Are network properties consistent indicators of synchronization?

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Abstract. - We investigate the collective dynamics of bi-stable elements connected in different network topologies, ranging from rings and small-world networks, to random and deterministic scale-free networks. We focus on the correlation between network properties and global stability measures of the synchronized state, in particular the average critical coupling strength $\langle \epsilon_c \rangle$ yielding transition to synchronization. Further we estimate the robustness of the synchronized state by finding the minimal fraction of nodes f_c that need to be perturbed in order to lose synchronization. Our central result from these synchronization features is the following: while networks properties can provide indicators of synchronization within a network class, they fail to provide consistent indicators across network classes. For instance, the heterogeneity of degree does not consistently impact synchronization, as evident through the stark difference in the synchronizability of rings vis-a-vis small-world and star networks, all of which have same average degree and deviation around the mean degree in the limit of large networks. Further we demonstrate that clustering coefficient is also not a consistent feature in determining synchronization. This is clear through the similarity of synchronization properties in rings with significantly different clustering coefficients, and the striking difference in synchronization of a star network and a ring having the same clustering coefficient. Even characteristic path length, which is of paramount importance in determining synchronization, does not provide a one-to-one correspondence with synchronization properties across classes. Namely, synchronization is significantly favoured in networks with low path lengths within a network class. However, the same characteristic path length in different types of networks yields very different $\langle \epsilon_c \rangle$ and f_c.

Synchronization of complex networks has attracted wide research interest, from fields as diverse as ecology and sociology to power grids and climatology [1–4]. Collective spatiotemporal patterns emerging in dynamical networks are determined by the interplay of the dynamics of the nodes and the nature of the interactions among the nodes. So it is of utmost relevance to ascertain how connection properties impact synchronization, and this important issue has attracted much research effort, though still not attained clarity [5–9].

In broad terms, the concept of synchronization provides a general approach to the understanding of the collective behavior of coupled dynamical systems, and the term has expanded in usage to include several types of correlated collective behaviour, such as complete synchronization, generalized synchronization, phase synchronization, lag synchronization, and anticipated synchronization. The unifying thread in all the different types of synchronization is the emergence of a consistent relationship between the variables of a set of dynamical systems, and this can arise in contexts ranging from simple first-order phase oscillators to complex chaotic systems. Further, these types of synchronization apply to both mono-stable, as well as multi-stable systems [10]. However, while synchronization in mono-stable systems is commonly studied, synchronization of multi-stable systems is still not that well explored. One underlying reason for this, is that the concept of a linearly stable synchronized state is inadequate to capture the collective dynamics of multi-stable systems, and consequently even the simple phenomenon of complete synchronization necessitates global measures in order to adequately understand the collective behaviour in systems with co-existing attractors [11, 12].

Here we consider the collective dynamics of a group of

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bi-stable elements connected in different network topologies, ranging from regular rings and small-world networks on one hand, to deterministic scale-free and random scale-free networks on the other. Bi-stable systems are relevant in a variety of fields, ranging from relaxation oscillators and multivibrators, to light switches and Schmitt triggers. The basic question we will address is the following: are there features of the underlying connection network that provide *consistent markers* for the emergence of *complete synchronization* and the robustness of the synchronized state? Specifically, complete or identical synchronization is the phenomena where the difference between the variables of the constituent dynamical systems in the network is zero. Since we consider bi-stable systems, complete synchronization here naturally implies that all nodes in the network evolve to the same well.

In particular, we consider the system of N coupled bi-stable elements given as:

$$\dot{x_i} = F(x_i) + \epsilon \frac{1}{k_i} \sum_j (x_j - x_i) \tag{1}$$

Here i is the node index ($i = 1, \dots N$) and ϵ is the coupling constant, reflecting the strength of coupling. The j in Eqn. 1 gives the node index of the set of k_i "neighbours" of the i^{th} node, with the set depending on the topology of the underlying connectivity. The function $F(x)$ gives rise to a double well potential, with two stable states x^*_- and x^*_+. For instance one can choose

$$F(x) = x - x^3$$

yielding two stable steady states $x^*_\pm$ at $+1$ and -1, separated by an unstable steady state at 0. So complete synchronization in this network implies that the dynamical variable x_i evolves to the same well, for all i (namely all $x_i > 0$, or all $x_i < 0$, in this particular case).

