1) $P = P_r + P_b = \frac{\rho_r c^2}{3}$, $\rho = \rho_r + \rho_b$ ⇒ ~~$\left(\frac{dP}{da}\right)\left(\frac{d\rho}{da}\right) = \frac{c^2}{3}\left(\frac{d\rho_r}{da}\right)\left(\frac{d\rho_r}{da} + \frac{d\rho_b}{da}\right)$~~

a) ~~$\rho_r = \rho_{r,0} a^{-4}$, $\rho_b = \rho_{b,0} a^{-3}$ ⇒ $c_s^2 = \frac{c^2}{3}\left[-4\rho_{r,0} a^{-5} / (-4\rho_{r,0} a^{-5} - 3\rho_{b,0} a^{-4})\right]$~~

$\Rightarrow \frac{dP}{da} = \frac{c^2}{3}\frac{d\rho_r}{da} \Rightarrow \frac{dP}{da} \Big/ \frac{d\rho}{da} = \frac{c^2}{3} \cdot \frac{d\rho_r}{da} \Big/ \left(\frac{d\rho_r}{da} + \frac{d\rho_b}{da}\right)$

$\rho_r = \rho_r' a^{-4}$, $\rho_b = \rho_b' a^{-3} \Rightarrow c_s^2 = \frac{c^2}{3}\left[(-4\rho_r' a^{-5})/(-4\rho_r' a^{-5} - 3\rho_b' a^{-4})\right]$

$= \frac{c^2}{3}\left[1 + \frac{3}{4}\frac{\rho_b'}{\rho_r'} a\right]^{-1}$ From slides, $\frac{\rho_r}{\rho_m} = \frac{2.48 \cdot 10^{-5}(1+z)}{\Omega_m h^2}$.

Thus, $\frac{\rho_b'}{\rho_r'} = \frac{\Omega_b h^2}{2.48 \cdot 10^{-5}(1+z)}$, at $z = 0$, $a = \frac{1}{1+0} = 1$, $c_s^2 = \frac{c^2}{3}\left[1 + \frac{3}{4}\frac{\Omega_b h^2}{2.48 \cdot 10^{-5}}\left(\frac{1}{1+z}\right)\right]^{-1}$

$\Rightarrow \boxed{c_s = \frac{c}{\sqrt{3}}\left[1 + 3.02 \cdot 10^4 \, \Omega_b h^2 / (1+z)\right]^{-1/2}}$ with $\Omega_b h^2$ and $z = 1000$,

the bracketed factor $\simeq 0.790$, so $\boxed{c_s \text{ within } 30\% \text{ of } \frac{c}{\sqrt{3}}}$

b) From the notes, $\frac{\rho_r}{\rho_m} = \frac{2.48 \cdot 10^{-5}(1+z)}{\Omega_m h^2}$, ρ_r ⇒ $\frac{\rho_r}{\rho_m} = \frac{4.17 \cdot 10^{-5}(1+z)}{\Omega_m h^2} = 1$

$\Rightarrow z_{eq} = \Omega_m h^2 / 4.17 \cdot 10^{-5} - 1 \Rightarrow \boxed{z_{eq} = 2.4 \cdot 10^4 \, \Omega_m h^2 - 1}$ ~~W/~~ values

c) $t = \int_z^\infty \frac{dz}{H(z)} \frac{1}{1+z} \Rightarrow dt = \frac{dz}{H(z)} \cdot \frac{1}{1+z}\Big|_z^\infty = \frac{-dz}{H(z)} \cdot \frac{1}{1+z} \Rightarrow s = \int_0^{t^*} dt\, c_s(z)(1+z)$

$= \frac{c}{\sqrt{3}} \int_\infty^{z^*=1000} -dz \cdot \frac{1}{H(z)} \frac{(1+z)}{(1+z)} = \frac{-c}{\sqrt{3}} \int_0^{z^*=1000} \frac{dz}{H(z)}$; This is the exact integral from HW 3 #2. Recycling the integral with the new bounds:

$$\boxed{s = \frac{c}{\sqrt{3}H_0}\left[\frac{2\sqrt{\Omega_R (z^*)^2 + (\Omega_M + 2\Omega_R) z^* + \Omega_M + \Omega_R}}{\Omega_M (z^*+1)} - \frac{2\sqrt{\Omega_R}}{\Omega_M}\right]}\ ;\ z^* = 1000$$

The radiation contribution can't be ignored since the universe is radiation dominated at $z \gtrsim 3000$.

