Graphics Transformations

Translate, Scale, Rotate

Homogeneous Coordinates

Affine Transformation Matrices

Combining Transformations

Shape Model Class

Combining Operations

We need to represent basic transformations:

- **T**ranslate
- **R**otate
- Scale

We want the ability to combine a series of these transformations together.

- e.g. translate, then scale, then translate some more, then rotate.

We should also be able to build up a complicated transformation and save it for later reuse (e.g. “window” example).

Matrix Representation

- Goal: Represent each 2D transformation with a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- Multiply matrix by column vector ⇔ apply transformation to point

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrix Representation

- Transformations can be combined by multiplication
  - transformations are associative

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- We can multiply transformation matrices together

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} aei + bgi + afk + bhk & aej + bgj + ael + bgl \\ cei + dgi + cfk + dhk & cej + dgj + cfl + dhl \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

  - This single matrix can then be used to transform many points
  - Can be sent to a GPU to speed the process

Can a 2 x 2 Matrix Represent All 2D Transformations?

- 2D **Scale** around (0,0)?

$$\begin{aligned} x' &= x \times s_x \\ y' &= y \times s_y \end{aligned} \Leftrightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- 2D **Rotate** around (0,0)?

$$\begin{aligned} x' &= x\cos(\theta) - y\sin(\theta) \\ y' &= x\sin(\theta) + y\cos(\theta) \end{aligned} \Leftrightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- 2D Mirror about Y axis?

$$\begin{aligned} x' &= -x \\ y' &= y \end{aligned} \Leftrightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

How to represent **Translate**?

Can a 2 x 2 Matrix Represent All 2D Transformations? No.

- 2D Translation?

$$x' = x + t_x \quad y' = y + t_y \Leftrightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

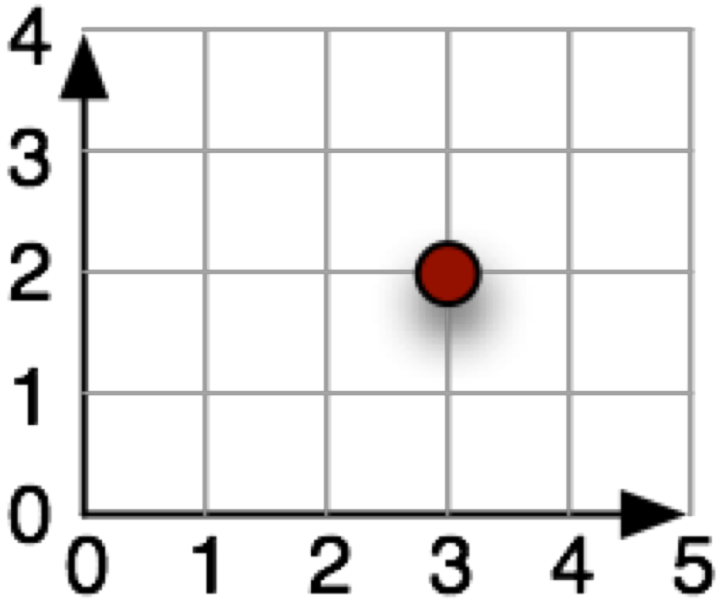
- Maybe this?

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & t_x/y \\ t_y/x & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Problem: Only works for a specific point. Can't create a general 2x2 matrix to transform a model

Homogeneous Coordinates

- Solution: add an extra component **w** to each coordinate
- **[x, y, w]**^{T} represents a point at location **[x/w, y/w]**^{T}
- many Homogeneous points for same Cartesian point



$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 7.5 \\ 5.0 \\ 2.5 \end{bmatrix} \quad \ldots$$

(divide by *w* to get Cartesian coords)

Homogeneous Coordinates

- represent coordinates in 2 dimensions with a 3-tuple (as a 3 × 1 column matrix)
- why? …

$$\begin{bmatrix} x \\ y \end{bmatrix} \Leftrightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

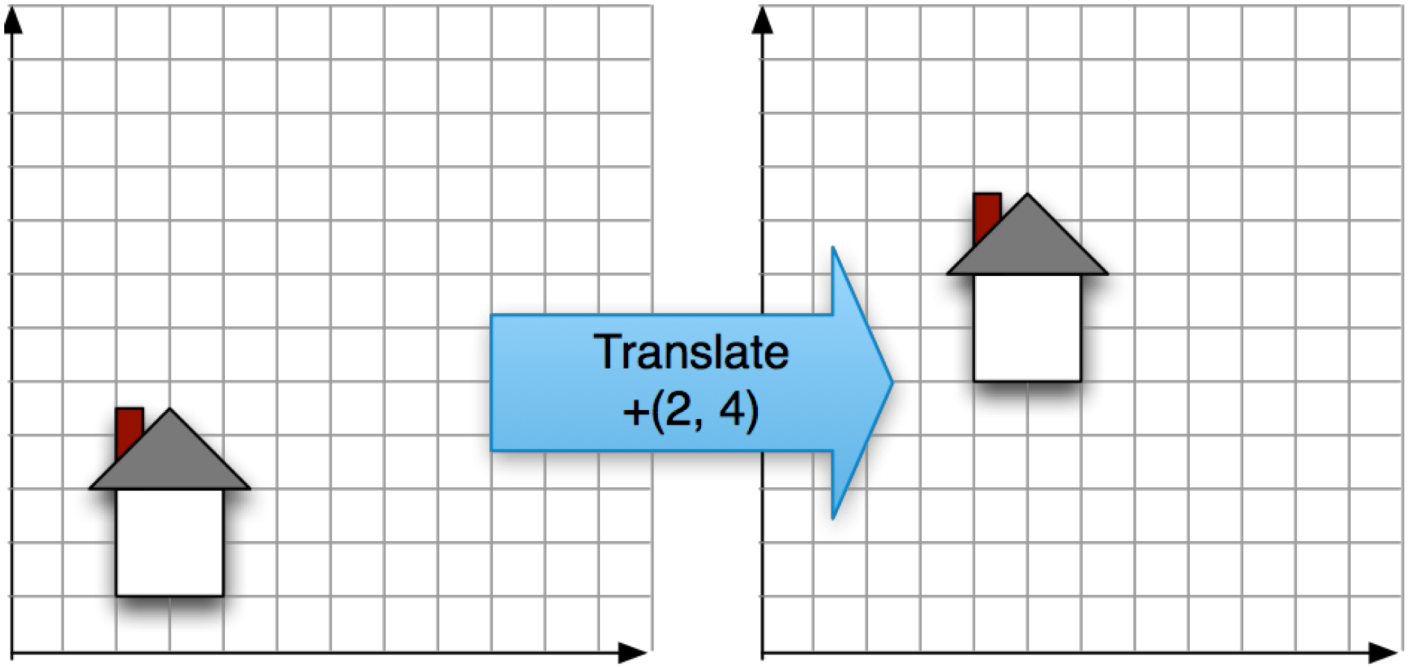
… need 3 columns in our transformation matrix

