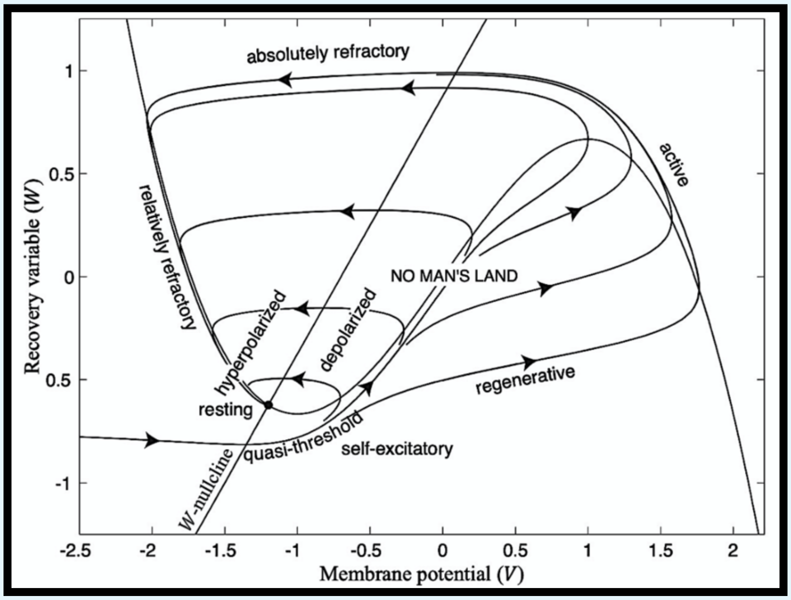
BT6270
COMPUTATIONAL NEUROSCIENCE
ASSIGNMENT 2

By Anshul Bagaria, BE21B005



Assignment Description

Simulating and Understanding the FitzHugh-Nagumo neuron model.

Question

Simulate the two variable FitzHugh-Nagumo Model using the following equations:

$\frac{dv}{dt} = f(v) - w + Im$ Where $f(v) = v(a-v)(v-1)$

$\frac{dw}{dt} = bv - rw$

Take a=0.5; Also choose b, r = 0.1.

Use single forward Euler Integration: $\frac{dv}{dt} = \frac{\Delta v}{\Delta t}$

This implies:

$$\Delta v(t) = v(t+1) - v(t) = [f(v(t)) - w(t) + I(t)] * \Delta t$$

Assumptions:

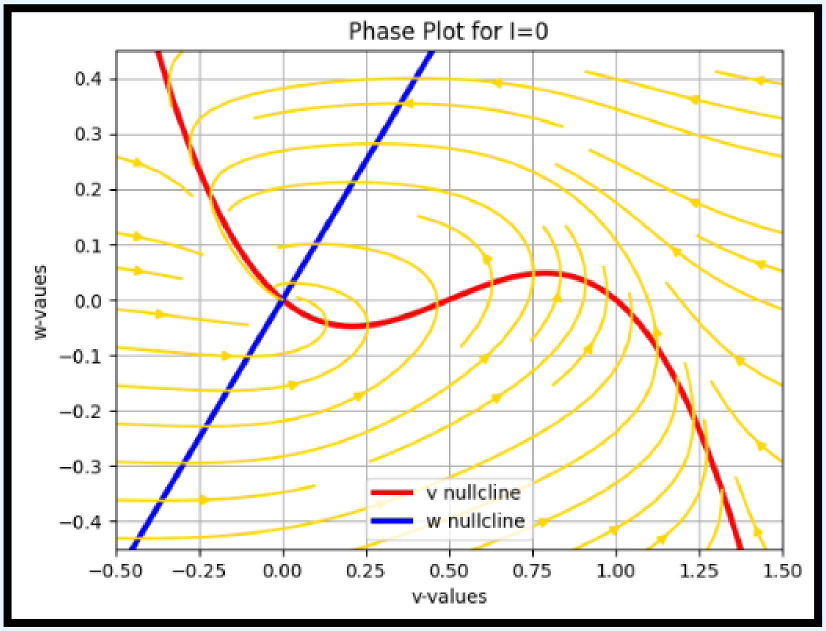
The following assumptions were made with respect to **FitzHugh-Nagumo Model**.

- It is a 2-neuron model, which is based on the limiting certain conditions in Hodgkin Huxley Model.
- The time scales of m, n and h are not all of same order. This is what gives rise to a 2 variable neuron model.
    - **a.** The time scale of m is much smaller than h, n
    - **b.** Also, h varies too slowly compared to m, n
- The model includes a recovery variable, w, which represents the inactivation and recovery processes on ion channels.
- The model reproduces the threshold behavior, where a certain level of input current is needed to initiate an action potential. This threshold concept is crucial in understanding excitable cells.
- The model describes the relationship between external stimulus (I (ext.)) and the cell's response, which is valid for characterizing how excitable cells respond to various inputs.
- The function f (v) = v*(a-v)*(v-1) used in the model introduces non-linearity, but it's a simplification of the complex dynamics of ion channel gating and membrane potential non-linearity.
- For getting the values of I_1, I_2, I have calculated the values mathematically on paper, correct up to 2 decimal places.
- For the various conditions of V (0), I have taken my values satisfying those specific conditions.
    - **a.** For example, for V (0) < a, I have taken V (0) as 0.4.
    - **b.** Also, for V (0) > a, I have taken V (0) as 0.6.
- Also, for the last case of bi-stability, I have considered one case that satisfies the bi-stability condition, i.e., the intersection of v-null-cline and w-null-cline has three real roots.
  However, we could have many more solutions for different values taken.

