EULER CIRCUITS AND THE KÃŰNIGSBERG BRIDGE PROBLEM

ANSHU SINGH (185035)*, National Institute of Technology, Hamirpur
ANSHUDHAR KUMAR SINGH(185032), National Institute of Technology, Hamirpur
ANSHIKA (185041), National Institute of Technology, Hamirpur

The Seven Bridges of KÃűnigsberg is a historically notable problem in mathematics. Its negative resolution by Leonhard Euler in 1736[1] laid the foundations of graph theory and prefigured the idea of topology.

The city of KÃűnigsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islandsâĂŤKneiphof and LomseâĂŤwhich were connected to each other, or to the two mainland portions of the city, by seven bridges. The problem was to devise a walk through the city that would cross each of those bridges once and only once.

Euler proved that the problem has no solution. The difficulty he faced was the development of a suitable technique of analysis, and of subsequent tests that established this assertion with mathematical rigor.

Additional Key Words and Phrases: Euler circuit, kÃűnigsberg bridge problem, graph theory, research paper summary

1 INTRODUCTION

Modern graph theory has seen many developments throughout the centuries, yet the remarkable beginning of graph theory was a âĂŸfeeble glanceâĂŹ which Leonard Euler directed towards the geometry of position. Euler undertook the development of mathematical formulation of now-famous as The Konigsberg Bridge Problem in his paper Commentarii Academiae Scientiarum Imperialis Petropolitanae in 1736.

2 KÃŰNIGSBERG

The story started in the town of KÃűnigsberg, Prussia on the banks of Pregel River. In the Middle Ages, KÃűnigsberg became a very important city and trading center. The healthy economy allowed the people of the city to build seven bridges across the river. The river divided the city into four regions. And according to lore, the people of the city decide to create a game with the goal being to devise a way in which they could walk around the city, through each bridge exactly once.

Though the problem has the appearance of an interesting puzzle, it does not involve measurements nor calculations. As first stated as LeibnizâĂŹs âĂIJgeometry of positionâĂİ , Euler suggested that the solution to the problem of the bridges only included position and hence was an example of the geometry of position. Euler introduces and utilizes now called graph theory in solving this famous problem. From such deceptively frivolous origins, graph theory grew into a powerful and deep mathematical theory.

3 EULER'S METHODOLOGY

Euler states the problem as follows :

âĂIJin KÃűnigsberg in Prussia, there is an island A , called the Kneiphof ; the river which surrounds it is divided into two branches, as can be seen in Fig. [1], and these branches are crossed

*All team members contributed equally to this documentation.

Authors' addresses: Anshu Singh (185035), 185035@nith.ac.in, National Institute of Technology, Hamirpur; Anshudhar Kumar Singh(185032), 185032@nith.ac.in, National Institute of Technology, Hamirpur; Anshika (185041), 185041@nith.ac.in, National Institute of Technology, Hamirpur.

