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Preventing catastrophes in spatially extended systems through dynamic switching of random interactions

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Abstract. In this paper, we review and extend the results from our recently published work [*Scientific Reports* (*Nature*) **4**, 4308] on taming explosive growth in spatially extended systems. Specifically, we consider collections of relaxation oscillators, which are relevant to modelling phenomena ranging from engineering to biology, under varying coupling topologies. We find that the system witnesses unbounded growth under regular connections on a ring, for sufficiently strong coupling strengths. However, when a fraction of the regular connections are dynamically rewired to random links, this blow-up is suppressed. We present the critical value of random links necessary for successful prevention of explosive growth in the oscillators for varying network rewiring time-scales. Further, we outline our analysis on the possible mechanisms behind the occurrence of catastrophes and how the switching of links helps to suppress them.

Keywords. Complex networks; dynamic links; blow-ups; catastrophe.

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1. Introduction

In nonlinear dissipative dynamical systems, a famous example of self-sustained oscillations is the Van der Pol oscillator, originally proposed by the Dutch electrical engineer Balthasar van der Pol [1], who found self-sustained oscillations in an electical circuit employing vacuum tubes. Along with his team he also studied the case of external forcing of this oscillator [2] and reported that at certain drive frequencies an irregular noise was heard which later came to be characterized as deterministic chaos.

The prototypical Van der Pol oscillator, with nonlinear damping, is governed by the second-order differential equation:

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \tag{1}$$

where x is the dynamical variable and μ is a parameter determining the nature of the dynamics. The relaxation oscillations arising in this system, for $\mu > 0$, are characterized by slow asymptotic behaviour and sudden discontinuous jumps. The Van der Pol equation is very relevant in modelling phenomena in physical and biological sciences. Extensions of such oscillators have been used to model the electrical activity of the heart and action potentials of neurons [3]. The equation has also been utilized in seismology to model the two plates in a geological fault [4].

Usually, isolated elements do not commonly occur in nature. Rather, most natural phenomena involve dynamical systems coupled through interactions. So, a natural but important question which arises is 'what is the collective behaviour of a distributed system composed of several such coupled relaxation oscillators?' To model such large interactive systems one has to consider three principal features [5]:

(i) *Functional form of the interaction*: Commonly used types of coupling include linear, nonlinear, global (namely mean-field), diffusive, pulsatile, etc.
(ii) *Topology of connections*: Here one considers issues such as connectedness, spatial structure, range and topology of the web of connections (e.g., short-range, long-range, modular, regular graphs, random networks etc.).
(iii) *Time dependence of the interaction*: This feature has been studied in [6] and most earlier works focussed on time-invariant coupling. However, it is indeed very relevant to ask whether the structure of the spatial connections evolve or change or adapt with time.

Depending upon the particular nature of the considered phenomenon, one can build appropriate models by specifying the above features of the interactions. However, it is also very useful to study the problem in a general framework, by investigating different broad classes of interactions and topologies. Such general studies are important for understanding the dynamical consequences of each interaction class and offer perspectives on how different spatiotemporal patterns emerge and are sustained in nature.

In the subsequent sections we shall review our key results [7] on networks of relaxation oscillators, and also present some extensions of our work.

2. Network of relaxation oscillators

We start with a generic network constituted of nonlinear dynamical elements at the nodes and a coupling term modelling the interaction between the elements. The isolated (uncoupled) dynamics at each node of the network is given by $\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X})$, where $\mathbf{X}$ is a an m-dimensional state vector of the dynamical variables and $\mathbf{F}(\mathbf{X})$ is a velocity field that is typically nonlinear. So the evolution of such a system is given by following equation:

$$\dot{\mathbf{X}}_i = \mathbf{F}(\mathbf{X}_i) + \varepsilon \sum_{j=1}^{N} J_{ij}\mathbf{H}(\mathbf{X}_i, \mathbf{X}_j), \quad i = 1, ..., N, \tag{2}$$

where J_{ij} are the elements of a connectivity matrix, which typically need not be symmetric [8]. The coupling strength is given by ε and $\mathbf{H}(\mathbf{X}_i, \mathbf{X}_j)$ is the coupling function determined by the nature of interactions between dynamical elements i and j.

