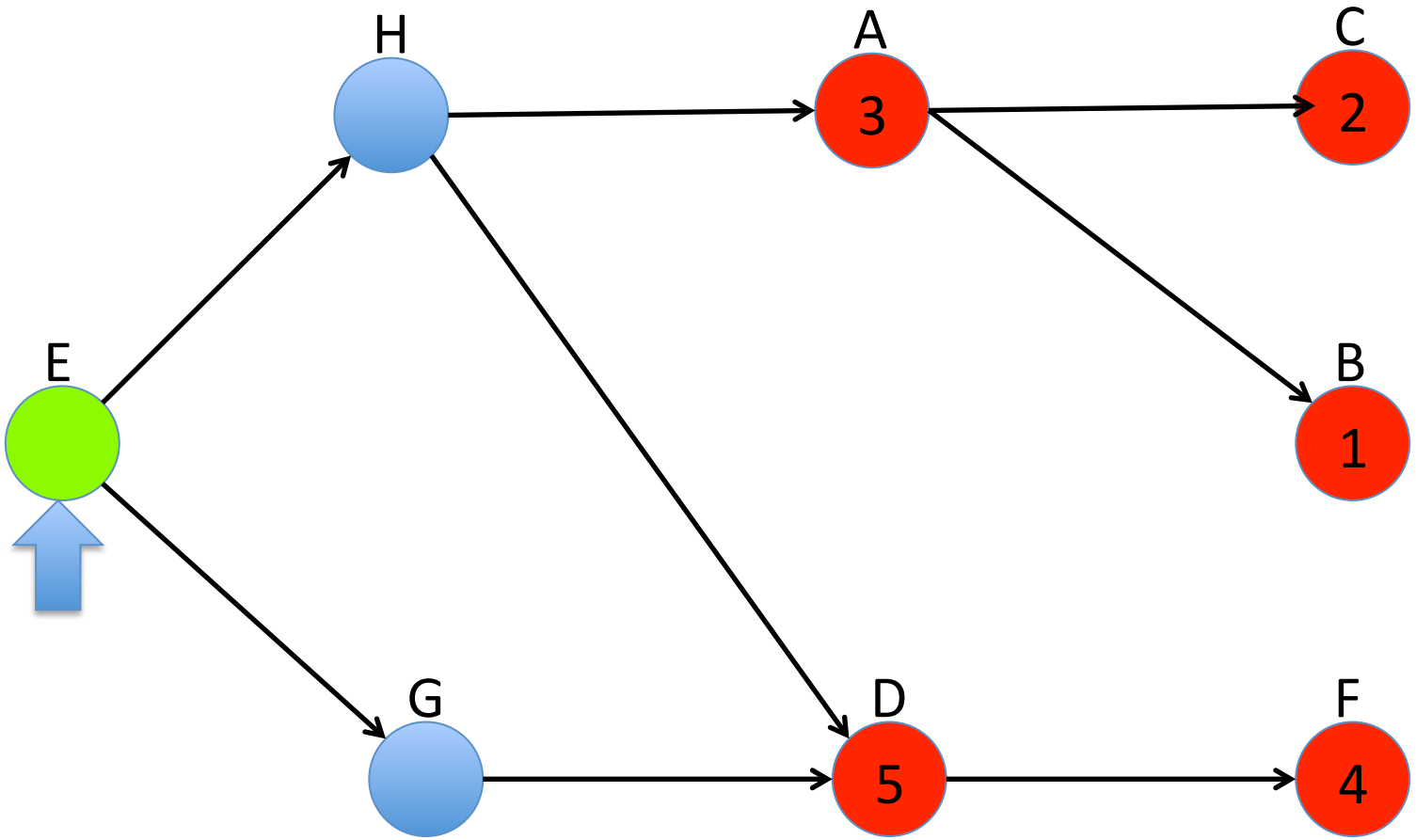
TS Algorithm #2 (Using DFS) Simulation 1



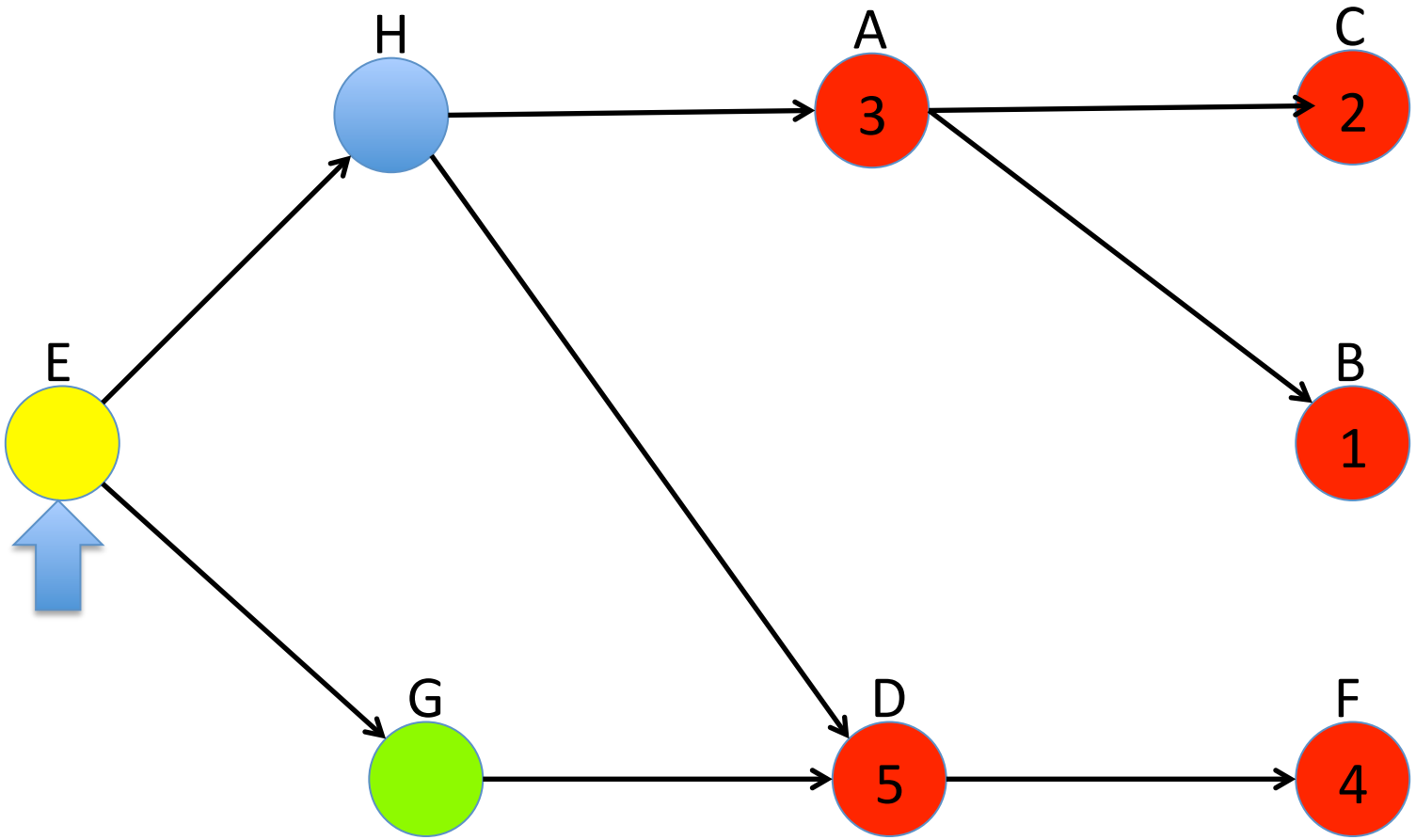
TS Algorithm #2 (Using DFS) Simulation 1



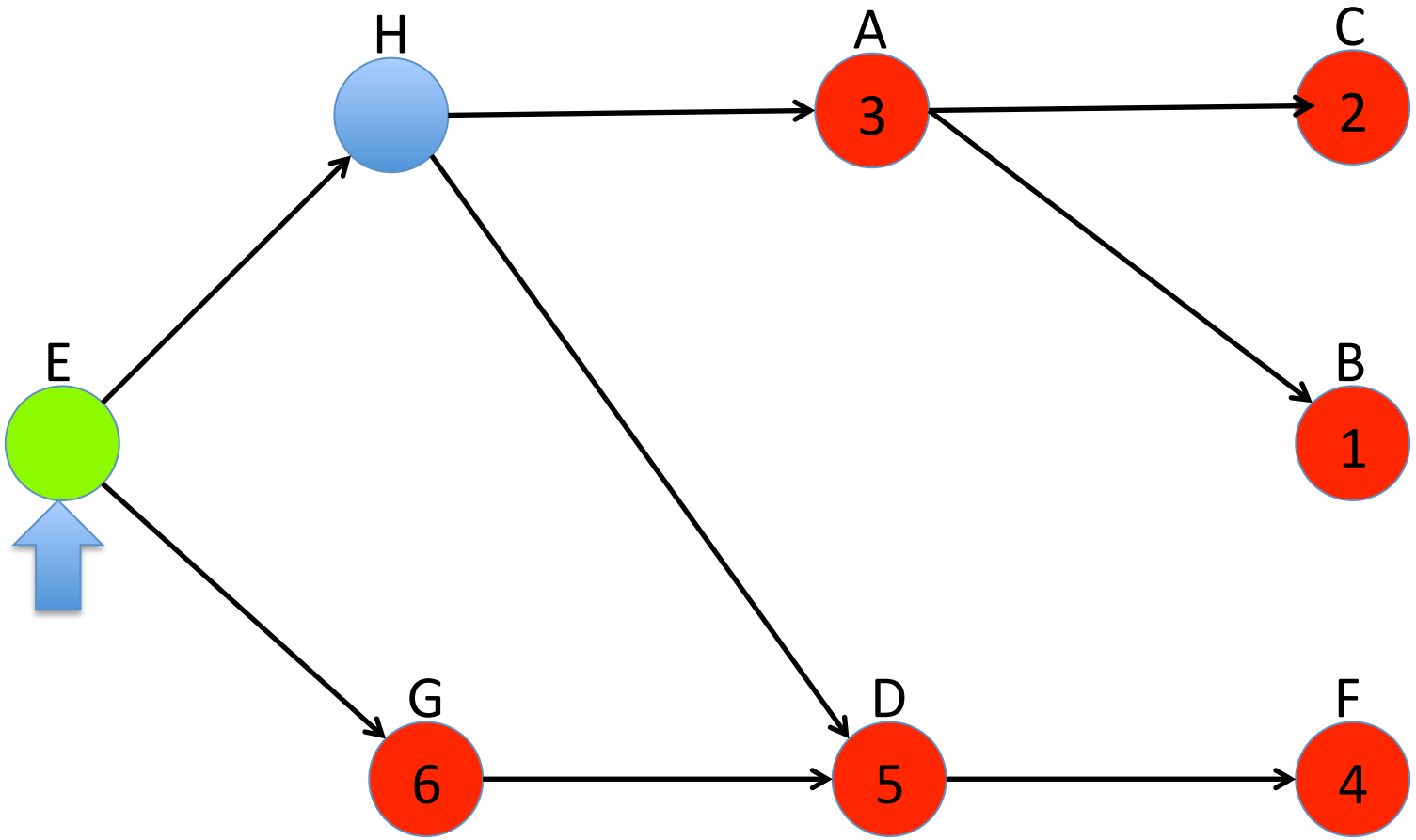
TS Algorithm #2 (Using DFS) Simulation 1



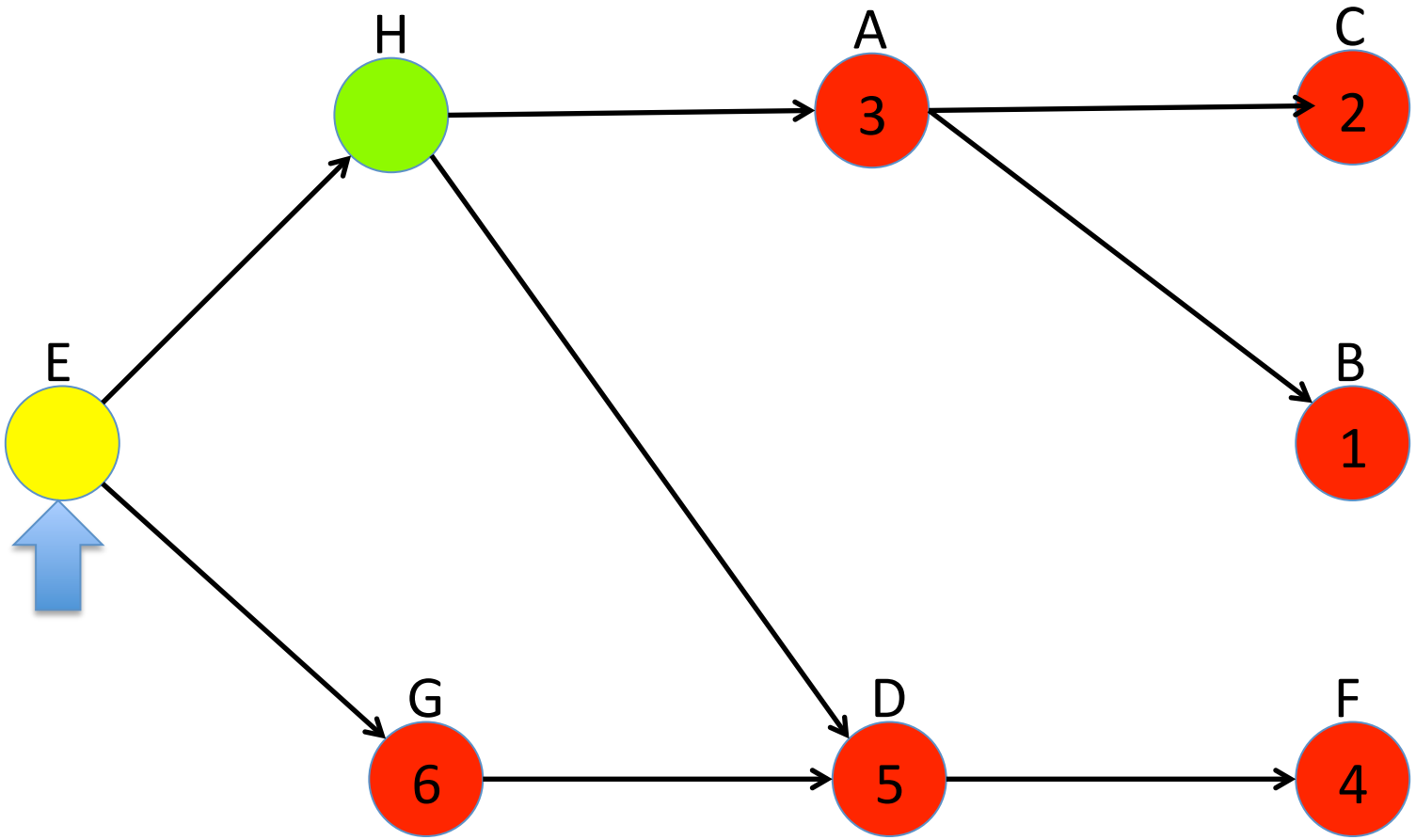
TS Algorithm #2 (Using DFS) Simulation 1



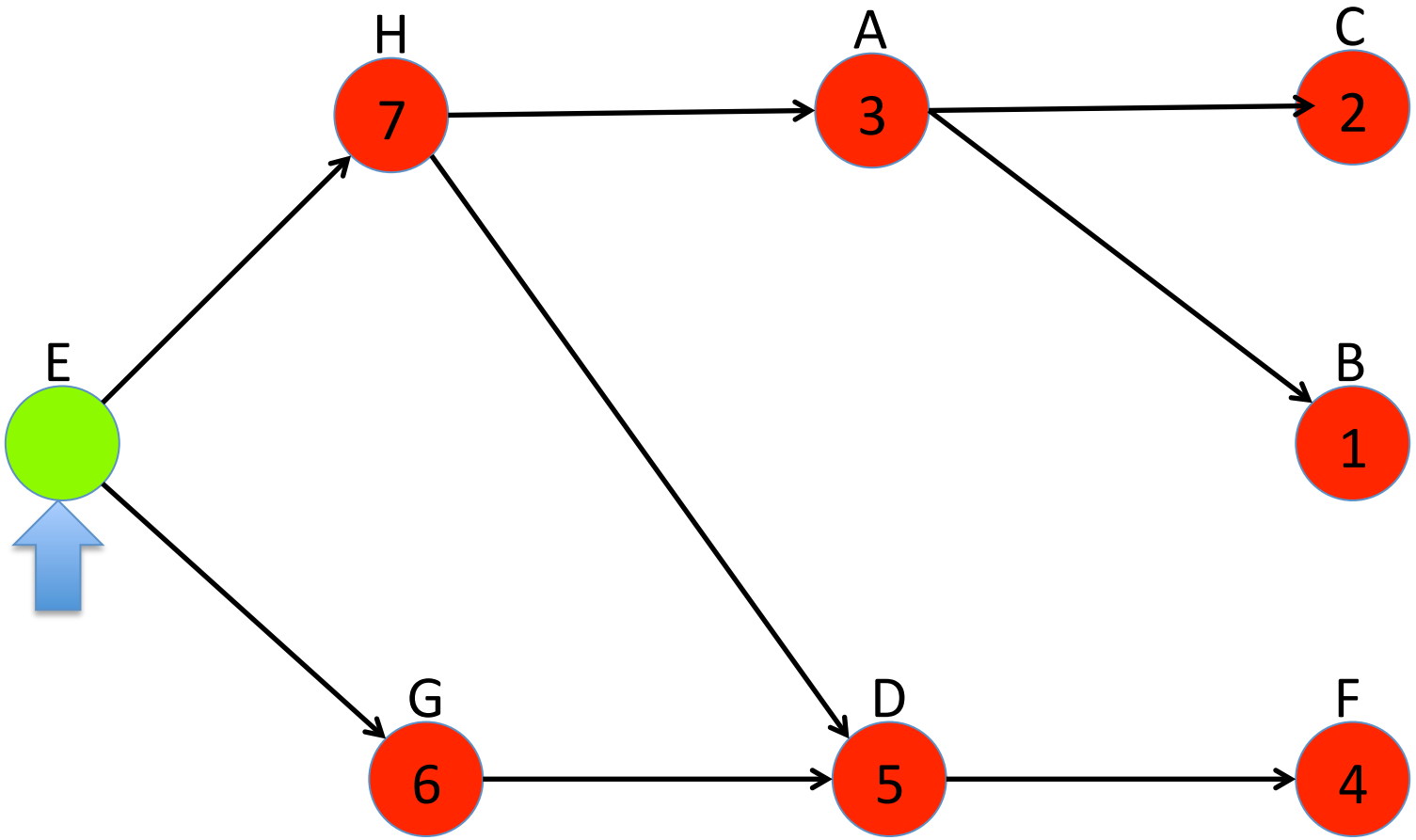
TS Algorithm #2 (Using DFS) Simulation 1



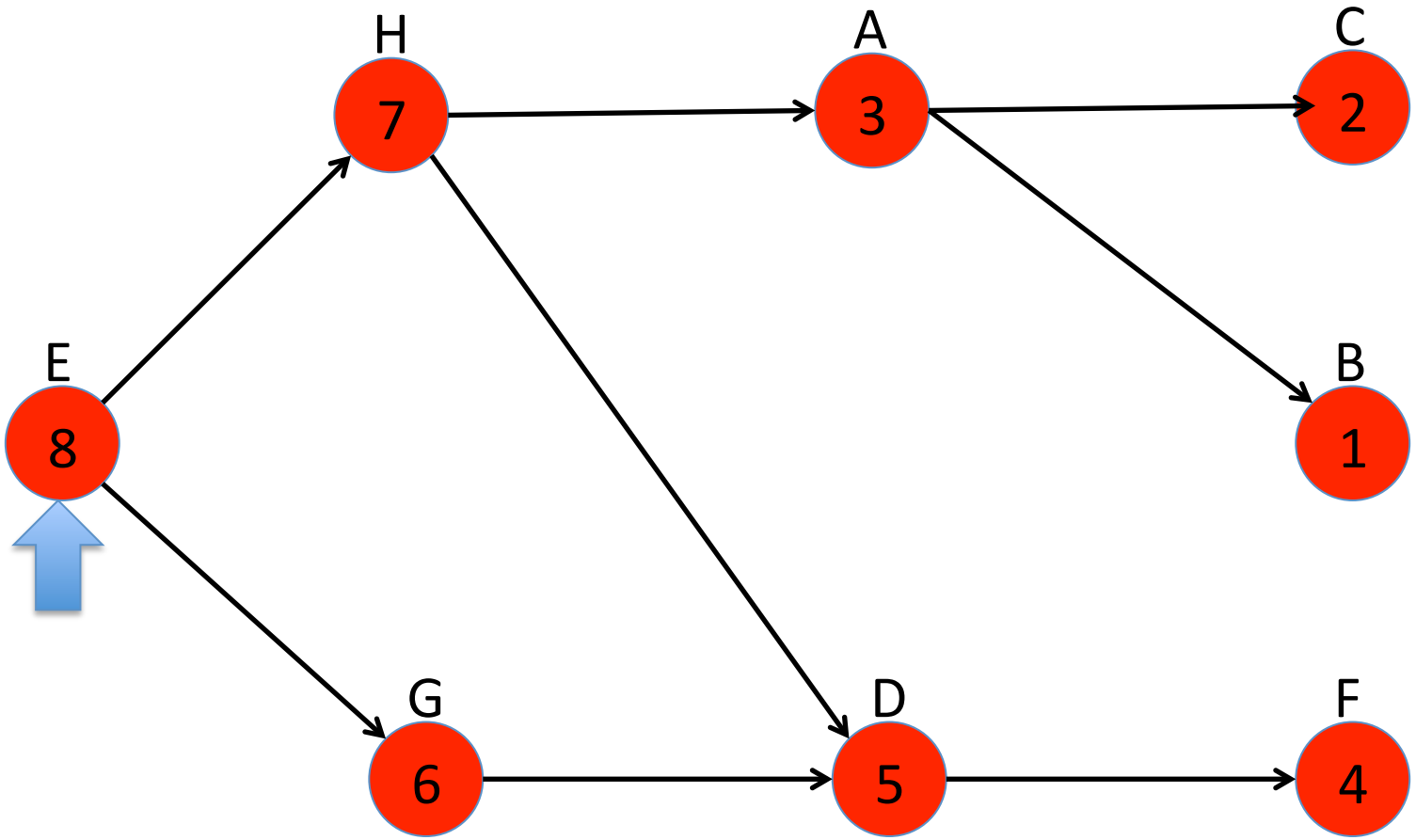
TS Algorithm #2 (Using DFS) Simulation 1



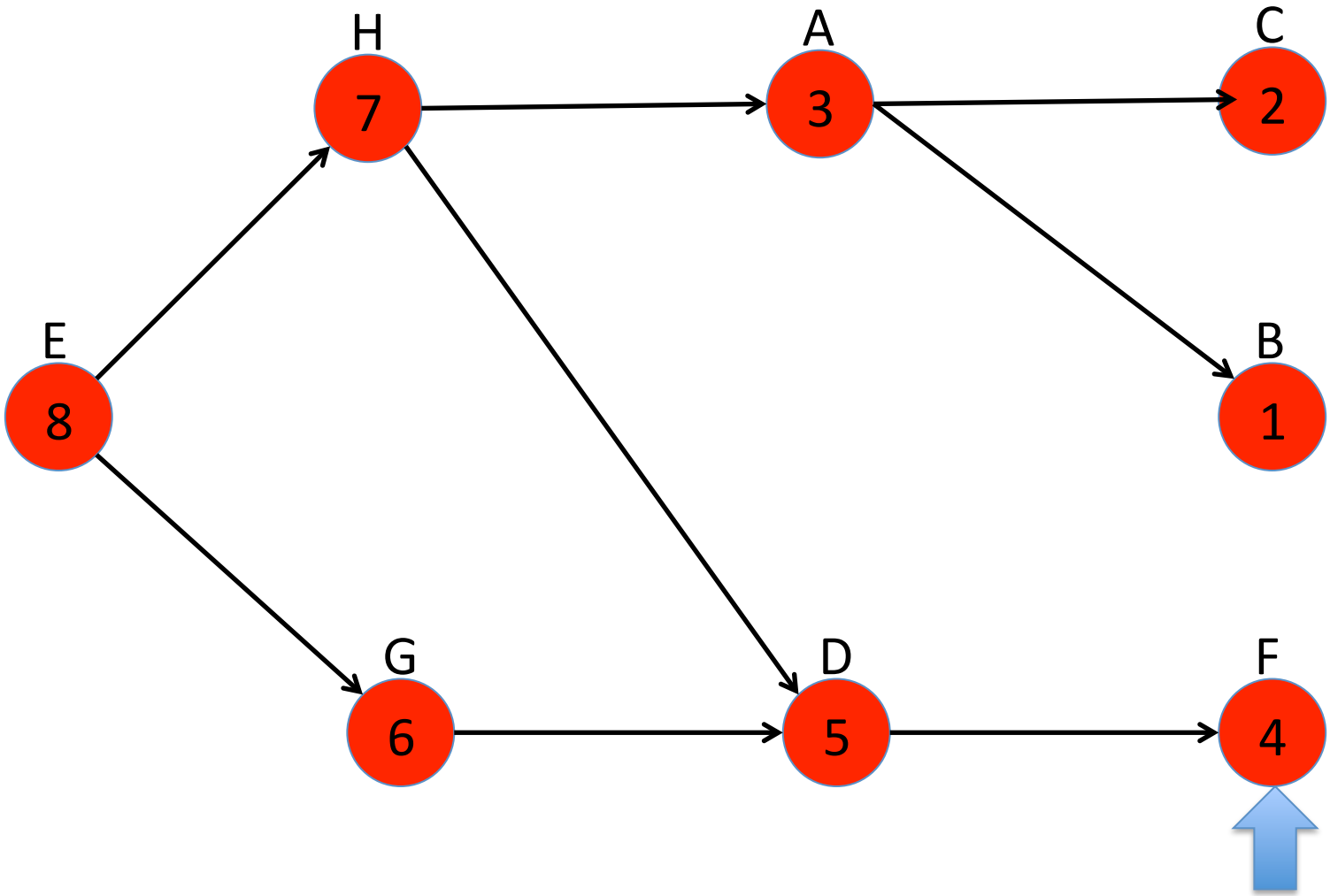
TS Algorithm #2 (Using DFS) Simulation 1



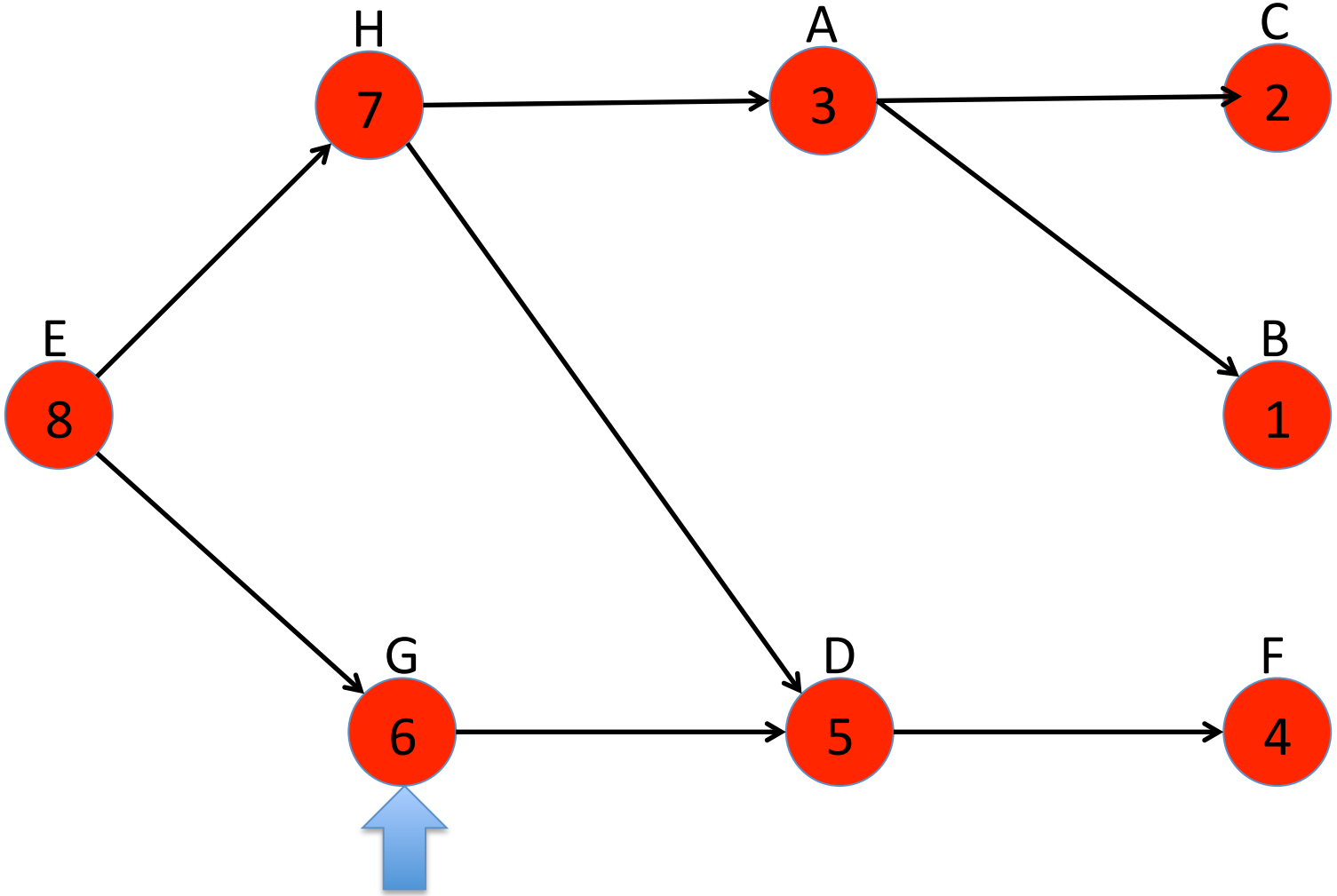
TS Algorithm #2 (Using DFS) Simulation 1