3 x 3 Translation Matrix

- Now we can represent 2D translation with a 3x3 matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$

3 x 3 Translation Matrix Example



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+2 \\ y+4 \\ 1 \end{bmatrix}$$

3 x 3 Scale Matrix

- 2 x 2 Scale Matrix

$$\begin{aligned} x' &= x \times s_x \\ y' &= y \times s_y \end{aligned} \Leftrightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- 3 x 3 Scale Matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \cdot s_x \\ y \cdot s_y \\ 1 \end{bmatrix}$$

Rotation and Scale

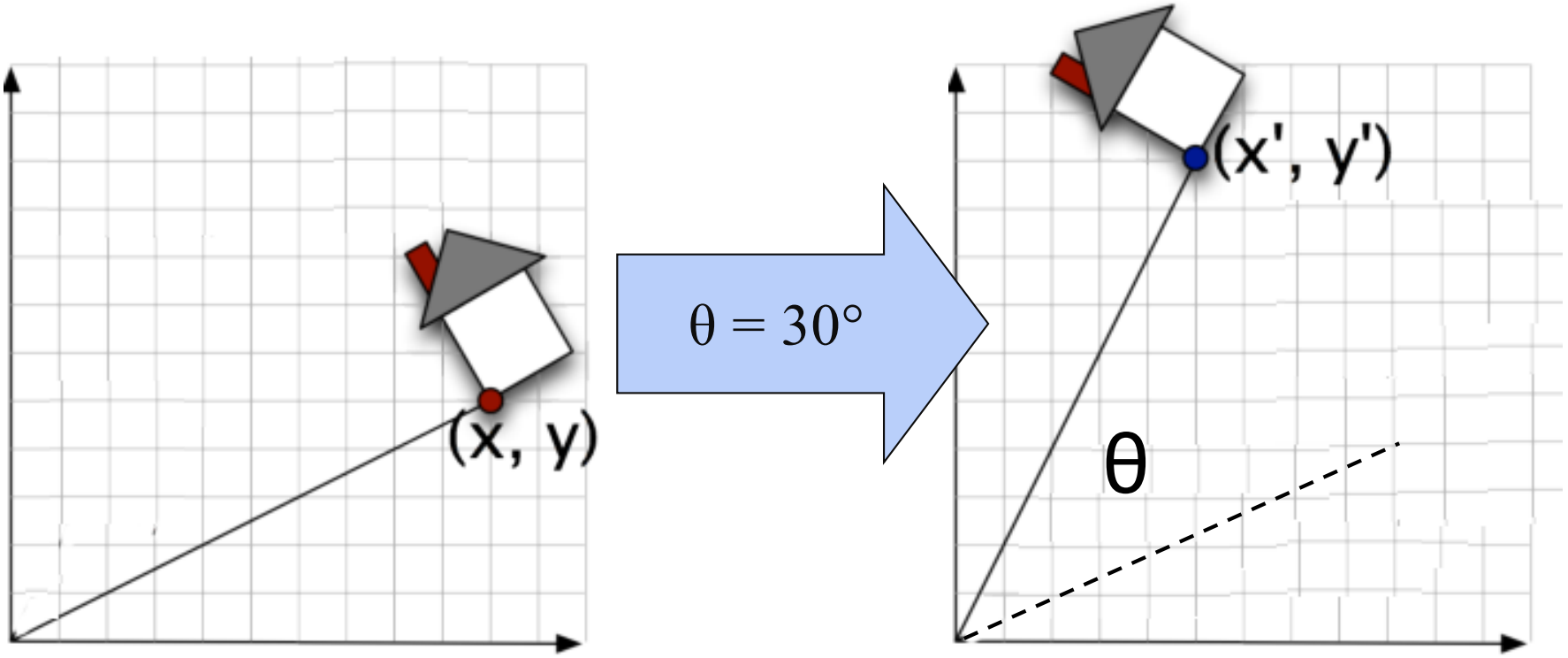
- 3 x 3 Scale Matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \cdot s_x \\ y \cdot s_y \\ 1 \end{bmatrix}$$

- 3 x 3 Rotation Matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \\ 1 \end{bmatrix}$$

3 x 3 Rotation Matrix Example



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} cos(30) & -sin(30) & 0 \\ sin(30) & cos(30) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} cos(30)x - sin(30)y \\ sin(30)x + cos(30)y \\ 1 \end{bmatrix}$$

