1. a) $P(A \mid B, C) = \dfrac{P(A \cap B \cap C)}{P(B \cap C)} = \dfrac{8}{13}$

- $P(A \cap B \cap C) = \dfrac{8}{20}$ $\left[\dfrac{\#\text{ Red}}{\#\text{ total}}\right]$
- $P(B \cap C) = \dfrac{13}{20}$ $\left[\dfrac{\#\text{ red} + \#\text{ green}}{\#\text{ total}}\right]$

b) $P(A \mid B, C) = P(A)$ [Since they are conditionall independent]

$P(A \mid B, C) = \dfrac{8}{13}$

$P(A) = \dfrac{8}{20}$ $\left[\dfrac{\#\text{ Red}}{\#\text{ total}}\right]$

$\therefore$ Event A is conditionally dependent on B and C

c) Baye's Theorem : $P(A \mid B) = \dfrac{P(B \mid A) \cdot P(A)}{P(B)}$

$P(B \mid A) = 1$

$P(A) = \dfrac{8}{20}$

$P(B) = \dfrac{\#\text{red} + \#\text{ green}}{\#\text{ total}} = \dfrac{13}{20}$

- $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{8}{13}$
- $\dfrac{P(B \mid A) \cdot P(A)}{P(B)} = \dfrac{\frac{8}{20}}{\frac{13}{20}} = \dfrac{8}{13}$

$\therefore$ $P(A \mid B) = \dfrac{P(B \mid A) \cdot P(A)}{P(B)}$ Baye's Theorem for event A and B

2.

a) C = class of Heads

x = observing 5 consecutive heads.

$C = 0.5$, Prior probability $P(C = 0.5) = 0.9$

$C = 0.6$, Proir probability $P(C = 0.6) = 0.1$

$\Rightarrow P(x \mid C = 0.5) = 0.5^5$

$\Rightarrow P(x \mid C = 0.6) = 0.6^5$

$\Rightarrow P(x) = 0.0359 \quad (P(0.5) \times P(x \mid 0.5) + P(x \mid 0.6) \times P(x \mid 0.6))$

Now for the posterior,

- $P(0.5 \mid x) = \dfrac{P(0.5) \times P(x \mid 0.5)}{P(x)}$

$= \dfrac{0.9 \times 0.5^5}{0.0359} = \boxed{0.7834}$

- $P(0.6 \mid x) = \dfrac{P(0.6) \times P(x \mid 0.6)}{P(x)} =$

$= \dfrac{(0.1)(0.6)^5}{0.0359} = \boxed{0.2166}$

b) likelyhood of heads $= p$

likely hood of tails $= 1 - p$

Given : Prior distribution : Beta (2,2) distribution

x = 10 tails

Prior distribution : $P(c) = p(1-p)$

$p(x \mid c) = (1-p)^{10}$

$P(c \mid x) = (1-p)^{10} \times p(1-p) = p \cdot (1-p)^{11}$

Since, $(1-p)^{10} = P(x \mid c)$ and $P(1-p) = P(c)$

We can represent $P(c \mid x)$ as $P(x \mid x) \cdot P(c)$

$\therefore$ Posterior distribution = Beta (2, 12) distribution.

[$\because$ posterior distribution $= (p^2 - 1) \cdot (p^{12} - 1)$]

3 Likely hood function : $L(\theta) = \prod_{i=1}^{n} f(x_i | \theta)$

1) $\mathcal{L}(\mu\theta | x) = -\frac{N}{2} \log(2\pi) - N \log \sigma - \frac{\Sigma t (x^t - \mu)^2}{2\sigma^2}$

Where $\mathcal{L}$ is the log likelyhood of the function.