We explore the behaviour of these bi-stable dynamical elements coupled in five distinct network classes, namely:

(i) Ring: Here each node has degree K (K even) and is connected to $K/2$ nearest neighbors on either side. Namely, $k_i = K$ for all nodes i in Eqn. 1, with connections existing between node i and j if $0 < |i - j| \bmod (N - 1 - \frac{K}{2}) \leq \frac{K}{2}$. The clustering coefficient C in a ring is independent of size, and its value for $K = 2$ is zero, while for $K = 4$ it is 0.5. The characteristic path length L is $\frac{N}{2K}$, and it increases linearly with size N.

(ii) Small-World network: This is constructed via the Watts-Strogatz algorithm [13]. Namely, we start from a ring with vertices having degree K (as in 1 above), and then rewire links to random non-local nodes with probability p. Here $k_i = K$ in Eqn. 1, with j being nearest neighbours with probability $1 - p$ and random non-local nodes with probability p. For $K = 2$, the clustering coefficient C varies from 0 at $p = 0$, to around 0.02 at $p \sim 1$. The characteristic path length varies from ~ 25 at $p = 0$ to around 5.5 at $p \sim 1$.

(iii) Random Scale-Free: this is constructed via the Barabasi-Albert preferential attachment algorithm, with the number of links of each new node denoted by parameter m [14]. The network is characterised by a fat-tailed degree distribution. The clustering coefficients C and characteristic path lengths L, depends on m. For instance, for $m = 1$, we have $C = 0.00$, $L = 4.67$, for $m = 2$ we have $C = 0.14$, $L = 2.99$, for $m = 3$ we have $C = 0.16$, $L = 2.58$, and for $m = 4$ we have $C = 0.19$, $L = 2.35$.

(iv) Deterministic Scale-Free: this has the particular hierarchical structure, generated iteratively for different orders (denoted by g) [15]. Here the order g, which is the only relevant network parameter, determines the number of nodes in the network, with $N = 3^g$. Here the set of neighbours for node i are all the nodes directly connected by an edge. It can be shown analytically that the degree distribution $P(k)$ is a power-law here, as in the Random Scale-Free case. Namely $P(k) \sim k^{-\gamma}$ with exponent $\gamma = \frac{\ln 3}{\ln 2}$ [15]. In Deterministic Scale-Free networks, the clustering coefficient is zero, and the characteristic path length depends on the order g. For instance, for $g = 3$ (namely a network of size $N = 27$) the characteristic path length is ~ 2.8, while for $g = 6$ (namely a network of size $N = 729$) the characteristic path length is ~ 5.3.

(v) Star network: here all the peripheral nodes are attached to one hub. The clustering coefficient of this network is zero, and its path length tends to 2 for large networks.

The focus of our investigation is the asymptotic state of the coupled bi-stable elements, starting from a random initial state which can be very far from the synchronized state. Our attempt is to *determine which connection topologies, typically, are most conducive to attracting all the elements to the same well.* Note that synchronization of a system depends on the particular realization of the connection network, and so systems with the same network parameters may yield different synchronization transitions [5, 6]. The measure of relevance for a network class would then be some averaged quantity, reflecting the typical behaviour arising from a generic random initial state and network realization.

So, in this work, we estimate the average critical coupling strength characteristic of a network, denoted by $\langle \epsilon_c \rangle$. This is the coupling strength after which synchronization occurs for a specific initial state and a specific realization of the network, averaged over a large range of initial states and network realizations. Smaller values of this global averaged quantity $\langle \epsilon_c \rangle$ indicates that the synchronization transition occurs at weaker coupling strengths on an average, namely the system is typically more easily synchronized.

Figs. 1-3 shows the synchronization error, given by the time and ensemble-averaged mean square deviation of the spatial profile, for different network classes: rings, small-world networks, Random Scale-Free networks, Deterministic Scale-Free networks and star networks. The central observations are the following:

(i) The Star network synchronizes most readily, with the synchronization transition occurring at very low coupling strength, irrespective of the size of the network (cf. Fig. 1).

