BT6270
COMPUTATIONAL NEUROSCIENCE
ASSIGNMENT 3

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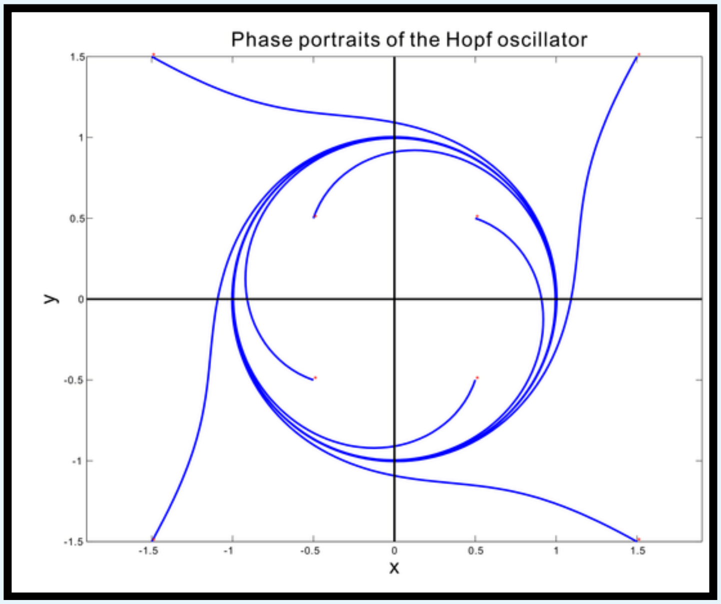


Assignment Description

*Simulating and Understanding the **Coupling** types on **Hopf Oscillators**.*

Question

Consider two **Hopf oscillators** (equation given below) coupled together. Calculate the **coupling coefficients (w12 and w21)** required to achieve a given phase difference in the between the oscillators. Two types of coupling are to be considered:-

a. **Complex Coupling** – The phase differences to be achieved are -47˚ and 98˚. Consider $\mathbf{w_1 = w_2 = 5}$.

b. **Power Coupling –** The normalized phase differences to be achieved are -47˚ and 98˚. Consider $\mathbf{w_1 = 5}$ and $\mathbf{w_2 = 15}$.

Assumptions:

The following assumptions were made with respect to coupling the Hopf oscillators.

- The coupling coefficients, w_{12} and w_{21}, each have two components,:
  i. **A** being the magnitude of coupling coefficient
  ii. **Θ** being the angle of coupling coefficient
- Since the value of A is not given, it is a hyperparameter. Its value doesn't seem to change the behavior of the function in any way. It just regulates the rate at which the phase-locking occurs.
- To avoid complexity, the supercritical Hopf regime is employed for the entire coupling of Hopf oscillators.
- We are limiting our scope of θ dynamics by assuming **A** to be constant.
- In the complex coupling scenario, we randomly choose the value of **A** and evaluate the coupling coefficient using the values given in the question.
- In the power coupling we make a couple of assumptions related to the values of **A** and **ϕ**, both of which have been highlighted in their respective sections.

Typical Scenario:

We usually represent a coupled pair and a network of Hopf oscillator in the supercritical regime, which can be defined as follows:

Complex state variable representation

$$\dot{z} = z(\mu + i\omega - |z|^2)$$

Cartesian coordinate representation

$$\dot{x} = -\omega y + x(\mu - r^2)$$

$$\dot{y} = +\omega x + y(\mu - r^2)$$

Polar coordinate representation

$$\dot{r} = \mu r - r^3$$

But, in our question, we already have these equations set up conditionally. So, we do not need to worry about the choice of Hopf oscillator.

Given:

The Hopf oscillator equations are given as follows,

$$\dot{x} = -y + \mu x(1 - x^2 - y^2)$$

$$\dot{y} = -x + \mu y(1 - x^2 - y^2)$$

In polar coordinates, the equations become,

$$x = r\cos(\theta) \quad y = r\sin(\theta)$$

$$\dot{r} = \mu r(1 - r^2)$$

$$\dot{\theta} = 1$$

Case a: Complex Coupling

When two Hopf oscillators with identical natural frequencies are coupled through complex coefficients, they can exhibit an oscillation at a particular angle similar to the angle of complex coupling coefficient.

