Undecidability

Thursday, April 4th

Outline For Today

1. Beyond Intractability: Undecidability
2. What to do with NP-complete Problems
3. Important Things to Remember In the Final Exam
4. Algorithmic Topics Beyond CS 341

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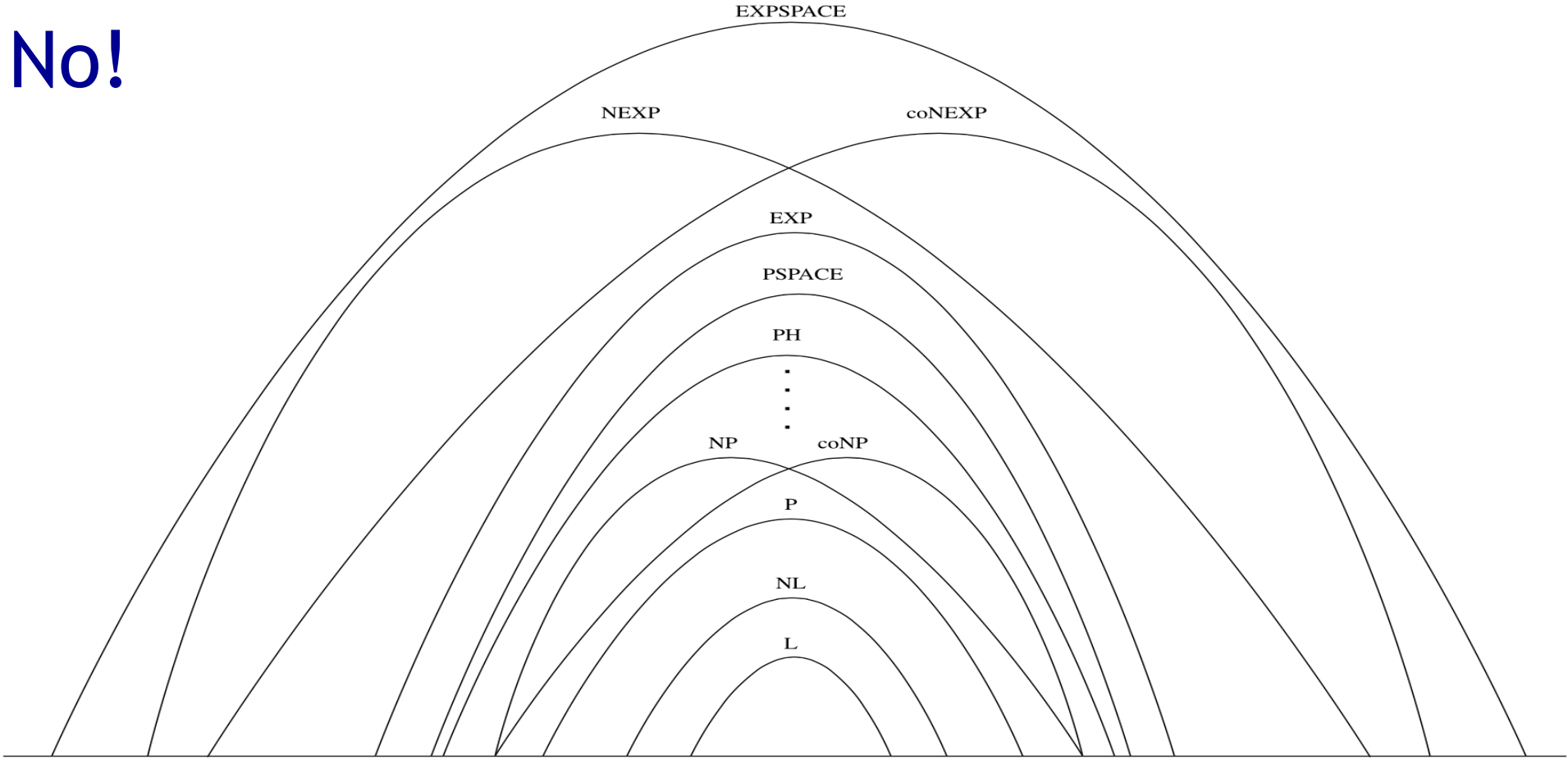
Are NP-complete Problems The Most Difficult Problems That Exist?

No!



And many more classes that is not in this picture.

However, all of these problems are *solvable*.

Unsolvable (Undecidable) Problems

- There are also unsolvable problems.
- We prove problem A is unsolvable by showing that:
  - The existence of an algorithm that solves A

    leads to a contradiction (paradox).

