Real Analysis

by Anshula Gandhi

I love real analysis — the math of formalizing calculus. You might or you might not be into that.

Either way, this book is for you, and I hope you'll like analysis by the end of this book.

This book aims to teach analysis by asking and answering the philosophical questions that gave rise to each analysis concept.

I'll be adding new pages weekly. While pages are being added, I hope you'll feel free to skip around and read whatever looks interesting to you.

Chapter 0

Introduction

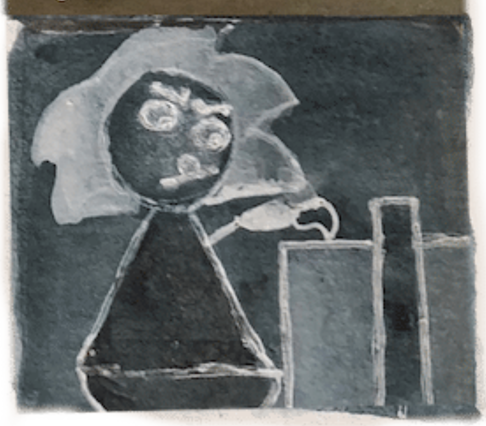
What is real analysis?

Why do analysis?

Disclaimer: Analysis isn't technically "useful," in that:
1) You can't really apply it to the real world, and
2) It probably won't help you land a job.

But despite analysis being useless in a job application, it is useful in that:

1) The hand-waving of calculus can lead to paradoxes. Analysis fixes that by proving calculus rigorously.



2) Analysis presents an opportunity to think and prove things in an entirely new way.

3) Analysis offers escape from reality - a chance to philosophize about problems that have nothing to do with your everyday life.

To echo Richard Feynman on physics, analysis "is like sex: sure it may give some practical results, but that's not why we do it."

DIGRESSION

Why do analysis books rarely use pictures?

Analysis came about as part of a movement in the 1700s. Some mathematicians wanted math to be more "pure" and abstract, and rely less on the "crutches" of figures and diagrams.

It was something of a challenge, perhaps, to define math without relying on figures

And that's probably why most analysis texts shy away from visuals: because a big point of analysis was to not need visuals anymore.

Chapter 1

Real and Complex Numbers


How do you prove that something doesn't exist?
Proving something doesn't exist can be a lot harder than proving something does exist.
A sociologist, Jerry Lembcke, ran into that problem.
SOCIOLOGY
He'd heard stories about antiwar protestors spitting on Vietnam War soldiers as they returned home to America.
PTHEW
But as Lembcke looked into the phenomenon, he could find no single instance of this spitting.
But he couldn't say for sure that "the spitting never happened."
If he only had to prove the spitting happened, he would just have to find one account of it.
I was spit on signed a Veteran
But how could he prove that nobody ever spat on a war veteran?
It would be impossible to interview every single Vietnam War veteran – dead and alive.
SIGH
The sociologist admitted that he couldn't say the "spitting protestor" phenomenon was untrue. He could only say he found no evidence it was true.

How do you prove non-existence in math?

So, people say "you can't prove a negative statement." Not true. It's hard. But in math, it's possible with a "proof by contradiction."

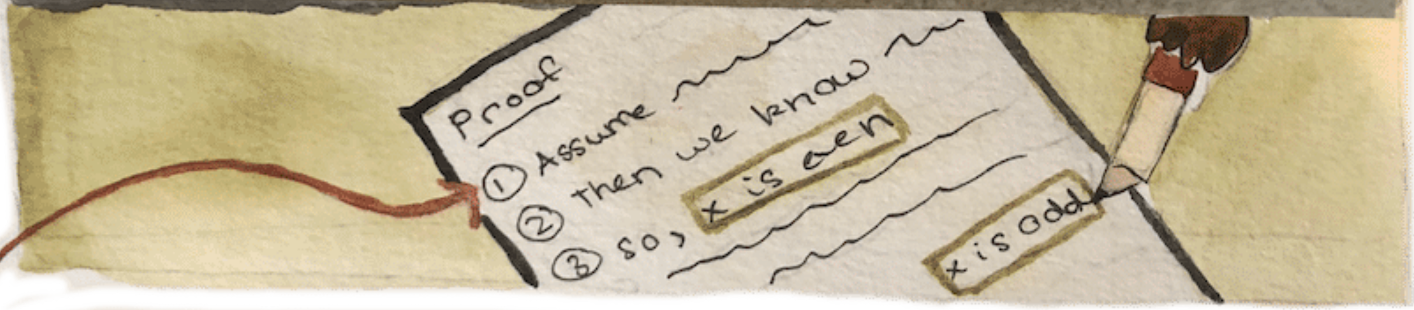
We start by assuming something <u>does</u> exist, and reason through it.

