

Noise Enhanced Activity in a Complex Network

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Abstract – We consider the influence of local noise on a generalized network of populations having positive and negative feedbacks. The population dynamics at the nodes is nonlinear, typically chaotic, and allows cessation of activity if the population falls below a threshold value. We investigate the global stability of this large interactive system, as indicated by the average number of nodal populations that manage to remain active. Our central result is that the probability of obtaining active nodes in this network is significantly enhanced under fluctuations. Further, we find a sharp transition in the number of active nodes as noise strength is varied, along with clearly evident scaling behaviour near the critical noise strength. Lastly, we also observe noise induced temporal coherence in the active sub-network, namely, there is an enhancement in synchrony among the nodes at an intermediate noise strength.

Introduction. – Recent years have witnessed a rapidly growing interest in network research primarily due to its wide applicability in modeling complex systems. From computer science and mathematics to physical, chemical, biological and social sciences [1–3], researchers are using ideas from network theory to gain understanding of large interactive dynamical systems. One of the classic approaches in this direction was the work by May where he analyzed the problem of the stability of a generalized ecosystem by considering a random network of different species. He proved the influential result that, as the network becomes sufficiently complex, the system gets unstable and the probability of survival becomes vanishingly small [4]. May assumed an a priori equilibrium state of the system and used local stability analysis to obtain his results. More recently, persistent activity, and the size of the asymptotic active sub-network, have been investigated as indicators of the global stability of a model ecosystem [5], and this criterion also yielded the same dependence on the complexity as May’s analysis.

In another direction, in recent years, several studies have been reported on the effect of noise in nonlinear systems. New counterintuitive phenomena emerging from the interplay of noise and nonlinearity, such as stochastic resonance [6], noise enhanced stability [7] and noise delayed extinction [8] have been observed. Motivated by these, we revisit the important problem global stability of complex systems to include the *role of noise*, since noise is ubiquitous in such systems. Our principal question is the fol-

lowing: do fluctuations allow a larger, or smaller, number of active populations to exist in a complex web? Namely, can stochastic influences actually yield a larger number of active nodes in the eco-network, on an average? Such questions are relevant, in general, to complex networks subject to fluctuations, where the evolution of the state of the nodes is nonlinear and allows cessation of activity.

Deterministic Network Model. – In population dynamics, a generic ecosystem can be modeled as a complex network, where each node represents a population and the interaction between the nodes are modeled through links which determine the strength and nature of the mutual interaction. Namely, one considers an ensemble of distributed populations which interact according to predefined rules, which account for various types of interactions, such as mutualism, predator-prey, competition, etc.

Specifically, here we consider N populations evolving in a complex web, where the nodal population dynamics is represented by a local nonlinear map f , and the interactions are given in most general terms by an interaction matrix or *Community Matrix* \mathbf{J} [9], whose elements J_{ij} represent the effect of species j on species i . The nature and strength of interaction between node i and node j is given by the sign and magnitude of the element (J_{ij}) in the interaction matrix \mathbf{J} . We consider the most general case where the coefficients can be asymmetric ($J_{ij} \neq J_{ji}$) and can be either positive or negative.

So the dynamical state of each node i ($i = 1, \dots, N$) at time (or generation) n is denoted by $x_i(n)$, which represents the scaled i^{th} population, and its time evolution under the prototypical Lotka-Volterra type interaction [5, 10, 11] is given by

$$x_i(n+1) = f \left[x_i(n) \left(1 + \sum_j J_{ij} x_j(n) \right) \right] \quad (1)$$

where f represents the local on-site dynamics, and connectivity matrix \mathbf{J} represents the positive and negative feedback amongst the nodal populations.

In this work we choose a prototypical map f , modeling population growth of species with non-overlapping generations, given by a modified Ricker (Exponential) Map as follows :

$$f(x) = \begin{cases} x e^{r(1-x)} & \text{if } x > x_{threshold}, \\ 0 & \text{otherwise,} \end{cases} \quad (2)$$

where growth rate r is the nonlinearity parameter yielding behaviour that ranges from fixed points and periodic cycles to chaos. Here $x_{threshold}$ is a threshold value, typically very small ($\ll 1$), giving the minimum population density necessary for any further activity to occur. Namely when the population density falls below this level, there is extinction [12]. Further note that very large population density is also detrimental, as the population dynamics maps large x back to the extinction zone, leading to inactivity in subsequent generations.