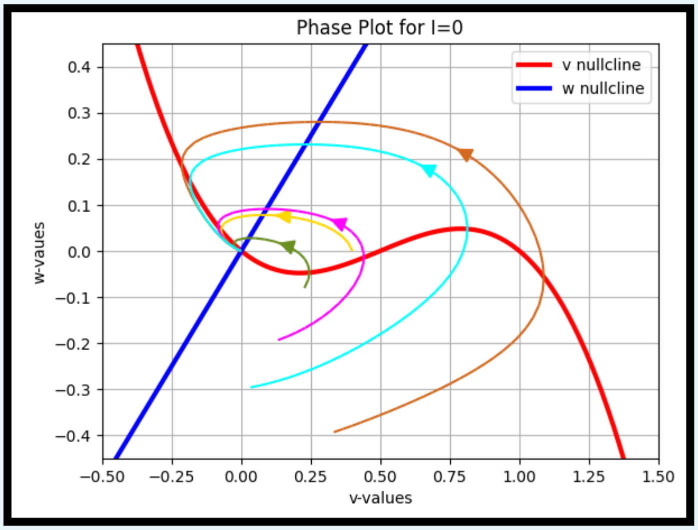
<u>Case 1:</u> $I_{ext.} = 0$

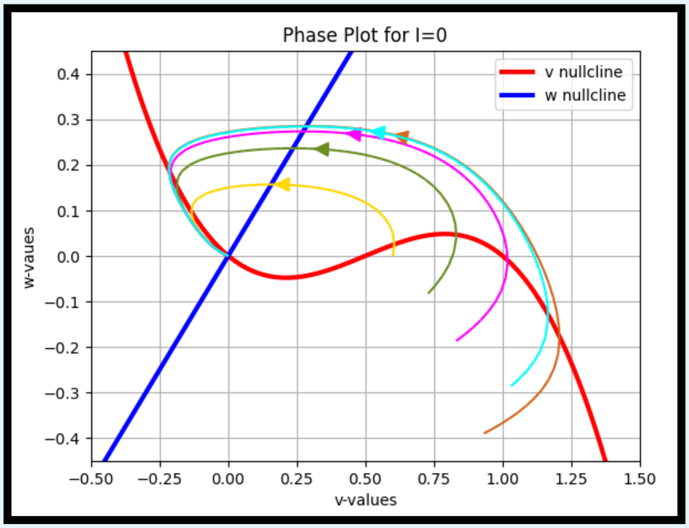
We are in the first region of the v - null cline.

a) ***Draw a Phase plot superimposed***



The above figure is the Phase plot of the system when I_{ext}=0. The stationary point so obtained is a stable point.

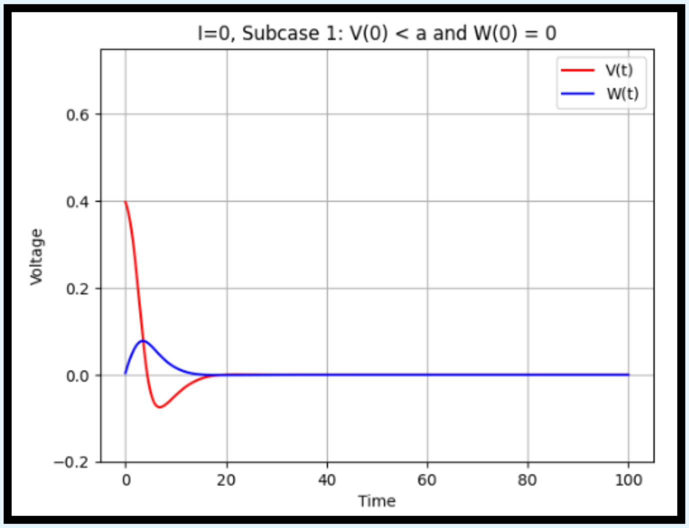




Analyzing the trajectories by using initial points, we can see that even for the perturbations in initial point, we approach the equilibrium at [0, 0]. Hence, the point is a stable fixed point.

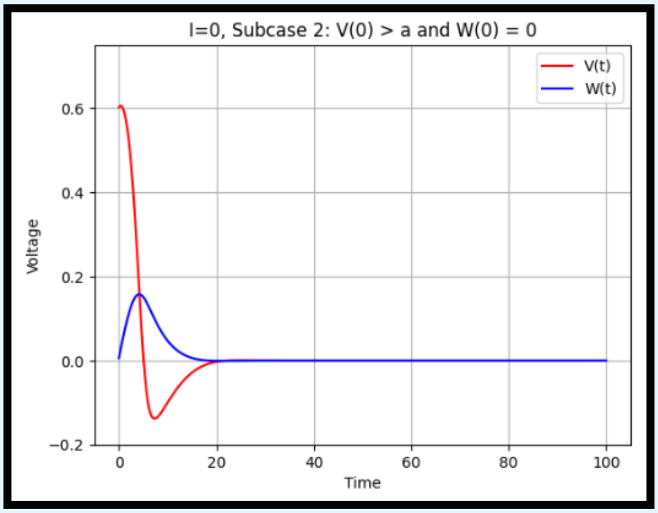
b)*Plot V(t) vs t and W(t) vs t and also show the trajectory on the phase plane for the cases below:*

i. **V(0) < a and W(0) = 0**



For an I_{ext} value of 0, no action potentials are observed.
In this scenario, where V (0) < a, with sub-threshold pulse injections, no action potentials are observed.

ii. **V(0) > a and W(0) = 0**



For an I_{ext} value of 0, no action potentials are observed.
In this scenario, where V (0) > a, with sub-threshold pulse injections, no action potentials are observed.

<u>Case 2:</u> Choose some current value $I_1 < I_{ext.} < I_2$ where it exhibits oscillations. Find the values of I_1 and I_2.

We are in the second region of the v - null cline.

In order to get the I_1 and I_2, we need to do some calculations.

We know that the v null cline is a cubic function, with two extreme points, one of which is a minima, while the other is the maxima.

Also, the w null cline is a straight line, which can cut the v null cline any number of times from 1 to 3.

The value of I_1 corresponds to when the w null cline cuts the v null cline at its minima.

The value of I_2 corresponds to when the w null cline cuts the v null cline at its maxima.

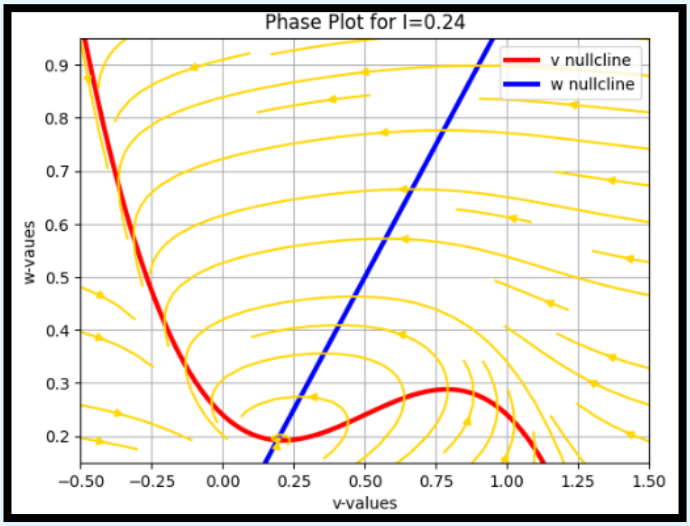
So, basically, if we find the minima and maxima of v null cline and look at the condition of I_{ext} for which the w null clines intersect v null cline at the minima and maxima, then we can find the values of I_1 and I_2.

