Lecture 16:

Minimum Spanning Trees

Kruskal's Algorithm

Tue, March 12th

Outline For Today

1. Graph Theory Part 1: Trees & Properties of Trees
2. Minimum Spanning Trees
3. Kruskal’s Algorithm
4. Graph Theory Part 2: Cuts & Properties of Cuts
5. Correctness Proof of Kruskal’s Algorithm

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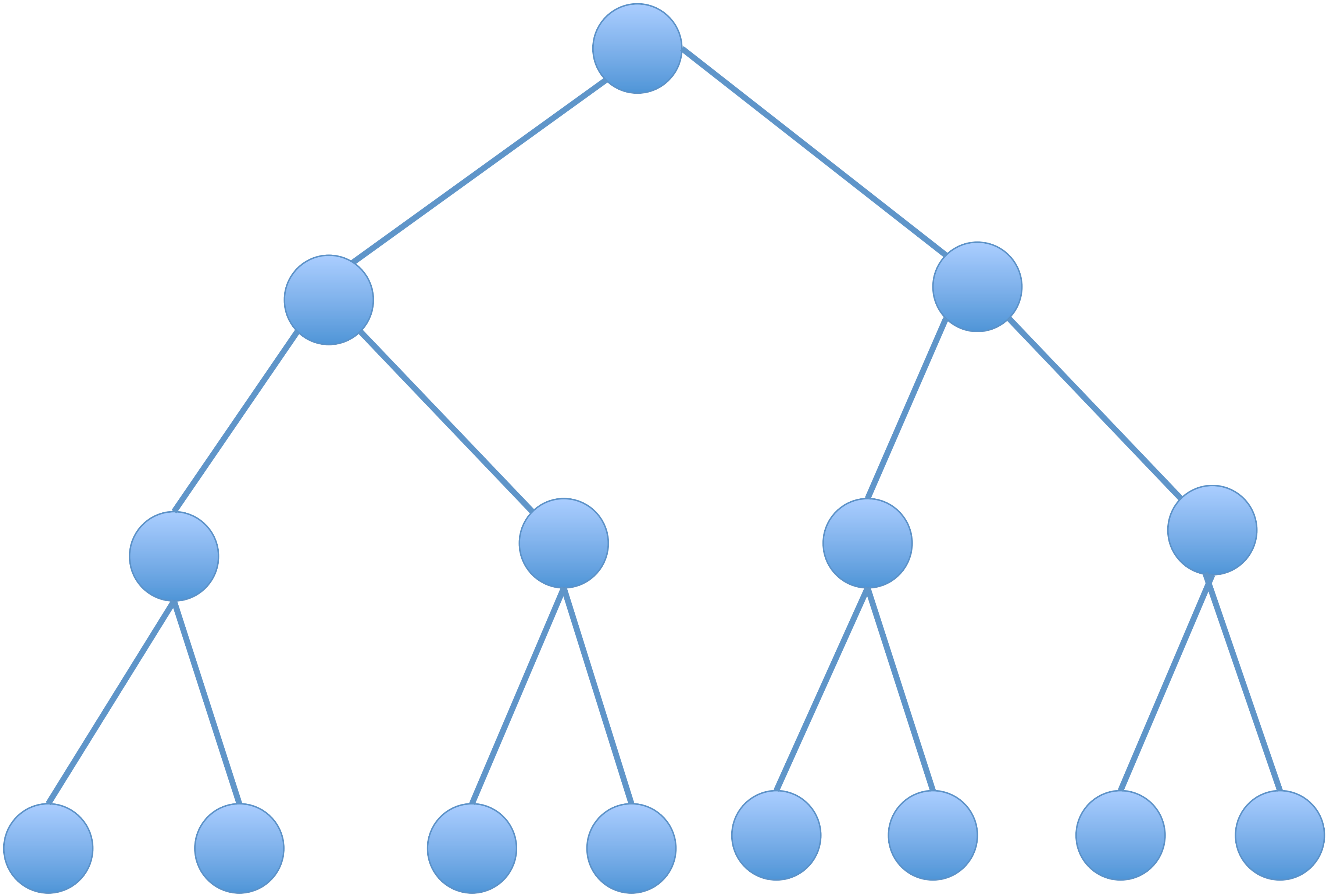
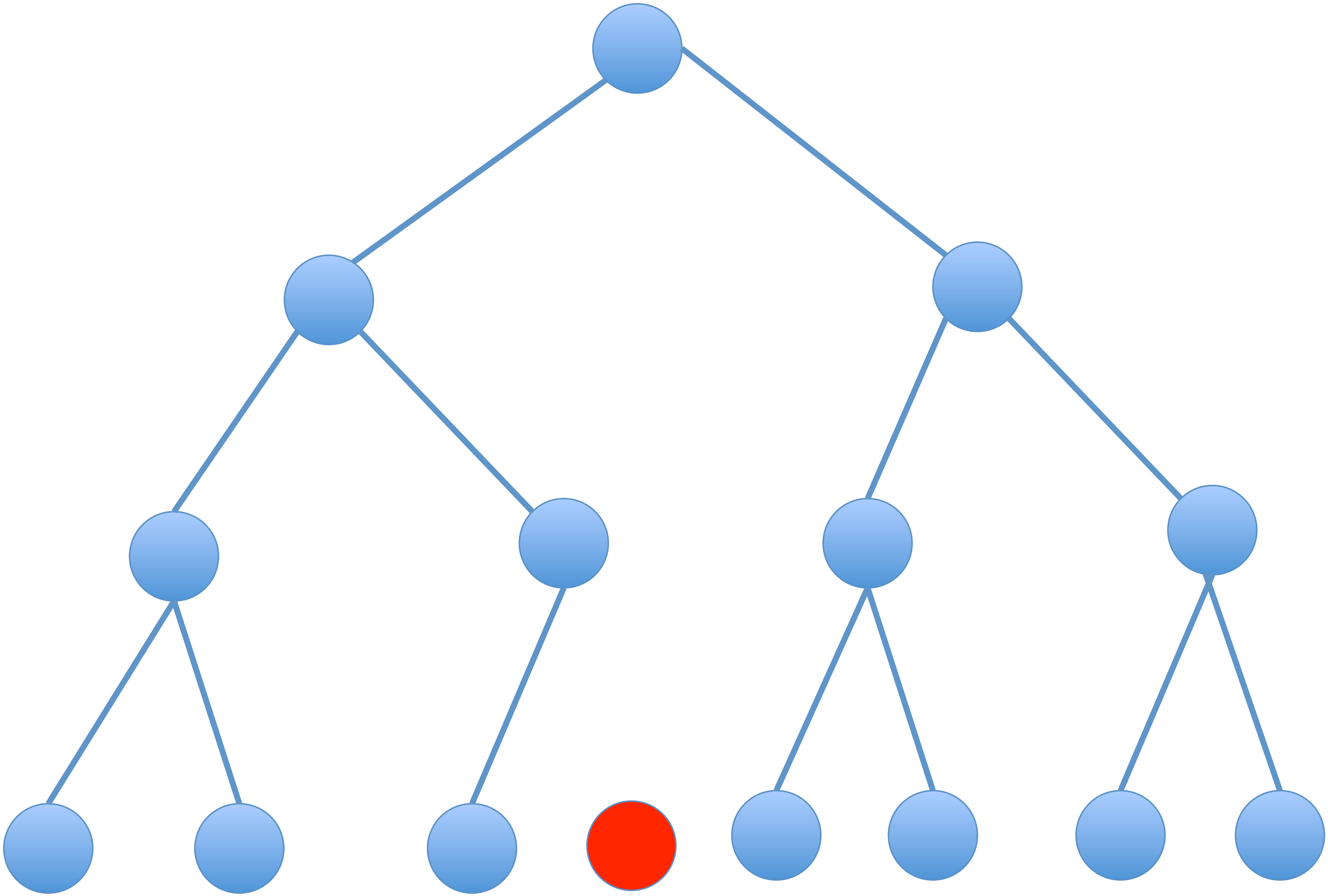
Tree

- Definition: An undirected graph G(V, E) is a tree iff
  1. G is connected
  2. G is acyclic

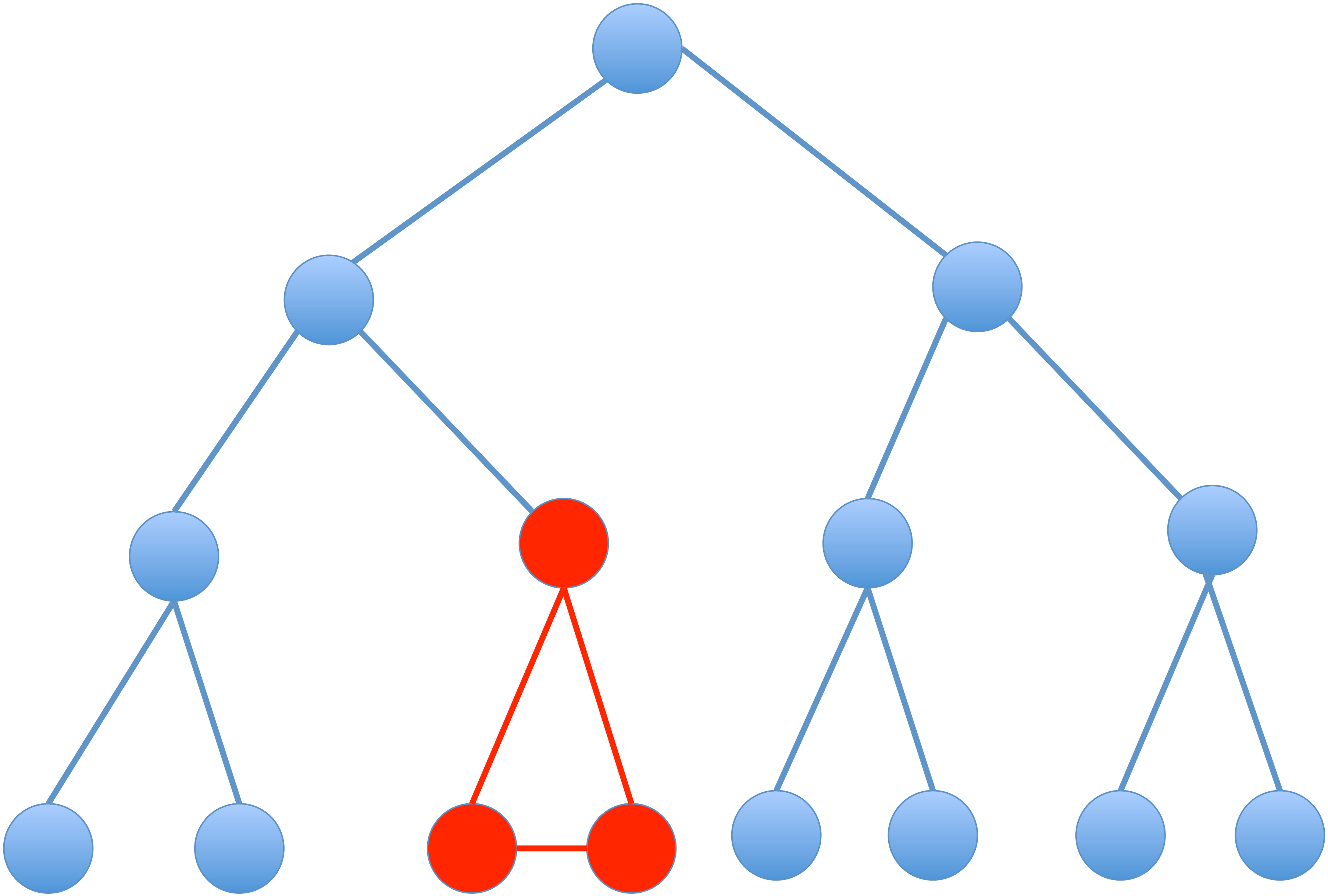
Example: Tree (1)



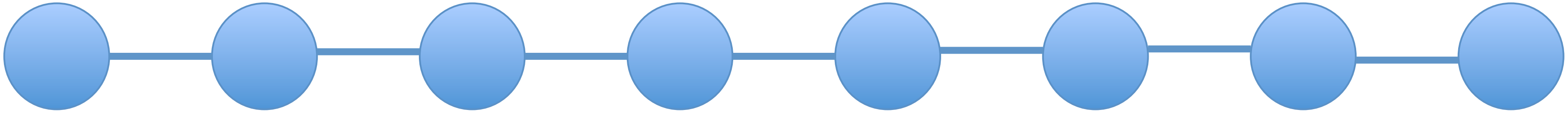
Example: Not a Tree (1)



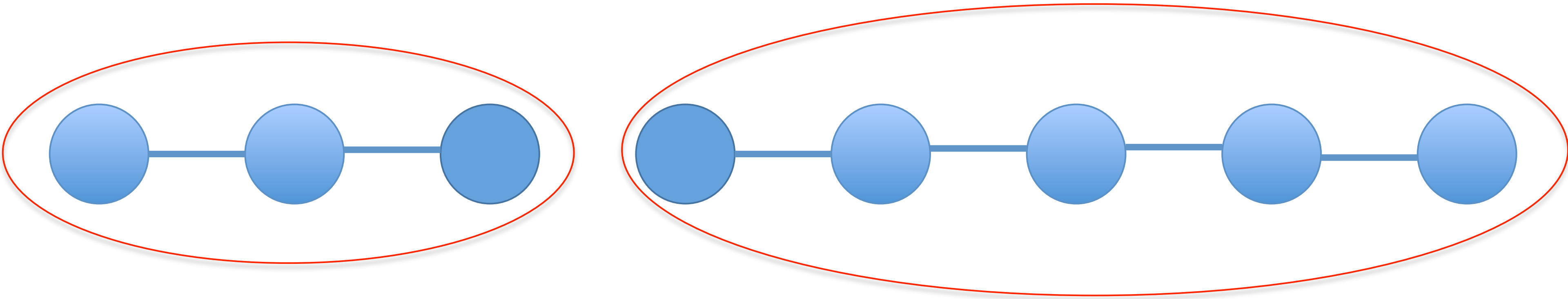
Example: Not a Tree (2)



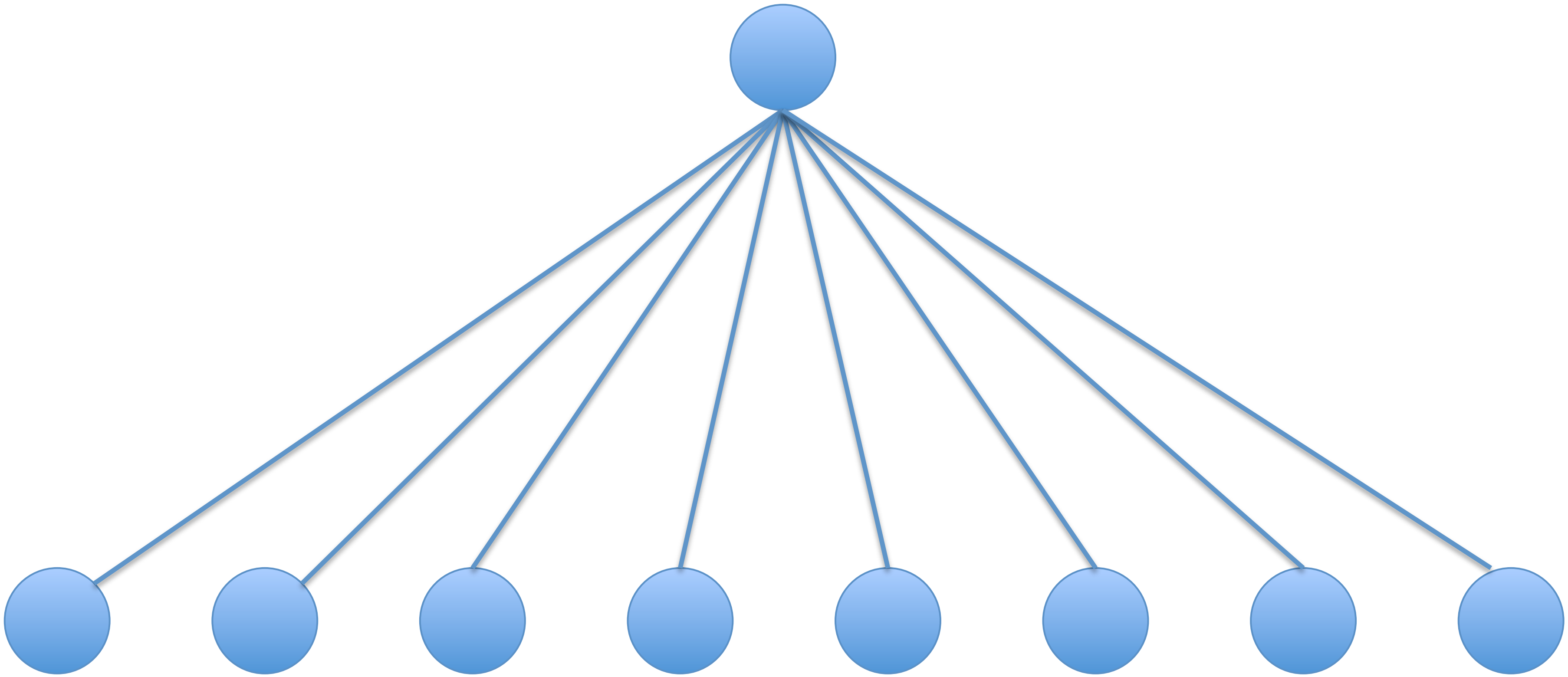
Example: Tree (2)



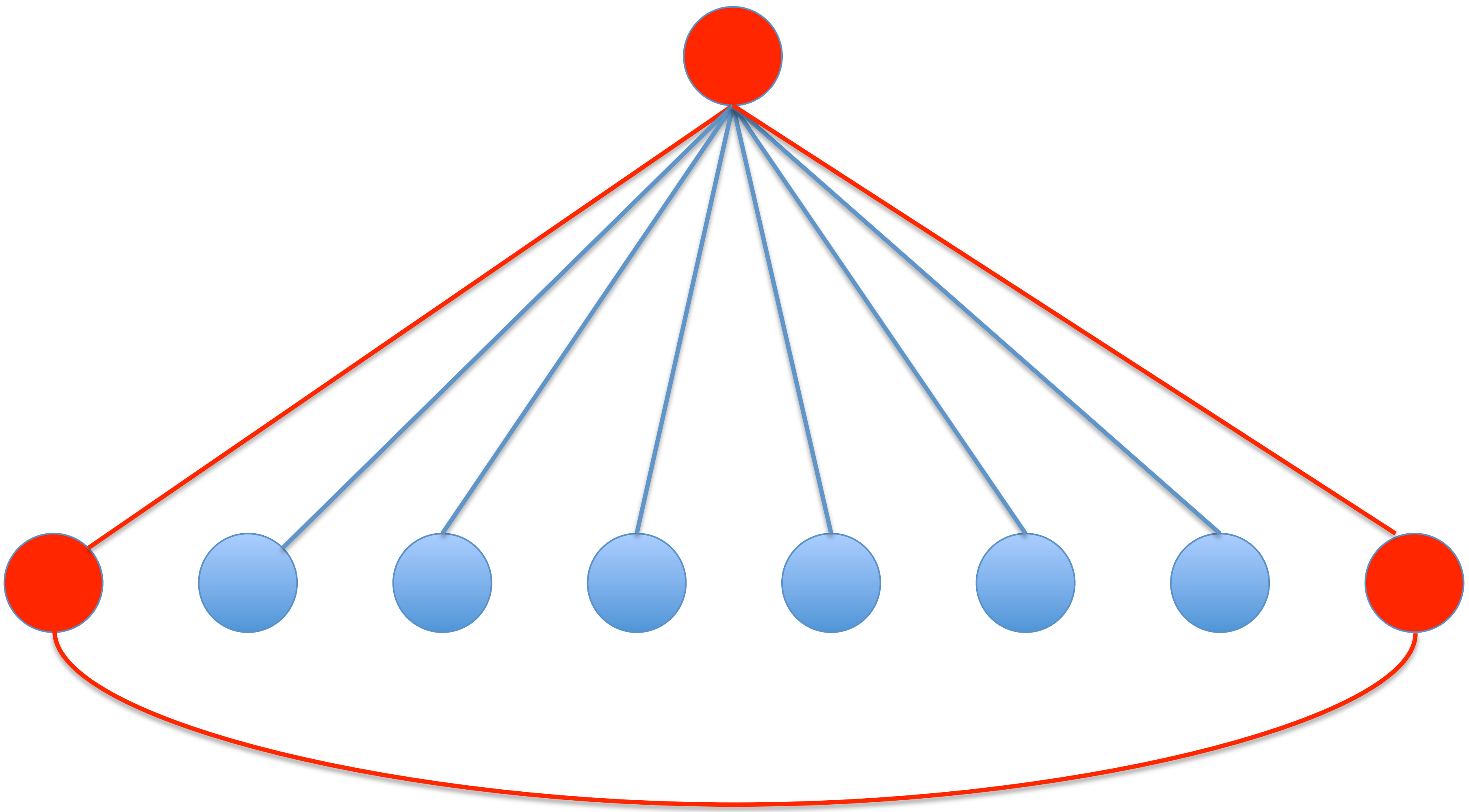
Example: Not a Tree (3)



Example: Tree (3)



Example: Not a Tree (3)



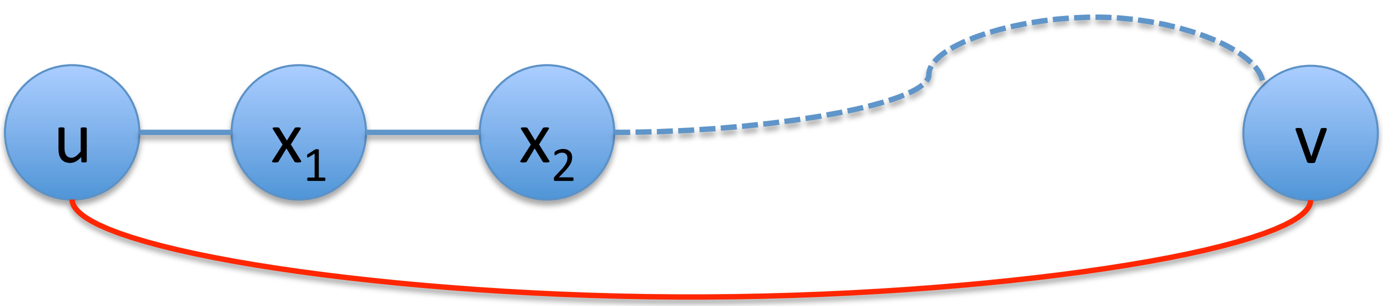
Breaking *a Tree Lemma*

Removing any edge (u, v) from a tree T

disconnects T!