We first review our results obtained for coupled Van der Pol oscillators [1], characterized by nonlinear damping, and governed by the second-order differential equation given by eq. (1). We take $\mu > 0$ which yields relaxation oscillations characterized by slow asymptotic behaviour and sudden discontinuous jumps. Associating $\dot{x} = y$ gives

$$\begin{aligned}\dot{x} &= f(x, y) = y,\\ \dot{y} &= g(x, y) = -\mu(x^2 - 1)y - x.\end{aligned} \tag{3}$$

Now we consider a ring of N nonlinearly coupled Van der Pol oscillators [9–11], namely a specific form of eq. (2) with $m = 2$, $\mathbf{X}_i = \{x_i, y_i\}$, $\mathbf{F} = \{f, g\}$ and $\mathbf{H}(\mathbf{X}_i, \mathbf{X}_j) = \{f(x_j, y_j) - f(x_i, y_i), g(x_j, y_j) - g(x_i, y_i)\}$ given as follows:

$$\begin{aligned}\dot{x}_i &= f(x_i, y_i) + \frac{\varepsilon}{2}[f(x_{i-1}, y_{i-1}) + f(x_{i+1}, y_{i+1}) - 2f(x_i, y_i)],\\ \dot{y}_i &= g(x_i, y_i) + \frac{\varepsilon}{2}[g(x_{i-1}, y_{i-1}) + g(x_{i+1}, y_{i+1}) - 2g(x_i, y_i)].\end{aligned} \tag{4}$$

Here index i specifies the site/node in the ring, with the nodal onsite dynamics being a Van der Pol relaxation oscillator. Note that this form of coupling has been explored insufficiently in the existing literature.

Starting with this regular ring, we consider increasingly random networks formed as follows: one begins with a regular lattice, such as the ring of regular nearest-neighbour interactions mentioned above, and then with probability p we replace ('rewire') the regular links with random connections. So when p is non-zero, random non-local connections exist alongside regular local links, and such networks have widespread relevance [12]. Note that our coupling occurs on a degree preserving directed network, i.e., the number of connections for each oscillator is the same, whether random or regular (and is specifically 2 in our case).

Additionally, we consider the scenario where the links may vary in time, i.e., the links may be dynamic. This implies that the specific set of nearest-neighbour links that get rewired to random nodes varies from time to time. So the underlying web of connections switches over time [13–15], and such time-varying connections are expected to be widely prevalent in response to environmental influences or internal adaptations [16,17].

We explored two different methods for changing the connectivity of the network. The first algorithm involves the periodic switching of links in the network. Here we start with a ring with N nodes, where each node has two links corresponding to its two nearest neighbours. These regular links of the nodes are replaced with probability p and rewired to some randomly selected distant nodes. In the network, on an average then, a fraction $(1 - p)$ links are to the nearest neighbours and fraction p to random nodes [12]. The above process is repeated after a time interval r, i.e., the network changes to this new configuration after time r. When the connections switch, some nearest-neighbour links of the previous network configuration may change to random links and vice versa.

So the network switches from one configuration to another periodically, keeping the qualitative nature of the connectivity matrix invariant. In other words, r gives the time-scale of network change and larger values of r imply that the network changes become more infrequent, with the limit of $r \to \infty$ corresponding to the standard well-studied case of quenched links.

Periodic switching of links occurs in situations where the connections are determined by some global external periodic influence. However, this is not the most realistic scenario, as the interaction patterns may not change periodically in time. In such cases we must consider a probabilistic model of link switching, such as in [15,18]. So a second algorithm is introduced here, where the links switch randomly and asynchronously in time.