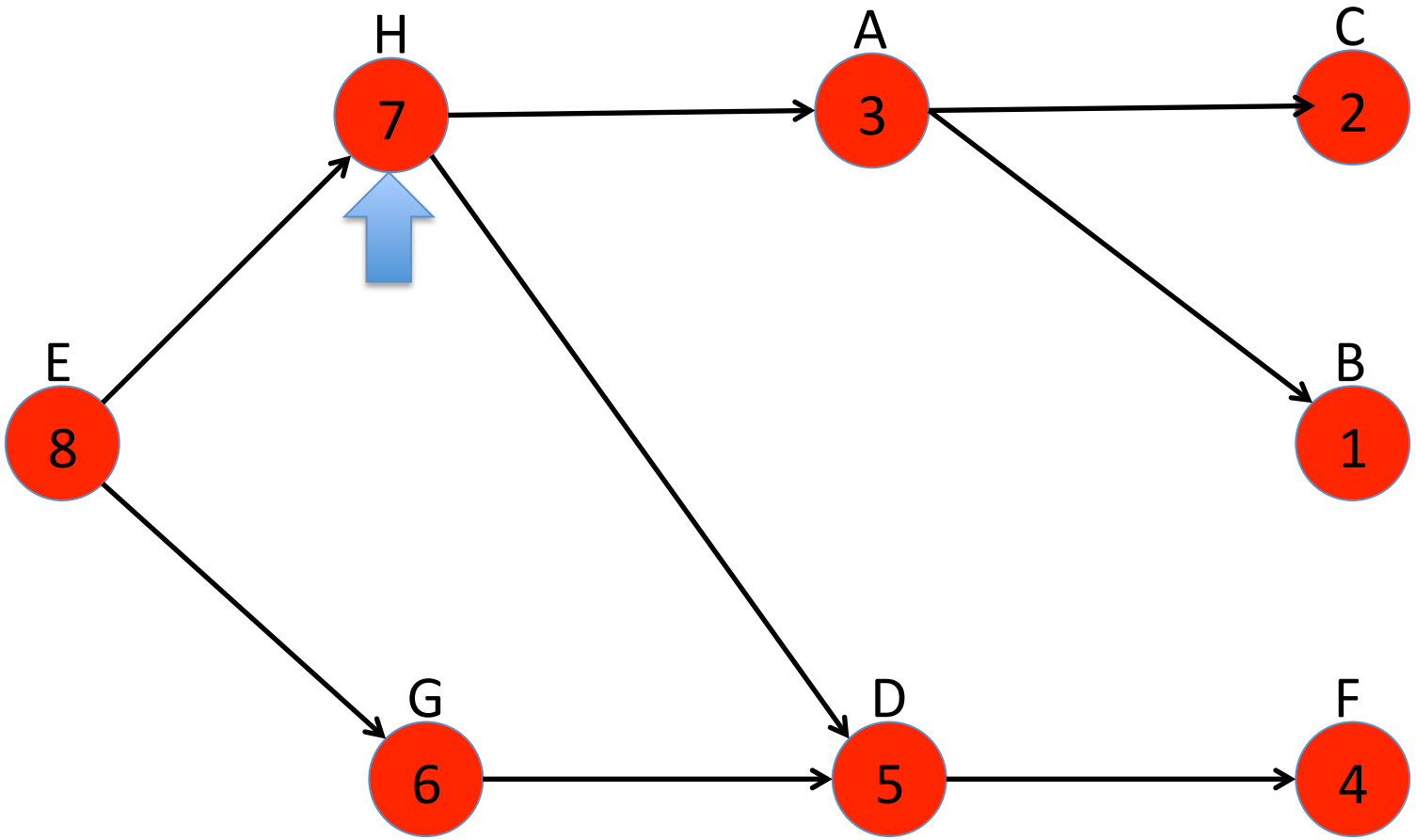
TS Algorithm #2 (Using DFS) Simulation 1



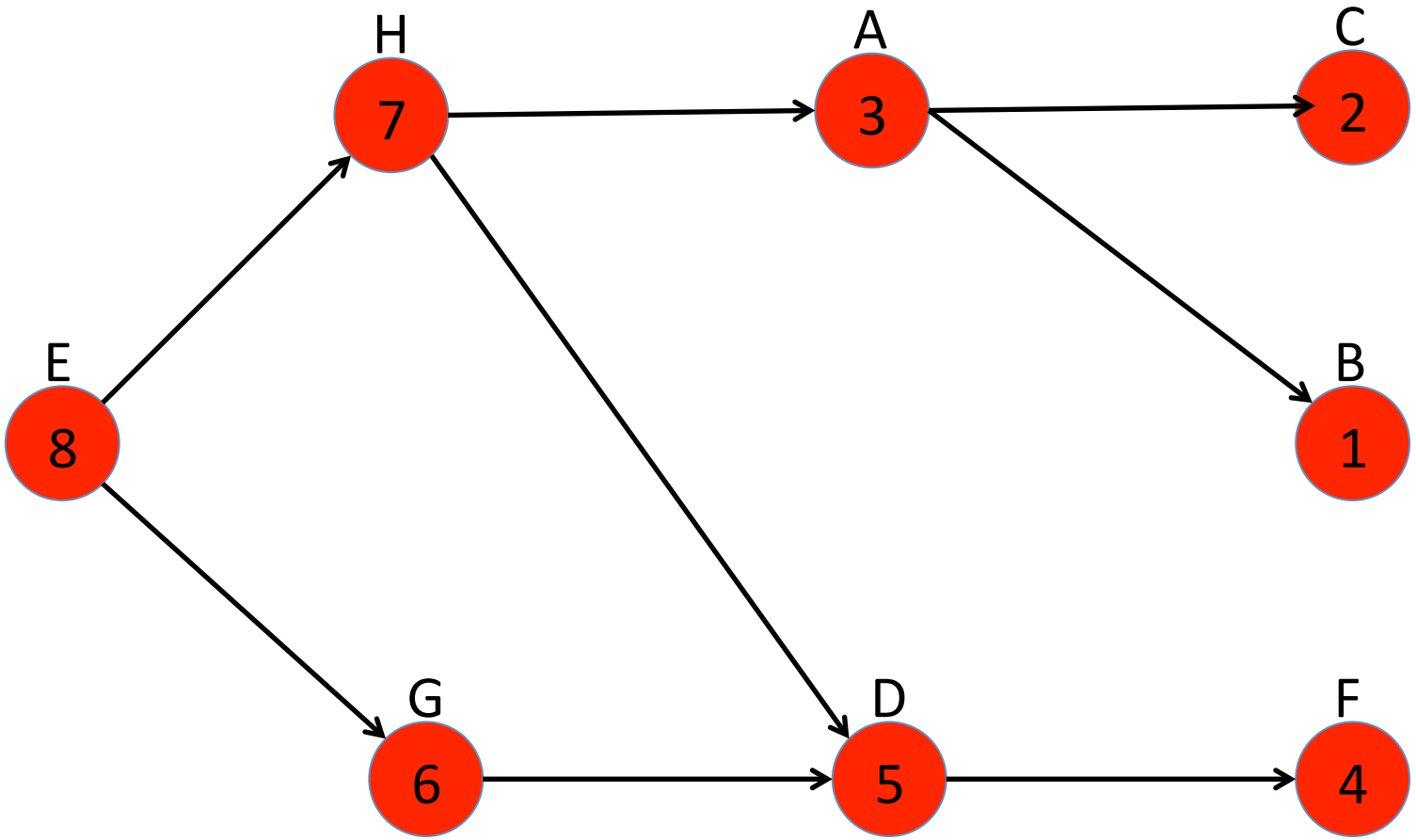
TS Algorithm #2 (Using DFS) Simulation 1



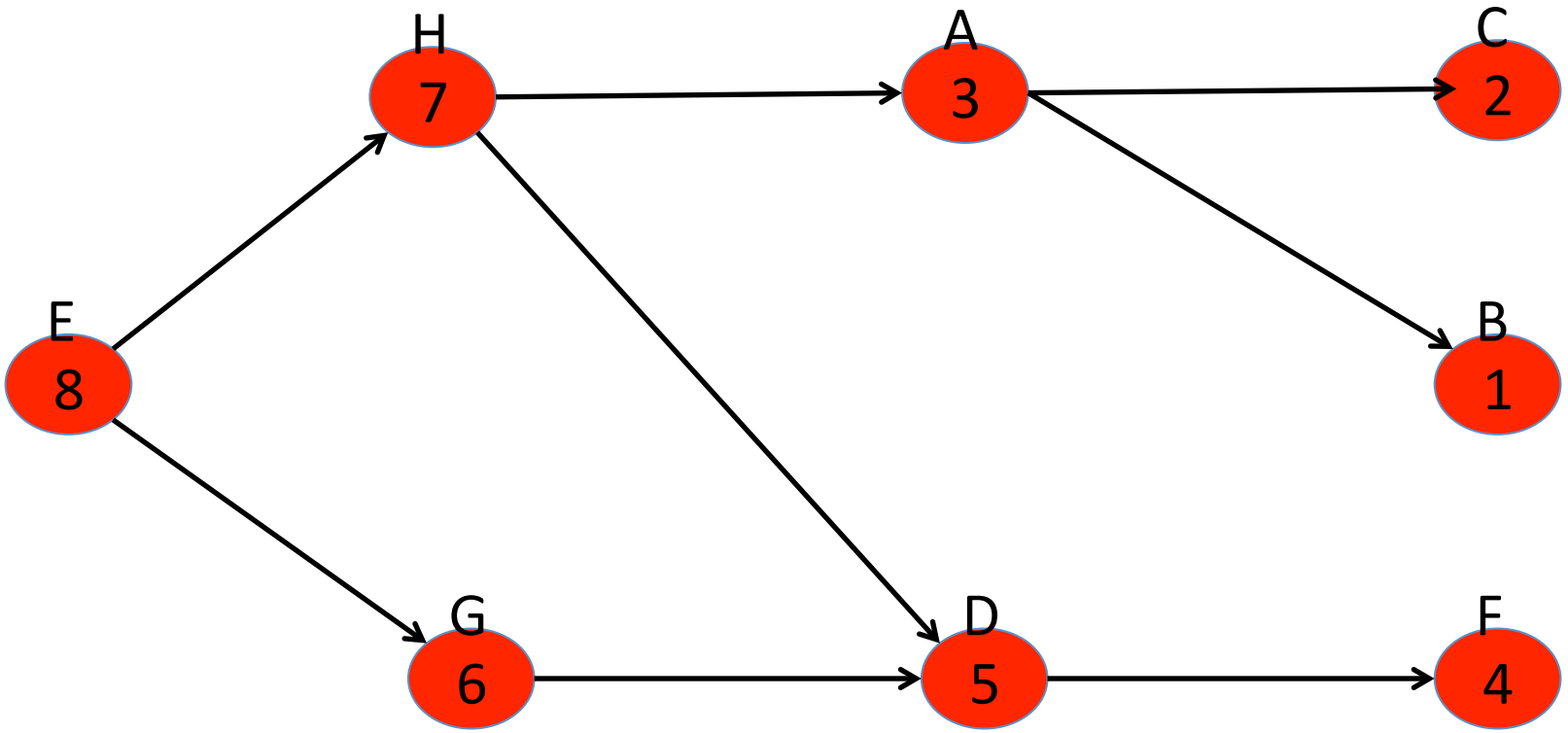
TS Algorithm #2 (Using DFS) Simulation 1

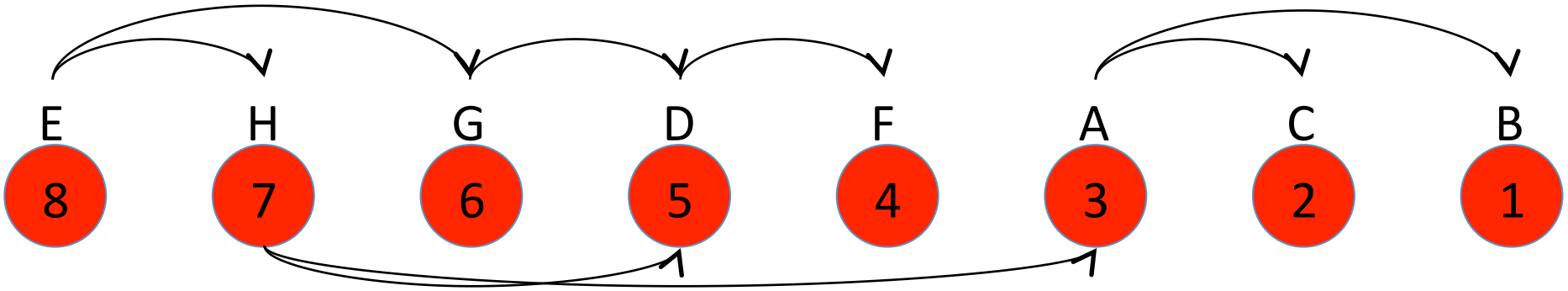


TS Algorithm #2 (Using DFS) Simulation 1

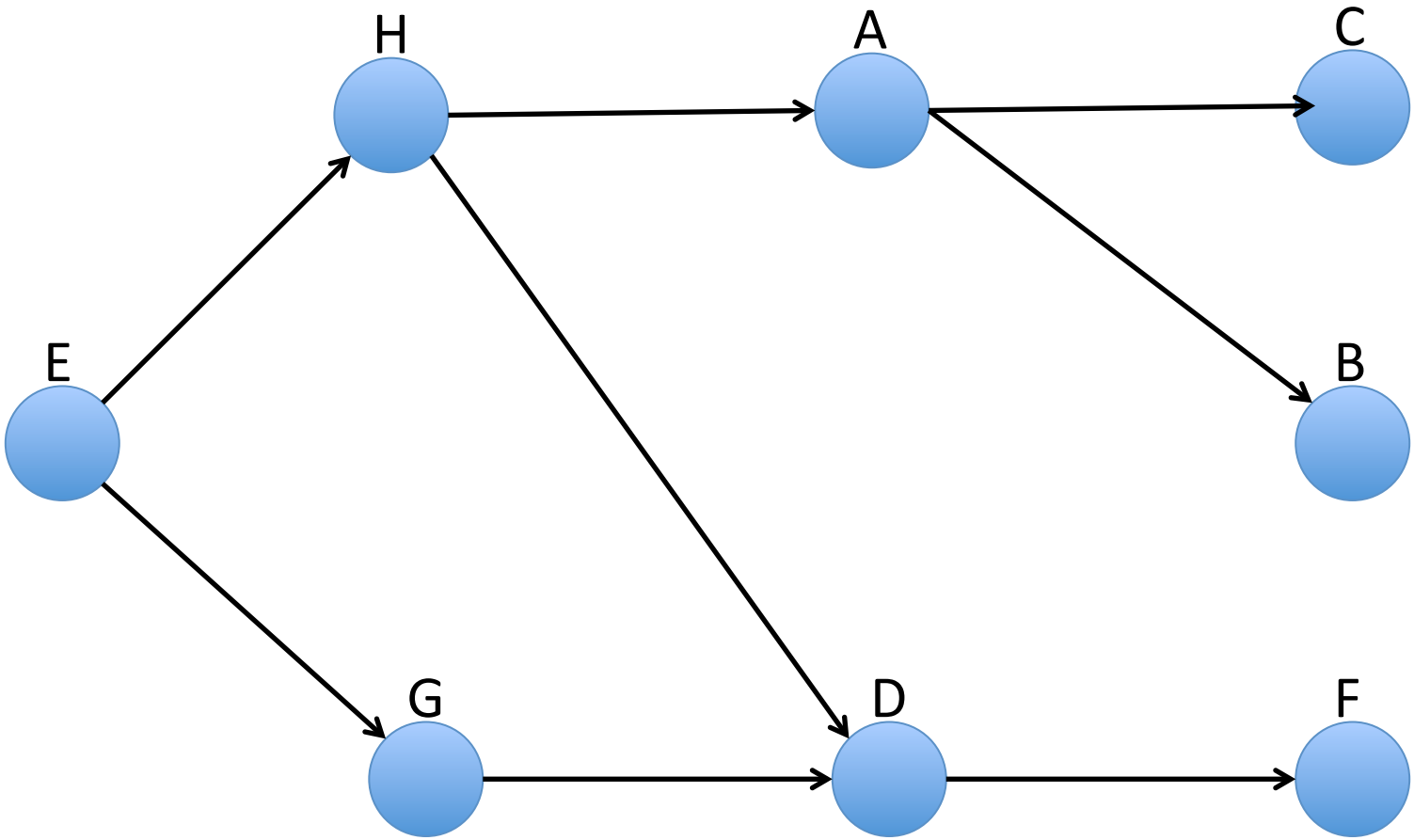


Order By Decreasing Finishing Times (1)

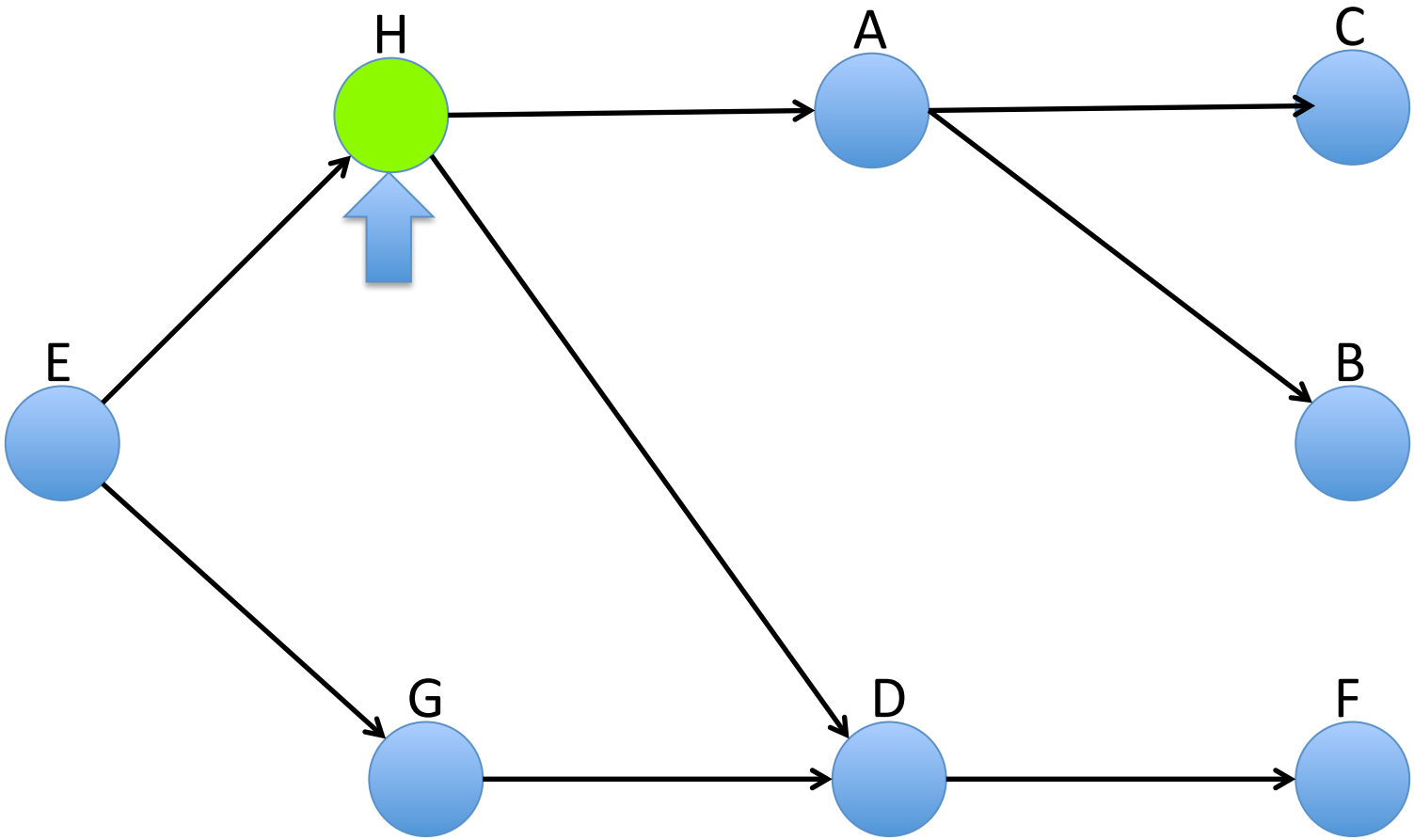




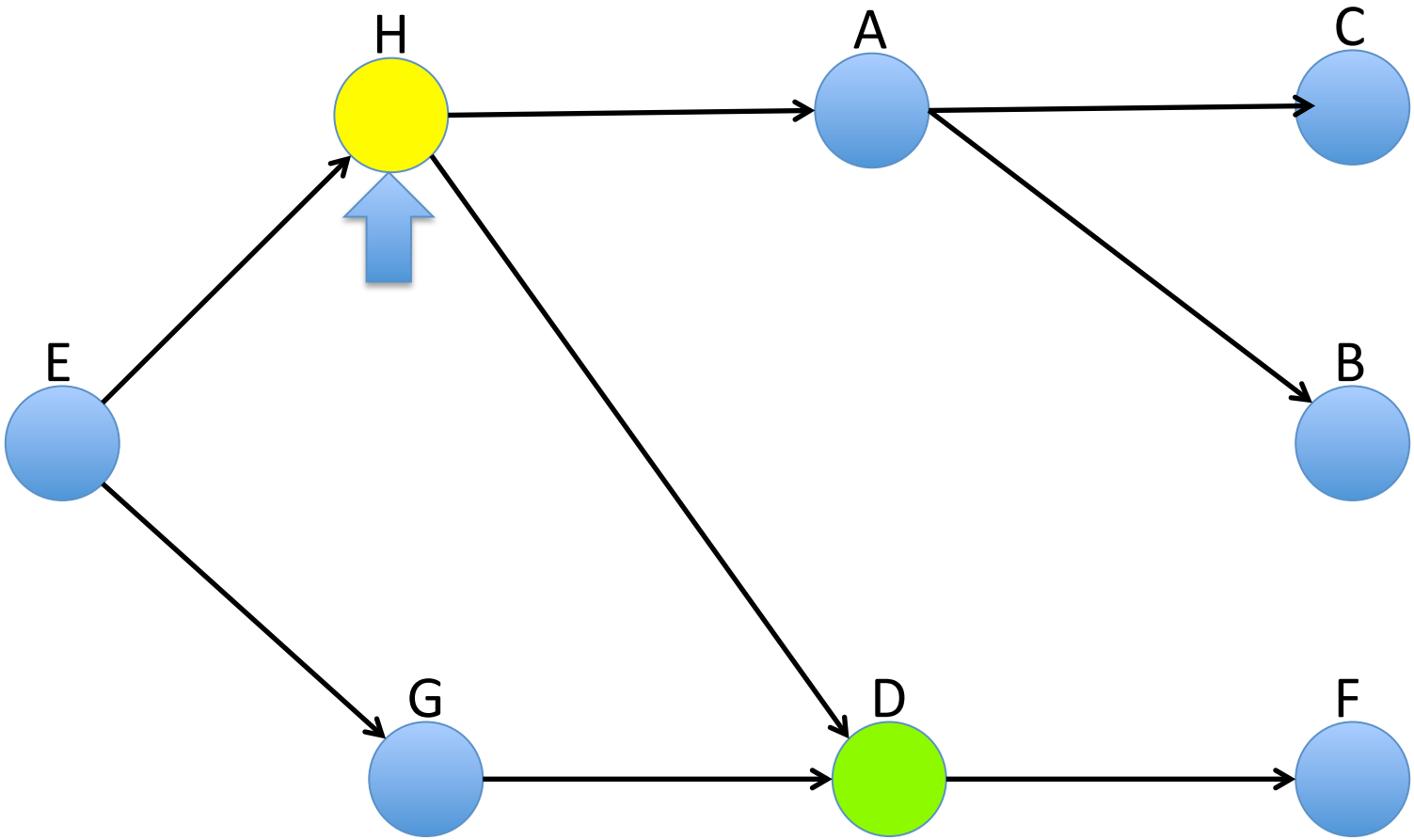
TS Algorithm #2 (Using DFS) Simulation 2



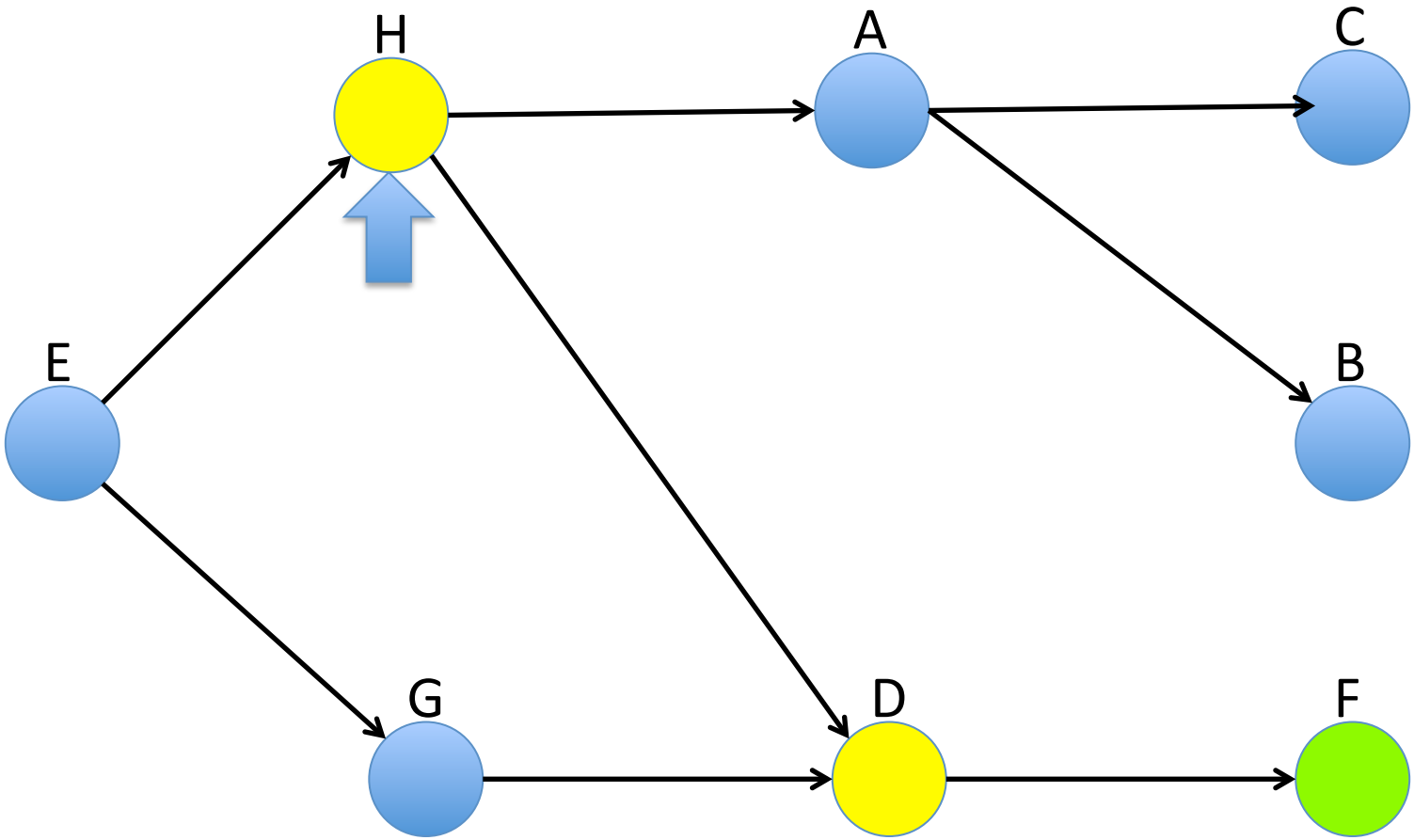
TS Algorithm #2 (Using DFS) Simulation 2