d) Since radiation dominates at high z and matter at low z, then z_* being lower than z_{eq} implies that recombination occurs when the universe is matter-dominated. From before: $\Omega_r \propto (1+z)^4$ and $\Omega_m \propto (1+z)^3$, so the rad density falls quickly after matter starts dominating.

e) For $\Omega_m = 1$, $\Omega_k = 0 \Rightarrow S[r(z^*)] = R(z^*) = \frac{2c}{H_0}\left[1 - \frac{1}{\sqrt{1+z^*}}\right]$

from the notes, with $\Omega_m = 1$, $s =$

$$\frac{c}{\sqrt{3}\,H_0}\left[\frac{2\sqrt{\Omega_R(z^{*2}+2z^*+1)+\Omega_M(z^*+1)}}{\Omega_M(z^*+1)} - \frac{2\sqrt{\Omega_R}}{\Omega_M}\right] = \frac{2c}{\sqrt{3}H_0}\left[\sqrt{\Omega_R}\left(\sqrt{1+\frac{1}{\Omega_R(z^*+1)}}-1\right)\right]$$

$$\cancel{\Rightarrow \frac{s}{S[r]} = \frac{1}{\sqrt{3}}\,\frac{[1/\sqrt{1+z^*} - 1 + \ldots]}{[1 - 1/\sqrt{1+z^*}]}}$$

At z^*_{eq}, $\Omega_R(1+z^*_{eq})^4 = \Omega_M(1+z^*_{eq})^3 \Rightarrow \Omega_R = 1/(1+z_{eq}) \simeq 4.17 \cdot 10^{-5} \cdot 1/h^2$

$$\frac{s}{S[r]} = \frac{1}{\sqrt{3}}\left[\sqrt{\Omega_R}\left(\sqrt{1+\frac{1}{\Omega_R(1+z^*)}}-1\right)\right] \Big/ \left[1 - 1/\sqrt{1+z^*}\right]$$ with $h = 0.5 \simeq 0.0127$

$\Rightarrow \ell = \frac{4S[r]}{s} \Rightarrow \boxed{\ell \simeq 316}$ with $h = 0.3$, $\boxed{\ell \simeq 401}$ with $h = 0.7$, $\boxed{\ell \simeq 283}$ So ℓ doesn't vary more than ~30%.

See code for f) and g)

2) a) Mean free path $l \sim 1/\sigma n$, $n = e^-$ # density and σ = Thompson scattering cross section. From Longair 12.55, $N_e = 11 \Omega_B h^2 (1+z)^3 m^{-3} \simeq 0.22 (1+z)^3 m^{-3}$, $\sigma \simeq 6.65 \cdot 10^{-29} m^2 \Rightarrow l \sim 6.84 \cdot 10^{28} m \cdot 1/(1+z)^3$

$\Rightarrow \boxed{l = 2.2 \cdot 10^6 \text{ Mpc} \cdot (1+z)^{-3}}$ $\quad c/H(z) = c/H_0 \sqrt{\Omega_R (1+z)^4 + \Omega_m (1+z)^3}$

$= \frac{c}{H_0} \left[\Omega_m (1+z)^3 \left(1 + \frac{\Omega_R}{\Omega_m}(1+z)\right)\right]^{-1/2}$, $\frac{\Omega_R}{\Omega_m} = \frac{\rho_R (1+z)^{-4}}{\rho_M (1+z)^{-3}}$, at $z = z_{eq}$,

$\rho_R = \rho_M \Rightarrow \frac{\Omega_R}{\Omega_M} = 1/1+z_{eq} \Rightarrow \frac{c}{H(z)} = \frac{c}{100h \frac{km/s}{Mpc}} \cdot \frac{1}{\sqrt{\Omega_m}} \cdot (1+z)^{-3/2} \cdot 1/\sqrt{1+\frac{(1+z)}{(1+z_{eq})}}$

$= \frac{3 \cdot 10^5}{100} \cdot \frac{1}{\sqrt{\Omega_m h^2}} \text{ Mpc} (1+z)^{-3/2} \cdot \left[1 + \frac{1+z}{1+z_{eq}}\right]^{-1/2}$, for early universe, $z \simeq 1+z$

and with $\Omega_m h^2 = 0.14$, $\boxed{\frac{c}{H(z)} = 8020 \text{ Mpc} (1+z)^{-3/2} \left[1 + \frac{(1+z)}{(1+z_{eq})}\right]^{-1/2}}$

b) Since $l \sim 1/n$, $l \sim 1/f \Rightarrow l' = l/f$. At $z = 1000$, taking $z_{eq} \simeq 3000$, $l = 0.0022$ Mpc; $\frac{c}{H(z)} = 0.219$ Mpc $= l' = \frac{l}{f}$

