Real Analysis

by Anshula Gandhi

1. What is this book?

"What is the point of this?"

...is a common question in math classes for a reason.

In math, we are often presented with the tools that answer questions, without being given the questions themselves.

For example, in math class, I learned what a "field" was, but had no idea why we used it, until I realized it helped answer the question "what is a number?"

The purpose of this book is to present the guiding philosophical questions that have led to core mathematical concepts in real analysis.

After all, math is like philosophy, but with answers.

I love real analysis - the math of formalizing calculus. You might or you might not be into that.

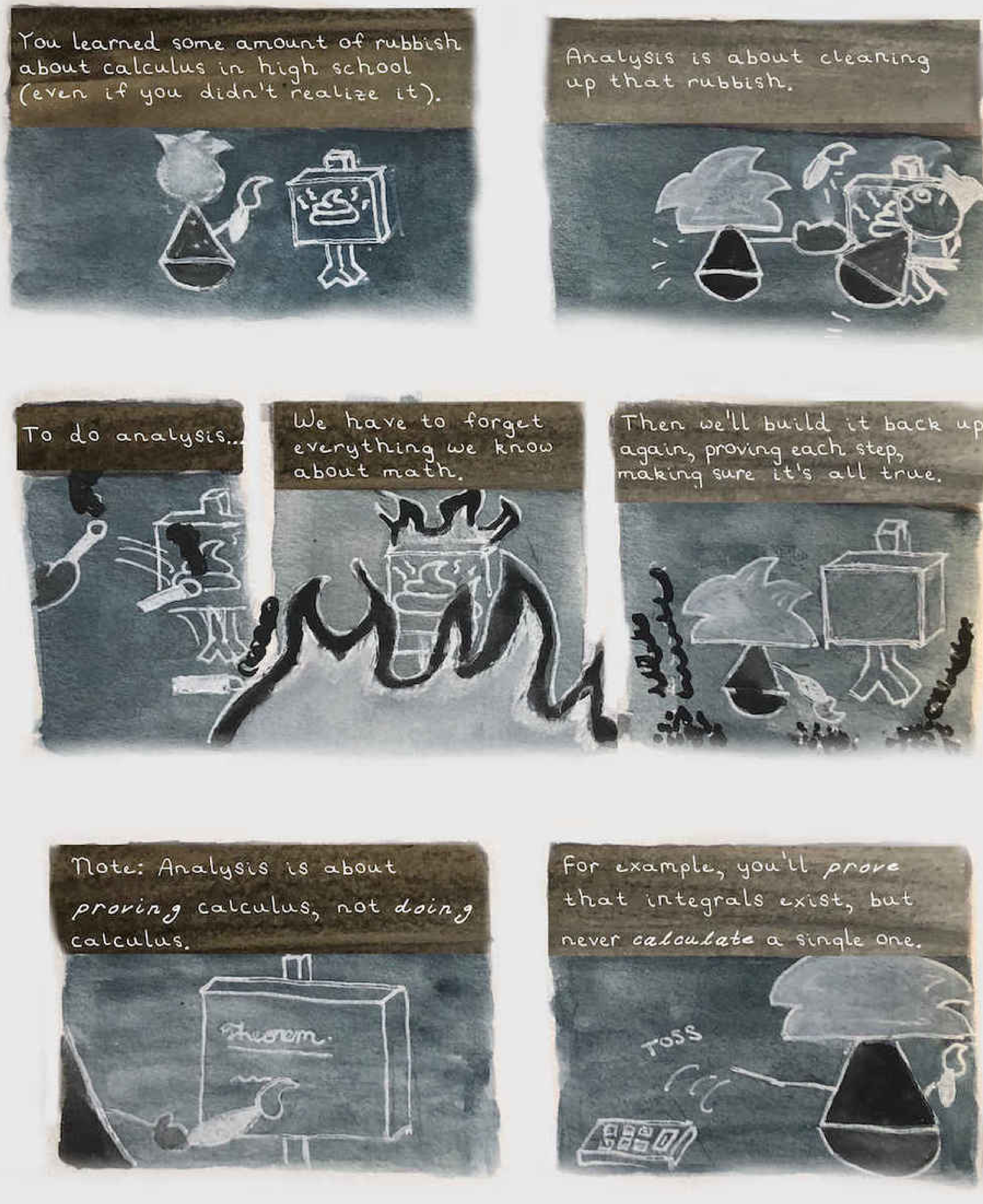
Either way, this book is for you, and I hope you'll like analysis by the end of this book. I hope it's something you'd want to read if you're curious about math, even if you don't have to pass a real analysis class. But if you are taking a class, this book should teach what you'd be tested on.