Affine Transformation Matrix

- Now we have Translation matrix, Rotation matrix and Scale matrix

$$T(t_x, t_y) = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \qquad R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad S(s_x, s_y) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- This 3 x 3 matrix is an **Affine Transformation matrix**
  - **it can express any combination of translate, rotate, and scale**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & B & C \\ D & E & F \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} Ax + By + C \\ Dx + Ey + F \\ 1 \end{bmatrix}$$

Matrix Composition

- Transformations can be combined by matrix multiplication

$$p' = T(t_x, t_y) \cdot R(\theta) \cdot S(s_x, s_y) \cdot p$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

transformations are applied from “right-to-left”,
(call post multiplication)

$$p' = A \cdot B \cdot C \cdot p$$
$$p' = (A \cdot (B \cdot (C \cdot p)))$$

Matrix Multiplication

- Associative: A(BC) = (AB)C
- **Not Commutative:** AB ≠ BA
- Order of transformations matters (but not always …)

(blackboard …)

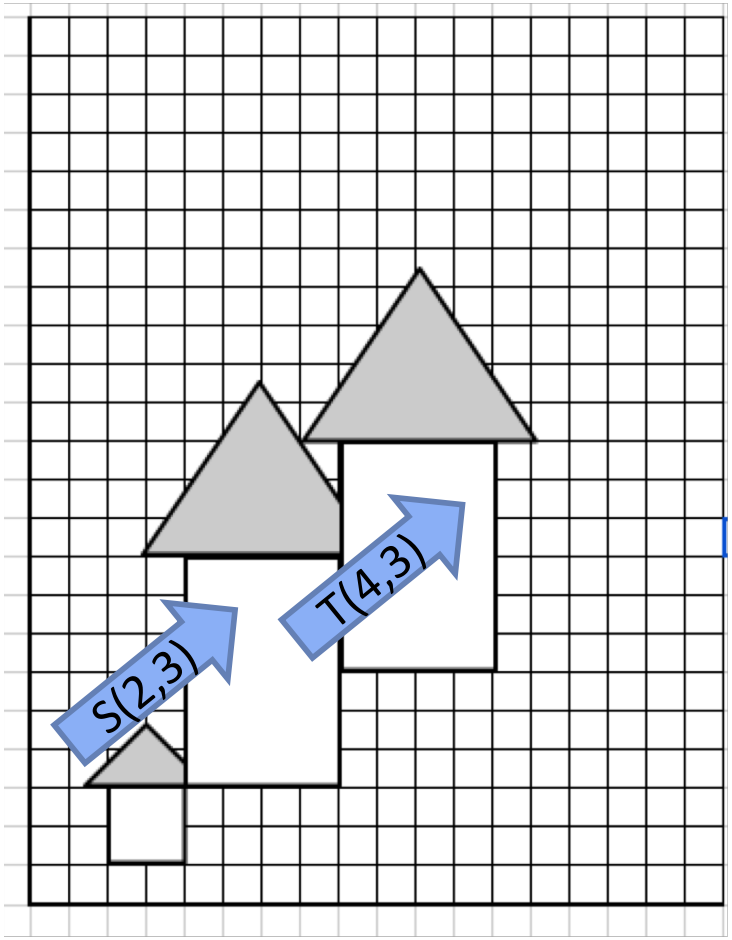
$$A = T(4,3) \cdot S(2,3) = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = S(2,3) \cdot T(4,3) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix Multiplication

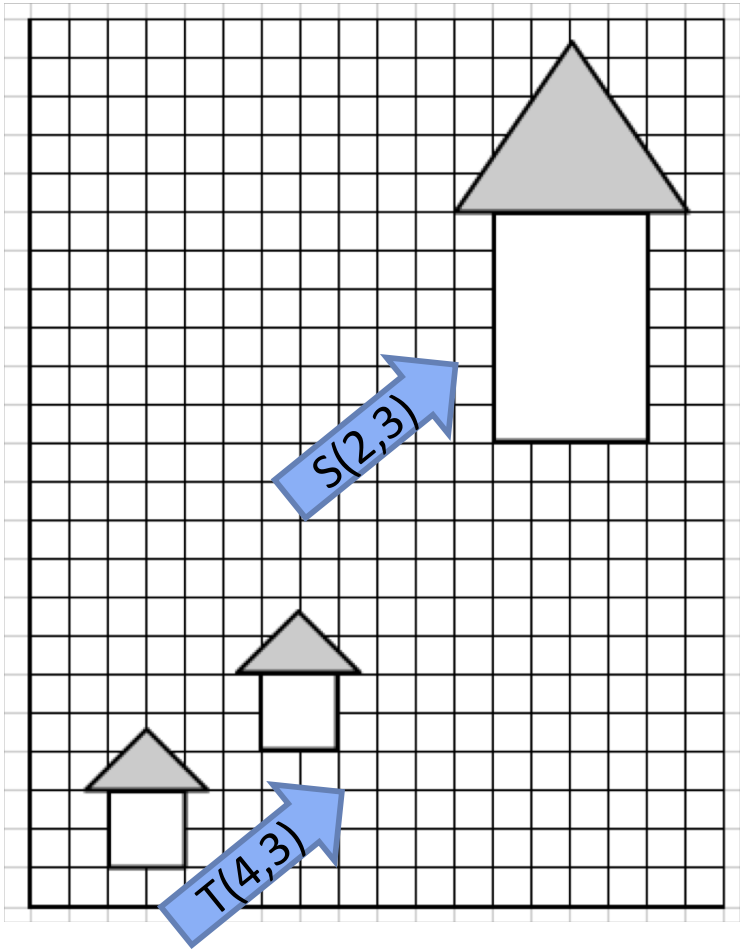
- $A = T(4,3) \cdot S(2,3)$

  $p' = A \cdot p = T(4,3) \cdot S(2,3) \cdot p$



$B = S(2,3) \cdot T(4,3)$

$p' = B \cdot p = S(2,3) \cdot T(4,3) \cdot p$



“Reading” an Affine Transformation Matrix

- You can get some idea of transformations in a matrix,
  … “reading” the translation, rotation, and scale is not always easy