If we find a contradiction within our reasoning, we must conclude that thing <u>does not</u> exist.

But why?

When two people contradict each other, you know one of them has got to be wrong.

Similarly, in math, if two of the lines in your proof contradict each other, then there's a lie in there somewhere.



And if every step you took in the proof was correct, then the lie can't be in any of the steps you took. The lie must be way back in the very first assumption you made.


How do you prove irrationality?
When an ancient Greek discovered that some numbers are irrational*...
*Irrational numbers are those that can't be written as a ratio of integers (e.g. pi).
He was exiled. (It was heresy to say numbers were so disorderly.)
But how could he have proven irrational numbers exist?
Did he line up all the infinite fractions in the world, and compare each to a number he thought was irrational?
6'0
5'10
5'8
5'6
2/5
42/3
6/7
1/2
OUR GUY
No... nope... not that one...
He couldn't have. He'd have to have checked all infinite fractions in the world before ensuring none of them was his guy.
OUR GUY
√2
That method of proof would be impossible.
Instead, we could use proof by contradiction.

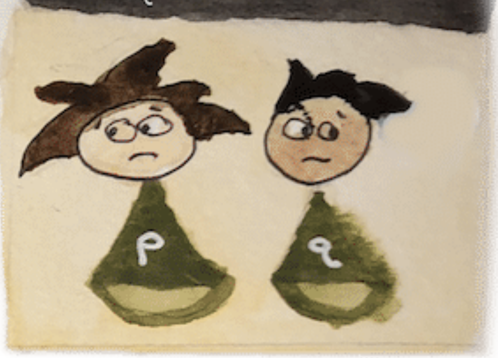
Is the square root of two irrational?

First, let's declare that $\sqrt{2}$ exists.
(We should be explicit about the rules we're going by.)

Let's assume, for the purpose of contradiction, that $\sqrt{2}$ is rational. (It's not.)

Then there must exist a fraction $\frac{p}{q}$ that equals $\sqrt{2}$.

Let's use a p and q that have no common factors (so that $\frac{p}{q}$ is simplified.)



Now we'll show the contradiction: even though we chose p and q to share no common factors, they always end up sharing a factor of two.

Let's interrogate p and q separately.

Let's prove p is divisible by two.

$$\sqrt{2} = \frac{p}{q}$$

square

$$2 = \frac{p^2}{q^2}$$

move q^2

$$p^2 = 2q^2$$

logic

p^2 is even

more logic

p is even

Let's prove q is divisible by two.

$$\sqrt{2} = \frac{p}{q}$$

since p is even, we can rewrite p as 2k (where k is some integer)