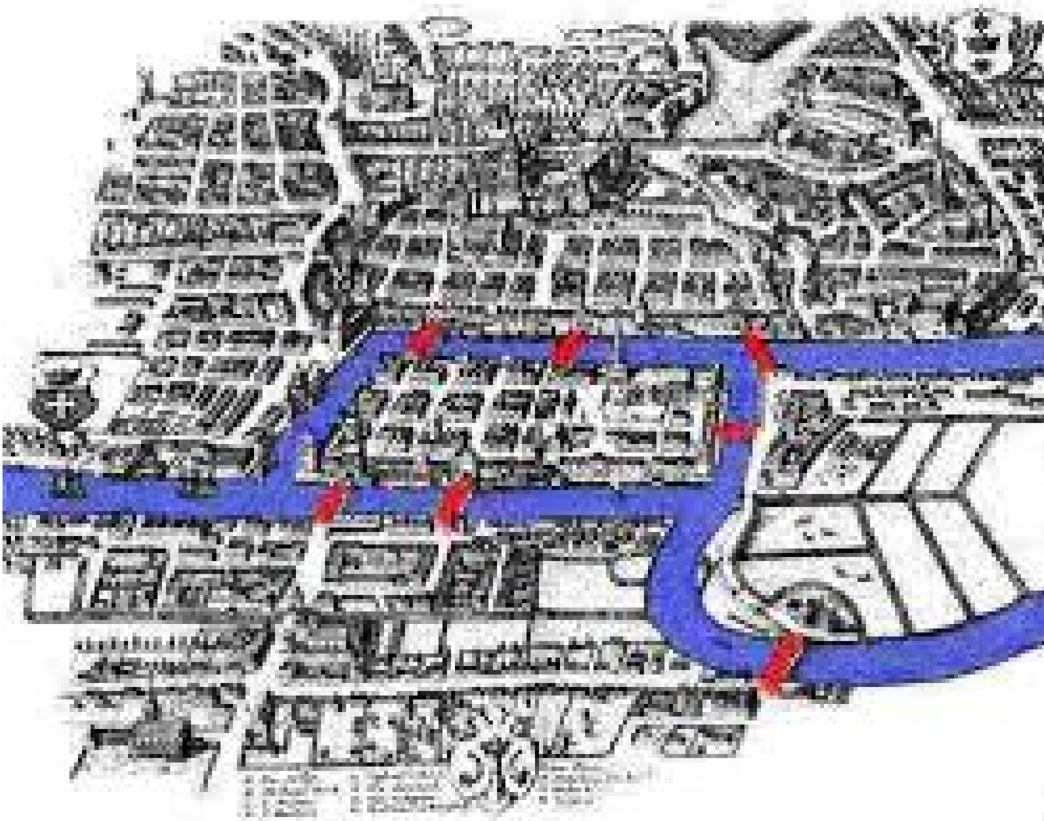

Fig. 1. The city of KÃűnigsberg.

by seven bridges, a,b,c,d,e,f and g . Concerning these bridges, it was asked whether anyone could arrange a route in such a way that he would cross each bridge once and only once. âĂİ

He first generalised the problem as :
âĂIJ whatever be the arrangement and division of the river into branches, and however many bridges there be, can one find out whether or not it is possible to cross each bridge exactly once? âĂİ

He began his analysis by replacing the land with points and bridges with line segments. He started with first ruling out the choice of making exhaustive lists of all possible routes as this task was too difficult as well as laborious. Hence, he first tries to find if there is such a path or not?

He then starts to name the land using capital letters and writes the route as follows;
âĂIJIf a traveller goes from A to B over bridge a or b , I write this as AB âĂŤ where the first letter refers to the area the traveller is leaving, and the second refers to the area he arrives at after crossing the bridge. Thus, if the traveller leaves B and crosses into D over bridge f , this crossing is represented by BD , and the two crossing AB and BD combined I shall denote by the three letters ABD , where the middle letter B refers to both the area which is entered in the first crossing and to

the one which is left in the second crossing.âĂİ

And hence by this method it was observed that for representing n bridges we need (n+1) points. If we can do that it means such a route will be possible.

And hence the modern day definition of walk i.e. a sequence of alternating vertices and edges in which both the order of vertices and edges used are specified. And if no edge is repeated, it is said to be path and if the terminal and initial vertex are equal, the path is said to be circuit.

The problem was therefore reduced to finding a sequence of letters. But before this euler turned to a sub problem that is, if there is such a sequence even possible?

For this purpose he turned to a rule where he only considered two areas, and tried to find out occurrences of landmass for different numbers of bridges.

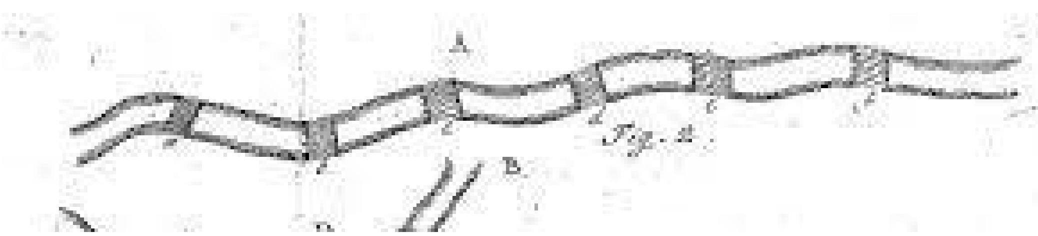

Fig. 2. City with two landmasses.

In the figure above, Letter A would occur one time if one bridge is considered, whether he starts from the A or not. If three bridges are considered, then A will occur twice in sequence and thrice if number of bridge is five and so on.

Hence, if bridges are odd in number then, letter will occur (n+1)/2 times.