We again start with a ring with N nodes, with fraction p being random links and fraction $(1 - p)$ being the nearest-neighbour connections. Now we consider that each individual node i has probability p_{r} of changing its links in a specified time interval τ. If the links of a certain node are selected for rewiring, they rewire such that with probability p they connect to random neighbours and with probability $(1-p)$ they connect to nearest neighbours.

In this algorithm, the nodes change their links independently and stochastically. The probability of the new links being random or regular is determined by p as in the first algorithm (and as in the standard small-world scenario). So, while in the first algorithm link changes occur globally throughout the network, in the second algorithm uncorrelated changes occur at the local level.

In order to get the dynamics of the complete system, we solved the $2N$ coupled non-linear ordinary differential equations (ODE) given by eqs (3) and (4). We obtained the shapes and sizes of the limit cycles arising in this network, for system sizes ranging from $N = 10$ to 10^3, under varying fractions of random links p $(0 \leq p \leq 1)$. We investigated a large range of network switching time periods r $(0.01 \leq r \leq 1)$ for periodically switched networks, and $0 \leq p_{\mathrm{r}} \leq 1$ for stochastically switched networks ($\tau = 0.001$). We describe now the central results obtained from our extensive numerical simulations [7].

3. Spatiotemporal patterns in a regular ring

First, we describe our principal observations for coupling on a ring with two nearest neighbours (i.e., $p = 0$) under increasing coupling strengths.

For very weak coupling, the system shows no regular spatiotemporal pattern. As coupling is increased, regular travelling wave-like behaviour develops. As coupling gets stronger, and approaches a critical value ε_{c}, the regularity of the pattern breaks up (see figure 1 for representative examples of these spatiotemporal patterns). When the coupling exceeds the critical value $(\varepsilon > \varepsilon_{\mathrm{c}})$, the amplitude of the oscillations grows explosively in an unbounded fashion, i.e., there is a blow-up in the system. Representative values characterizing this blow-up are: the amplitude grows from $O(1)$ to $O(10^4)$ in an interval of time as short as $\sim 10^{-3}$.

We also observed that for small system sizes $(N < 10)$, some special solutions exist and systems with odd number of oscillators are relatively more stable. For example, it was found that for $N = 3$ the system does not blow-up for coupling strengths upto 0.66 and similarly for $N = 5$ and 7 the system is stable up to higher coupling strengths than a small even-numbered ring of oscillators. However, as the network becomes larger $(N > 10)$ we get qualitatively the same results.

3.1 *Effect of increasing coupling range*

To further understand the mechanism behind explosive growth, we explored the case of long-range interactions where a node interacts with k nearest neighbours (where $1 < k \leq N$).