The connectivity matrix \mathbf{J} in Eqn. 1 is a matrix where an element is non-zero with probability C ($0 \leq C \leq 1$), i.e. C represents the connectivity of the system. The diagonal entries $J_{ii} = 0$, and this indicates that in the absence of interactions the local nonlinear map (Eq. (2)) completely determines the dynamical state of each node. The nonzero entries in the matrix are chosen from a normal distribution with mean 0 and variance σ^2 .

Also note that the local dynamical map belongs to the class of maps defined over the semi-infinite interval $[0, \infty)$, rather than a finite, bounded interval (such as the commonly used logistic map). This allows us to explore arbitrary distributions of coupling between nodes, unlike maps bounded in an interval, which are well behaved only for restrictive coupling schemes.

Now, consider the initial states of all the N nodes to be randomly distributed about $x = 1$. In the course of evolution of the network, if a nodal population below certain minimum sustainable population $x_{threshold}$, the node stops being active and subsequently has no interaction with the rest of the network. It was observed in [5, 11], that as a result of interactions, the number of active nodes (i.e. nodes i with $x_i > x_{threshold}$) decreased rapidly and eventually attained a steady state [13]. This can be rationalized by the fact that in the initial stages the population at each node undergoes strong fluctuations due to interactions with other nodes coupled to it, resulting in the

cessation of the activity of a large number of nodes [14]. Within a very short time, the effective number of interacting nodes decreases due to extinction and consequently the intensity of such fluctuations is also reduced. The number of active nodes in the asymptotic steady state, denoted as N_{active} , was found to be independent of N [5]. So, the evolution of this complex network leads to an active sub-network whose size does not scale with the size of the initial network, i.e. the asymptotic state is characterized by a macroscopic quantity that is non-extensive. The nonextensivity for the active subnetwork has significant implications, as it indicates that there exists a characteristic, rather small, size for a globally stable web of populations [15].

Stochastic Nodal Dynamics in the Network. –

Now, most real ecosystems cannot be modeled by deterministic networks alone, as noise is ubiquitous. The stochasticity in models are necessitated by the fact that habitats are typically open systems subject to external influences such as migrations. Furthermore there are fluctuations in population size due to random demographic events [16]. So it is of considerable interest to ascertain if the emergent active sub-networks found in deterministic systems are robust against small perturbation or noise. *Further, it would be very interesting to ascertain if noise aids, or hinders, the average activity of the web.*

In order to investigate these questions we study the effects of stochasticity in the network above, by considering the evolution equation (eq. (1)) under a random *additive noise* $\xi(t)$. Here $\xi(t)$ is a gaussian white noise with zero mean and correlation function given by $\langle \xi_i(t) \xi_j(t') \rangle = \eta \delta(t-t') \delta_{ij}$ ($i, j = 1, 2, \dots, N$), where η governs the strength of noise. Therefore, the evolution of the local nodal populations in the network is now governed by the equations:

$$x_i(n+1) = f \left[x_i(n) \left(1 + \sum_j J_{ij} x_j(n) \right) \right] + \xi_i(t) \quad (3)$$

In order to understand the macroscopic or collective response of the system to noise at the microscopic level, we calculate the the number of active nodes (i.e. nodes with $x > x_{threshold}$) in the network averaged over a long time, and further averaged over different realizations of the system. We denote this time and ensemble average of the number of active nodes as $\langle N_{active} \rangle$. In this work, we calculate this quantity for networks with varying local dynamics and connectivities, under a wide range of noise strengths.

Effect of Nodal Noise on the Activity of the Network:

Fig. 1 shows how the average number of active nodes varies as the strength of noise at the nodes increases. It is clearly evident that there is very sharp transition at noise strengths $\sim \eta_c$. Below this critical noise strength