HALTING Problem (1936, Church-Turing)

- Input: Computer Program A & an input D to A
  - All input represented as binary bits
  - Note D can be any input (including an algorithm)
- Output: YES: if A halts and returns on D

  NO: if A doesn't halt (gets into an infinite loop)

Claim: Halting Problem is Undecidable

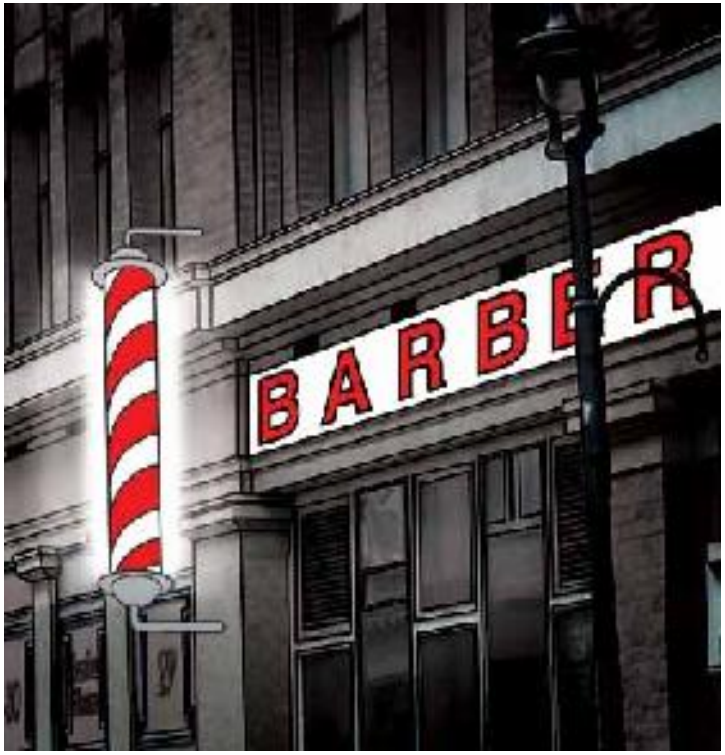
i.e.: there cannot be an algorithm that solves HALTING

Existence of such an alg. leads to a contradiction/paradox!

A Fun Example of a Paradox (Bertrand Russell, 1901)

- Self-referential statements: A trick to achieve paradoxes

"Suppose there is a village in which the barber shaves those and only those who don't shave themselves. That is:



Case 1: if person A shaves himself →
the barber does not shave A.

Case 2: If A does not shave himself →
the barber shaves A."

Q: Does the barber shave himself?

If yes → barber doesn't shave himself

If no → barber shaves himself

Contradiction either way! Such a barber cannot exist!

Cases When We Can Know If A Halts Or Not

Example 1:

```
procedure A(D):
   while (true); print ("still running");
```

Never halts on any D!

Example 2:

```
procedure A(D):
   return 5;
```

Always halts on every D!

But cannot have an alg that will decide if an algorithm A will halt or not on an arbitrary (A, D)!

Proof that HALTING is Undecidable

Suppose, for contradiction, ∃ an alg. M that solves HALTING

Suppose w.l.o.g M returns YES if (A, D) halts and NO o.w.

```
procedure M(A, D):
   if A halts on D: return YES;
   else return NO
```

Then we can design the following algorithm OPP-M:

```
procedure OPP-M(A, D):
   // if A halts on D, go into an infinite loop
   if M(A, D) == YES: while (true) { … };
   else return 1; // else halt
```

Let's Make OPP-M Self Referential

Note: Some algs take as an input another algs (or program)

Ex: compilers take as input source codes of other programs.

Design OPP-M-S, for self, that takes one input:

```
procedure OPP-M-S(A):
   // if A halts on A, go into an infinite loop
   if M(A, A) == YES: while (true) { … };
   else return 1; // else halt
```

Consider OPP-M-S on Input OPP-M-S

```
procedure OPP-M-S(OPP-M-S):
   // if OPP-M-S halts on OPP-M-S, loop forever
   if M(OPP-M-S, OPP-M-S) == YES: while (true) {…};
   else return 1; // else halt
```

If OPP-M-S halts on OPP-M-S

→ OPP-M-S loops forever on OPP-M-S (contradiction)!

If OPP-M-S does not halt on OPP-M-S

→ OPP-M-S halts on OPP-M-S (contradiction)!

Q.E.D

Conclusion: ∄ an algorithm M that solves HALTING!

Conclusion: HALTING is Unsolvable/Undecidable

Note however that HALTING is a well-defined computational problem!

That is we know every program either halts on a particular input or it does not!

Yet we cannot write an algorithm that solves HALTING (no matter how much time/memory/network/electricity we give to the algorithm).