$$\sqrt{2} = \frac{(2k)}{q}$$

square

$$2 = \frac{4k^2}{q^2}$$

move q^2

$$q^2 = 2k^2$$

logic

q^2 is even

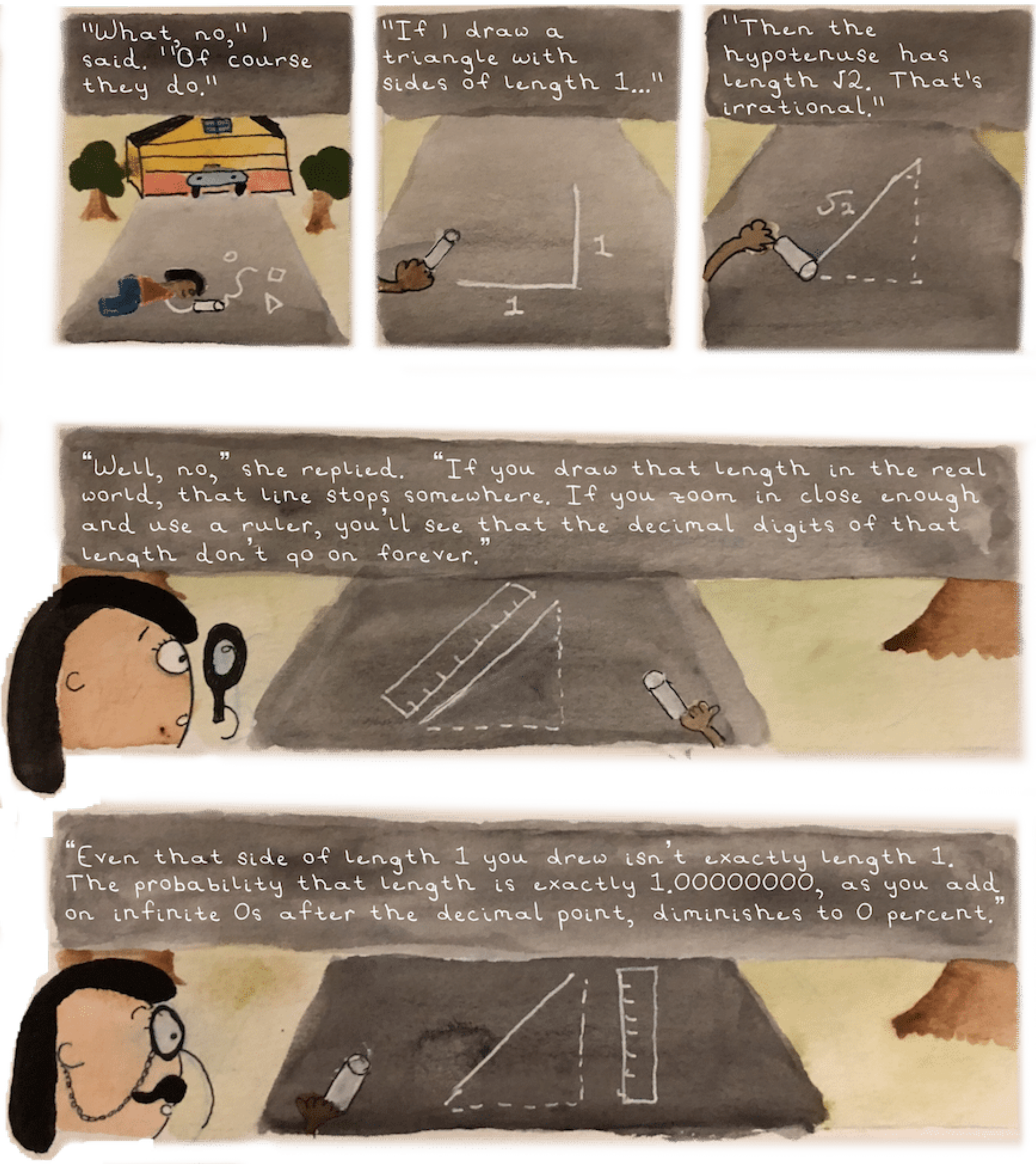
more logic

q is even

* Remember, we chose a p and q that shared no common factors.

But do irrational numbers actually exist?

A friend once told me that irrational numbers don't exist in the real world.