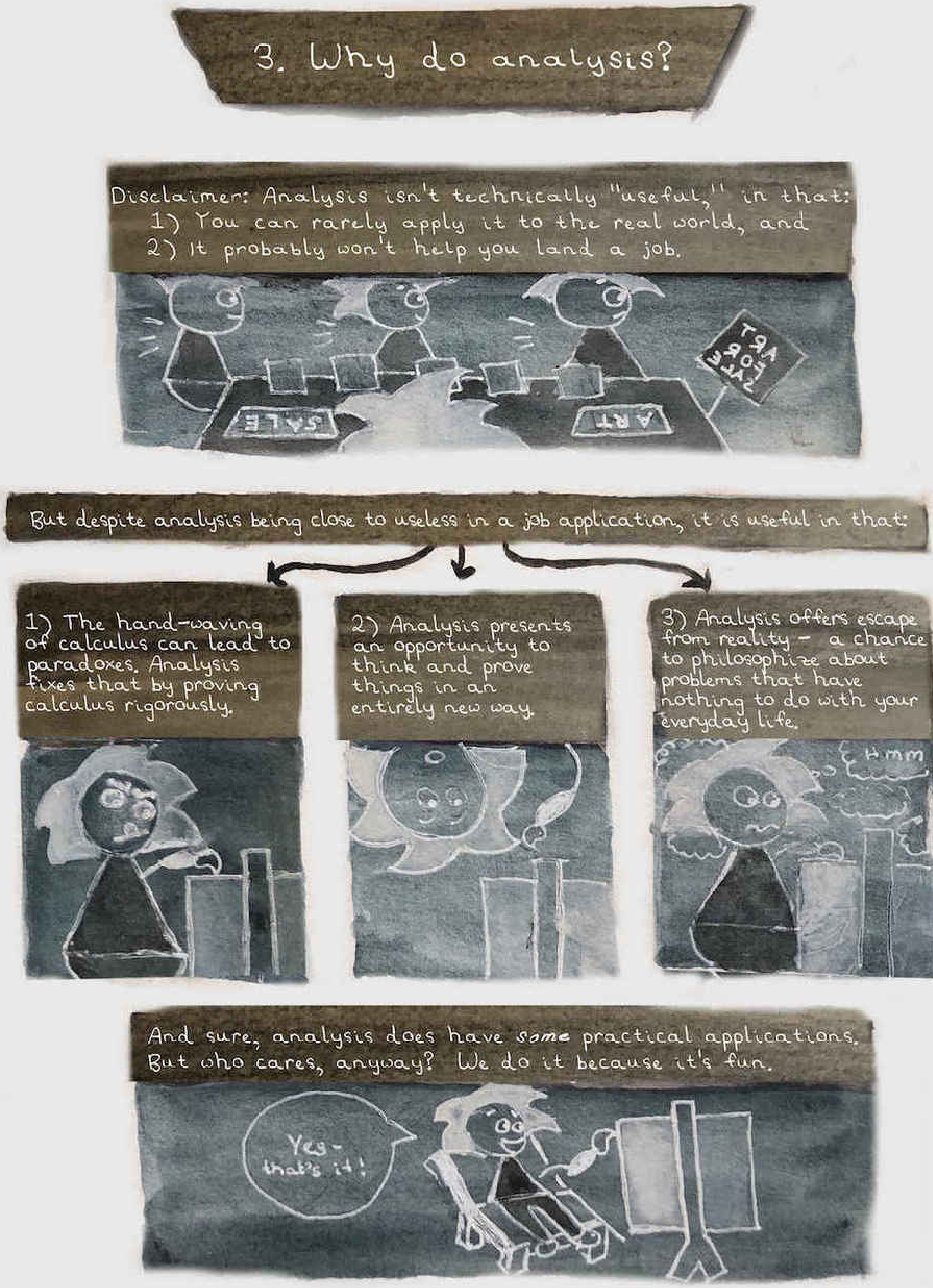
I'll be adding new pages weekly. While pages are being added, I hope you'll feel free to skip around and read whatever looks interesting to you.