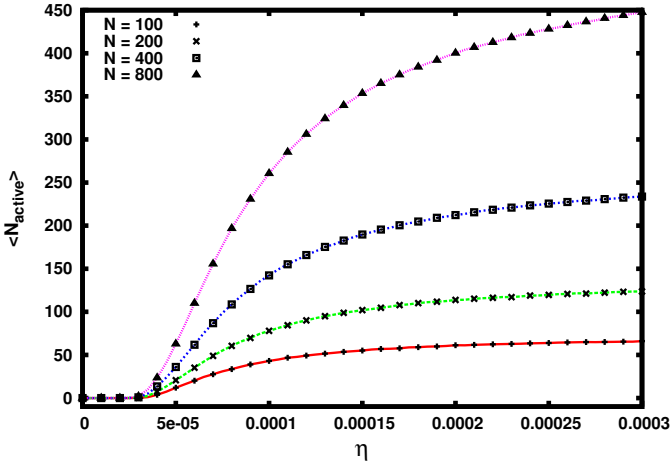


Fig. 1: Variation in the average number of active nodes $\langle N_{active} \rangle$ (see text), with respect to increasing noise strength, indicating a sharp rise in the average number of active nodes after a critical noise strength, η_c . System sizes are 100, 200, 400 and 800 and threshold population size (cf. Eqn. 2), $x_{threshold} = 0.0001$. Here the network connectedness, $C = 1$, mean interaction strength is 0 and standard deviation, $\sigma = 0.1$.

(i.e. $\eta < \eta_c$), noise has no discernable effect on network activity, while above η_c there is a *very significant jump in the number of active nodes in the network*. In the noise regime much larger than η_c , the system settles to a non-equilibrium steady state where a constant number of active nodes is maintained on an average. Namely, at high enough noise strengths $\eta \gg \eta_c$, the mean number of active nodes saturates to an asymptotic value. So clearly there is a *noise induced transition*, from a system where the emergent active sub-network is very small to a system with a large active sub-network.

In order to account for the effect of network size on the nature of the transition, we have done the finite size scaling for a range of system sizes. As evident from Fig 2, the transition curves for all system sizes collapse to a single curve, allowing us to calculate the value of the critical noise strength η_c in the thermodynamic limit. The scaling function near the critical region is given by:

$$P \sim N^{\alpha/\beta} \Theta(N^{-1/\beta}, (\eta - \eta_c)) \quad (4)$$

where, $\eta_c = 0.00003 \pm 0.000001$, $\alpha/\beta = 0.1 \pm 0.02$ and $1/\beta = 0.94 \pm 0.005$. Note that the critical noise strength η_c is significantly smaller than the threshold of survival $x_{threshold}$ of the individual populations.

Also notice that the number of active nodes saturates to a steady mean size for high noise strengths. However, in contrast to the deterministic network, we now observe that the size of the asymptotic active sub-network depends on N . Namely, under reasonable noise, the network settles down to an active sub-network that scales with the size of the original network, unlike the deterministic case where the network settled down rapidly to an active sub-network

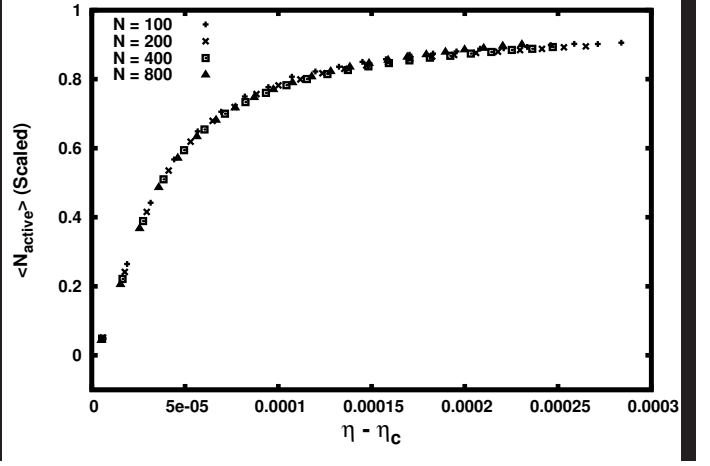


Fig. 2: Scaling function for which the relation between the average number of active nodes, $\langle N_{active} \rangle$, and noise strength collapse on a single curve for all system sizes, N . (See Eqn. 4.)

of small characteristic size, for all system sizes.

To further understand the behavior of the population dynamics at a node under the influence of noise, we analyze this problem as a system exploring a state space with a zone of inactivity, bounded by $x_{threshold}$. Now in the deterministic model (Eqn. (1)), when a population enters the inactive zone, it cannot leave the region. So the extinction threshold density $x_{threshold}$ acts as an absorbing boundary condition, leading to large-scale extinctions and very small persisting active sub-networks. However in the presence of noise, populations can be pushed out of this zone, i.e. noise allows populations to escape the inactive region and revive to become active again.