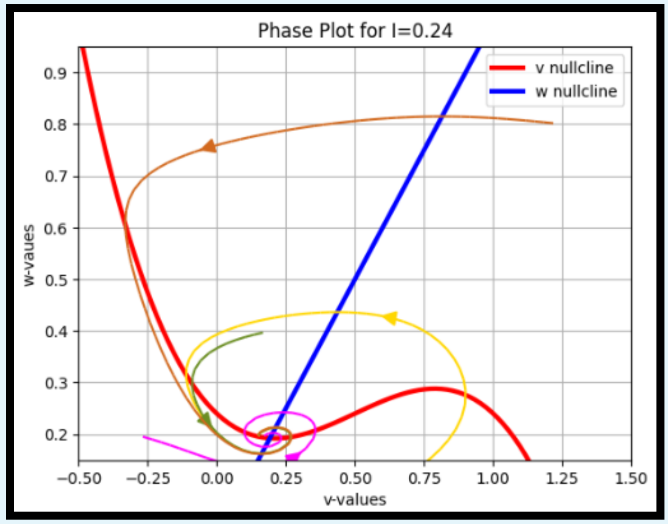
On solving the 2 equations in the above described manner, we get the values of I_1 and I_2 to be:

- **$I_1 = 0.26$**
- **$I_2 = 0.74$**

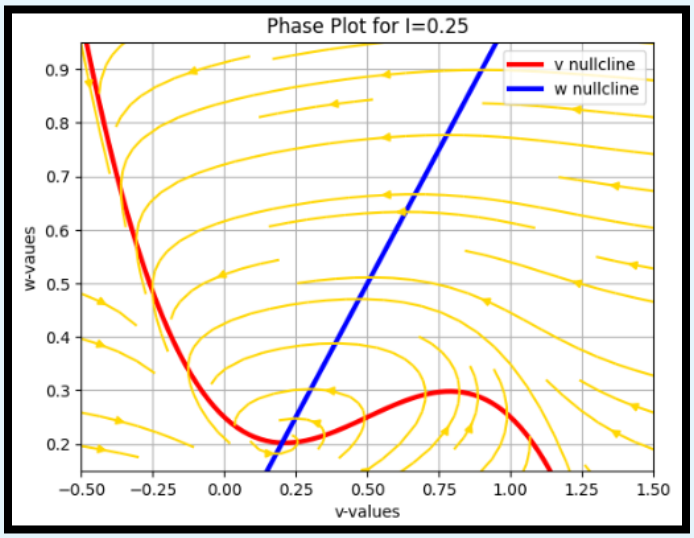
The above can be verified by looking at the transition in the phase plot, as we move from stable point to a limit cycle.

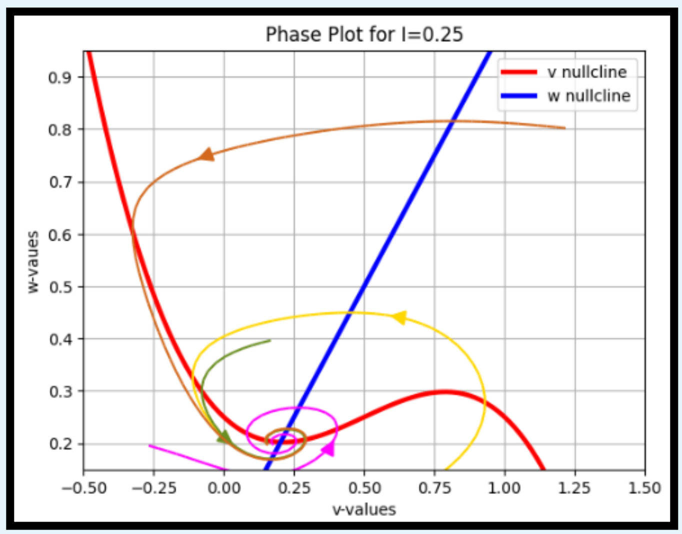
a) *Draw a Phase plot for some values of I_{ext}*



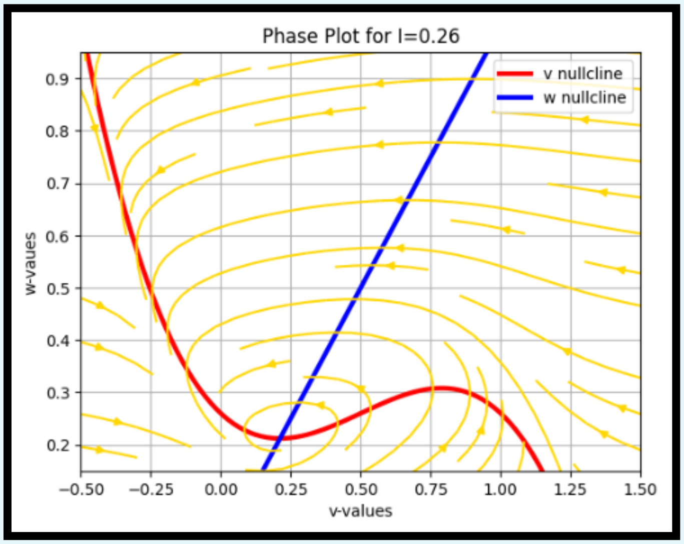


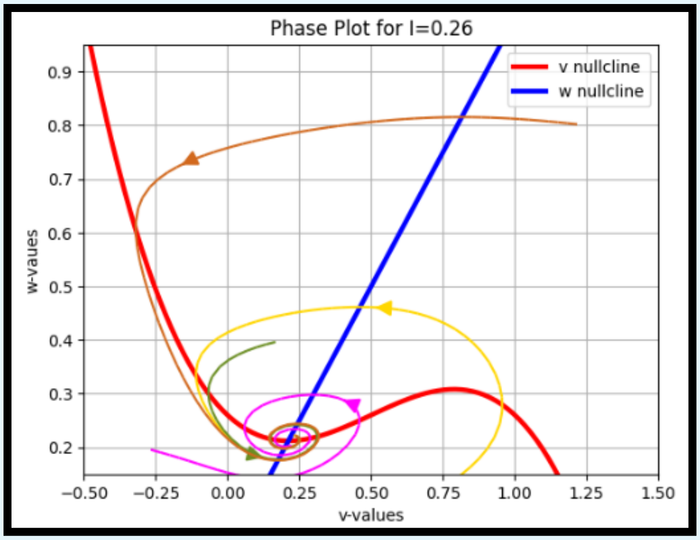
If we look at the phase plot and analyze the intersection point for I = 0.24 (< I_1), we can see that the point acts as a stable point.





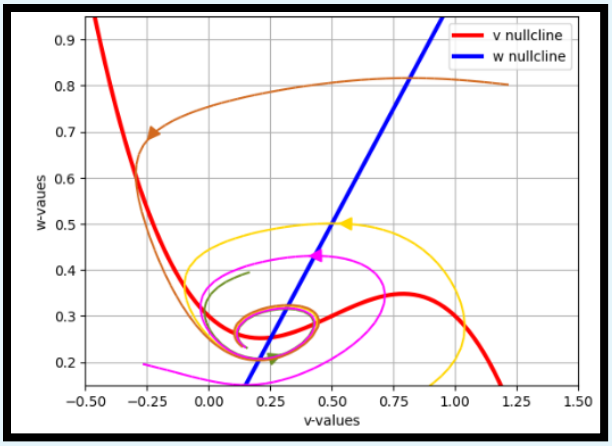
If we look at the phase plot and analyze the intersection point for I = 0.25 (<I_1), we can see that the point still acts as a stable point.