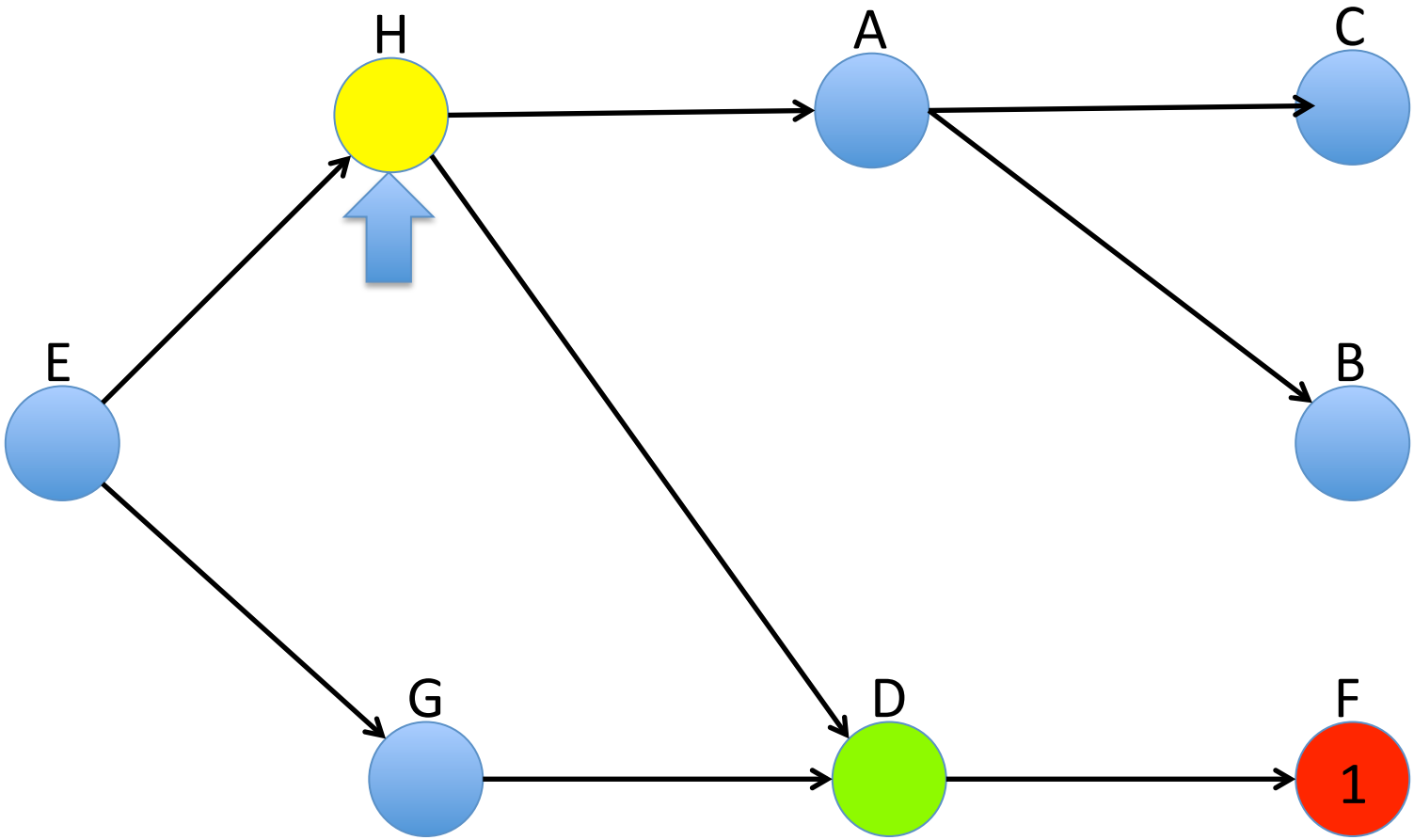
TS Algorithm #2 (Using DFS) Simulation 2



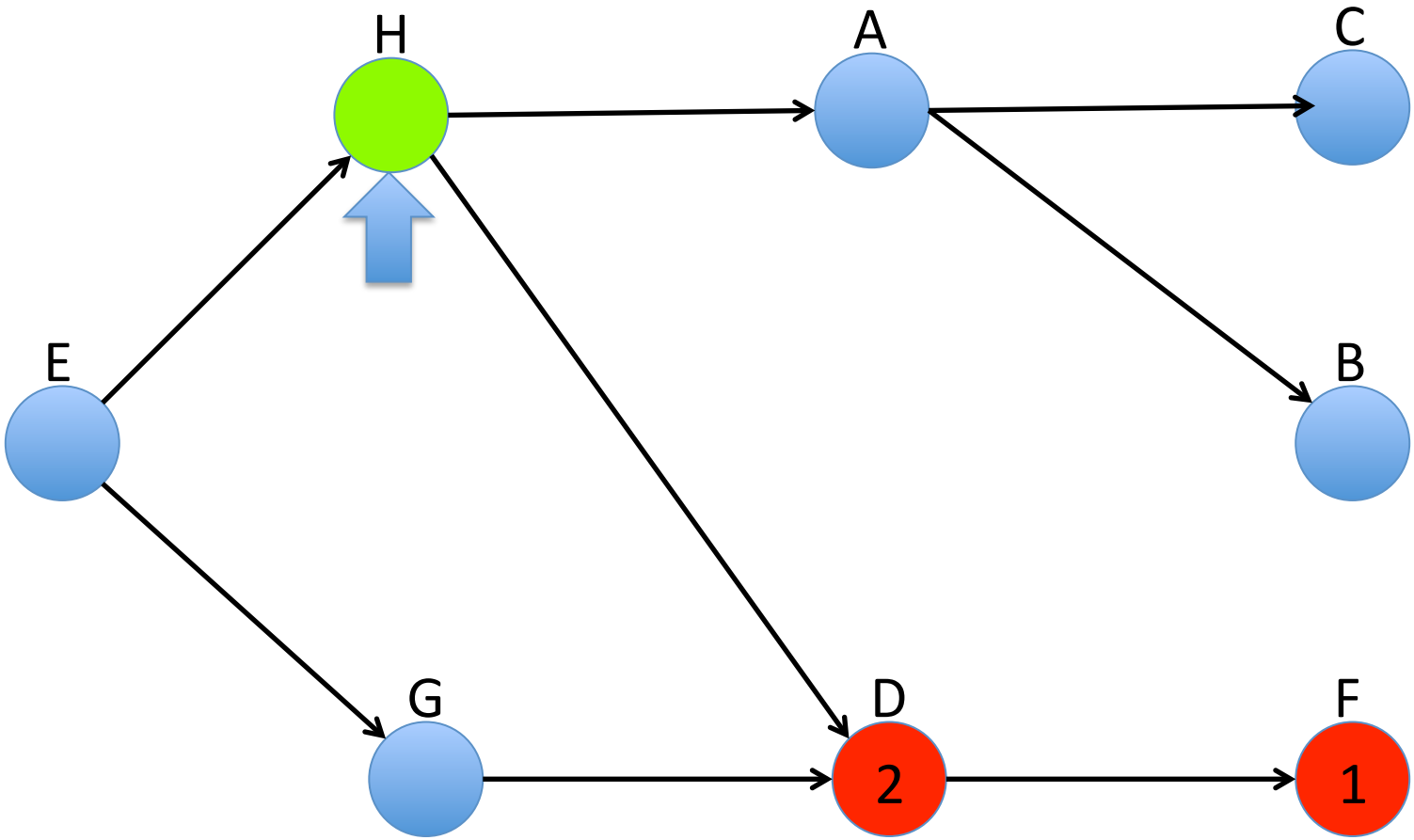
TS Algorithm #2 (Using DFS) Simulation 2



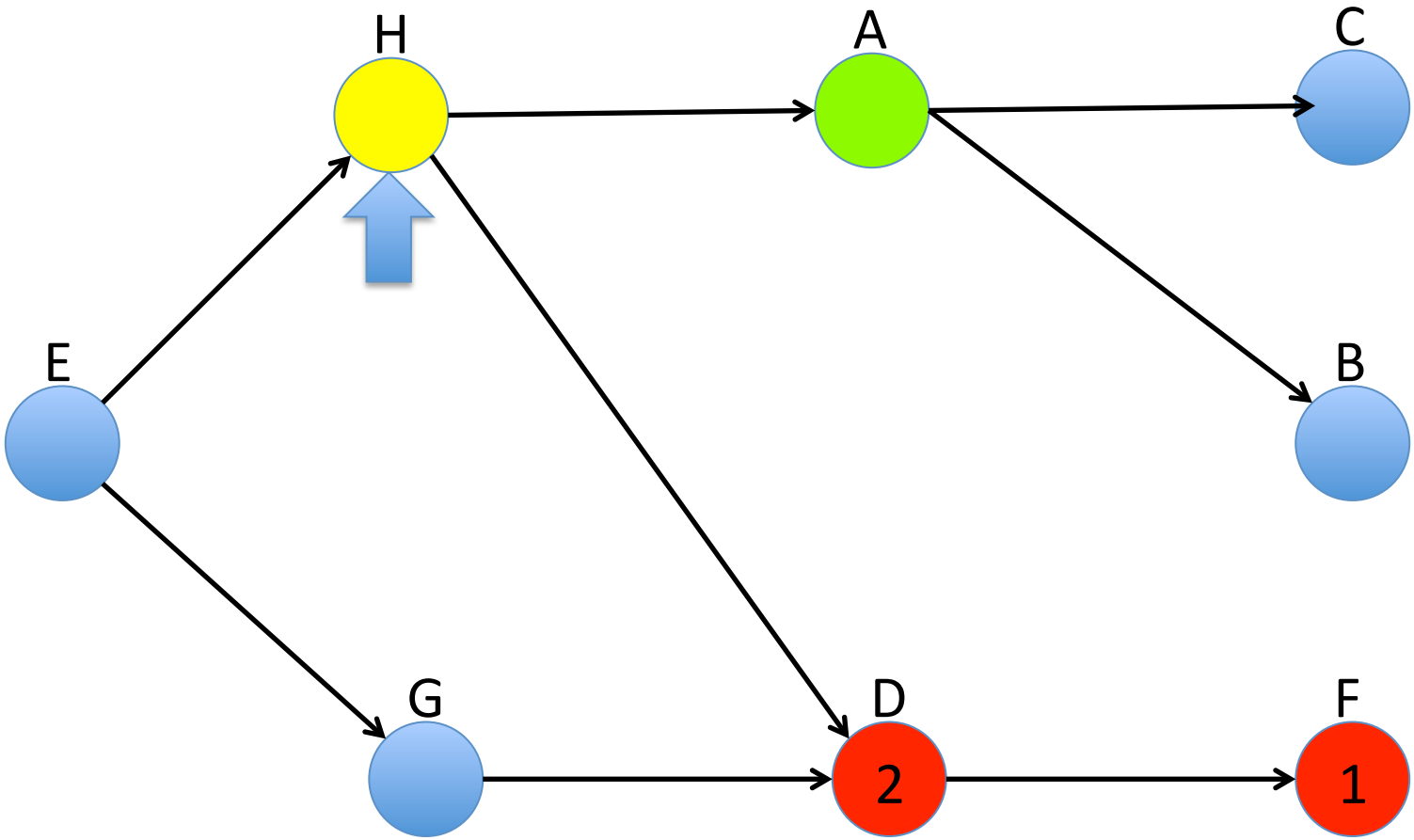
TS Algorithm #2 (Using DFS) Simulation 2



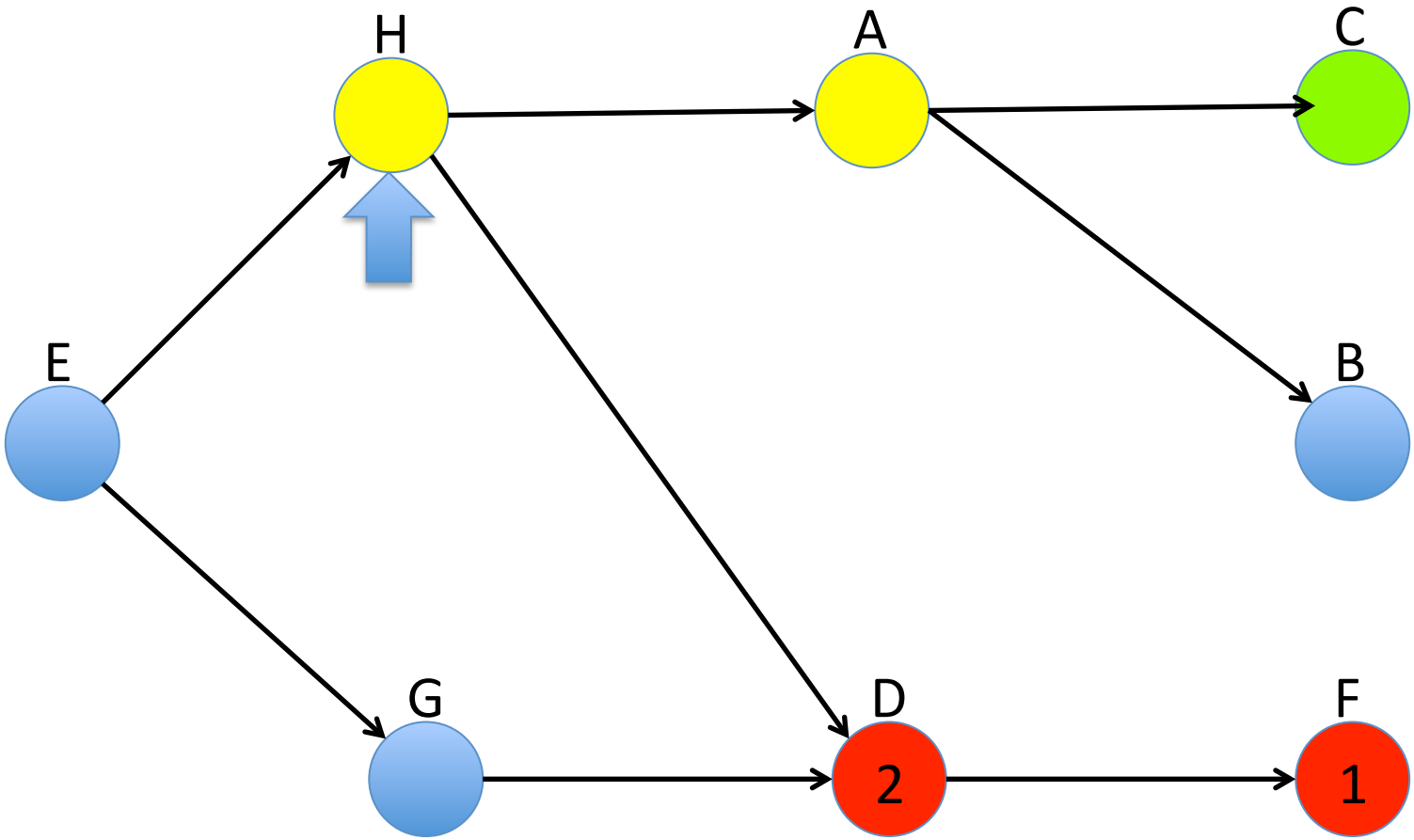
TS Algorithm #2 (Using DFS) Simulation 2



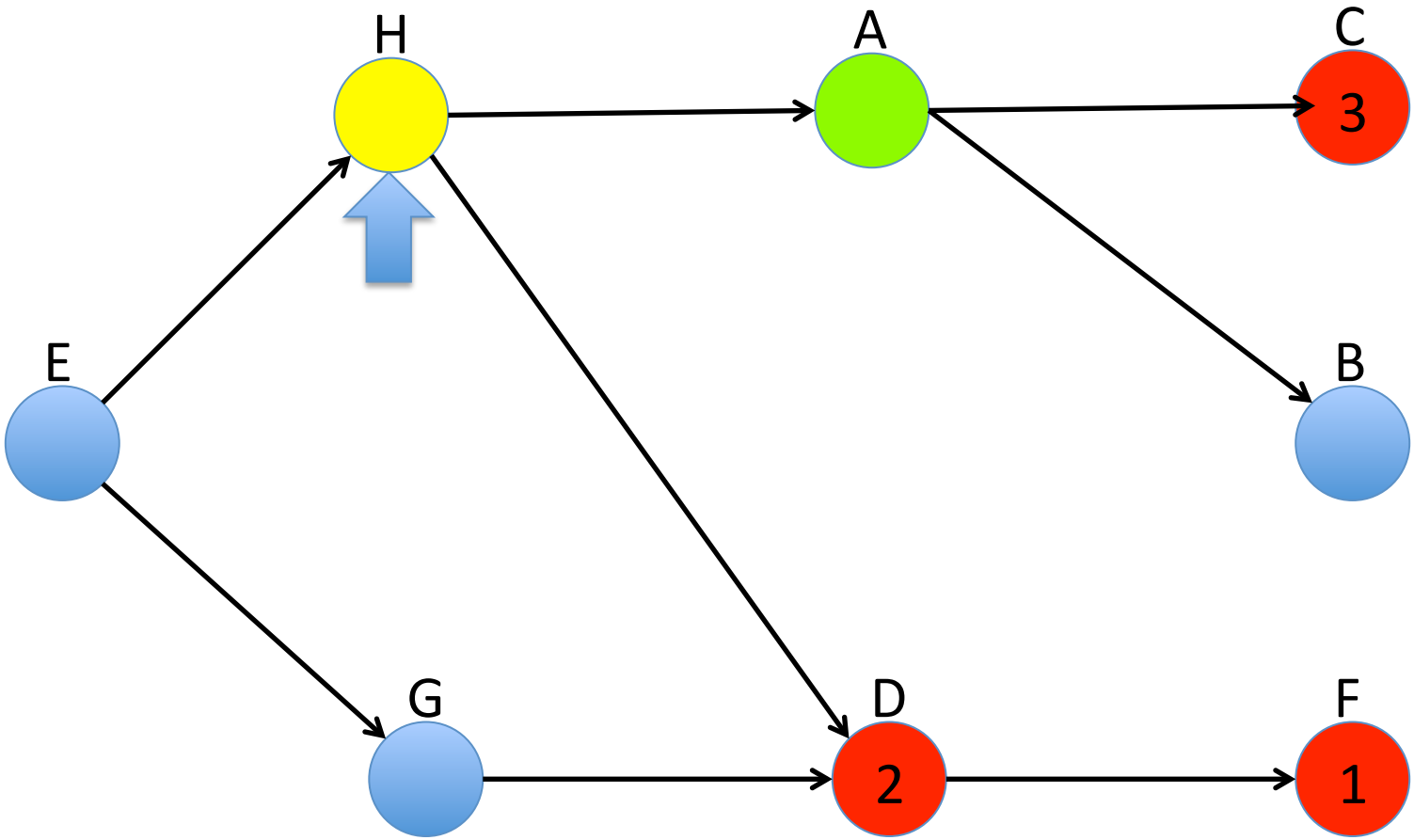
TS Algorithm #2 (Using DFS) Simulation 2



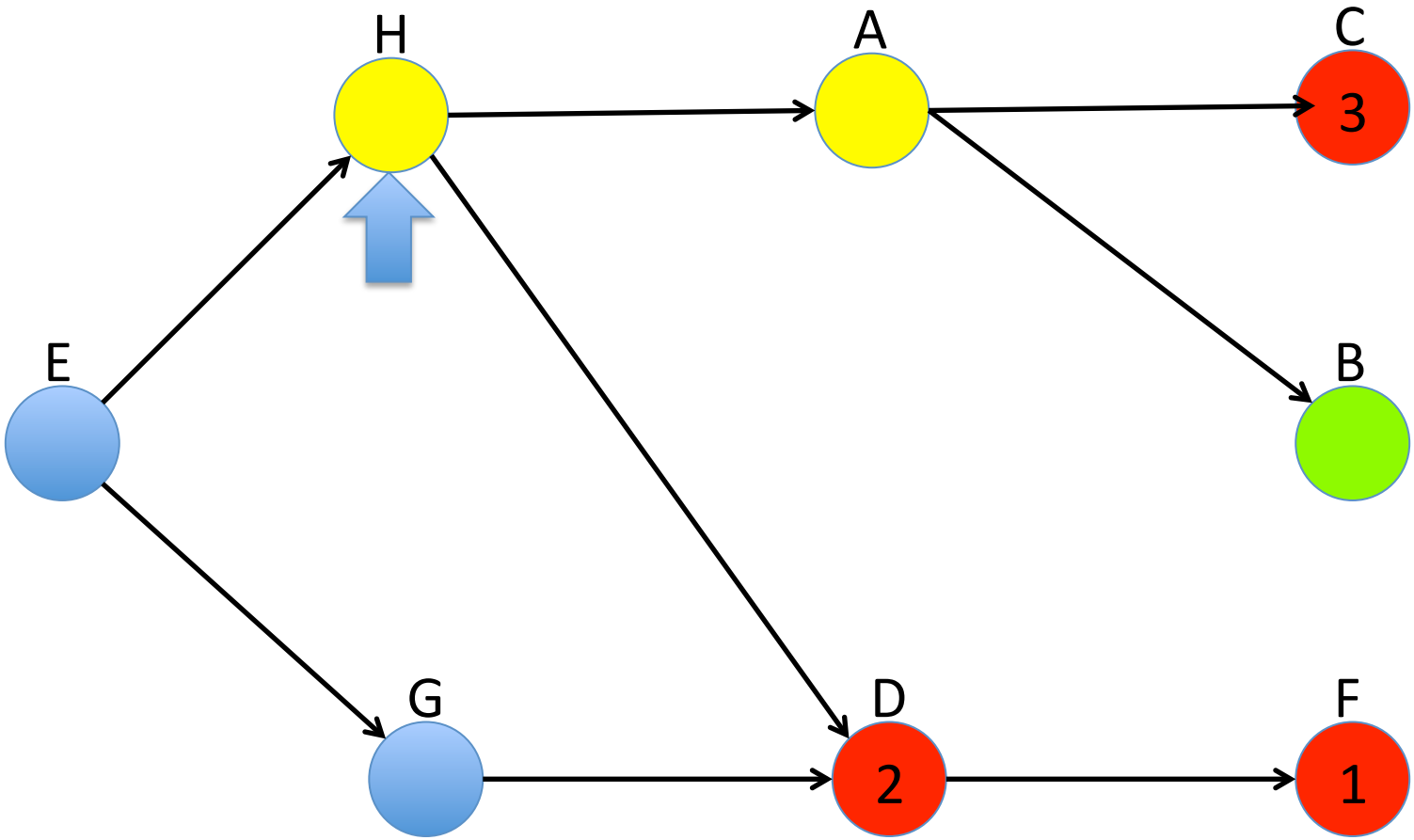
TS Algorithm #2 (Using DFS) Simulation 2



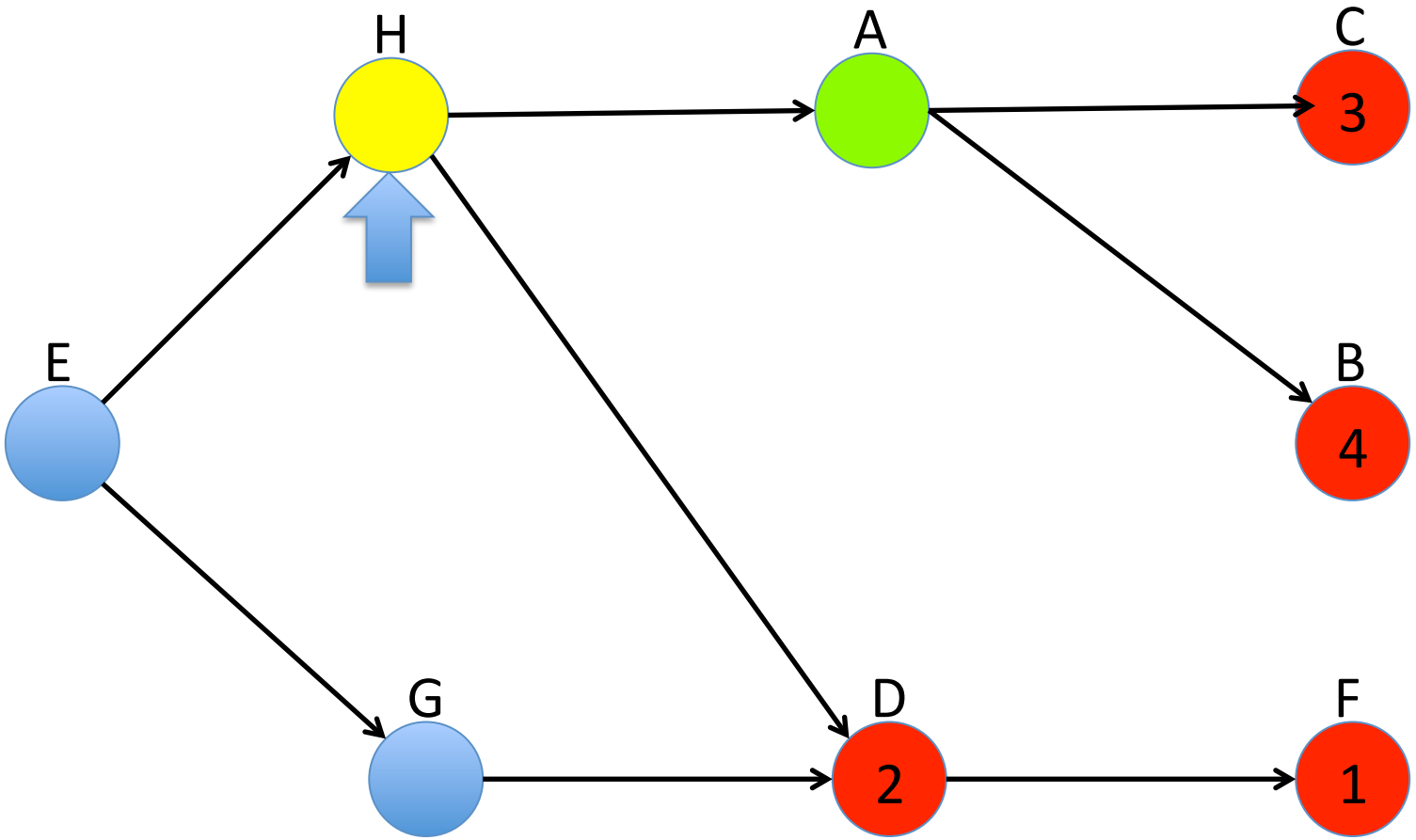
TS Algorithm #2 (Using DFS) Simulation 2