$$\dot{z_1} = z_1(\mu + i\omega - |z_1|^2) + W_{12}\, z_2$$

$$\dot{z_2} = z_2(\mu + i\omega - |z_2|^2) + W_{21}\, z_1$$

Where

$$z_1 = r_1 e^{i\theta_1}, \qquad z_2 = r_2 e^{i\theta_2}$$

And

$$W_{12} = Ae^{i\varphi}, \qquad W_{21} = Ae^{-i\varphi}$$

W_{12}, W_{21} are the coupling coefficient. The polar coordinate system representation for the same is:

$$\dot{r_1} = (\mu - r_1^2)r_1 + Ar_2 cos(\theta_2 - \theta_1 + \varphi)$$

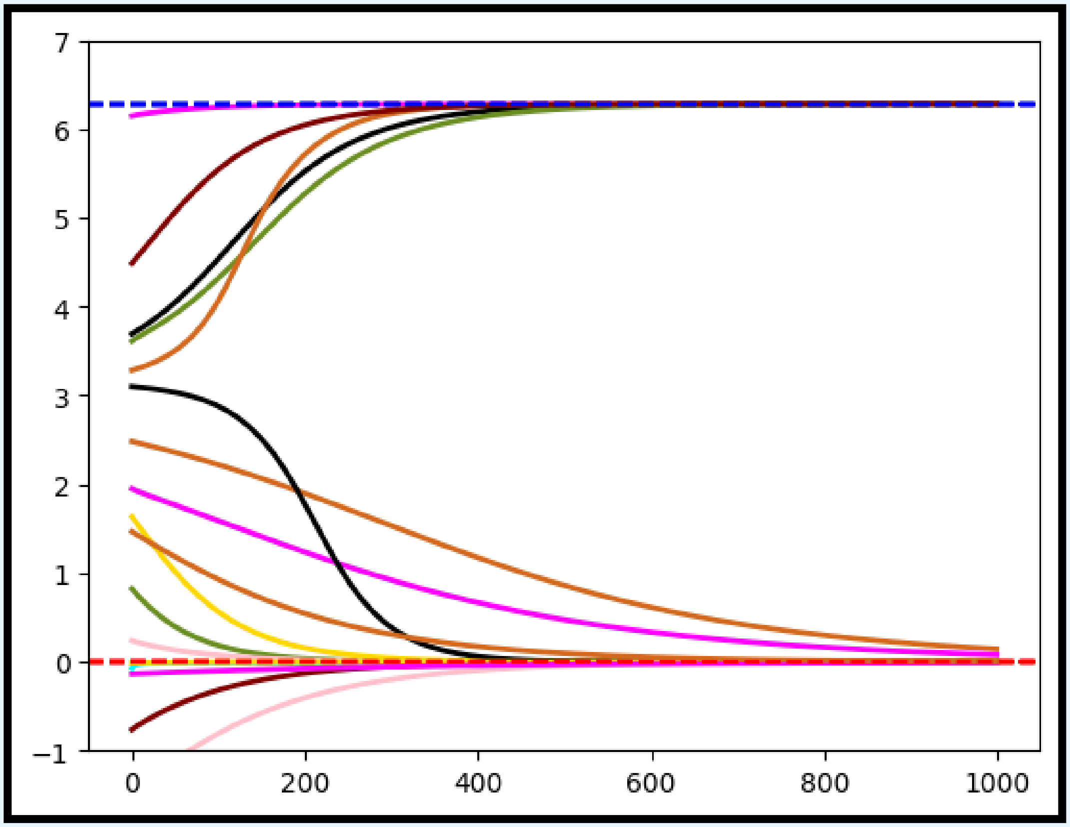
$$\dot{\theta_1} = \omega_1 + A * \frac{r_2}{r_1} * sin(\theta_2 - \theta_1 + \varphi)$$

$$\dot{r_2} = (\mu - r_2^2)r_2 + Ar_1 cos(\theta_1 - \theta_2 - \varphi)$$

$$\dot{\theta_2} = \omega_2 + A * \frac{r_1}{r_2} * sin(\theta_1 - \theta_2 - \varphi)$$

Now, using these representative equations for the two Hopf oscillators, we can model the complex coupling, to find the coupling coefficient.

I made use of a function to represent the dynamics of the two oscillators influenced by complex coupling, as shown in the above equations. I then make use of Euler integration to numerically integrate the system's state variables. The simulation is conducted over a specific period of time. The plots plotted after the simulation, offer insights into the dynamics of coupled oscillatory systems.