Doesn't mean we cannot solve it on some inputs: e.g., compilers report some infinite loop cases

Proving Other Problems are Undecidable by "Reductions"

Just like we spread intractability among NP-complete problems

we can also spread undecidability through reductions.

HALT-ALL : Reducability among Undecidable Problems

- Input: A Computer Program A
- Output: YES: if A halts on all possible inputs

  NO: if A doesn't halt on all inputs, i.e., ∃ an input D on which A doesn't halt.

Claim: HALT-ALL is Undecidable

Reducing HALTING to HALT-ALL

Goal: Take an instance of Halting problem (A, D) and convert to an equivalent HALTING-ALL problem A` s.t.

A halts of D ↔ A` halts on all instances

```
procedure A`(input x)
   // ignore input x
   A(D); return YES;
```

Notice the input to A` is ignored.

← If A` halts on all instances then A halts on D

→ If A halts on D; then A` halts on all intances

Therefore: HALTING-ALL is Undecidable

Note about Reductions Among Undecidable Problems

Our reduction from HALTING-HALT-ALL took constant time.

But in general when proving that a problem C is undecidable through a reduction from a known undecidable problem C`, the reduction does NOT have to be poly-time.

It can actually take any (finite amount of) time.

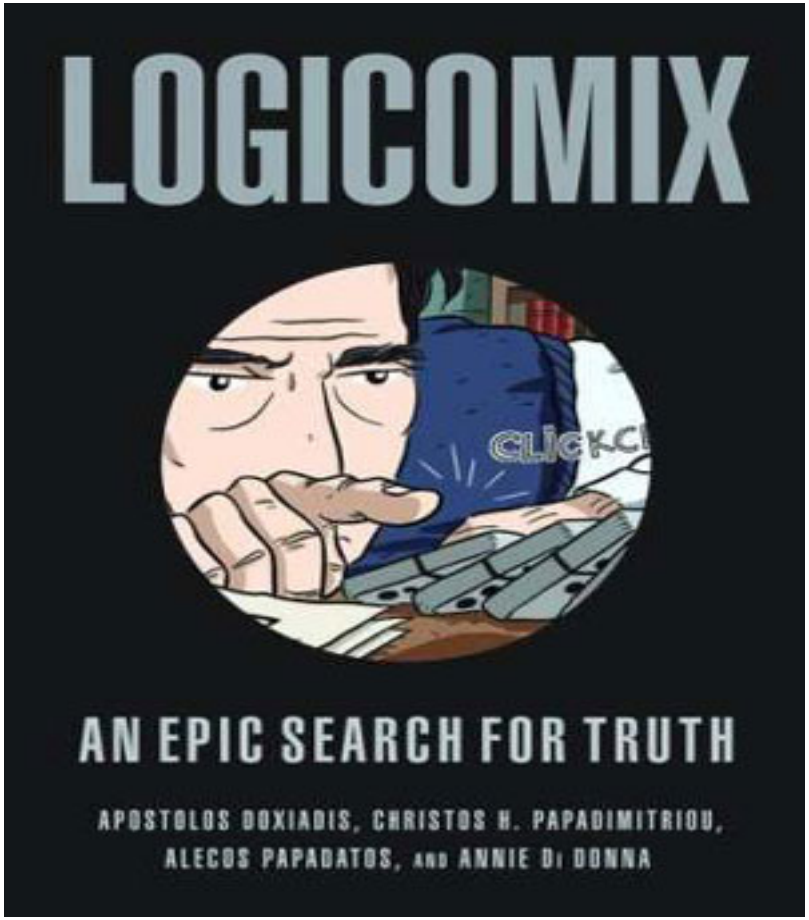
But recall when proving NP-completeness through reductions, the reductions HAVE TO BE poly-time.

Logicomix

Book by Christos Papadimitriou (theoretical CS @Berkeley)

About how Russell and other logicians of his time (1920s & 30s),

e.g., Hilbert, Whitehead, Godel, contributed to the birth of CS



CS 341 Diagram

Fundamental *(& Fast)* Algorithms to Tractable Problems

- MergeSort
- Strassen's MM
- BFS/DFS
- Dijkstra's SSSP
- Kosaraju's SCC
- Kruskal's MST
- Floyd Warshall APSP
- Topological Sort
- …

Common Algorithm Design Paradigms

- Divide-and-Conquer
- Greedy
- Dynamic Programming

Mathematical Tools to Analyze Algorithms

- Big-oh notation
- Recursion Tree
- Master method
- Substitution method
- Exchange Arguments
- Greedy-stays-ahead Arguments

Intractable Problems

- P vs NP
- Poly-time Reductions
- Undecidability

Other (Last Lecture)

