An Algebraic Formulation of Graph Reconstruction Conjecture Revisited:

Submitted in partial fulfilment of the requirements for the degree of

Master of Science
in
MATHEMATICS AND COMPUTING
By

SUDHANSHU RAWAT
Admission No. 18MS0063

Under the kind guidance of

Dr. DINABANDHU PRADHAN
ASSISTANT PROFESSOR

DEPARTMENT OF MATHEMATICS AND COMPUTING
INDIAN INSTITUTE OF TECHNOLOGY (INDIAN SCHOOL OF MINES)
DHANBAD
APRIL, 2020

CERTIFICATE

This is to certify that Mr. Sudhanshu Rawat (Admission No. 18MS0063 , student of M.Sc. (Mathematics and Computing), Department of Mathematics and Computing, Indian Institute of Technology (Indian School of Mines), Dhanbad have worked under my guidance and completed their dissertation entitled "An Algebraic Formulation of Graph Reconstruction Conjecture" in partial fulfilment of the requirement for award of degree of M.Sc. (Mathematics and Computing), Department of Mathematics and Computing from Indian Institute of Technology (Indian School of Mines), Dhanbad. This work has not been submitted for any other degree, award, or distinction elsewhere to the best of my knowledge and belief. They are solely responsible for the technical data and information provided in this work.

Dr. Dinabandhu Pradhan
Assistant Professor and Guide
Department of Mathematics and Computing
Indian Institute of Technology
(Indian School of Mines), Dhanbad.

FORWARDED BY:
Prof. G. N. Singh
Head of the Department,
Department of Mathematics and Computing
Indian Institute of Technology
(Indian School of Mines), Dhanbad

Declaration

The dissertation titled " An Algebraic Formulation of Graph Reconstruction Conjecture." is a presentation of my Paper Revision research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions, as may be applicable. Every effort is made to give proper citation to the published/unpublished work of others, if it is referred to in the dissertation.
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Sudhanshu Rawat
M.Sc. (M & C)
Department of Mathematics and Computing
Admission No:18MS0063

Forwarded by:
Dr. Dinabandhu Pradhan
Assistant Professor
Department of Mathematics and Computing
IIT (ISM) Dhanbad

Acknowledgement

It gives me immense pleasure to express to our deep sense of gratitude and indebtedness to our respected teacher and supervisor, Dr. Dinabandhu Pradhan, Department of Mathematics and Computing, Indian School of Mines, Dhanbad, for his valuable guidance, untiring efforts, timely suggestions and constant inspiration throughout the period of our research pursuit. The knowledge and experience that we gained from him will always enlighten our future life. Working under his supervision and guidance always remained a matter of deep satisfaction.
We must express our sincere thanks to Prof. G.N. Singh, Head of the Department of Mathematics and Computing, Indian School of Mines, Dhanbad, for encouragement and providing us all necessary facilities for completion of the project work.
I would also like to thank our classmates and friends for their motivation and help.
Last but not the least, I would like to express our deep sense of gratitude to our parents and families whose continuous motivation helped us to carry out our dissertation work successfully.

Contents

1 Introduction

There are many unsolved problems in the Mathematics. The following problem is one of them:-

Can we determine the Graphs uniquely by their sub-graphs?

In graph theory, Ulam and Kelly proposed a conjecture which says that Graphs are uniquely determined by their sub-graphs. So if two of at least 3 vertices have same multi-sets of vertex-deleted subgraphs then the graphs are called isomorphic that is, we can reconstruct any such graph up to isomorphism from the multi-set(also called deck) of its vertex-deleted subgraphs.

Brendan Mckay has verified the conjecture for all undirected simple graphs with at most 11 vertices. Further Kelly has proved this conjecture for many specific calss of graphs like as disconnected graphs,trees and regular graphs. Although B. Bellas have shown in his paper that it is sufficient to reconstruct almost every graph having three subgraphs in its multiset(deck). The conjecture fails for directed graphs. In 1981 P.K. Stock Meyer proved this by constructing number of family of non-reconstructible digraphs.

There are two very famous lemmas which are very useful in reconstruction on the Graphs. One is Kelly's lemma and the other one in Kocay's lemma.

Kelly's lemma states that if, say s(F, G) be the number of subgraphs of Graph G isomorphic to F and let H be any reconstruction of G then for $v(F) \quad < \quad v(G)$

$$s(F, H) = s(F,\ G)$$

There is a condition in Kelly's lemma as $v(F) < v(G)$ and to overcome that restriction up to some extent Kocay's lemma comes into picture which informally says that if $\mathcal{F}$ be the sequence of graphs $\mathcal{F} = (F_1, F_2, F_3, ..., F_m)$, $\forall F_i \leq v(G)$ such that

$$\sum_{H} cv(\mathcal{F}, H)\ s(H, G)$$

is reconstructible.