(ii) Small-World networks with high degree of randomness in connections (namely high p) also yield synchronization at low coupling strengths. Fig. 2 shows the average synchronization error as a function of coupling strength, for varying fractions of random links p in the Small-World network. It is clearly evident that as p increases, synchronization occurs at increasingly weaker coupling strengths. Namely, the critical coupling $\langle \epsilon_c \rangle$ after which synchronization ensues, decreases with increasing p. So it is clear that a *greater degree of randomness in networks, where characteristic path lengths are significantly shorter, assists synchronization.*

(iii) Rings and Deterministic Scale-Free networks and Random Scale-Free networks with $m = 1$ (namely, where a new node has one link to the existing nodes) *do not yield synchronization* even for very high coupling strengths (cf. Fig. 3).

(iv) However, Random Scale-Free networks with $m \geq 2$ (namely, where a new node has two or more links to the existing nodes), allows stable synchronization, even for low coupling strengths (cf. Fig. 3).

So the synchronization transition is markedly different in Random Scale-Free networks having the same qualitative degree distribution, but different path lengths, as evident through the lack of synchronization of Random Scale-Free networks with $m = 1$ vis-a-vis the efficient synchronization of Random Scale-Free networks with $m \geq 2$. This indicates that degree distribution does not influence synchronization [5], though the characteristic path length needs to be sufficiently small in order to give rise to a stable synchronized state.

Additionally there is also very notable difference in the synchronization properties of Deterministic Scale-Free networks and Random Scale-Free networks with higher m, both of which have power-law degree distributions but markedly different characteristic path lengths. This further corroborates that, while *shorter characteristic path lengths assists synchronization*, the *qualitative nature of the degree distribution is not the key feature that determines synchronization.*

Further, the heterogeneity of degree also does not consistently influence synchronization either [7]. This is clearly inferred through the stark difference in the synchronization of Rings and Small-World networks (where the mean square deviation of the degree is zero) and Star networks (where the mean square deviation of the degree tends to zero in the limit of large N).

Comparative synchronizability of different network classes:

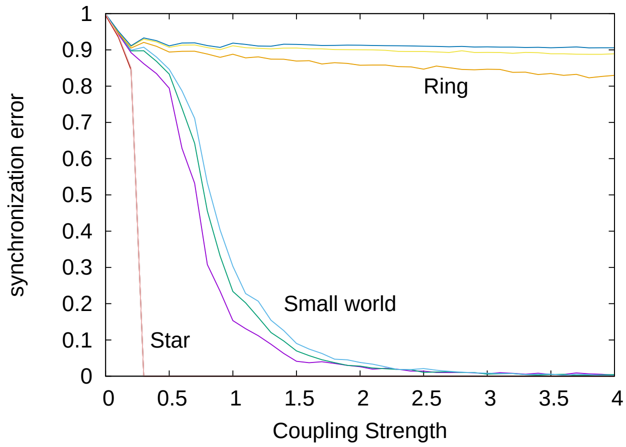


Fig. 1: Synchronization error for varying coupling strengths, for a system of coupled bi-stable elements given by Eqn.1, with different underlying connection networks: regular Ring with $K = 2$ ($N = 100, 250, 500$ shown in orange, yellow, navy blue respectively), Small-World network with $K = 2$ and $p = 0.5$ ($N = 100, 250, 500$ shown in violet, green, blue respectively) and Star (for $N = 100, 250, 500$ shown in red, black, pink respectively).

Inspite of a wide range of studies over the years, no clear picture has emerged on what class of networks synchronize most readily [5–9]. Here we address this issue by attempting to compare the average critical coupling strength of different network classes (cf. Fig. 4). Further, we will try to correlate this with the characteristic path length, which emerged as the network property which most strongly impacts synchronization.

Fig. 5 shows the dependence of $\langle \epsilon_c \rangle$ of different classes of networks on characteristic path length [16]. It is clearly evident that, within a particular network class, decreasing path length improves synchronization. However there is a caveat: across network classes the comparison does not hold. First, there does not appear to be a critical characteristic path length, below which synchronization is ensured. Nor is there a one-to-one correspondence between path length and ease of synchronization. For example, Small-World networks with high p have greater characteristic path lengths than certain Ring and the Deterministic Scale-Free networks. Yet these Ring and Deterministic Scale-Free networks are much harder to synchronize and consequently have significantly higher $\langle \epsilon_c \rangle$.