One can calculate the fraction of time spent by the system in the inactive region ($[0 : x_{threshold}]$), F_τ , in the deterministic case, and in the presence, of noise. Representative results are displayed in Fig. 3 and these suggest that as noise strength varies there is a very sharp fall in F_τ . The inset shows the variation of F_τ with growth parameter r of the local dynamics, in the presence and in the absence of noise. The results suggest that the noise-free system spends most of the time in the inactive zone, as once the node goes inactive, it remains inactive due to the nature of the predator-prey type interaction. On the other hand, when there are external fluctuations, the system spends a significantly less amount of time in the inactive region, which results in much enhanced average activity.

Also note that in the particular example shown in inset of Fig. 3, with noise strength, $\eta = 0.002$, the fraction of time spent in the extinction zone is about 30 %, for growth rate parameter $r = 4$ in the local population dynamics. This implies that about 70 % of the nodes in the network are active, on an average, in the saturation limit. This estimate agrees very well, quantitatively, with results from simulations shown in Fig. 1.

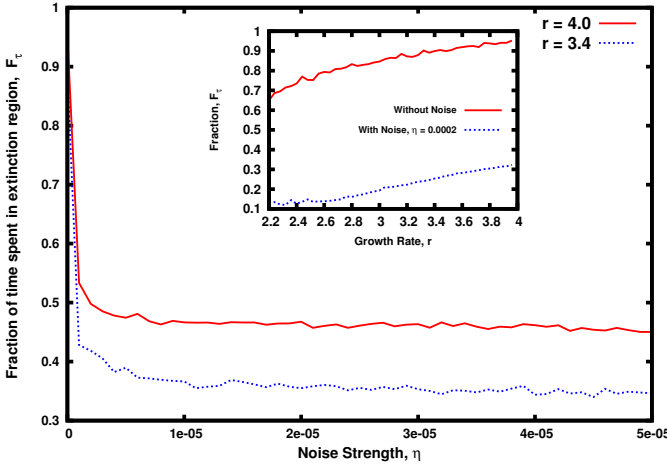


Fig. 3: Variation of the fraction of time spent by the system in the inactive region $[0 : x_{threshold}]$ with respect to noise strength η (main) and with respect to growth rate r in the local population dynamics given by Eqn. 2 (inset). Here the network consists of 100 nodes, subjected to gaussian white noise, and we leave a transience of 10000 steps.

Influence of a sub-set of noisy nodes on network activity. — Now, all the nodes in the network are not necessarily influenced by noise at all instants of time. In fact, at any given time there may be only a certain fraction of noisy nodes, and this fraction too may change with time depending on global influences. In this section, we consider a network where a different set of random nodes are influenced by noise at every instant of time, with the total number of noisy nodes being fixed. That is, dynamically, a certain fraction of the network is subject to fluctuations.

Fig. 4 shows the transition from a small number of active nodes to a high number of active nodes, as the noise strength increases, for a system with different number of nodes subject to fluctuations. It is clear that the average number of active nodes saturates to a high mean value, at noise strengths larger than η_c , with the maximum $\langle N_{active} \rangle$ depending on the fraction of noisy nodes in the system.

The interesting question here is the following: *What fraction of the network needs to be subject to noise in order to obtain the noise induced transition to high activity?* It is evident from Fig. 4, that even when just a *single* node in the network is subject to noise, the activity of the network is significantly enhanced.

The dependence of the asymptotic $\langle N_{active} \rangle$ on the fraction of noisy nodes in the network is displayed in Fig. 5, for representative cases. It is clear from the figure that the asymptotic value of the average number of active nodes in the network rises with the increase in number of noisy nodes.

In order to find the functional dependence of asymptotic value of $\langle N_{active} \rangle$ on fraction of noisy nodes, F_n , we study the behavior for different network sizes as shown in Fig.