$$C = \begin{bmatrix} \cos\theta & -\sin\theta & 4 \\ \sin\theta & \cos\theta & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 2\cos\theta & -3\sin\theta & 4 \\ 2\sin\theta & 3\cos\theta & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 2\cos\theta & -3\sin\theta & 4\cos\theta - 3\sin\theta \\ 2\sin\theta & 3\cos\theta & 4\sin\theta - 3\cos\theta \\ 0 & 0 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} \cos\theta & -\sin\theta & 4\cos\theta - 3\sin\theta - 4 \\ \sin\theta & \cos\theta & 4\sin\theta - 3\cos\theta - 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} 2\cos\theta & \sin\theta & 4 \\ \sin\theta & 3\cos\theta & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

NOTE: G is NOT a concatenation of only R(θ), T(4,3), and S(2,3) in any order.

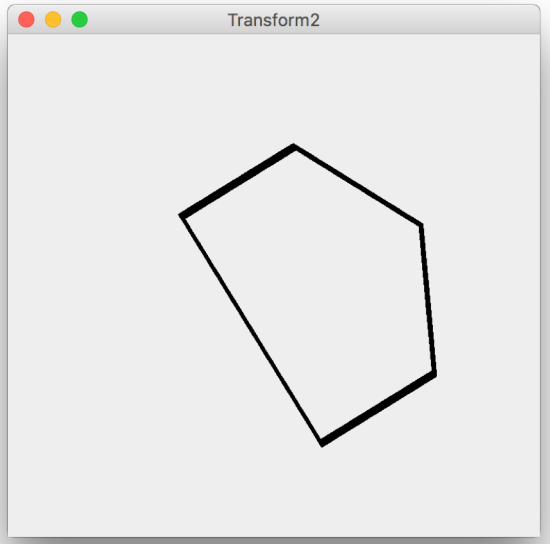
Transform2.java

```
// the house shape model (centred at top left corner)
private Polygon s = new Polygon(new int[] {-50, 50, 50, 0, -50},
                                new int[] {75, 75, -25, -75, -25}, 5);
...


// the shape will get transformed when rendered
g2.translate(M.x, M.y); // T
g2.rotate(Math.toRadians(45)); // R
g2.scale(2, 1); // S

g2.setStroke(new BasicStroke(3));
g2.drawPolygon(s.xpoints, s.ypoints, ...);
```

$$p' = T \cdot R \cdot S \cdot p$$

$$p' = (T \cdot (R \cdot (S \cdot p)))$$

$$p' = (T \cdot R \cdot S) \cdot p$$

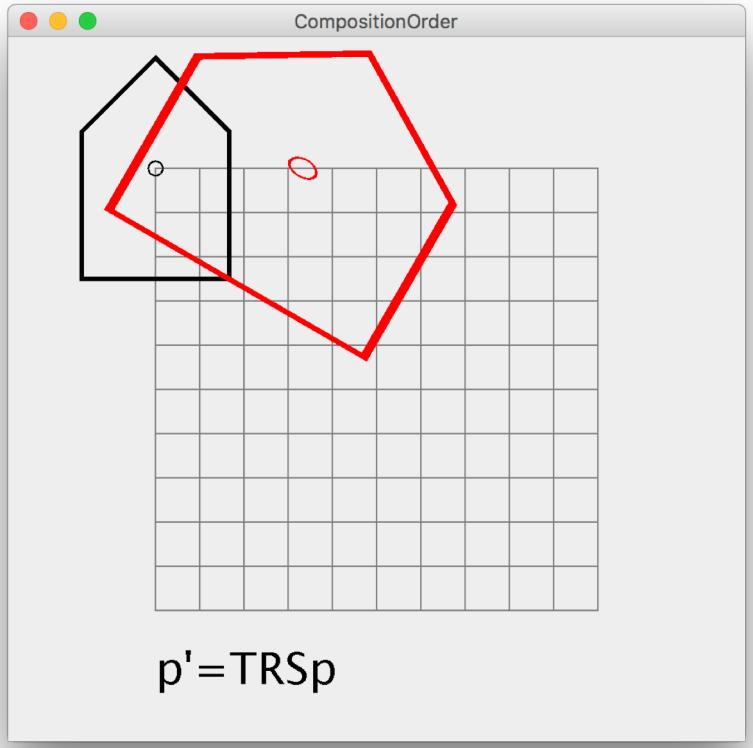


Transformation Composition Order

- Example:

$$p' = T \cdot R \cdot S \cdot p$$
$$p' = (T \cdot (R \cdot (S \cdot p)))$$
$$p' = (T \cdot R \cdot S) \cdot p$$