$\Rightarrow f = l / \frac{c}{H(z)} \Rightarrow \boxed{f \simeq 0.01}$

c) For a random walk with N steps, the RMS distance is $\sqrt{N} \cdot l$, where l is the avg. distance per step aka mean free path. Since we want to integrate over redshift, since that's how l varies. After doing some googling, I see that the right way to set up integral is $\lambda^2 = \int_0^t \frac{c\,dt}{\lambda} \frac{\lambda^2}{a^2}$.

In our notation, $r^2 = \int_0^t c\,dt\,\ell(1+z)^2 = c\int_0^{t_0} \frac{dt(1+z)^2}{n_e \sigma_T}$

Using the dz/dt conversion from earlier and n_e,

$= \frac{c}{\sigma_T}\int_{z^*}^{\infty} \frac{dz}{H(z)(1+z)}(1+z)^2 \cdot \frac{1}{0.22\,m^{-3}}(1+z)^{-3} = c(2.2\cdot 10^6\,Mpc)\int_{z^*}^{\infty}\frac{dz}{(1+z)^2 H(z)}$,

$H(z) = H_0\sqrt{\Omega_R(1+z)^4 + \Omega_m(1+z)^3} = H_0(1+z)^2\sqrt{\Omega_R + \frac{\Omega_m}{(1+z)}}$

$\Rightarrow r^2 = \frac{c}{H_0}('') \int_{z^*}^{\infty} \frac{dz}{(1+z)^4\sqrt{\Omega_R + \frac{\Omega_m}{(1+z)}}}$; call $u = \frac{1}{(1+z)} \Rightarrow du = \frac{-1}{(1+z^2)}$

$\Rightarrow = '' \int_{z^*}^{\infty} \frac{-u^2\,du}{\sqrt{\Omega_R + \Omega_m u}}$ Plugging into integral calculator:

$= -''\left[\frac{2(\Omega_m u + \Omega_R)^{5/2}}{5\Omega_m^3} - \frac{4\Omega_R(\Omega_m u + \Omega_R)^{3/2}}{3\Omega_m^3} + \frac{2\Omega_R^2\sqrt{\Omega_m u + \Omega_R}}{\Omega_m^3}\right]\Bigg|_{z^*}^{\infty}$

At $z=\infty$, $u=0$; cancelling out the Ω_R's, we get

$\frac{-''}{\Omega_m^3}\left[\frac{2}{5}\Omega_R^{5/2}\left(1+\frac{\Omega_m}{\Omega_R}\cdot\frac{1}{1+z}\right)^{5/2} - \frac{4}{3}\Omega_R^{5/2}\left(1+\frac{\Omega_m}{\Omega_R}\frac{1}{1+z}\right)^{3/2} + 2\Omega_R^{5/2}\left(1+\frac{\Omega_m}{\Omega_R}\frac{1}{1+z}\right)^{1/2}\right]_{z^*}^{\infty}$

with $\frac{\Omega_m}{\Omega_R} = 1+z_{eq}$: $= -''\frac{\Omega_R^{5/2}}{\Omega_m^3}\left[\frac{2}{5}\left(1+\frac{1+z_{eq}}{1+z}\right)^{5/2} - \frac{4}{3}('')^{3/2} + 2('')^{1/2}\right]_{z^*}^{\infty}$

$= \underbrace{\frac{-c}{H_0\sqrt{\Omega_m}}}(2.2\cdot 10^6\,Mpc)\cdot\frac{1}{(1+z_{eq})^{5/2}}\left[\frac{2}{5}\left(1+\frac{1+z_{eq}}{1+z^*}\right)^{5/2} - \frac{4}{3}\left(1+\frac{1+z_{eq}}{1+z^*}\right)^{3/2} + 2('')+\frac{16}{15}\right]$

$= 8020\,Mpc$ from before; plugging in $z_{eq} = 3000$, $z^* = 1000$:

$\Rightarrow r^2 = 181\,Mpc^2 \Rightarrow \boxed{r = 13.4\,Mpc}$

d) (see code) $\Omega_m = 0.2$, $\Omega_\Lambda = 0.6 \Rightarrow D = 16{,}100\,Mpc$;

$\boxed{\Theta = 3\text{ arcmin}}$