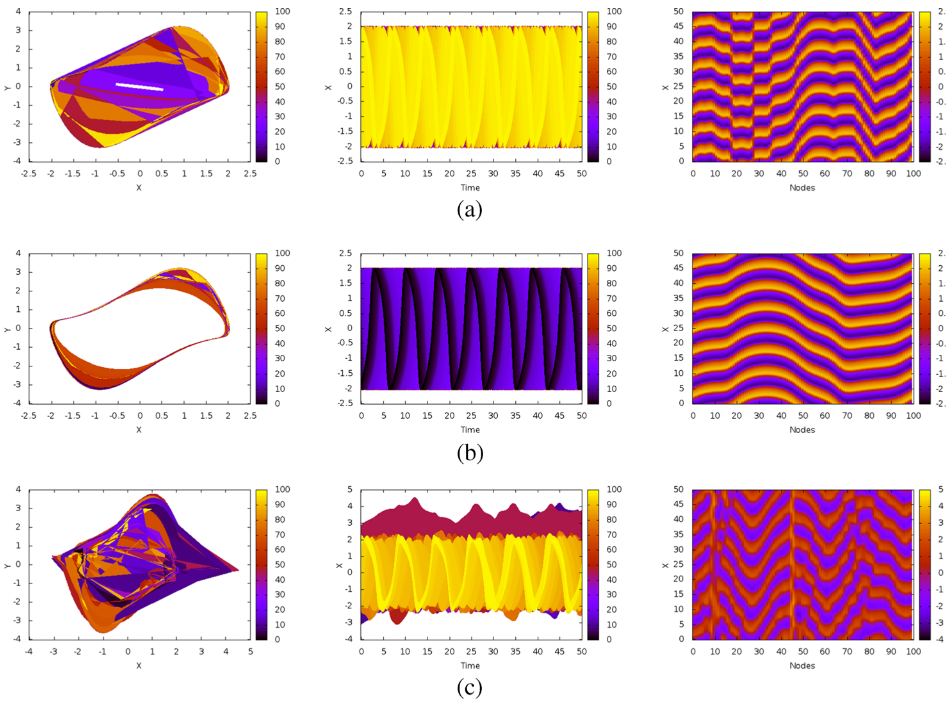
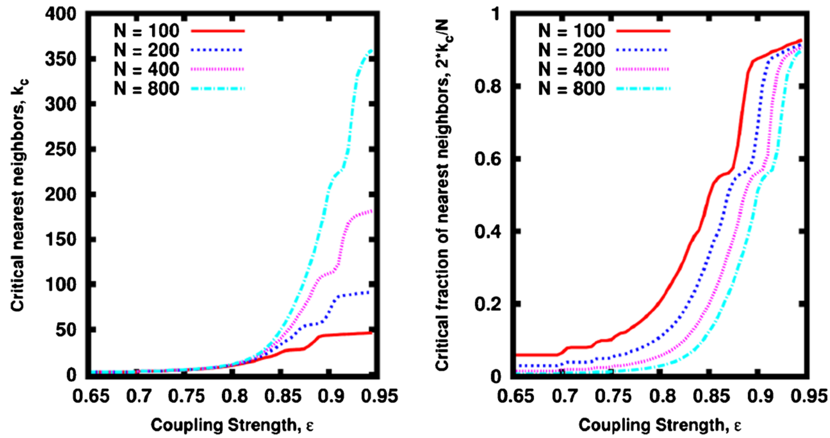


Figure 1. Left-most figures correspond to phase-space plots of all the van der Pol oscillators coupled to nearest neighbours on a ring with $\mu = 1.5$ ($i = 1, 2, \ldots, 100$), with the x-axis representing position, the y-axis representing velocity and the different colours representing different oscillators; middle figures correspond to the time evolution of position $x^i(t)$ of all the oscillators, with different colours representing the different oscillators; right-most figures correspond to density plots of the spatiotemporal evolution, with the site indices on the x-axis and time along the y-axis, and the magnitude of the variables $x_i(t)$ represented by the colour scale. The coupling strengths are **(a)** 0.007, **(b)** 0.3, **(c)** 0.5. For coupling strengths greater than the critical value $\varepsilon_c \sim 0.5$, the dynamics of all the oscillators is unbounded.

From figure 2, it is clear that long-range interactions (i.e., increasing k) stabilize the network and the unbounded growth is suppressed. Besides, it appears that for low coupling strengths, k_c, the minimal number of neighbours necessary for preventing blow-ups is independent of system size N, while for high coupling strengths the fraction k_c/N is independent of the network size N.

4. Spatiotemporal behaviour of the oscillators under random links

A very different behaviour from that described earlier, ensues when the links are rewired randomly. The results for periodically switched networks are shown in figure 3, for representative values of p and r, with coupling strengths greater than critical value ε_c. For

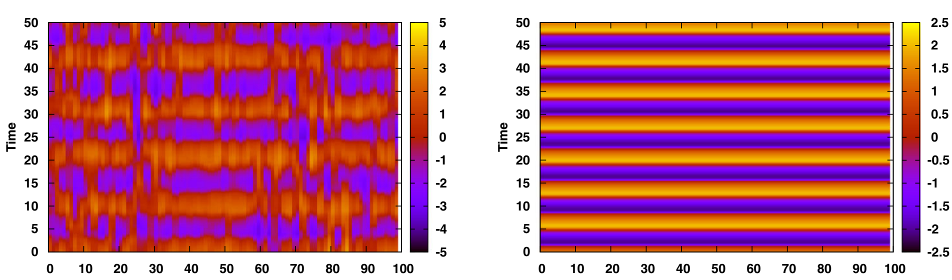


(a) (b)