Why?

No u⤳v path can exist! Assume it did…



Contradicts acyclicity of T!

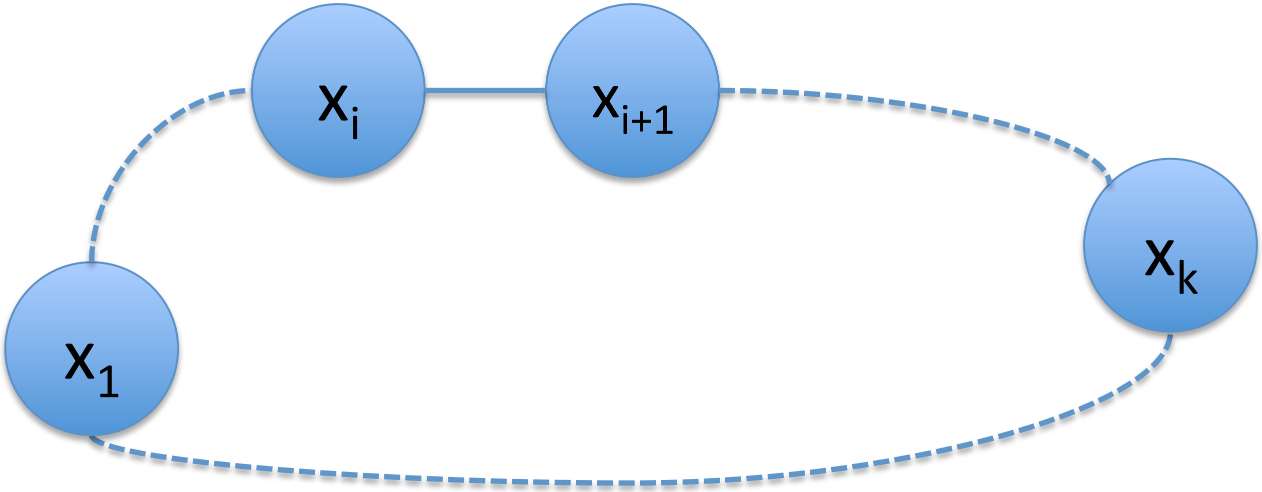
Reverse is Also True

Let G be a connected graph and assume removing any edge from G disconnects it.

Then G is acyclic and hence a tree.

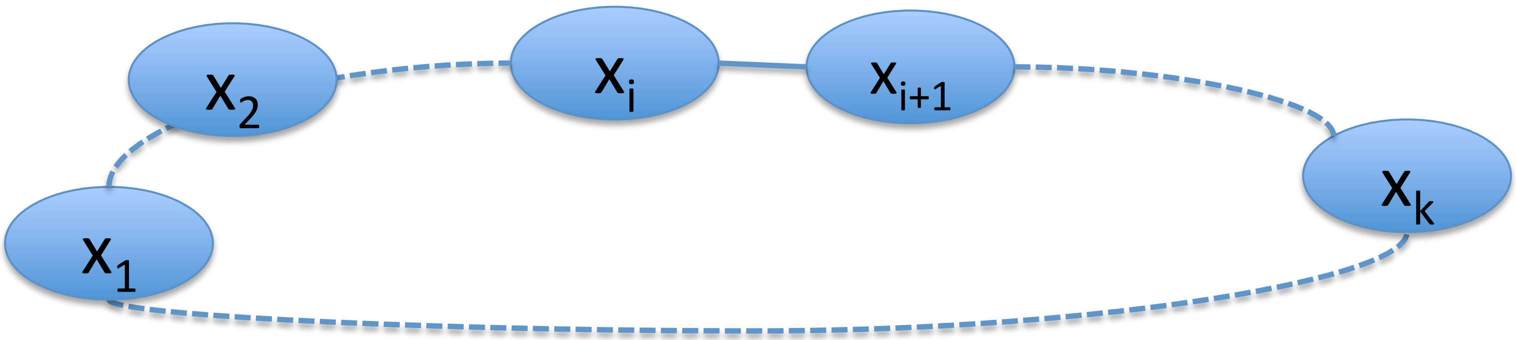
Why?

Claim: G cannot contain a cycle C

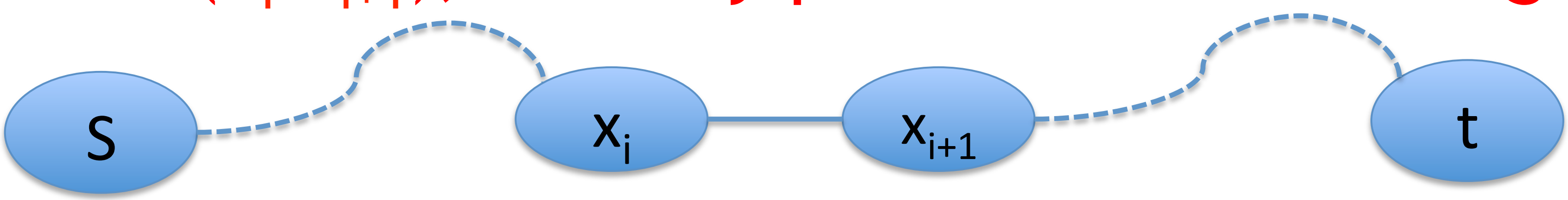


Proof That G Cannot Contain a Cycle C

***Breaking a Cycle Lemma**: Removing **any edge** from a cycle cannot disconnect a connected graph!*



Take (x_i x_{i+1}), and any path P that was using it



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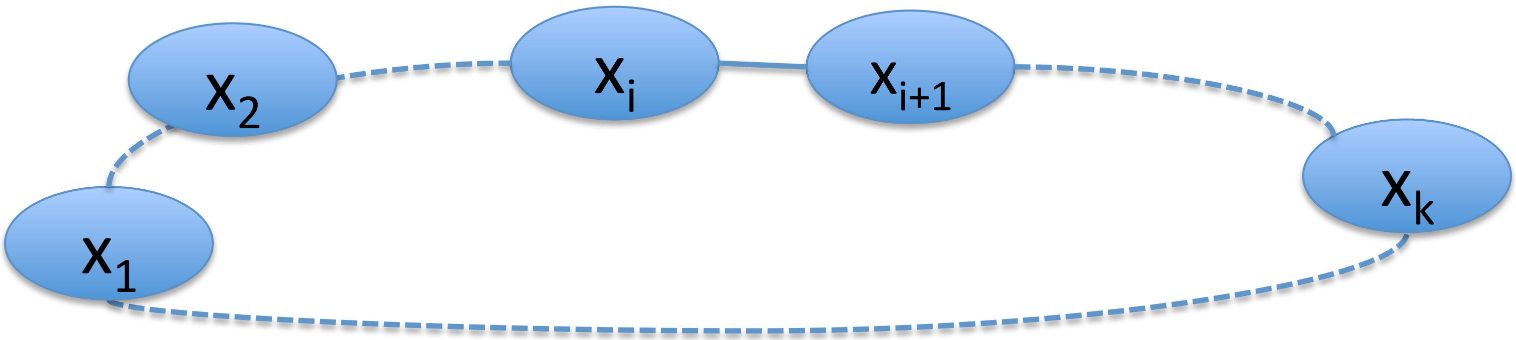

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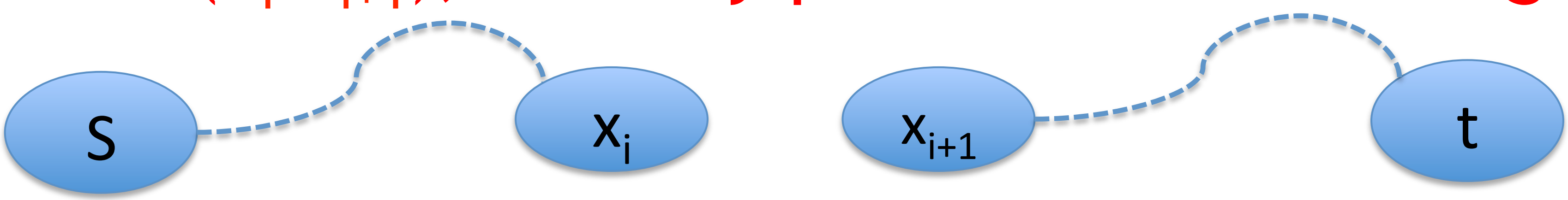
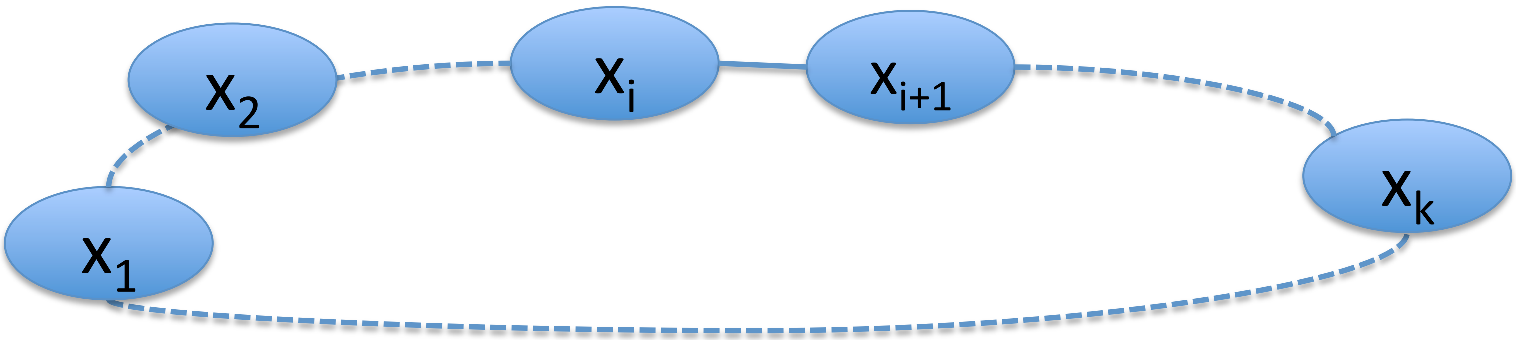
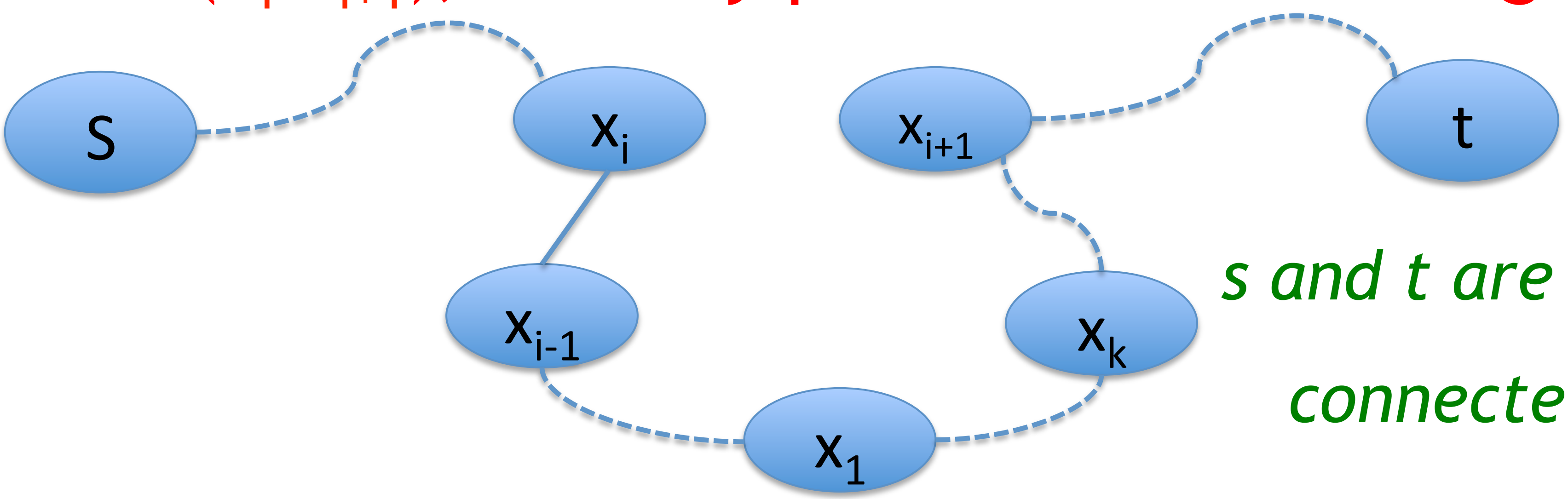


Proof That G Cannot Contain a Cycle C

Breaking a Cycle Lemma*: Removing* ***any edge*** *from a cycle cannot disconnect a connected graph!*



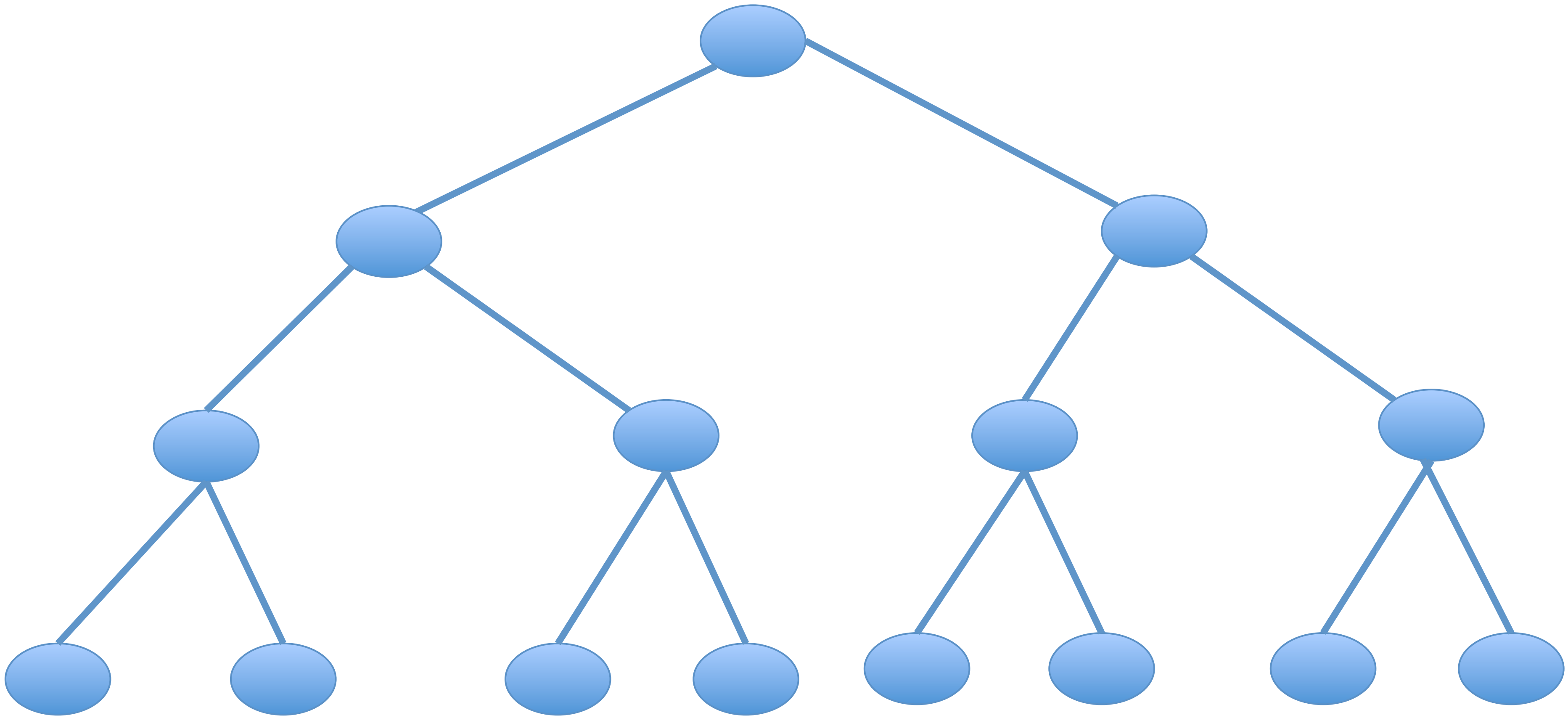
Take $(x_i\ x_{i+1})$, and any path P that was using it



s and t are still connected!

Cycle Creation Lemma

Adding any edge (u, v) to a tree T creates a cycle!



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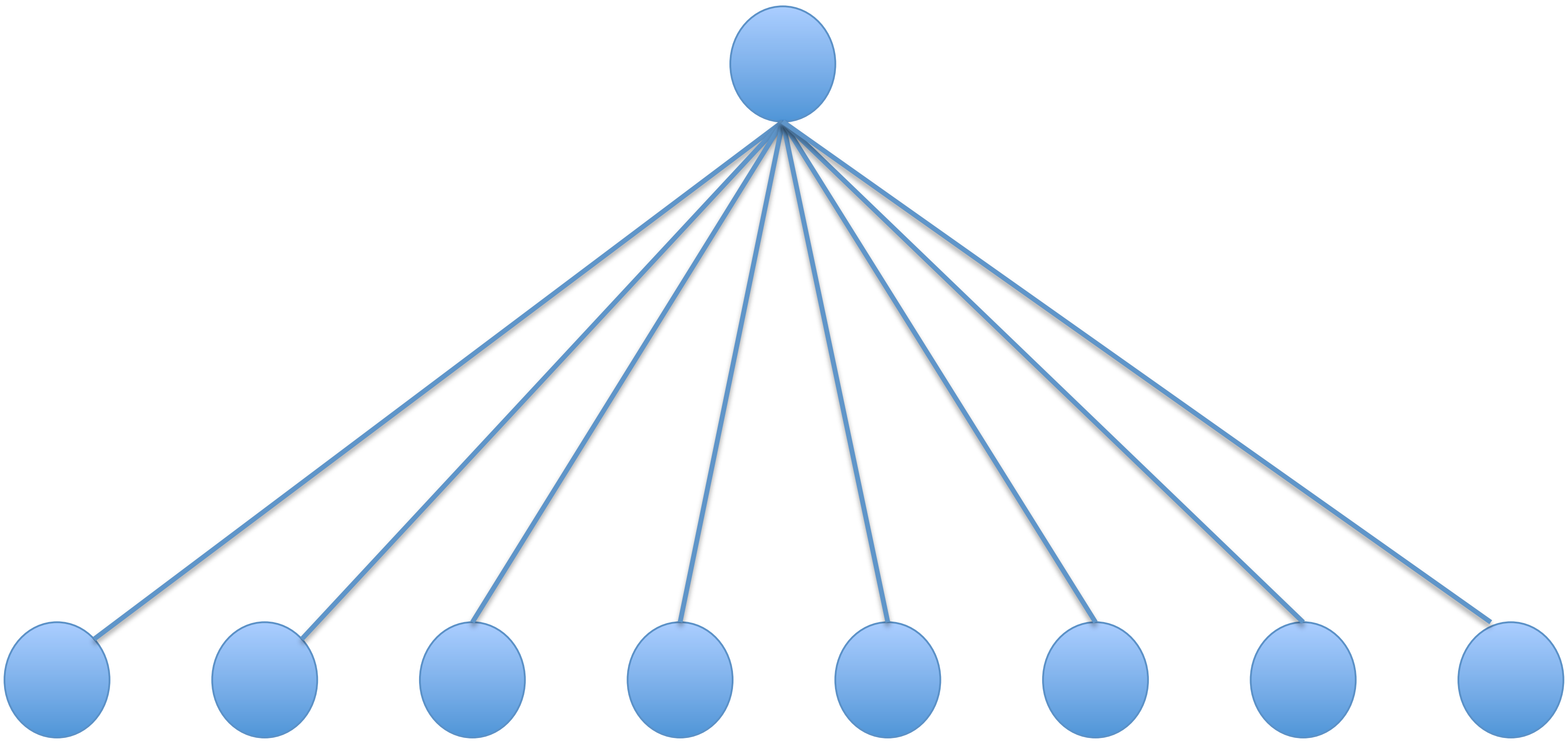
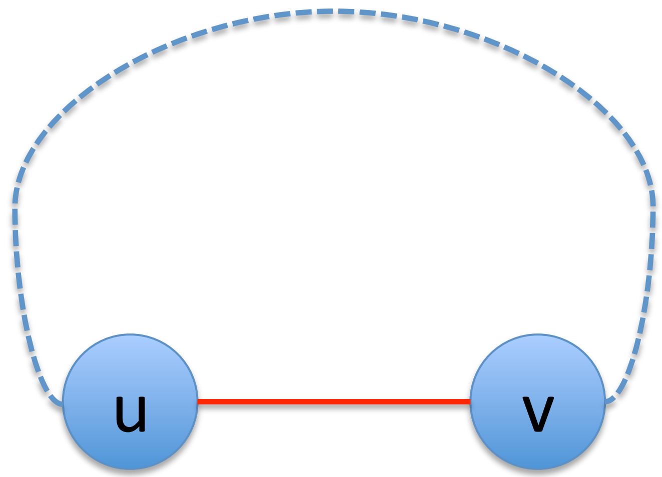


Cycle Creation Lemma

Adding any edge (u, v) to a tree T creates a cycle!

Proof: B/c T is connected, ∃ path P from u to v.



adding (u, v) closes the cycle.

Theorems

Do all by induction as exercise.

1) Every tree of n vertices contains exactly n-1 edges!

2) Every n-1 acyclic set of edges among n nodes is a tree => i.e., they connect V!

3) Every n-1 set of edges that connects n vertices is a tree => i.e., they are acyclic!