It is proved that the equations which are obtained by applying Kocay's lemma, provide very important information about the reconstruction of the graph. Further these equations also provide the total number of non reconstructible graphs on n vertex.

Let M be the matrix of coefficients of the equations and if $d(n)$ is the number of multiset (deck) obtained from graph having n vertex then the rank of the matrix M is the lower bound of the number of decks that is $d\ (n) \geq Rank(M)$. If the coefficient matrix of the equations is full ranked then all graphs on n vertices are reconstructible.

If coefficient matrix of equations is full ranked then all graphs on n vertices are reconstructible. Therefore as a summary, we say that the restriction in Kocay's lemma is removed. It is proved that the number of independent equations is precisely the number of the distinct deck on the vertices. Hence for the discrepancy between the number of distinct decks and the number of distinct graphs an algebraic characterization based on the Kocay's lemma, if needed it was to be false. If we take the ratio of the independent equations and the number of distinct graphs it is found to be large. There is a total different mathematical perspective was presented by Munkhin in his paper. He discussed the reconstruction problem in the context of orbit algebra and also he mentioned Ulam's conjecture in the algebraic terms.

2 SOME BASIC DEFINITIONS AND LEMMAS

2.1 Definition (Graph)

A Graph G is defined as a triple (V, E, ϕ) where V is the set of vertices and is also denoted as $V(G)$, E is the set of edges and is also denoted by $V(G)$ and ϕ is the relationship between them. There are many specific classes of graphs such as directed graphs, undirected graphs, Trees, regular graphs etc. Here we consider only finite, simple, undirected graphs for this article.

2.2 Definition (Graph Isomorphism

Two graphs H and G are called isomorphic to each other if there exists a one-one mapping $f : V(G) \to \; V(H)$ and $g : E(G) \to E(H)$ such that an edge e and a vertex v are incident in G iff the edge $g(e)$ and the vertex $f(v)$ are incident in H. If F and G are isomorphic to each other then they can be denoted as $G \cong H$. This is the definition in case of undirected graph. The definition of graph isomorphism in case of directed graphs can be stated as follows:- A vertex v is head/tail (the vertex at which the edge is incident is known as head and the from where the an edge is originated is known as tail) of an edge e iff $f(v)$ is the head/tail of $g(e)$ respectively.

2.3 Definition (class of graphs)

A Graph which is closed under isomorphism is known as class of graphs. And if a class of graphs contains finitely many isomorphic classes then it is said to be finit class of graphs.

2.4 Definition (Reconstruction)

If G the graph and v be the vertex of graph G. A sub-graph of G obtained by deleting vertex and all edges incident on vertex is known as vertex-deleted subgraph of graph G. A vertex-deleted subgraph is denoted by $G - v$.

A graph H is called reconstruction of graph G if there exists an injective map from vertex set of H to vertex set G $i.e.f : V(G) \to V(H)$ such that vertex-deleted subgraph of H and G ($G-v$ and $H - f(v)$) are isomorphic. It is denoted as $H \sim G$ where '$\sim$' is equivalence relation. If every reconstruction of graph G is isomorphic to G the graph G is called reconstructible.
In sort if $H \sim G \implies H \cong G$.

If $\mathcal{T}(H) = \mathcal{T}(G)$ $\forall$ reconstructions of G then $\sqcup(G)$ is called recognizable. If $\mathcal{C}$ denotes the class of graphs and $G \in \mathcal{C}$ then $\mathcal{C}$ is called recognizable. And if every $G \in \mathcal{C}$ is reconstructible then $\mathcal{C}$ is also reconstructible.

2.5 lemma (kelly's lemma)

let $H \sim G)$ and if F is any other graph satisfying the condition $v(F) < v(G)$ then

$$s(F, G) = s(F, H)$$

i.e. number of subgraphs of G isomorphic to F are equal to the number of subgraphs of H isomorphic to F.

2.6 Definition (Cover)

A sequence $(G_1, G_2, ..., G_m)$ subgraphs of G such that $\forall G_i \cong F_i$, where $\mathcal{F} = (F_1, F_2, ..., F_m)$ is the sequence of graphs, and $\bigcup G_i = G$, $1 \leq i \leq m$ is called cover if G by $\mathcal{F}$ and is denoted by $c(\mathcal{F}, G)$.