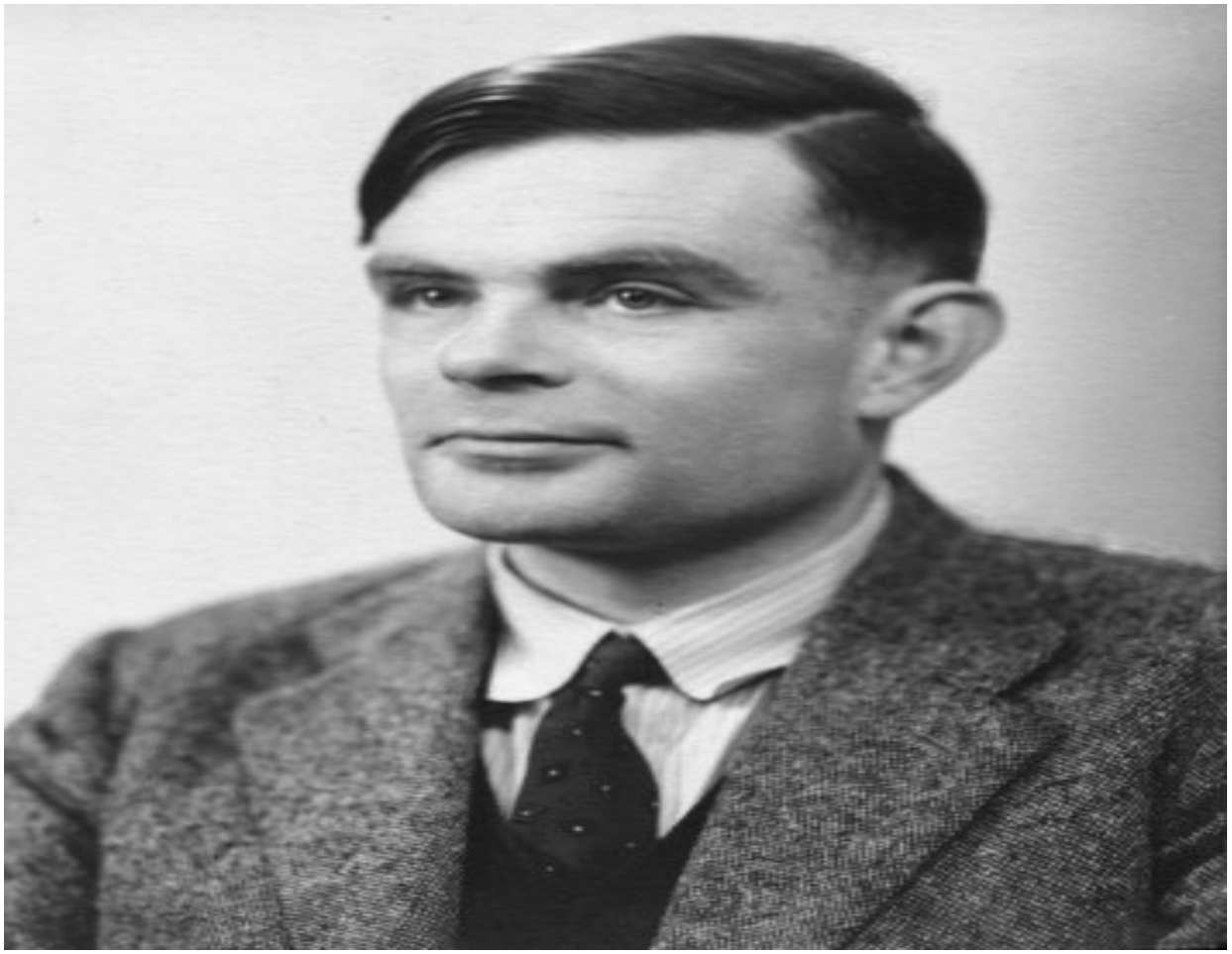
- Randomized/Online/ Parallel Algorithms

End of Official Curriculum

The rest will be about what to do when you see an Intractable (NP-complete) problem

But first a little bit of history…

Alan Turing: Founder of Computer Science



Turing's Answer To What Computation Is (1936)

"On Computable Numbers, with an Application to the Entscheidungsproblem":

"We may compare a man in the process of computing a real number to a machine which is only capable of a finite number of conditions"

Turing's Answer To What Computation Is

Computing is normally done by writing certain symbols on paper. We may suppose this paper is divided into squares like a child's arithmetic book. The behavior of the computer at any moment is determined by the symbols which he is observing, and his " state of mind " at that moment. We may suppose that there is a bound B to the number of symbols or squares which the computer can observe at one moment. If he wishes to observe more, he must use successive observations. We will also suppose that the number of states of mind which need be taken into account is finite.

