**THE EUROPEAN
PHYSICAL JOURNAL B**

Regular Article

Noise enhanced activity in a complex network

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Abstract. We consider the influence of local noise on a generalized network of populations having positive and negative feedbacks. The population dynamics at the nodes is nonlinear, typically chaotic, and allows cessation of activity if the population falls below a threshold value. We investigate the global stability of this large interactive system, as indicated by the average number of nodal populations that manage to remain active. Our central result is that the probability of obtaining active nodes in this network is significantly enhanced under fluctuations. Further, we find a sharp transition in the number of active nodes as noise strength is varied, along with clearly evident scaling behaviour near the critical noise strength. Lastly, we also observe noise induced temporal coherence in the active sub-network, namely, there is an enhancement in synchrony among the nodes at an intermediate noise strength.

1 Introduction

Recent years have witnessed a rapidly growing interest in network research primarily due to its wide applicability in modeling complex systems. From computer science and mathematics to physical, chemical, biological and social sciences [1–3], researchers are using ideas from network theory to gain understanding of large interactive dynamical systems. One of the classic approaches in this direction was the work by physicist turned zoologist, Robert May, where he analyzed the problem of the stability of a generalized ecosystem by considering a random network of different species. He proved the influential result that, as the network becomes sufficiently complex, the system gets unstable and the probability of survival becomes vanishingly small [4]. May assumed an a priori equilibrium state of the system and used local stability analysis to obtain his results. More recently, persistent activity, and the size of the asymptotic active sub-network, have been investigated as indicators of the global stability of a model ecosystem [5], and this criterion also yielded the same dependence on the complexity as May's analysis.

In another direction, in recent years, several studies have been reported on the effect of noise in nonlinear systems. New counterintuitive phenomena emerging from the interplay of noise and nonlinearity, such as stochastic resonance [6], noise enhanced stability [7] and noise delayed extinction [8] have been observed. Motivated by these, we revisit the important problem of global stability of complex systems to include the *role of noise*, since noise is ubiquitous in such systems. Our principal question

is the following: do fluctuations allow a larger, or smaller, number of active populations to exist in a complex web? Namely, can stochastic influences actually yield a larger number of active nodes in the eco-network, on an average? Such questions are relevant, in general, to complex networks subject to fluctuations, where the evolution of the state of the nodes is nonlinear and allows cessation of activity.

2 Deterministic network model

In population dynamics, a generic ecosystem can be modeled as a complex network, where each node represents a population and the interaction between the nodes are modeled through links which determine the strength and nature of the mutual interaction. Namely, one considers an ensemble of distributed populations which interact according to predefined rules, which account for various types of interactions, such as mutualism, predator-prey, competition, etc.

Specifically, here we consider N populations evolving in a complex web, where the nodal population dynamics is represented by a local nonlinear map f, and the interactions are given in most general terms by an interaction matrix or *Community Matrix* **J** [9], whose elements J_{ij} represent the effect of species j on species i. The nature and strength of interaction between node i and node j is given by the sign and magnitude of the element (J_{ij}) in the interaction matrix **J**. We consider the most general case where the coefficients can be asymmetric ($J_{ij} \neq J_{ji}$) and can be either positive or negative.

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So the dynamical state of each node i ($i = 1, \ldots, N$) at time (or generation) n is denoted by $x_i(n)$, which represents the scaled ith population, and its time evolution under the prototypical Lotka-Volterra type interaction [5,10,11] is given by

$$x_i(n+1) = f\left[x_i(n)\left(1 + \sum_j J_{ij} x_j(n)\right)\right], \quad (1)$$

where f represents the local on-site dynamics, and connectivity matrix $\mathbf{J}$ represents the positive and negative feedback amongst the nodal populations.

In this work we choose a prototypical map f, modeling population growth of species with non-overlapping generations, given by a modified Ricker (exponential) map as follows:

$$f(x) = \begin{cases} xe^{r(1-x)} & \text{if } x > x_{threshold}, \\ 0 & \text{otherwise,} \end{cases} \quad (2)$$

where growth rate r is the nonlinearity parameter yielding behaviour that ranges from fixed points and periodic cycles to chaos. Here $x_{thresold}$ is a threshold value, typically very small ($\ll 1$), giving the minimum population density necessary for any further activity to occur. Such a survival threshold is well documented in ecology literature and goes by the name of *Allee Effect* [12]. Namely when the population density falls below this level, there is extinction[1]. Further note that very large population density is also detrimental, as the population dynamics maps large x back to the extinction zone, leading to inactivity in subsequent generations.