Similarly on observing the even cases, letter A will occur n/2 times if it started from other end and (n/2 + 1) times if start from A. (n refers to the number of bridges)

And hence, the table for KÃűnigsberg bridges will be as follows : No of Bridges : 7;

Table 1. For KÃűnigsberg bridges

Region	No of Bridges	No of times one region must occur
A	5	3
B	3	2
C	3	2
D	3	2

However, 3+2+2+2 = 9, which is more than 8, so the journey is impossible .
Now the same logic applied to a rather following figure, the table yield is as follows :

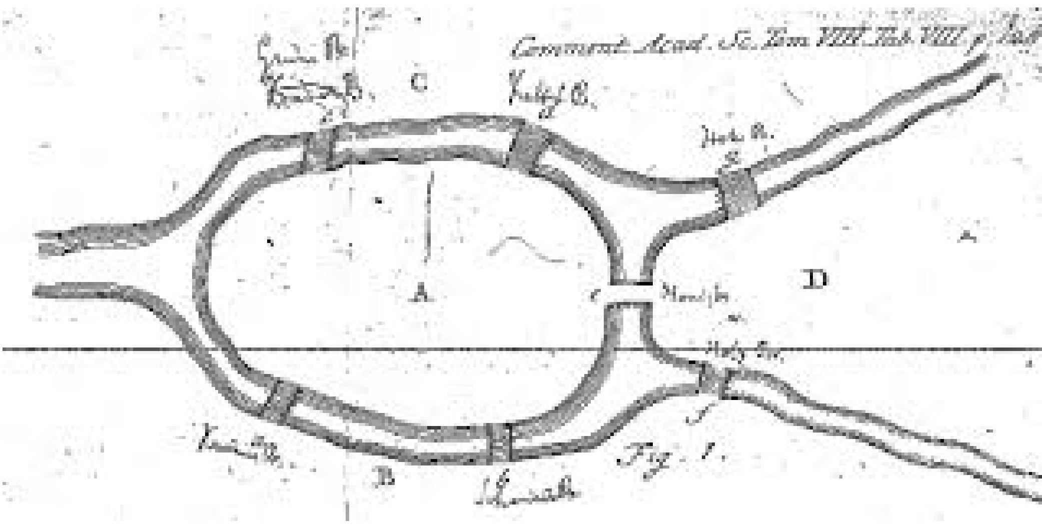

Fig. 3. Euler's Imaginary city

Table 2. For Fig.2 bridges

Region	No of Bridges	No of times one region must occur
A*	8	4
B*	4	2
C*	4	2
D	3	2
E	5	3
F*	6	3

By addition, we get 16 which equals to the number of bridges plus one, which means journey is, in fact, possible.

Also, if the total number of appearances is equal to the number of bridges plus one, journey should start from region in which there are odd number of bridges which leads to it. And if the number is equal to the number of bridges than it must be started from the even region.

4 EULERÂĂŹS CONCLUSIONS

In paragraph 16, Euler points out that the total of the numbers listed directly to the right of the landmasses adds up to twice the total number of bridges. This fact later becomes known as the handshaking lemma.

In Paragraph 17, Euler goes on to state that the sum of all the bridges leading to each region is even, since half of this number is equal to the total number of bridges. However, this is impossible if there are an odd number of landmasses with an odd number of bridges. Therefore, Euler proves that if there are some odd numbers attached to land masses, there must be an even number of these landmasses.

Euler then explains that it is obvious that if there are two landmasses with an odd number of bridges then the journey will always be possible if the journey starts in one of the regions with an odd number of bridges. This is because if the even numbers are halved, and each of the odd ones are increased by one and halved, the sum of these halves will equal one more then the total number of bridges. However, if there are four or more landmasses with an odd number of bridges, then it is impossible for there to be a path. This is because the sum of the halves of the odd numbers plus one along with the sum of all of the halves of the even numbers will make the sum of the third column greater than the total number of bridges plus one. Therefore, Euler just proved that there can be at most two landmasses with an odd number of bridges.

Euler gives the three guidelines that someone can use to figure out if a path exists using each bridge once and only once. First, he claimed if there are more than two landmasses with an odd number of bridges, then no such journey is possible. Second, if the number of bridges is odd for exactly two landmasses, then the journey is possible if it starts in one of the two odd numbered landmasses. Finally, Euler states that if there are no regions with an odd number of landmasses then the journey can be accomplished starting in any region. After stating these three facts, Euler concludes his proof with Paragraph 21, which simply states that after one figures out that a path exists, they still must go through the effort to write out a path that works. Euler believed the method to accomplish this was trivial, and he did not want to spend a great deal of time on it. However, Euler did suggest concentrating on how to get from one landmass to the other, instead of concentrating on the specific bridges at first.

5 TASK 1

Sketch the diagram of a graph with 5 vertices and 8 edges to represent the bridge crossing problem in the following figure.