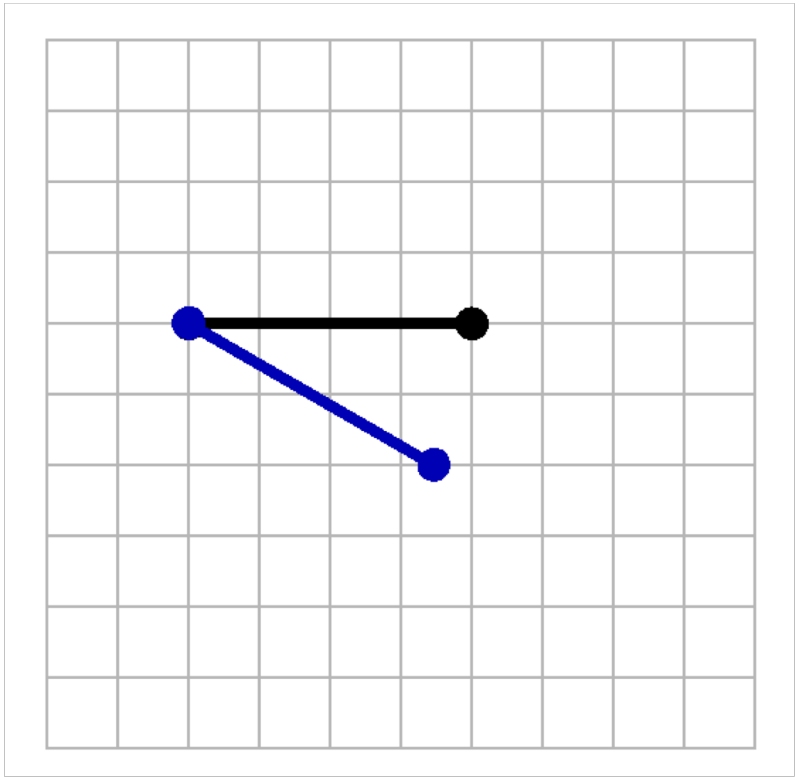
CompositionOrder.java



BarExercise.java

- Rotate the black bar about it's left end by 30°
  - (after rotating, it should be in the blue bar position)

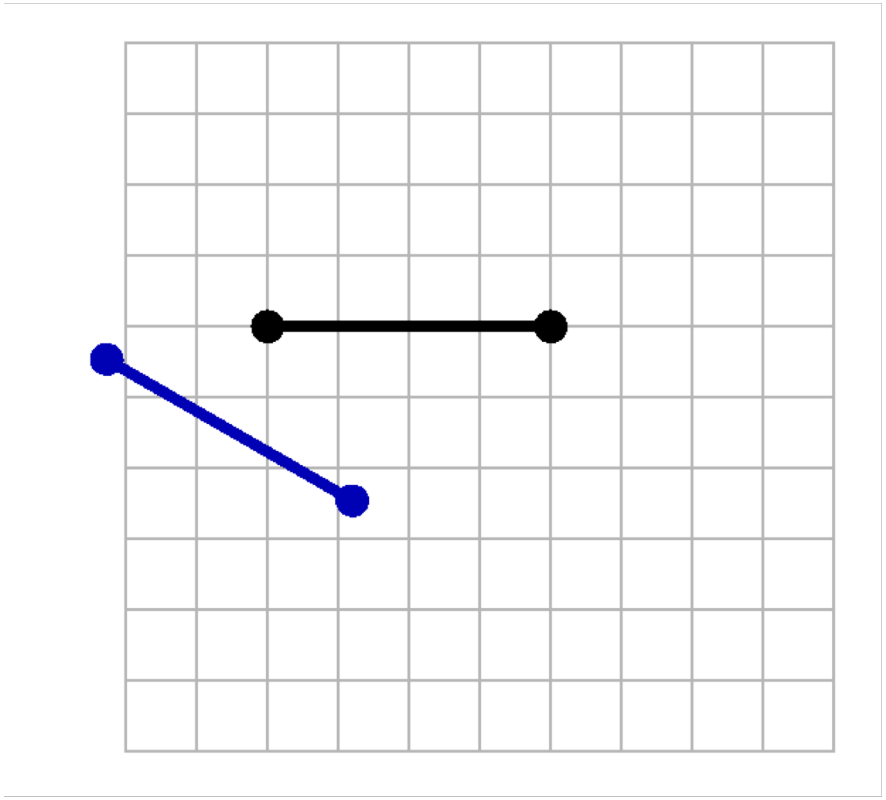
```
// left end (50, 100), right end (150, 100)
drawBar(g2, 50, 100, 150, 100);
```



Exercise: Attempt 1 (Wrong)

- Just rotate it?

```
g2.rotate(Math.toRadians(30));
drawBar(g2, 50, 100, 150, 100);
```

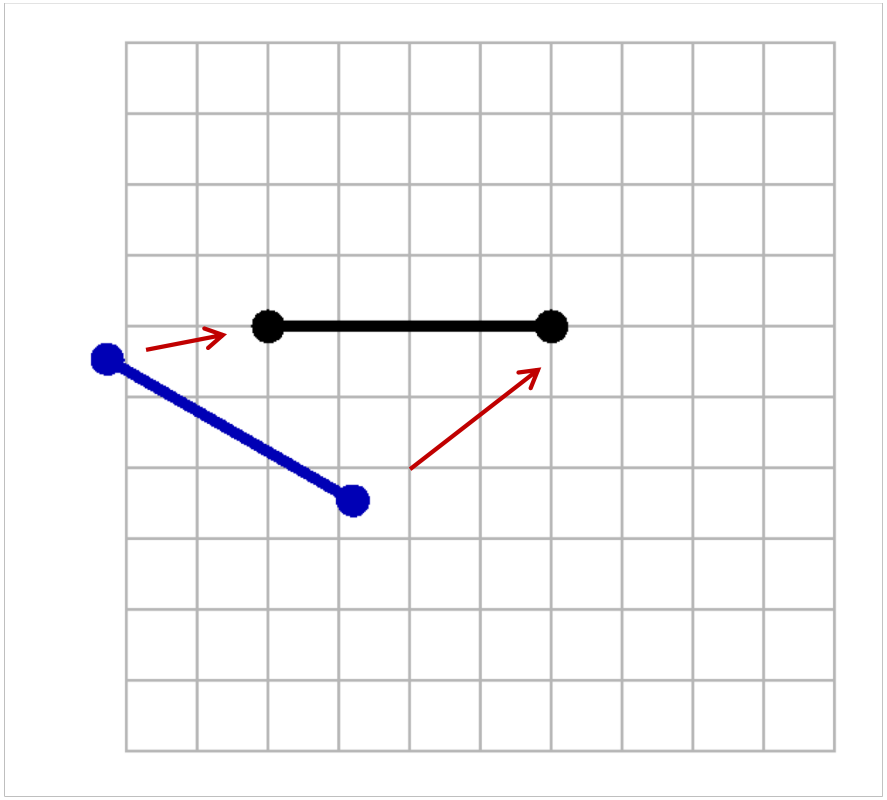


… why didn't this work?