2.7 lemma (kocay's lemma)

If F is a graph having n vertices then

$$\sum_H cv(\mathcal{F}, H)s(H, G)$$

is reconstructible for any sequence of grpahs $\mathcal{F}$ satisfying condition $v(F_i) < n)$.
Note: if $\mathcal{C}_n$ is the class of graphs on n vertices and for every $v(F_i) < n$ then summation

$$\sum_{H \in c_n/\cong} s(\mathcal{F}, H)s(H, G)$$

is reconstructible, then $\mathcal{C}_n$ satisfies Kocay's lemma.

2.8 Proposition

Let $s(H,G)$ be reconstructible $\forall G \in \mathcal{C}_n$ where $\mathcal{C}_n$ is class of graphs on n vertices then and $\forall H \in \mathcal{C}_n$ (H is n-vertex graph) then $\mathcal{C}_n$ satisfies Kocay's lemma.
Proof:
$\because$ $s(H,G)$ is reconstructible,
$\therefore$ if $\mathcal{F}$ be the sequence of graphs satisfying condition $v(F_i) < n$ then,

$$\prod_{i=1}^{n} s(F_i,G) = \sum_{H \in c_n/\cong} s(\mathcal{F},H)s(H,G) + \sum_{H \notin c_n/\cong} s(\mathcal{F},H)s(H,G)$$

where $\sum\limits_{H \notin c_n/\cong} s(\mathcal{F},H)s(H,G)$ is reconstructible.
Now after rearranging the terms we get $\sum\limits_{H \in c_n/\cong} s(\mathcal{F},H)s(H,G)$
Hence $\mathcal{C}_n$ satisfies Kocay's lemma. and this completes the proof of above proposition.

Further $s(H,G)$ is reconstructible if G is any disconnected graph. There are finite and recognizable classes such as planer graphs, trees satisfy Kocay's lemma. If $\mathcal{C}_n$ is recognizable and finite class on n-vertex which satisfies kocay's lemma then the study of equation obtained by applying Kocay's lemma on $\mathcal{C}_n$ can be done.

Let $\mathcal{F} = (F_1, F_2, ...F_m)$ be the sequence of graphs satisfying the condition $v(F_i) < n \ \forall 1 \le i \le m$. Further let $\bar{G} \sim G \in \mathcal{C}_n$, $\mathcal{C}_n$ is reconstructible and also $\bar{G} \in \mathcal{C}_n$ then,

$$\sum_{H \in c_n/\cong} s(\mathcal{F},H)s(H,\bar{G}) = \mathfrak{k}_{\mathcal{F},R}$$

where $\mathfrak{k}_{\mathcal{F},R}$ is constant depending on sequence $\mathcal{F}$ and reconstruction class R.

2.9 Definition

Suppose $\mathcal{W} = \{y \in \mathbb{R}^{|\mathcal{C}_n/\cong|} | M_{\xi,\mathcal{C}_n/\cong}.y \equiv 0\}$ where ξ is the family of sequence of graphs that is $\xi = (\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_p)$ and $M_{\xi,\mathcal{C}_n/\cong}$ is the matrix whose rows are indexed by the sequence $\mathcal{F}_i, 1 \le i \le p$ and the columns of that matrix are fixed by different isomorphism classes of graphs in $\mathcal{C}_n$, is the subspace of vector space $\mathbb{R}^{|\mathcal{C}_n/\cong|}$ over $\mathbb{R}$ so the constant $\alpha(\mathcal{C}_n)$ is defined as:

$$\alpha(\mathcal{C}_n) = | \mathcal{C}_n/\cong| - | \mathcal{C}_n/\sim|$$

In the next section we will discuss some theorem which are based on the rank of matrix which are obtained from Kocay's lemma.

3 THE RANK OF MATRIX

If the matrix has large rank then it will have few non-reconstructible classes. Before moving to the actual theorem let us first discuss the following lemma which plays an important role in the proof of theorem:-

3.1 Lemma:

$$dim(\mathcal{W}) \geq \alpha(\mathcal{C}_n)$$

Proof:

CASE 1: If $\alpha(\mathcal{C}_n) = 0$ then the proof is trivial as $dim(\mathcal{W}) \geq 0$ is always true.