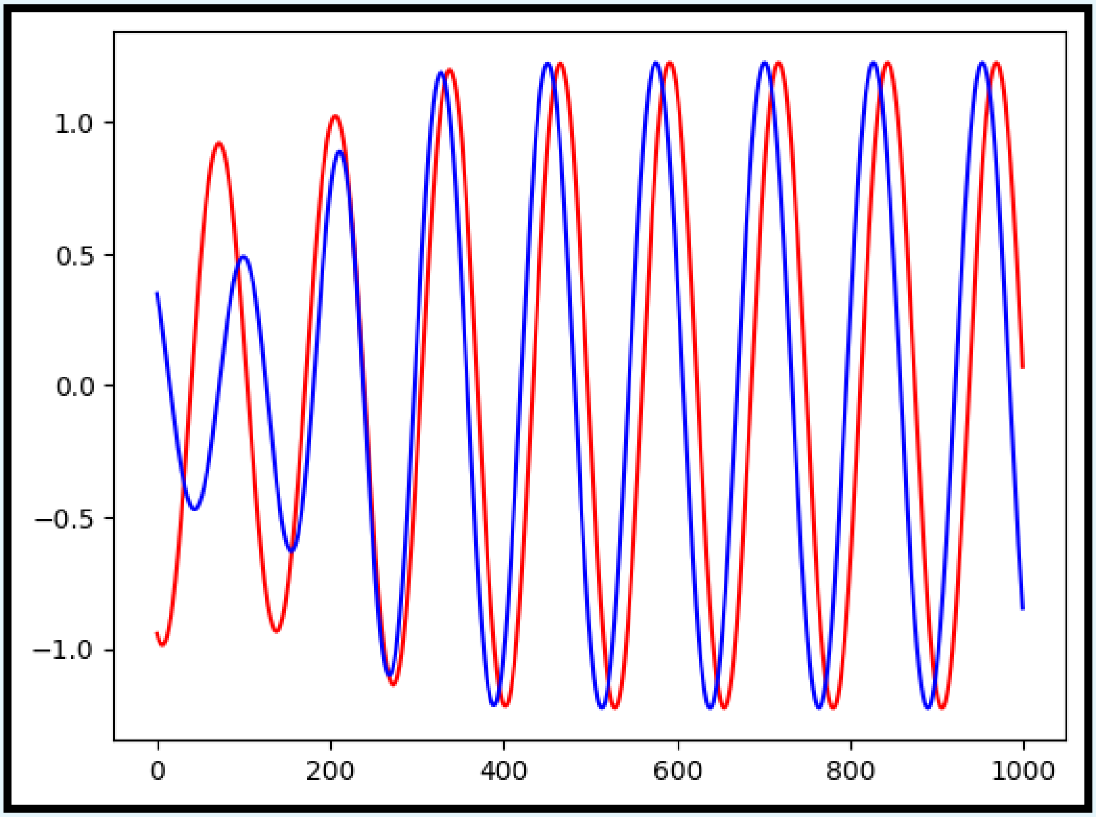
The above is a phase difference plot, where the

- **x axis represents time**,
- **y axis represents** [$\boldsymbol{\theta_1 - \theta_2 - \varphi}$]
- $\boldsymbol{\varphi = -47^\circ}$

The dotted red line represents **0**. The dotted blue line represents **2π**.

We can see in the above plot that, at steady state, $\theta_1 - \theta_2$ approaches any of the solutions $\boldsymbol{2n\pi + \theta}$, depending on the initial conditions [$\theta_1(0)$ and $\theta_2(0)$], whereas, as already stated, the magnitude of the complex coupling coefficient determines the rate at which the phase-locking occurs. Let Ψ represent the value $\theta_1 - \theta_2$, then Ψ_{ss} is the steady state value for the same. We can see that Ψ_{ss} attains the solution of $2n\pi + \theta$.

Here, all the initial starting values for the different parameters have been chosen randomly.



The above is a variance plot, where the

- **x axis represents time**,
- **y axis represents** $\boldsymbol{real(z(t)) = rcos(\theta_i)}$

This plot depicts the variation of x (t) with respect to the real part of z (t) at various time instants.

We can observe, that irrespective of their starting points the two Hopf oscillators end up oscillating at the rate.