Exercise: Fix Attempt 1 (Wrong)

- Rotate it but fix with translations …

```
// add g2.translate(x, y) HERE?
g2.rotate(Math.toRadians(30));
// or maybe add g2.translate(x, y) HERE?
drawBar(g2, 50, 100, 150, 100);
```

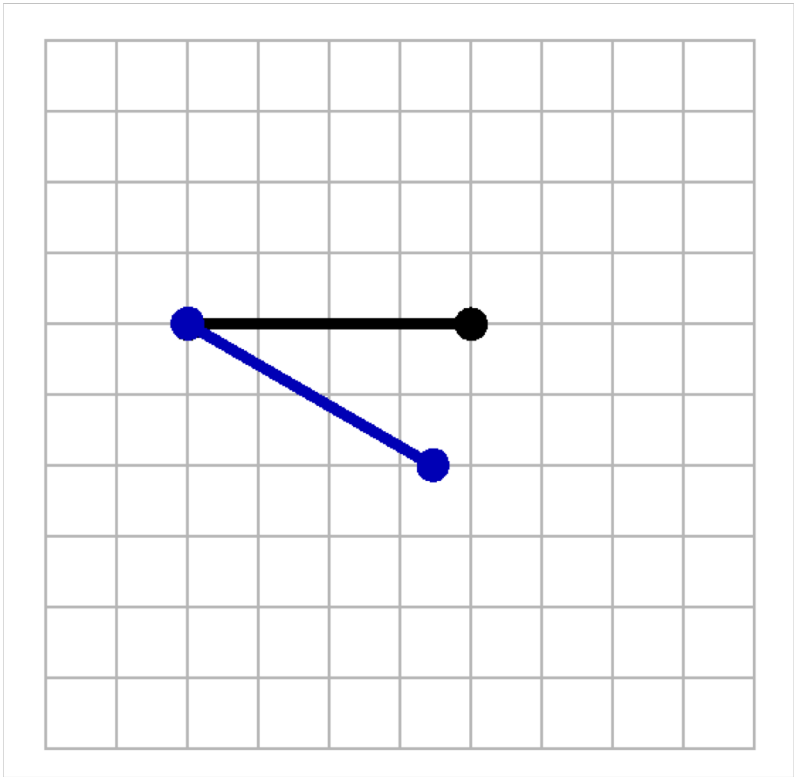


… try to correct with a translation?
but how much?

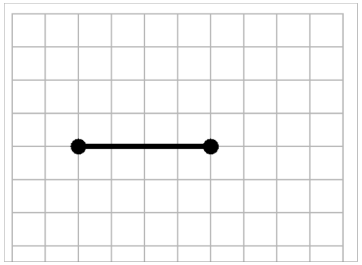
Exercise: Answer

- Scaling and rotation are matrices *are always about (0,0)*
- Need to translate to origin, rotate it, then translate back

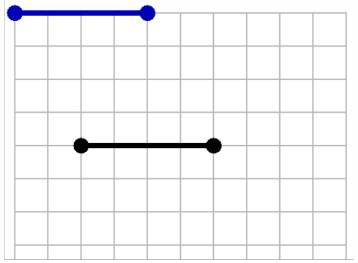
```
g2.translate(50, 100);
g2.rotate(Math.toRadians(30));
g2.translate(-50, -100);
drawBar(g2, 50, 100, 150, 100);
```



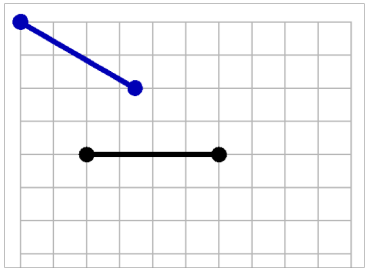
Illustration

1 

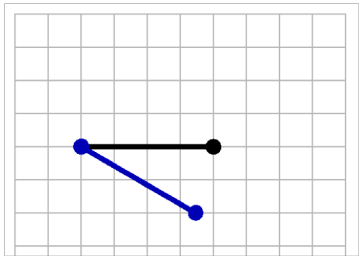
$$p' = I \cdot p$$

2 

$$p' = T(-50, -100) \cdot p$$

3 

$$p' = R(30) \cdot T(-50, -100) \cdot p$$

4 

$$p' = T(50,100) \cdot R(30) \cdot T(-50, -100) \cdot p$$

Shape.java

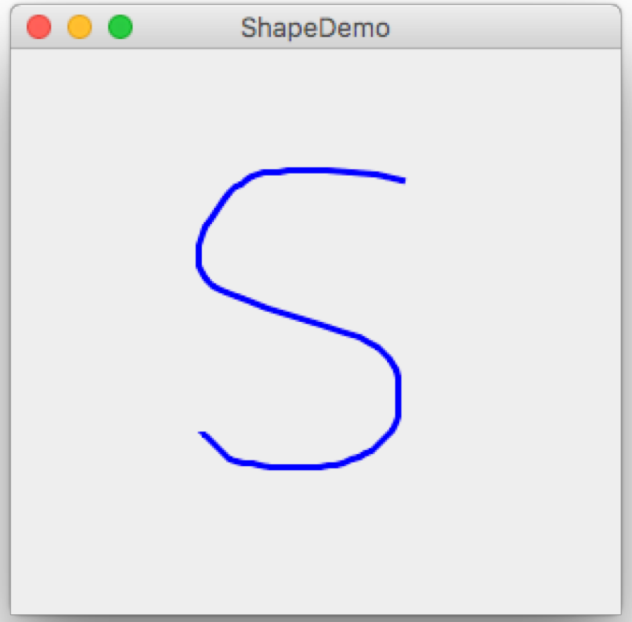
```
// simple shape model class
class Shape {

   // shape points
   ArrayList<Point2d> points;

   // shape type
   Boolean isClosed = true;
   Boolean isFilled = true;

   // drawing attributes
   Color colour = Color.BLACK;
   float strokeThickness = 3.0f;

   // shape's transform
   ...
```