Fig. 6 shows the dependence of $\langle \epsilon_c \rangle$ of different classes of networks on clustering coefficient. It is observed that within a network class, increasing clustering coefficient aids synchronization. However the sensitivity of synchronization on clustering coefficient in the different network is markedly different. For instance, in Random Scale-Free networks increasing the clustering coefficient has very little effect on the critical coupling strength, while in Small-World networks the effect is stronger. The lack of one-to-one correspondence is very pronounced for the case of

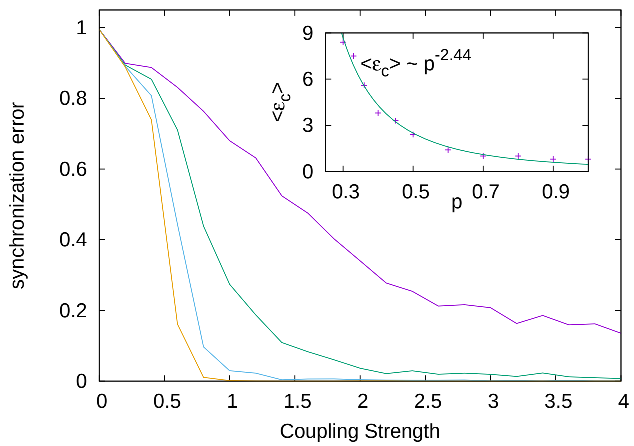
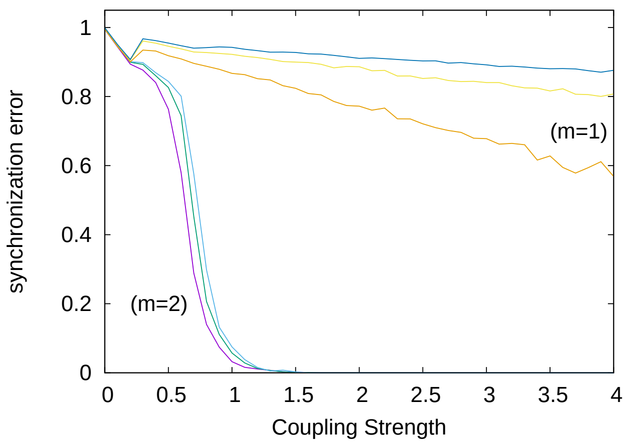
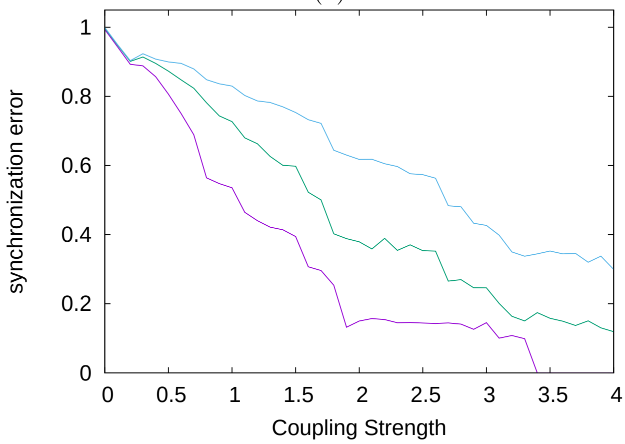


Fig. 2: Synchronization error as a function of varying coupling strengths, for a system of coupled bi-stable elements given by Eqn.1, in Small-World networks having different fractions of random links p ($p = 0.2, 0.4, 0.6, 0.8$ for violet, green, blue, orange respectively). Here $N = 100$. Inset: Dependence of the average critical coupling strength for the onset of synchronization $\langle \epsilon_c \rangle$, on the fraction of random links p in Small-World networks. The points are obtained from numerical simulations and the solid line displays the best fit curve.