So, while the zero of a Ricker map is highly unstable and acts as a repellor (with $|f'| > 1$), the modified Ricker map has a small "extinction" zone that is stable, i.e. once the trajectory enters this region it gets trapped there forever, as all values in that zone are mapped to zero (with $f' = 0$ there). So an inactive node will stay that way, until it gets sufficient additive impulse to be pushed outside this region.

The connectivity matrix $\mathbf{J}$ in equation (1) is a matrix where an element is non-zero with probability C ($0 \leq C \leq 1$), i.e. C represents the connectivity of the system. The diagonal entries $J_{ii} = 0$, and this indicates that in the absence of interactions the local nonlinear map (Eq. (2)) completely determines the dynamical state of each node. The nonzero entries in the matrix are chosen from a normal distribution with mean 0 and variance σ^2.

Also note that the local dynamical map belongs to the class of maps defined over the semi-infinite interval $[0, \infty)$, rather than a finite, bounded interval (such as the commonly used logistic map). This allows us to explore arbitrary distributions of coupling between nodes, unlike maps bounded in an interval, which are well behaved only for restrictive coupling schemes.

Now, consider the initial states of all the N nodes to be randomly distributed about $x = 1$. In the course of evolution of the network, if a nodal population below certain minimum sustainable population $x_{threshold}$, the node stops being active and subsequently has no interaction with the rest of the network. It was observed in references [5,11], that as a result of interactions, the number of active nodes (i.e. nodes i with $x_i > x_{threshold}$) decreased rapidly and eventually attained a steady state [13][2]. The number of active nodes in the asymptotic steady state, denoted as N_{active}, was found to be typically small and independent of N [5]. So the evolution of this complex network leads to an active sub-network whose size did not scale with the size of the initial network, i.e. the asymptotic state was characterized by a macroscopic quantity that is non-extensive. The nonextensivity for the active subnetwork has significant implications, as it indicates that there exists a characteristic, rather small, size for a globally stable web of populations [14].

3 Stochastic nodal dynamics in the network

Now, most real ecosystems cannot be modeled by deterministic networks alone, as noise is ubiquitous. The stochasticity in models is necessitated by the fact that habitats are typically open systems subject to external influences such as migrations. Furthermore there are always fluctuations in the population size due to random demographic events [15], and these could be well-modelled by adding a stochastic term to the deterministic evolution dynamics given by equation (2). Generally then, it is of considerable interest to ascertain if the emergent active sub-networks found in deterministic systems are robust against small perturbation or noise. *In particular, it would be very interesting to ascertain if noise aids, or hinders, the average activity of the web.*

In order to investigate these questions we study the effects of stochasticity in the network above, by considering the evolution equation (Eq. (1)) under a random *additive noise* $\xi(t)$. Here $\xi(t)$ is a Gaussian white noise with zero mean and correlation function given by $\langle \xi_i(t)\xi_j(t') \rangle = \eta\delta(t - t')\delta_{ij}$ ($i, j = 1, 2, \ldots, N$), where η governs the strength of noise. Therefore, the evolution of the local nodal populations in the network is now governed by the equations:

$$x_i(n+1) = f\left[x_i(n)\left(1 + \sum_j J_{ij} x_j(n)\right)\right] + \xi_i(t). \quad (3)$$

In order to understand the macroscopic or collective response of the system to noise at the microscopic level, we calculate the number of active nodes (i.e. nodes with $x > x_{threshold}$) in the network, averaged over a long time,

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[1] In addition to the biological motivation, setting this threshold also helps in making the numerical results more robust to effects of finite precision.