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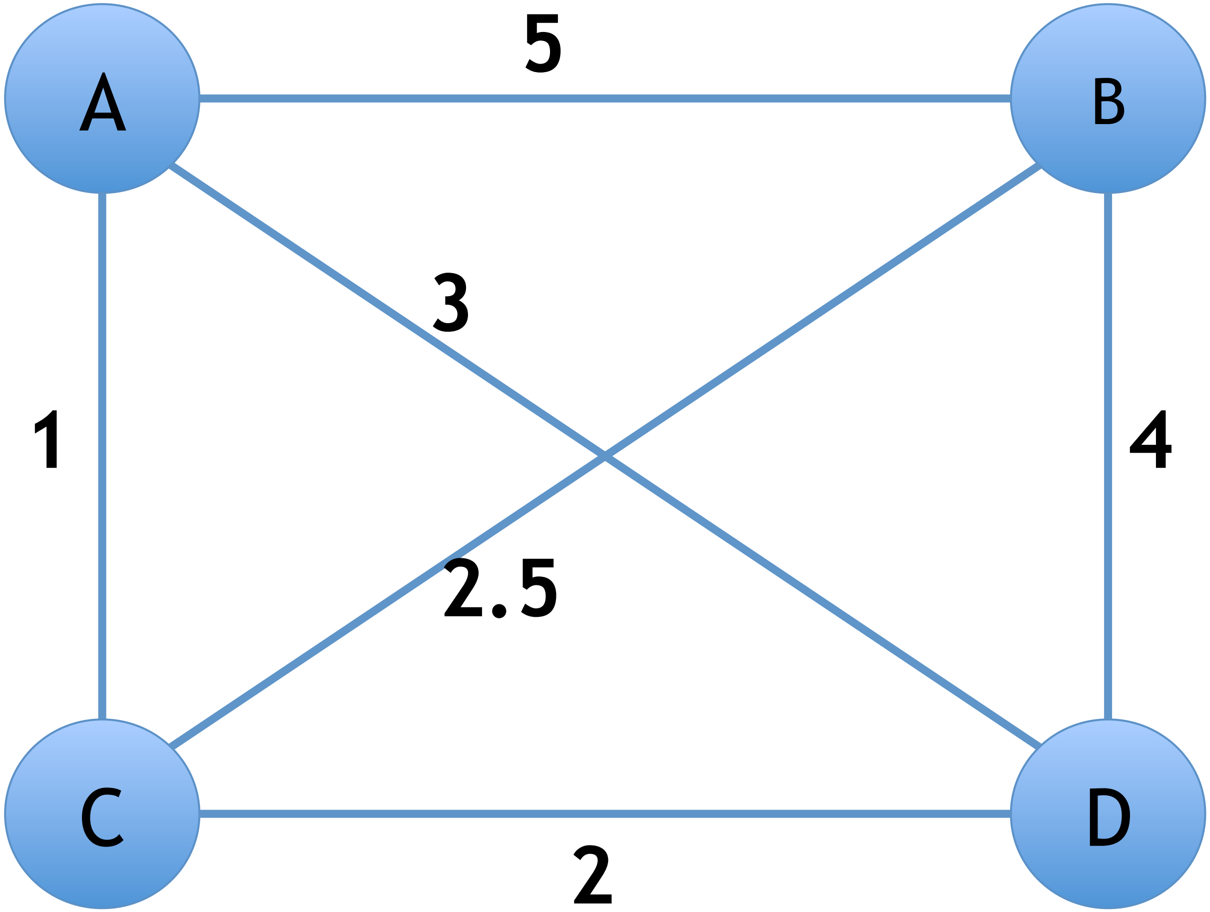
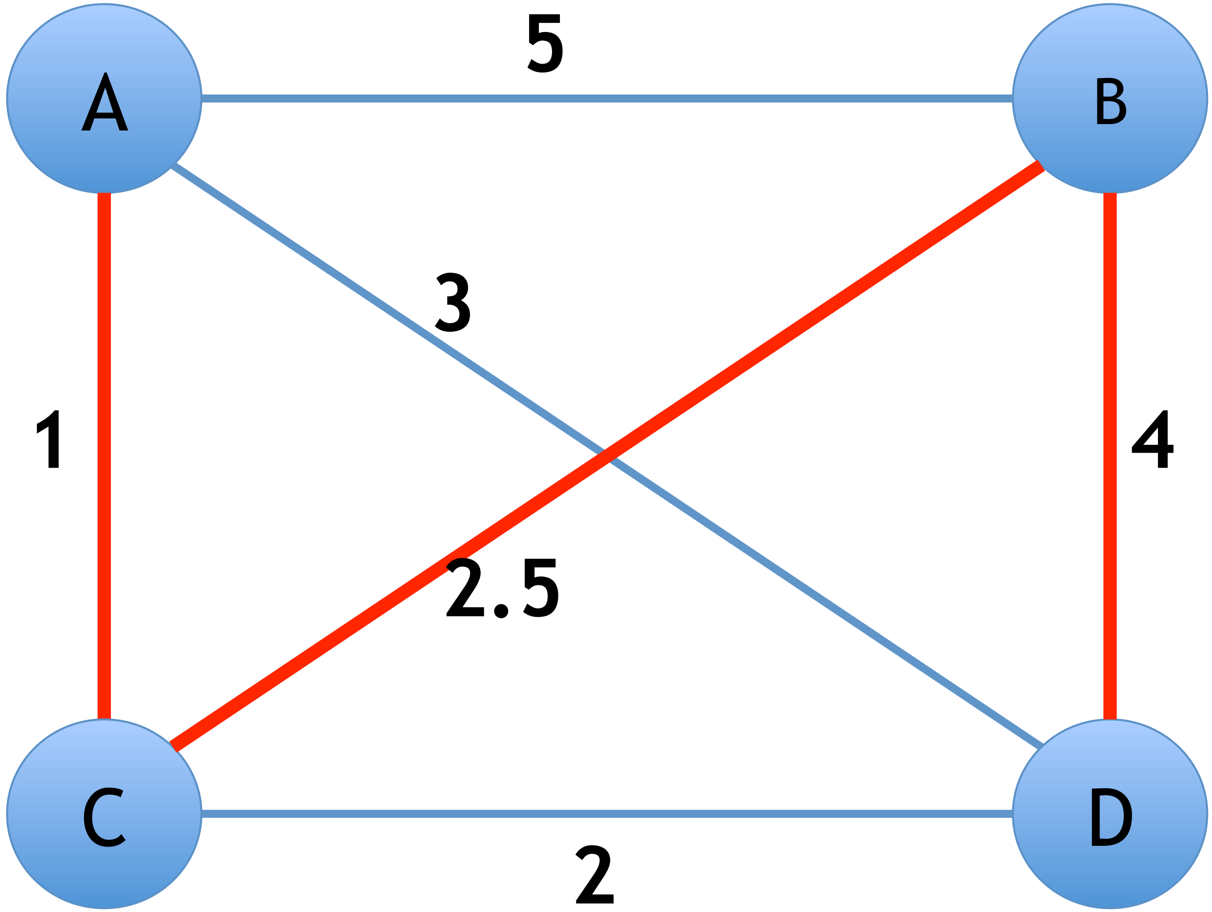
Minimum Spanning Tree

- Input: undirected & connected G(V, E) and arbitrary edge weights
- Output: A tree T* of V such that w(T*) ≤ any other T of V
  - w(T) = sum of the weights of all n-1 edges in T
  - “spanning” tree of G(V, E) means T* has to connect all of V
- Assumptions:
  1. G is connected (minor)
  2. Edge weights are distinct (all of our algorithms work w/o this assumption)

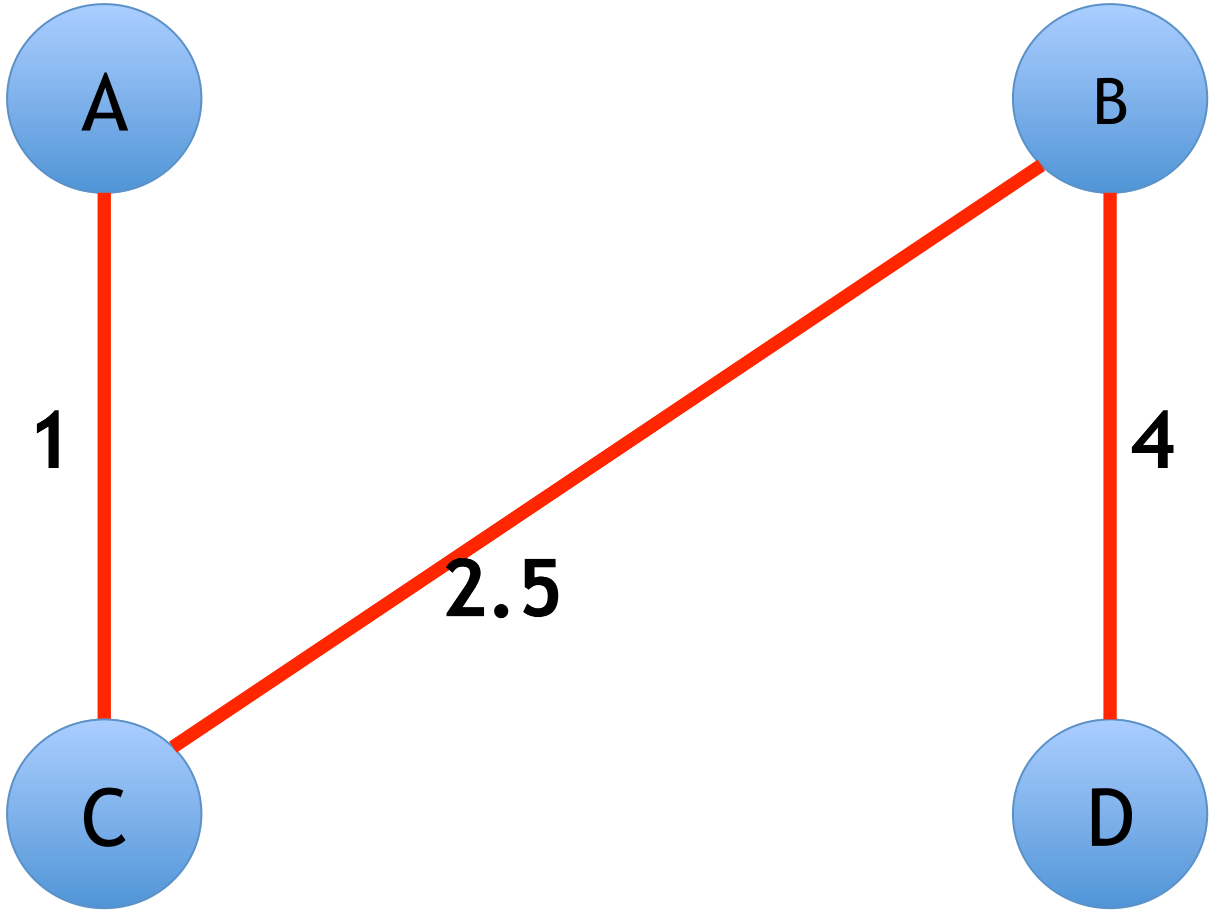
Spanning Tree Example



Spanning Tree Example

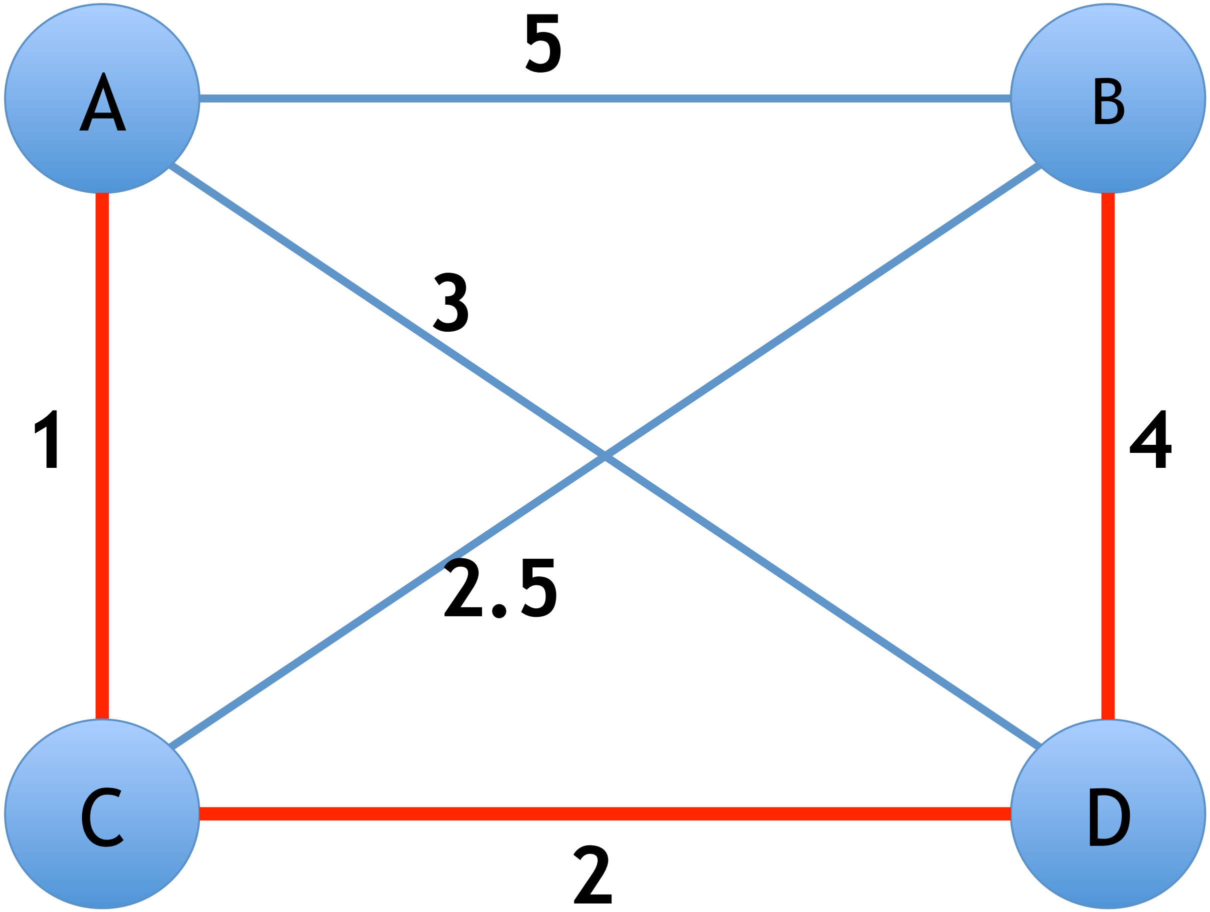


Spanning Tree Example

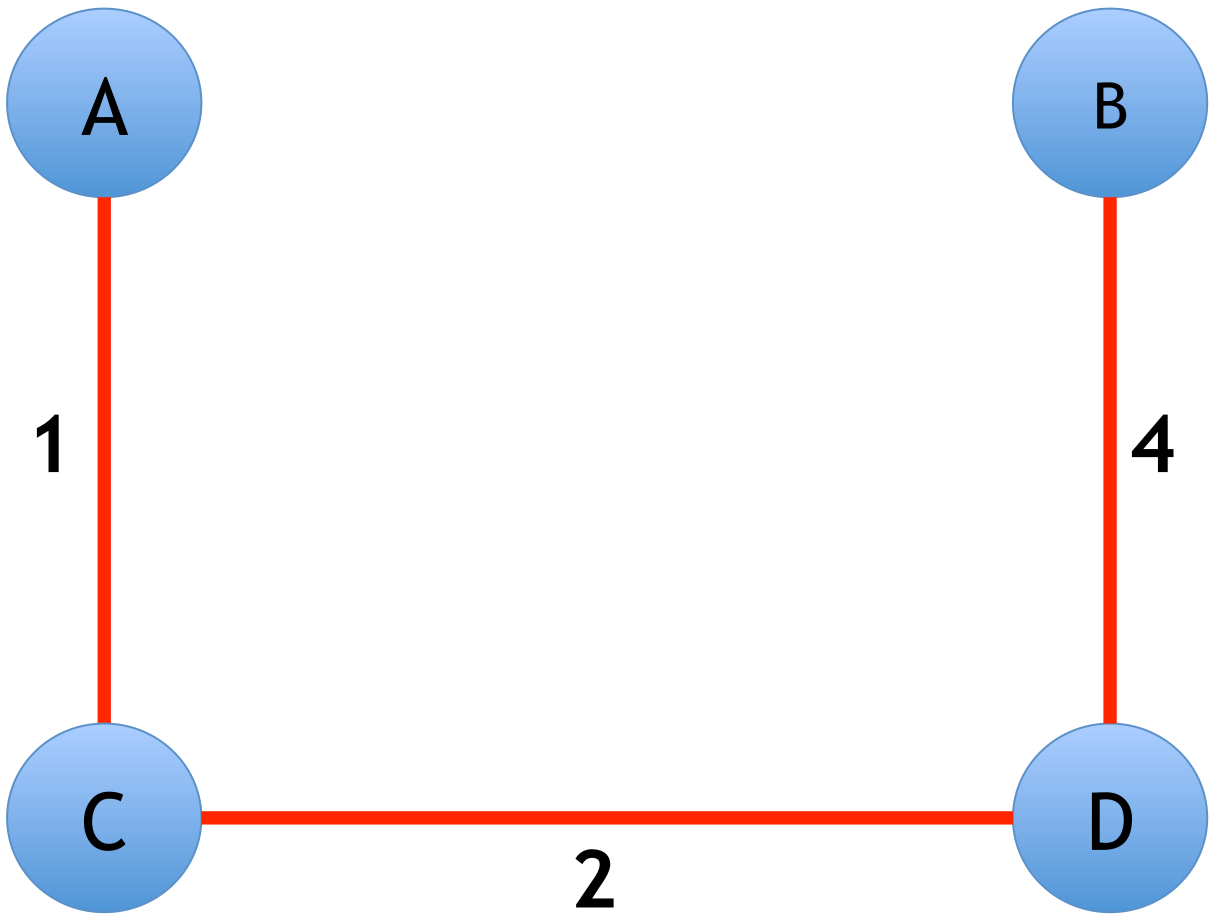


$W((A,C), (C,B), (B,D)) = 1 + 2.5 + 4 = 7.5$

Spanning Tree Example

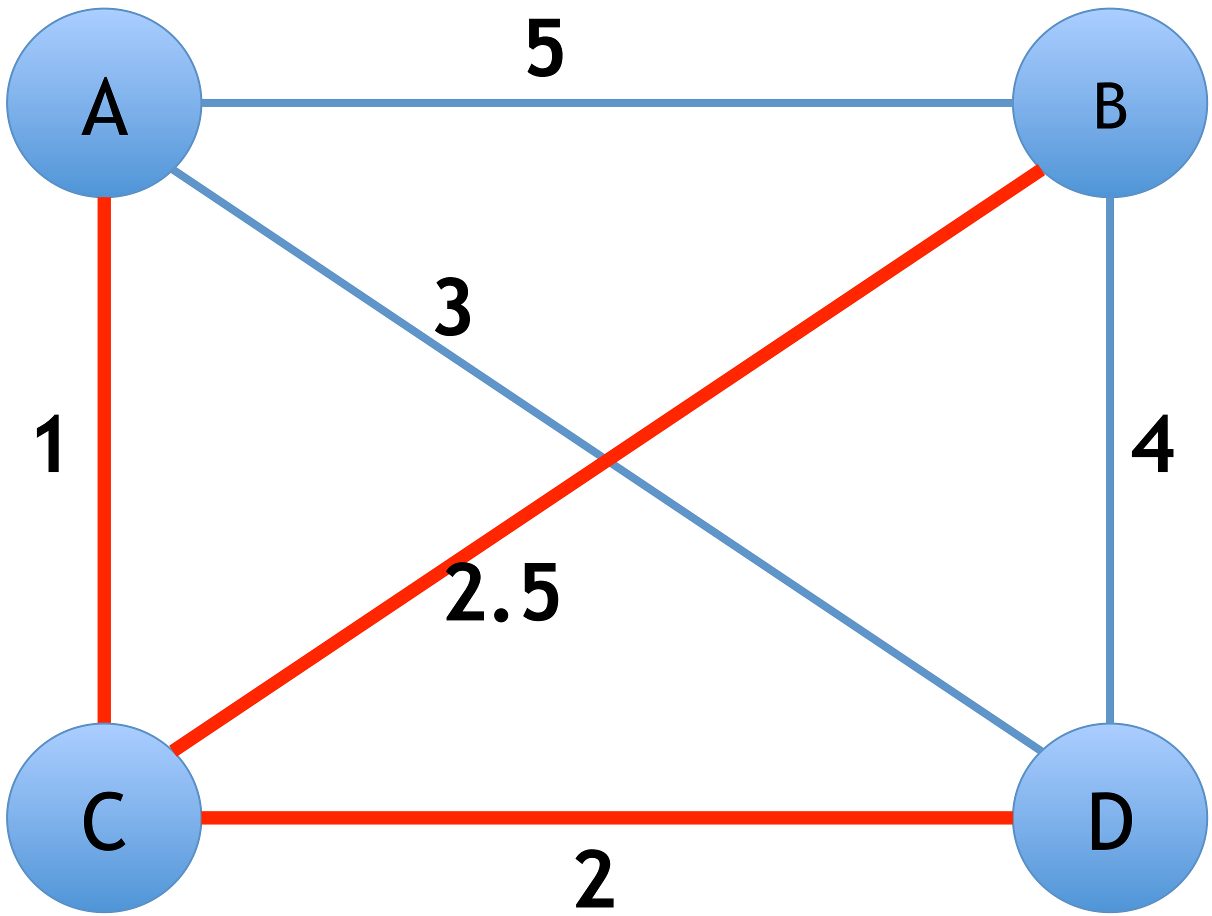


Spanning Tree Example

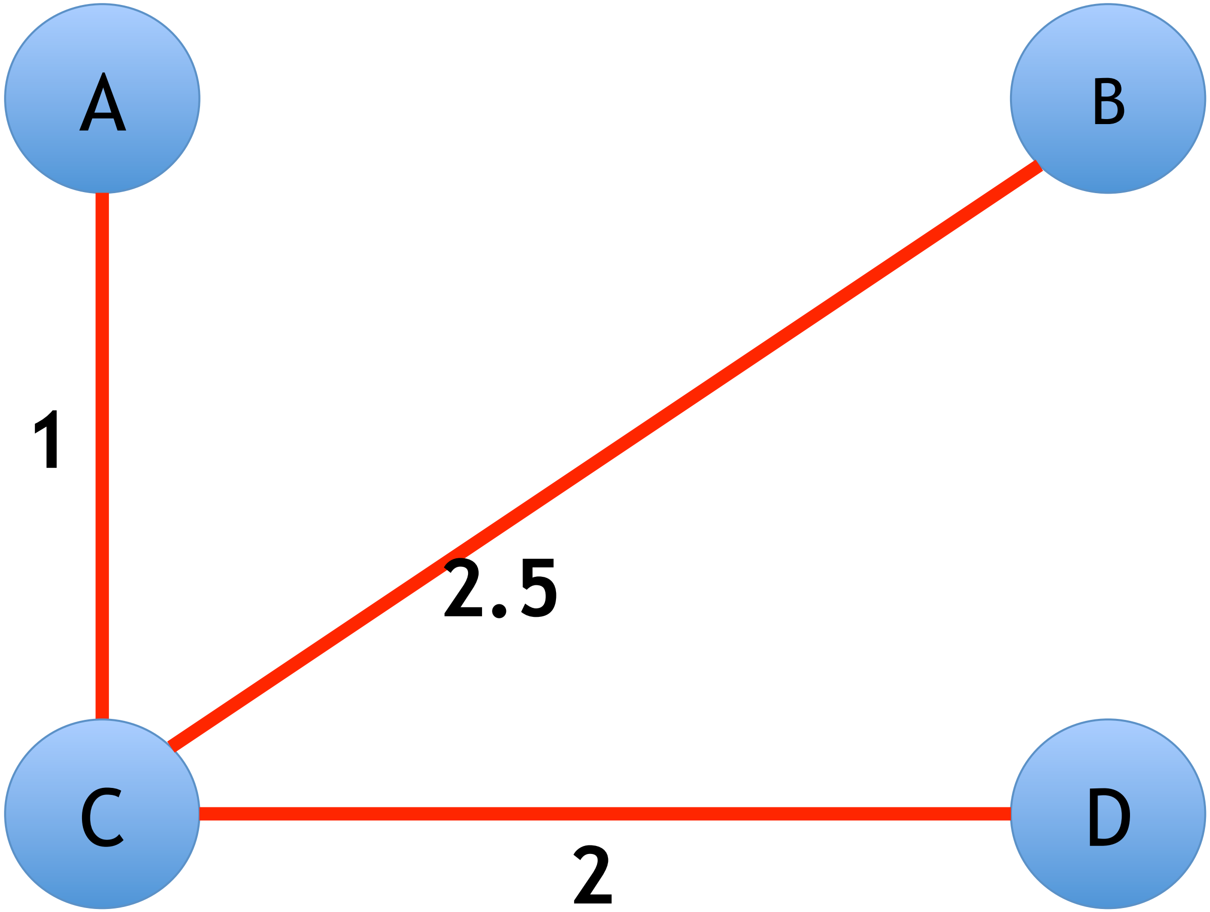


W((A,C), (C,D), (B,D)) = 1 + 2 + 4 = 7

Spanning Tree Example



Spanning Tree Example



W((A,C), (C,D), (C,B)) = 1 + 2 + 2.5 = 5.5

MST Applications

- Designing all kinds of networks:
  - datacenter networks
  - road networks
  - phone networks
- Circuit Design
- Clustering
- Image Segmentation

…

> 110 years old problem. Can find publications from 1906.