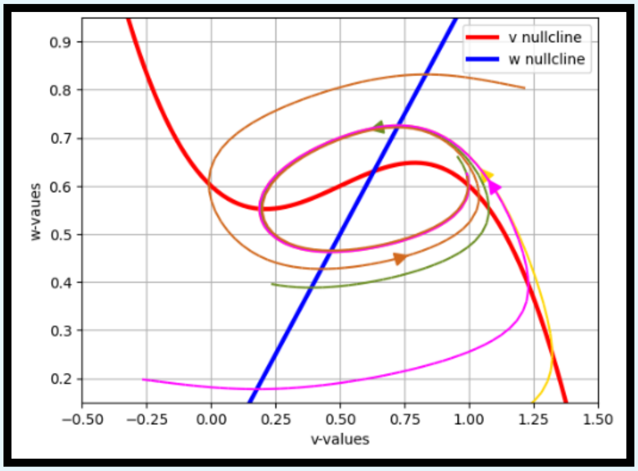


In the above scenario, we reach the value of I_1. However, we see the oscillation happening very close to the fixed point. On plotting the V (t) vs t and W (t) vs t for this value, we observe that the frequency is gradually decreasing.

This is indicative of the fact that though mathematically, we have entered the limit cycle region, there might be certain physical constraints that are yet to be overcome.

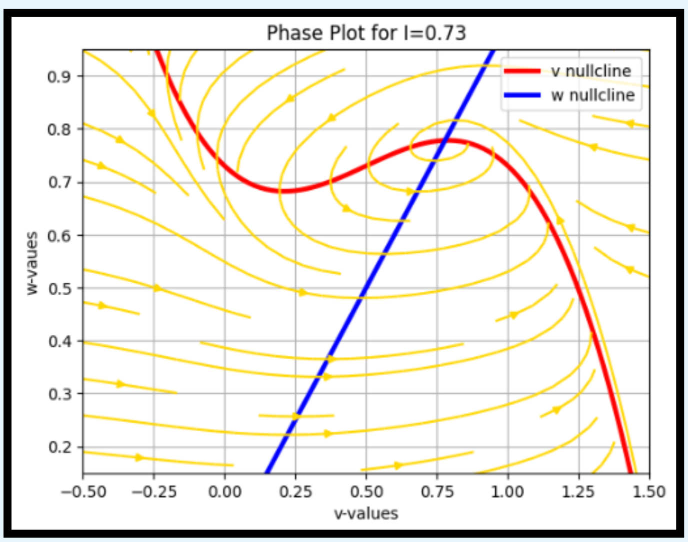


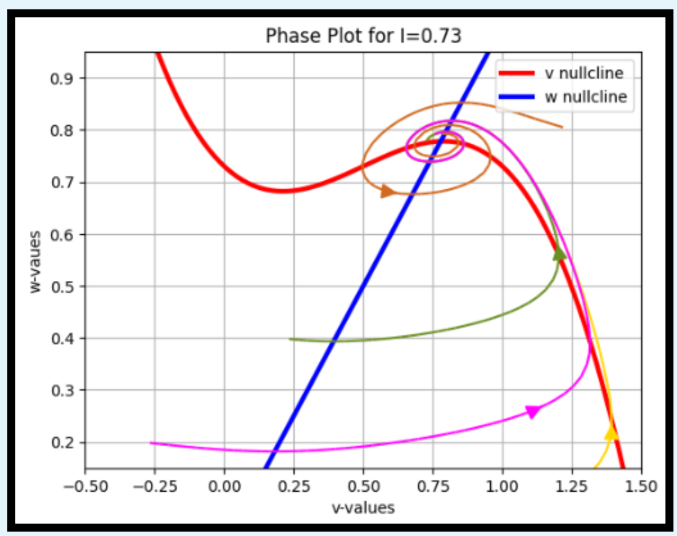
In this at I = 0.35, we can clearly see the oscillation about a point distant from the intersection point. Also, on analyzing the graphs V (t) vs t and W (t) vs t ahead, we will notice that the frequency of oscillations becomes constant from here on. This is indicative of the fact that the pure limit cycle region has been entered. Here, any starting point chosen, will eventually oscillate about a fixed ring.



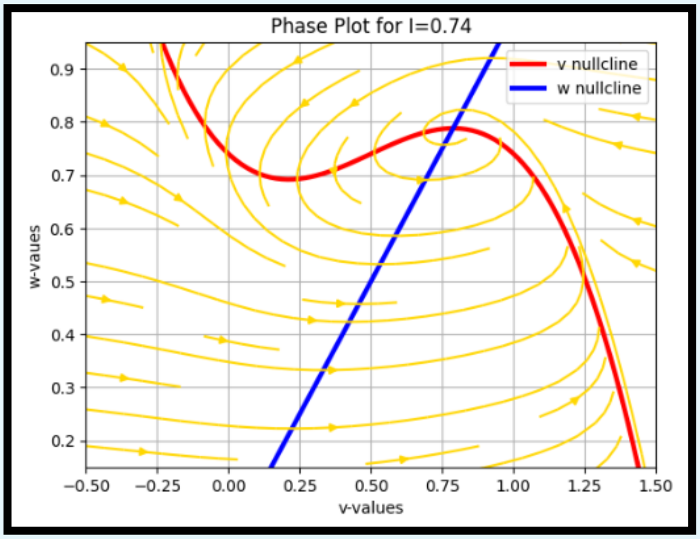
In this at I = 0.66, we can clearly see the oscillation about a point distant from the intersection point. Also, on analyzing the graphs V (t) vs t and W (t) vs t ahead, we will notice that the frequency of oscillations becomes decaying from henceforth. This is indicative of the fact that the pure limit cycle region has been exited at this point. Till here, any starting point chosen, will oscillate about a fixed ring.

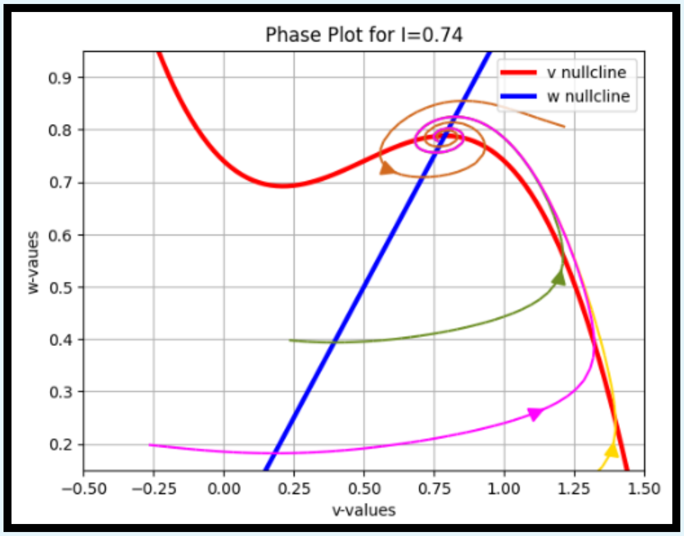
Beyond this point, the frequency of oscillations start decaying slightly.