Figure 2. (**a**) Minimum number of nearest neighbours k_c necessary for obtaining bounded behaviour at varying coupling strengths, for different network sizes; (**b**) critical fraction of nearest neighbours $2k_c/N$ as a function of coupling strength, for different network sizes. The topology remains static throughout the evolution.

these coupling strengths, the limit cycles grew explosively for regular coupling. However, it is clearly evident that the blow-up has been prevented for $p > 0$, and all the limit cycles remain confined in a bounded region of phase space. Also note that we obtain qualitatively the same results when the links are switched stochastically (as in the second algorithm for network change).

We also explored this behaviour over different values of the parameter μ. Representative results are shown in figure 4, which displays the minimum fraction of random links



(a) (b)

Figure 3. Spatiotemporal evolution of the oscillators under random links, with the site indices on the x-axis, time along the y-axis and the magnitude of the variables $x_i(t)$ ($i = 1, \ldots N$), represented by the colour scale, for network switching time period (**a**) $r = 0.1$; (**b**) $r = 0.01$. Here $N = 100$, $\mu = 1.5$, fraction of random links $p = 0.6$ and coupling strength $\varepsilon = 0.6 > \varepsilon_c$. Note that regular coupling ($p = 0$) yields unbounded dynamics at this coupling strength.

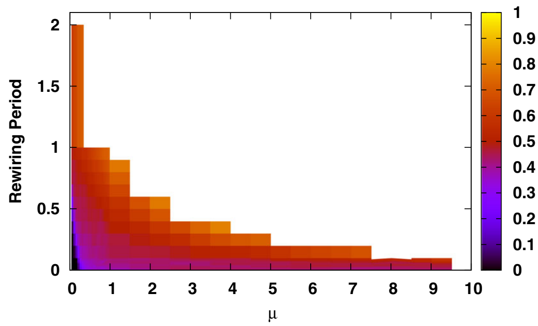


Figure 4. Variation of the minimum fraction of random links p_c necessary to suppress blow-ups in a network of coupled van der Pol oscillators (represented by the colour scale), as a function of μ (see eq. (3)) and rewiring period r. The coupling strength $\varepsilon = 0.6$. In the white regions of the figure the blow-ups cannot be prevented through dynamic random connections.

necessary to tame the unbounded growth in the system. It is evident from the figure that for higher μ, even when the links switch rapidly, the range over which the dynamics remains bounded is quite small. So the nonlinearity of the local dynamics determines how fast the random links need to be switched in order to suppress the unbounded growth. Clearly, the faster the network changes, the smaller is the fraction of random links necessary to effect prevention of blow-ups (i.e., p_c is smaller the for smaller rewiring time period r).

5. Understanding unbounded solutions

In order to understand the unbounded behaviour, we analyse the dynamics of an oscillator in the system, as a sum of the nodal dynamics and the coupling contributions denoted by $c_1(t)$ and $c_2(t)$, where the amplitudes of $c_1(t)$ and $c_2(t)$ are proportional to the coupling strength ε:

$$\dot{x} = y + \varepsilon c_1(t), \tag{5}$$

$$\dot{y} = -\mu(x^2 - 1)y - x + \varepsilon c_2(t). \tag{6}$$

When $c_1 = 0$ and $c_2 = 0$, the equations reduce to a single Van der Pol oscillator, with a stable limit cycle and an unstable fixed point at the origin ($x^* = 0$, $y^* = 0$). When $c_1(t) \neq 0$ and $c_2(t) = 0$, the velocities couple, and this form of coupling arises mostly in mechanical systems. Conversely, when $c_1(t) = 0$ and $c_2(t) \neq 0$, the coupling is through acceleration, and this form of coupling has been found in relaxation oscillators of chemical, electrical and biological systems.