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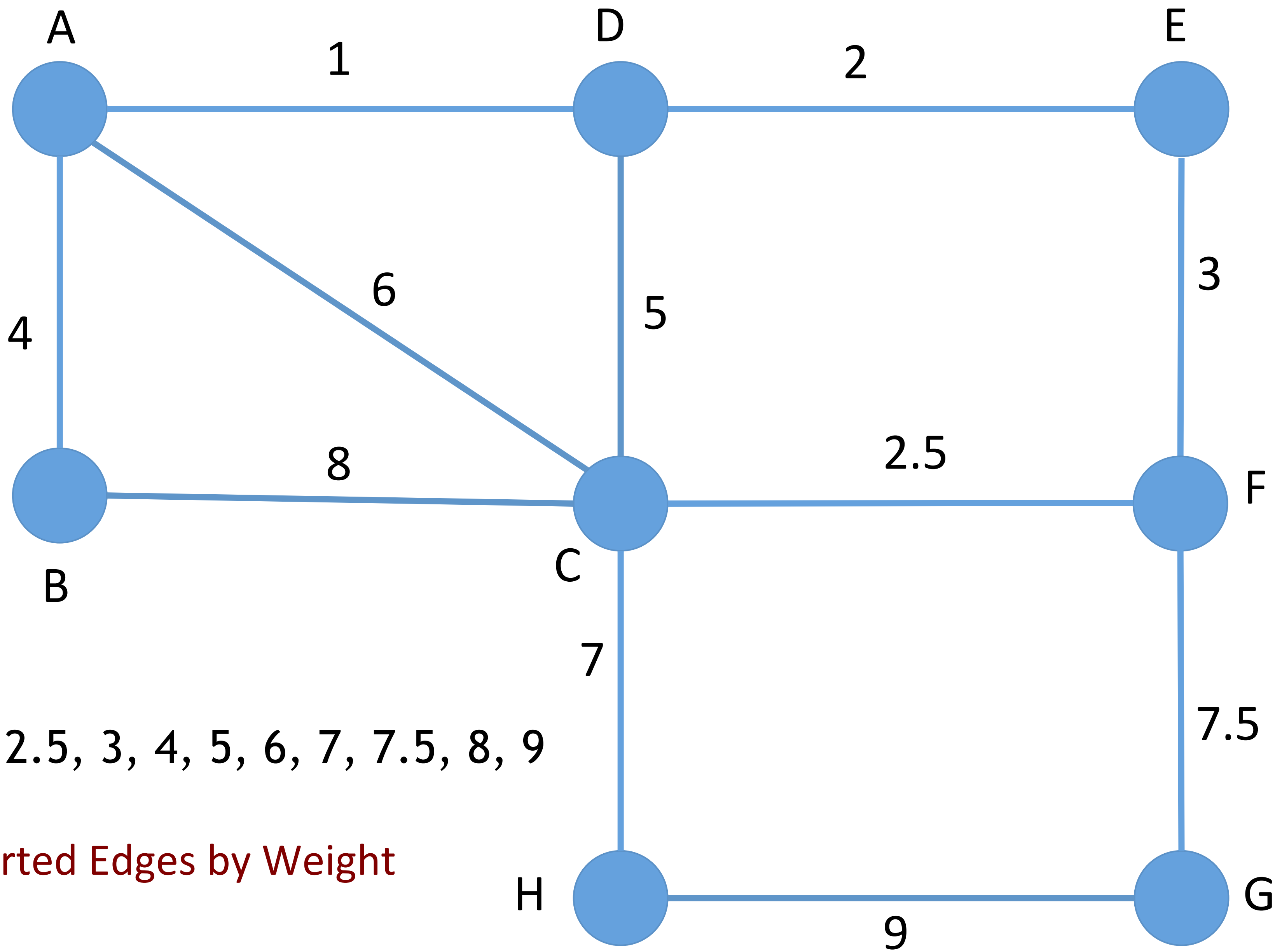
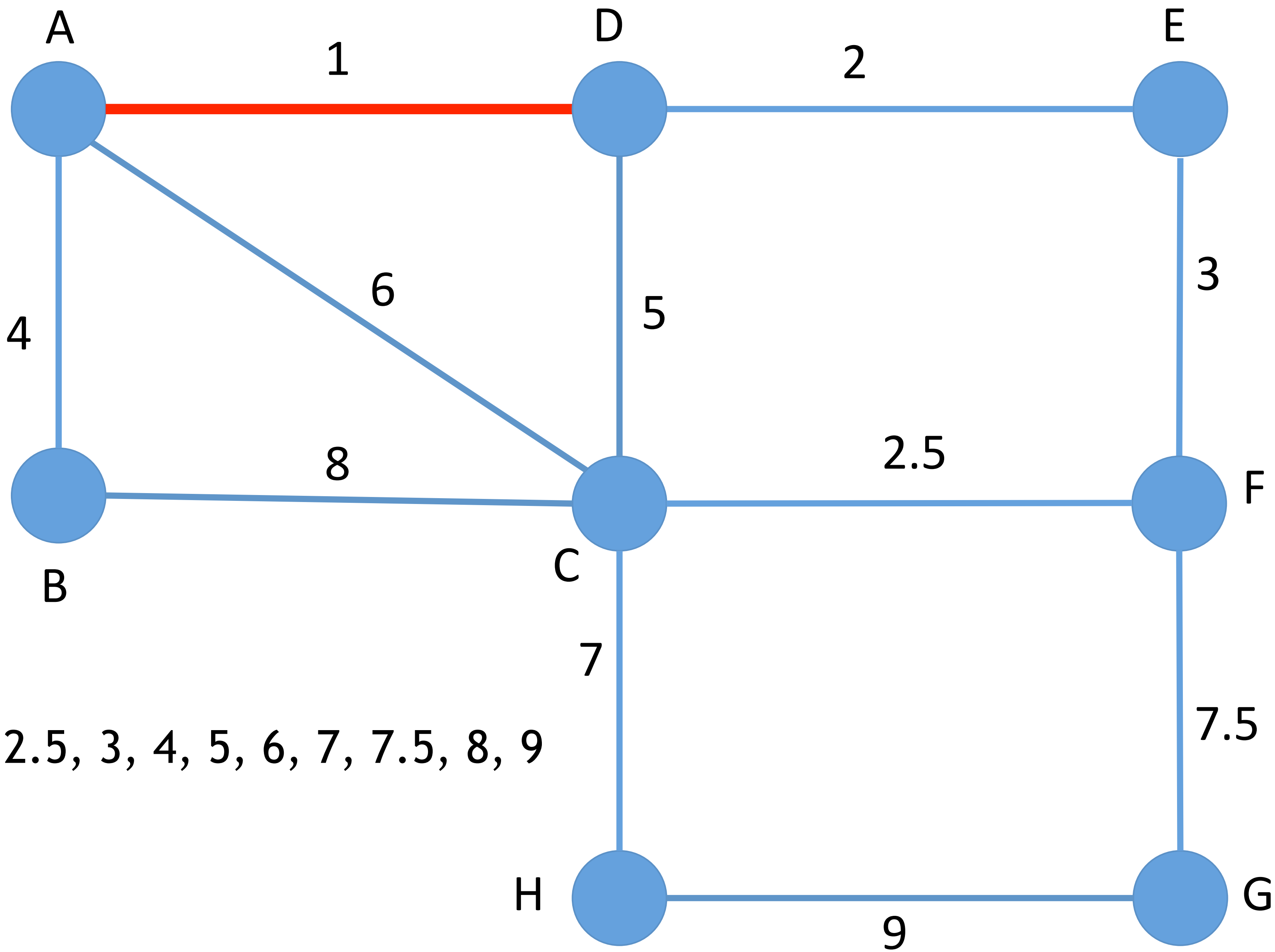
Kruskal’s Algorithm Simulation



1, 2, 2.5, 3, 4, 5, 6, 7, 7.5, 8, 9

Sorted Edges by Weight

Kruskal's Algorithm Simulation



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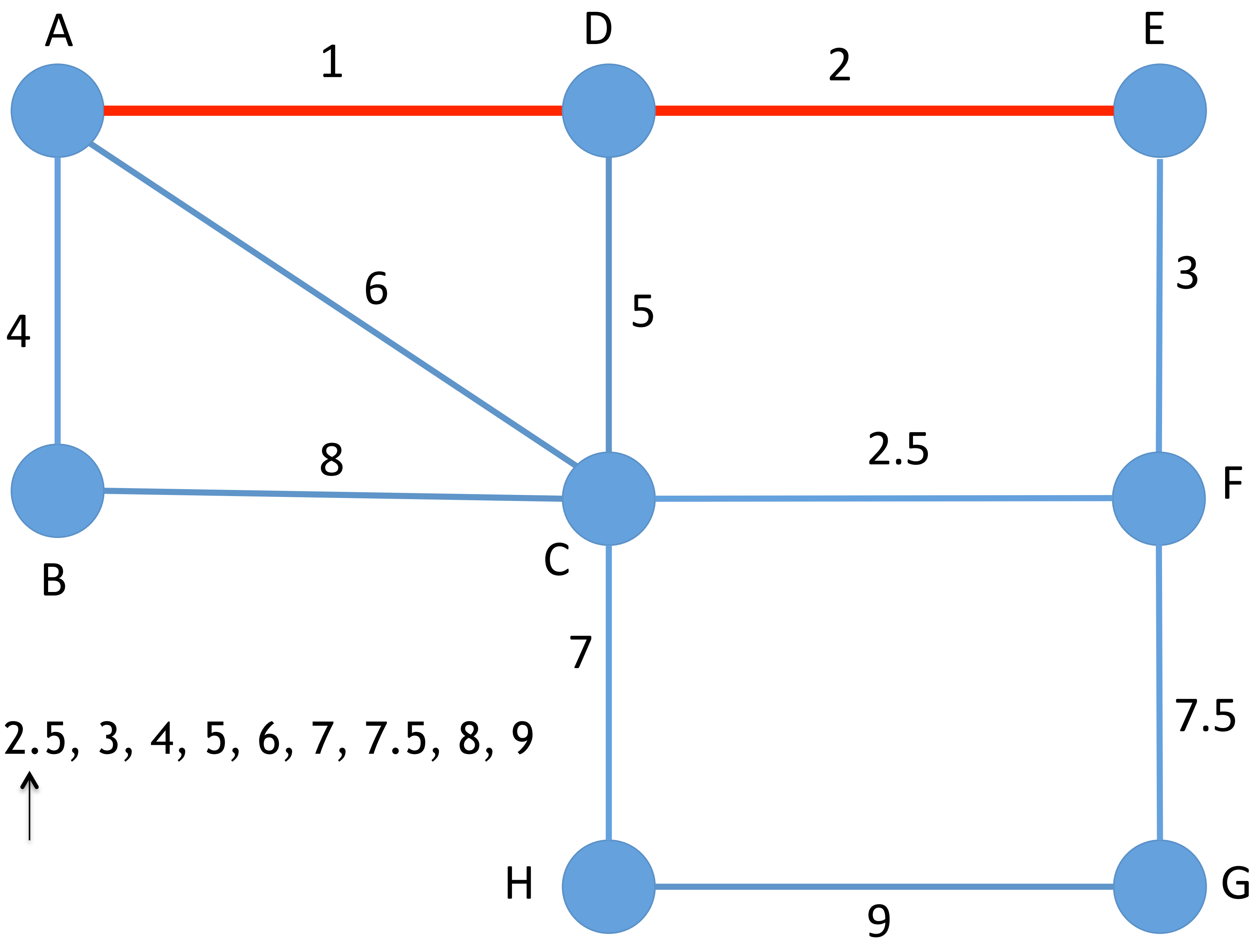

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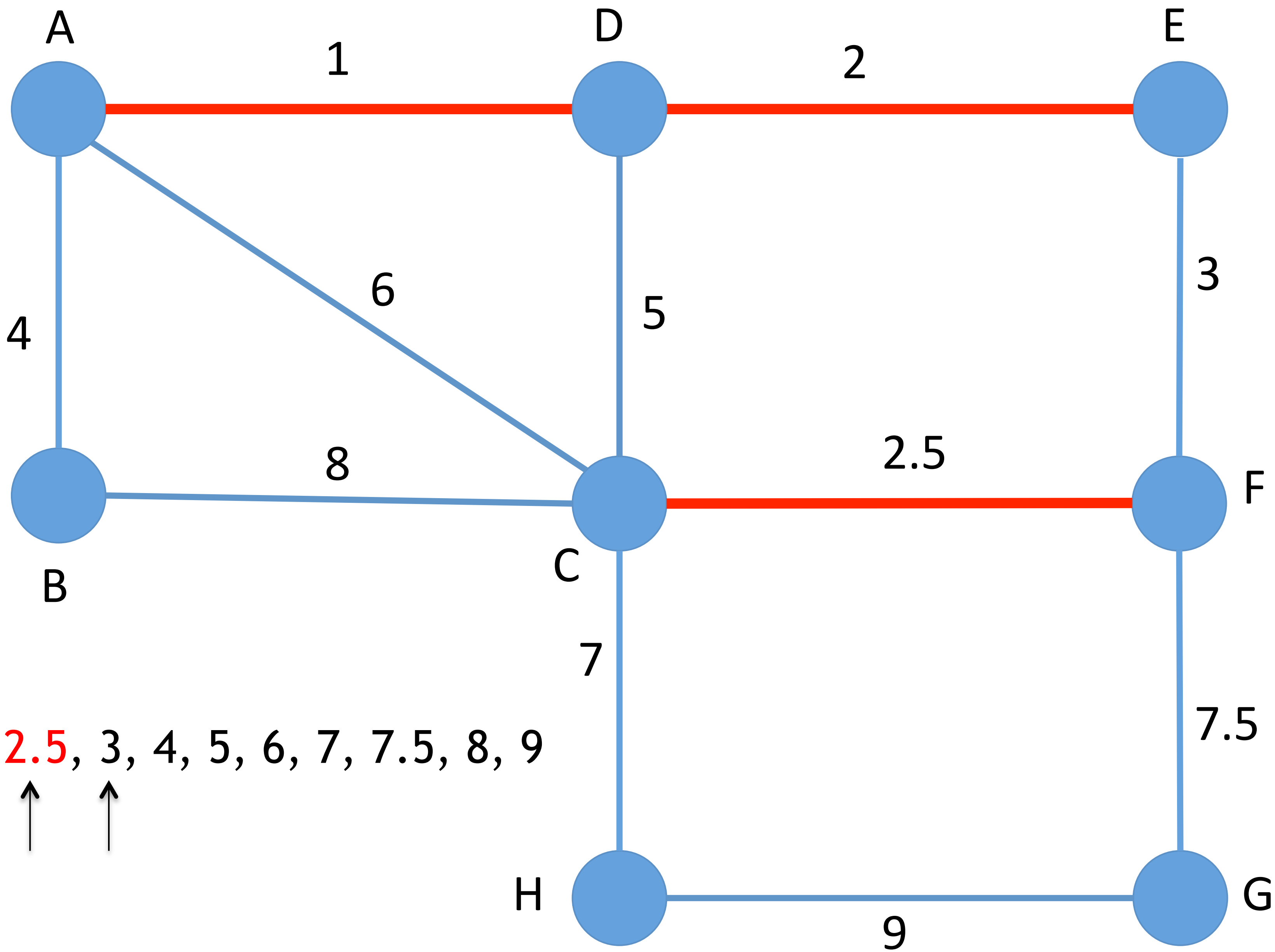

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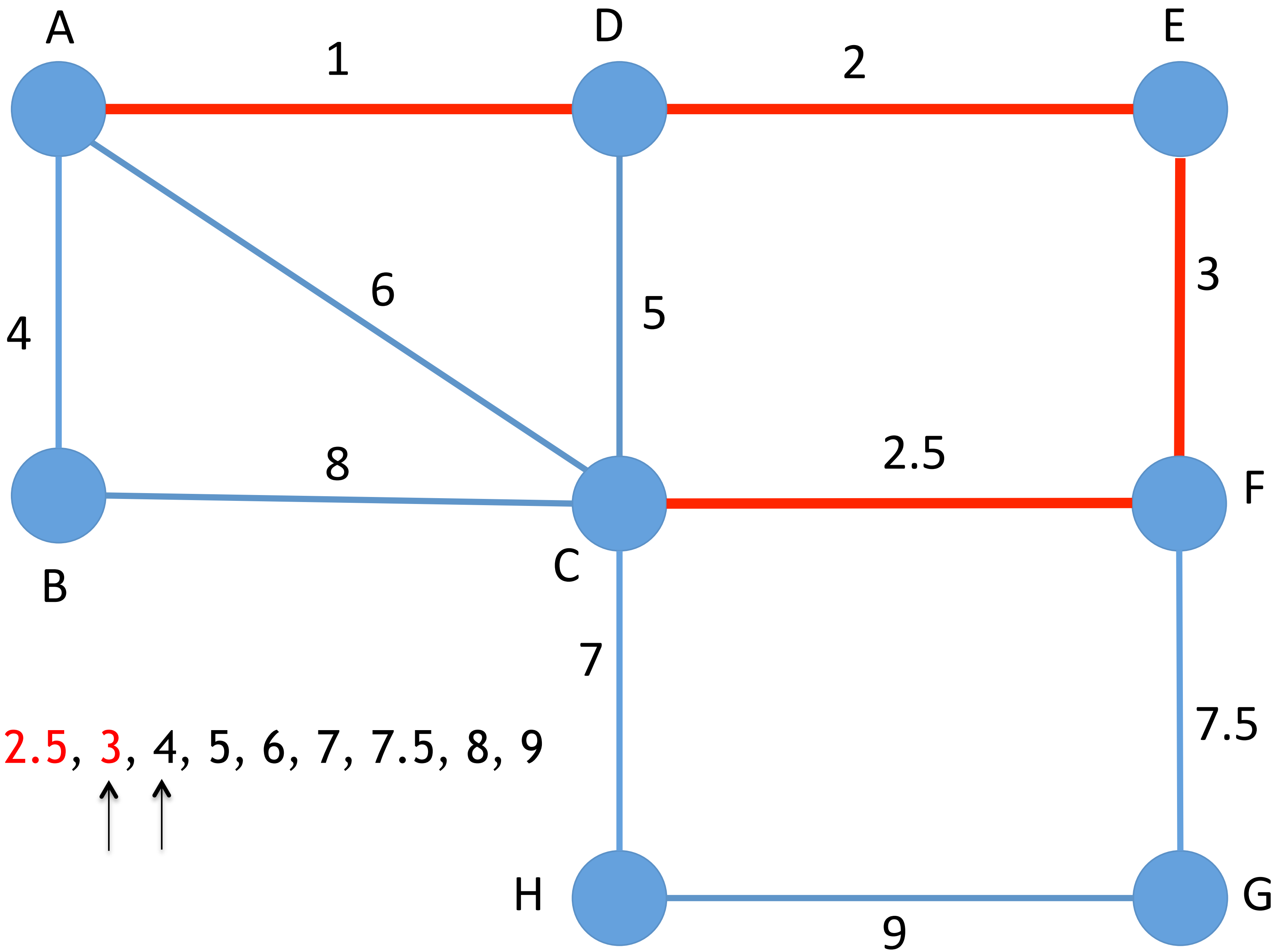