Chapter 0

Introduction

2. What is real analysis?




3. Why do analysis?
Disclaimer: Analysis isn't technically "useful," in that:
1) You can rarely apply it to the real world, and
2) It probably won't help you land a job.
ART FOR SALE
SALE
ART
But despite analysis being close to useless in a job application, it is useful in that:
1) The hand-waving of calculus can lead to paradoxes. Analysis fixes that by proving calculus rigorously.
2) Analysis presents an opportunity to think and prove things in an entirely new way.
3) Analysis offers escape from reality – a chance to philosophize about problems that have nothing to do with your everyday life.
Hmm
And sure, analysis does have *some* practical applications. But who cares, anyway? We do it because it's fun.
Yes – that's it!

DIGRESSION

4. Why do analysis books rarely use pictures?

Analysis came about as part of a movement in the 1700s. Some mathematicians wanted math to be more "pure" and abstract, and rely less on the "crutches" of figures and diagrams.

It was something of a challenge, perhaps, to define math without relying on figures

And that's probably why most analysis texts shy away from visuals: because a big point of analysis was to not need visuals anymore.

Chapter 1

Real and Complex Numbers

5. How do you prove that something doesn't exist?

6. How do you prove non-existence in math?

So, people say "you can't prove a negative statement." Not true. It's hard. But in math, it's possible with a "proof by contradiction."

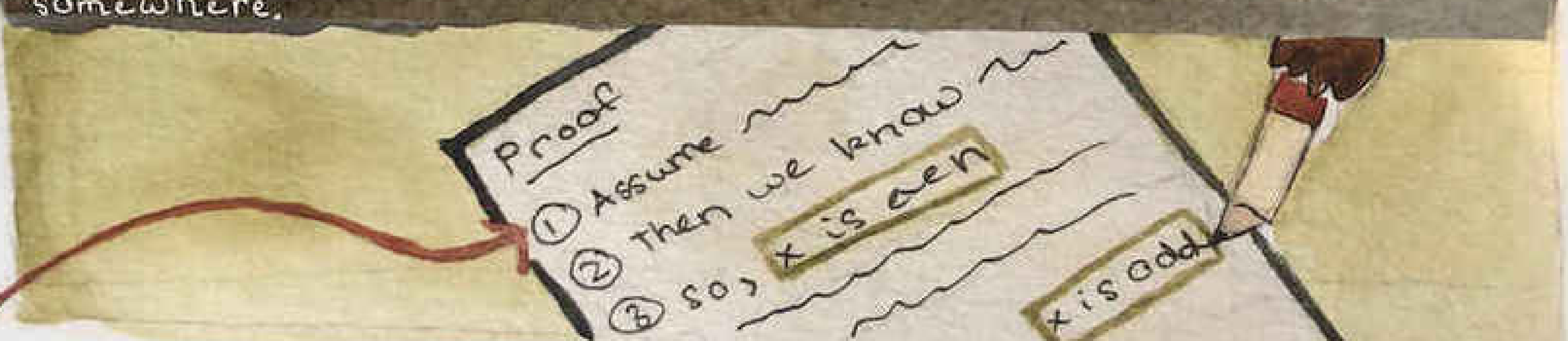
We start by assuming something *does* exist, and reason through it.

If we find a contradiction within our reasoning, we must conclude that thing *does not* exist.

But why?