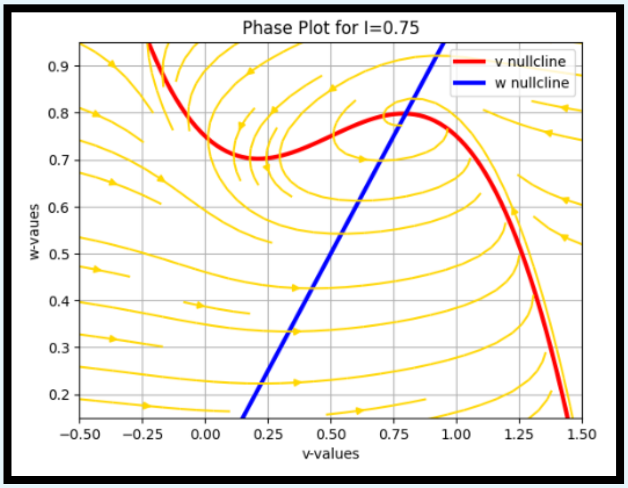
If we look at the phase plot and analyze the intersection point for I = 0.73 (< I_2), we can see that the point acts as a nearly stable point. It is almost converging to the point, yet not exactly converging.

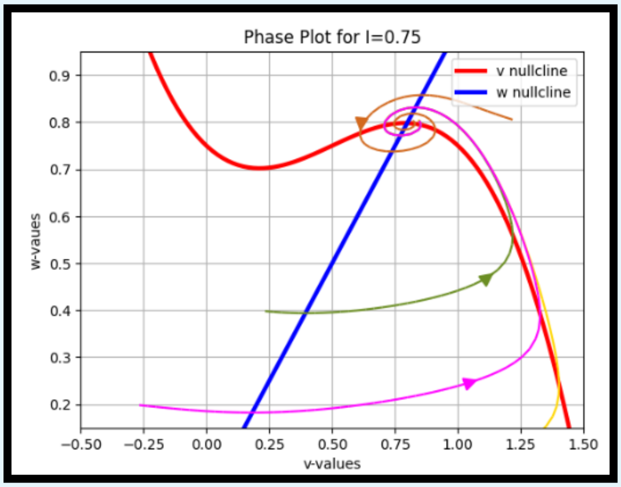




In the above scenario, we reach the value of I_2. However, we see the oscillation happening very close to the fixed point. On plotting the V (t) vs t and W (t) vs t for this value, we observe that the frequency has gradually started decreasing.

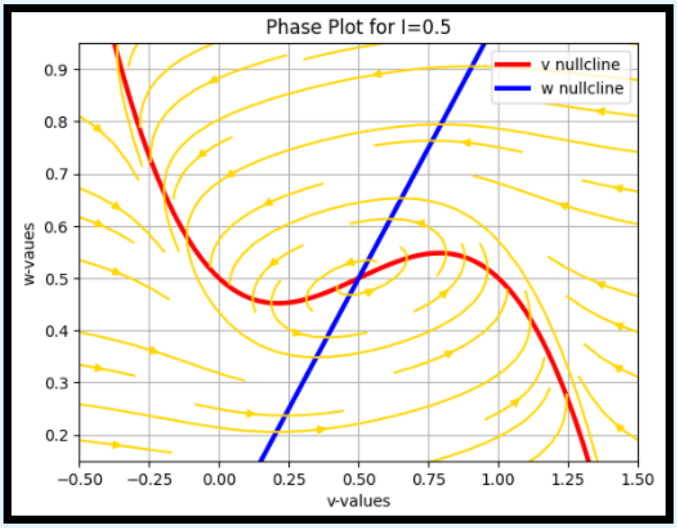
This is indicative of the fact that though mathematically, we have should have exited the limit cycle region here, there might be certain physical constraints owing to which we exited the limit cycle region early on itself.

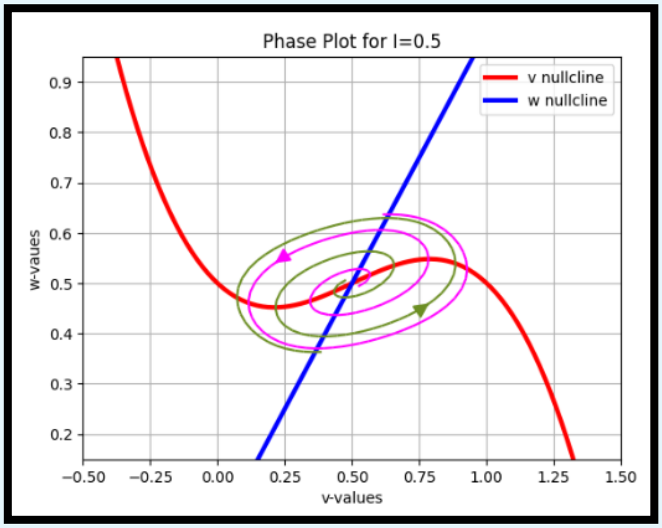




If we look at phase plot and analyze the point for I = 0.75 ($>I_2$), we can see that the intersection point acts as a stable point.

b) ***Show that the fixed point is unstable, i.e., for a small perturbation there is a no return to the fixed point (show the trajectory on the phase plane) – also show the limit cycle on the phase plane***





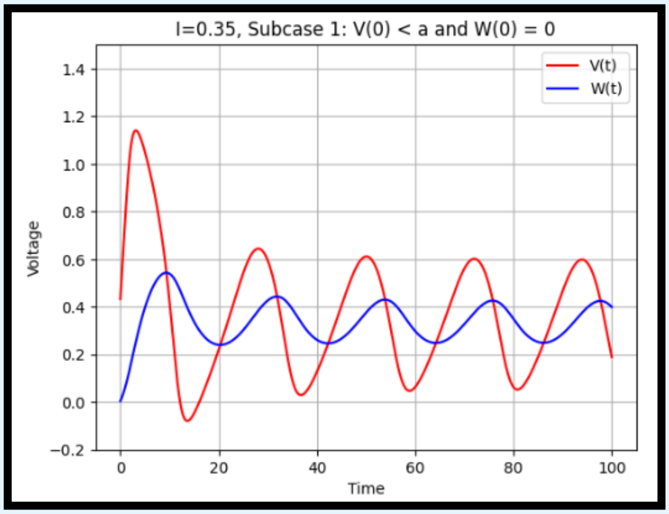
In order to exhibit the fixed point as unstable, I have chosen an arbitrary value of I_{ext} as 0.5. Close to this point I have taken two additional points for which I have done the phase analysis.