CASE 2: If $\alpha(\mathcal{C}_n) \neq 0$ then let us consider $R_1, R_2, \ldots, R_k \in \mathcal{C}_n/\cong$ as non-reconstructible reconstruction classes in $\mathcal{C}_n$. Further suppose $R_{i,j}$ be the reconstruction classes in R_i $1 \leq i \leq k$, $1 \leq j \leq q_i$ where $q_i = \mid R_i/\cong \mid > 1$. Let us consider $G_{i,j}$ as representation graph from $R_{i,j}$.

Now let us define a vector $v^{i,j}$ in $\mathbb{R}^{|\mathcal{C}_n/\cong|}$ for each $G_{i,j}$. And we index its entries by unlabelled graph H in isomorphic class of $\mathcal{C}_n$, which is defined as follows:-

$$v^{i,j} = s(H, G_{i,j}) - s(H, G_{i,1})$$

As we can see that to complete the proof of this lemma we have to show that this vector has following two properties:-

(i) $\forall i \in \{1, 2, ..., p\}$ and $\forall j \in \{1, 2, ...q_i\}$ vector $v^{i,j} \in \mathcal{W}$ and

(ii) If we consider vector set $U = \{v^{i,j} | 1 \leq i \leq p, 2 \leq j \leq q_i\}$ then vectors in set are non-zero and linearly independent, $\mid U \mid = \alpha(\mathcal{C}_n$.

Proof (i):

Since $G_{i,j}$ and $G_{i,1}$ are reconstructions of each others. $\mathcal{C}_n$ satisfies Kocay's lemma, therefore,

$$\sum_{H \in \mathcal{C}_n/\cong} s(\mathcal{F}, H)\ \ s(H, G_{i,j}) = \sum_{H \in \mathcal{C}_n/\cong} s(\mathcal{F}, H)\ \ s(H, G_{i,1})$$

so $\forall$ row $M_{\mathcal{F}}$ of matrix $M_{\xi, \mathcal{C}_n/\cong}$,

$$M_{\mathcal{F}}.v^{i,j} = \sum_{H \in c_n/\cong} s(\mathcal{F}, H) s(H, G_{i,j}) - s(H, G_{i,1}) = 0$$

$$\implies M_{\mathcal{F}}.v^{i,j} = 0$$

$$\implies M_{\xi, \mathcal{C}_n/\cong}.v^{i,j} = 0$$

hence, $v_{i,j} \in \mathcal{W}$, $\forall i \in \{1, 2, ..., p\}$ and $\forall j \in \{1, 2, ...q_i\}$

Proof (ii):

Let $u^1, U^2, \ldots, u^{\alpha(\mathcal{C}_n)}$ be ordered vectors in set U. If we prove vector u_1 is nonzero, vector u_k is nonzero $\forall k\ 2 \leq k \leq \alpha(\mathcal{C}_n)$, and linearly independent of $u_1, u_2, \ldots, u_{k-1}$ then we can show that the vectors in set U are linearly independent.

Now suppose for some $i \in \{1, 2, \ldots, p\}$ and $j \in \{2, 3, \ldots, q_i\}$ $u_l = v_{i,j}$. Note that $\mathcal{C}_n$ is recognizable $G_{i,j} \in R_i/\cong$.

$$\because R_i \in \mathcal{C}_n/\cong \implies G_{i,j} \in \mathcal{C}_n/\cong$$

Further, $\because j \geq 2 \implies G_{i,j}$ and $G_{i,1}$ are isomorphic to each other,
and also, $G_{i,j}, G_{i,1} \in R_i/\cong$ but they do not belong to same isomorphic class.

$$i.e.\ e(G_{i,j}) = e(G_{i,1})$$

$\therefore u^l(G_{i,j}) = v^{i,j}(G_{i,j})$

$\implies u^l(G_{i,j}) = s(G_{i,j}, G_{i,j}) - s(G_{i,j}, G_{i,1}) = 1 - 0 = 1$

Now suppose $u_k = v_{\bar{i},\bar{j}}, 1 \leq\ k \leq l$,

Then to prove $u^k(G_{i,j}) = 0$

. $\because k \leq l \implies e(G_{\bar{i},\bar{j}}) = e(G_{\bar{i},1})$ and $G_{\bar{i},\bar{j}}$ & $G_{\bar{i},1}$ are reconstruction of each other,
$\implies e(G_{\bar{i},\bar{j}}) = e(G_{\bar{i},1})$, Now,

$e(G_{\bar{i},\bar{j}}) < e(G_{\bar{i},1}) \implies u_k(G_{i,j}) = v^{\bar{i},\bar{j}}(G_{i,j})$

$\implies u_k(G_{i,j}) = s(G_{i,j}, G_{\bar{i},\bar{j}}) - s(G_{i,j}, G_{\bar{i},1}) = 0 - 0 = 0$

$\implies u_k(G_{i,j}) = 0$

and if $e(G_{\bar{i},\bar{j}}) = e(G_{\bar{i},1})$,

then $s(G_{i,j}, G_{\bar{i},\bar{j}}) = 0$ $\{\because G_{i,j}$ & $G_{\bar{i},\bar{j}}$ are non isomorphic and having equal number of edges.

and, $s(G_{i,j}, G_{\bar{i},1}) = 0$ $\{\because j \geq 2$.