[2] The steady state value of $\langle N_{active} \rangle$ depends on the choice of $x_{threshold}$. If $x_{threshold}$ is sufficiently high there would be mass extinction and $\langle N_{active} \rangle \rightarrow 0$ with time.
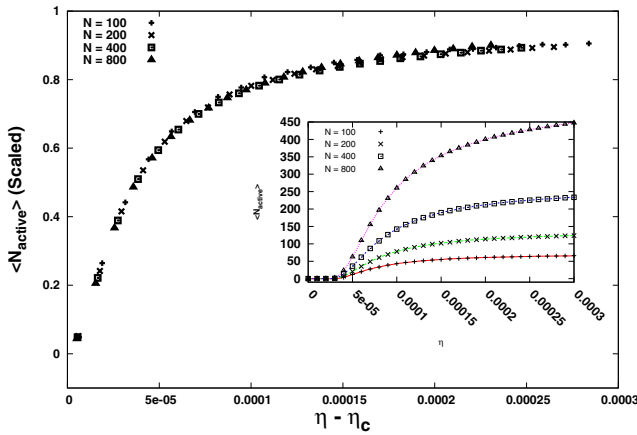
Fig. 1. Scaling of the average number of active nodes $\langle N_{active} \rangle$ (see text), with respect to increasing noise strength, indicating a sharp rise in the average number of active nodes after a critical noise strength, η_c. Inset shows the above variation for different system sizes namely, $N = 100, 200, 400$ and 800. Here $x_{threshold} = 0.0001$, $C = 1$ and $\sigma = 0.1$.

and further averaged over different realizations of the system (with the initial population densities randomly distributed in the interval $[0.5 : 1.5]$). We denote this time and ensemble average of the number of active nodes as $\langle N_{active} \rangle$. In this work, we calculate this quantity for networks with varying local dynamics, under a wide range of noise strengths.

Effect of nodal noise on the activity of the network

Inset of Figure 1 shows how the average number of active nodes varies as the strength of noise at the nodes increases. It is clearly evident that there is very sharp transition at noise strengths $\sim \eta_c$. Below this critical noise strength (i.e. $\eta < \eta_c$), noise has no discernable effect on network activity, while above η_c there is a *very significant jump in the number of active nodes in the network*. In the noise regime much larger than η_c, the system settles to a non-equilibrium steady state where a constant number of active nodes is maintained on an average. Namely, at high enough noise strengths $\eta \gg \eta_c$, the mean number of active nodes saturates to an asymptotic value. So clearly there is a *noise induced transition*, from a system where the emergent active sub-network is very small to a system with a large active sub-network. That is, the consequence of noise is not a small change in the network activity; rather, stochastic influences, even when reasonably small, make a very large difference in the activity of the network of populations.

In order to account for the effect of network size on the nature of the transition, we have done the finite size scaling for a range of system sizes. As evident from Figure 1, the transition curves for all system sizes collapse to a single curve, allowing us to calculate the value of the critical noise strength η_c in the thermodynamic limit. The scaling function near the critical region is given by:

$$P \sim N^{\alpha/\beta} \Theta \left(N^{-1/\beta}, (\eta - \eta_c) \right), \qquad (4)$$

where, $\eta_c = 0.00003 \pm 0.000001$, $\alpha/\beta = 0.1 \pm 0.02$ and $1/\beta = 0.94 \pm 0.005$. Note that the critical noise strength η_c is significantly smaller than the threshold of survival $x_{threshold}$ of the individual populations.

Also notice that the number of active nodes saturates to a steady mean size for high noise strengths. However, in contrast to the deterministic network, we now observe that the size of the asymptotic active sub-network depends on N. Namely, under reasonable noise, the network settles down to an active sub-network that scales with the size of the original network, unlike the deterministic case where the network settled down rapidly to an active sub-network of small characteristic size, for all system sizes.

To further understand the behavior of the population dynamics at a node under the influence of noise, we analyze this problem as a system exploring a state space with a zone of inactivity, bounded by $x_{threshold}$. Now in the deterministic model (Eq. (1)), when a population enters the inactive zone, it cannot leave the region. So the extinction threshold density $x_{threshold}$ acts as an absorbing boundary condition, leading to large-scale extinctions and very small persisting active sub-networks. However in the presence of noise, populations can be pushed out of this zone, i.e. noise allows populations to escape the inactive region and revive to become active again. Note that this apparent reversibility of extinction under the influence of noise is reminiscent of recolonization of empty patches through migration of species.

One can calculate the fraction of time spent by the system in the inactive region $([0 : x_{threshold}])$, F_τ, in the deterministic case, and in the presence, of noise. Representative results are displayed in Figure 2 and these suggest that as noise strength varies there is a very sharp fall in F_τ. The inset shows the variation of F_τ with growth parameter r of the local dynamics, in the presence and in the absence of noise. The results suggest that the noise-free system spends most of the time in the inactive zone, as once the node goes inactive, it remains inactive due to the nature of the predator-prey type interaction. On the other hand, when there are external fluctuations, the system spends a significantly less amount of time in the inactive region, which results in much enhanced average activity.