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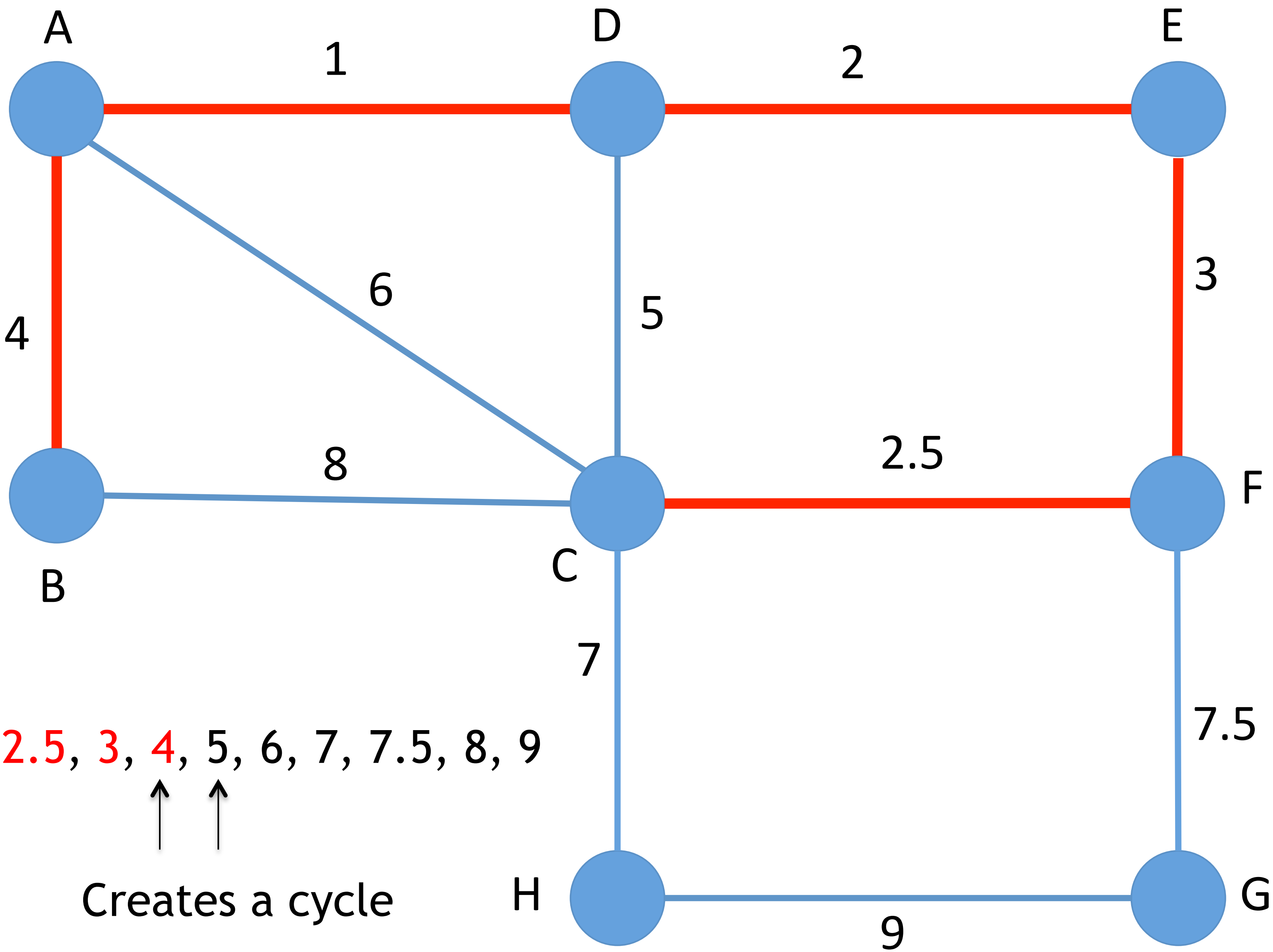

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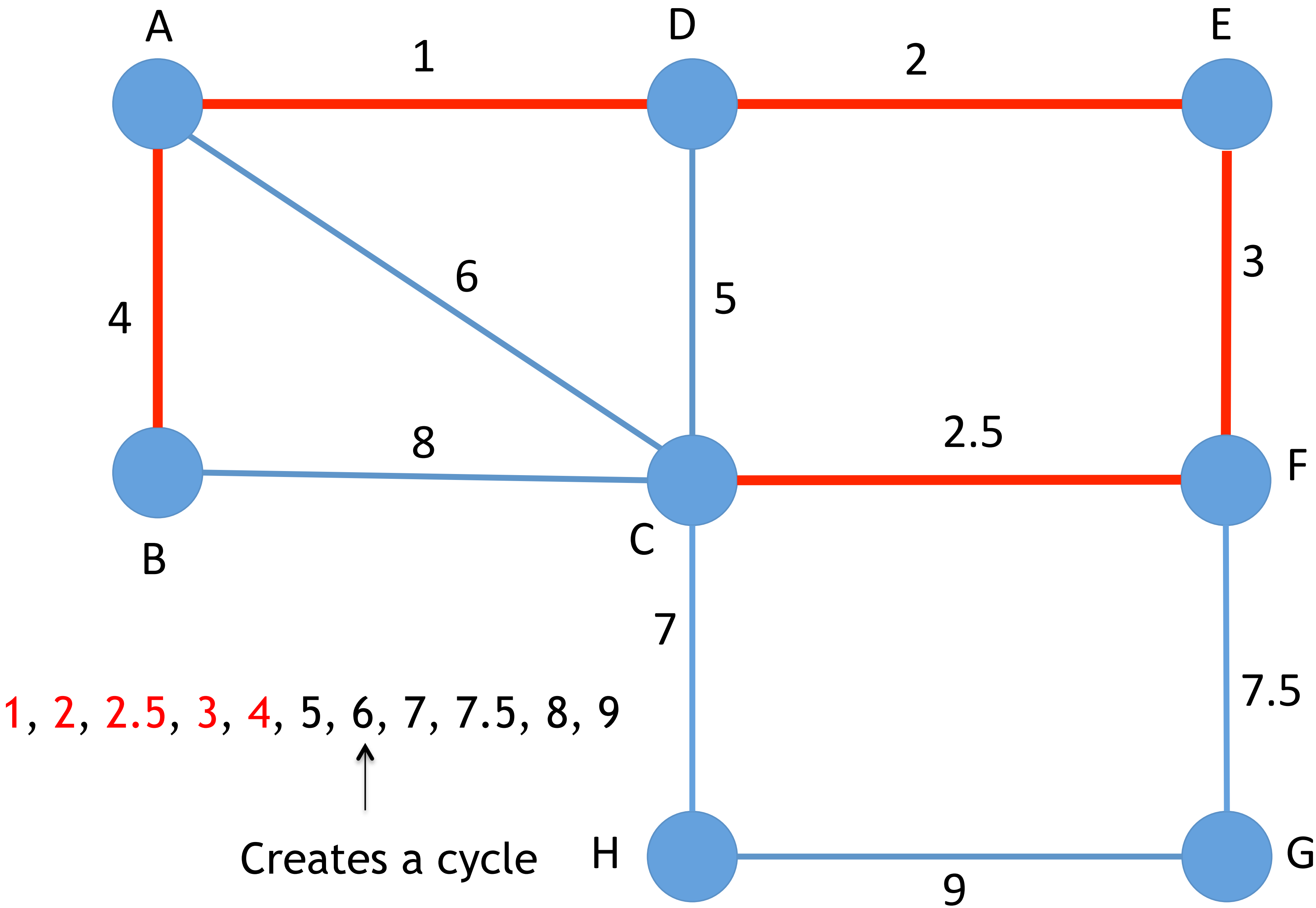
Creates a cycle

Kruskal's Algorithm Simulation



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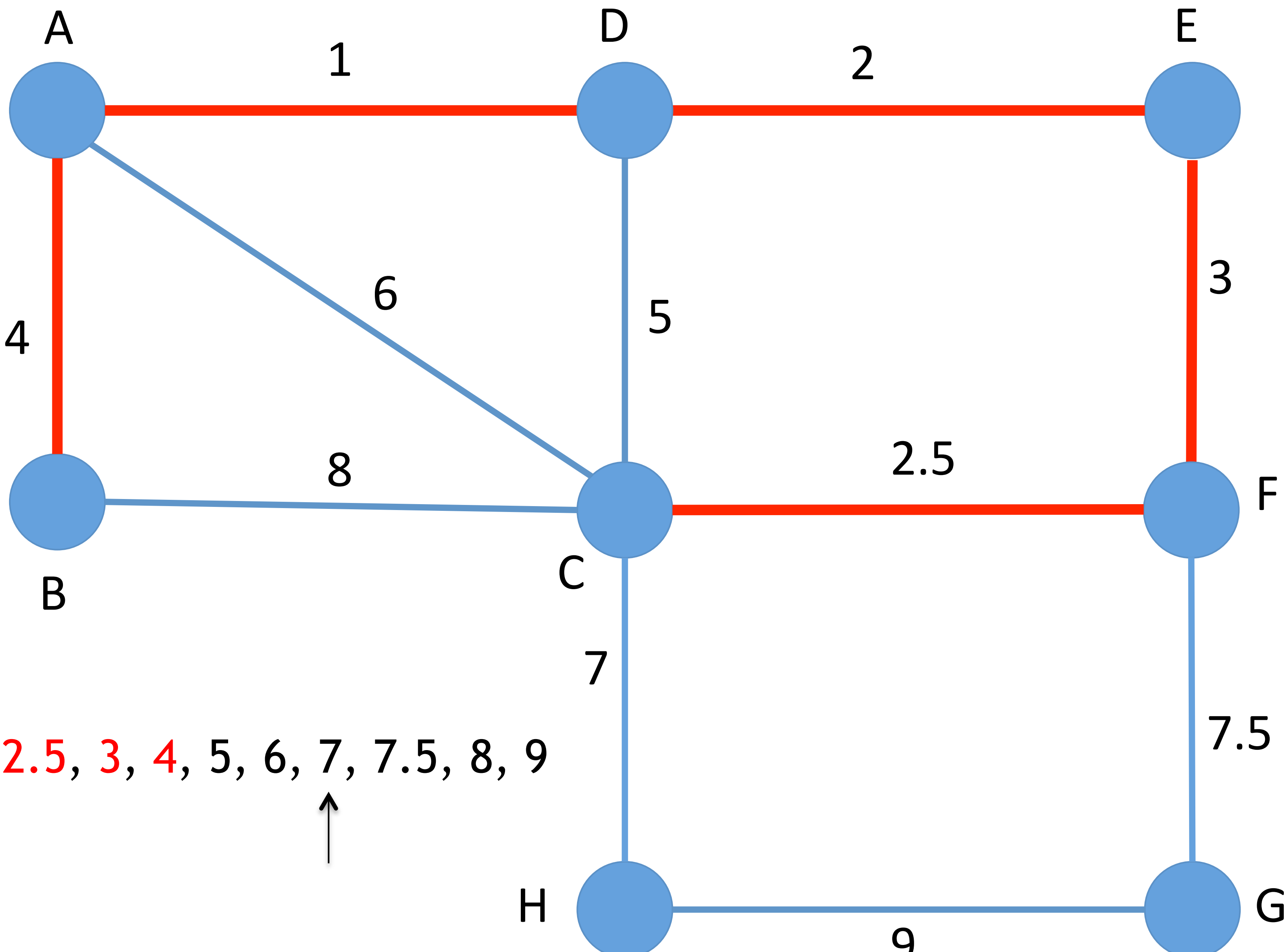

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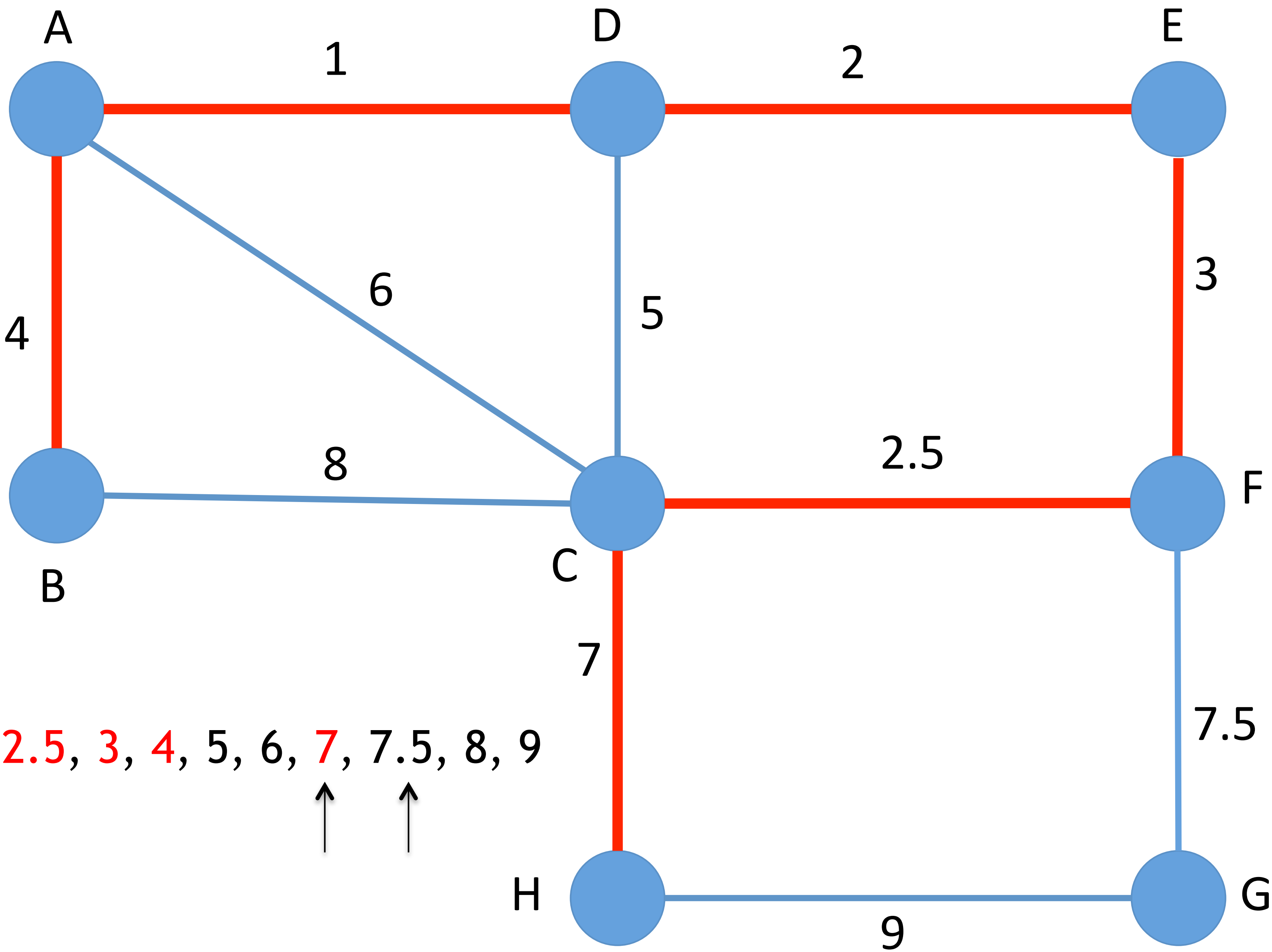

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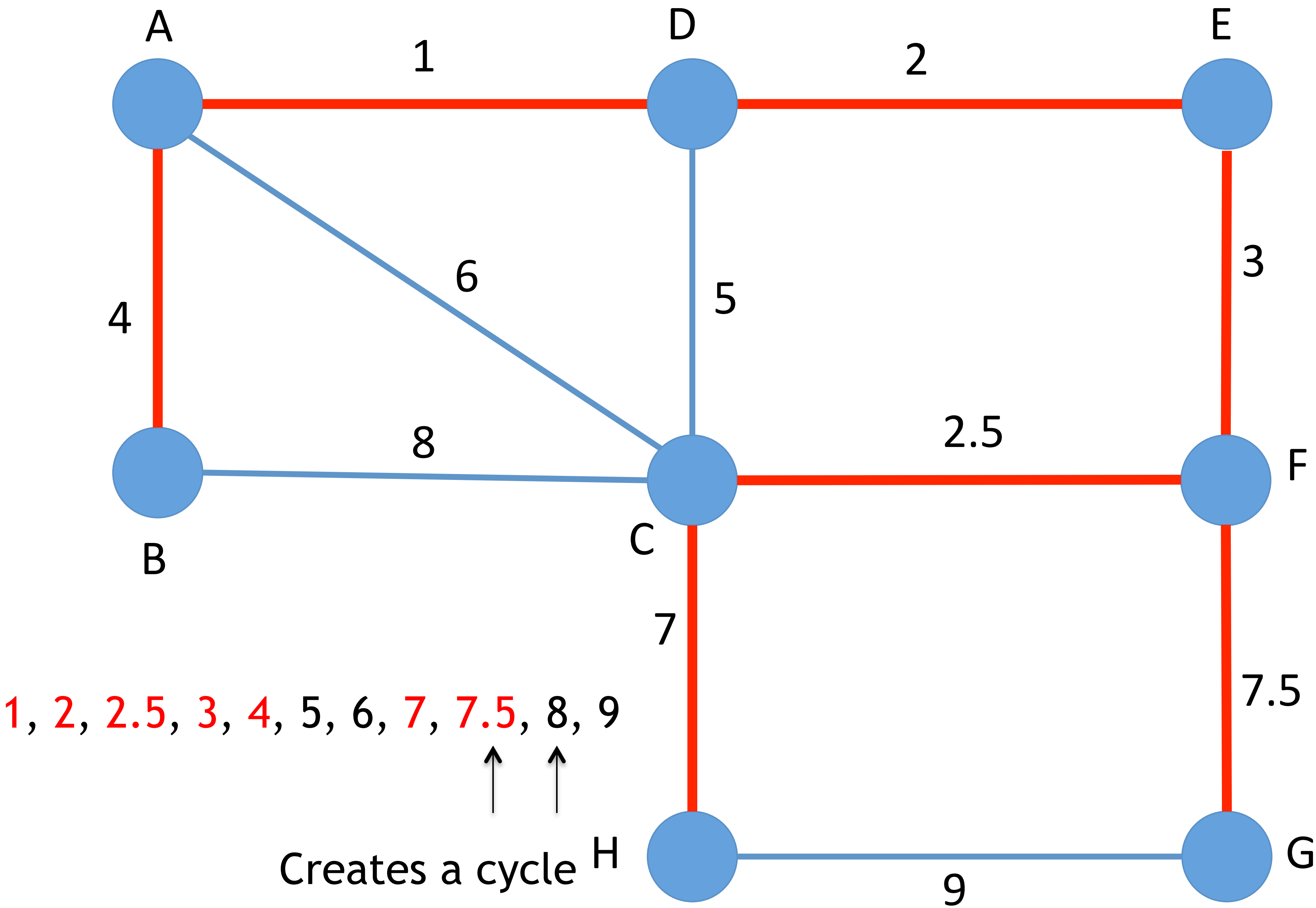

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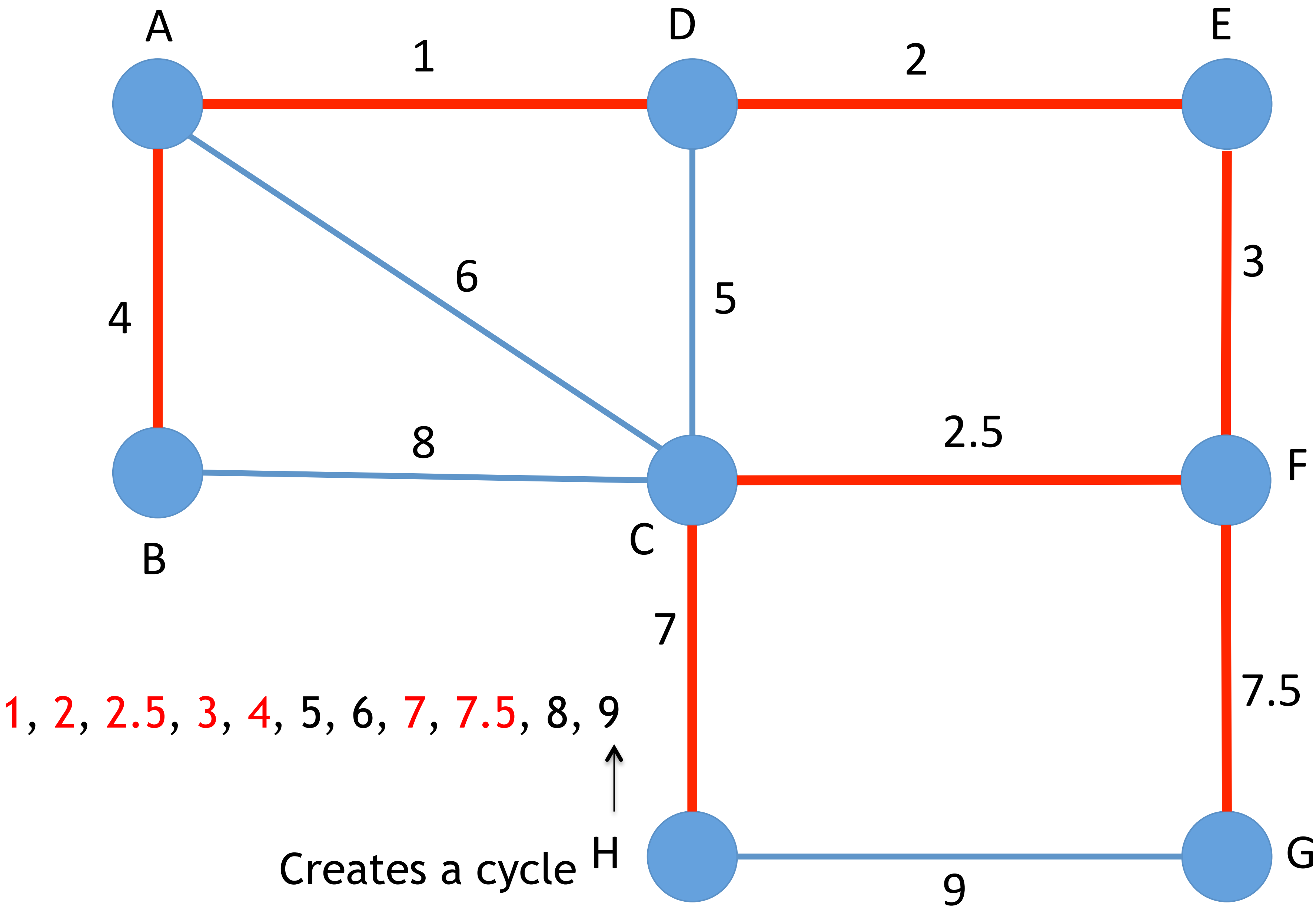

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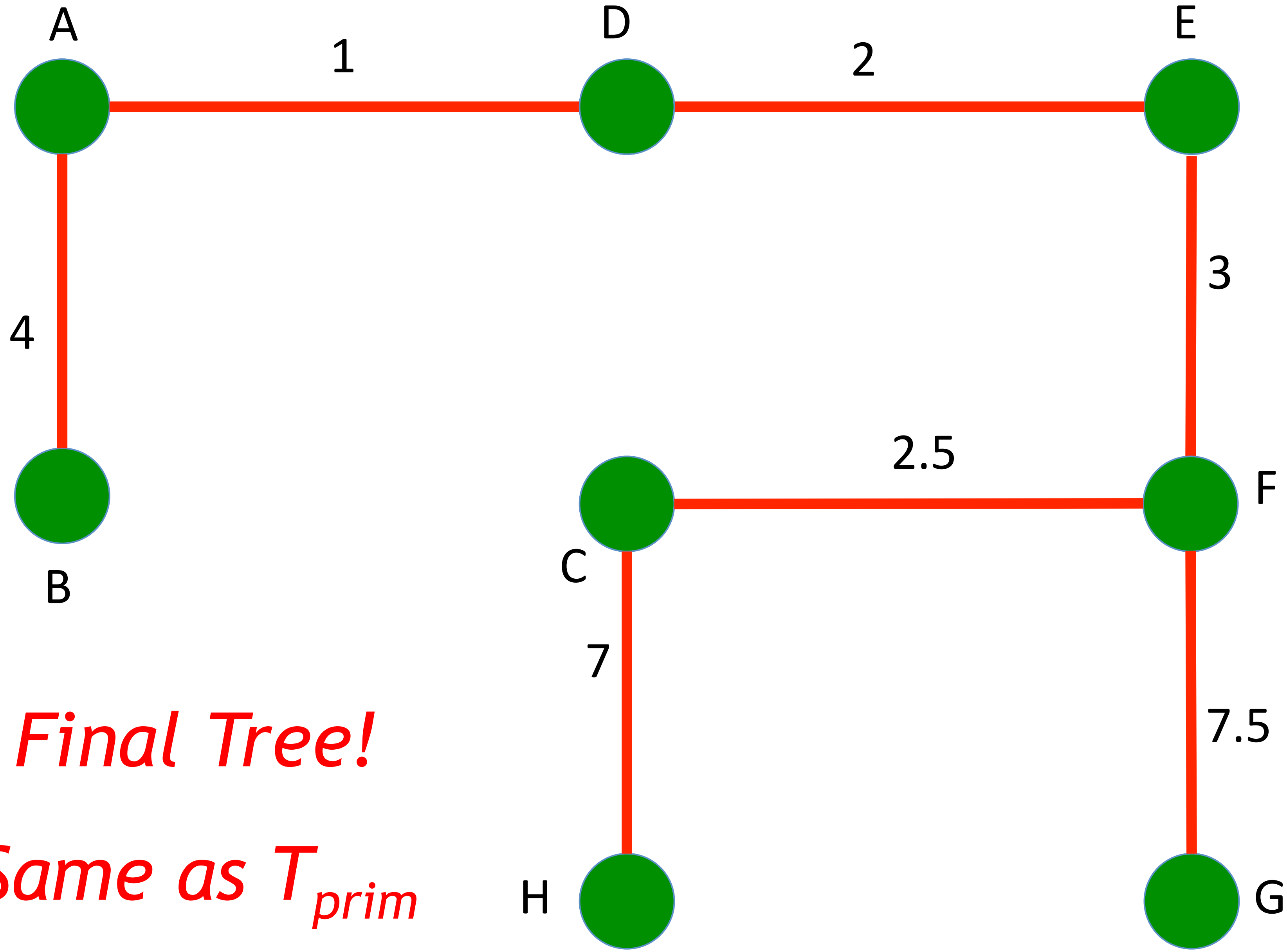

Final Tree!