$\implies u_k(G_{i,j}) = 0$.

In both cases $u_k(G_{i,j}) = 0 \implies$ vectors in U are linearly independent.

Now by definition, $\alpha(\mathcal{C}_n) =\mid \mathcal{C}_n/\cong\mid - \mid \mathcal{C}_n/\sim\mid$

$\implies \qquad \alpha(\mathcal{C}_n) = \sum i = 1^p(a_i - 1)$

$\implies \qquad \alpha(\mathcal{C}_n) =\mid U \mid$.

This completes the proof.

3.2 Theorem:

If $\mathcal{C}_n$ is the finite, recognizable class satisfying Kocay's lemma and ξ be the family of sequences of graphs. Let $M_{\xi,\mathcal{C}_n/\cong}$ the corresponding matrix then,

$$\mid \mathcal{C}_n/\sim\mid \geq rank(M_{\xi,\mathcal{C}_n/\cong})$$

Proof: Note that the Rank-Nullity theorem is defined as:-

$$Rank \quad + \quad Nullity \quad = \quad Number\ of\ colums$$

Where nullity is defined as the number of non-zero row of a matrix in Echelon form.
Hence by using Rank-Nullity theorem,

$$rank(M_{\xi,\mathcal{C}_n/\cong}) \ + \ dim(\mathcal{W}) \ = \mid \mathcal{C}_n/\cong\mid$$

Now using lemma 3.1 we get

$$rank(M_{\xi,\mathcal{C}_n/\cong}) \ + \ \alpha(\mathcal{C}_n) \ \leq \ rank(M_{\xi,\mathcal{C}_n/\cong}) \ + \ dim(\mathcal{W}) \ = \mid \mathcal{C}_n/\cong\mid$$

recall the definition of $\alpha(\mathcal{C}_n)$ as $\alpha(\mathcal{C}_n) \ = \mid \mathcal{C}_n/\cong\mid \ - \ \mid \mathcal{C}_n/\sim\mid$
we get,

$$rank(M_{\xi,\mathcal{C}_n/\cong}) + \ \mid \mathcal{C}_n/\cong\mid \ - \ \mid \mathcal{C}_n/\sim\mid \leq \mid \mathcal{C}_n/\cong\mid$$

$$\Longrightarrow \ rank(M_{\xi,\mathcal{C}_n/\cong}) - \ \mid \mathcal{C}_n/\sim\mid \leq 0$$

$$\Longrightarrow \ rank(M_{\xi,\mathcal{C}_n/\cong}) \ \leq \mid \mathcal{C}_n/\sim\mid$$

which completes the proof of the theorem.

4 Existence of Matrix with Optimal Rank

4.1 Theorem

Suppose $\mathcal{C}_n$ be the recognizable class satisfying Kocay's lemma, then $\exists\ \xi$, a family of sequences of graphs with corresponding matrix of covering numbers $M_{\xi,\mathcal{C}_n/\cong}$ such that,

$$rank(M_{\xi,\mathcal{C}_n/\cong}) = \mid \mathcal{C}_n/\sim \mid.$$

Proof:

let us consider ξ as family of all sequences of graphs of length n which are in-equivalent. Two sequences $\mathcal{F}_i$ & $\mathcal{F}_j$ are called inequivalent sequences if for every one-one onto map $f : \mathcal{F}_i \to \mathcal{F}_j\ \ \exists$ at least one graph $F \in \mathcal{F}_i$ for which $f(F) \ncong F$.

Our assumption is here that ξ has only sequence of length at least 2 because the covering number for sequence of length 1 is zero. Consider $M_{\xi,\mathcal{C}_n/\cong}$ as corresponding matrix. Let $\tilde{cv}(\mathcal{F}, G)$ denotes non-overlapping cover of graph G by sequence $\mathcal{F}$. let $\mathcal{F} = (F_1, F_2, \ldots, F_r) \in \xi$.