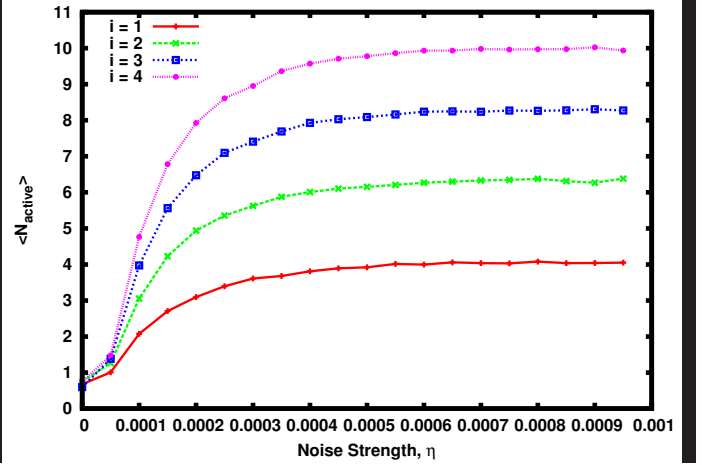


Fig. 4: Variation of the time averaged number of active nodes in the network of size 100, with respect to noise strength, with the number of nodes subject to noise being: (a) 1 (b) 2 (c) 3 and (d) 4. Note that $\langle N_{active} \rangle$ saturates to some maximum asymptotic value with increasing noise strength.

6. It is evident that the asymptotic (maximum) value of $\langle N_{active} \rangle$ varies as :

$$\langle N_{active} \rangle \sim F_n^\alpha \quad (5)$$

where, $\alpha = 0.675$.

Synchronization. — We now investigate the nature of the *local population dynamics* under the influence of noise. In particular, we study how noise affects the *synchrony of the nodal dynamics*. In order to quantify the degree of synchronization, we compute an average error function as the synchronization order parameter, Z_{sync} , defined as the mean square deviation of the instantaneous states of the nodes

$$Z = \frac{1}{N} \sum_{i=1}^N \{[x_i(t) - \langle x(t) \rangle]^2\} \quad (6)$$

where, $\langle x(t) \rangle$ is the space average at time t . This quantity Z , averaged over time n and over different initial conditions, is denoted as Z_{sync} .

We observe an interesting feature in the behavior of synchronization as noise strength is varied: For a particular intermediate value of noise strength, there is a significant fall in the synchronization error. This increase in synchrony is more pronounced for larger network sizes. So clearly, there is *noise induced temporal coherence* in the active sub-network, at optimal noise strengths [17]. Note that, quite unlike the scenario where synchronization is a result of the constituents being subject to common noise, here the nodes are driven by uncorrelated noise that varies from node to node.

Note: It may appear that synchronization error diverges as $\eta \rightarrow 0$, but this comes from the definition of Z_{sync} (Eq.6) as number of active nodes tend to zero when $\eta \rightarrow 0$.

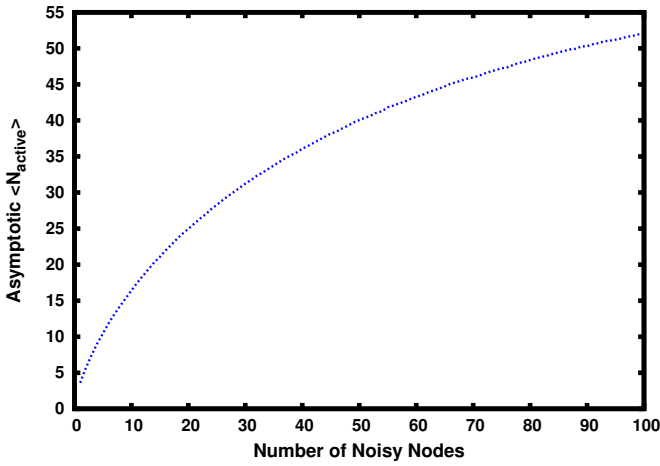


Fig. 5: Asymptotic $\langle N_{active} \rangle$ in the network of size 100 (see Fig. 4), as a function of the number of nodes subject to gaussian white noise where $x_{threshold} = 0.0001$.