Now, on differentiating $\frac{d\mathcal{L}}{d\mu}$ towards 0,
we get,

$$\frac{d\mathcal{L}}{d\mu} = -\frac{1}{2\sigma^2} \cdot 2 \cdot \Sigma t (x^t - \mu) = 0$$

$$\Rightarrow \frac{\Sigma_t x^t - N\mu)}{-\sigma} = 0$$

$$N\mu = \Sigma t\, x^t$$

$$\boxed{\mu = \frac{\Sigma t\, x^t}{N} = \text{MLE for } \mu}$$

Now assume $a = \sigma^2$

$$\Rightarrow \frac{d\mathcal{L}}{da} = 0 \Rightarrow \left(-\frac{N}{2} \log(2\pi) - N \log \sqrt{a} - \frac{\Sigma t (x^t - \mu)^2}{2a}\right) = 0$$

$$\Rightarrow N \log (a^{1/2}) - \frac{\Sigma t (x^t - \mu)^2}{2a} = 0$$

$$\Rightarrow \frac{-N}{a^{1/2}} \cdot \frac{1}{2} a^{-1/2} - \Sigma t \frac{(x^t - \mu)^2}{2} \cdot (a^{-1}) = 0$$

$$\Rightarrow \frac{-N}{2a} + \frac{\Sigma_t (x^t - \mu)^2}{2a^2} = 0$$

$$\Rightarrow a = \frac{\Sigma t (x^t - \mu)^2}{N}$$

$$\Rightarrow \sigma^2 \frac{\Sigma t (x^t - \mu)^2}{N} \Rightarrow \boxed{s^2 = \frac{\Sigma t (x^t - \mu)^2}{N}}$$

b) Given :

$x = [-5, 7, 9, 12, -17, 1, 4, 24, 0, 3]$

$$m = \frac{(-5+7+9+12+(-17)+1+4+24+0+3)}{10}$$

$$= \frac{38}{10} = 3.8$$

from the formula,

$$s^2 = \left[\frac{(-5-3.8)^2 + (7-3.8)^2 + (9-3.8)^2 + (12-3.8)^2 + (-17-3.8)^2 + (1-3.8)^2 + (4-3.8)^2 + (24-3.8)^2 + (0-3.8)^2 + (3-3.8)^2}{10}\right]$$

4.

a) Given : $L(p|x) = \log \prod_t p^{x^t} (1-p)^{1-x^t}$

$$= \sum x^t \log(p) + (N - \sum_t x^t) \log(1-p)$$

Assume $a = \sum_t x^t$

$$\Rightarrow L(p|x) = a \log(p) + (N-a) \log(1-p)$$

$$\frac{dL}{dp} = \frac{a}{p} + \frac{(N-a)(-1)}{1-p} \Rightarrow \frac{(1-p)a + (a-n)p}{p(p-1)}$$

$$\Rightarrow \frac{a - pa + pa - Np}{p(p-1)} = 0$$

$$\Rightarrow a - Np = 0 \quad \Rightarrow Np = a \Rightarrow \hat{p} = \frac{a}{N} = \frac{\sum_t x^t}{N}$$

b) Sample space = 32 possible rolls

$x = 1$, to satisfy the outcome of sum = 9 :

we have 4 outcomes (3,6), (5,4), (4,5) (6,3)

$x = 0$ = ~~32~~ 36 - 4 = 32 outcomes

$$\Rightarrow P = \frac{4}{36} \left[\frac{4 \text{ outcomes}}{\text{sample space}}\right] = \boxed{\frac{1}{9}}$$

5. Given: $P(X=5 \mid \theta=2) = 0.022$

$P(X=5 \mid \theta=4) = 0.105$

$\Rightarrow P(\theta=4) = 0.3$

$\Rightarrow P(\theta=2) = 0.7$

$$P(\theta \mid x) = \frac{P(x \mid \theta) \cdot P(\theta)}{P(x)}$$

- $P(x)=5 = P(X=5 \mid \theta=2)\, P(\theta=2) + P(X=5 \mid \theta=4) P(\theta=4)$

$\Rightarrow (0.022 \times 0.7) + (0.105 \times 0.3)$

$= \underline{\underline{0.0469}}$

- $P(\theta=2 \mid X=5) = \dfrac{P(X=5 \mid \theta=2) \cdot P(\theta=2)}{P(X=5)}$

$\Rightarrow \dfrac{0.022 \times 0.7}{0.0469} = \underline{\underline{0.32836}}$

- $P(\theta=4 \mid X=5) = \dfrac{P(X=5 \mid \theta=4) \cdot P(\theta=4)}{P(X=5)}$

$= \underline{\underline{0.671}}$

$\therefore$ As we have calculated,

$\theta = 4$ given 5 children $= 0.671$

$\theta = 2$ given 5 children $= 0.32836$