The recurrence relation for $cv(\mathcal{F}, G)$ is:-

$$cv(\mathcal{F}, G)\ =\ \sum_{k=2}^{r}\ \sum_{q\in\mathcal{Q}_r^k}\ \sum_{\mathcal{H}=(H_1,H_2,\ldots,H_k)} \gamma(\mathcal{H})\ \tilde{cv}(\mathcal{H}, G)\ \prod_{i=1}^{k} cv(\mathcal{F}|q^{-1}(i), H_i)$$

where $\mathcal{Q}_r^k$ denotes the set of all onto functions $\ q : \{1, 2, .., r\} \to \{1, 2, ..., k\},\ \ \ \mathcal{F}|q^{-1}(i)$ is the subsequence of $\mathcal{F}$ and $F_j \in \mathcal{F}|q^{-1}(i),\ \ j \in q^{-1}(i)$.

Every cover of G be $\mathcal{F}$ naturally implies to a partition of $\{1, 2, ..., r\}\ \ in\ \ k$ blocks for some $k \in [2, r]$.

$$i,\ j \text{ are in same partition} \implies G_1\ \ \&\ \ G_2 \text{ have same vertex set.}$$

The subsequence of $\mathcal{F}$ having indices $j \in q^{-1}(i)$ is denoted by $\mathcal{F}|q^{-1}(i)$. Now if we consider a cover of G by $\mathcal{H} = (H_1, H_2, ..., H_k)$ then it is non overlapping and we can cover every H_i by $\mathcal{F}_j$ in $cv(\mathcal{F}|q^{-1}(i), H_i)$ ways.

$\because$ we can not find such any cover of G by $\mathcal{F}$ that all G_i have same vertex set

$\therefore$ There is no need to consider trivial partition into single block.

If σ_1 & σ_2 are two mutually non isomorphic graphs and k_1 copies of σ_1 & k_2 copies of σ_2 are contained by sequence $\mathcal{H}$, then, the factor $\gamma(\mathcal{H}) = (\prod_i k_i!)^{-1}$.

After rearranging the term we get,

$$\tilde{cv}(\mathcal{F}, G)\ =$$

$$cv(\mathcal{F}, G)\ \ -\ \ \sum_{k=2}^{r-1}\ \sum_{q\in\mathcal{Q}_r^k}\ \sum_{\mathcal{H}=(H_1,H_2,\ldots,H_k)} \gamma(\mathcal{H})\ \tilde{cv}(\mathcal{H}, G)\ \prod_{i=1}^{k} cv(\mathcal{F}|q^{-1}(i), H_i)$$

In the above formula we can see that constant $cv(\mathcal{F}|q^{-1}(i), H_i)$ does not contains G.
and If $r = 2 \implies \tilde{cv}(\mathcal{F}, G) \ = \ cv(\mathcal{F}, G) \ - \ 0 \implies \tilde{cv}(\mathcal{F}, G) \ = \ cv(\mathcal{F}, G)$.

Hence by applying the above equations again and again to the terms that are containing non overlapping covering numbers we get,

$$\tilde{cv}(\mathcal{F}, G) \ = \ \sum_{\bar{\mathcal{F}}} \theta_{\mathcal{F}}(\bar{\mathcal{F}}) \ cv(\bar{\mathcal{F}}, G)$$

Where coefficient $\theta_{\mathcal{F}}(\bar{\mathcal{F}})$ show that they have been come from factors $cv(\mathcal{F}|q^{-1}(i), H_i)$ & $\gamma(\mathcal{H})$ is independent of G.
Hence in general,

$$\tilde{cv}(\mathcal{F}, \ *) \ = \ \sum_{\bar{\mathcal{F}}} \theta_{\mathcal{F}}(\bar{\mathcal{F}}) \ cv(\bar{\mathcal{F}}, \ *)$$
$$\implies rank(\tilde{M}_{\xi, \mathcal{C}_n/\cong}) \ \leq \ rank(M_{\xi, \mathcal{C}_n/\cong})$$

Now if we show that,
$rank(M_{\xi, \mathcal{C}_n/\cong}) \ = |\ \mathcal{C}_n/\sim|$
Then we done with proof of this theorem.
For that we construct a sub-matrix A of $rank(\tilde{M}_{\xi, \mathcal{C}_n/\cong})$. Further let $R_i \ = \ \mathcal{C}_n/\sim$, $i = 1, 2, ...$ we choose a reconstruction $G_i \in R_i/\cong$ arbitrarily for each reconstruction R_i. We will index the by the sequence $\mathcal{F}_i \simeq \ (G_i \ - \ v)$ and all other rows and columns of $\tilde{M}_{\xi, \mathcal{C}_n/\cong}$ And ultimately we will show,
matrix A has full rank $\implies rank(M_{\xi, \mathcal{C}_n/\cong}) \geq rank(A) \ = |\ \mathcal{C}_n/\sim|$
Let us define a partial order $\leq$ on $\mathcal{C}_n/\sim$ as $R_i \ \leq \ R_i$ if $\exists$ a one one onto map $f : V(G_i) \to V(G_j)$ such that $v \in V(G_i)$
$G_i \ - \ v \ \cong \ G_j \ f(v)$

First let us show that $\leq$ is partial ordered set on $\mathcal{C}_n/\sim$.