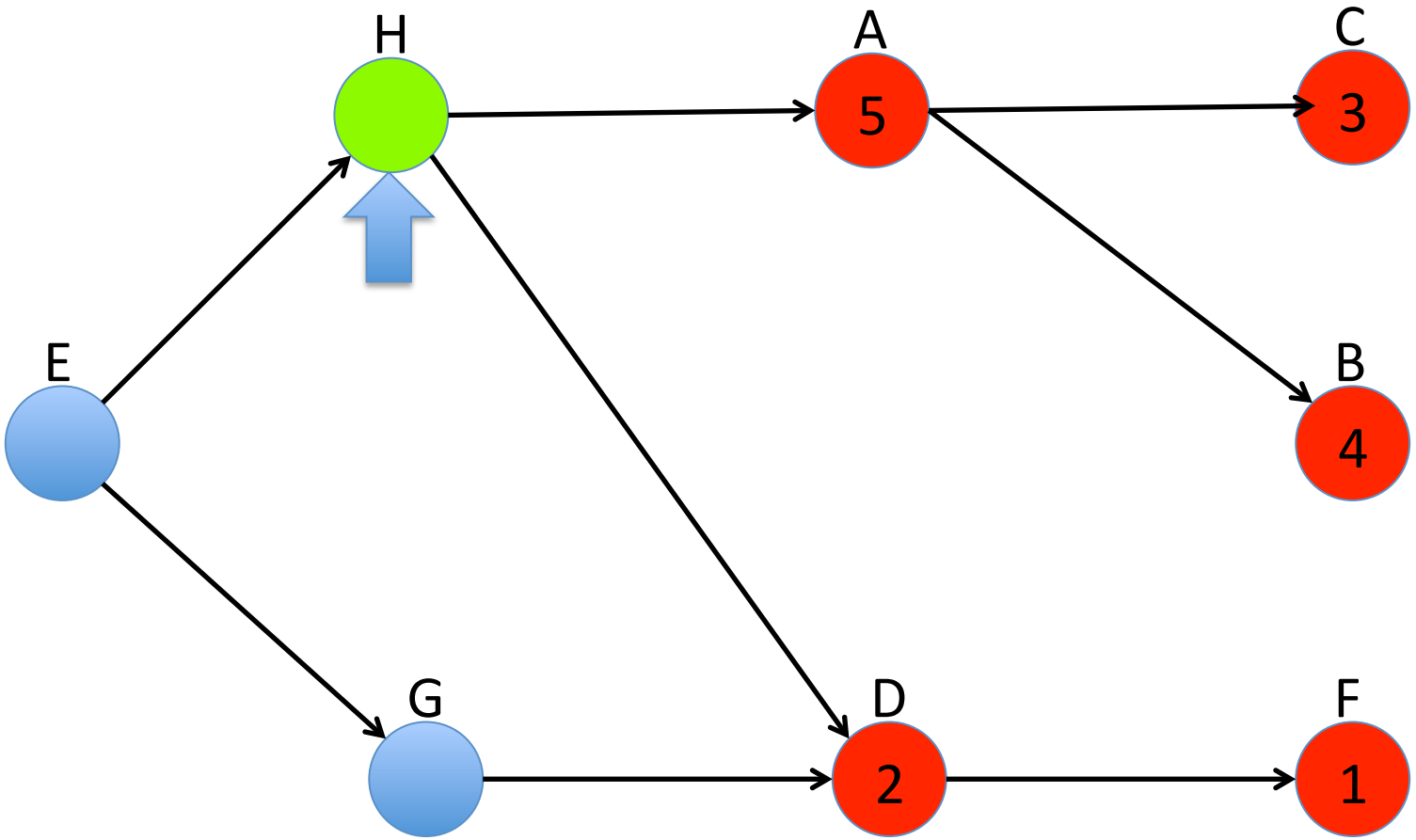
TS Algorithm #2 (Using DFS) Simulation 2



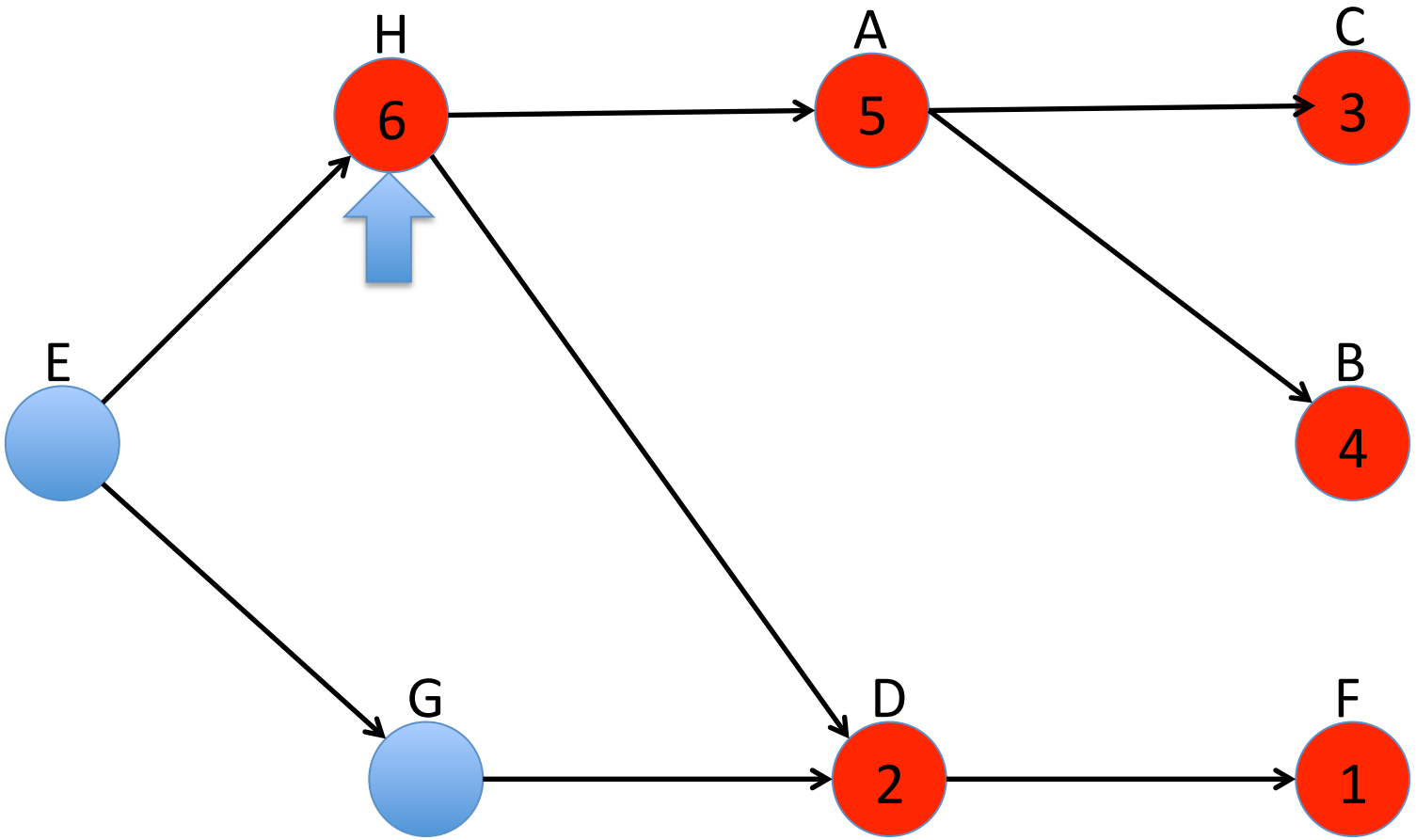
TS Algorithm #2 (Using DFS) Simulation 2



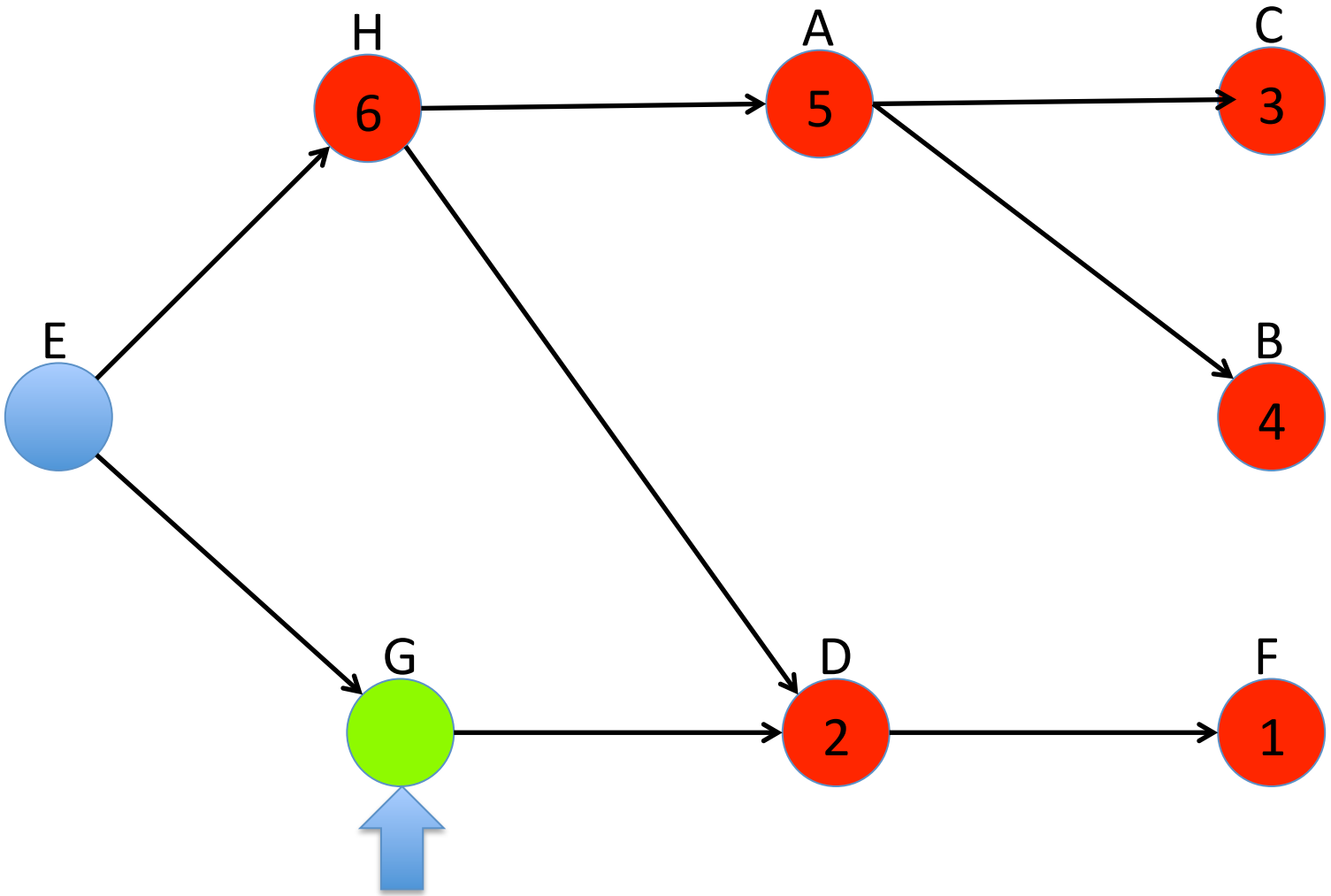
TS Algorithm #2 (Using DFS) Simulation 2



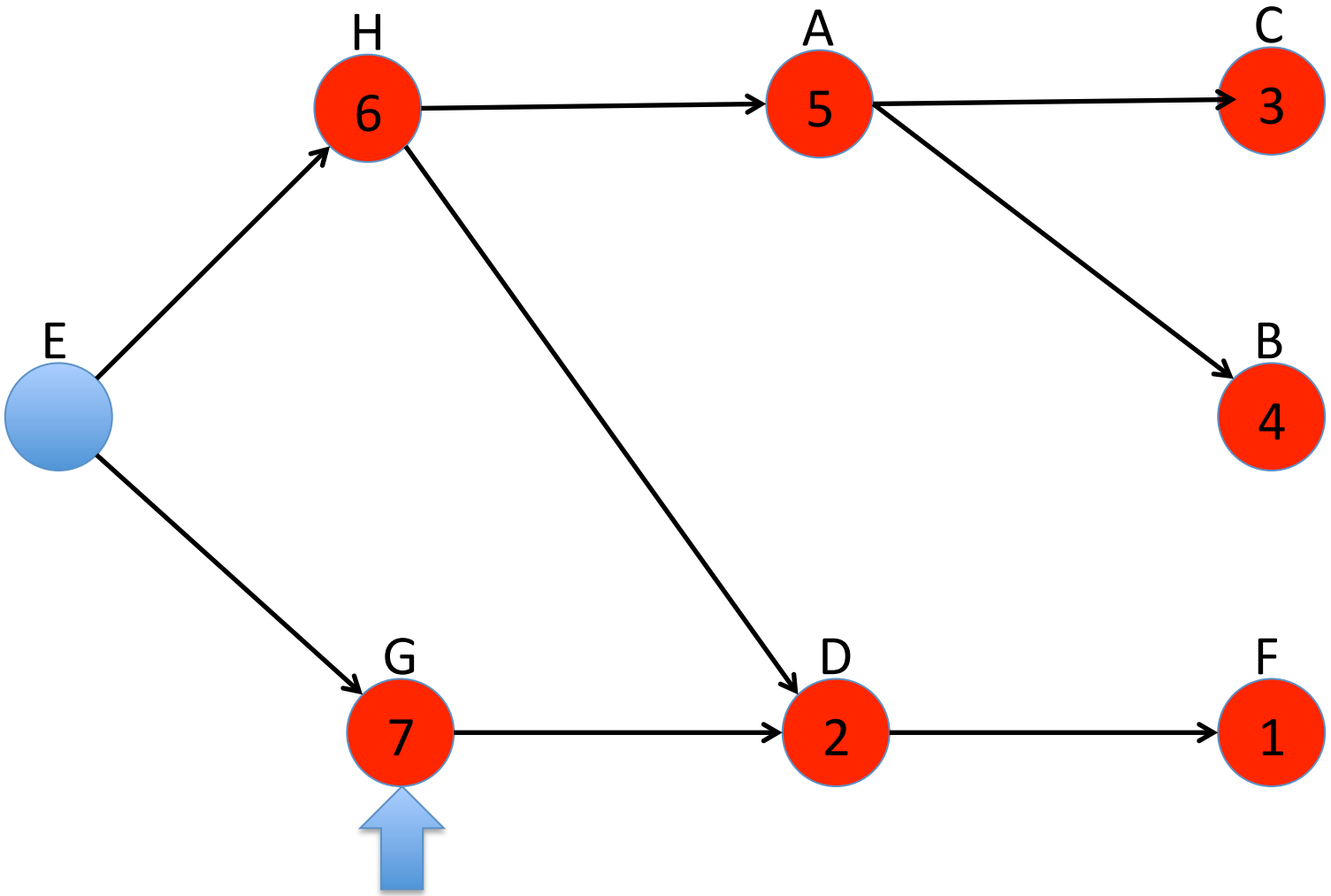
TS Algorithm #2 (Using DFS) Simulation 2



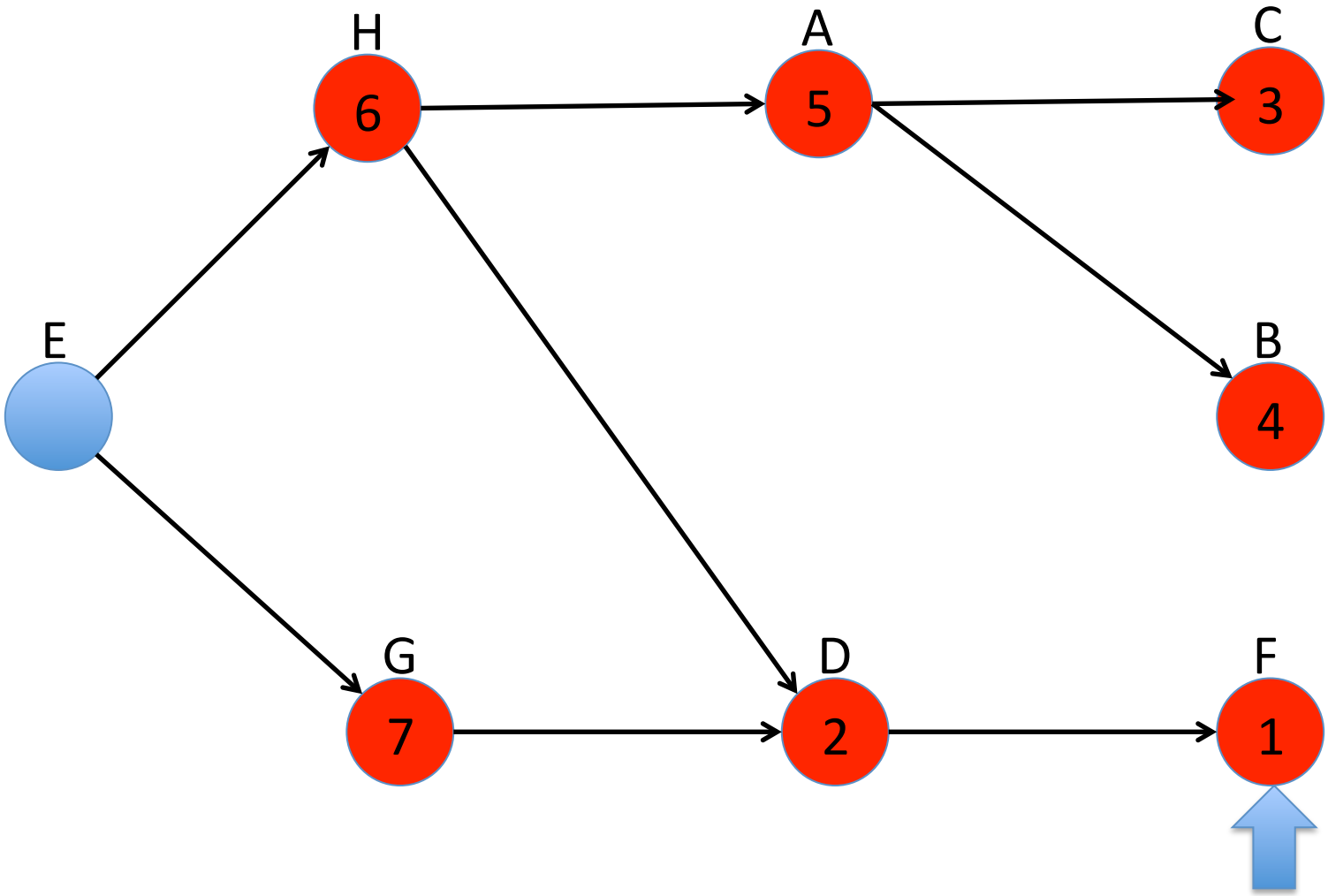
TS Algorithm #2 (Using DFS) Simulation 2



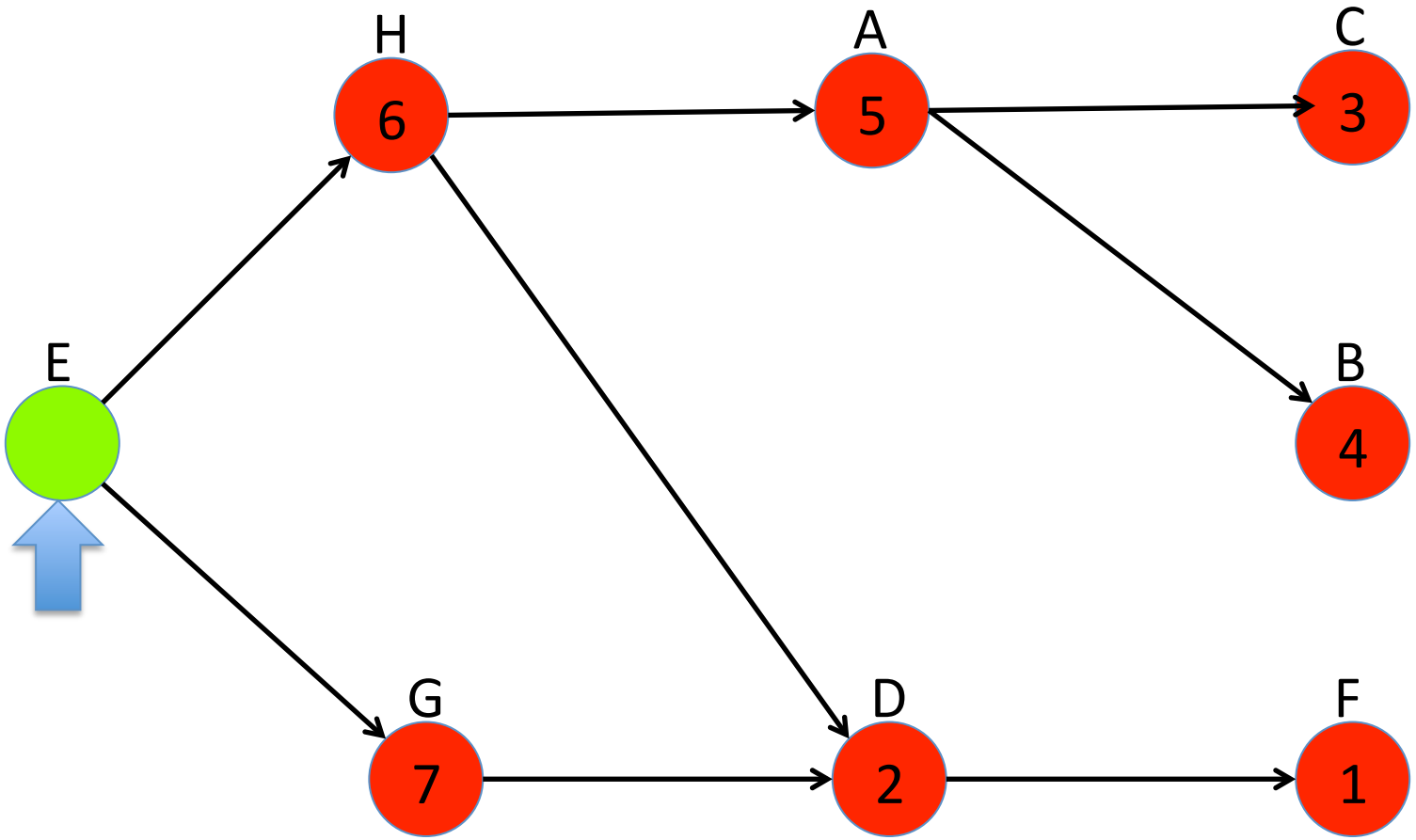
TS Algorithm #2 (Using DFS) Simulation 2



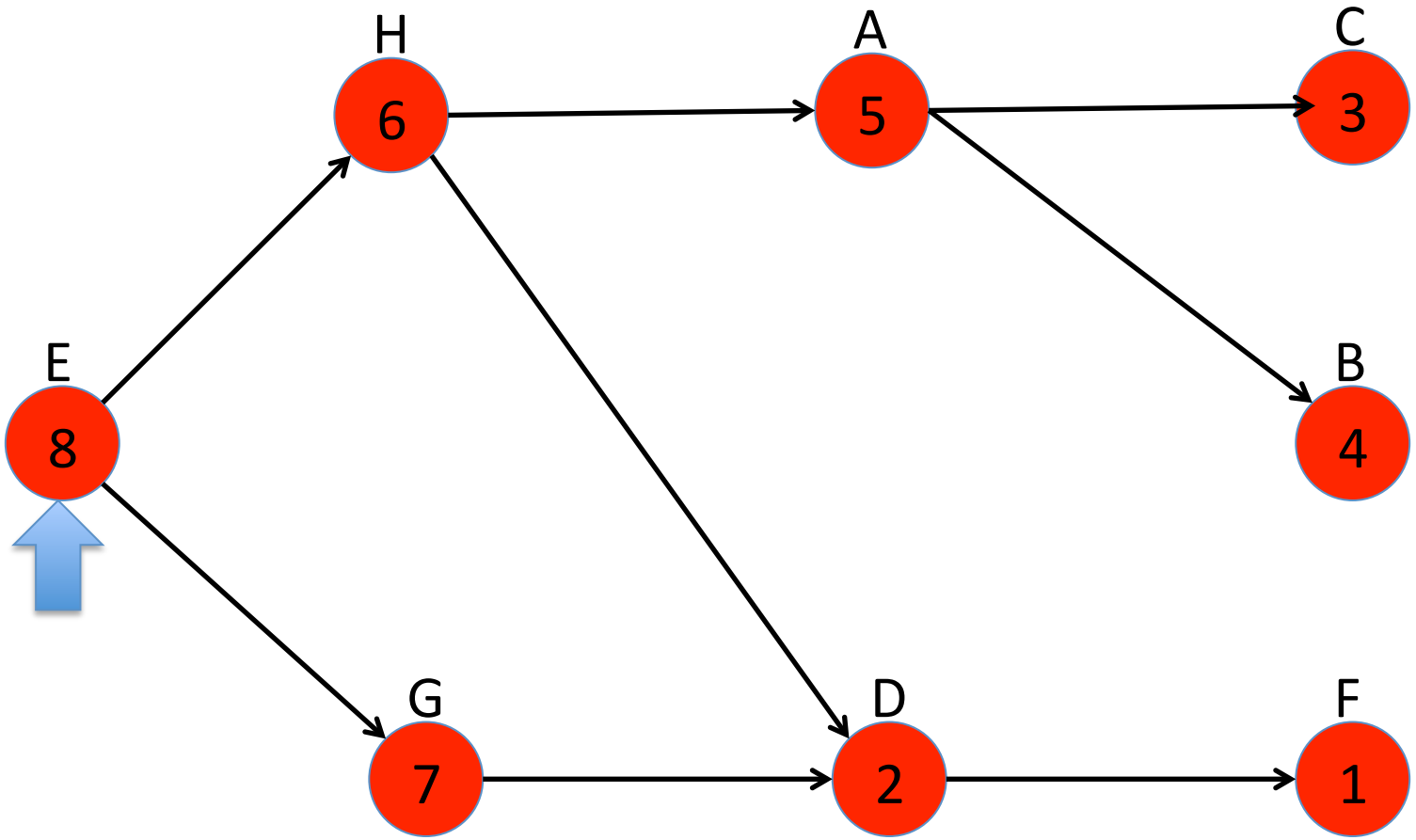
TS Algorithm #2 (Using DFS) Simulation 2



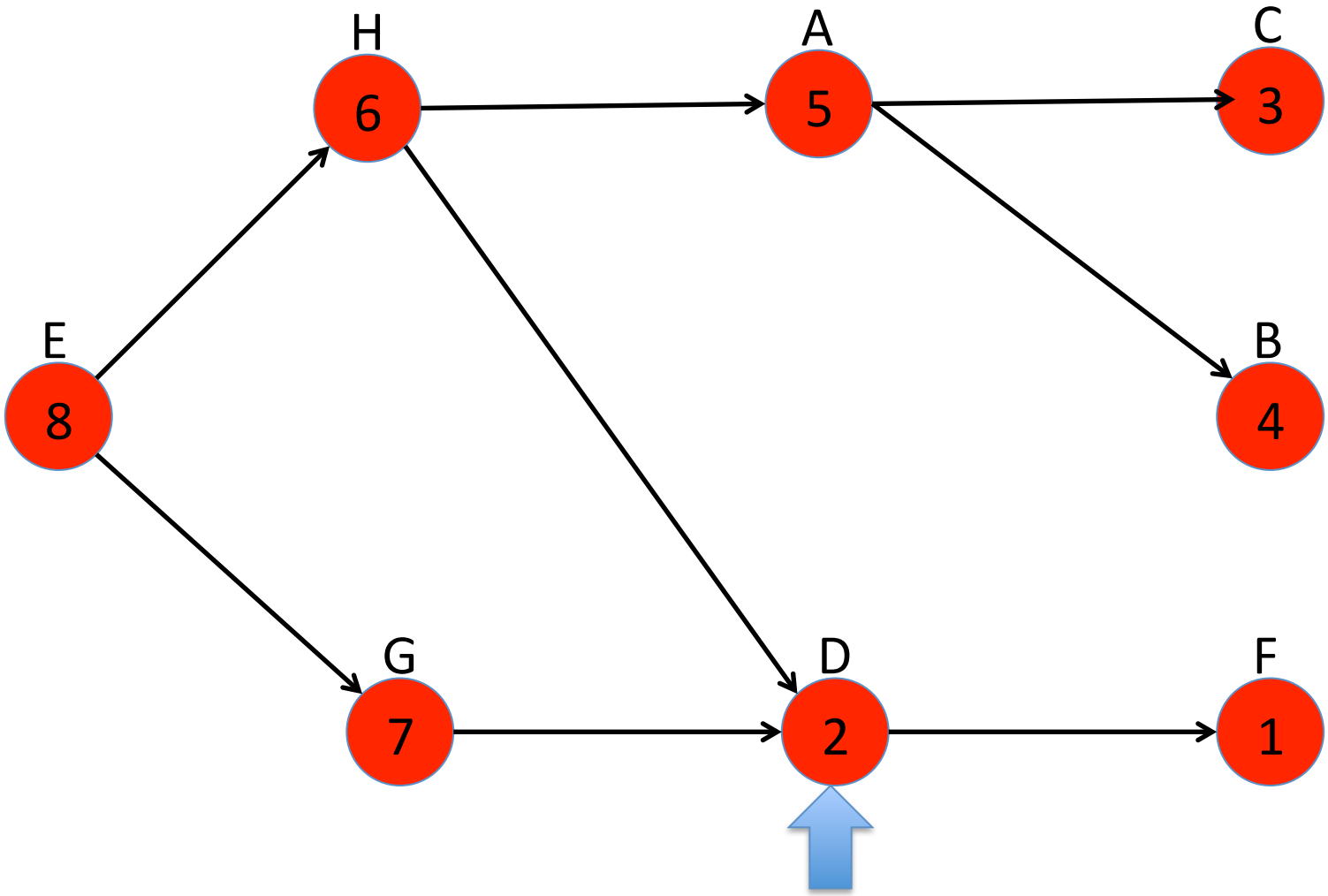
TS Algorithm #2 (Using DFS) Simulation 2