Turing's Answer To What Computation Is

Let us imagine the operations performed by the computer to be split up into "simple operations" which are so elementary that it is not easy to imagine them further divided. Every such operation consists of some change of the physical system consisting of the computer and his tape.

Turing's Answer To What Computation Is

Besides these changes of symbols, the simple operations must include changes to the observed squares. … I think it is reasonable to suppose that they can only be squares whose distance from the closest of the immediately previously observed squares does not exceed a certain fixed amount.

Computer Science

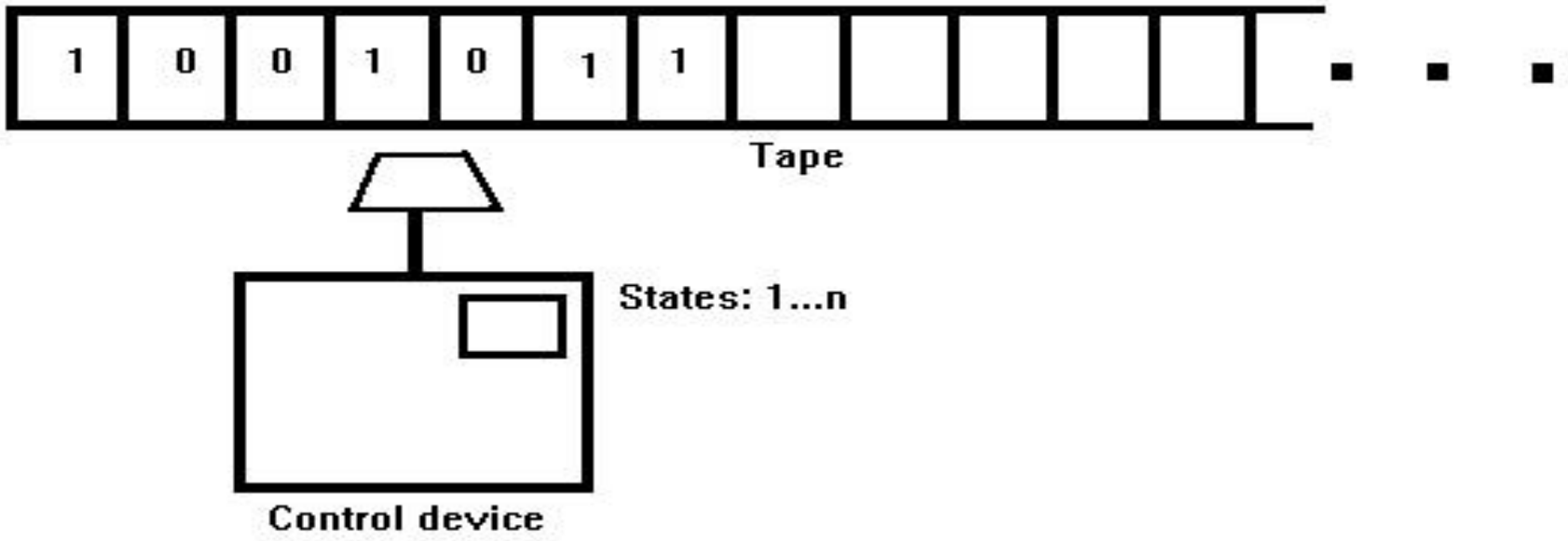
Studies the powers of machines.

Fundamental Question CS asks:

What is “computable” by machines?

Turing (along with Church and Godel) was the person who made computation something we can mathematically study.

The Turing Machine



It's interesting that as humans in our quest to understand what machines can do, we have been studying an abstract machine that in essence imitates a human being.

Church-Turing Thesis

Central Dogma of Computer Science:

Whatever is computable is computable by the Turing machine.

There is no proof of this claim.

Turing In Defense Of His Claim:

"All arguments which can be given are bound to be, fundamentally appeals to intuition, and for this reason rather unsatisfactory mathematically.

The arguments which I shall use are of three kinds.

(a) A direct appeal to intuition.

(b) A proof of the equivalence of two definitions *[referring to Godel's definition]*.

(c) Giving examples of large classes of numbers which are computable."

Algorithm = Turing Machine

When we say there is an algorithm computing shortest paths of a graph in O(mlog(n)) times we really mean:

There is a Turing Machine that computes the shortest paths of a graph in O(mlog(n)) operations.

Turing Machine Is The Most Powerful Machine

CS tries to understand the limits of TM.
We limit/extend TM and try to understand what can be computed by it.