Also note that in the particular example shown in inset of Figure 2, with noise strength, $\eta = 0.002$, the fraction of time spent in the extinction zone is about 30%, for growth rate parameter $r = 4$ in the local population dynamics. This implies that about 70% of the nodes in the network are active, on an average, in the saturation limit. This estimate agrees very well, quantitatively, with results from simulations shown in Figure 1.

4 Influence of a sub-set of noisy nodes on network activity

Now, all the nodes in the network are not necessarily influenced by noise at all instants of time. In fact, at any given time there may be only a certain fraction of noisy nodes,
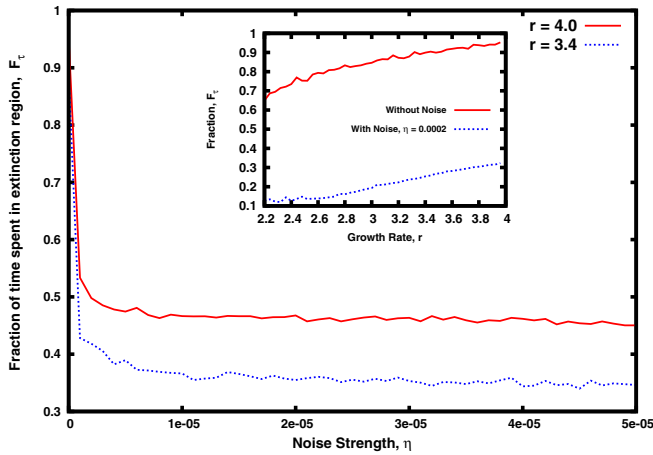
Fig. 2. Variation of the fraction of time spent by the system in the inactive region $[0 : x_{threshold}]$ with respect to noise strength η (main) and with respect to growth rate r in the local population dynamics given by equation (2) (inset). Here the network consists of 100 nodes, subjected to Gaussian white noise, and we leave a transience of 10 000 steps.
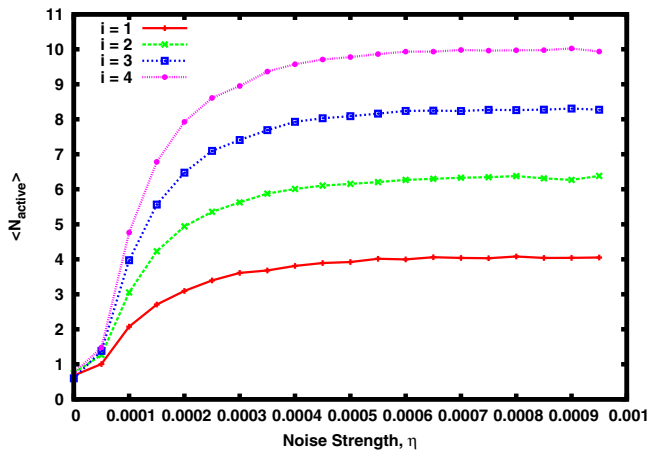

Fig. 3. Variation of the time averaged number of active nodes in the network of size 100, with respect to noise strength, with the number of nodes subject to noise being: (a) 1, (b) 2, (c) 3, and (d) 4. Note that $\langle N_{active} \rangle$ saturates to some maximum asymptotic value with increasing noise strength.

and this fraction too may change with time depending on global influences. In this section, we consider a network where a different set of random nodes are influenced by noise at every instant of time, with the total number of noisy nodes being fixed. That is, dynamically, a certain fraction of the network is subject to fluctuations.

Figure 3 shows the transition from a small number of active nodes to a high number of active nodes, as the noise strength increases, for a system with different number of nodes subject to fluctuations. It is clear that the average number of active nodes saturates to a high mean value, at noise strengths larger than η_c, with the maximum $\langle N_{active} \rangle$ depending on the fraction of noisy nodes in the system.
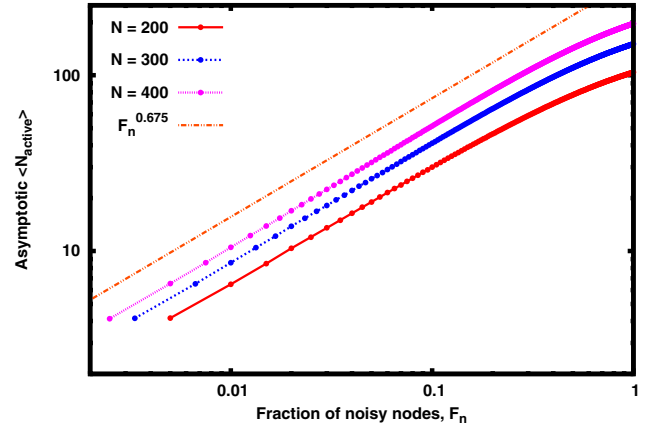