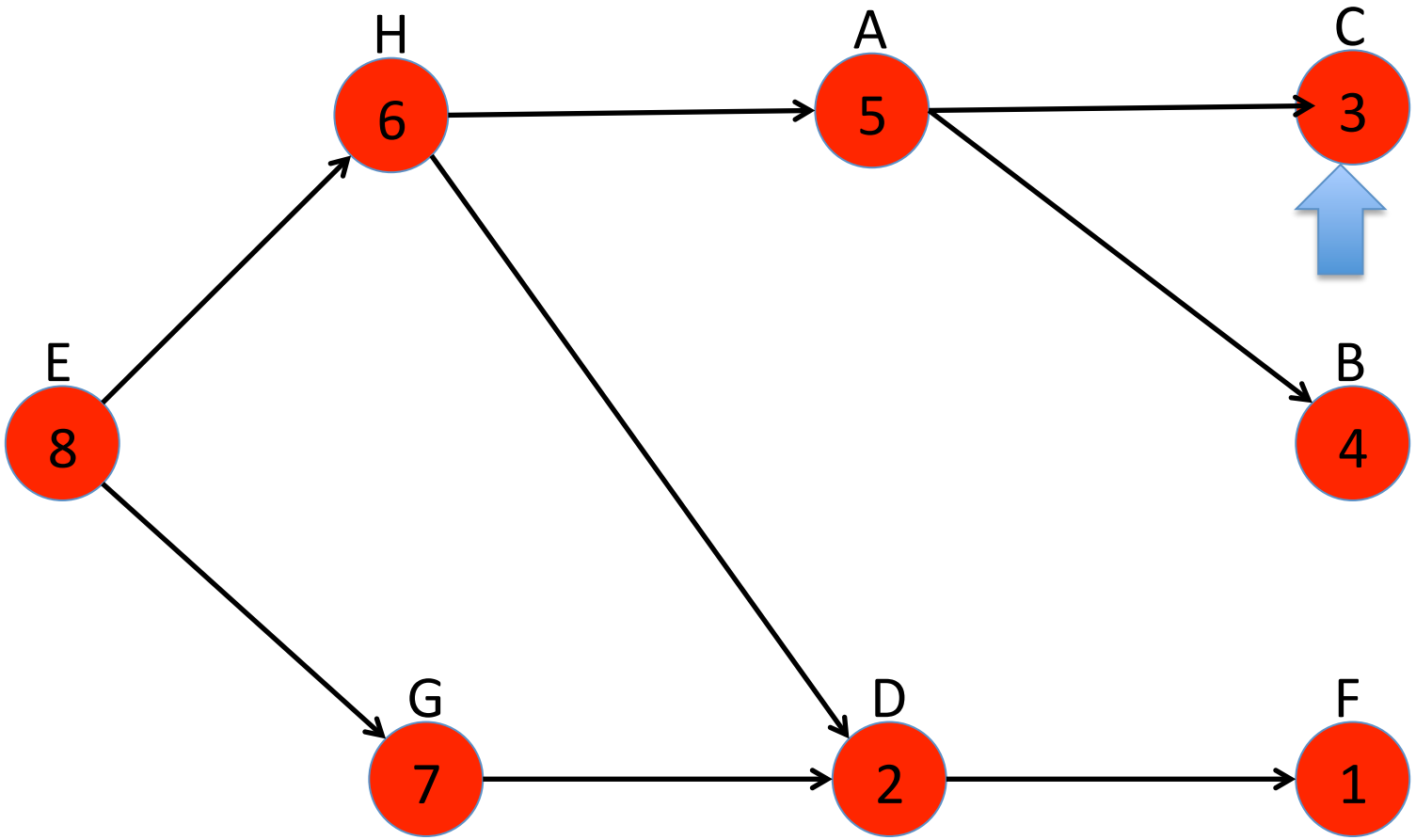
TS Algorithm #2 (Using DFS) Simulation 2



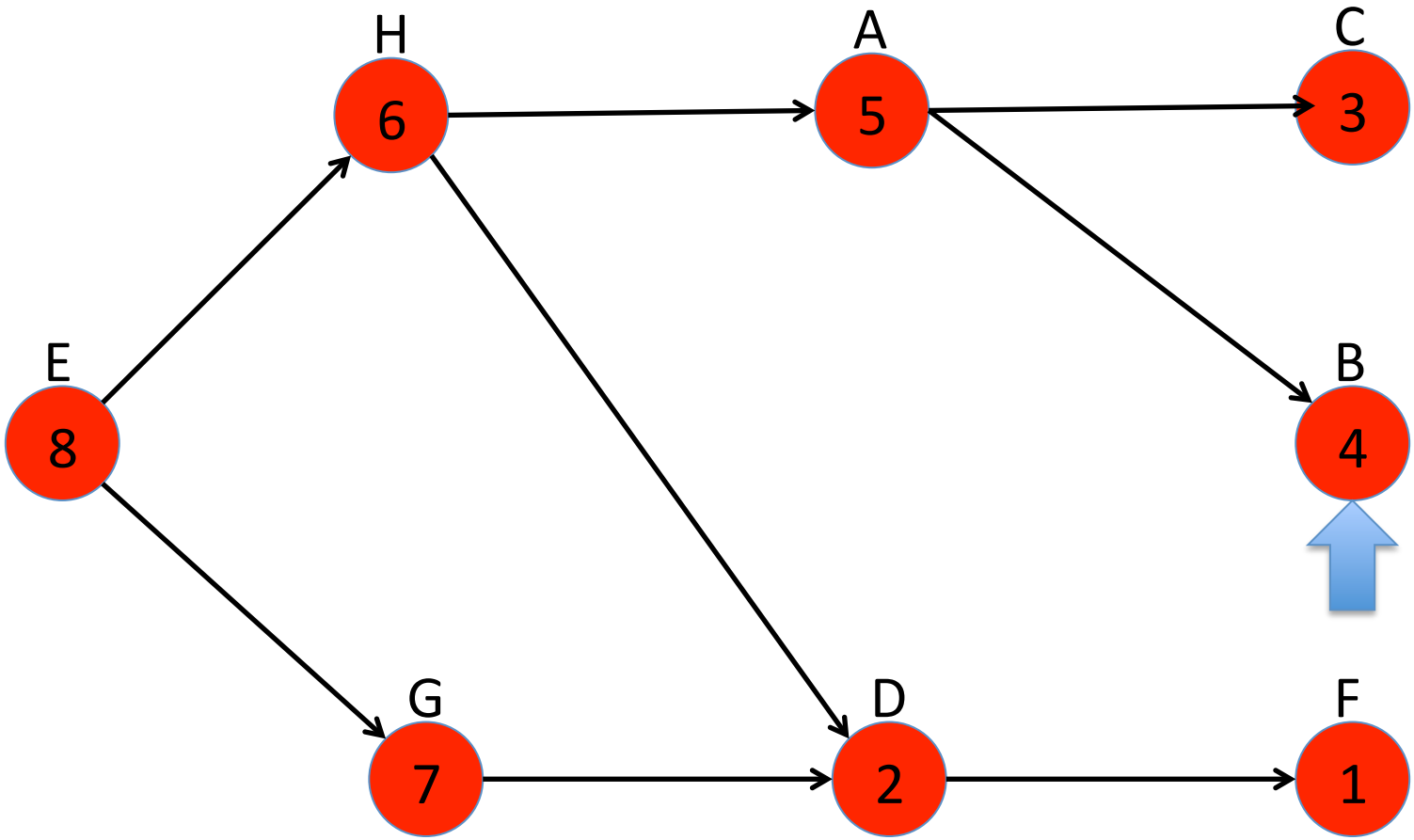
TS Algorithm #2 (Using DFS) Simulation 2



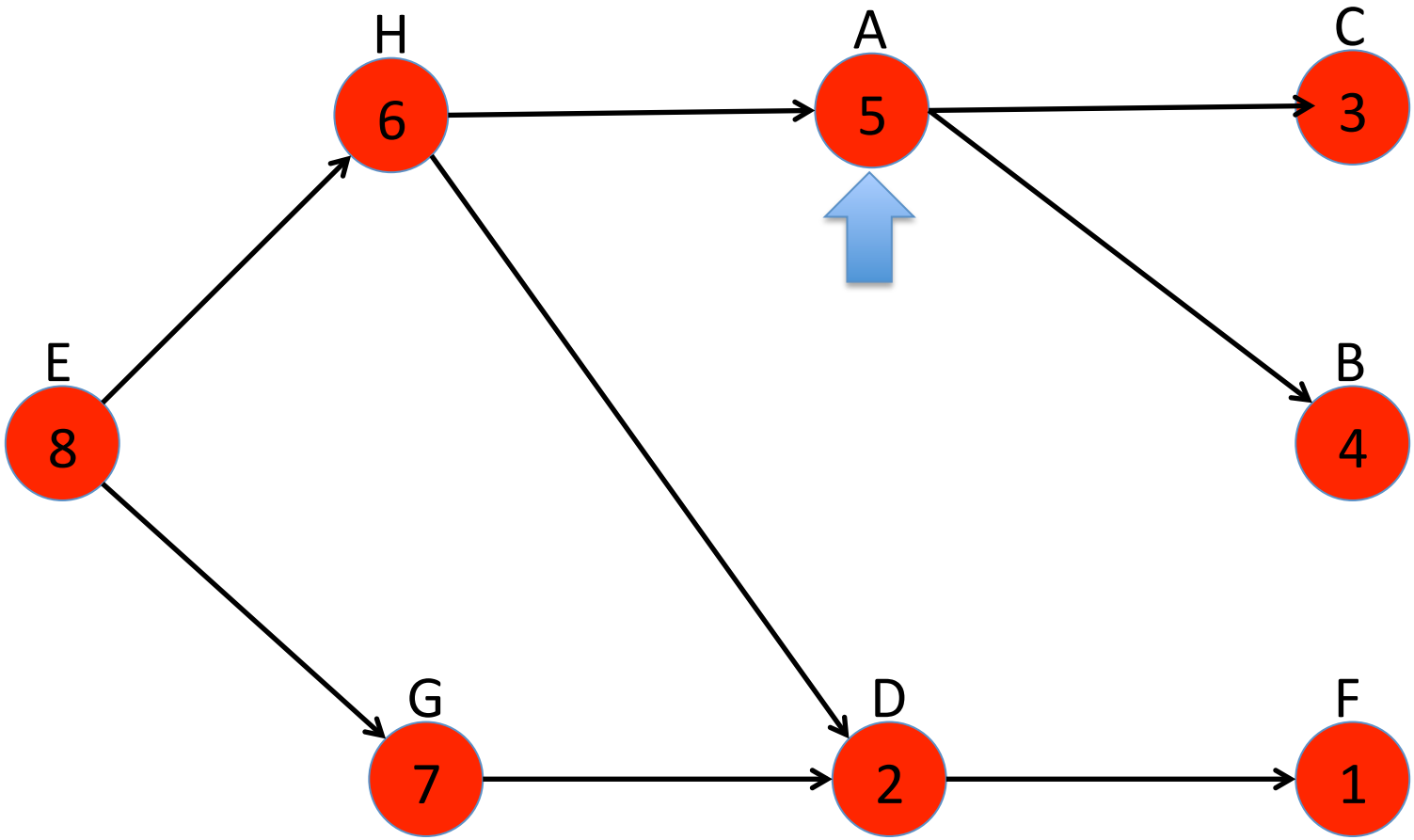
TS Algorithm #2 (Using DFS) Simulation 2



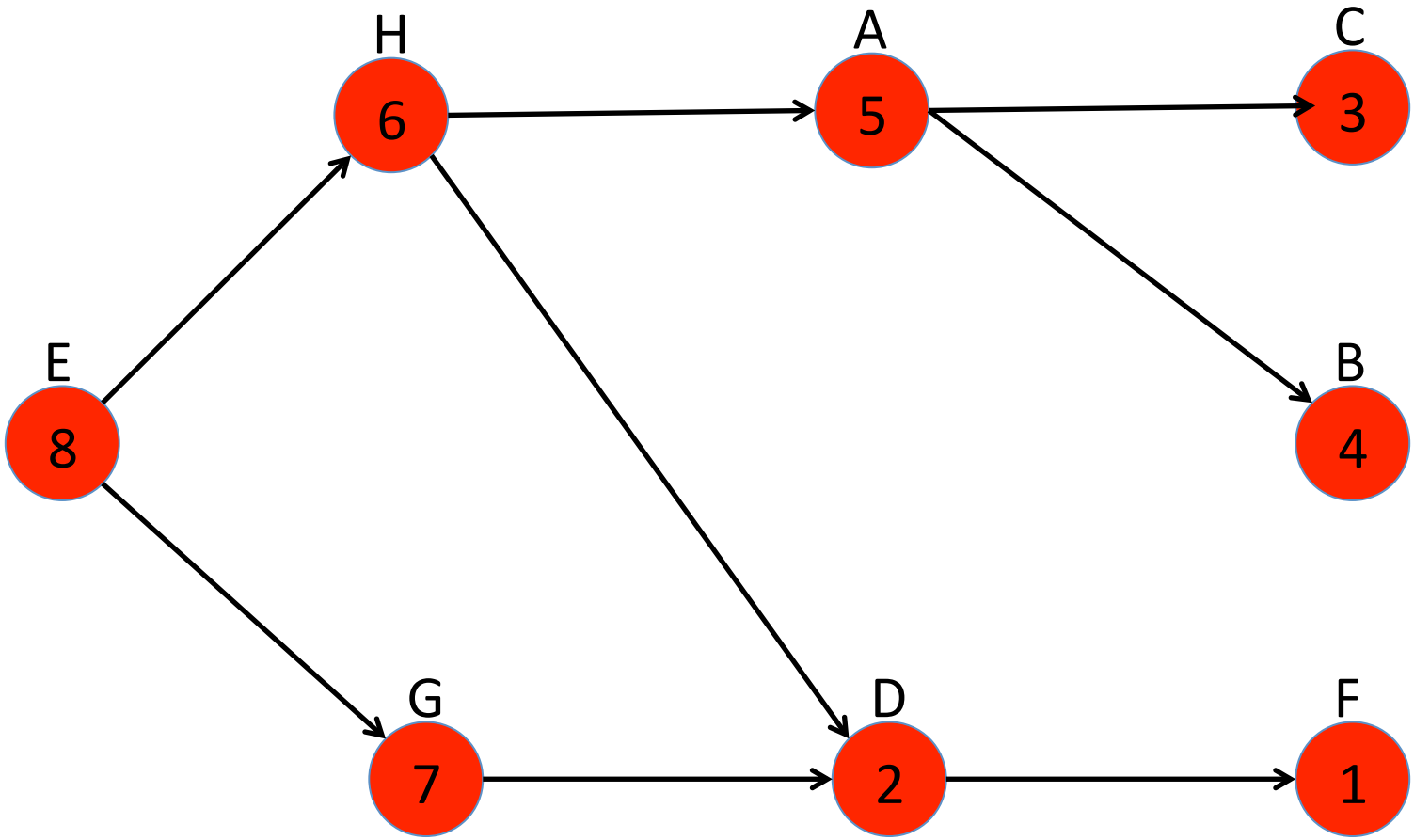
TS Algorithm #2 (Using DFS) Simulation 2



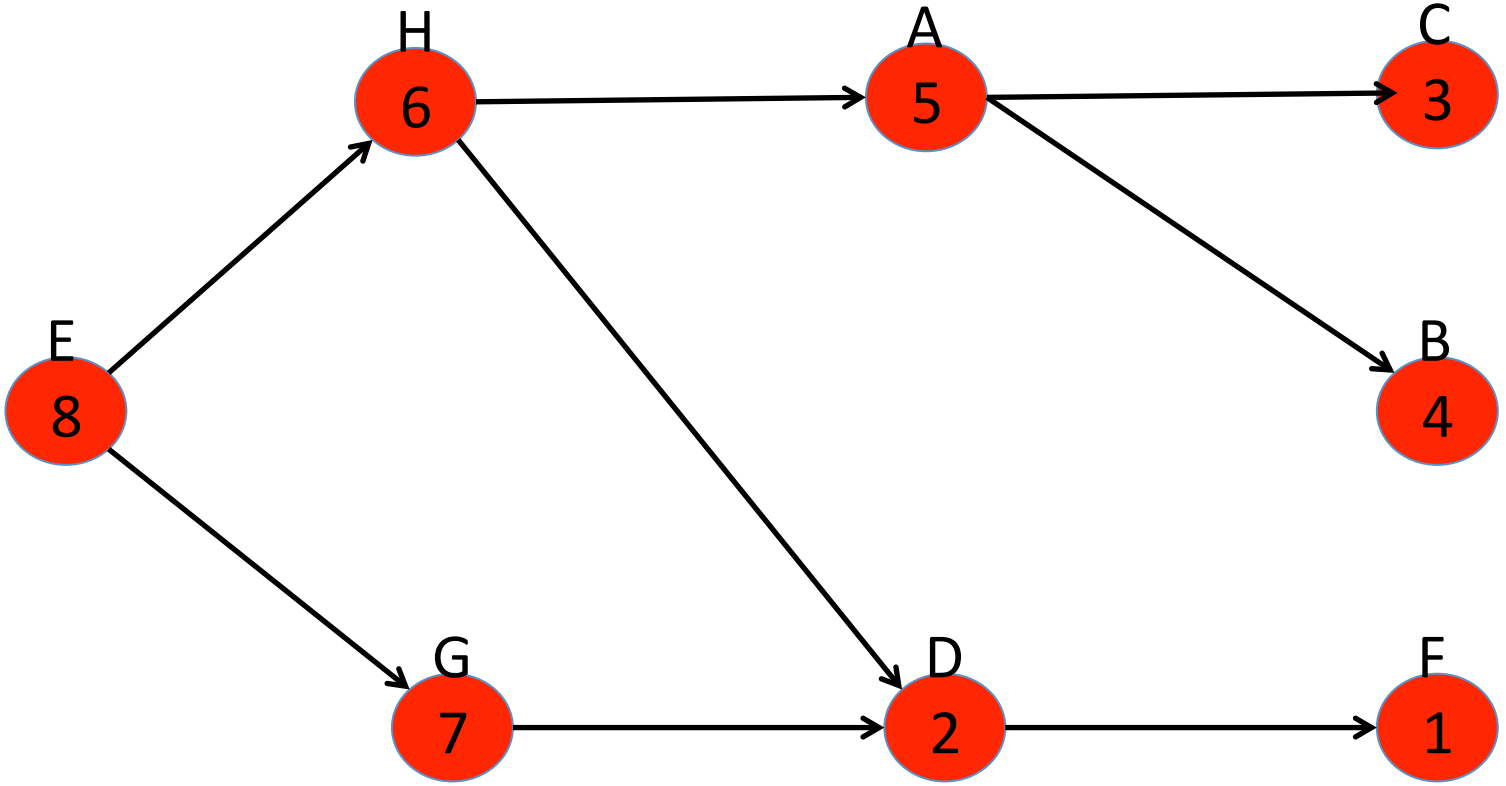
TS Algorithm #2 (Using DFS) Simulation 2

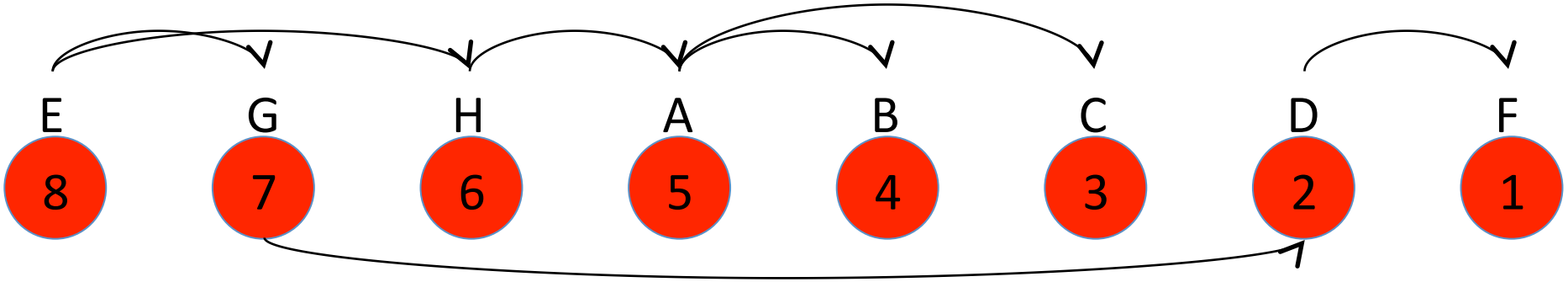


TS Algorithm #2 (Using DFS) Simulation 2

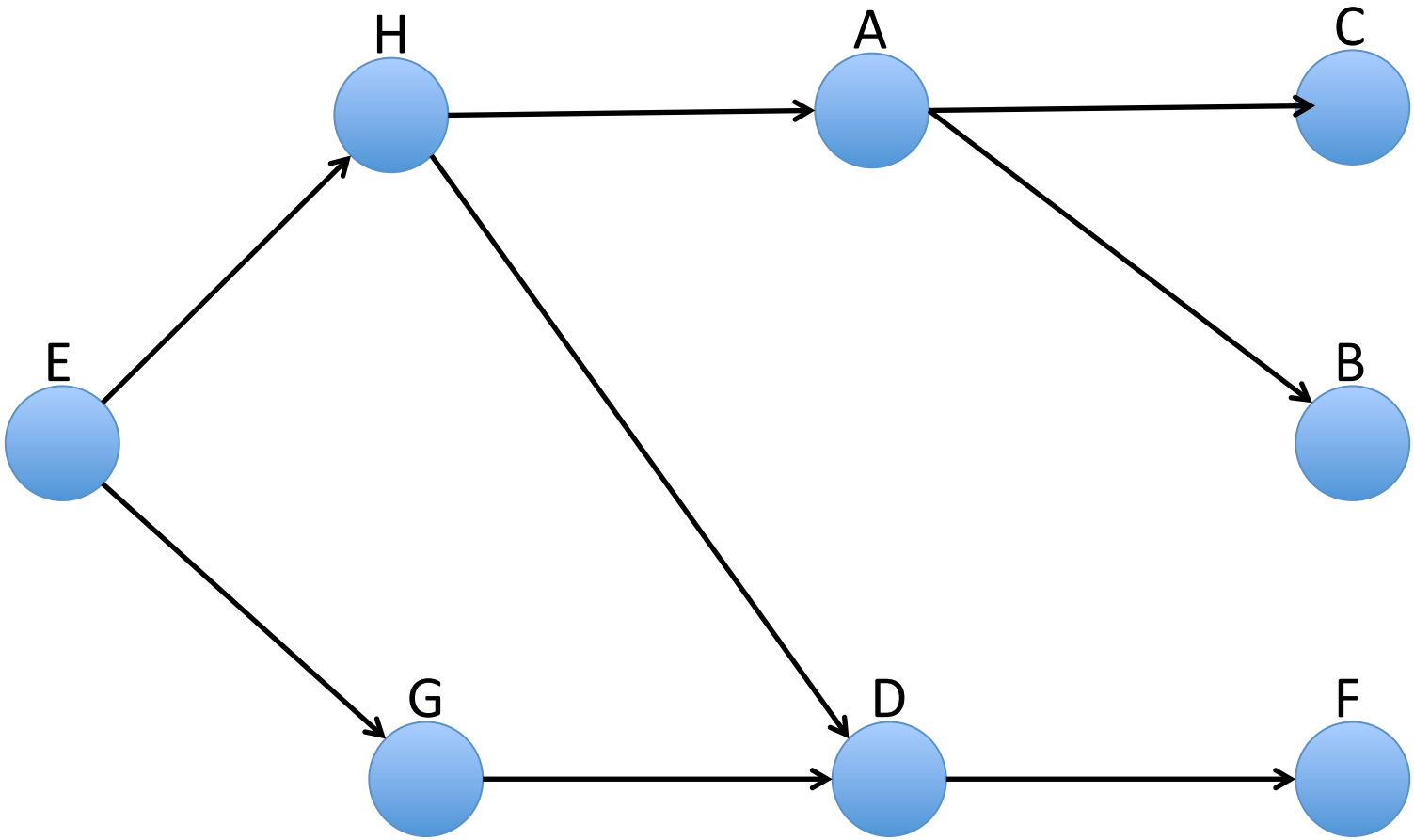


Order By Decreasing Finishing Times (2)