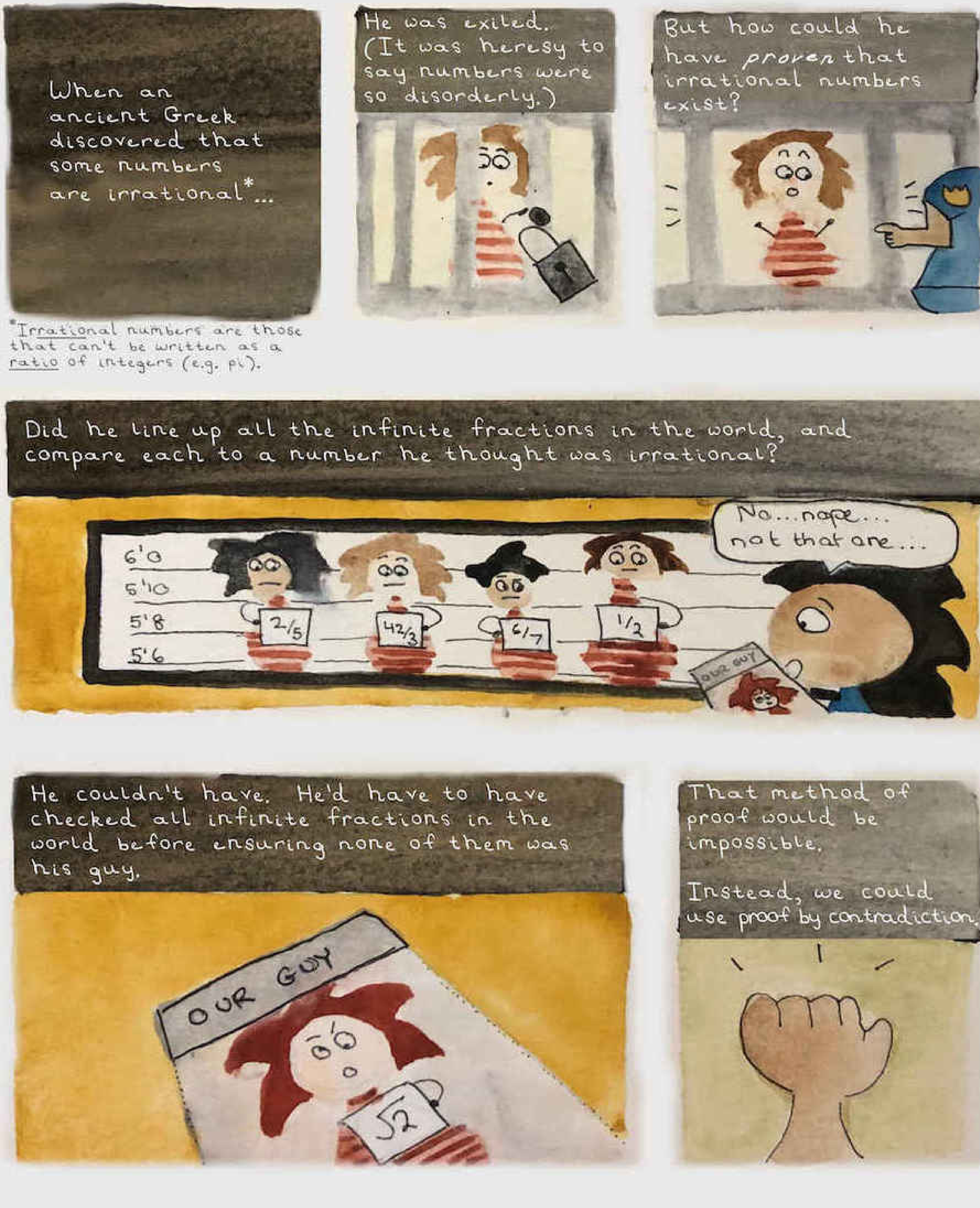
When two people contradict each other, you know one of them has got to be wrong.

Similarly, in math, if two of the lines in your proof contradict each other, then there's a lie in there somewhere.



And if every step you took in the proof was correct, then the lie can't be in any of the steps you took. The lie must be way back in the very first assumption you made.

7. How do you prove irrationality?



8. Is the square root of two irrational?

First, let's declare that $\sqrt{2}$ exists.
(We should be explicit about the rules we're going by.)