Fig. 4. Asymptotic $\langle N_{active} \rangle$ for different network of sizes, as a function of the fraction of nodes subject to Gaussian white noise where $x_{threshold} = 0.0001$.

The interesting question here is the following: *what fraction of the network needs to be subject to noise in order to obtain the noise induced transition to high activity?* It is evident from Figure 3, that even when just a *single* node in the network is subject to noise, the activity of the network is significantly enhanced.

In order to find the functional dependence of asymptotic value of $\langle N_{active} \rangle$ on the fraction of noisy nodes, F_n, we study the behavior for different network sizes as shown in Figure 4. It is evident that the asymptotic (maximum) value of $\langle N_{active} \rangle$ varies as:

$$\langle N_{active} \rangle \sim F_n^\nu \qquad (5)$$

where, $\nu = 0.675$.

One can argue that ν will be close to 1 by the following reasoning: the number of active nodes is directly proportional to the number of inactive nodes that are influenced by noise large enough to map them out of the extinction zone. On an average at any given instant a fraction F_n of nodes are influenced by noise, and so one expects $\langle N_{active} \rangle \sim F_n$ (i.e. $\nu \sim 1$). Notice however that the observed exponent ν is somewhat less than 1, signifying additional correlations that make the effect of noisy nodes stronger than that suggested by simple scaling arguments.

5 Synchronization

Now we investigate the following question: what is the nature of the dynamics of the surviving nodes? We find that the dynamics of the local population of the active nodes has two features.

(i) The evolution of the individual local populations remains chaotic. This is a significant departure from many models of ecology (such as the May-Wigner formalism) which implicitly investigates stability around a fixed equilibrium steady state.

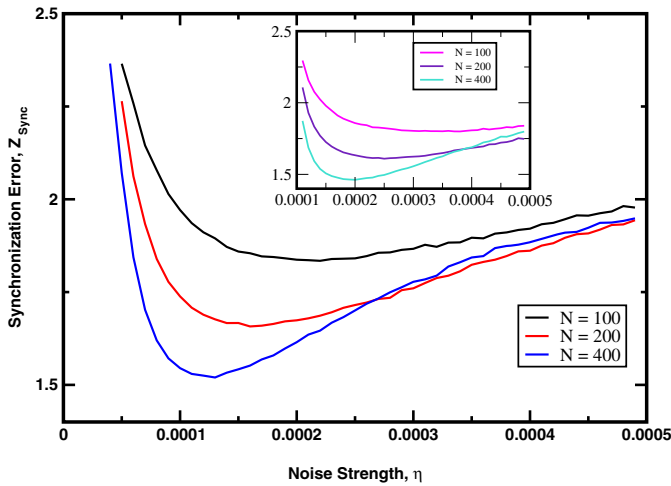(ii) The population dynamics is not synchronized, namely, there is *no synchrony of the nodal dynamics.*

Fig. 5. Synchronization error as a function of noise strength, for three different system sizes. Here Gaussian white noise is applied[3]. Inset shows the variation of synchronization error under the influence of uniform noise of varying strengths.

However, even though unsynchronized, we do find evidence of *subtle correlation* amongst the nodal populations at intermediate noise strengths. In order to quantify the degree of synchronization, we compute an average error function as the synchronization order parameter, Z_{sync}, defined as the mean square deviation of the instantaneous states of the nodes

$$Z = \frac{1}{N} \sum_{i=1}^{N} \left\{ [x_i(t) - \langle x(t) \rangle]^2 \right\}, \qquad (6)$$

where, $\langle x(t) \rangle$ is the space average at time t. This quantity Z, averaged over time n and over different initial conditions, is denoted as Z_{sync}.

We observe an interesting feature in the behavior of synchronization as noise strength is varied, as evident from Figure 5: For a particular intermediate value of noise strength, there is a significant fall in the synchronization error[4]. This increase in synchrony is more pronounced for larger network sizes. So clearly, there is *noise induced temporal coherence* in the active sub-network, at optimal noise strengths [17].