Shape.java

```
   ...

   public void draw(Graphics2D g2) {

      // call drawing functions
      g2.setColor(colour);
      if (isFilled) {
         g2.fillPolygon( ... );
      } else {
         g2.setStroke(new BasicStroke(strokeThickness));
         if (isClosed)
            g2.drawPolygon( ... );
         else
            g2.drawPolyline( ... );
        }
   }

   // let shape handle it's own hit testing
   public boolean hittest(double x, double y) { ... }
}
```

ShapeDemo.java

- using Shape.java
- setting attributes and drawing
- setting transform for shape



Shape.java (with scale transform)

```
// save the current g2 transform matrix
AffineTransform M = g2.getTransform();

// multiply in this shape's transform
g2.scale(scale, scale);

// call drawing functions
g2.setColor(colour);
if (isFilled) {
    g2.fillPolygon(xpoints, ypoints, npoints);
} else {
   // adjust stroke size using scale
   g2.setStroke(new BasicStroke(strokeThickness / scale));
   if (isClosed)
        g2.drawPolygon(xpoints, ypoints, npoints);
    else
        g2.drawPolyline(xpoints, ypoints, npoints);
}

// reset the transform to what it was before we drew the shape
g2.setTransform(M);
```

Useful Graphics2D methods

- Concatenates the current Graphics2D Transform with a rotation transform.

```
void rotate(double theta)
```

- Translates origin to (x,y), rotates, and translates origin (-x, -y).

```
void rotate(double theta, double x, double y)
```

- Concatenates the current Graphics2D Transform with a scaling transformation. Subsequent rendering is resized according to the specified scaling factors relative to the previous scaling

```
void scale(double sx, double sy)
```

- Concatenates the current Graphics2D Transform with a translation transform.

```
void translate(double tx, double ty)
```

- Returns/sets a copy of the current Transform in the Graphics2D context.

```
AffineTransform getTransform(),
void setTransform(AffineTransform Tx)
```

Java2D AffineTransform Class

- AffineTransform handles all matrix manipulations
  - A bit more control than Graphics2D
- Static Methods

```
static AffineTransform getRotateInstance(double theta)
static AffineTransform getRotateInstance(double theta,
                       double anchorx, double anchory)
static AffineTransform getScaleInstance(
                       double sx, double sy)
static AffineTransform getTranslateInstance(
                       double tx, double ty)
```

Java2D AffineTransform Class

- Concatenation methods

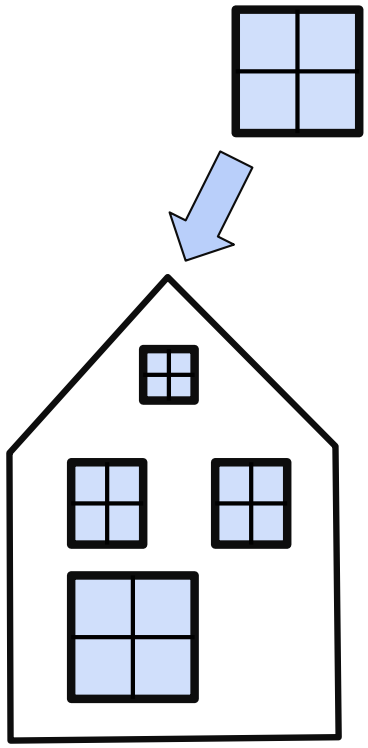
```
void rotate(double theta)
void rotate(double theta, double anchorx,
            double anchory)
void scale(double sx, double sy)
void translate(double tx, double ty)
void concatenate(AffineTransform Tx)
```