Let's assume, for the purpose of contradiction, that $\sqrt{2}$ is rational. (It's not.)

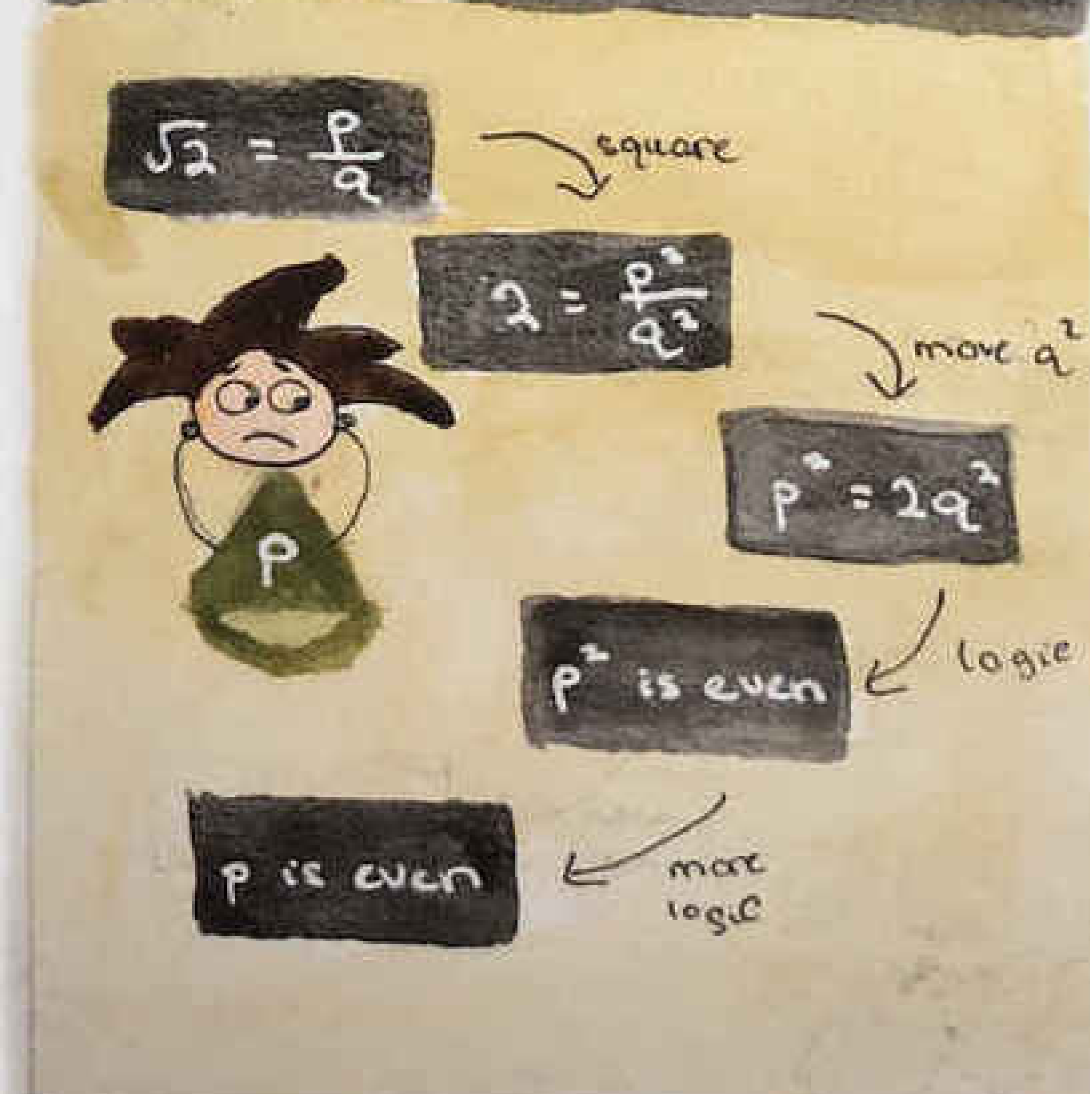
Then there must exist a fraction $\frac{p}{q}$ that equals $\sqrt{2}$.