Also, the above two plots have been made, considering the value of **φ** to be -47˚. On plotting the same nature of graphs for the value of **φ** to be 98˚, we notice that only the internal parameters have changed. However, the external nature, behavior and nature of the graph variations do not change. Hence, we get similar graphs for the 98˚ too.

Case b: Power Coupling

When two Hopf oscillators with arbitrary different natural frequencies are coupled through complex "power coupling" coefficients, they can exhibit an oscillation which cannot give insights on the usual phase difference, but gives insights upon defining a quantity called **normalized phase difference** as follows:

$$\Psi_{12} = -\Psi_{21} = \frac{\theta_1}{\omega_1} - \frac{\theta_2}{\omega_2}$$

Now, looking at the pair of Hopf oscillators coupling through power coupling:

$$\dot{z_1} = z_1(\mu + i\omega - |z_1|^2) + A_{12} e^{i\frac{\varphi_{12}}{\omega_2}} z_2^{\frac{\omega_1}{\omega_2}}$$

$$\dot{z_2} = z_2(\mu + i\omega - |z_2|^2) + A_{21} e^{i\frac{\varphi_{21}}{\omega_1}} z_1^{\frac{\omega_2}{\omega_1}}$$

Where

$$W_{12} = A_{12} e^{i\frac{\varphi_{12}}{\omega_1}}, \qquad W_{21} = A_{21} e^{i\frac{\varphi_{21}}{\omega_2}},$$

W_{12}, W_{21} are the coupling coefficient. W_{12} is the weight of power coupling from 2nd oscillator to the 1st oscillator. W_{21} is the weight of power coupling from 1st oscillator to the 2nd oscillator.

The polar coordinate system representation for the same is:

$$\dot{r_1} = (\mu - r_1^2) r_1 + A_{12}\, r_2^{\frac{\omega_1}{\omega_2}} \cos\left(\omega_1 \left(\frac{\theta_2}{\omega_2} - \frac{\theta_1}{\omega_1} + \frac{\varphi_{12}}{\omega_1\omega_2}\right)\right)$$

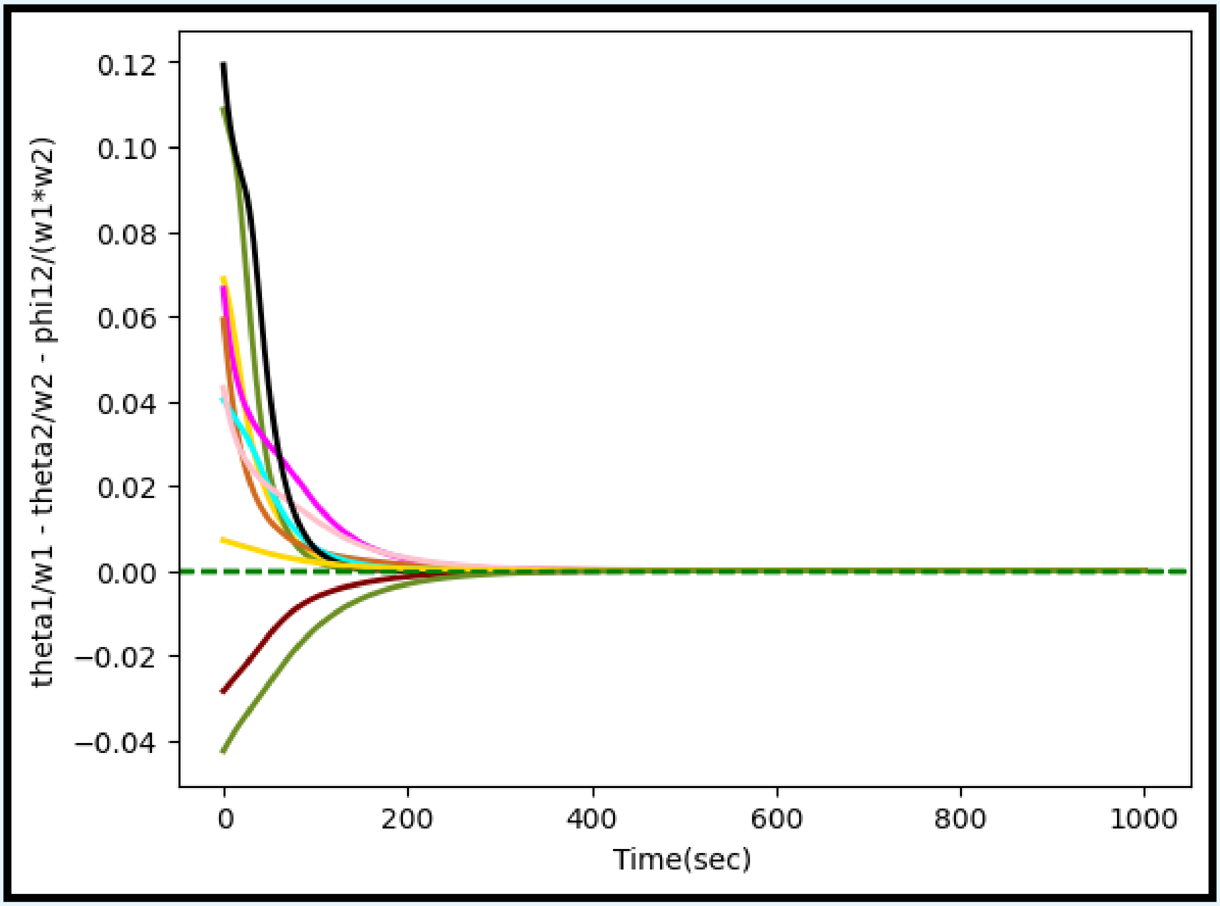
$$\dot{\theta}_1 = \omega_1 + A_{12} * \frac{r_2^{\frac{\omega_1}{\omega_2}}}{r_1} * \sin\left(\omega_1 \left(\frac{\theta_2}{\omega_2} - \frac{\theta_1}{\omega_1} + \frac{\varphi_{12}}{\omega_1\omega_2}\right)\right)$$