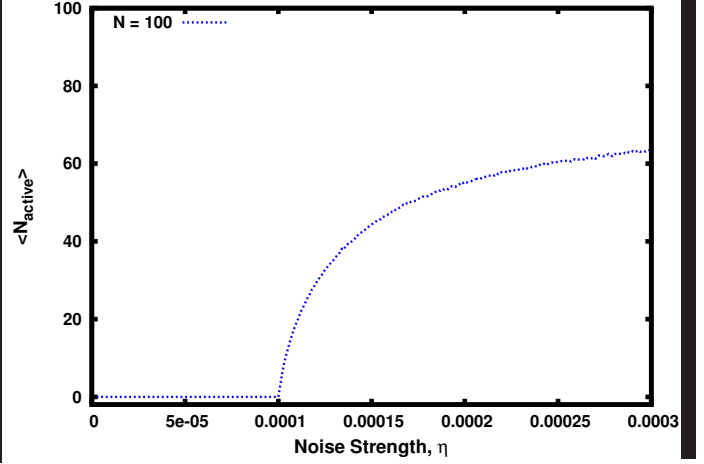


Fig. 8: Average number of active nodes as a function of η (see text) for the case of nodes subjected to uniformly distributed noise. Here the network size $N = 100$ and $x_{threshold} = 0.0001$.

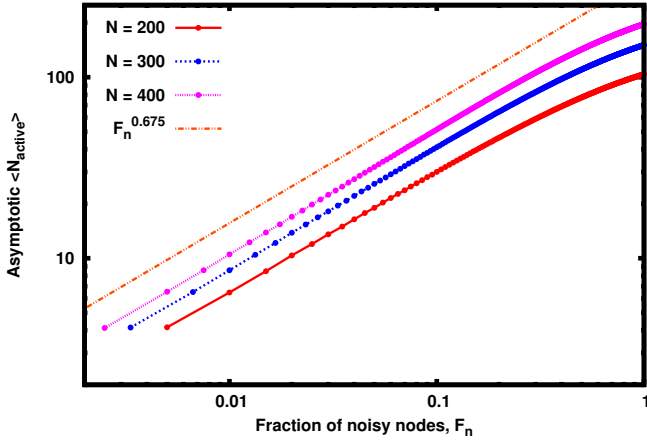


Fig. 6: Asymptotic $\langle N_{active} \rangle$ for different network of sizes, as a function of the fraction of nodes subject to gaussian white noise where $x_{threshold} = 0.0001$.

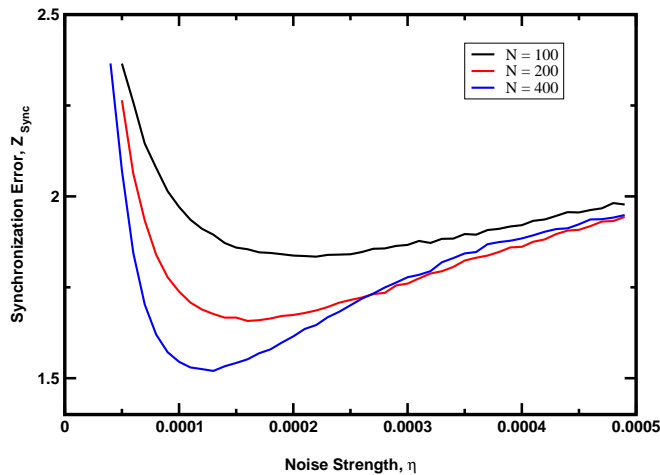


Fig. 7: Synchronization error as a function of noise strength, for three different system sizes. Here, gaussian white noise is applied.

Generality of our results. — In order to check the generality of our results we have investigated different kinds of noise and different local dynamics. Specifically, we demonstrated the increase of active nodes in the system under *uniform noise*, namely the noise at the nodes was drawn from a uniform distribution in the interval $[-\eta : \eta]$, where $\eta > 0$.

Again we find that the number of active nodes in the network increases sharply after a critical noise strength. As evident from Fig. 8, the critical noise strength for the case of uniform noise bounded in the interval $[-\eta : \eta]$ is equal to the extinction threshold, $x_{threshold}$. This is indeed expected, as the minimum perturbation to push a population out of the inactive zone, i.e. the minimum noise required to revive, is $x_{threshold}$.

We have also simulated the system for modified logistic growth at the nodes:

$$f(x) = \begin{cases} rx(1-x) & \text{if } x > x_{threshold}, \\ 0 & \text{otherwise,} \end{cases} \quad (7)$$

The behavior that emerges in this network, under a wide range of growth rates r , is qualitatively similar to that in the network of Ricker maps discussed above, as evident from Fig. 9.

Further, we investigated heterogeneous networks, with a range of growth rates at the nodal level. The central conclusion, namely the enhancement of activity in networks under stochastic influence, holds in such systems as well.