We rationalize this in the following manner: the basis for the correlations naturally arise from the interaction term. This interaction term has two features. When the number of active species is very small, such as in the low noise limit, there are very few terms in the sum, which is consequently small in magnitude. This accounts for reduced synchronization when noise is very small. However, there is also another trend. The interaction term, when averaged over a very large number of species, having positive and negative interaction weights J_{ij}, leads to a mean value

close to zero, by the central limit theorem. So at large noise strengths, when the number of active species is large, the interaction term again tends to zero, and synchronization is again reduced. So then, only in an intermediate range of noise, where the number of active species is neither too small, nor too large (such as near the transition point), is the interaction term most significant, leading to greatest coherence.

From the view point of general network dynamics this is also interesting: most earlier examples of noise induced coherence occurred where the constituents were subject to *common noise*. However, the noise in our network is uncorrelated from node to node, and so the source of correlation in our system is not as obvious as the spatially coherent drive provided by common noise.

6 Analysis

In this section we attempt to gauge the underlying reasons determining the nature of the noise-induced transition. Specifically, we consider the case of ricker growth at nodes with Gaussian additive noise as shown in Figure 1, where the system evolves according to equations (2) and (3). Since η can be negative or positive, the effect of noise is a *non-trivial* one, and at noise strengths $\eta > x_{threshold}$, there will be some nodes that will get pushed out of the extinction region due to noise, while some nodes will get mapped back to the extinction zone at the subsequent update. Here we will try to get a broad idea of the average network activity by considering all these effects. Namely, the fraction of inactive nodes at a particular noise strength η is given by:

$$\frac{N_{inactive}}{N} \sim P(\eta_t < x_{threshold}) \int_0^{x_{threshold}} \rho^*(x) dx$$
$$+ P(\eta_t < -x_{threshold}) \int_{x_{threshold}}^{\eta_t} \rho^*(x)\, dx \qquad (7)$$

where, ρ^* is the invariant measure of the coupled system and $P(\eta_t < x_{threshold})$ is the probability of noise η being less than the threshold population density. The first term in equation (7) corresponds to the fraction of nodes unable to get out of the extinction region due to sub-threshold noise strength. The second term corresponds to the fraction of nodes which were initially outside the extinction region, but got absorbed into the extinction region on application of noise. Note that noise is drawn from a Gaussian distribution with zero mean and correlation function given by $\langle \xi_i(t)\xi_j(t^{'}) \rangle = \eta\delta(t - t^{'})\delta_{ij}$ $(i, j = 1, 2, \ldots, N)$, where η governs the strength of noise. The probabilities for the noise input to lie in different regions is calculated as follows:

$$P(\eta_t < x_{threshold}) = \frac{1}{2} \left\langle 1 + Erf\left(\frac{\phi^2}{2}\right) \right\rangle,$$
$$P(\eta_t < -x_{threshold}) = \frac{1}{2} \left\langle 1 - Erf\left(\frac{\phi^2}{2}\right) \right\rangle \qquad (8)$$

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[3] It appears that the synchronization error diverges as $\eta \to 0$. This arises from the definition of Z_{sync} (Eq. (6)), where the number of active nodes tend to zero when $\eta \to 0$.

[4] This is reminiscent of the phenomenon of stochastic resonance. See, for instance [16].

where, $Erf(x)$ is an error function of variable x given by:

$$\mathbf{Erf(x)} = \frac{\mathbf{2}}{\sqrt{\pi}} \int_{\mathbf{0}}^{\mathbf{x}} \mathrm{e}^{-\mathbf{u^2}} \, d\mathbf{u} \quad \text{and} \quad \phi \sim \frac{\mathbf{x_{threshold}}}{\eta}.$$

Now the calculation of $\rho*$ for this high-dimensional coupled stochastic system is an intractable task, and does not lend itself to closed form solutions. So here we approach this problem from a different perspective. Since we are more interested in the qualitative behavior of the system, we assume these integrals to be finite constants whose values are related to values obtained in a single map[5] in some appropriate limit. These constants are defined below as:

$$\int_{0}^{x_{threshold}} \rho^*(x) \, dx = C_1,$$

$$\int_{x_{threshold}}^{\eta} \rho^*(x) \, dx = C_2. \tag{9}$$

Using equations (7)–(9), we get $N_{active} = N - N_{inactive}$ as,

$$\frac{N_{active}(\eta)}{N} = 1 - Erf\left(\frac{\phi^2}{2}\right)(C_1 - C_2) - (C_1 + C_2) \tag{10}$$

which can be re-written in a simple form as:

$$\frac{N_{active}(\eta)}{N} = \alpha - \beta \, Erf\left(\frac{\phi^2}{2}\right) \tag{11}$$

where, $\alpha = 1 - (C_1 + C_2)$ and $\beta = C_1 - C_2$.