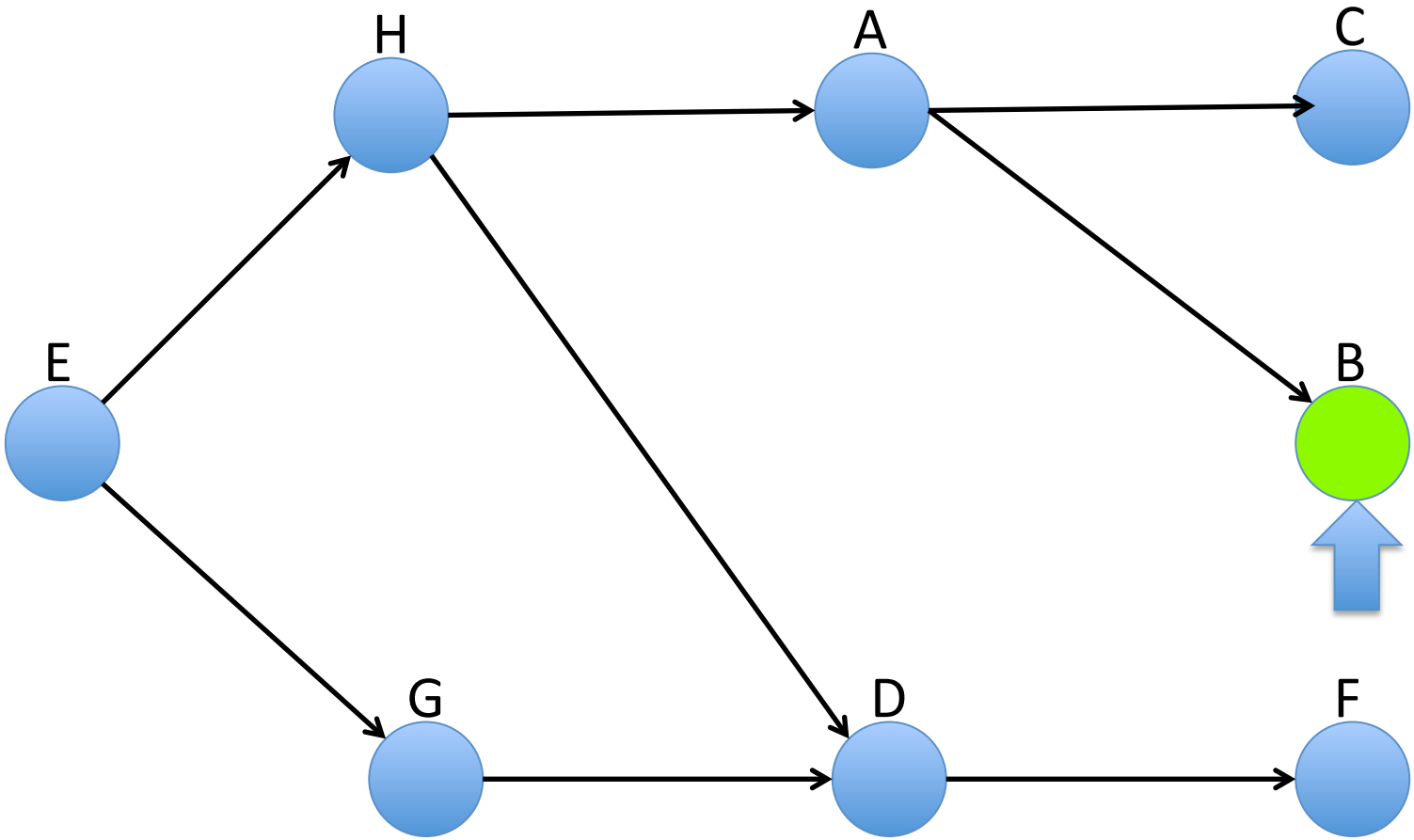




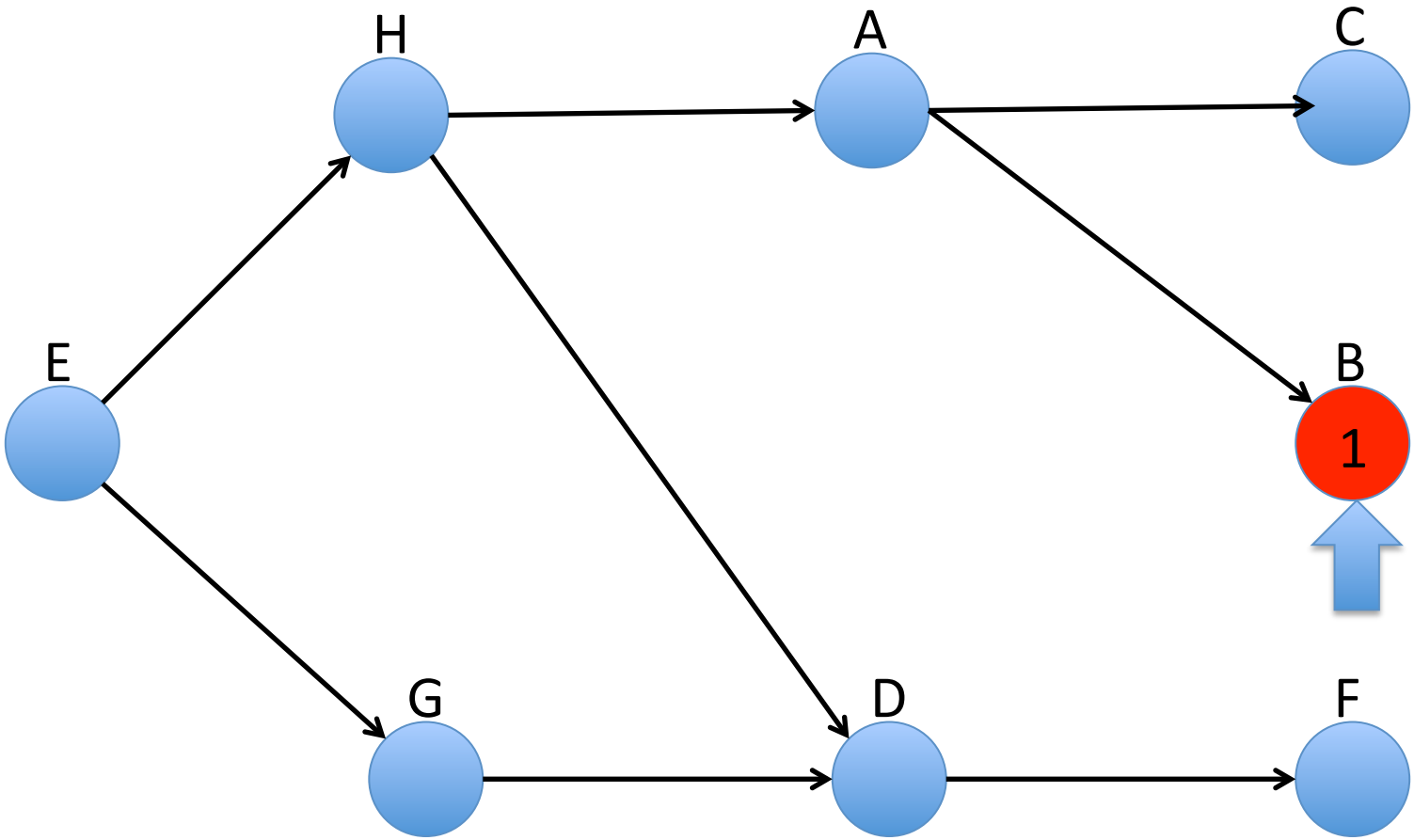
TS Algorithm #2 (Using DFS) Simulation 3



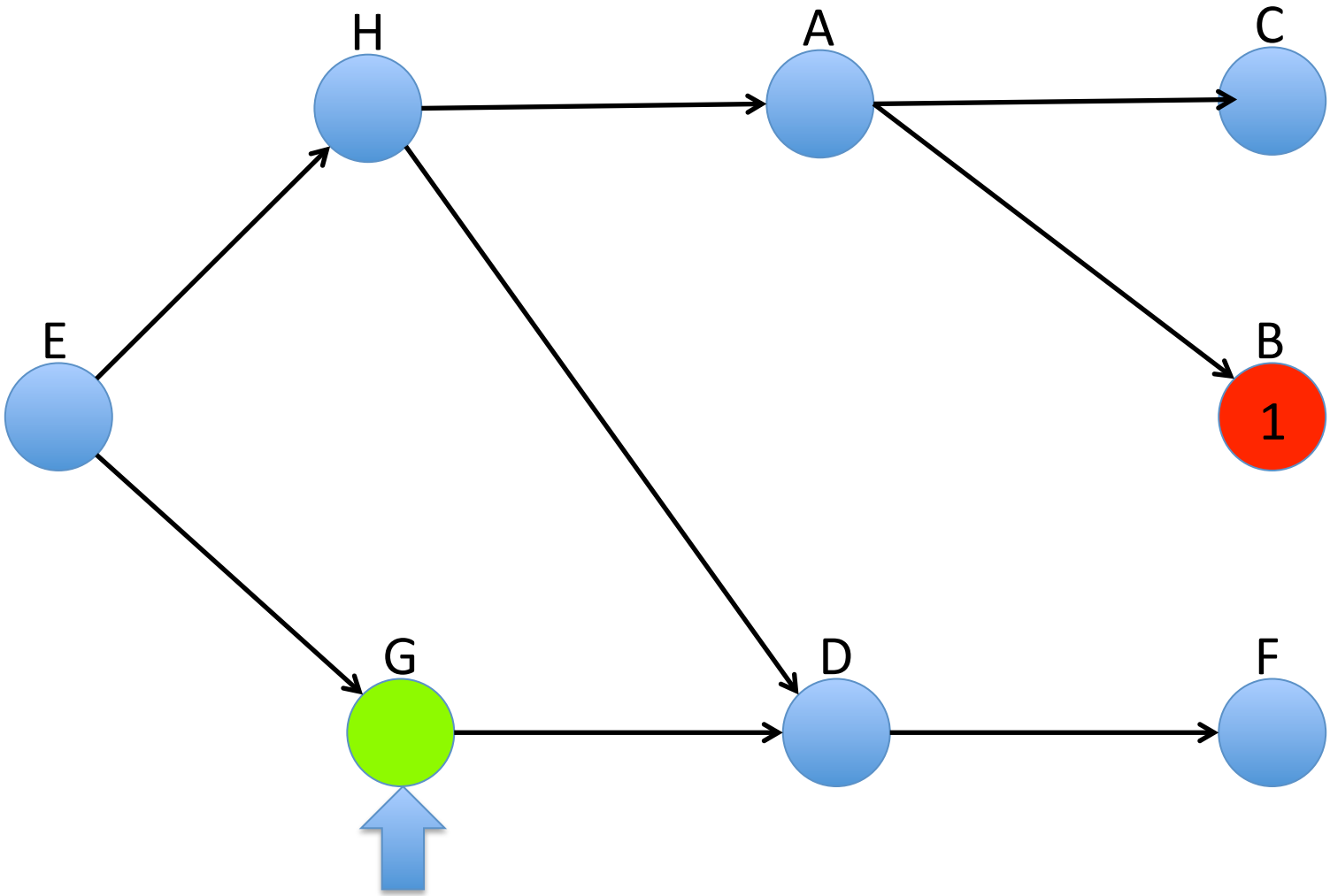
TS Algorithm #2 (Using DFS) Simulation 3



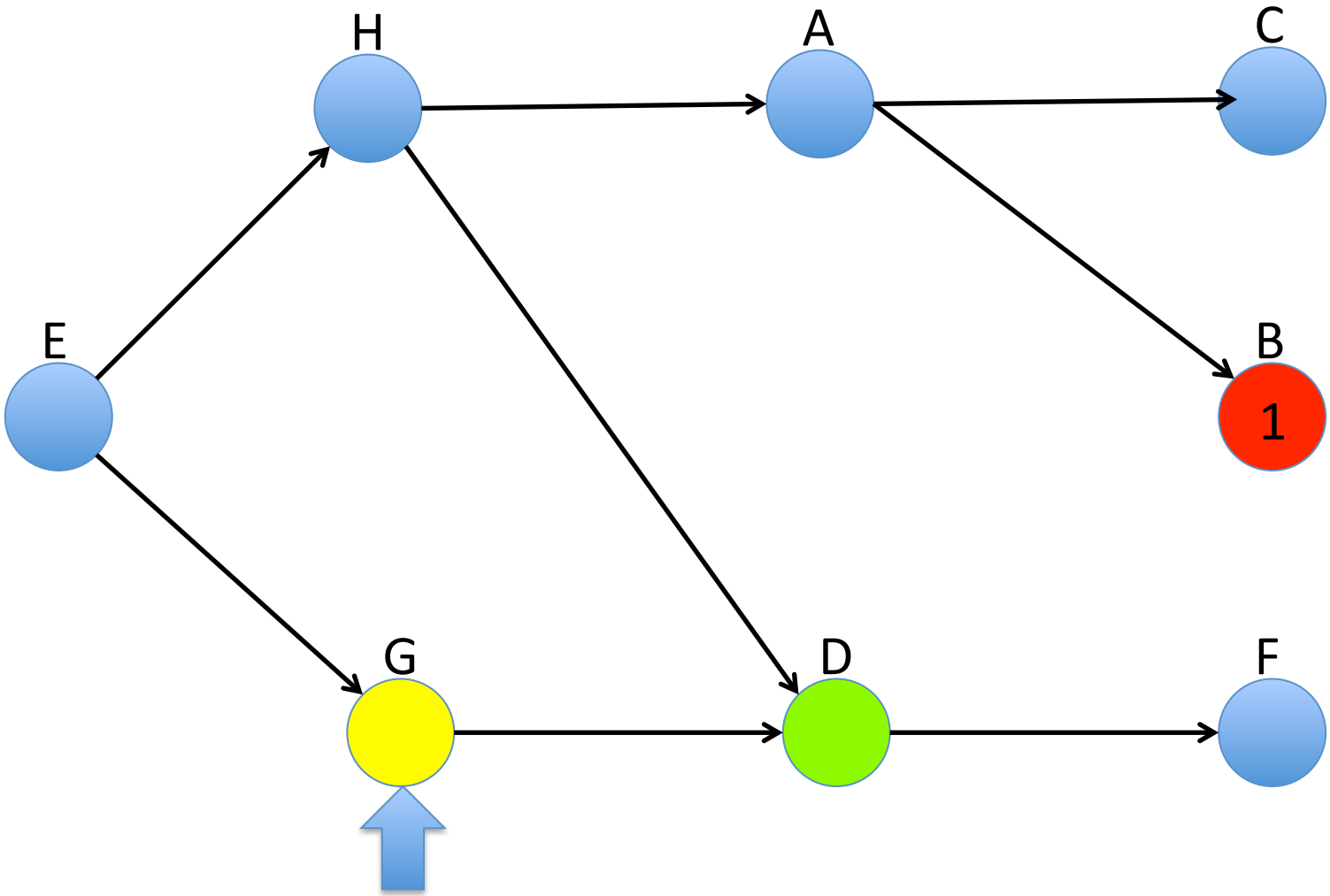
TS Algorithm #2 (Using DFS) Simulation 3



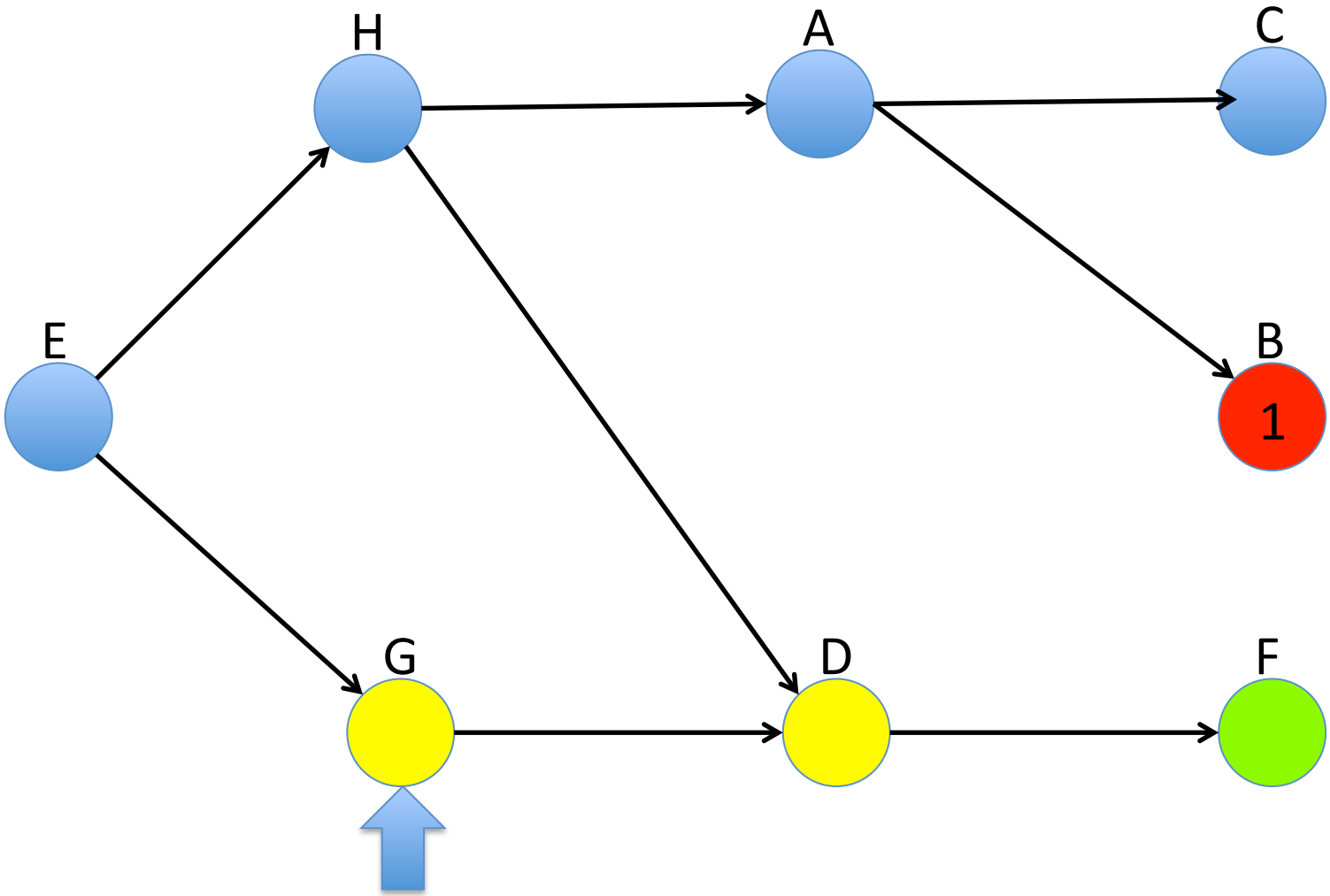
TS Algorithm #2 (Using DFS) Simulation 3



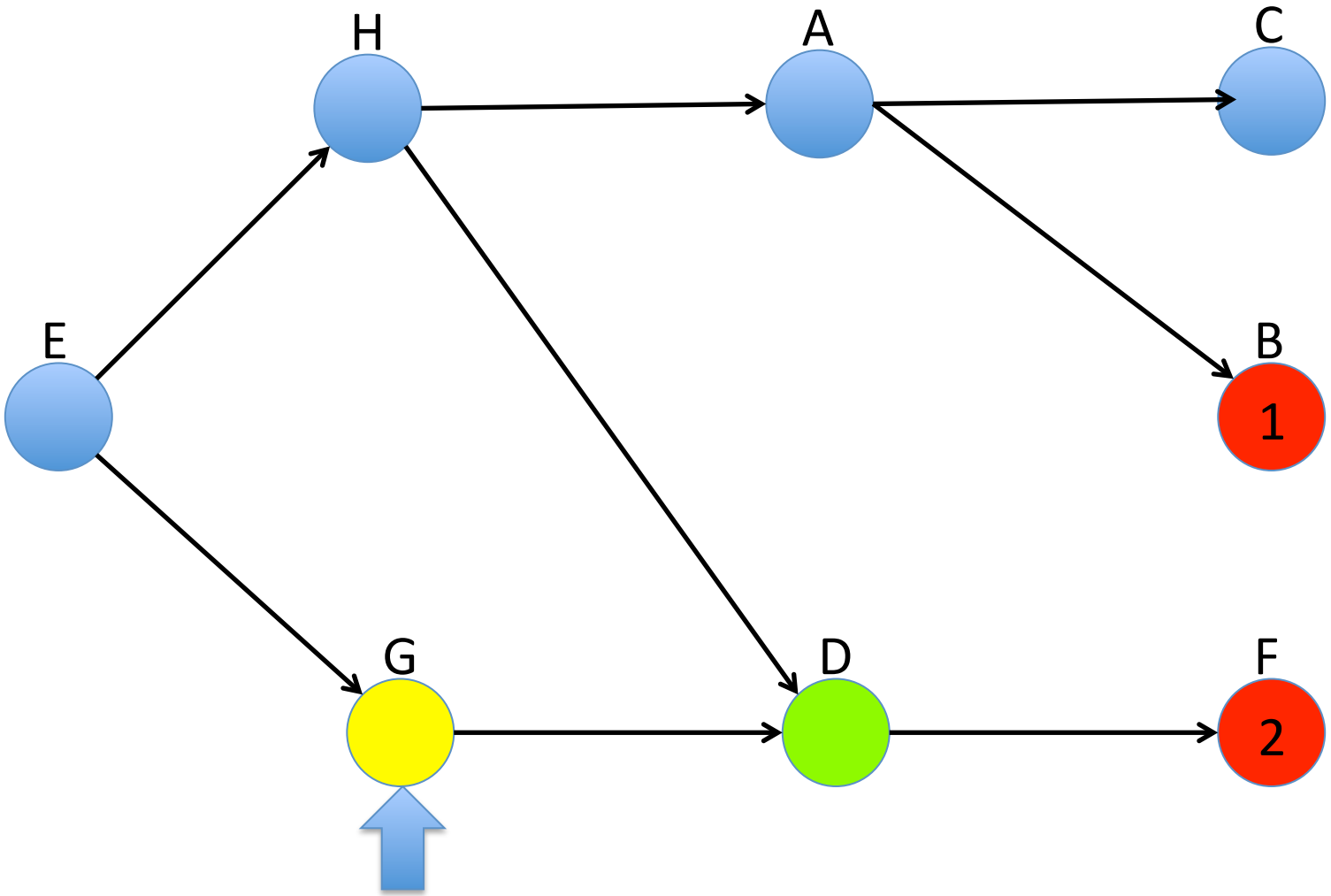
TS Algorithm #2 (Using DFS) Simulation 3



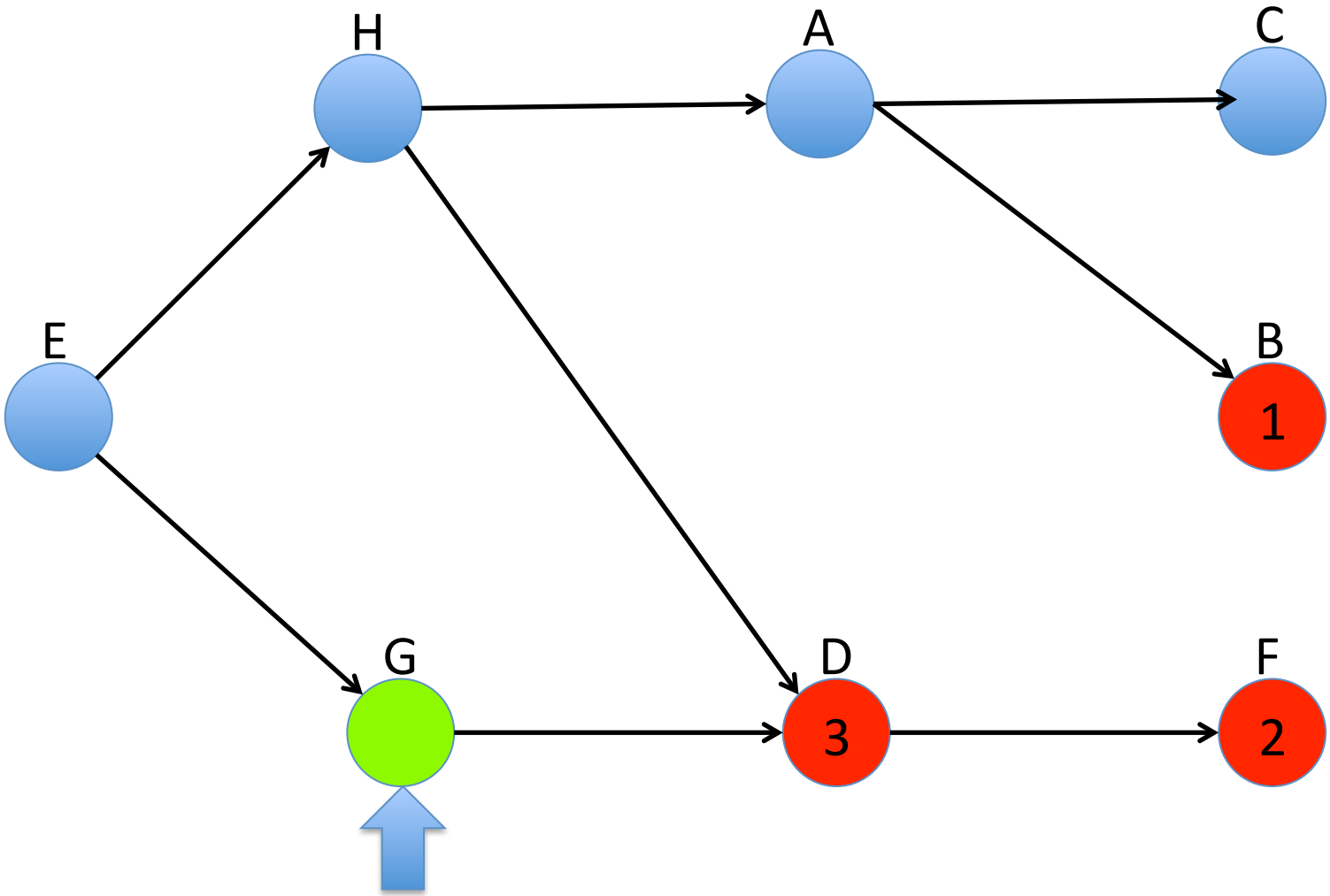
TS Algorithm #2 (Using DFS) Simulation 3



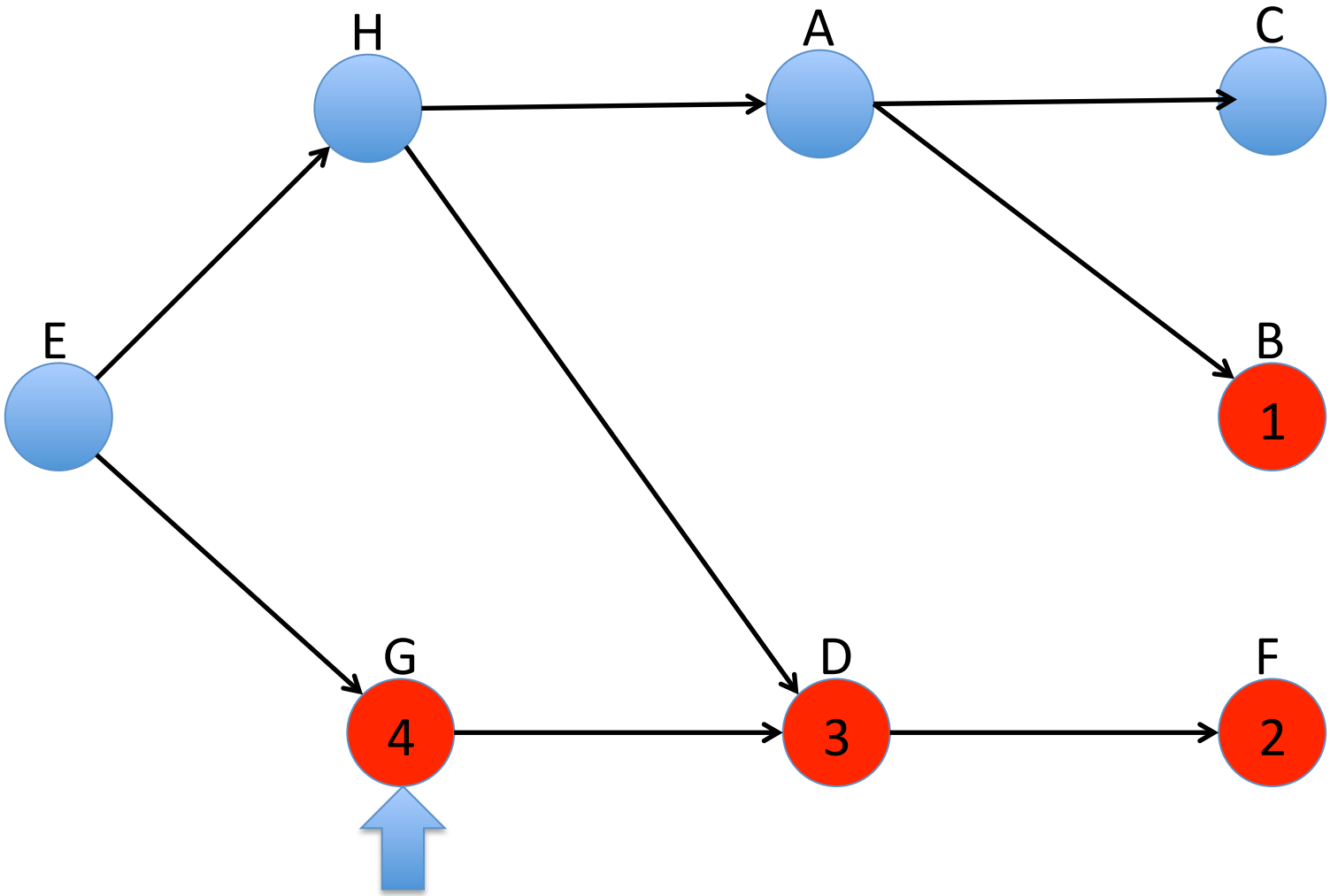
TS Algorithm #2 (Using DFS) Simulation 3