Analysis. — In this section we attempt to gauge the underlying reasons determining the nature of the noise-induced transition. Specifically, we consider the case of logistic growth as shown in Fig. 8 where the system evolves according to Eqn.[3,7]. Now, at noise strengths $\eta > x_{threshold}$, there will be some nodes that will get pushed out of the extinction region due to noise, while some nodes will get mapped back to the extinction region

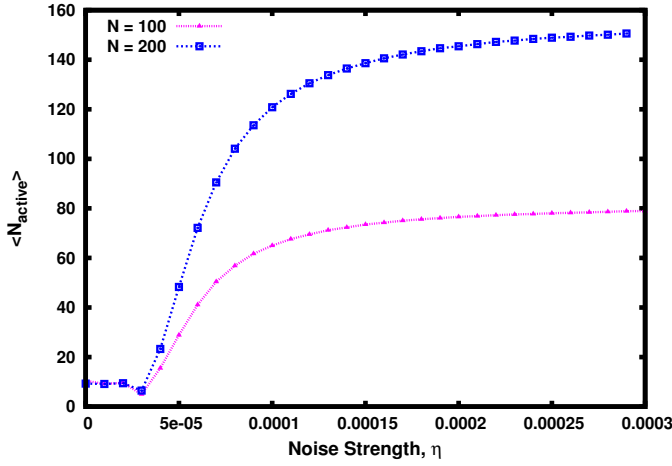


Fig. 9: Average number of active nodes as a function of η (see text) for the case of nodes subjected to gaussian white noise. Here we have a modified logistic map with $r = 4$ (Eqn. 7) at the nodal level and the network size $N = 100$, with $x_{threshold} = 0.0001$.

at the subsequent update. So here we will try to get a broad idea of the number of active nodes in the system at some particular noise strength η . Now the fraction of inactive nodes is given by:

$$\begin{aligned} \frac{N_{inactive}}{N} &\sim P(\eta < x_{threshold}) \int_0^{x_{threshold}} \rho^*(x) dx \\ &+ P(\eta < -x_{threshold}) \int_{x_{threshold}}^{\eta} \rho^*(x) dx \\ &+ \int_{x'}^1 \rho^*(x) dx \end{aligned} \quad (8)$$

where, ρ^* is the invariant measure of the coupled system and $P(\eta < x_{threshold})$ is the probability of noise η being less than the threshold population density. The first term in Eqn. 8 corresponds to the fraction of nodes unable to get out of the extinction region due to sub-threshold noise strength. The second term corresponds to the fraction of nodes which were initially outside the extinction region, but got absorbed into the extinction region on application of noise. The last term represents the fraction of nodes that map onto the extinction region due to the nature of the local dynamics (x' represents the inverse image of the logistic map corresponding to $x_{threshold}$). Note that noise is drawn from a uniform distribution bounded in the interval $[-\eta : \eta]$ here. The probabilities for the noise input to lie in different regions is calculated as below:

$$\begin{aligned} P(\eta < x_{threshold}) &= \frac{\eta + x_{threshold}}{2\eta}, \\ P(\eta > x_{threshold}) &= \frac{\eta - x_{threshold}}{2\eta}, \\ P(\eta < -x_{threshold}) &= \frac{x_{threshold} - \eta}{2\eta} \end{aligned} \quad (9)$$

Now the calculation of ρ^* for this high-dimensional coupled stochastic system is an intractable task, and does

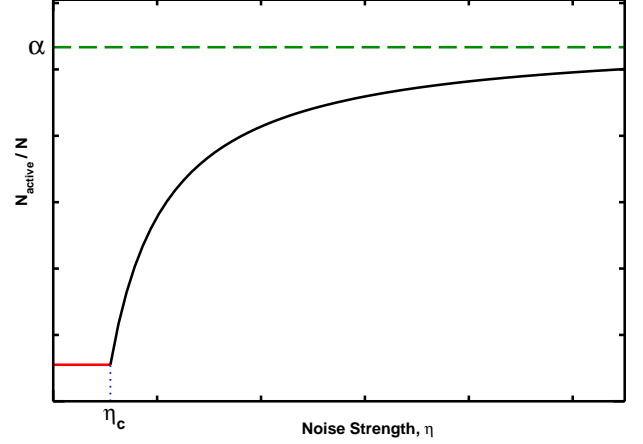


Fig. 10: Figure shows the plot of Eqn. 12 depicting the behavior of fraction of active nodes with respect to noise strength. Parameters α, β are taken to be 0.1, 0.0009 respectively. Region $\eta < \eta_c$ represents the region where noise has no discernable effect on the activity of the network. Here the noise is drawn from an uniform distribution.