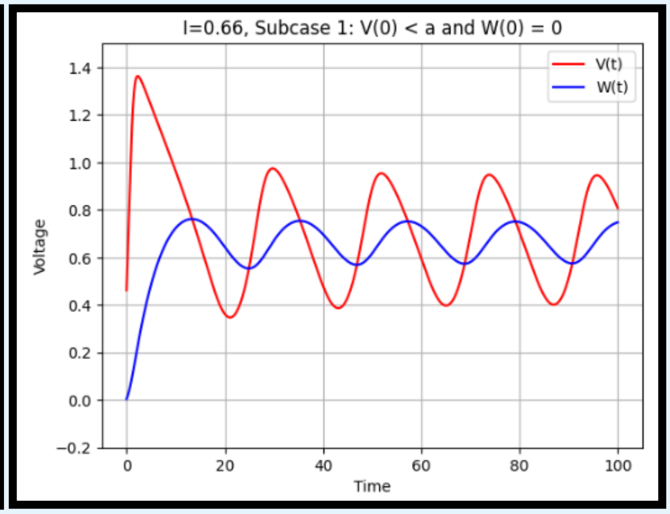
In the image on the right, we can see that the two points originating in close proximity to the fixed point, move away from the point and eventually start oscillating in some orbit. This is essentially the limit cycle behavior, where the points converge in some specific locus. Also, I have considered the point of intersection for the phase analysis. However, the point remains at the same location.

The above properties indicate that the fixed point is an unstable point, perturbation from which leads to it never returning back to the fixed point, and exhibiting a limit cycle behavior.

c) ***Plot V(t) vs t and W(t) vs t***

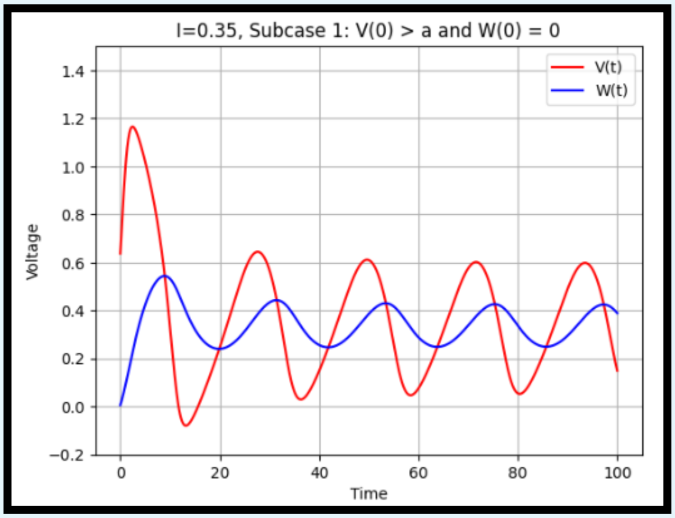
i. V(0) < a and W(0) = 0

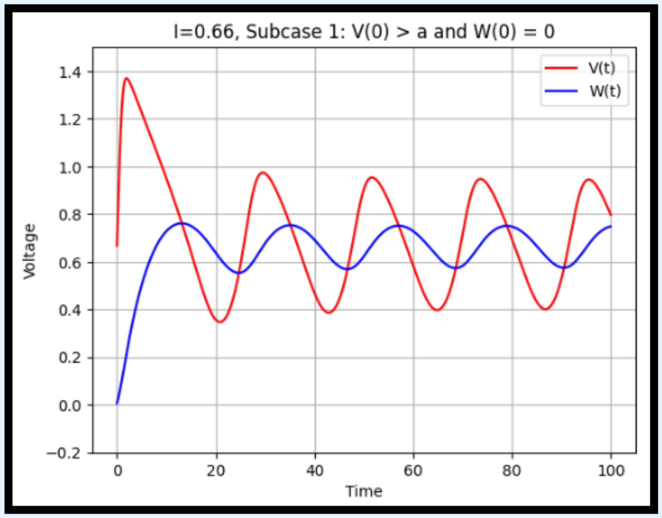




Sustained oscillations are observed here, in both the extremes of observed range for V (0) < a

ii. V(0) > a and W(0) = 0





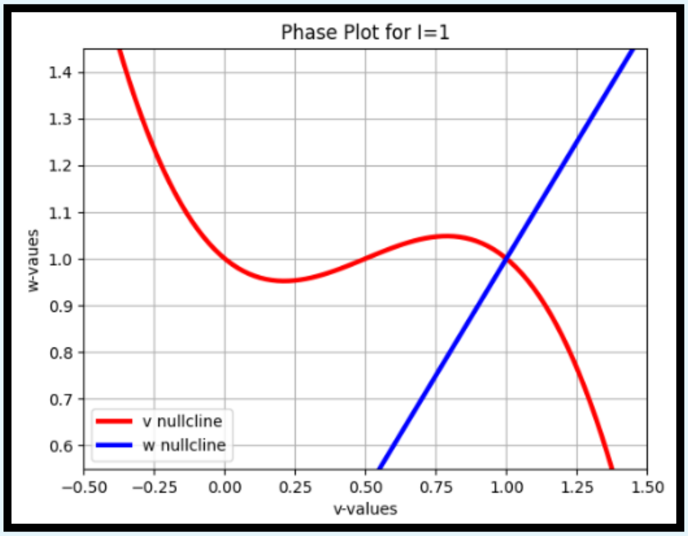
Sustained oscillations are observed here, in both the extremes of observed range, for V (0) > a

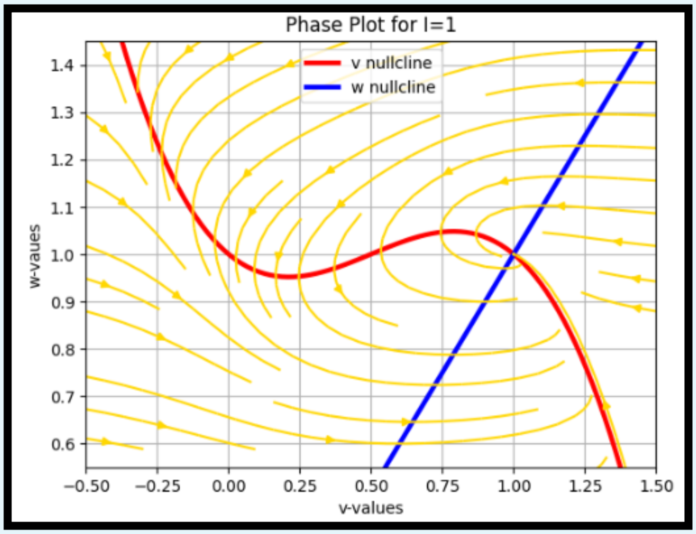
(However, we do not see sustained oscillations at the boundaries described in case of mathematical extremities)

Case 3: Choose some $I_{ext.} > I_2$

We are in the third region of the v - null cline.

a) ***Draw a Phase plot for some sample value of I_{ext}***





If we look at the phase plot and analyze the intersection point for I = 1 (> I_2), we can see that the point acts as a stable point.

Region 3 is very much like the region 1.