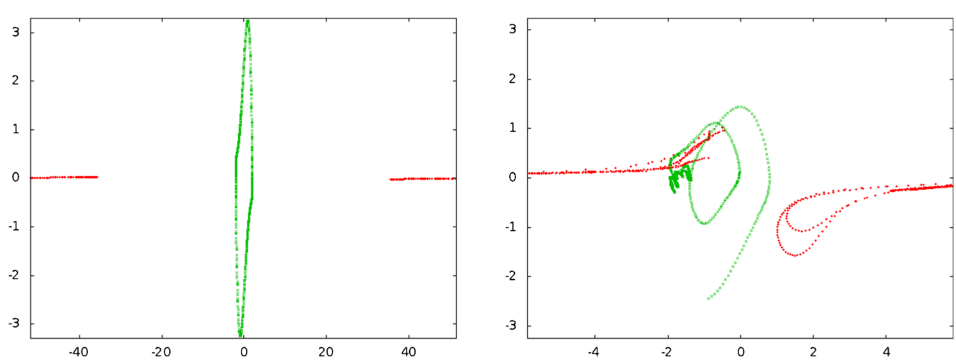
Now, in order to broadly understand the global dynamics of the coupled system ($c_1 \neq 0$, $c_2 \neq 0$), we treat the effective dynamics of each oscillator in the network as a relaxation oscillator with fluctuating parameters c_1 and c_2 [19]. We first see what happens when $c_1 \neq 0$ (with c_1, c_2 taken to be static). This yields the fixed point solutions:

$$x^* = \frac{1 \pm \sqrt{1 - 4\mu\varepsilon^2 c_1(c_2 - \mu c_1)}}{2\mu\varepsilon c_1},$$
$$y^* = -\varepsilon c_1. \tag{7}$$

This implies that two fixed point solutions are obtained for each oscillator in the coupled system. For very weak coupling ($\varepsilon \to 0$) the additional fixed point remains at infinity. As coupling strength increases, this fixed point migrates towards the limit cycle, and at some critical coupling strength, collides with it, undergoing a saddle node bifurcation. This leads to the destruction of the attracting limit cycle and consequent blow-up. The parameters c_1 and c_2 are actually fluctuating, and sufficiently large fluctuations in c_1 and c_2 results in a collision leading to an annihilation of the limit cycle [19]. As the magnitude of fluctuations is governed directly by coupling strength, this implies that blow-ups will occur for sufficiently large ε (see figure 5).

In this scenario, the underlying mechanism that controls the blow-up is the following: when the links are changed rapidly (r is small or p_r is large), or when the number of elements a node is coupled to increases (large k), to first approximation the terms $c_1(t)$ and $c_2(t)$ in eq. (6) averages to zero, as it is a sum of uncorrelated random inputs. This effectively reduces eq. (6) to that of a single Van der Pol oscillator for which there is no blow-up. Alternately, one can argue that for rapid random switching of coupling connections, the deviations from the single oscillator dynamics is essentially a sum of uncorrelated contributions, which to first approximation average out to zero.

Finally, we would like to mention that the current understanding of the underlying cause of blow-up is not very clear and points to many open problems that demand further research work. One of the leading directions of future research would be to classify global bifurcations in high-dimensional systems, whose degrees of freedom is neither too small (for which we have sufficient understanding now), nor too large (such that it can be understood in the framework of statistical mechanics). The other line of research that may be of importance from the applied point of view, is to find out the type of interactions or the coupling forms that are more prone to instabilities and catastrophes in dynamical networks.



(a) (b)

Figure 5. The figure shows the trajectory of a representative limit cycle (in green) and a sequence of points (x^*, y^*) (in red) obtained from eq. (7) under varying c_1 and c_2. The sequence of values of c_1 and c_2 are obtained from the state vector of the system at different times. In (**a**) we have low coupling strength ($\varepsilon = 0.1$) and the saddle and limit cycle are far apart in phase-space. In (**b**) coupling strength $\varepsilon = 0.5$, and (x^*, y^*) collide with the limit cycle, and the trajectory becomes unbounded.