$$\dot{r_2} = (\mu - r_2^2) r_2 + A_{21}\, r_1^{\frac{\omega_2}{\omega_1}} \cos\left(\omega_2 \left(\frac{\theta_1}{\omega_1} - \frac{\theta_2}{\omega_2} + \frac{\varphi_{21}}{\omega_1\omega_2}\right)\right)$$

$$\dot{\theta}_2 = \omega_2 + A_{21} * \frac{r_1^{\frac{\omega_2}{\omega_1}}}{r_2} * \sin\left(\omega_2 \left(\frac{\theta_1}{\omega_1} - \frac{\theta_2}{\omega_2} + \frac{\varphi_{21}}{\omega_1\omega_2}\right)\right)$$

Now, using these representative equations for the two Hopf oscillators, we can model the power coupling, to find the coupling coefficient.

I made use of a function to represent the dynamics of the two oscillators influenced by power coupling, as shown in the above equations. I then make use of Euler integration to numerically integrate the system's state variables. The simulation is conducted over a specific period of time. The plots plotted after the simulation, offer insights into the dynamics of coupled oscillatory systems.



The above is a phase difference plot, where the

- **x axis represents time**,
- **y axis represents** $[\frac{\theta_1}{\omega_1} - \frac{\theta_2}{\omega_2} - \frac{\varphi_{12}}{\omega_1\omega_2}]$
- $\boldsymbol{\varphi_{12} = -47^\circ}$