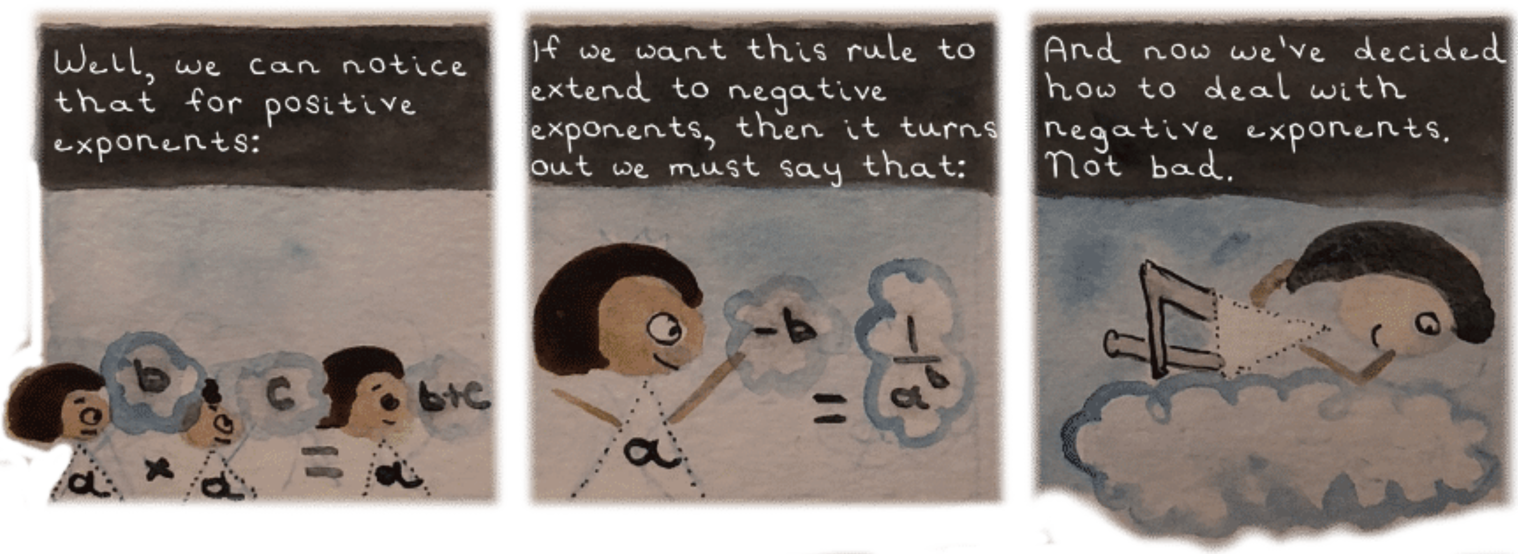



She's right.
Like a lot of analysis concepts, infinite decimals exist perfectly only in our heads.
So why bother with all of this stuff if it all only exists in our heads?
"Well, why do we do philosophy?" asked a frend.
"It's not always useful. And it's sometimes ridiculous."
"But it forces you to view things from a different perspective, and get a fuller understanding of your assumptions."
And that's exactly what makes analysis so cool. It's an entirely different way of thinking.

What does it mean to raise a number to the negative power?

Really – what can it possibly mean to raise something to the negative power?*

*Yes, we learned in school that a^{-b} equals $1/a^b$. But does that make any sense?



Bibliography

(By Chapter)

3 For more on the origins of abstract mathematics, see the
book Duel at Dawn: Heroes, Martyrs, and the Rise of Modern Mathematics by Amir Alexander.

5 Lembke's book on the Vietnam War spitting story is called The Spitting Image: Myth, Memory and the Legacy of Vietnam. I owe this reference to Thalia Rubio at MIT.

7 For more on the Greek mathematician exiled for asserting that some numbers are irrational, see the fascinating Ted Ed video: "A Brief History of Banned Numbers."

9 Thank you to Amanda Sobel at MIT for the wonderful comparison of analysis to philosophy.

10 I learned about the reasoning behind the definition of negative exponents from the book Burn Math Class: And Reinvent Mathematics for Yourself by Jason Wilkes.