(i) $\mathcal{C}_n/\sim, \leq)$ is reflective (Trivial)
(ii) $\mathcal{C}_n/\sim, \leq)$ is transitive (Trivial) (iii) Let $f : V(G_i) \to V(G_j)$ is bijection
$\therefore \ \forall v \in V(G_i)$
$e(G_i \ - \ v) \ \leq \ e(G_j \ - \ f(v))$
let $f : V(G_i) \to V(G_j)$ is also a bijection
therefore, we can define a bijective composition map,

$gof : V(G_i) \to V(G_j)$ such that

$$\forall v \in V(G_i), \quad G_i \ - \ v \ \cong \ G_j \ gof(v)$$
$$\implies e(G_i \ - \ v) \ \leq \ e(G_j \ - \ f(v) \ \leq \ e(G_j, \ (gof)(v))$$
$$\implies e(G_i \ - \ v) \ \leq \ e(G_j, \ (gof)(v))$$

Taking summation over v on both side,

$\sum_{v} e(G_i \; - \; v) \; \leq \; \sum_{v} e(G_j, \; (gof)(v)).$

$\because$ gof is bijection,

$\implies e(G_i \; - \; v) \; = \; e(G_j, \; (gof)(v))$

$implies G_i \; - \; v \; \cong \; G_j \; f(v) \quad i.e. R_i = R_j$

Hence $(\mathcal{C}_n/\sim, \leq)$ is antisymmetric.
Therefore $\leq$ is partial ordered set on $\mathcal{C}_n/\sim$.

Now $\tilde{cv}(\mathcal{F}_i, \; G_j) \; > \; 0 \implies R_i < R_j$
Matrix A is upper triangular and also $\tilde{c}(\mathcal{F}_i, \; G_j) \; > \; 0 \quad \forall G_i$

A is full ranked matrix. $\implies rank(A) \; = |\, \mathcal{C}_n/\sim|$

Now we can apply the theorem 3.2 because $\mathcal{C}_n$ is recognizable and satisfying kocay's lemma.
$\therefore \quad rank(A) \; \leq \; rank(\tilde{M}_{\xi, \mathcal{C}_n/\cong}) \; \leq \; rank(M_{\xi, \mathcal{C}_n/\cong}) \; \leq |\, \mathcal{C}_n/\sim|$

$\implies |\, \mathcal{C}_n/\sim| \leq \; rank(M_{\xi, \mathcal{C}_n/\cong}) \; \leq |\, \mathcal{C}_n/\sim|$

$\implies rank(M_{\xi, \mathcal{C}_n/\cong}) \; = |\, \mathcal{C}_n/\sim|$

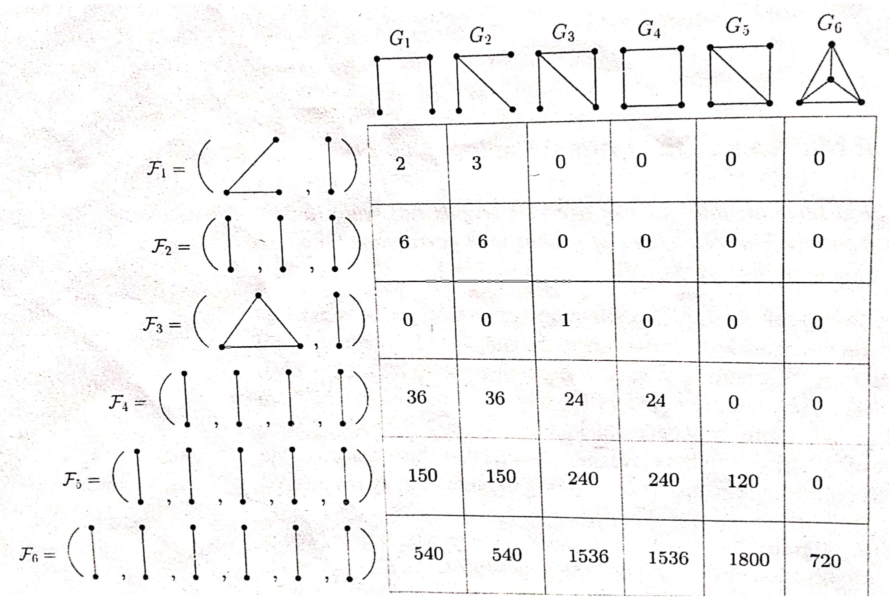
This completes the proof of this theorem.