Same as T_{prim}

Kruskal's Algorithm Pseudocode (1956)

```
procedure kruskal(G(V, E)):
  sort E in order of increasing weights
  rename E so w(e1) < w(e2) < … < w(em)
  T = {} // final tree edges
  for i = 1 to m:
    if T ∪ ei=(u,v) doesn't create cycle
      add ei to T
  return T
```

O(mlog(n))

O(log(n)) by union-find data structure
(Verify as an exercise)
So in total: O(mlog(n))

Total Runtime:
O(mlog(n))

Correctness?

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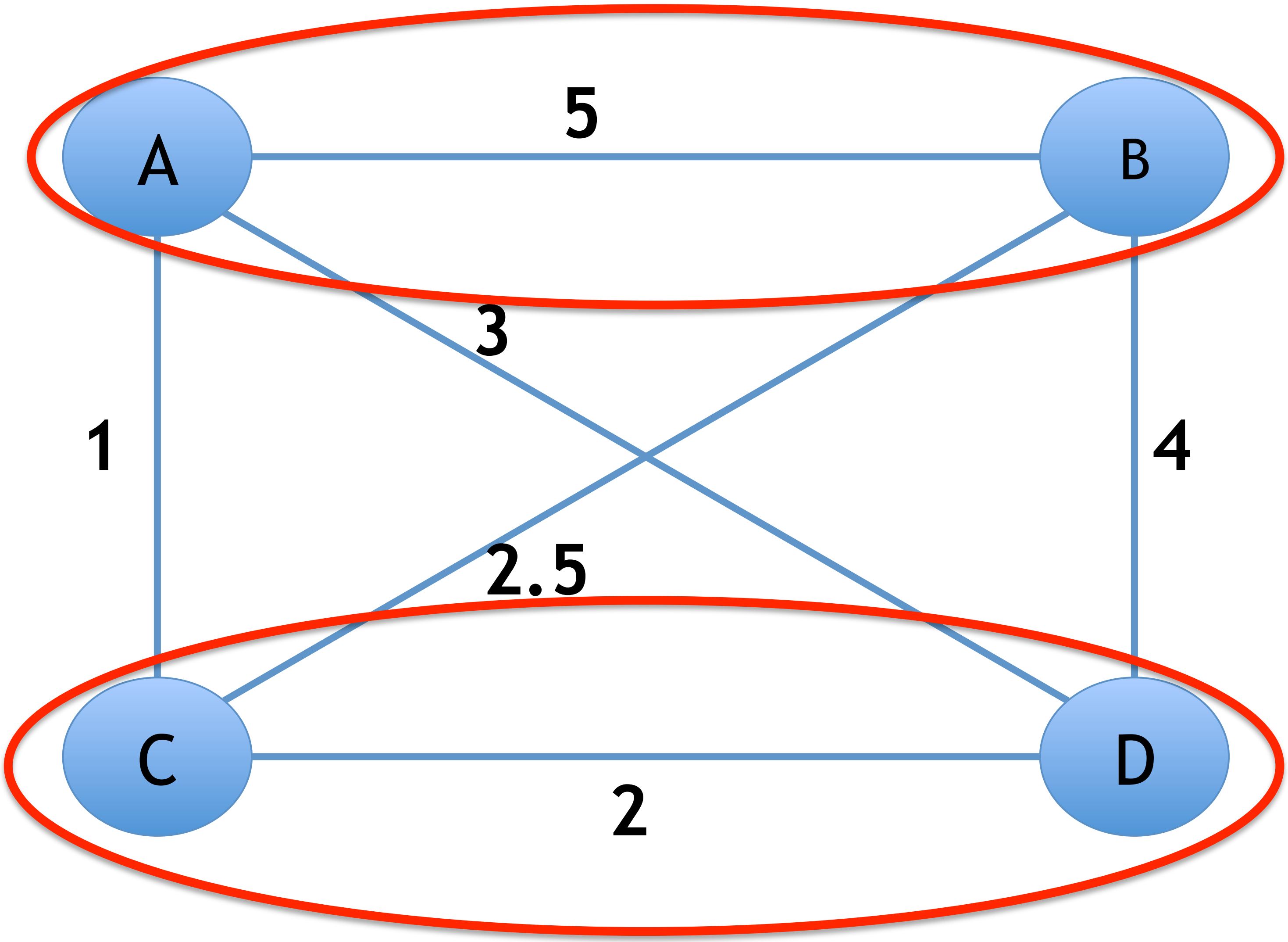
Cuts

- A cut of G(V, E) is a ***slice*** of V into 2 non-empty sets



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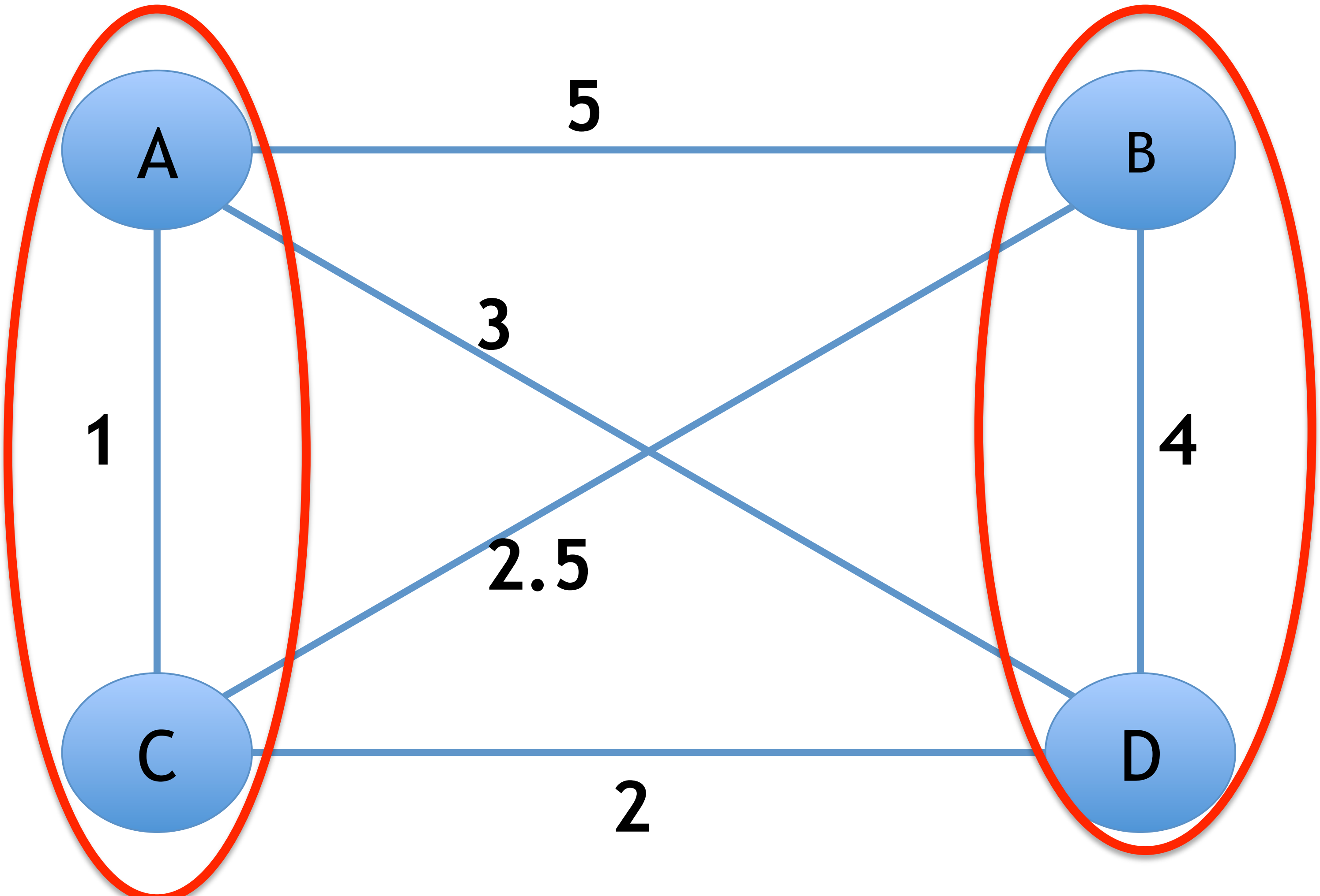

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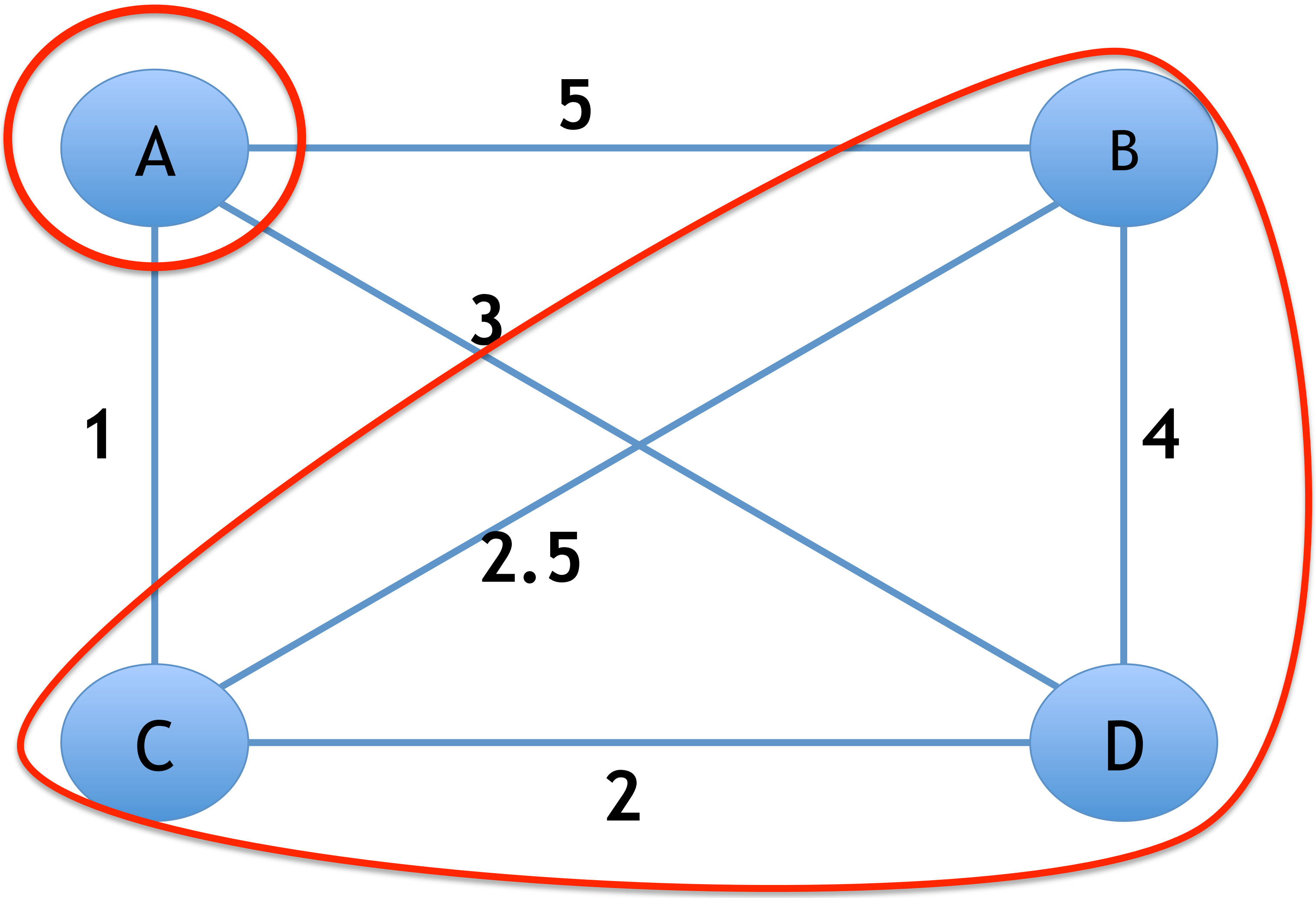


Empty Cut Lemma

A graph G(V, E) is disconnected $\Leftrightarrow$ *∃ cut (X, Y) with no crossing edges*

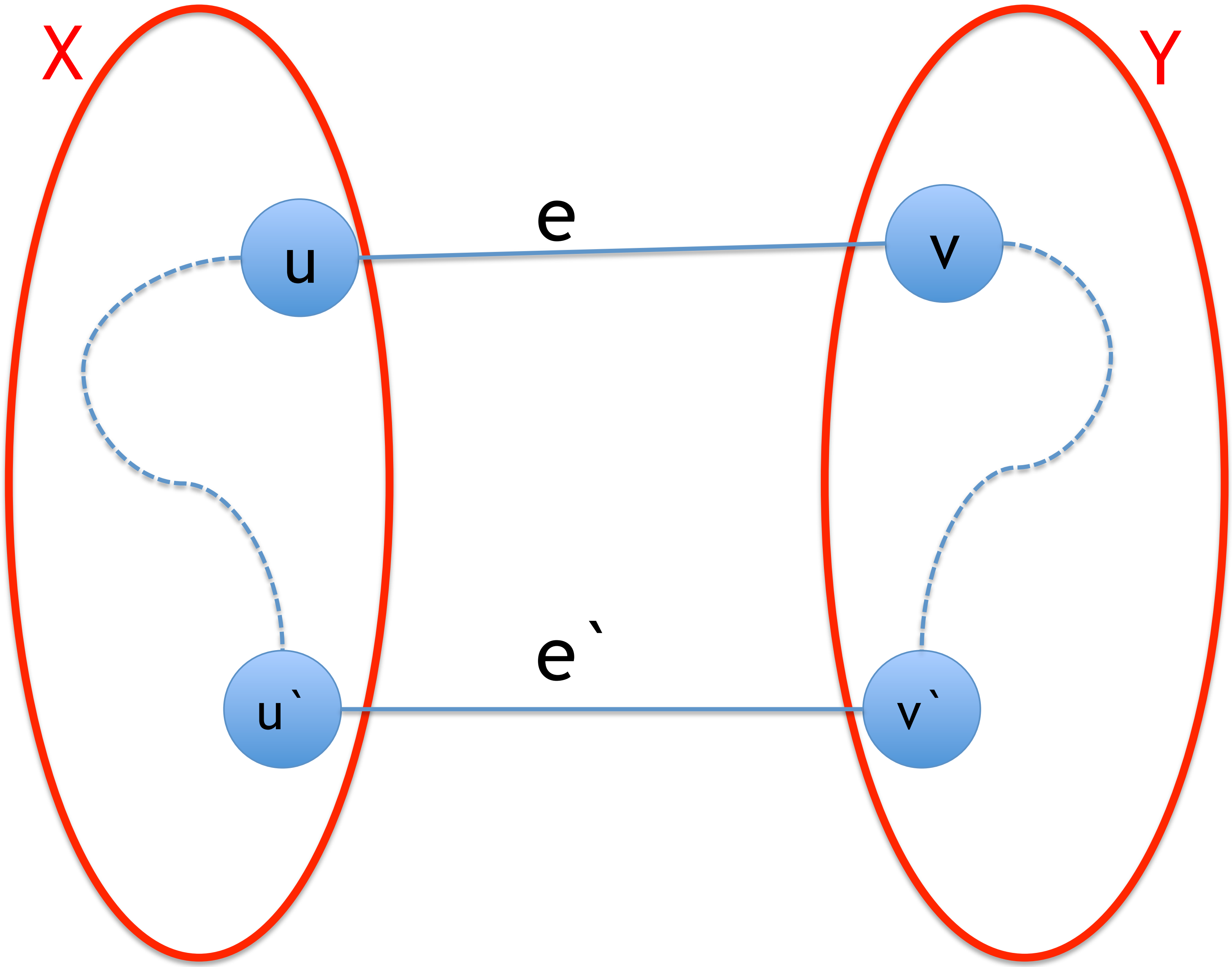
[prove as exercise]

Double Cut Crossing Lemma of Cycles

Suppose a cycle $C \subseteq E$ has an edge e crossing a cut (X, Y)

Claim: Then there is another e` ≠ e of C that also crosses (X, Y)

Double Cut Crossing Lemma of Cycles

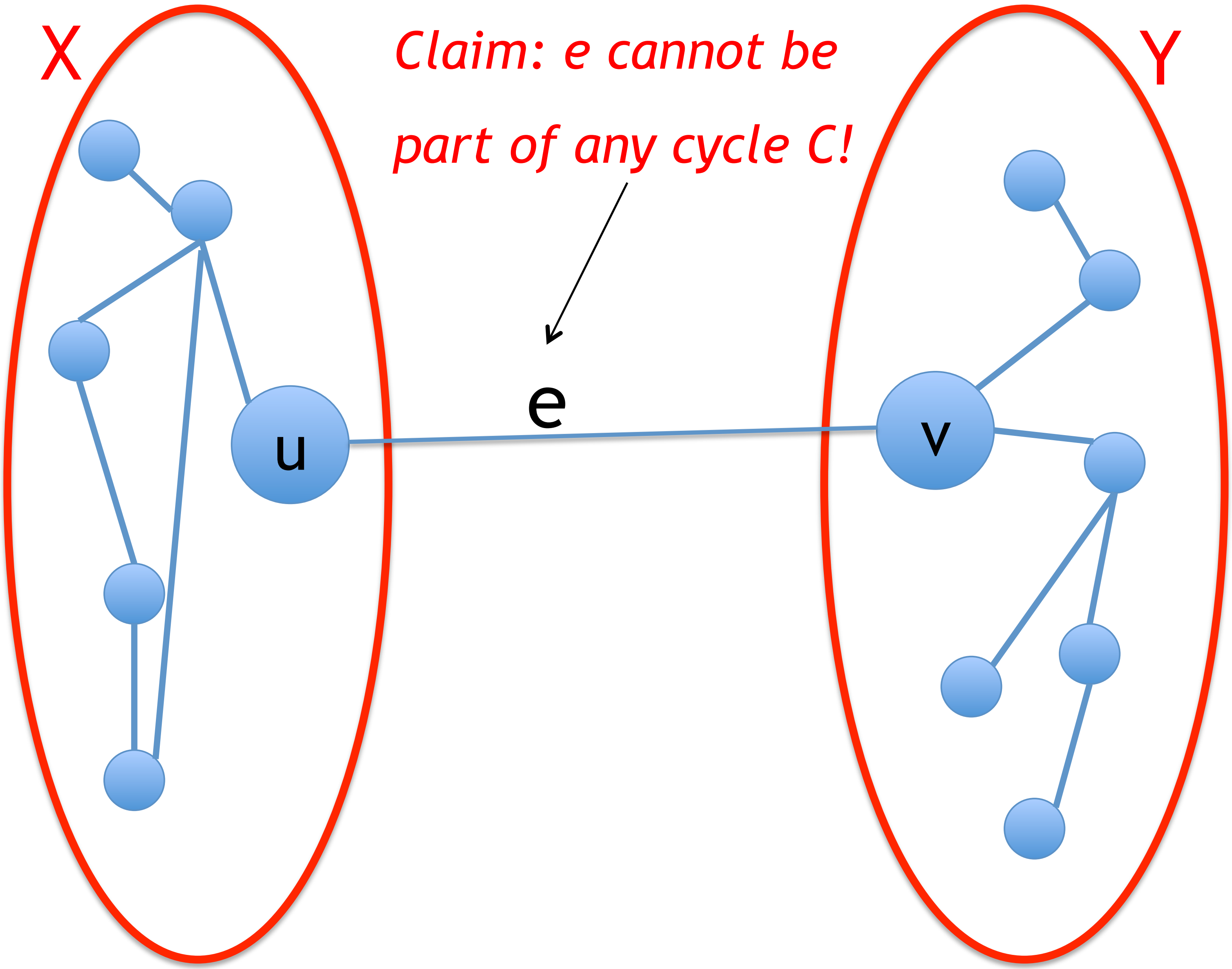


Lonely Cut Edge Corollary

Suppose there is a cut (X, Y) which has only one edge e crossing it

Claim: e cannot be part of any cycle C!