- Limit the # times it's head is allowed to move left/right and it changes states to a polynomial. => poly-time algs
- What if the machine had access to a random source => randomized algorithms.
- What if there were multiple heads on the tape => parallel algorithms
- Limit the length of its tape. => space-efficient algs
- What if the head was only allowed to move right => streaming algorithms
- …

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Your Problem is NP-complete. Now What?

- Option 1: Focus to special-case inputs.
  - Ex: Independent Set is NP-complete.
  - Focusing on line graphs, had a O(n) DP alg.
- Option 2: Find an approximate answer.
- Option 3: Be exponential time but better than brute-force search.
  - 0-1 Knapscak O(nW) runtime DP algorihm.
- Option 4: Heuristics: fast algorithms that are not always correct (or even approximate)
- Option 5: Mix some of these options

Options 1&3: Restrict Input & Be exponential but better than brute force search (0/1 Knapsack)

- Input: n items
  - values for items v_1, …, $v_n \geq 0$
  - sizes for items w_1, …, $w_n \geq 0$
  - knapsack capacity $W \geq 0$
- Output: subset $S \subseteq \{1, 2, \ldots, n\}$ items s.t.

$$\max \sum_{i \in S} v_i$$

$$s.t. \sum_{i \in S} w_i \leq W$$

Recall Fact: 0/1 Knapsack is NP-complete

Restrict w_i and W to be Integers

- Input:
  - n items
  - values for items $v_1, \ldots, v_n \geq 0$
  - sizes for items $w_1, \ldots, w_n \geq 0$ & w_i are **INTEGERS**
  - knapsack capacity $W \geq 0$ & W is an **INTEGER**
- Output: subset $S \subseteq \{1, 2, \ldots, n\}$ items s.t.

$$\max \sum_{i \in S} v_i$$

$$s.t. \sum_{i \in S} w_i \leq W$$

0-1 Knapsack with integral w_i and W DP Alg

$K_{(i, c)}$: opt. knapsack for the first i items and cap c.

$$K_{(i, c)} = \max \begin{cases} K_{(i-1, c)} \\ K_{(i-1, c - w\_i)} + v_i \end{cases}$$

```
procedure DP-Knapsack-1(n, W):
 Base Cases: A[0][i] = 0
 for i = 1,2,…,n:
    for c = 1, …, W:
    A[i][c] = max{A[i-1][c], A[i-1][c-w_i]+v_i}
 return A[n][W]
```

Options 1& 3: "Good" Exponential Runtime on restricted input

Runtime: $O(nW)$

Brute-Force Search: $\Omega(2^n)$

Observation: This is polynomial in n and W.

Q: Is Knapsack then tractable?!

A: No! B/c we're still exponential in input size.

Input size n + log(W) and W is exponential in log(W).

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Final Information

- Date: April 18th
- Practice final will be posted on Learn (no solutions most likely, ask us)
- Same format as the practice final
- Coverage: Everything including Undecidability

Things To Remember In The Final and After (1)

- Get the indices in your DP algs right!

```
procedure DP:
  for i = 1…n
    for j = 1…n
      A[i][j] = A[i][j+1]
```

A[i][j+1] is not yet computed, so cannot access it

Usually this is not a problem at all, but make sure you are accessing already solved problem.

Things To Remember In The Final and After (2)

- Know the runtimes of every fundamental algorithms we learned

Things To Remember In The Final and After (3)

- Remember the math tools:
  - Big-Oh notation
  - Recurrences:
    - Master Method
    - Substitution Method
    - Recursion Tree Method
  - Basic math sequences we learned
  - …

Things To Remember In The Final and After (4)

- 2 Things to Remember in NP-completeness Proofs
- Suppose you'll prove problem X is NP-Complete

1. *First you have to show that X is in NP:*
2. *Get the direction of your reduction correct:*

   You reduce a known NP-complete problem Y to X

   Not X to Y.

Things To Remember In The Final and After (5)

- If you believe your algorithm is wrong, practice on skills to construct very very simple counter examples

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Randomization: Max Cut Problem

- Input: Undirected G(V, E)
- Output: cut (X, Y) with max # of crossing edges

Fact: Max-Cut is Intractable (NP-complete)

$\Rightarrow$ can't hope to solve it exactly within reasonable (poly-time) amount of time

Another Fact: Min-Cut is poly-time

Takeaway: Don't be deceived by how similar two problems look like! One may be tractable and the other intractable!

Simple Randomized Algorithm

```
procedure Approximate-MaxCut(G(V,E)):
    let L, R be 2 empty sets
    for each vertex v:
        toss a fair coin
        if heads: put v into L
        if tails: put v into R
```

This is a ½ approximation to OPT.