6. Synchronization of the bounded state

Now we address the following question: when random coupling prevents a blow-up, does it yield a state that is synchronized, or not? Interestingly, different patterns emerge under different time-scales of network change. Networks in which the underlying links change rapidly yield a synchronized state. However, slow network changes give bounded dynamics that is not synchronized [7]. This was demonstrated quantitatively in [7] through a synchronization order parameter, defined as the mean square deviation of the instantaneous states of the nodes, averaged over time and over different initial conditions.

This relaxation oscillator network yields three kinds of dynamical states: (i) bounded synchronized motion; (ii) bounded unsynchronized dynamics and (iii) unbounded dynamics. The nature of the dynamics depends on the interplay of the coupling strength, fraction of random links and frequency of network switching.

In addition, for stochastic switching of links we found some interesting scaling relations between the fraction of random links p and the link rewiring probability p_{r} [7]. First, we found that the critical coupling strength ε_{c} beyond which blow-ups occur (figure 6), scales with the link rewiring probability p_{r} and fraction of random links p as

$$\varepsilon_{\mathrm{c}} \sim (pp_{\mathrm{r}})^{\beta}, \tag{8}$$

where $\beta = 0.119 \pm 0.001$. This scaling relation implies that as the links change more frequently, and there is larger fraction of random connections, the range over which bounded dynamics is obtained becomes larger. Further, we note that the quantity that occurs in the scaling relation is the product pp_{r}. So the emergent phenomenon is the same if this product remains the same, even though individually p and p_{r} may differ.

Furthermore, the minimum fraction of random links p_{c} necessary to suppress blow-ups, at a particular coupling strength, varies with link switching probabilities p_{r} as

$$p_{\mathrm{c}} \sim (p_{\mathrm{r}})^{\delta}. \tag{9}$$

Representative values of the exponent are: $\delta = -1.04 \pm 0.008$ for $\varepsilon = 0.7$ and $\delta = -0.98 \pm 0.008$ for $\varepsilon = 0.9$.

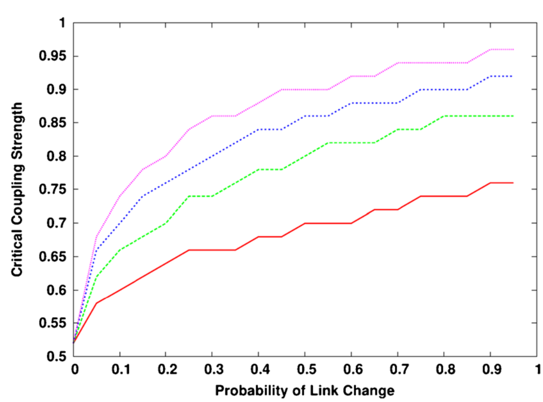


Figure 6. Variation of the critical coupling strength at which blow-up occurs, ε_{c}, with respect to probability of link change p_{r}, for fraction of random links (p) equal to 0.1 (red), 0.3 (green), 0.5 (blue) and 0.8 (pink).