(a)



(b)

Fig. 3: Synchronization error for varying coupling strengths, for a system of coupled bi-stable elements given by Eqn.1, with different underlying connection networks: (a) Random Scale-Free for $N = 100, 250, 500$ shown in violet, green, blue for $m = 1$ and $N = 100, 250, 500$ shown in orange, yellow, navy blue for $m = 2$ respectively, and (b) Deterministic Scale-Free (for $N = 81, 243, 729$ shown in violet, green, blue respectively).

clustering coefficient, as evident through the extremely different synchronization properties of Star networks and Rings with nearly same clustering coefficients, and almost same synchronization properties of Rings with very different clustering coefficients. Further that symmetry of the connection graph, as reflected by the order of its automorphism group, is also not a consistent indicator of the synchronization efficiency. This is evident from the fact that the ring which has more symmetry than small-world networks is less synchronizable than small-world systems, while a star network with much more symmetry than small-world networks is much more synchronizable.

Lastly, we probe if the commonly used measures of synchronizability obtained through the linear stability analysis can offer a more consistent indicator of $\langle \epsilon_c \rangle$. Since the average critical coupling strength is obtained by sampling a large set of initial conditions, typically far from the synchronized state, it provides a global indicator of the stability of the synchronized state. So at the outset it is not clear how local measures can capture these global trends. In order to ascertain the correlation of local synchronizability measures and the average critical coupling, we calculated the synchronizability measure obtained through master stability analysis. Following the formalism outlined in Ref. [17], the synchronizability index of a system of size N can be assessed by the spread of eigen-values of the network Laplacian L (defined as $L = D - A$, where D is diagonal of row sums of the network adjacency matrix A). Ordering the eigen-values as: $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \lambda_N$, where λ_1 has multiplicity 1 and all other eigen-values are strictly positive, the synchronizability index is given by the ratio $\frac{\lambda_2}{\lambda_N}$, with larger values of this ratio implying greater propensity for synchronization.

The results are displayed in Fig. 7, and it is evident again that there is *no one-to-one correspondence* between synchronizability (as defined above) and the average critical coupling. For instance, we observe that a Small-World network and a ring, with the *same* value of synchronizability have *different* average critical coupling strengths. So the relationship between synchronizability and the average critical coupling strength is not one-to-one, and *same values of synchronizability do not guarantee that the network will synchronize at similar coupling strengths on an average*. Further, the dependence of $\langle \epsilon_c \rangle$ on synchronizability shows different trends for different network classes, with $\langle \epsilon_c \rangle$ decreasing much faster with respect to synchronizability for some networks (such as Small-World networks), and quite slowly for others (such as Deterministic Scale-Free networks). All this suggests that linear stability

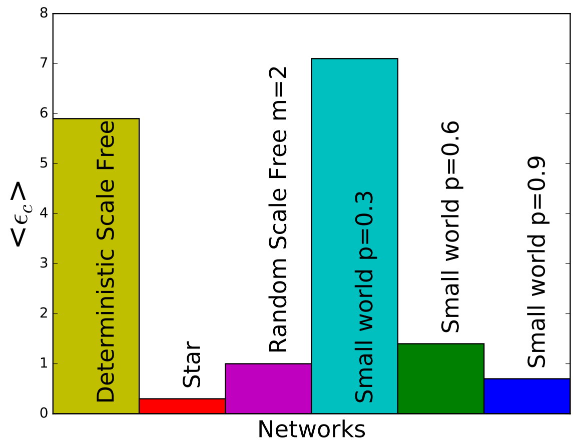


Fig. 4: Average critical coupling $\langle \epsilon_c \rangle$ for different classes of networks. Ring and Random Scale-Free networks with $m = 1$ do not synchronize at all for $\epsilon \leq 10$, and thus do not appear in the figure. Here system size is $N = 243$ for all networks, and the threshold synchronization error is considered to be 0.02. Low $\langle \epsilon_c \rangle$ (namely, low bars) indicate greater ease of synchronization.