5 Some Results and Conclusion

5.1 Some Results:

(i) In the theorem 3.2 we see saw that $(M_{\xi,\,\mathcal{C}_n/\cong})\ \leq |\,\mathcal{C}_n/\sim|$

Now under the same hypothesis if $rank(M_{\xi,\,\mathcal{C}_n/\cong})\ = |\,\mathcal{C}_n/\sim|$ then we can claim that every graph in $\mathcal{C}_n$ is recognizable.



	G_1	G_2	G_3	G_4	G_5	G_6
$\mathcal{F}_1$	2	3	0	0	0	0
$\mathcal{F}_2$	6	6	0	0	0	0
$\mathcal{F}_3$	0	0	1	0	0	0
$\mathcal{F}_4$	36	36	24	24	0	0
$\mathcal{F}_5$	150	150	240	240	120	0
$\mathcal{F}_6$	540	540	1536	1536	1800	720

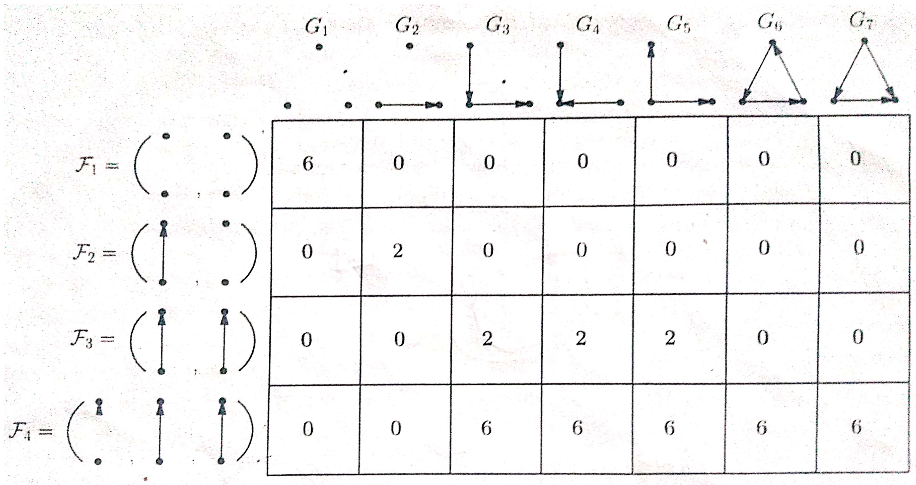
figure 1.

The result can be verified by the figure 1. There are six sequences of graphs and six connected graphs on 4 vertices. If there is 0 in i^{th} and j^{th} column it means there is no way to cover that graph by the graphs in sequence. And it is full ranked matrix that proves out claim to be true.

(ii) We can not improve the rank of matrix by adding more sequence of graphs.

To verify this result let us consider a non-trivial example of directed graphs as shown in figure 2.

We can observe the matrix of covering numbers on three vertices of directed graphs.



	G_1	G_2	G_3	G_4	G_5	G_6	G_7
$\mathcal{F}_1$	6	0	0	0	0	0	0
$\mathcal{F}_2$	0	2	0	0	0	0	0
$\mathcal{F}_3$	0	0	2	2	2	0	0
$\mathcal{F}_4$	0	0	6	6	6	6	6

figure 2.

Note that seven distinct graphs have been shown on 4 reconstruction classes where G_1 & G_2 are reconstructible graphs while G_3, G_4 & G_5 belong to the same reconstruction class. Similarly, G_6 & G_7 belong to the same reconstruction class. But the rank of matrix is still four and thai is the number of reconstruction classes. That verifies our result.

5.2 Conclusion

After going through all these results and theorems discussed above we can conclude that the equations which are obtained by applying Kocay's lemma provide not only the information about reconstruction of G but also they provide total number of graphs that are non-reconstructible. Further, we saw that the rank of matrix can not be improve by adding more sequence of graphs. Also we can say if the matrix is full ranked then every graph in that class is reconstructible.

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