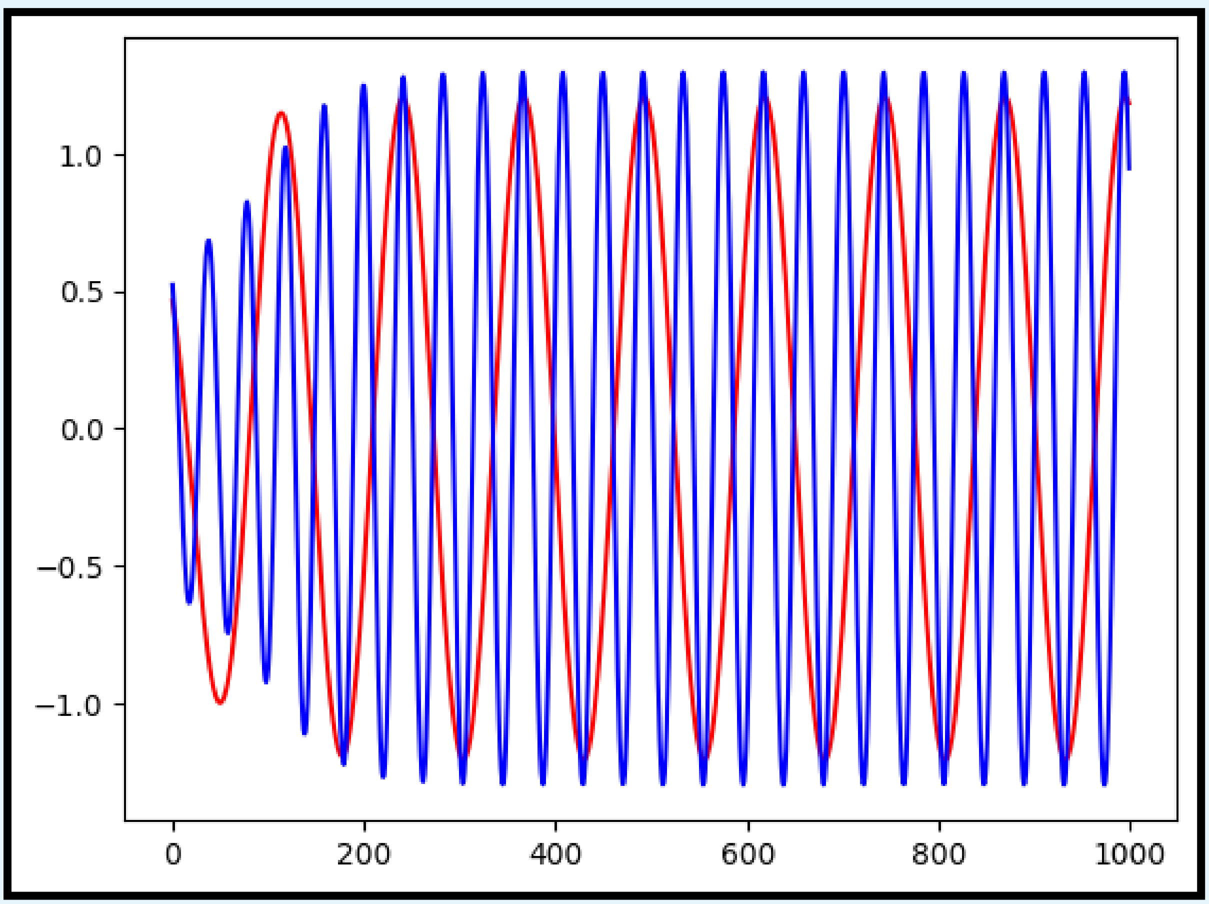
The dotted red line represents **0**

In the case of power coupling there are certain parameters that we have assumed. The assumptions are:

- $\mathbf{A_{12}} = \boldsymbol{A_{21}}$
- $\boldsymbol{\varphi_{12}} = -\boldsymbol{\varphi_{21}}$

We can see in the above plot that, at steady state, $\frac{\theta_1}{\omega_1} - \frac{\theta_2}{\omega_2}$ approaches the solution $\frac{\varphi_{12}}{\omega_1\omega_2}$, irrespective of the initial conditions [$\theta_1(0)$ and $\theta_2(0)$], whereas, as already stated, the magnitude of the power coupling coefficient determines the rate at which the phase-locking occurs. Let Ψ represent the value $\frac{\theta_1}{\omega_1} - \frac{\theta_2}{\omega_2}$, then Ψ_{ss} is the steady state value for the same. We can see that Ψ_{ss} attains the value of $\frac{\varphi_{12}}{\omega_1\omega_2}$.

Here, all the initial starting values for the different parameters have been chosen randomly.



The above is a variance plot, where the

- **x axis represents time**,
- **y axis represents** $\boldsymbol{real(z(t)) = rcos(\theta_i)}$
- $\boldsymbol{\varphi_{12} = -47^\circ}$

This plot depicts the variation of x (t) with respect to the real part of z (t) at various time instants.

We can observe that the individual Hopf oscillators retain their original frequencies, yet their normalized phase differences reaches a constant value, which is a function of the external frequency.

Also, the above two plots have been made, considering the value of $\boldsymbol{\varphi_{12}}$ to be -47˚. On plotting the same nature of graphs for the value of $\boldsymbol{\varphi_{12}}$ to be 98˚, we notice that only the internal parameters have changed. However, the external nature, behavior and nature of the graph variations do not change. Hence, we get similar graphs for the 98˚ too.