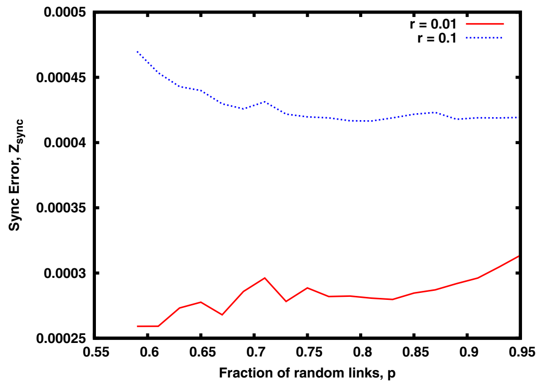
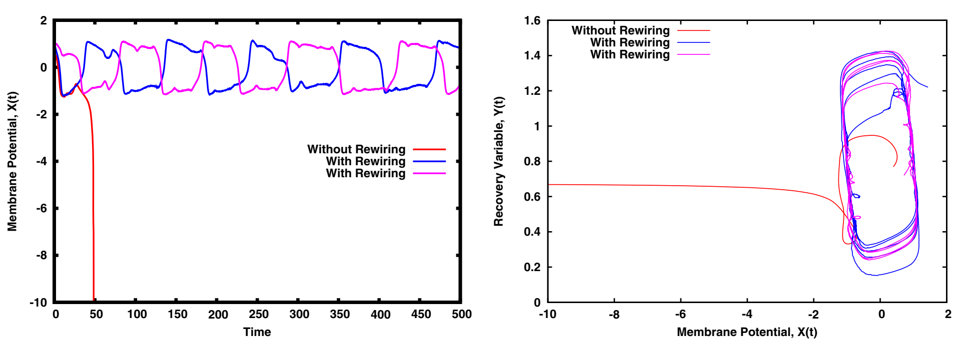


Figure 7. Synchronization order parameter Z_{sync}, defined as the mean square deviation of the instantaneous states of the nodes, averaged over time and over different initial conditions, as a function of the fraction of random links, for link changing time periods $r = 0.1$ (slow changing network), 0.01 (rapidly changing network).

Lastly, we checked the generality and scope of our results by studying the behaviour of a network of Stuart–Landau oscillators given by

$$\dot{z} = (1 + i\omega - |z|^2)z, \tag{10}$$

where ω is the frequency and $z(t) = x(t) + iy(t)$. Figure 7 shows representative results. It is clear that networks that change faster prevent blow-ups over a larger range of p. That is, even when fraction of random links p is small, blow-ups are prevented in fast changing networks, while slow-varying networks need a larger fraction of random links to



(a) (b)

Figure 8. (**a**) Phase portrait and (**b**) time evolution of the membrane potential for three representative FitzHugh–Nagumo model neurons in the network given by eq. (11), with fraction of random links $p = 0.65$, time period of changing the links $r = 0.01$ and coupling strength $\varepsilon = 0.52$ (cyan and blue curves). The case of a static regular network, exhibiting blow-up, is shown alongside as well (red curve). Note that the bounded orbits obtained under dynamic random links are not synchronized (cyan and blue curves).

suppress unbounded growth. Further, dynamic links lead to more synchronized states, as evident from the much smaller synchronization errors in fast time-varying networks.

We also studied nonlinearly coupled networks of FitzHugh–Nagumo oscillators modelling neuronal populations, where the dynamics of the membrane potential x and the recovery variable x are described by [20,21]

$$\begin{aligned} \dot{x} &= x - x^3/3 - y + I, \\ \dot{y} &= \Phi(x + a - by), \end{aligned} \tag{11}$$

with I being the magnitude of stimulus current (see figures 8a and 8b). We also explored the heterogeneous systems, such as the case where the nonlinearity parameter μ was distributed over a range of positive values for the local Van der Pol oscillators. Our extensive numerical simulations showed qualitatively similar behaviour for all of the above systems [7]. This strongly suggests that the phenomenon of prevention of blow-ups through dynamic random links is quite general.

7. Conclusions

In summary, we have reviewed our recent results on the dynamics of a collection of relaxation oscillators under varying coupling topologies, ranging from a regular ring to a random network. Our central result is the following: the coupled system experiences unbounded growth under regular ring topology for sufficiently strong coupling strength. However, when some fraction of the links are dynamically rewired to random connections, this blow-up is suppressed and the system remains bounded. So our results suggest an underlying mechanism by which complex systems can avoid a catastrophic blow-up. Further from the standpoint of potential applications, our observations indicate a method to control and prevent blow-ups in coupled oscillators that are commonplace in a variety of engineered systems.

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