Take Away 1: Solving max-cut exactly is impossible but we have a very simple linear time approximate randomized algorithm!

Take Away 2: Randomized Algorithms can be very elegant.

Linear Programming (CS 466, CO 250/251/351)

- Optimization Problem of following structure:

$$\text{maximize } x_1 + x_2$$

$$\text{subject to}$$

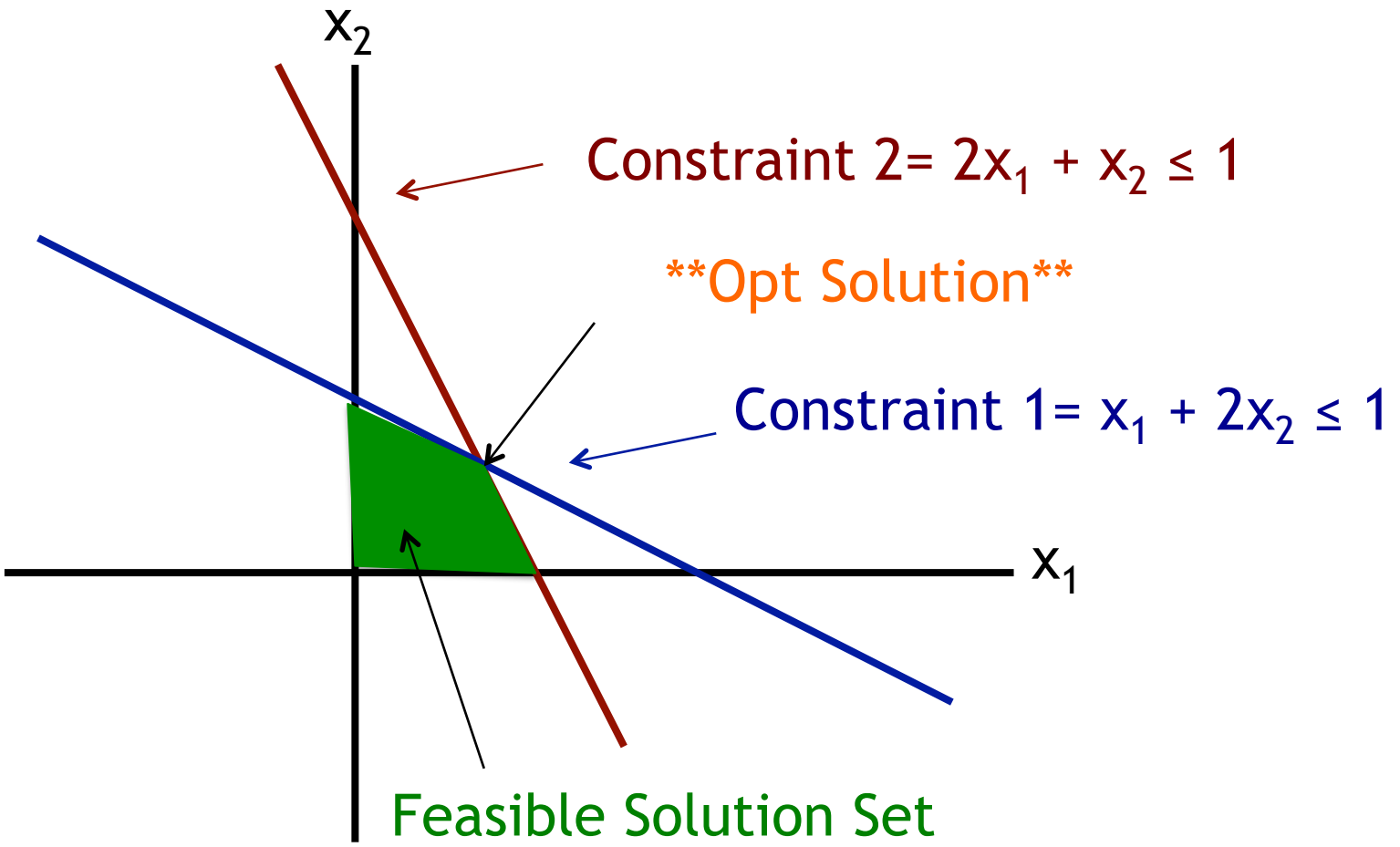
$$x_1 + 2x_2 \leq 1$$

$$2x_1 + x_2 \leq 1$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

Geometric Interpretation



Linear Programming Applications

- Tons! Lots and lots of problems can be solved or approximated with LP!
  - Vertex Cover
  - Set Cover
  - Load Balancing
  - Lots of problems in manufacturing/operations research/finance, etc..
- Covered in CS 466

Invented By George Dantzig.