not lend itself to a closed form solutions. So here we approach this problem from a different perspective. Since we are more interested in the qualitative behavior of the system, we assume these integrals to be finite constants whose values are related to values obtained in single map [18] in some appropriate limit. These constants are defined below as:

$$\begin{aligned} \int_0^{x_{threshold}} \rho^*(x) dx &= C_1, \\ \int_{x_{threshold}}^{\eta} \rho^*(x) dx &= C_2, \\ \int_{x'}^1 \rho^*(x) dx &= C_3 \end{aligned} \quad (10)$$

Using Eqns. 8,9 & 10, we get $N_{active} = N - N_{inactive}$ as,

$$\frac{N_{active}(\eta)}{N} = 1 - \left(\frac{C_1 + C_2}{2} \right) \frac{x_{threshold}}{\eta} + \frac{C_2 - C_1}{2} + C_3 \quad (11)$$

which can be re-written in a simple form as:

$$\frac{N_{active}(\eta)}{N} = \alpha - \frac{\beta}{\eta} \quad (12)$$

where, $\alpha = 1 + \frac{C_2 - C_1}{2} + C_3$ and $\beta = \frac{(C_1 + C_2)x_{threshold}}{2}$.

It is evident that our analysis (Fig. 10) matches well with the results from numerical simulations (Fig. 8).

Multiplicative Noise. – Lastly, we also studied the evolution of this complex network under multiplicative noise :

$$x_i(n+1) = f \left[x_i(n) \left(1 + \sum_j J_{ij} x_j(n) \right) + \xi_i(t) \right] \quad (13)$$

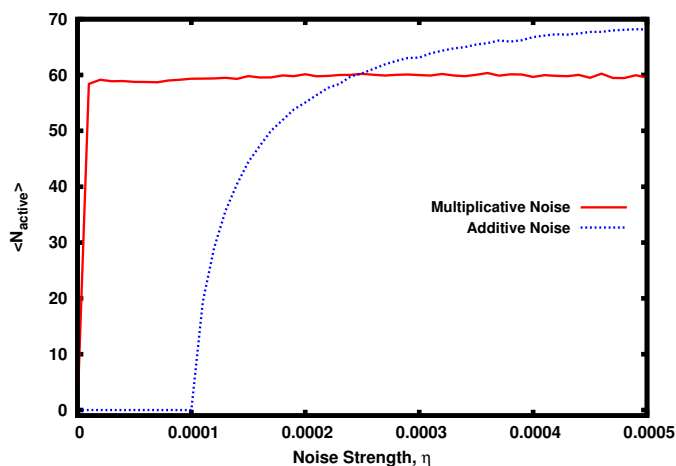


Fig. 11: Average number of active nodes as a function of noise strength. The two curves represent results for multiplicative and additive noise. Network size is 100, activity threshold is $x_{\text{threshold}} = 0.0001$. Noise here is drawn from a uniform distribution in the interval $[-\eta : \eta]$.

Here $\xi(t)$ is a uniform noise in an interval $[-\eta : \eta]$ with zero mean and correlation function is given by $\langle \xi_i(t) \xi_j(t') \rangle = \eta \delta(t - t') \delta_{ij}$ ($i, j = 1, 2, \dots, N$), where η governs the strength of noise.

While in the case of additive noise with uniform distribution (cf. Fig. 8) the critical noise needed for enhancing activity in the system was exactly the extinction threshold, for the case of multiplicative noise we observe that the critical noise is far below $x_{\text{threshold}}$. In fact η_c tends to zero, as evident from Fig. 11, indicating the extreme sensitivity of network activity to multiplicative noise [19].

Conclusions. — In summary, we have investigated the role of stochasticity on the global stability of complex networks. In this study, very general networks, incorporating positive and negative interactions of the generalized Lotka-Volterra type, were considered. The population dynamics at the nodal level was typically chaotic, and allowed cessation of activity if the population density fell below a certain threshold value.

Our central result is the following: the probability of obtaining active nodes in the network is *significantly enhanced under noise*. Further, we find a sharp transition in the number of active nodes as noise strength is varied, along with clearly evident scaling behaviour near the critical noise strength.

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