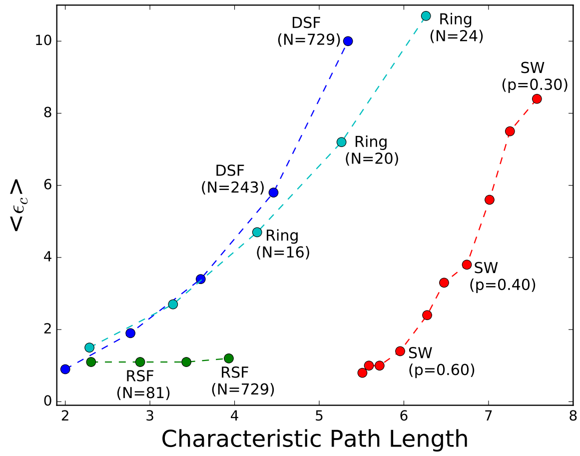


Fig. 5: Average critical coupling $\langle \epsilon_c \rangle$ vs characteristic path length of different network classes: Deterministic Scale-Free (DSF) in blue, Random Scale-Free (RSF) with $m = 2$ in green, Small-World (SW) in red and Ring with $K = 2$ in cyan. Here the system size of the DSF networks and RSF networks vary from 27 to 729, and the system size of the Ring varies from 8 to 24. SW networks with varying p are shown, for network size $N = 100$.

approaches, like Master Stability Function analysis, also does not offer a complete picture for global measures of synchronizability.

Robustness of the Synchronized State:

To check the robustness of the synchronized state, we now investigate the number of elements that need to be perturbed in order to push the system away from the synchronized state. In order to gauge this we calculate a measure, denoted as f-node basin stability, given as follows: we initialize the network to its synchronized stable state, with a small spread in the values of x_i, all of which are in the same well. We then perturb f number of randomly chosen nodes such that the values of x_i for these nodes are strongly perturbed and kicked to the basin of attraction of the other well. We then ascertain whether all the elements return to the original well after this perturbation. We repeat this "experiment" over a large sample of perturbed nodes and perturbation strengths, and find the fraction of times the system manages to revert to the synchronized state. This measure is quite analogous to Basin Stability measures [11, 12], and is indicative of the robustness of the synchronized state to localized perturbations in the network.

It can be observed in figure 8 that the synchronized state of bi-stable elements coupled in a ring is robust if the number of nodes perturbed is less than approximately 10%. Similar robustness holds for bi-stable elements coupled in a Deterministic Scale-Free network. If the number of nodes perturbed is more than about 15%, one almost never obtains a state where all nodes are in the original well. In contrast, it can be seen that the synchronized state of bi-stable elements coupled in a Small-World network, or a Random Scale-Free network with $m = 2$, is robust even when the number of nodes perturbed is as high as 40-45%.

Further we can obtain the critical number of nodes f_c needed, on an average, to destroy synchronization, namely the value of f for which the Basin Stability falls below a threshold value (taken to be ≈ 0.9 here). Fig. 8(c) displays this as a function of characteristic path length for the Small-World case. It is apparent that the number of nodes that need to be perturbed increases significantly with increasing characteristic path length. However, again, across network classes there is no one-to-one correspondence between critical fraction f_c and the characteristic path length. This is clear from the comparison of Random Scale-Free networks and Small-World networks, which have similar f_c, but very different characteristic path lengths. Further, Deterministic Scale-Free networks having shorter characteristic path lengths than Small-World networks have considerably lower f_c, namely less robust synchronized states.

Varying Nodal Dynamics:

In order to demonstrate the generality of these observations, we also explored different networks of bi-stable *synthetic genetic networks*, where the nodal dynamics was given by [18–20]:

$$F(x) = \frac{m(1 + x^2 + \alpha\sigma_1 x^4)}{1 + x^2 + \sigma_1 x^4 + \sigma_1\sigma_2 x^6} - \gamma_x x \qquad (2)$$