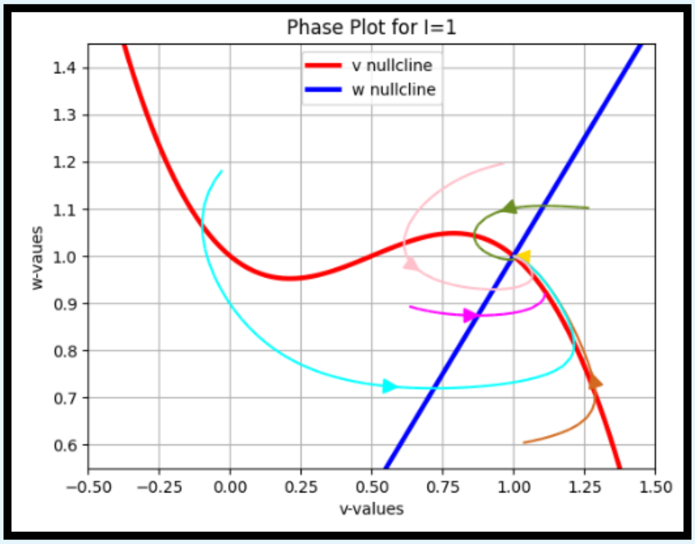
The fixed points in region 3 too are stable fixed points, i.e., for small perturbations from the fixed points in region 3, we shall return back to the same fixed point.

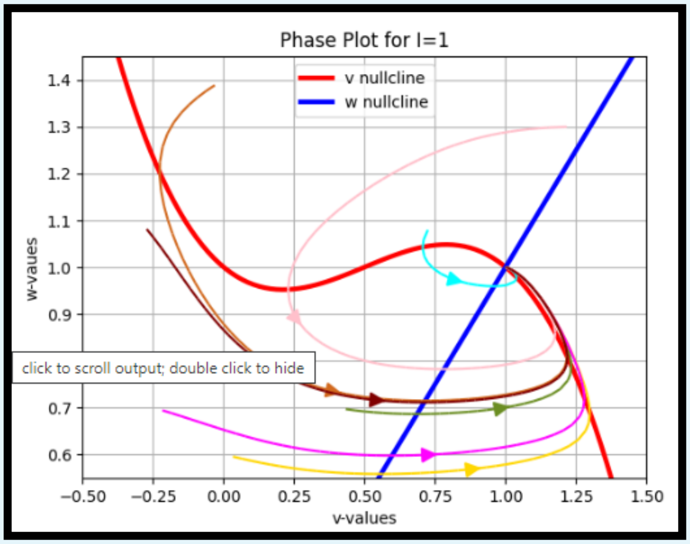
Why is the Region 3 much like Region 1?

This is because the 4 regions into which the phase plot is divided by the intersection of v null cline and w null cline behaves the same way in the cases of Region 1 and Region 3.

However, the reverse the case in Region 2. Hence region 1 and region 3 behave in a similar fashion, while region 2 is unique in terms of its behavior.

b)*Show that the fixed point is stable, i.e., for a small perturbation there is a return to the fixed point (show the trajectory on the phase plane)*





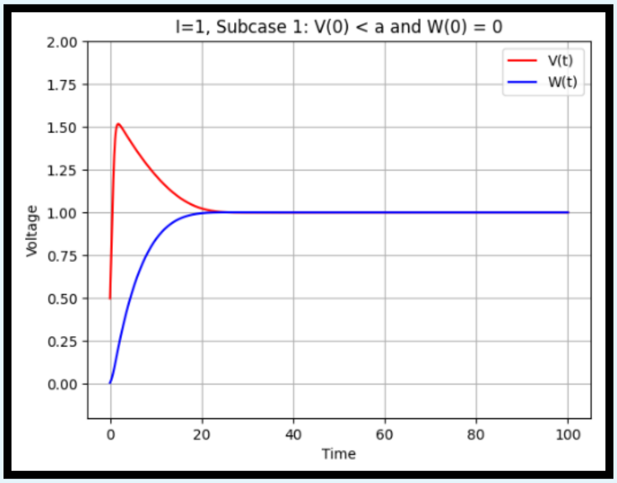
In order to exhibit the fixed point as stable, I have chosen an arbitrary value of I_{ext} as 1. Close to this point I have taken several additional points for which I have done the phase analysis.

In the images on left and right, we can see that any points originating in close or far proximity to the fixed point, move towards the point and eventually converge at the equilibrium point. This is essentially a stable point behavior, where the points converge. Also, I have considered the point of intersection for the phase analysis. The point remains at the same location.

The above properties indicate that the fixed point is a stable point, perturbation from which leads to it always returning back to the fixed point, and exhibiting a stable point behavior.

c) ***Plot V(t) vs t and W(t) vs t***

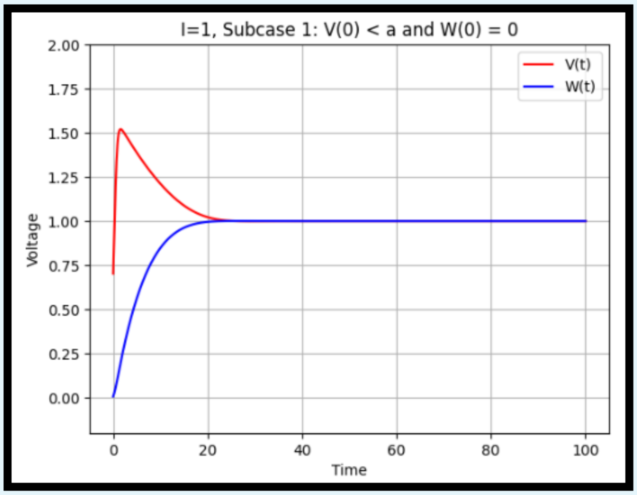
i. V(0) < a and W(0) = 0



For an I_{ext} value greater than I_2, the voltage is seen to rise and then stay at the high value.

In this scenario, V (0) < a, with sub-threshold pulse injections, depolarization in the action potential can be observed.

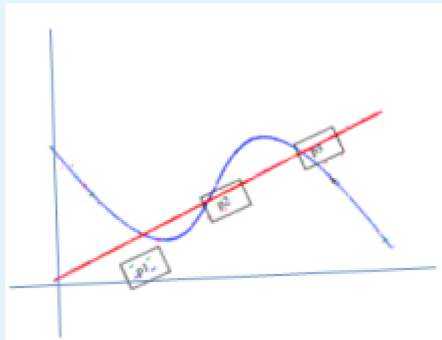
ii. V(0) > a and W(0) = 0



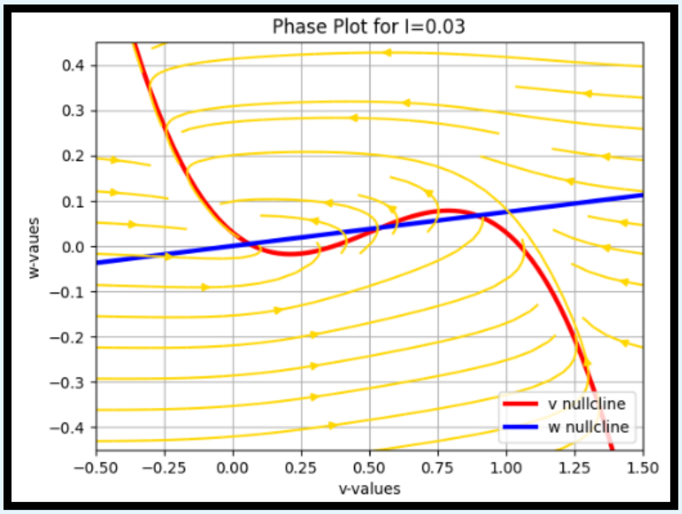
For an I_{ext} value greater than I_2, the voltage is seen to rise and then stay at the high value.