1. **Case:** $\eta \ll x_{threshold}$
   As $\eta \to 0$, $\phi \to \infty$ which would yield $N_{active} = \alpha - \beta$ as $Erf(\infty) = 1$.
2. **Case:** $\eta \gg x_{threshold}$
   As $\eta \to \infty$, $\phi \to 0$ and we obtain $N_{active} = \alpha$ as $Erf(0) = 0$.

Now we get some idea of the bounds for N_{active} vs. η and we also know that $Erf(x)$ has sigmoidal curve. From this much of information it is not hard to see that qualitative nature of N_{active} (coming from error functions) is similar to inset of Figure 1. So it is evident that our analysis matches well with the results from numerical simulations.

Following the similar procedure we have obtained the expression for $N_{active}(\eta)$ for uniform additive noise, which is an example of *bounded noise* [18]:

$$\frac{N_{active}(\eta)}{N} = \alpha - \frac{\beta}{\eta} \tag{12}$$

where,

$$\alpha = \frac{1}{N} + \frac{C_2 - C_1}{2} \quad \text{and} \quad \beta = \frac{(C_1 + C_2)x_{threshold}}{2}.$$

Except for the constants C_1 and C_2, we have obtained a closed form functional relation (Eq. (12)) between N_{active} and η as calculation of probability integrals

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[5] The invariant measure for single logistic map with $r = 4$ has a well known closed form expression given by $\rho(x) = \frac{1}{\Pi \sqrt{x(1-x)}}$.
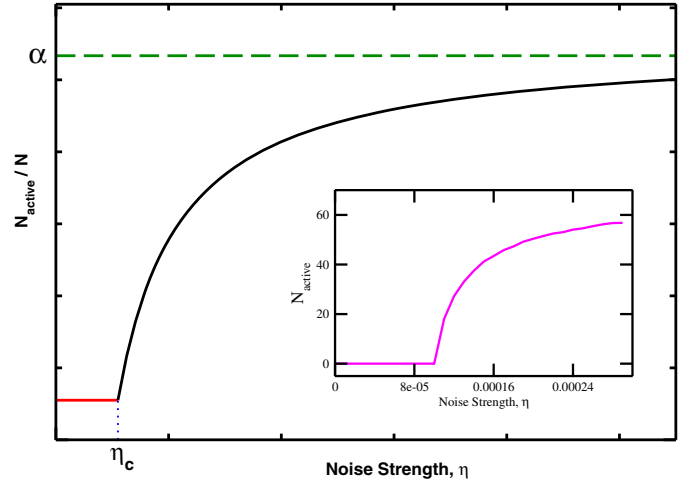

Fig. 6. Plot of analytical result (Eq. (12)) showing the behavior of fraction of active nodes with respect to noise strength. Parameters α, β are taken to be 0.5415, 0.2259, respectively. Region $\eta < \eta_c$ represents the region where noise has no discernable effect on the activity of the network. Here the noise is drawn from a uniform distribution. Inset shows the corresponding behavior from numerical simulation.

is straightforward in case of uniform noise. Figure 6 shows the qualitative match between analytical and numerical result.

7 Generality of our results

In order to check the generality of our results we have investigated different kinds of noise and different local dynamics. Specifically, we demonstrated the increase of active nodes in the system under *uniform noise*, namely the noise at the nodes was drawn from a uniform distribution in the interval $[-\eta : \eta]$, where $\eta > 0$.

Again we find that the number of active nodes in the network increases sharply after a critical noise strength. As evident from Figure 6 (inset), the critical noise strength for the case of uniform noise bounded in the interval $[-\eta : \eta]$ is equal to the extinction threshold, $x_{threshold}$. This is indeed expected, as the minimum perturbation to push a population out of the inactive zone, i.e. the minimum noise required to revive, is $x_{threshold}$.