where x is the concentration of the repressor. The nonlin-

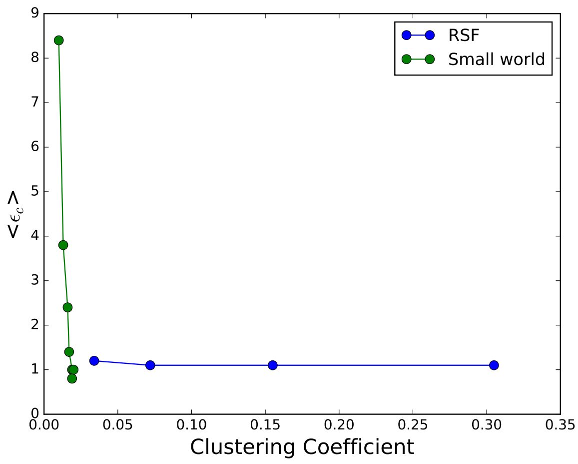


Fig. 6: Average critical coupling $\langle \epsilon_c \rangle$ vs clustering coefficient for the case of (a) Random Scale-Free (RSF) networks (blue) and (b) Small-World (SW) networks (green).

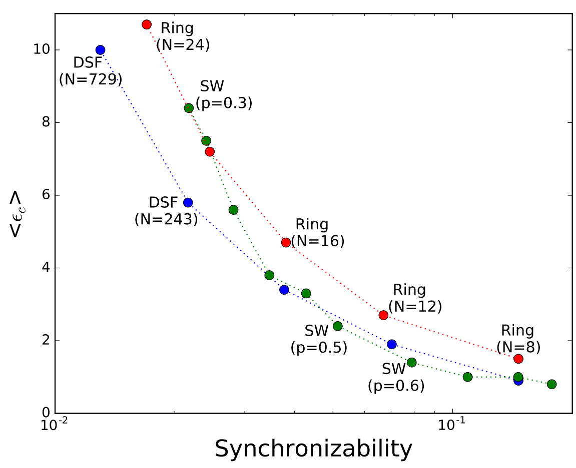


Fig. 7: Average critical coupling vs Synchronizability ($\frac{\lambda_2}{\lambda_N}$) of different networks: Deterministic Scale-Free (DSF) in blue, Small-World (SW) in green and Ring in red. Here the system size of the DSF network ranges from 81 to 729, the system size of the Ring varies from 8 to 24 and the system size of the SW network is $N = 100$, with varying fractions p of random links ($p \in [0.3 : 0.9]$). Each node has two neighbors in the Ring and the Small-World network.

earity in this $F(x)$ leads to a double well potential, and different γ introduces varying degrees of asymmetry in the potential.

Further we studied different networks of a piece-wise linear bi-stable system, that can be realized efficiently in electronic circuits [21], given by:

$$F(x) = -\alpha x + \beta\ g(x) \tag{3}$$

with the piecewise-linear function $g(x) = x$ when $x_l^* \leq x \leq x_u^*$, $g(x) = x_l^*$ when $x < x_l^*$ and $g(x) = x_u^*$ when $x > x_u^*$, where x_u^* and x_l^* are the upper and lower thresholds respectively.

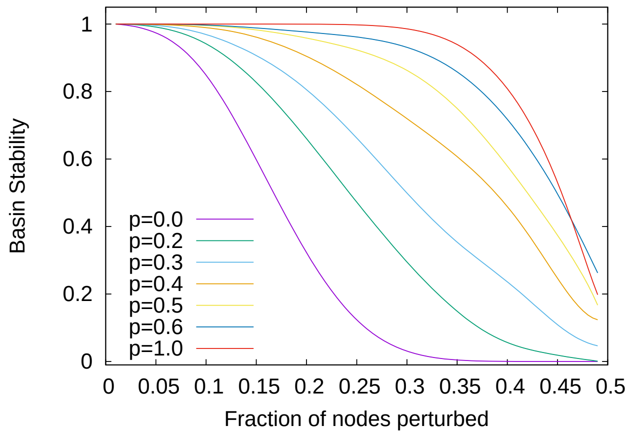
We simulated the coupled dynamics of these two bi-stable systems for different network topologies as well. We find that the qualitative trends in both these bi-stable systems is similar to that described above, indicating the generality of the central results presented here.

Conclusions:

In summary, we have explored the collective dynamics of bi-stable elements connected in different network topologies, ranging from rings and small-world networks, to random and deterministic scale-free networks. We have focussed on the correlation between network properties and global synchronization features. In particular, we have estimated the average critical coupling strength yielding transition to synchronization, a quantity indicating the ease of synchronization. Further we estimated the minimal number of nodes that need to be perturbed in order to lose synchronization, and this quantity indicates the robustness of the synchronized state.