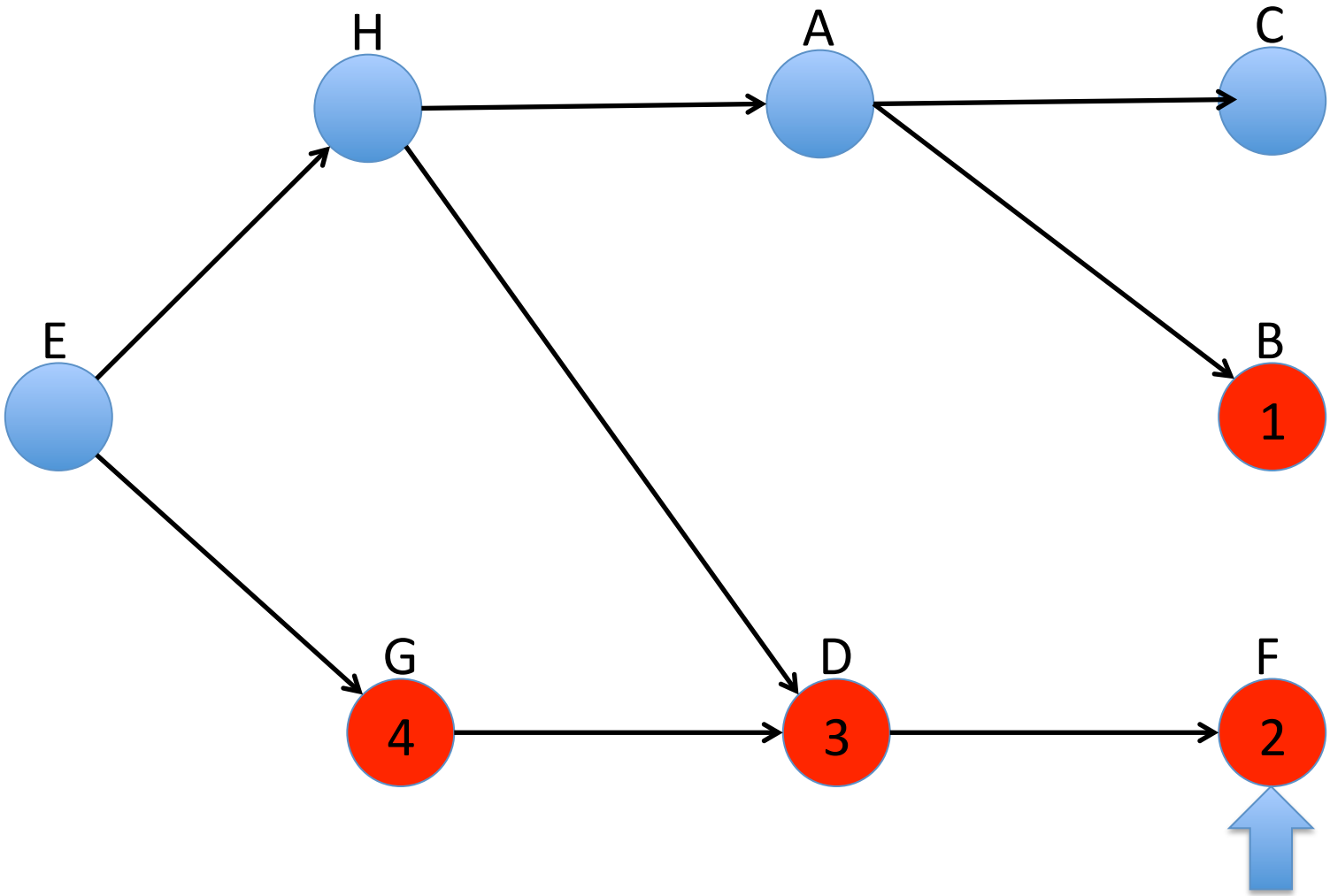
TS Algorithm #2 (Using DFS) Simulation 3



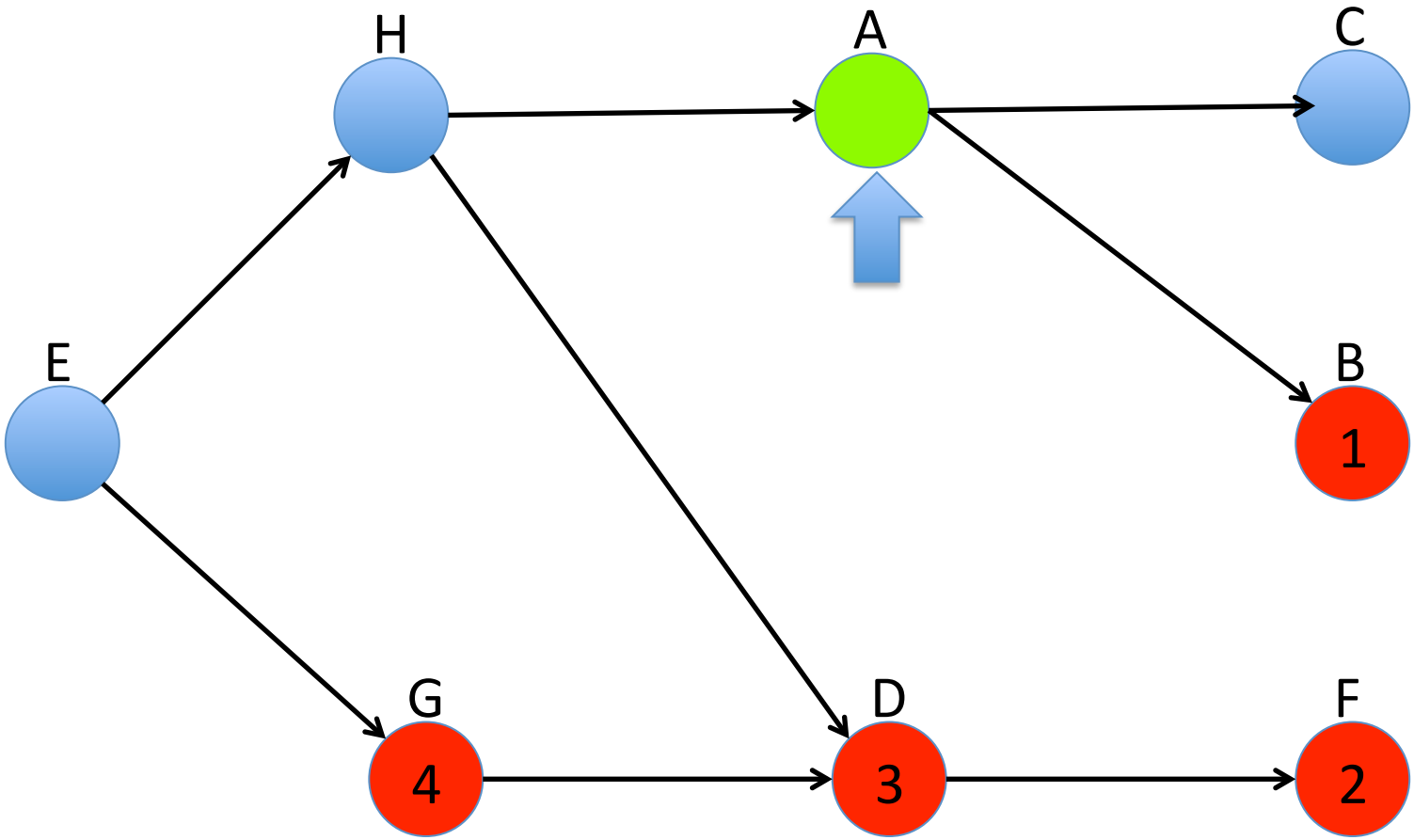
TS Algorithm #2 (Using DFS) Simulation 3



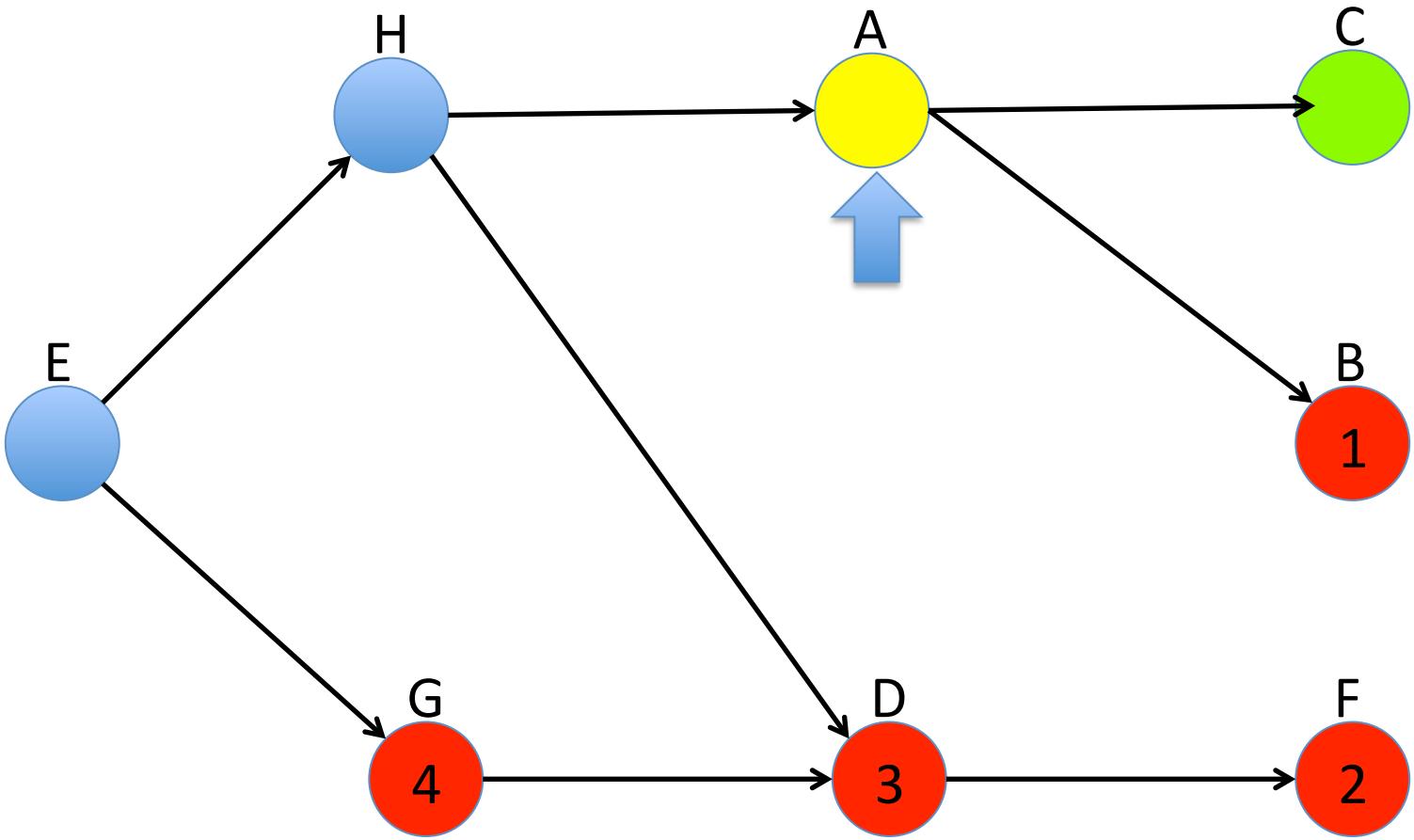
TS Algorithm #2 (Using DFS) Simulation 3



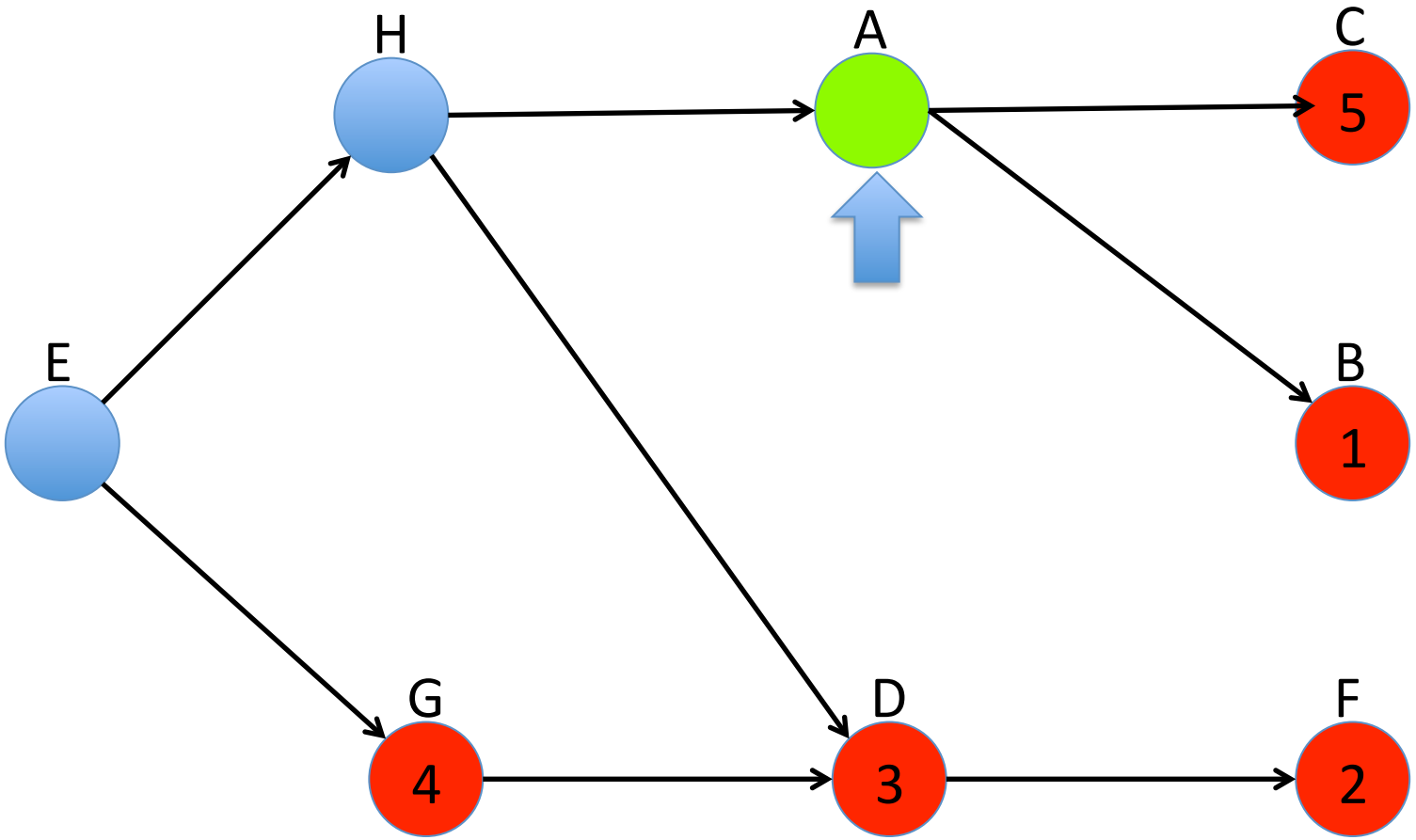
TS Algorithm #2 (Using DFS) Simulation 3



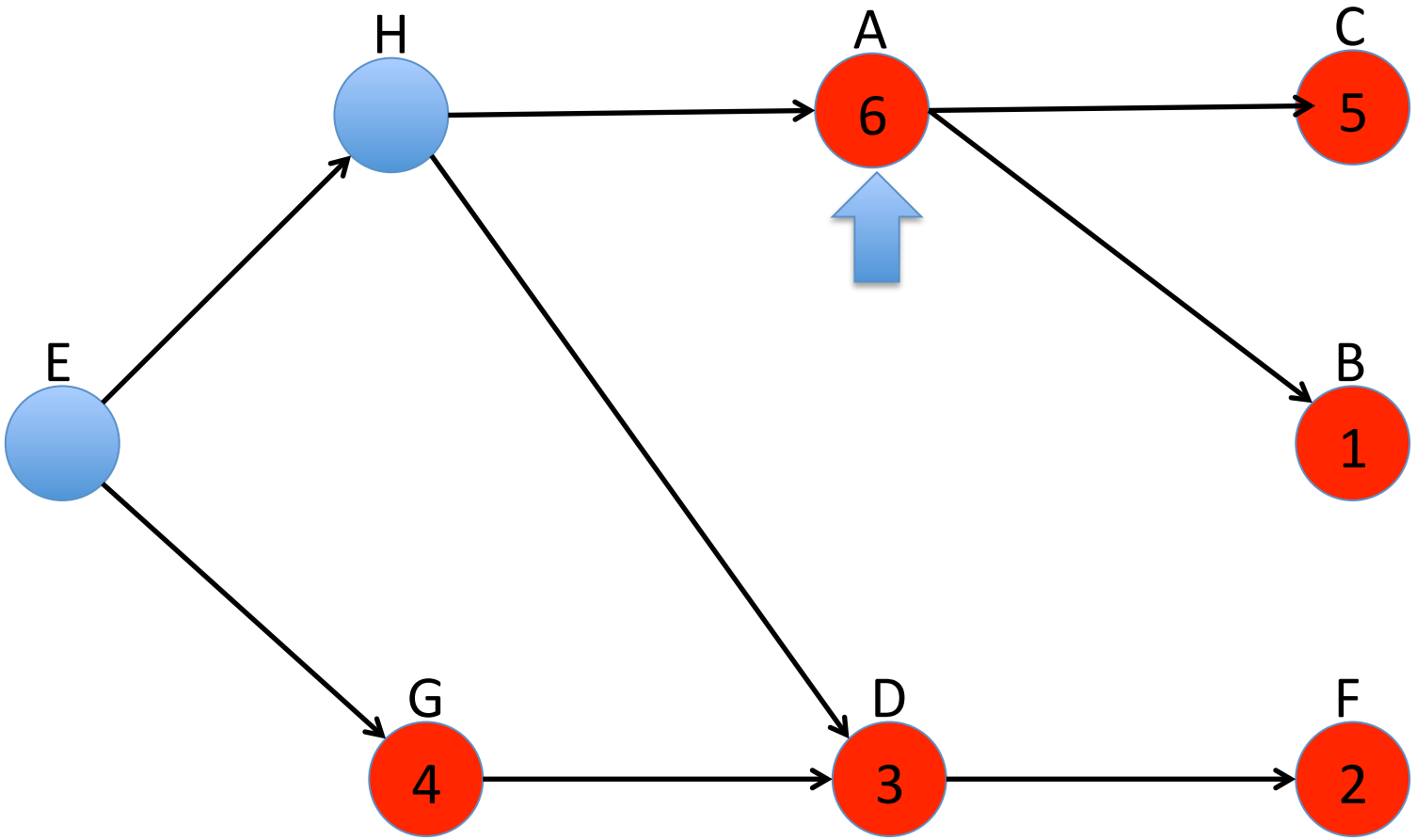
TS Algorithm #2 (Using DFS) Simulation 3



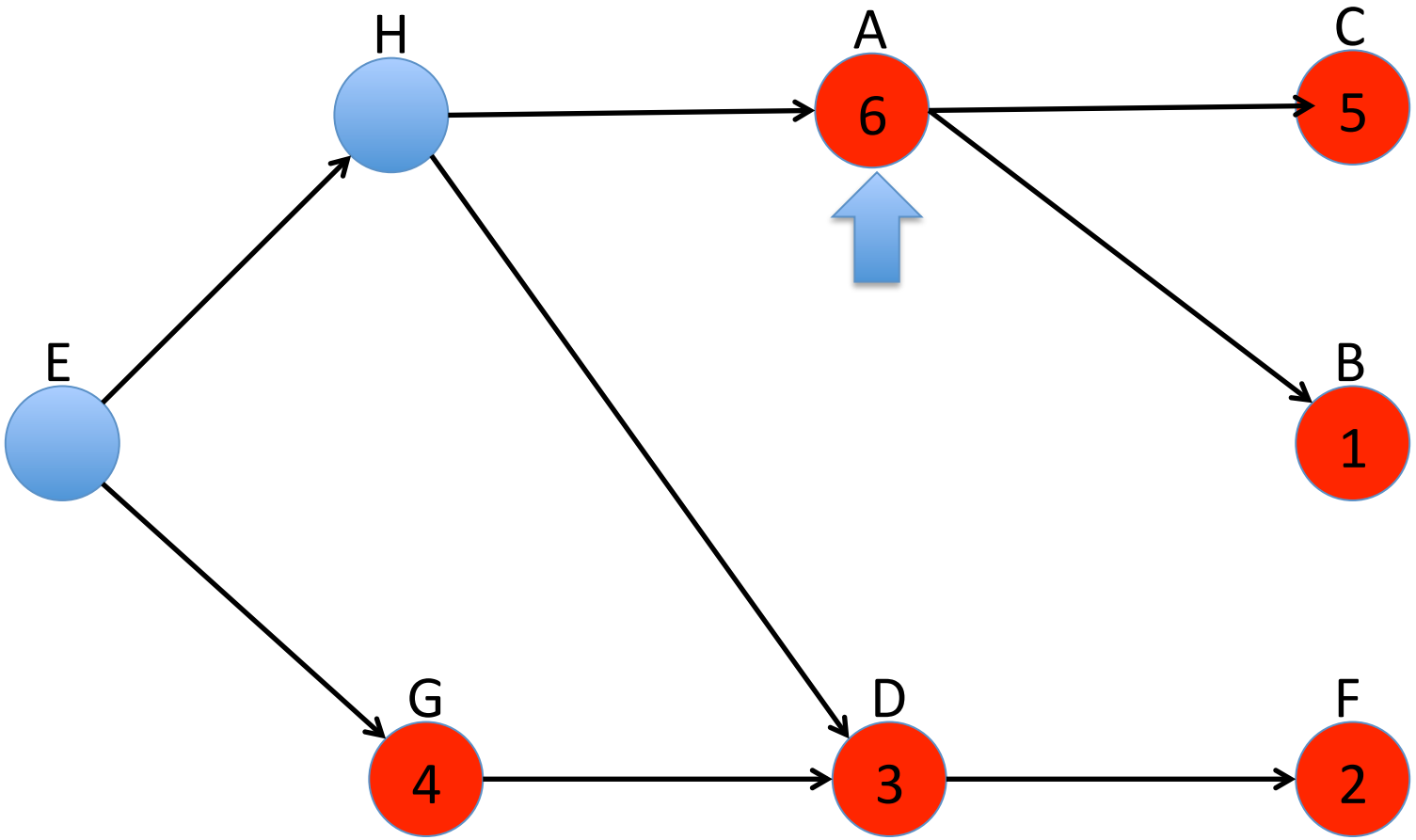
TS Algorithm #2 (Using DFS) Simulation 3



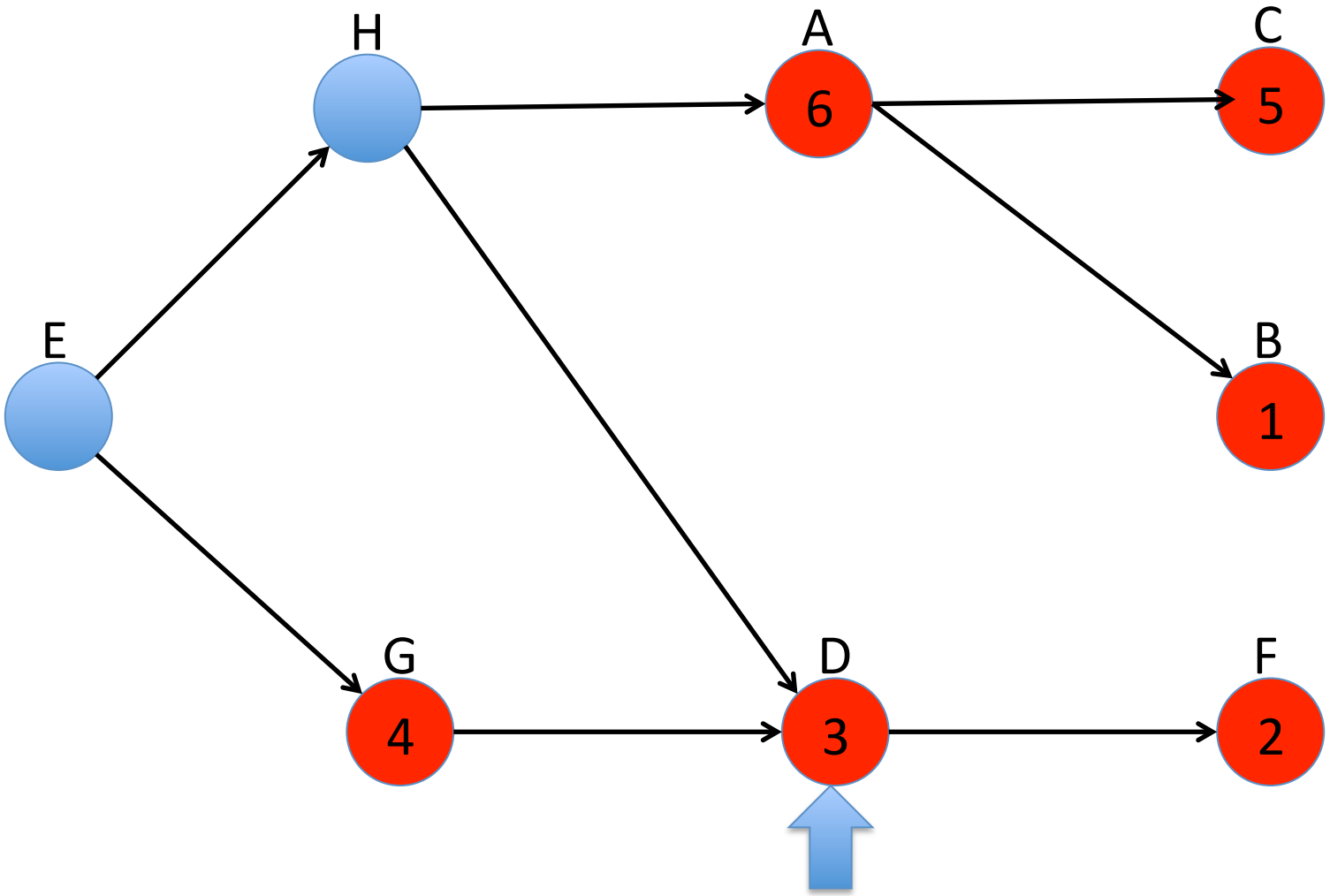
TS Algorithm #2 (Using DFS) Simulation 3



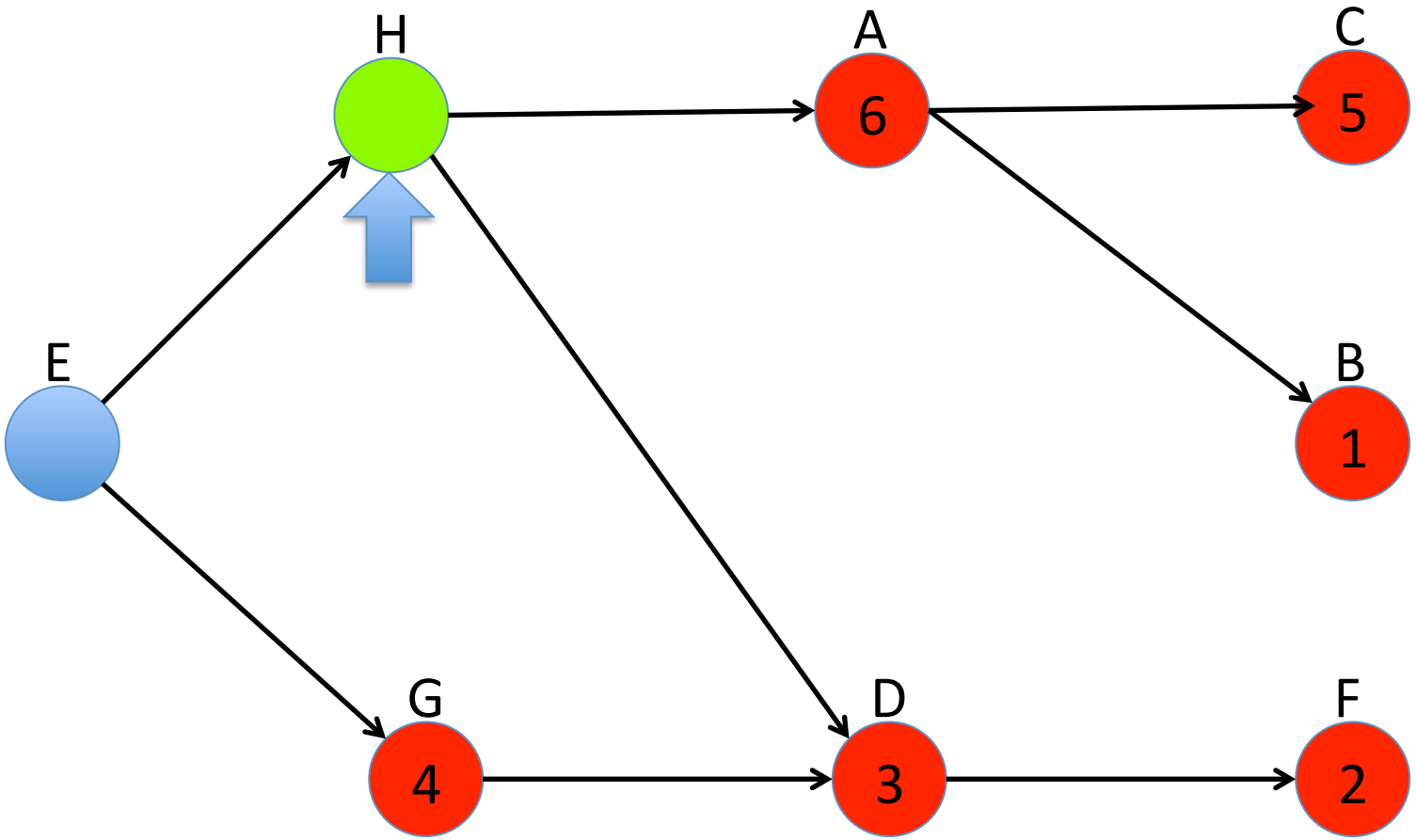
TS Algorithm #2 (Using DFS) Simulation 3