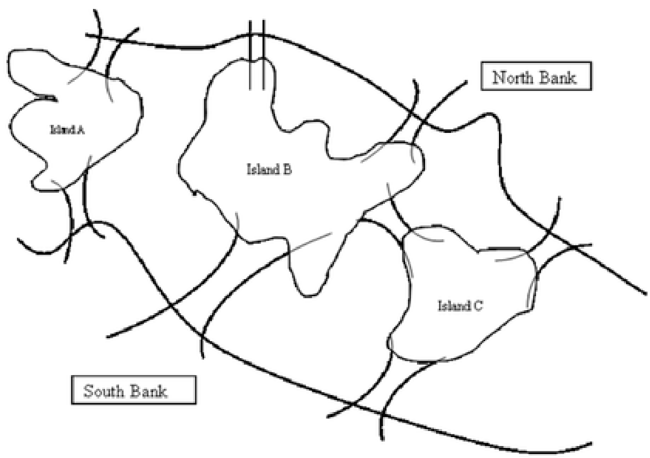

Fig. 4

For solving the problem we can assume the landmasses as the vertices and the bridges as the edges. Then we get the following graph :

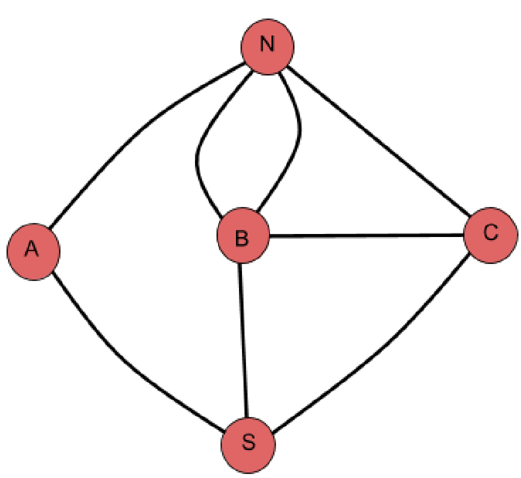

Fig. 5

6 TASK 2

For the bridge problem shown in Project TASK 1 above, how many capital letters (representing graph vertices) will be needed to represent an Euler path?

Solution)

From EulerâĂŹs paragraph 5, it says that we need number of bridges plus one letters for representing an Euler path.

Then according to that we should be needing (8+1) i.e. 9 letters to represent the path. Also from next paragraphs we can say there would be,
1 times A,
2 times letter B, C, N, S.

Hence, in total 9 letters.

7 TASK 3

In paragraph 8, Euler deduced a rule for determining how many times a vertex must appear in the representation of the route for a given bridge problem for the case where an odd number of bridges leads to the land mass represented by that vertex. Before reading further, use this rule to determine how many times each of the vertices A , B , C and D would appear in the representation of a route for the KÃűnigsberg Bridge Problem. Given EulerâĂŹs earlier conclusion (paragraph 5) that a solution to this problem requires a sequence of 8 vertices, is such a sequence possible? Explain.

Solution)

According to the observation recorded till paragraph 8, Euler found the number of occurrences of a letter by considering only one landmass as A (the other side of the bridge will be considered as B).

According to that rule, if one crosses one bridge that letter will occur 1 time, if there are three bridges, then the traveller will have a journey like ABAB or BABA, i.e. two times. Similarly for five bridges, it should be ABABAB or BABABA. Hence the letter should appear (n+1)/2 times.

Hence, for KÃűnigsberg problem, we will have, graph as :

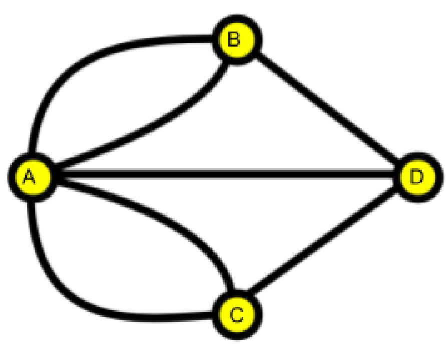

Fig. 6

Hence if we consider each vertices in the above graph, then vertex A have 5 brides, hence it should occur three times, and since each of the other vertices have three bridges attached they should occur two times each. Hence the total sequence should be of 3+2+2+2 = 9 digits. But the previous to this statement Euler already proved that the sequence should be one more than the number of bridges i.e. 8. Hence, such a sequence is not possible. Hence no Euler path is possible in the KÃűnigsberg problem.

8 REFERENCES

References used in preparing the document

https://google.com

https://medium.com/basecs/k%C3%B6nigsberg-seven-small-bridges-one-giant-graph-problem-2275d1670a1

https://www.maa.org/press/periodicals/convergence/leonard-eulers-solution-to-the-konigsberg-bridge-prol
~:text=Euler%20states%20that%20if%20the,like%20the%20K%C3%B6nigsberg%20Bridge%20problem

https://scilogs.spektrum.de/hlf/the-bridges-of-konigsberg/