Lonely Cut Edge Corollary



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For Correctness We Need To Prove 2 Things

1. Outputs a Spanning Tree T_{krsk}
2. T_{krsk} is a minimum spanning tree

1: Kruskal Outputs a Spanning Tree (1)

Need to prove T_{krsk} is spanning AND is acyclic

Acyclic is by construction of the algorithm.

Why is T_{krsk} spanning (i.e., connected)?

Recall Empty Cut Lemma:

A graph is not connected iff $\exists$ cut (X, Y) with no crossing edges

$\Rightarrow$ *If all cuts have a crossing edge -> graph is connected!*

1: Kruskal Outputs a Spanning Tree (2)

Consider any cut (X, Y)

Since input G is connected, ∃ edges crossing (X, Y)

Let e be the min-weight edge*

Kruskal inspects all edges

*Consider the time when Kruskal inspects e**

Claim: No edge crossing (X, Y) is in T_{krsk}

And adding e to T_{krsk} cannot create a cycle => Why?*

(because of lonely cut edge corallary)

Therefore Kruskal will add e, so for each cut, there is an edge in T_{krsk} crossing it => T_{krsk} spans V.*

Plan for 2: T_{krsk} is a *Minimum* Spanning Tree

1. "MST Cut Property" of Edges
2. Argue that the MST Cut Property Implies T_{krsk} is an MST

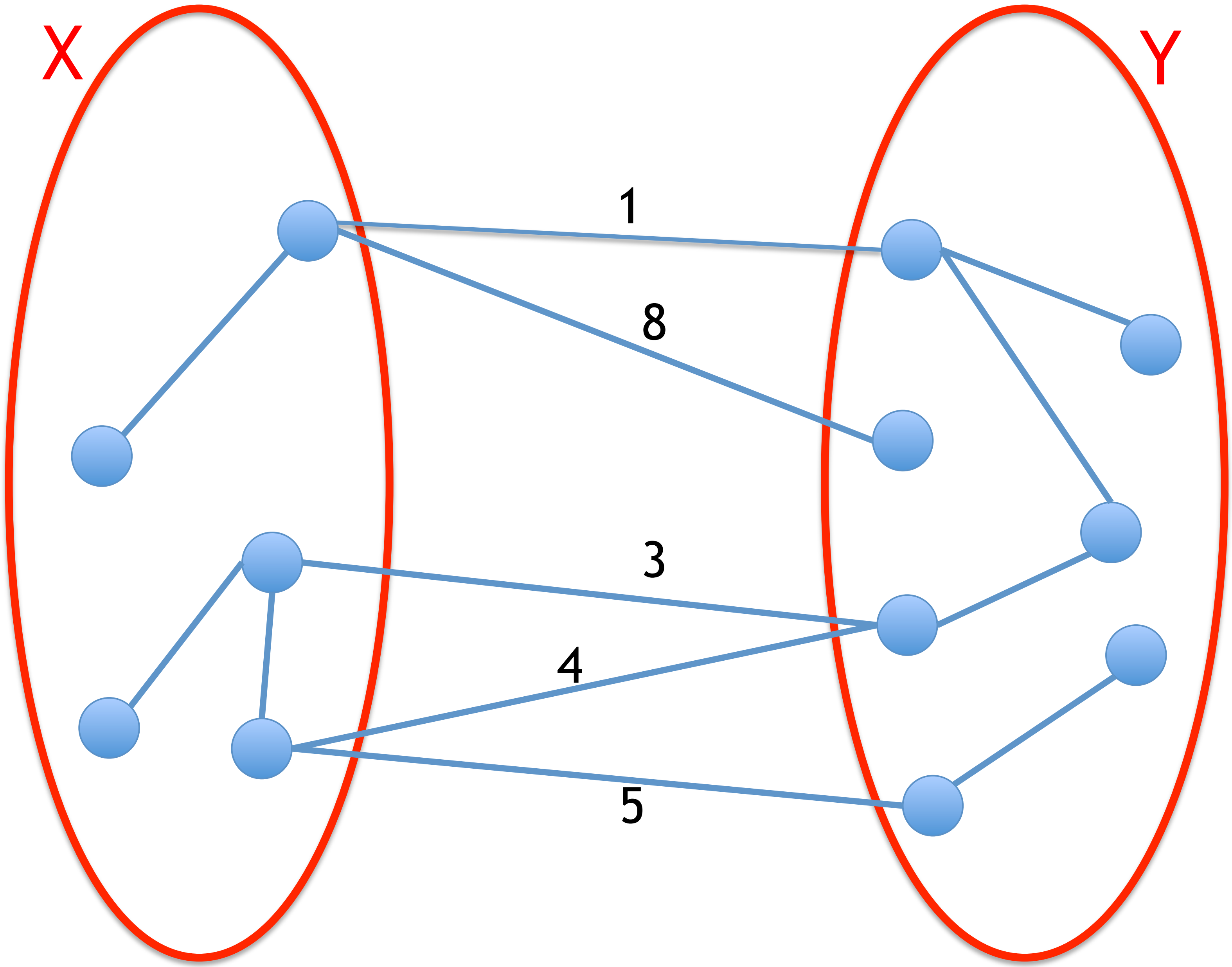
MST Cut Property of Edges

Consider any cut (X, Y) of G, and suppose e is the minimum edge crossing (X, Y)

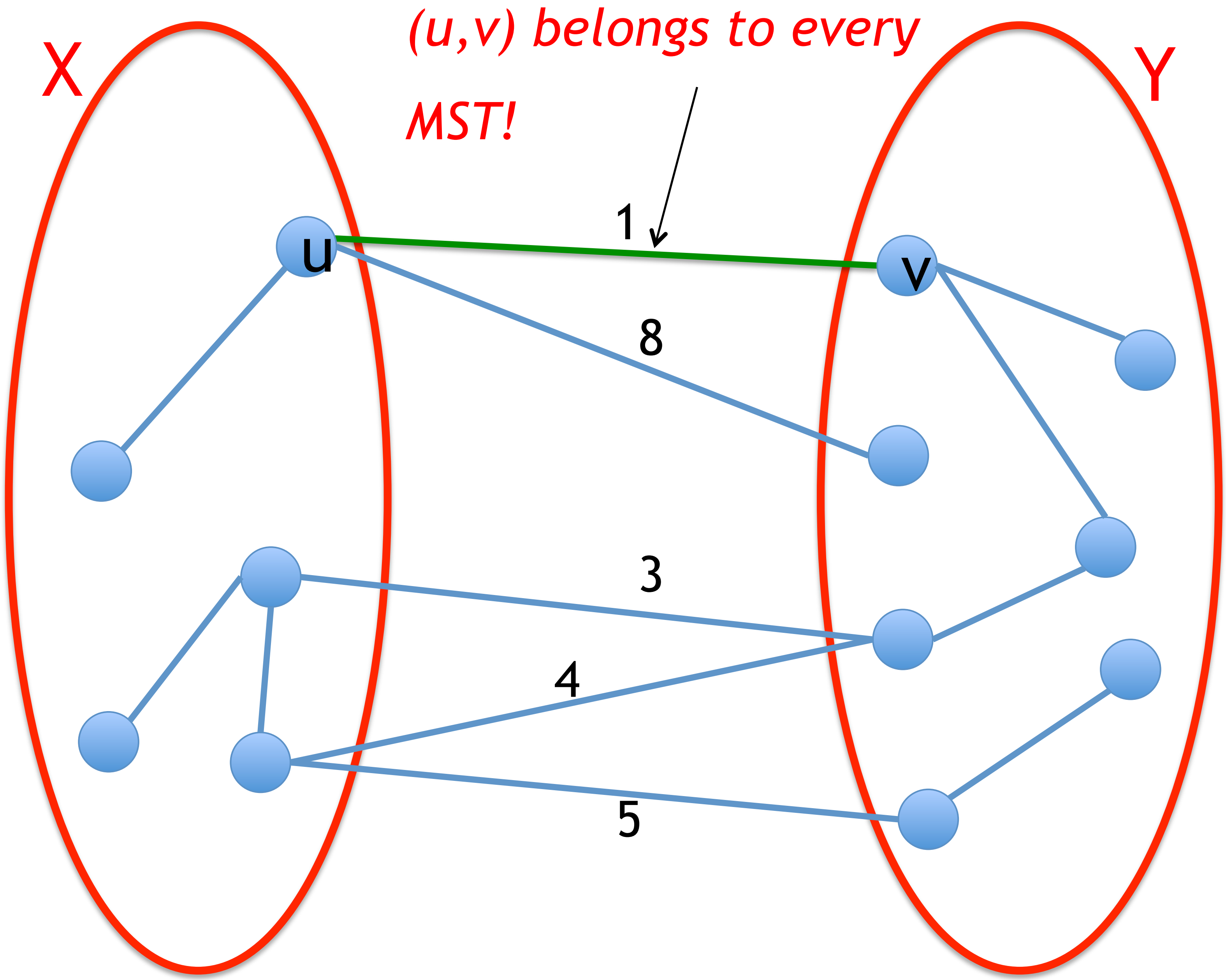
Claim: e belongs to every MST

(assuming edge weights are distinct)

MST Cut Property of Edges



MST Cut Property of Edges



Final Step: MST Cut Property => T_{krsk} is a MST

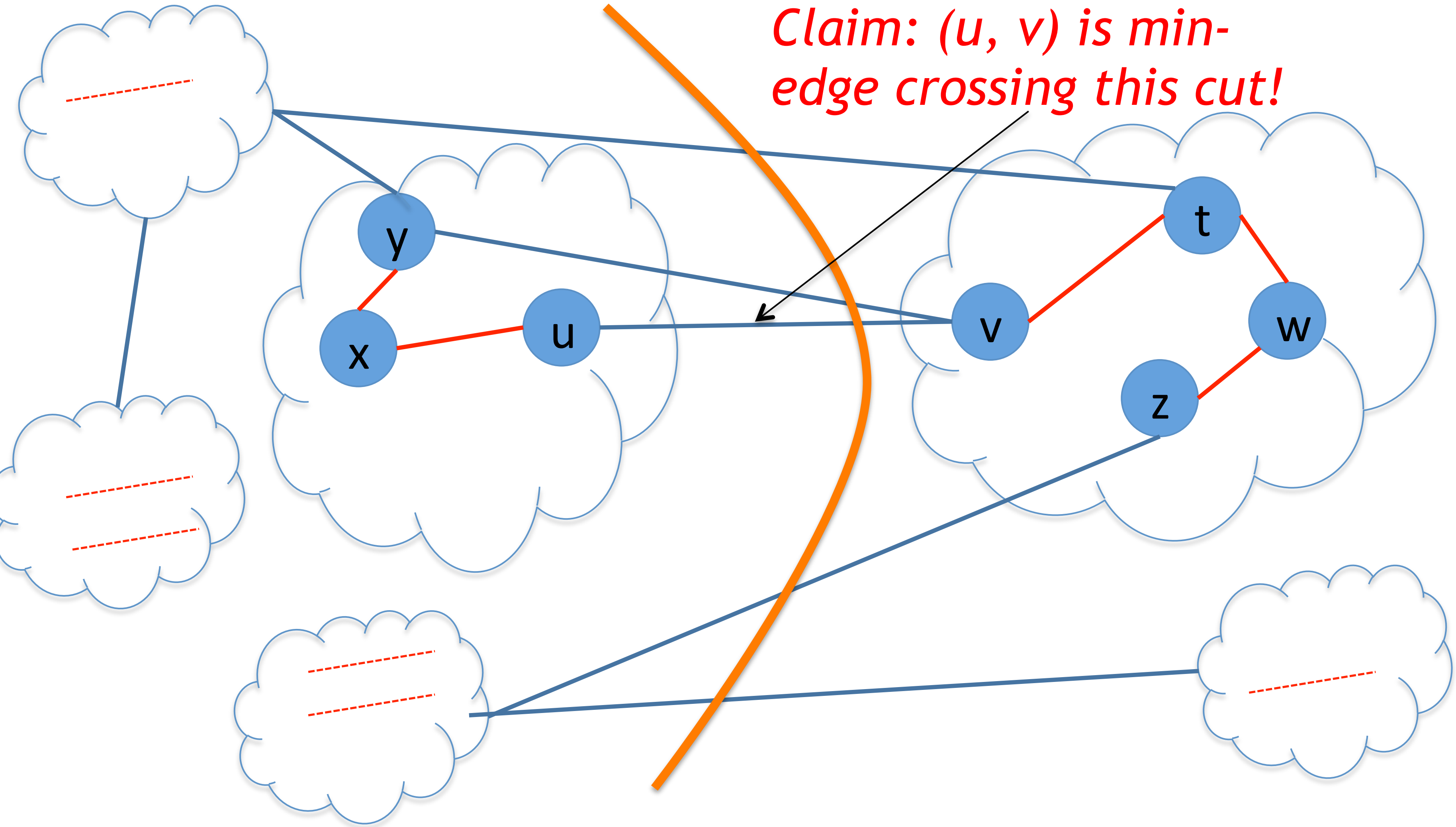
Argue that MST Cut Property Implies T_{krsk} is an MST

In particular we will argue that each edge (u, v) added to T_{krsk} is a min-edge in a cut (X, Y) in G.

MST Cut Property => T_{krsk} is a MST (1)

Let (u, v) be any edge added by Kruskal's Algorithm.

u and v are in different comp. (b/c Kruskal checks for cycles)



MST Cut Property => T_{krsk} is a MST (2)

(u, v) is the min edge crossing the cut b/c Kruskal looks at edges in increasing weights.

By the Cut Property => (u, v) is in every MST.

Therefore all edges added to T_{krsk} is in every MST

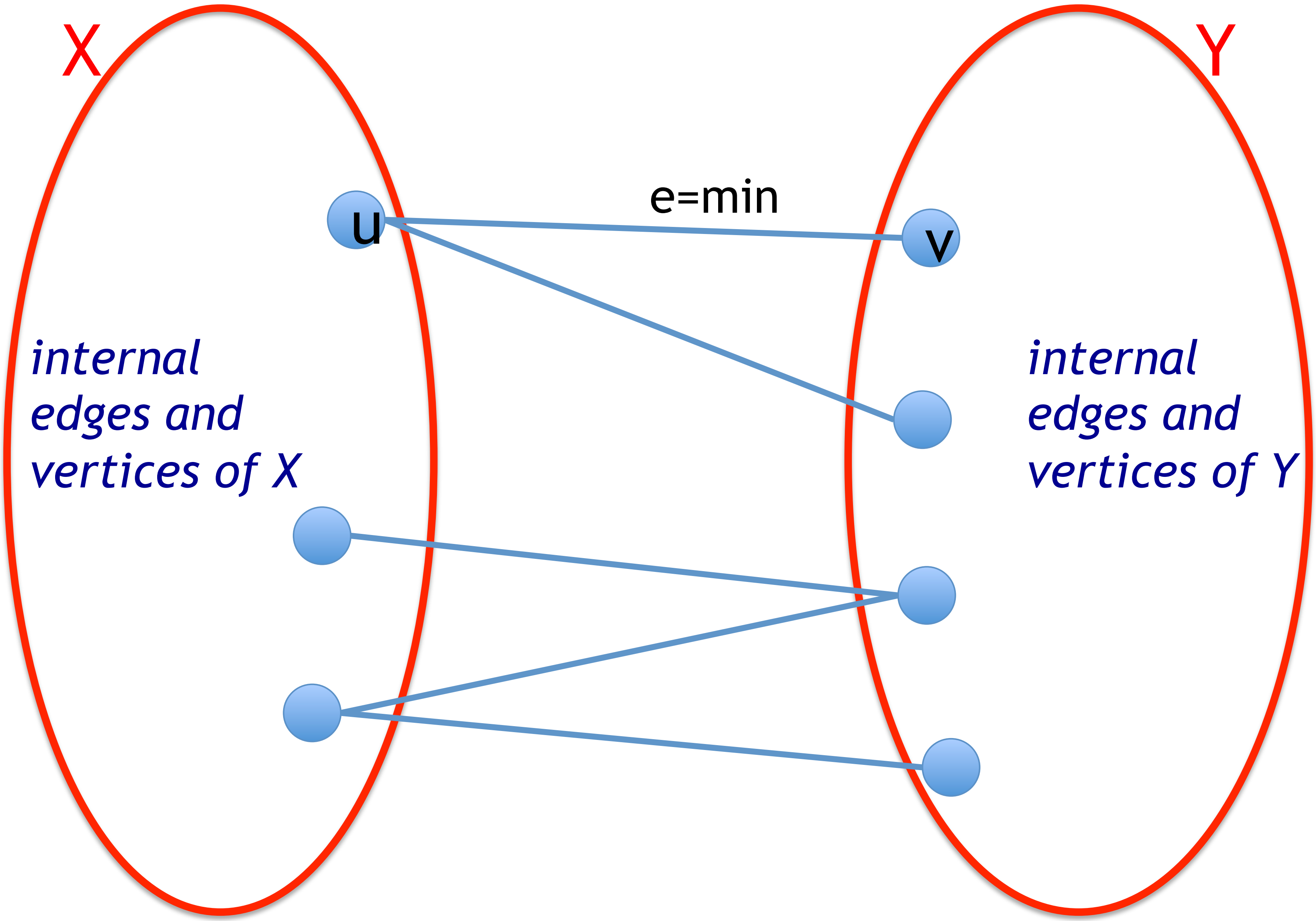
=> T_{krsk} is "the" MST.

Q.E.D.

Summary of Kruskal's Correctness

- First proved T_{krsk} is a spanning tree
  - Acyclicity was by definition
  - Connectedness followed from showing that there is an edge of T_{krsk} crossing any cut (X, Y) of G, which by the empty cut lemma implied that T_{krsk} was connected
- Second proved T_{krsk} was a *minimum* spanning tree by arguing that any edge (u, v) if T_{krsk} is a min-edge in some cut (X, Y)
- Note this version of Kruskal's Correctness Proof contains:
  - Exchange Argument: MST Cut Property
  - Greedy Stays Ahead Argument: Kruskal adds n-1 edges and all are justified.

Proof of MST Cut Property (Exchange Argument)



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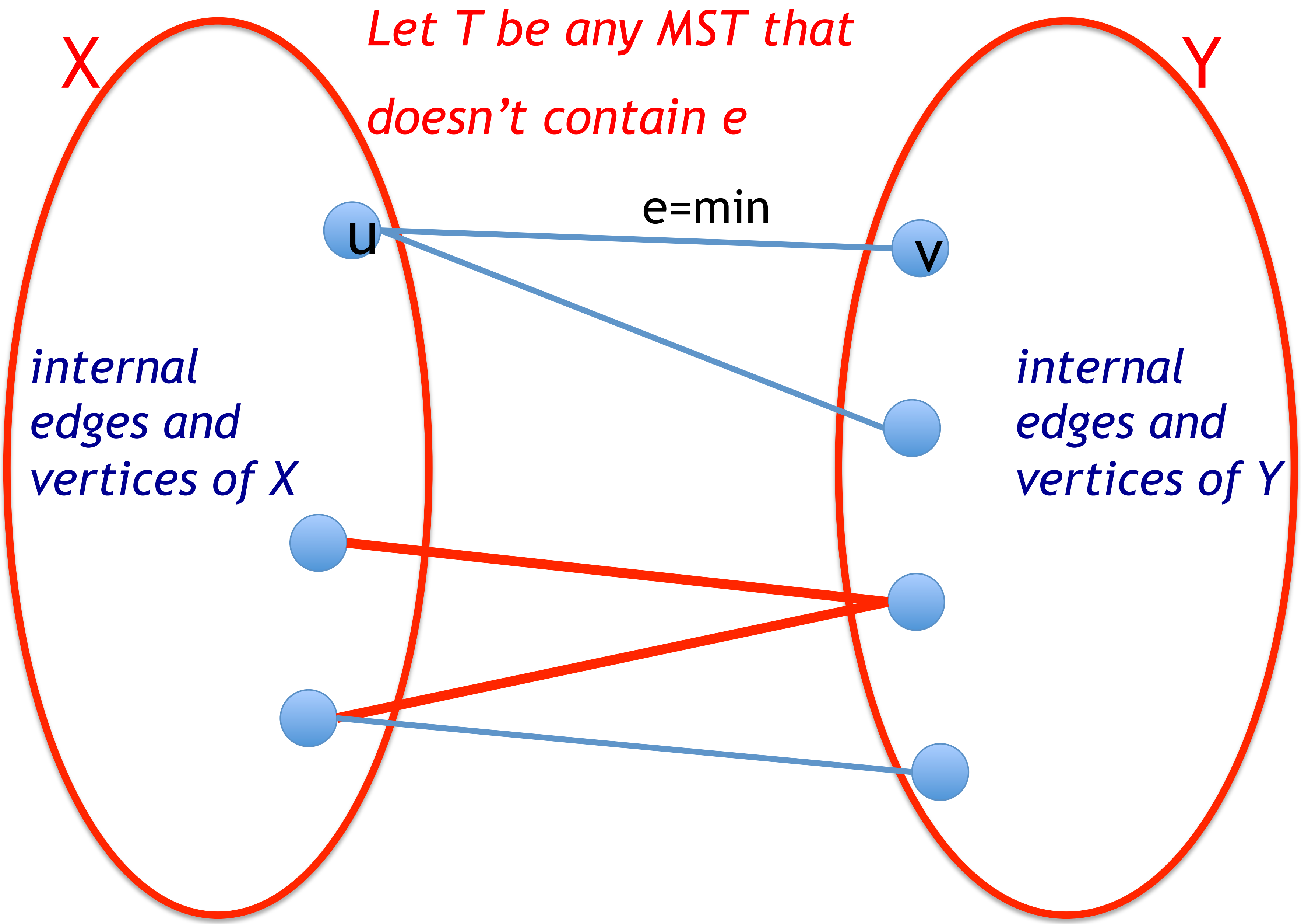