Our central result is that, while networks properties can provide indicators of synchronization within a network class, they fail to provide consistent indicators across network classes (cf. Table 1 for a summary). For instance, we demonstrate that clustering coefficient is not a key feature in determining synchronization. This is clear through the similarity of synchronization properties in rings with significantly different clustering coefficients, and the striking difference in synchronization of a star network and a ring having the same clustering coefficient. Even characteristic path length, which is of paramount importance in determining synchronization, does not provide a one-to-one correspondence with synchronization properties across classes. Namely, while synchronization is significantly favoured in networks with low path lengths within a network class, the same characteristic path length in different types of networks yields very different $\langle \epsilon_c \rangle$ and f_c. Further, there appears to be no critical minimal characteristic path length that ensures synchronization from generic random initial states. All this suggests that local properties determined by the connection network does not provide a complete picture of global measures of synchronization.

Our observations then have potential applications. For instance, if one needs to achieve synchronization in a network of bi-stable elements, such as electronic circuits, constrained by certain connection properties, our analysis can guide the choice of preferred topology [10]. More importantly, in the context of the general understanding of dynamical networks, our observations suggest important caveats to correlating network features to global dynamical phenomena.



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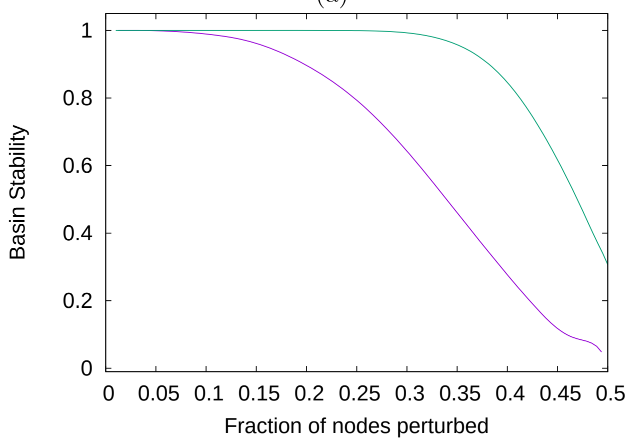

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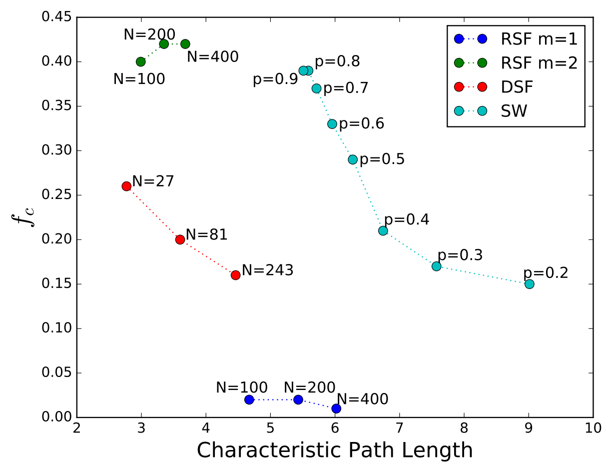

(c)

Fig. 8: Basin Stability (namely, fraction of perturbed systems that remain synchronized) vs fraction of nodes perturbed, in a system of N bi-stable elements coupled in (a) Small-World (SW) network for different values of fractions of random links; (b) Deterministic Scale-Free (DSF) network (violet) with $g = 4$, and Random Scale-Free (RSF) network (green) with $m = 2$ and $N = 100$. (c) Average critical fraction f_c of nodes that need to be perturbed in order to destroy the synchronized state, as a function of characteristic path length, for SW network of size $N = 100$ and varying fraction of random links p, RSF network of varying size N with $m = 1$ and $m = 2$ and DSF network of varying size N.

Easy to Synchronize	**Difficult to Synchronize**
Star	Deterministic Scale Free
Random Scale Free $m = 2$	Random Scale Free $m = 1$
Small world (high p)	Ring

Table 1: Ability of different network classes to synchronize, as reflected by global measures such as the average critical coupling strength $\langle \epsilon_c \rangle$.

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