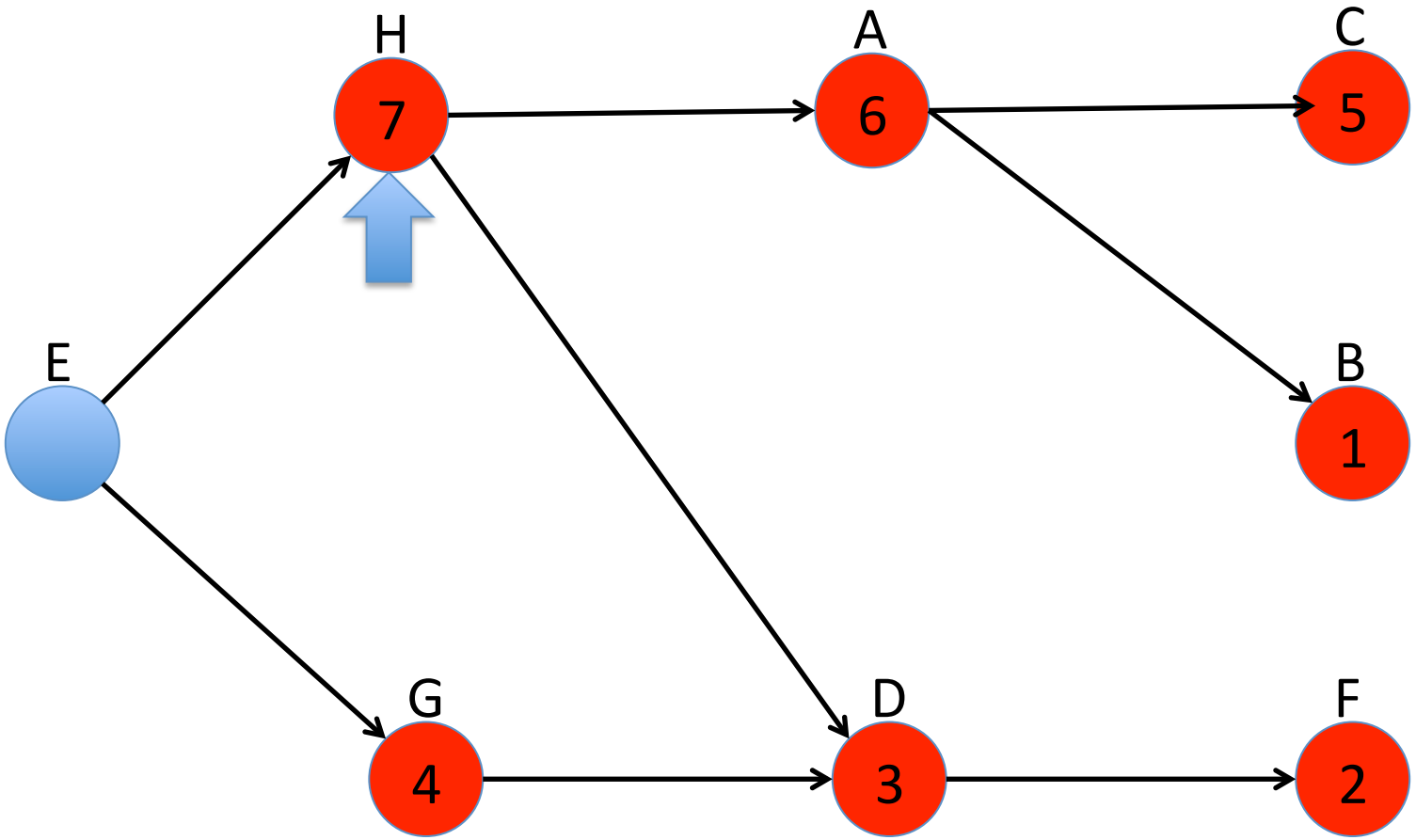
TS Algorithm #2 (Using DFS) Simulation 3



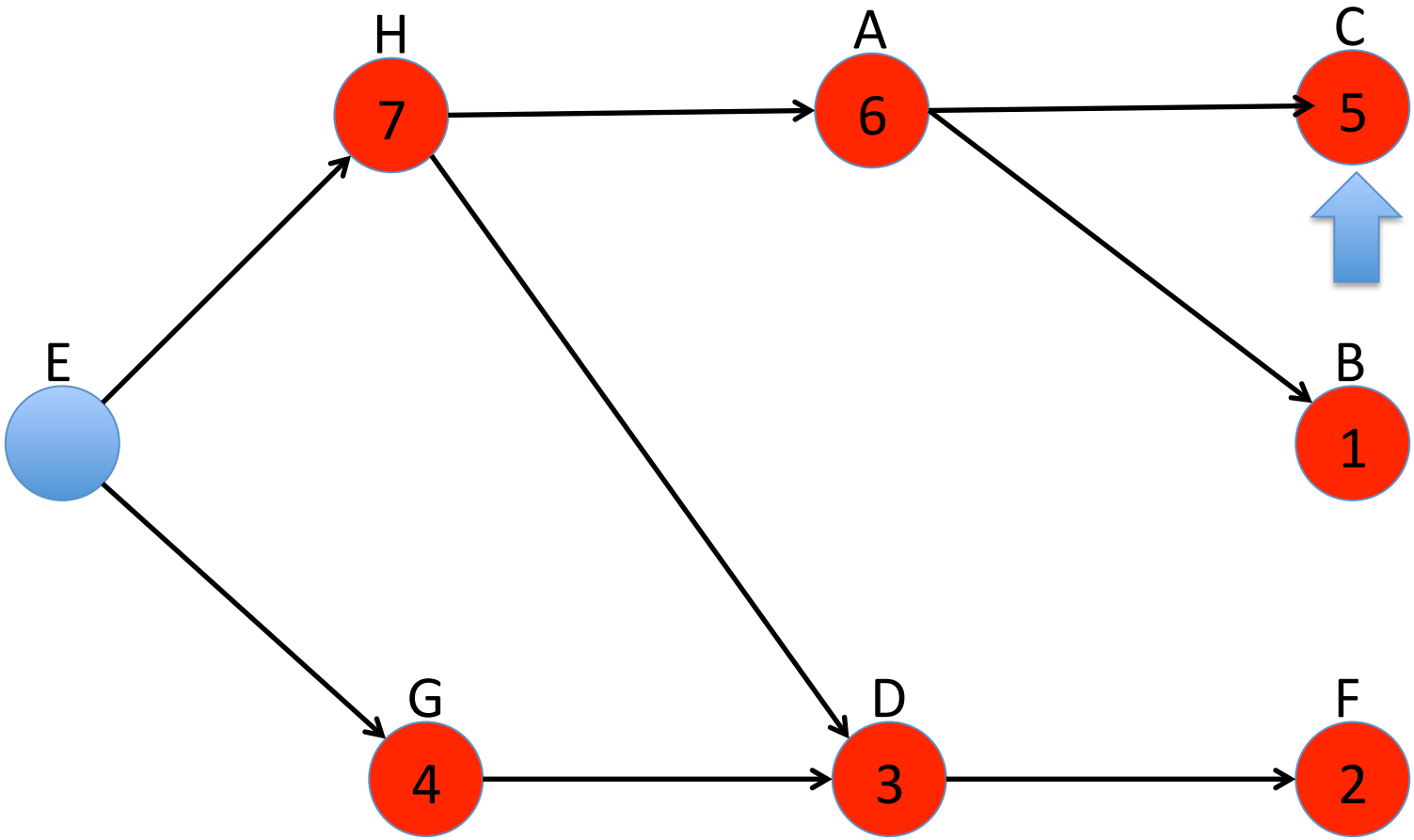
TS Algorithm #2 (Using DFS) Simulation 3



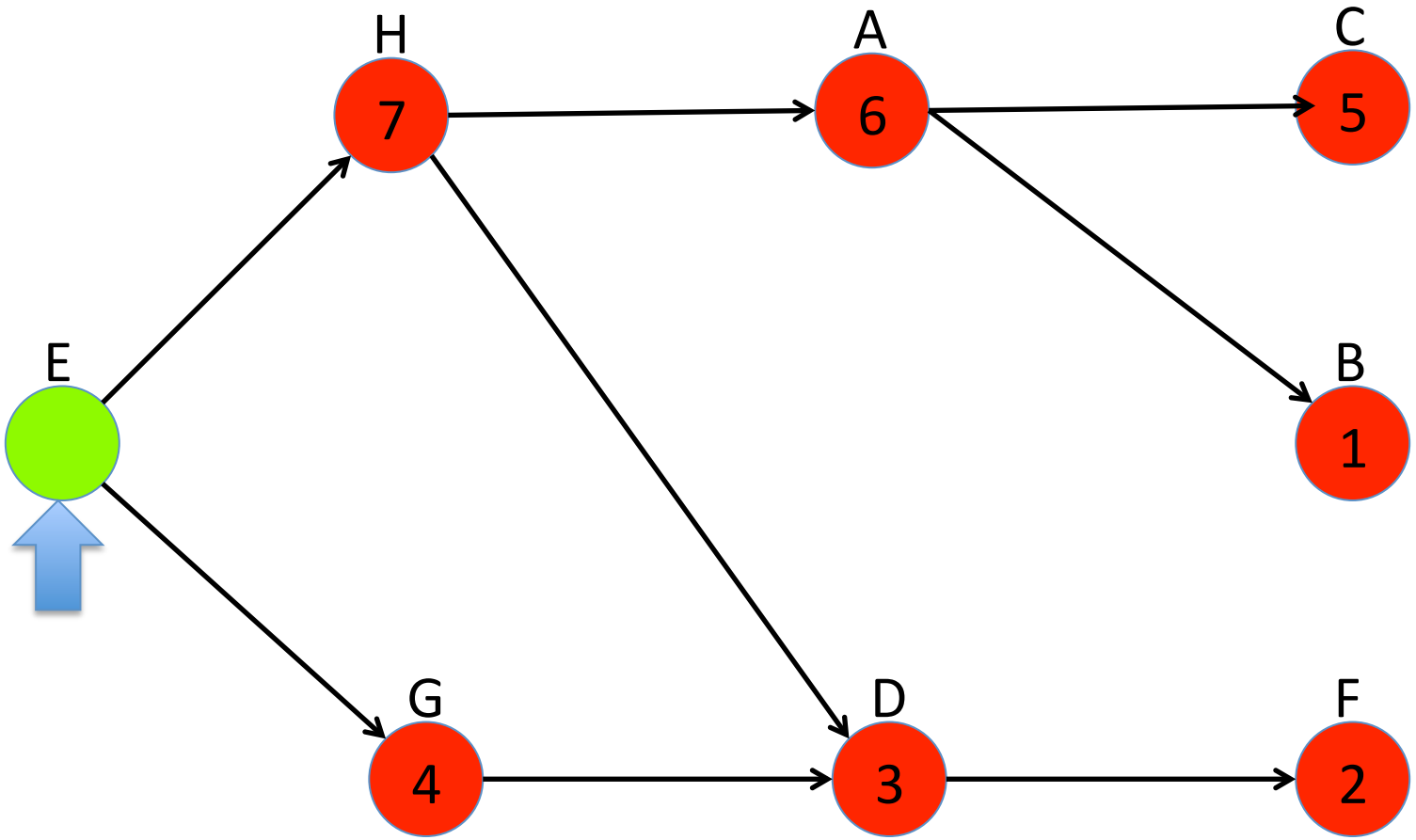
TS Algorithm #2 (Using DFS) Simulation 3



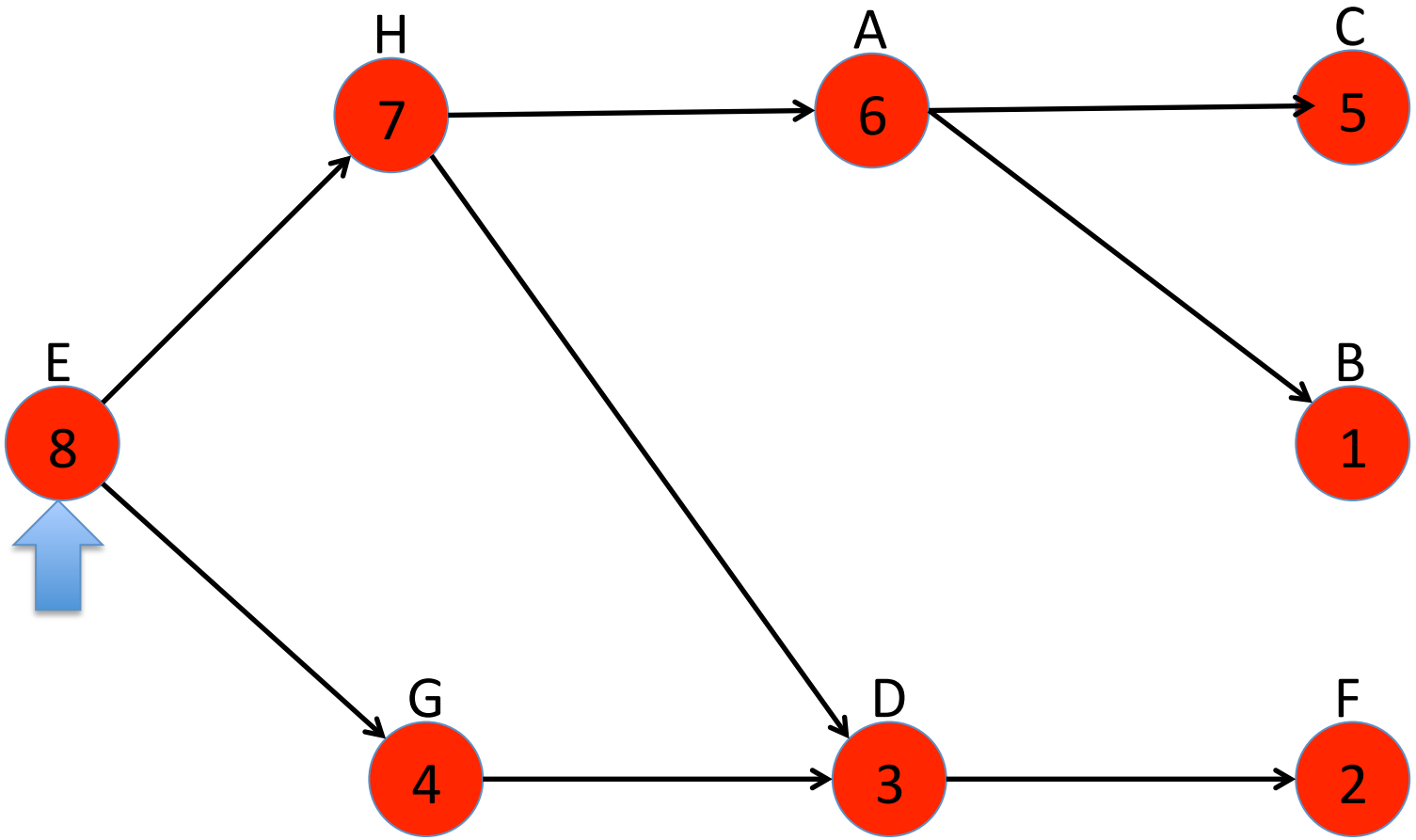
TS Algorithm #2 (Using DFS) Simulation 3



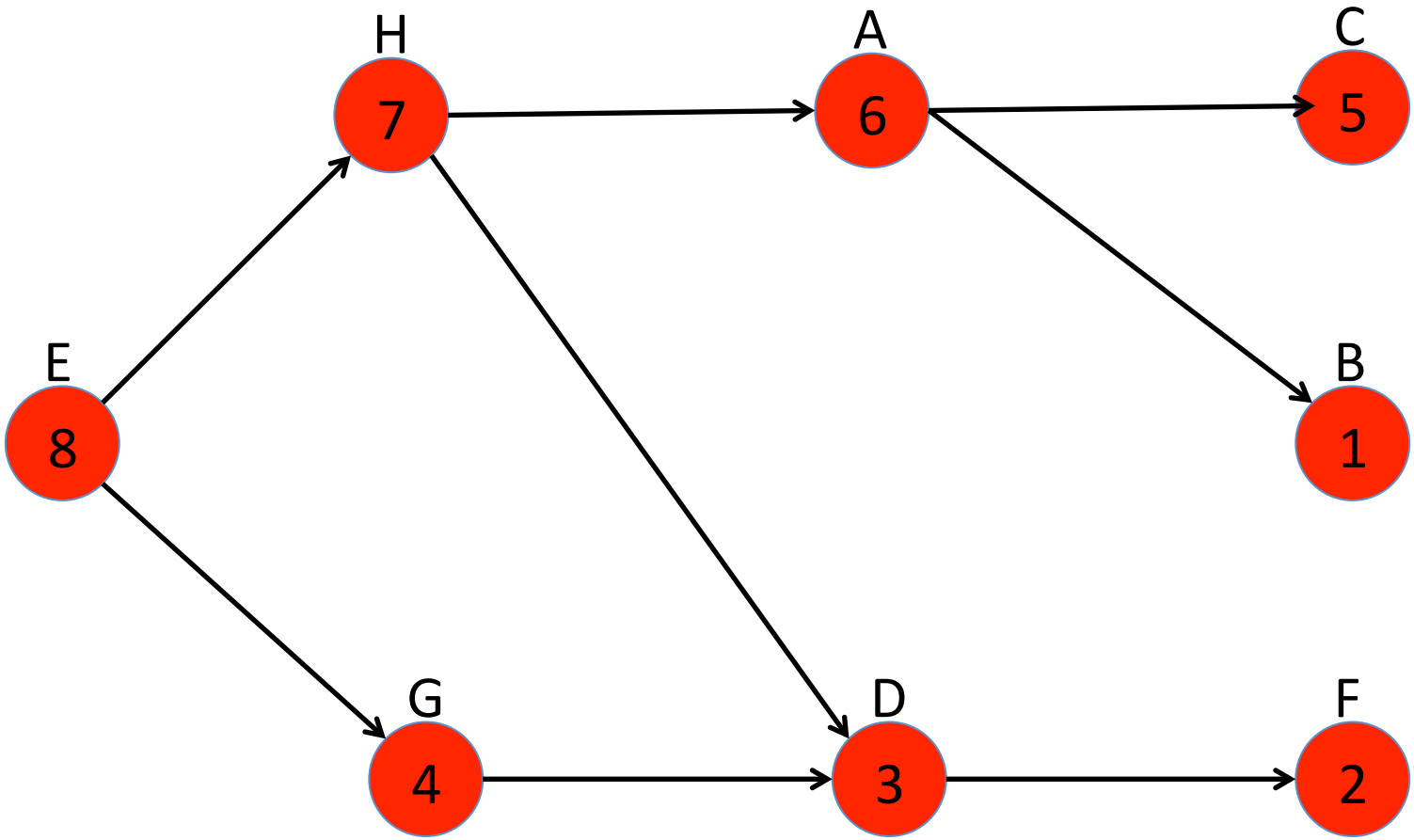
TS Algorithm #2 (Using DFS) Simulation 3



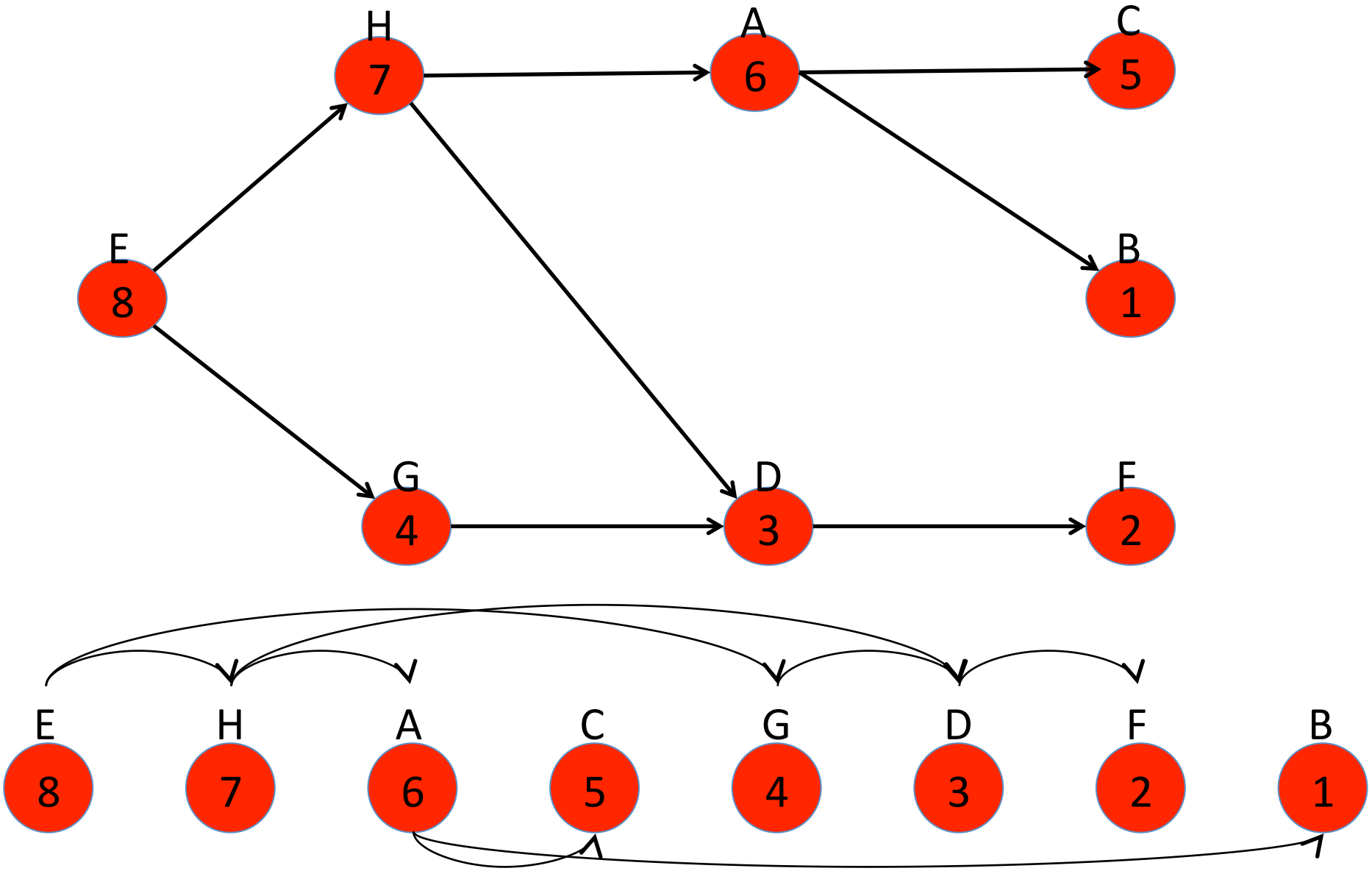
TS Algorithm #2 (Using DFS) Simulation 3



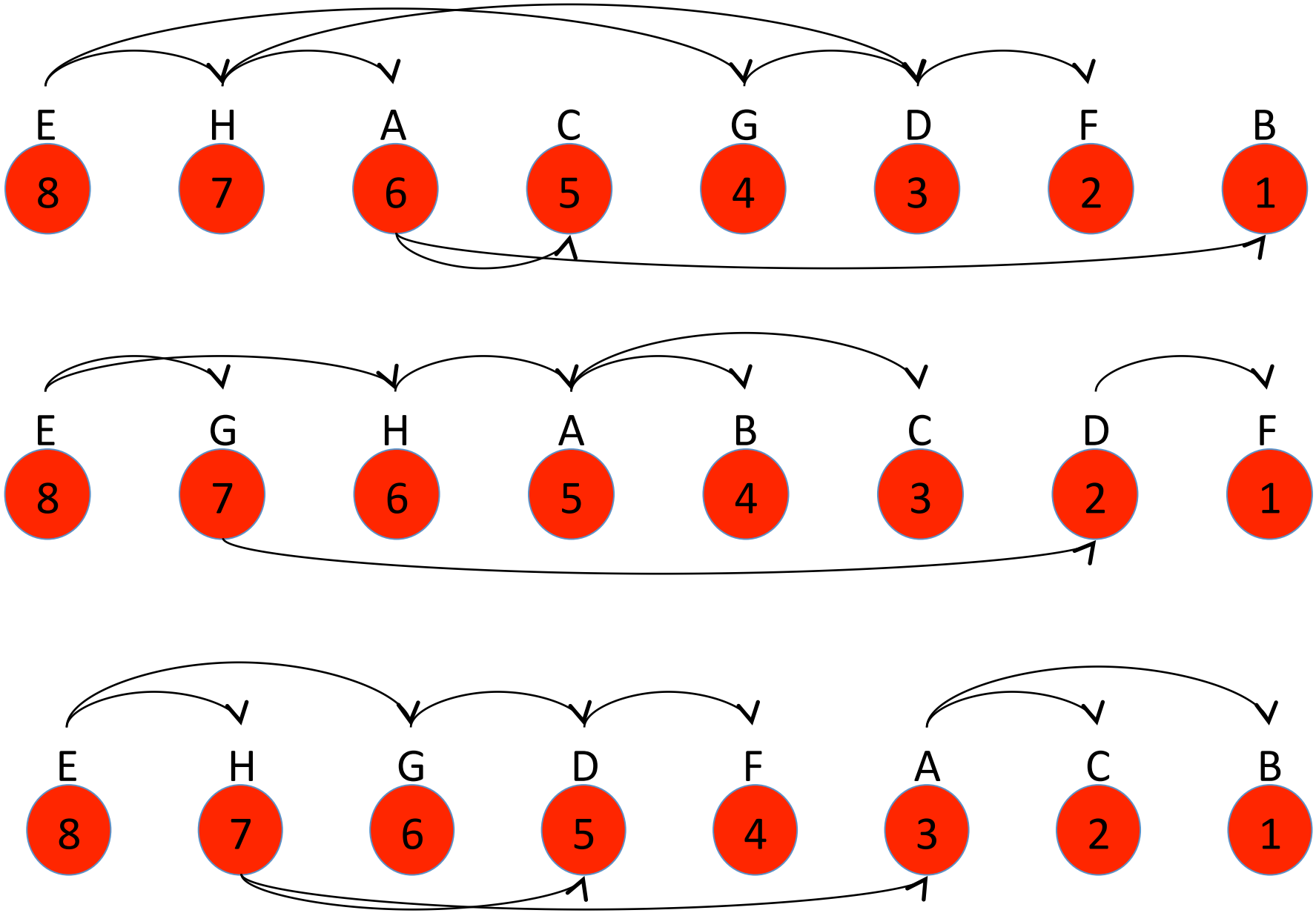
TS Algorithm #2 (Using DFS) Simulation 3



Order By Decreasing Finishing Times (3)



3 DFS Simulations 3 Topological Orders



Pseudocode TS Algorithm #2 (Using DFS)

```
procedure topologicalSort2(DAG G):
    run DFS(G) & put each finished vertex
               into an array in reverse order
```

Runtime = Runtime of DFS = O(n + m)

Correctness?

DFS PseudoCode With Finishing Time Keeping

```
global var t = 1;
procedure DFS(G):
  f: array of size n initialized to null
  for i = 1 to n:
     if V[i] is not yet visited:
        DFS(G, i)

procedure DFS(G, u):
   mark u as visited
   for all neighbors v of u:
      if v is not yet visited:
        DFS(G, v)
   f[u] = t; t++;
```

Key Claim About Finishing Times in DAGs

In a DAG if u → v, then f[u] > f[v]

Proof:

Break into 2 cases by whether u or v is visited first

Case 1: u is visited first. Then:

DFS(v) call will be made before DFS(u) finishes/
(b/c DFS call is made on every edge from u)
=> *f[u] > f[v]*

Case 2: v is visited first. Then:

Recall the graph is a acyclic
Therefore DFS cannot discover u from v.
Therefore v has to finish before even discovering u
=> f[v] < s[u] < f[u] (where s[u] is discovery time of u)
=> f[u] > f[v]

Extended Key Claim

In a DAG if $u \rightsquigarrow v$, then $f[u] > f[v]$

Proof: By induction on the length of the paths, where

Base case is simply Key Claim from previous slide

Prove it as an Exercise!

Note: Not needed for the correctness of Topological Sort with DFS finishing times but will be important next lecture.

Correctness of TS Algorithm #2

Correctness Proof: Immediate from Key Claim

if (u, v) exists, then f(u) > f(v)

we order according to decreasing f values

therefore u will be ordered before v