Algorithms/Theory Courses Beyond CS 341

- CS 466: Approximation & Randomized Algorithms
- CS 365/CS 664 : Complexity Theory
- CS 467: Intro to Quantum Information Processing
- CS 482: Comp. Techniques in Biological Sequence Analysis

For students who like theory: watch out for the 600 and 800 level courses thought by theory faculty:

Ian Munro, Eric Blais, Jeffrey Shallit, …

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Common Algorithm Design Paradigms

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Mathematical Tools to Analyze Algorithms

- Big-oh notation
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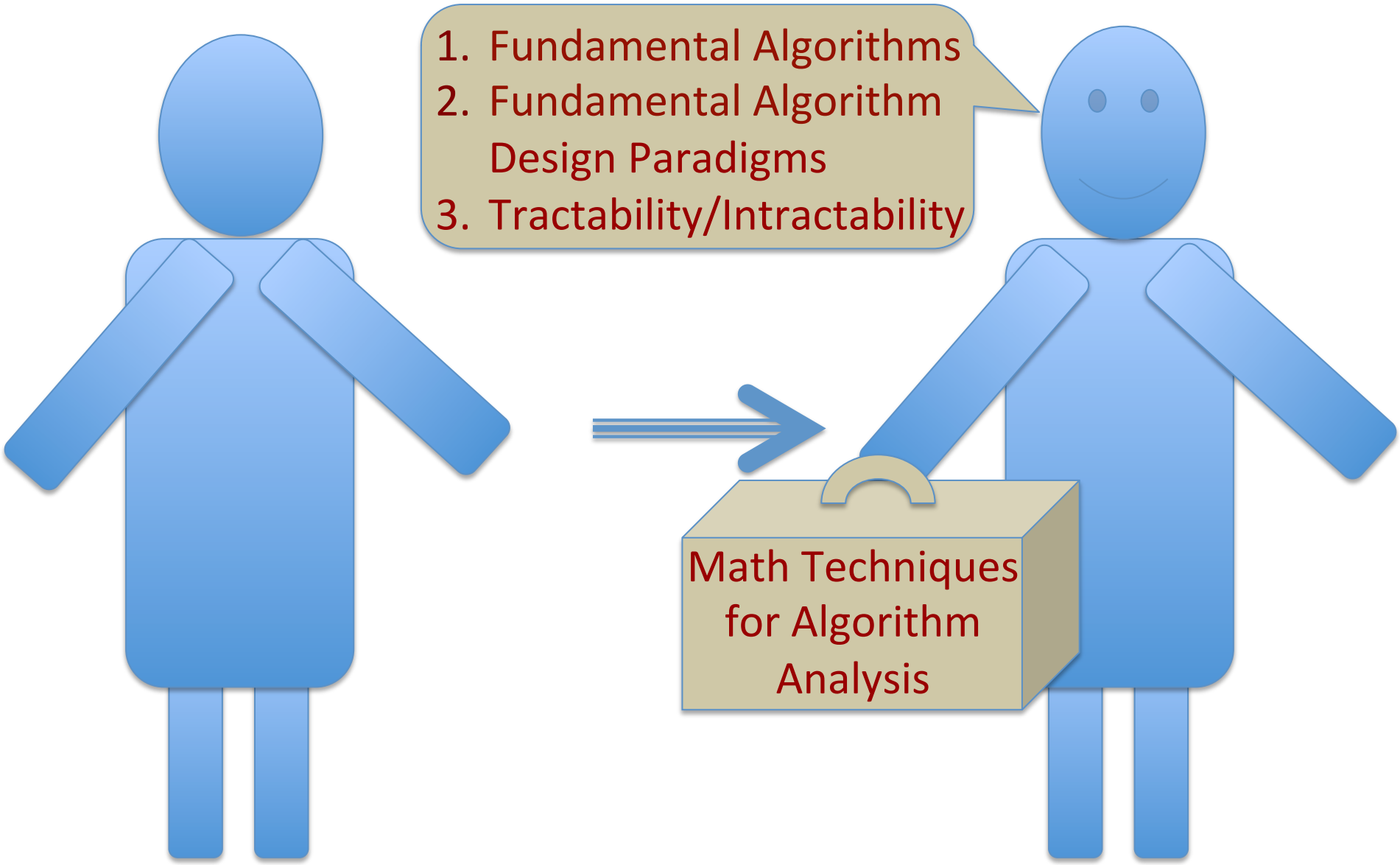
Intractable Problems

- P vs NP
- Poly-time Reductions
- Undecidability

Other (Last Lecture)

- Randomized/Online/ Parallel Algorithms

Before/After CS 341



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Thank you!

Good luck in the final!