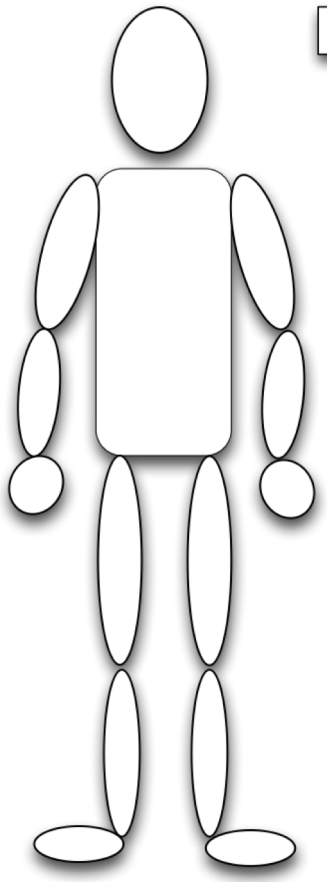
- Other Methods

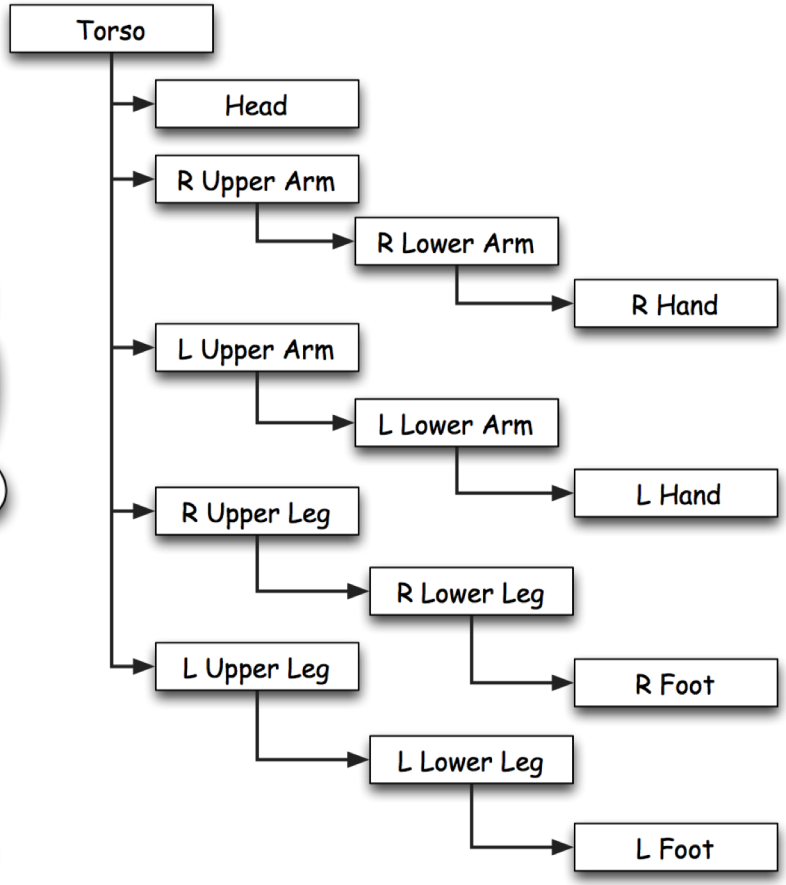
```
AffineTransform createInverse()*
void transform(Point2D[] ptSrc, int srcOff,
         Point2D[] ptDst, int dstOff, int numPts)
```

Scene Graphs

- Each part has a transform matrix
- Each part draws its children relative to itself

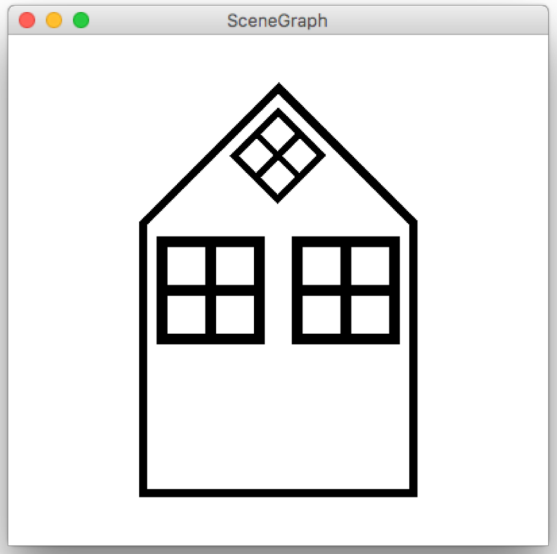


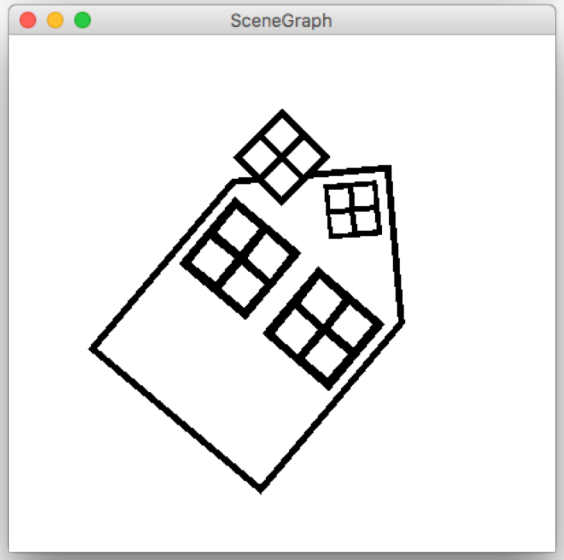




SceneGraph.java

```
// draw the house in centre of screen
g2.translate(getWidth() / 2, getHeight()/ 2);
g2.rotate(Math.toRadians(rotateBy));
g2.scale(scaleBy, scaleBy);

g2.drawPolygon(houseShape.xpoints, ... );

// save transform for later
AffineTransform save = g2.getTransform();

// in "house coordinates"
g2.translate(-25, 0); // centred 25 px
g2.scale(0.4, 0.4); // 40% house width
drawWindow(g2);

// translate to right 50 px
// (relative to last window)
g2.translate(50 / 0.4, 0);
drawWindow(g2);

// return to "House" coordinates
g2.setTransform(save);
drawWindow(g2, 0, -50, 45, 0.25);
```





SceneGraph.java

```
// draws 100 x 100 window shape centred at 0,0
void drawWindow(Graphics2D g2, double x, double y,
        double theta, double s) {

    // save the current g2 transform matrix
    AffineTransform save = g2.getTransform();

    // do the model to world transformation
    g2.translate(x, y);                  // T
    g2.rotate(Math.toRadians(theta)); // R
    g2.scale(s, s);                      // S

    // draws 100 x 100 window centred at 0,0
    drawWindow(g2);
    // reset transform to what it was before we drew the shape
    g2.setTransform(save);
}
```

Benefits of Using Transformations

- Allow reuse of objects in scenes
  - Can create multiple instances by translating model of object and re-rendering
- Allows specification of object in its own coordinate system
  - Don’t need to define object in terms of its screen location or orientation
- Simplifies remapping of models after a change
  - E.g. animation