We have also simulated the system for modified logistic growth at the nodes:

$$f(x) = \begin{cases} rx(1-x) & \text{if } x > x_{threshold}, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

The behavior that emerges in this network, under a wide range of growth rates r, is qualitatively similar to that in the network of Ricker maps discussed above.

Further, we investigated heterogenoeus networks, with a range of growth rates at the nodal level. The central conclusion, namely the enhancement of activity in networks under stochastic influence, holds in such systems as well.
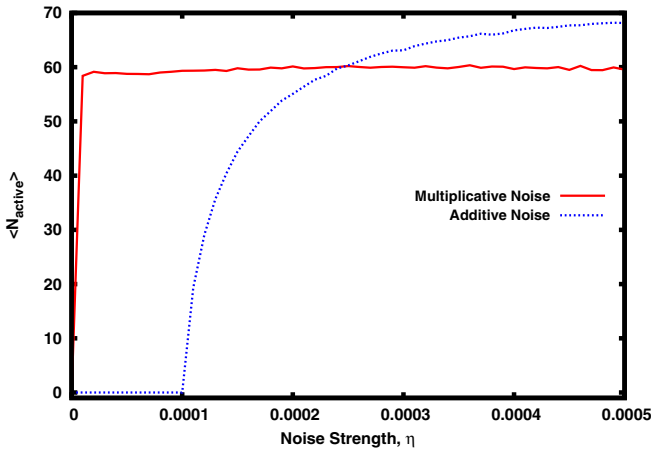
Fig. 7. Average number of active nodes as a function of noise strength. The two curves represent results for multiplicative and additive noise. Network size is 100, activity threshold is $x_{threshold} = 0.0001$. Noise here is drawn from a uniform distribution in the interval $[-\eta : \eta]$.

8 Multiplicative noise

Lastly, we also studied the evolution of this complex network under multiplicative noise:

$$x_i(n+1) = f\left[x_i(n)\left(1 + \sum_j J_{ij}x_j(n)\right) + \xi_i(t)\right]. \quad (14)$$

Here $\xi(t)$ is a uniform noise in an interval $[-\eta : \eta]$ with zero mean and correlation function is given by $\langle \xi_i(t)\xi_j(t') \rangle = \eta\delta(t - t')\delta_{ij}$ $(i, j = 1, 2, \ldots, N)$, where η governs the strength of noise.

While in the case of additive noise with uniform distribution (cf. inset of Fig. 6) the critical noise needed for enhancing activity in the system was exactly the extinction threshold, for the case of multiplicative noise we observe that the critical noise is far below $x_{threshold}$. In fact η_c tends to zero, as evident from Figure 7, indicating the extreme sensitivity of network activity to multiplicative noise [19].

The sensitivity of multiplicative noise arises from the nature of the nonlinear map close to threshold, where $f(x + \eta) > f(x) + \eta$, with η being the noise term. This holds true as the magnitude of $|f'|$ is larger than 1 and typically large. So clearly, multiplicative noise has a much greater effect on the iterated state of the system than additive noise.

9 Conclusions

In summary, we have investigated the effect of stochasticity on the global stability of complex networks. In this study, very general networks, incorporating positive and negative interactions of the generalized Lotka-Volterra type, were considered. The population dynamics at the nodal level was typically chaotic, and allowed cessation of activity if the population density fell below a certain threshold value.

Our central result is the following: the probability of obtaining active nodes in the network is *significantly enhanced under noise*. Further, we find a sharp transition in the number of active nodes as noise strength is varied, along with clearly evident scaling behaviour near the critical noise strength.

From the ecological perspective, our method made an attempt to address the classic problem of ecosystem stability from two different perspectives. One, our results are non-perturbative in nature, as we use global measures to gauge the stability of ecosystem in presence of large perturbations, rather than the usual local stability analysis in response to small perturbations. This is also relevant in the context of recent demonstrations [20,21] that global stability measures are more appropriate in identifying the dynamical behavior of high dimensional systems.

Secondly, we have studied the global activity of the network in the presence of stochasticity, for which not many results have been reported. Recently empirical evidence in dryland ecosystems [22], supported by numerical work, have found that noise plays a constructive role similar to the work reported here. Also in theoretical ecology much effort has been devoted to the identification of mechanisms that promotes stable coexistence (and consequent diversity) of species at equilibrium. In this context, our model suggests the possible phenomena of species coexistence in an open variable environment with an emergent non-equilibrium steady state having a large number of active sub-populations.

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