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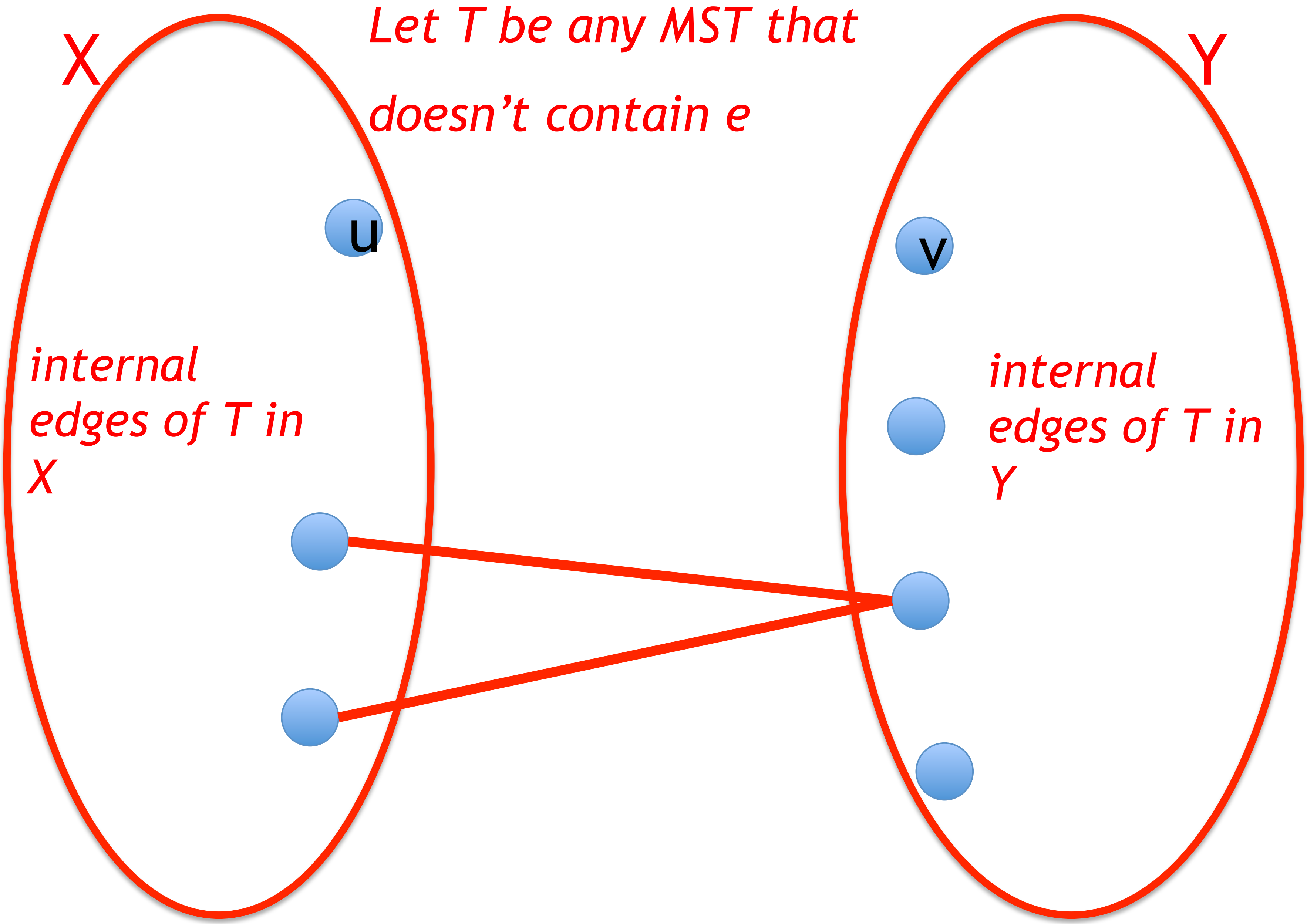
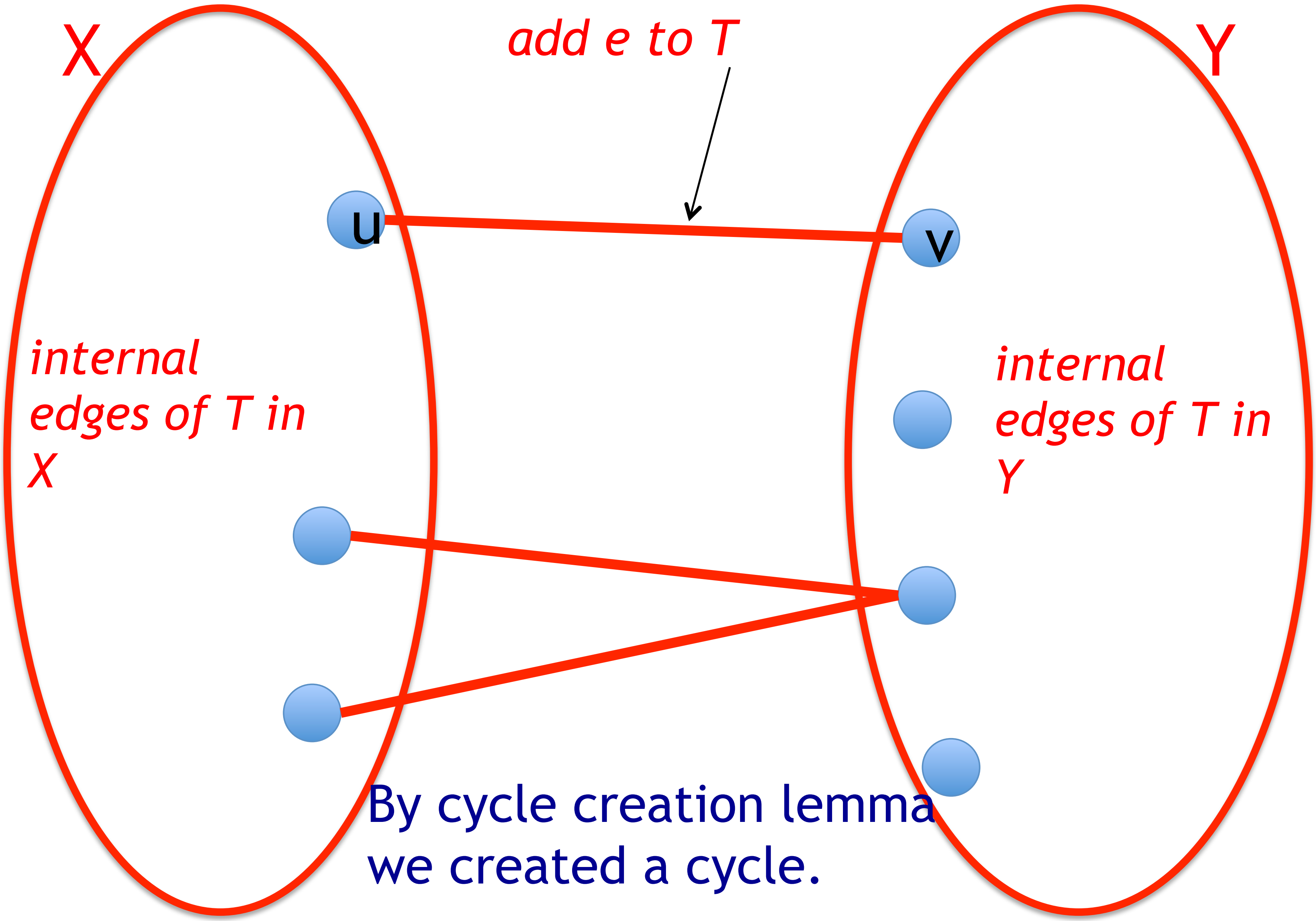
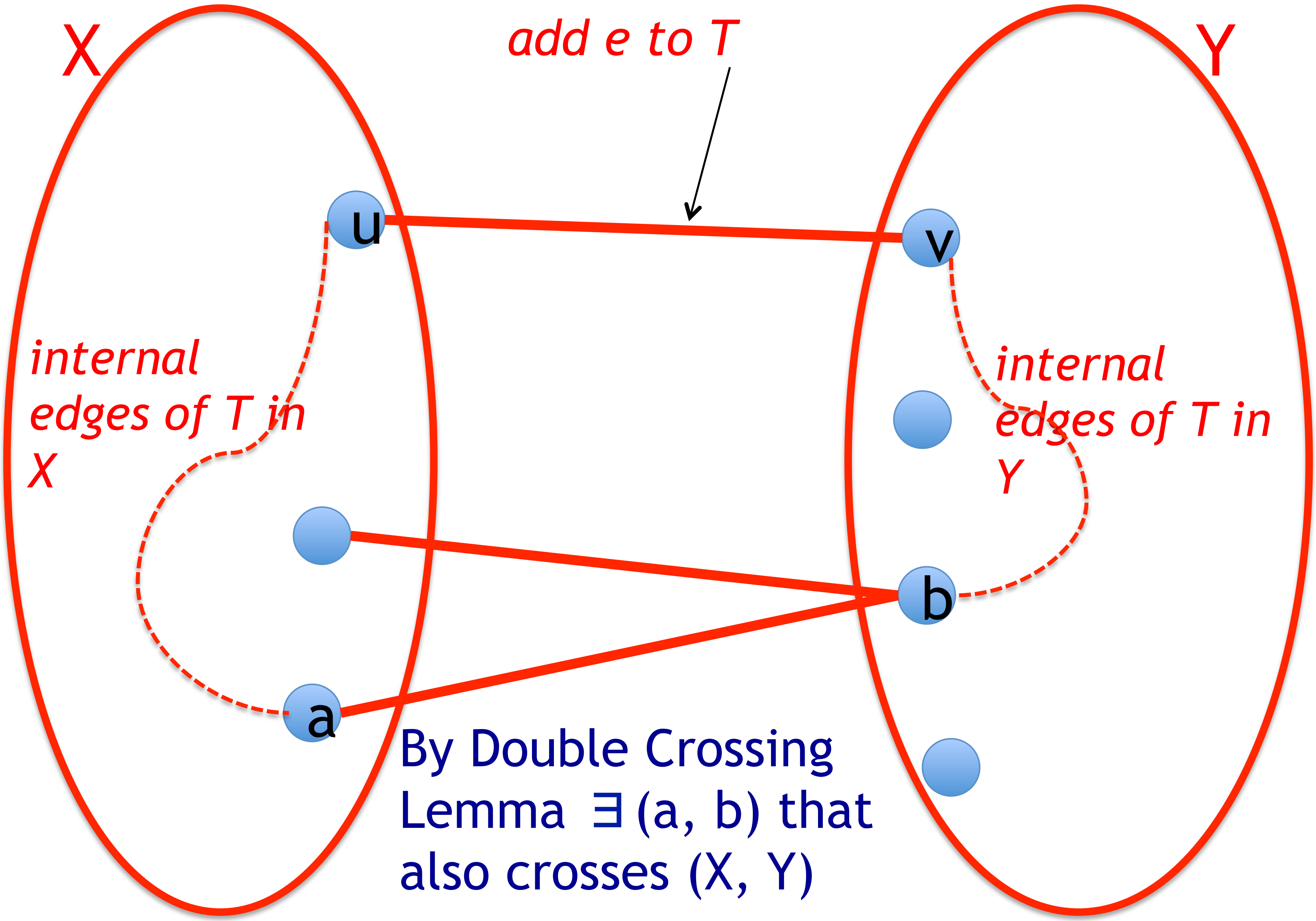


Proof of MST Cut Property (Exchange Argument)



By cycle creation lemma we created a cycle.

Proof of MST Cut Property (Exchange Argument)



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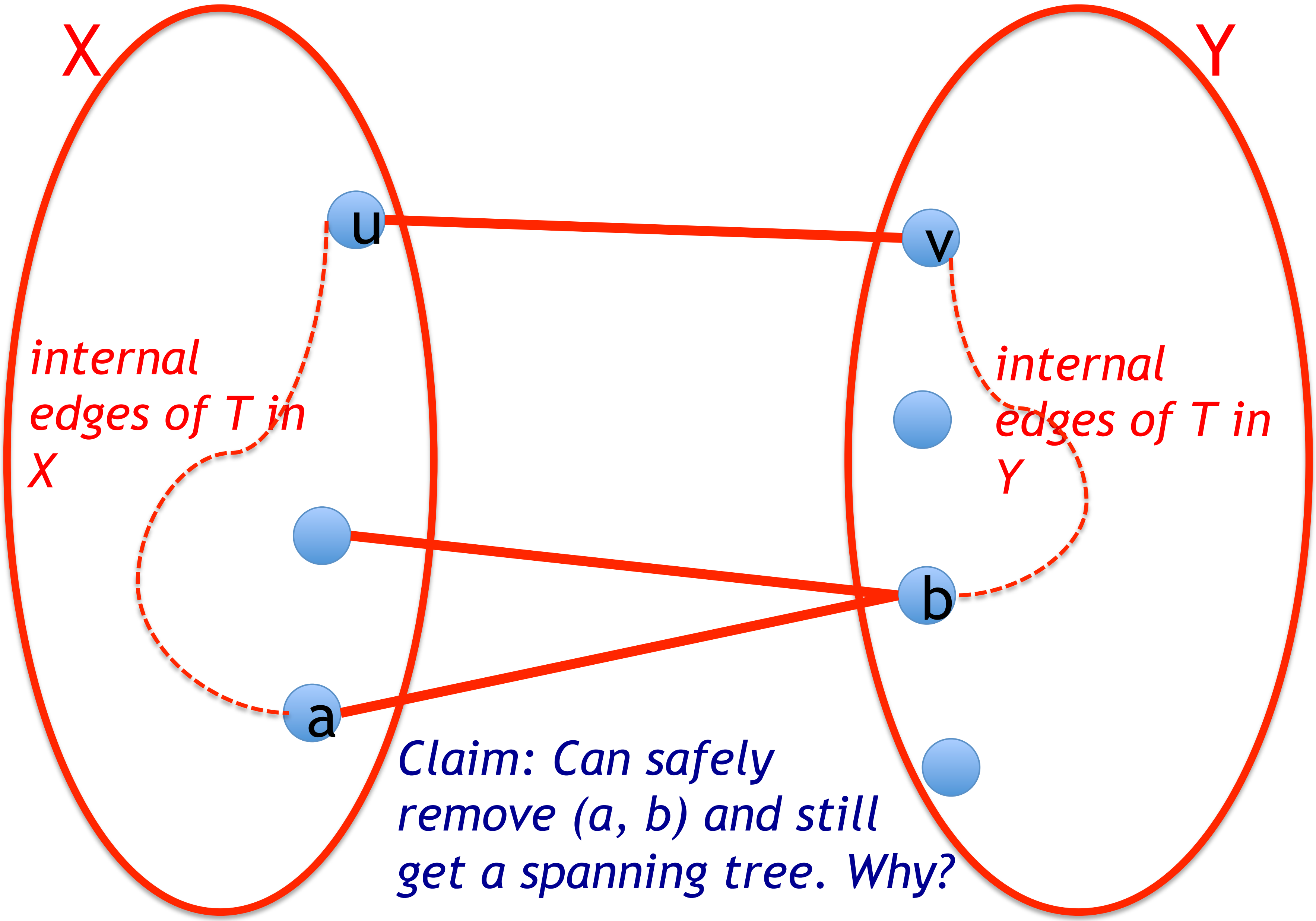

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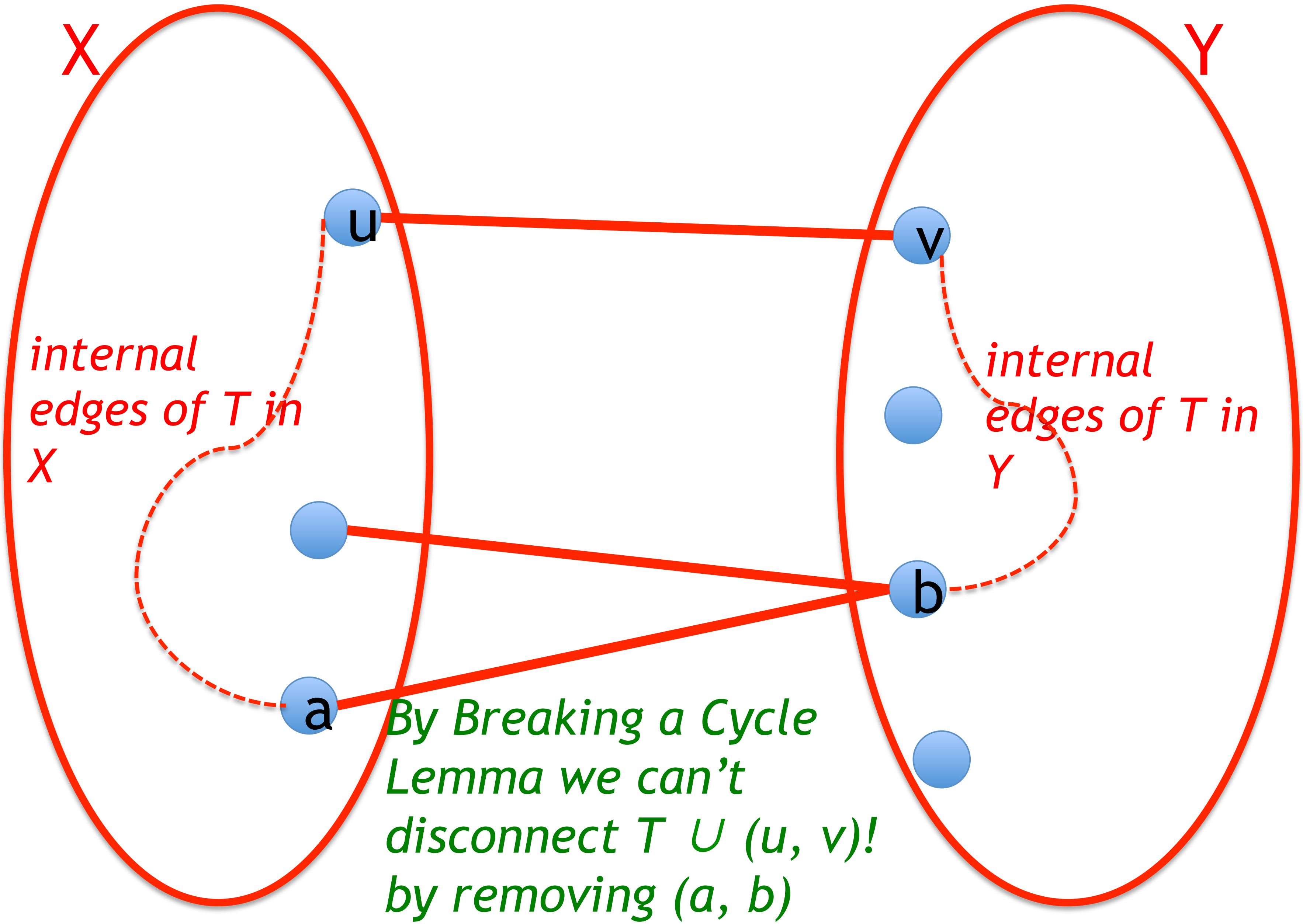

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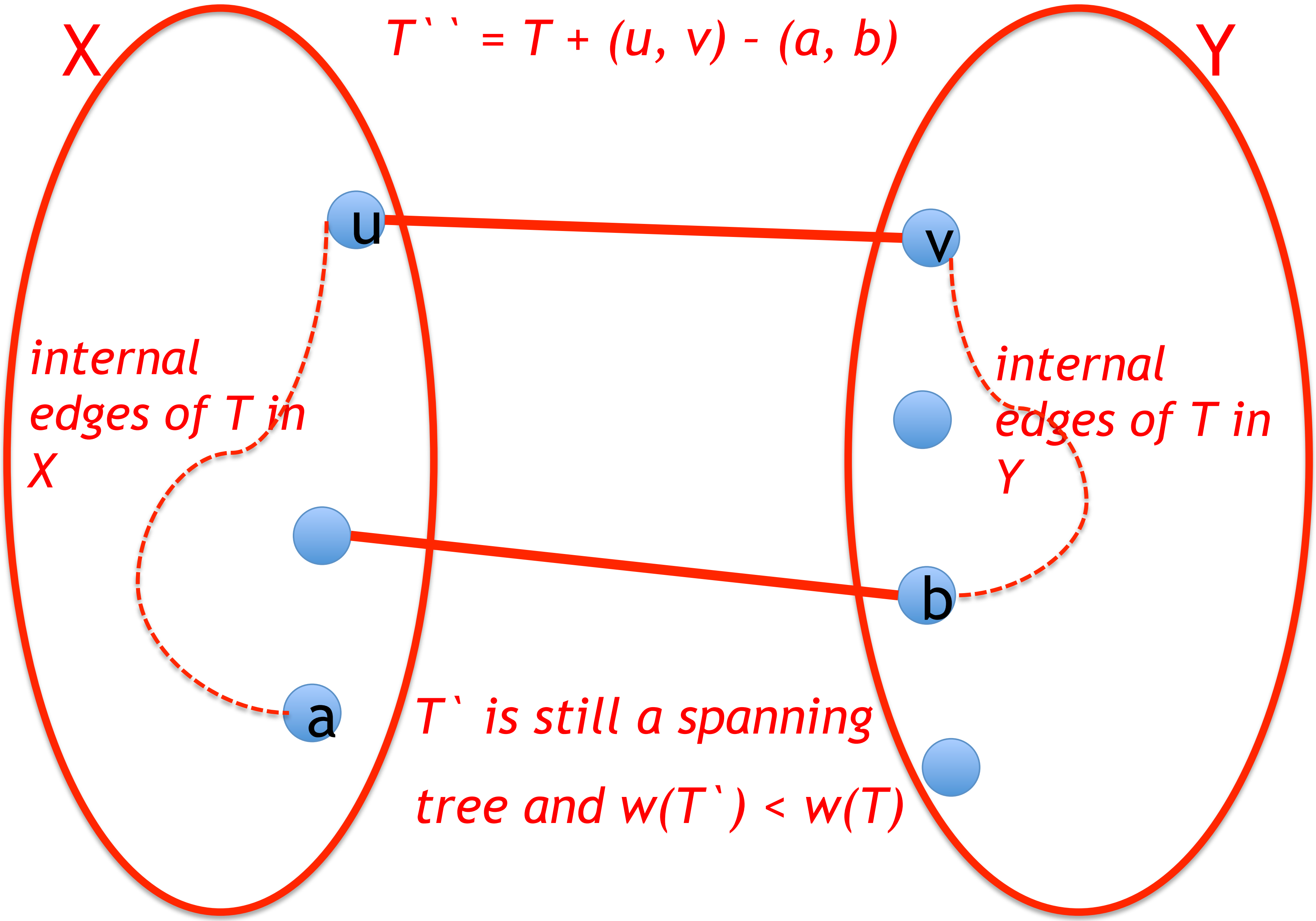

Proof of MST Cut Property (Exchange Argument)

- Started with (X, Y) cut, and e=(u, v): min edge crossing (X, Y)
- Took any minimum spanning tree T that did not include e.
- Added e to T => created a cycle C (by cycle-creation-lemma)
- By DCL, there is another edge e` that crosses (X, Y)
- Removed e` => T`
- By Breaking A Cycle Lemma: T` is still connected
- Therefore w(T`) < w(T) => T was not an MST.

=> Every MST has to include e!

Q.E.D.