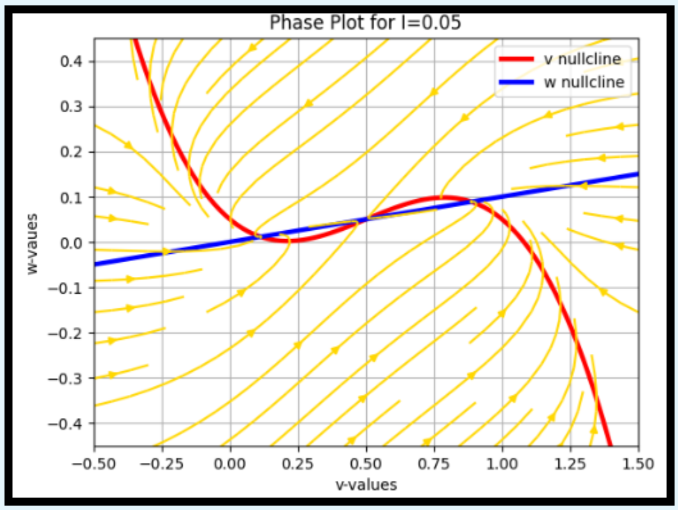
In this scenario, V (0) > a, with sub-threshold pulse injections, depolarization in the action potential can be observed.

<u>Case 4:</u> Find suitable values of I_{ext} and (b / r) such that the graph looks as phase plot shown below:



a) ***Redraw the Phase plot***



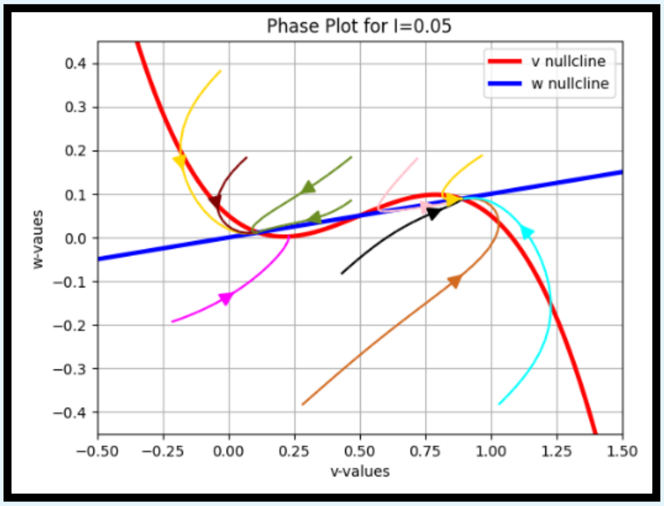


This is the phase plot for the scenario of bi-stability as depicted in the picture. In this case, we will have three intersecting points.

There can be a lot of scenarios for which we get this condition of bi-stability. I have shown above 2 such cases, where we encounter bi-stability.

Each of the three points is a fixed point. Consider the three points as P_1, P_2, and P_3. We can quantitatively analyze the stability at each of the three points.

b)*Show the stability of P_1, P_2, P_3*



First, let us look at the P_1 point. Any point in close proximity to it, converges to this point. This is indicative of the fact that the point P_1 is a stable point.

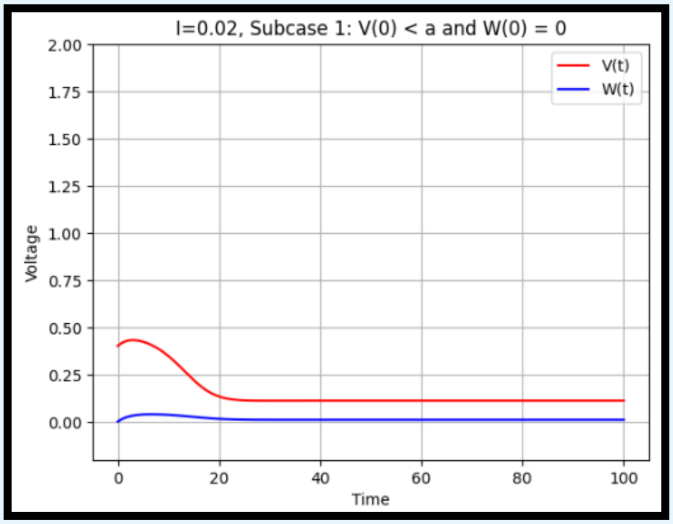
Now, let us look at the P_3 point. Any point in close proximity to it, converges to this point. This is indicative of the fact that the point P_3 is a stable point too.

Now, let us look at a slightly different case of P_2. Any point from the positive half space, with respect to the w null cline, comes closer to this point and then moves towards the point P_1. Any point from the negative half space, with respect to the w null cline, comes closer to this point and then moves towards the point P_2.

The phase plot at this point is curving down in one direction and curving up in the other direction. This is indicative of the fact that the point P_2 is a saddle point. These act as transition between the different stable points in the system.

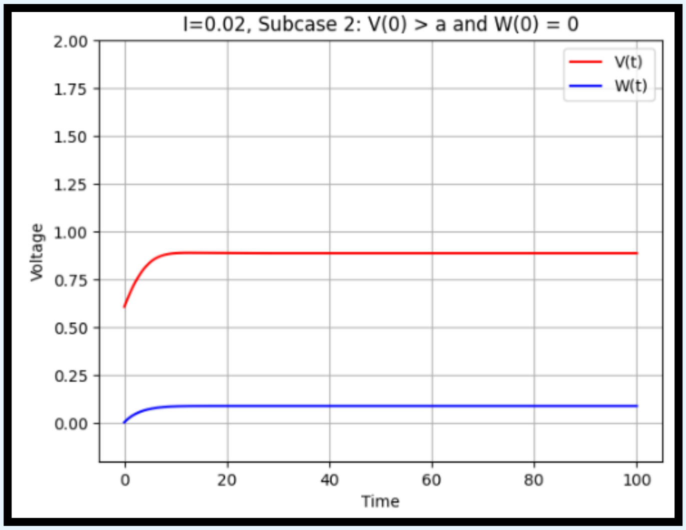
c) Plot V(t) vs t W(t) vs t

i. V(0) < a and W(0) = 0



Here, the neuron exists in a tonically down state. It is a phase of reduced neuronal activity characterized by a lower firing rate of neurons or a hyperpolarized resting membrane potential.

ii. V(0) > a and W(0) = 0



Here, the neuron exists in a tonically up state. It is a phase of increased neuronal activity characterized by a higher firing rate of neurons or a depolarized resting membrane potential.