Let's use a p and q that have no common factors (so that $\frac{p}{q}$ is simplified.)

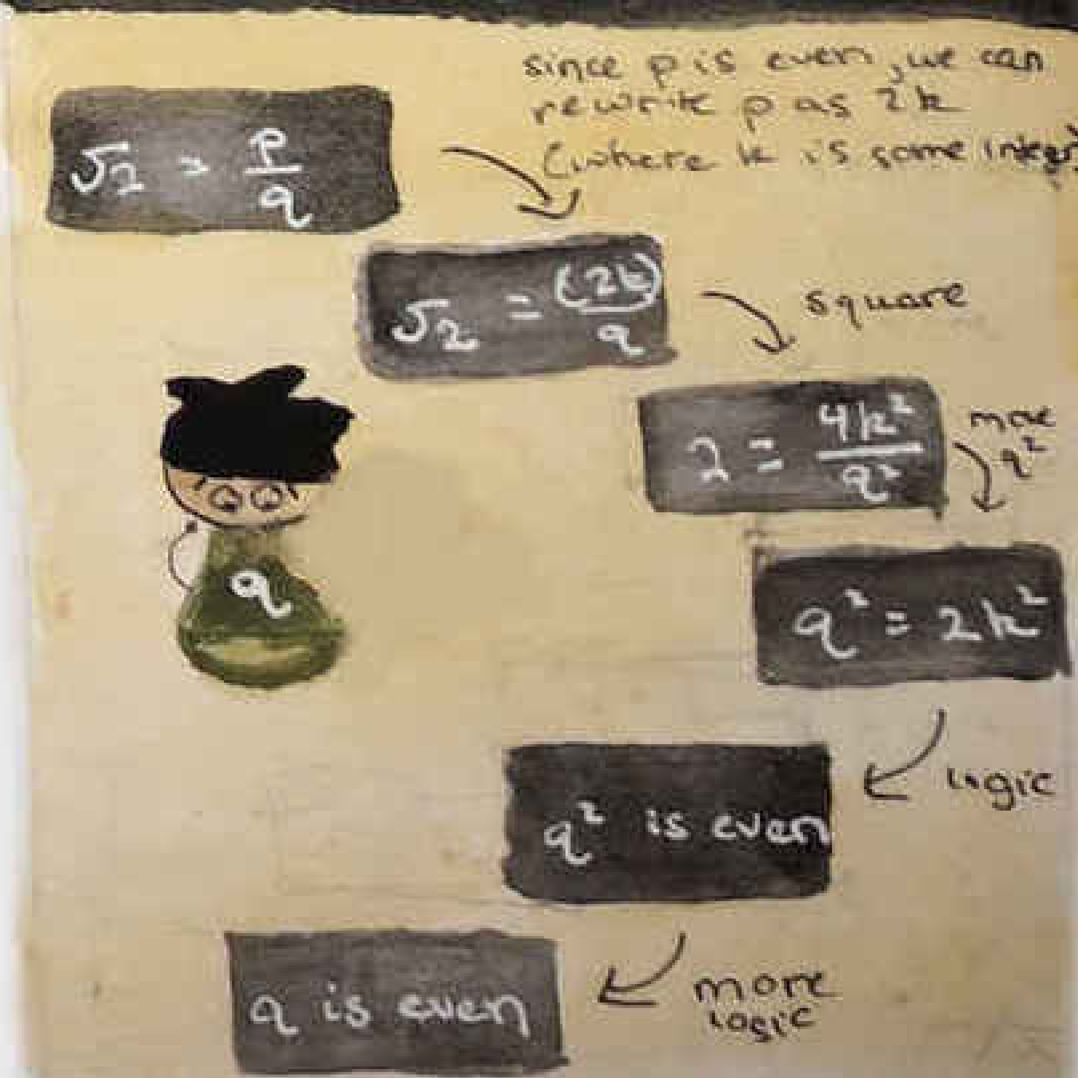
Now we'll show the contradiction: even though we chose p and q to share no common factors, they always end up sharing a factor of two.

Let's interrogate p and q separately.

Let's prove p is divisible by two.



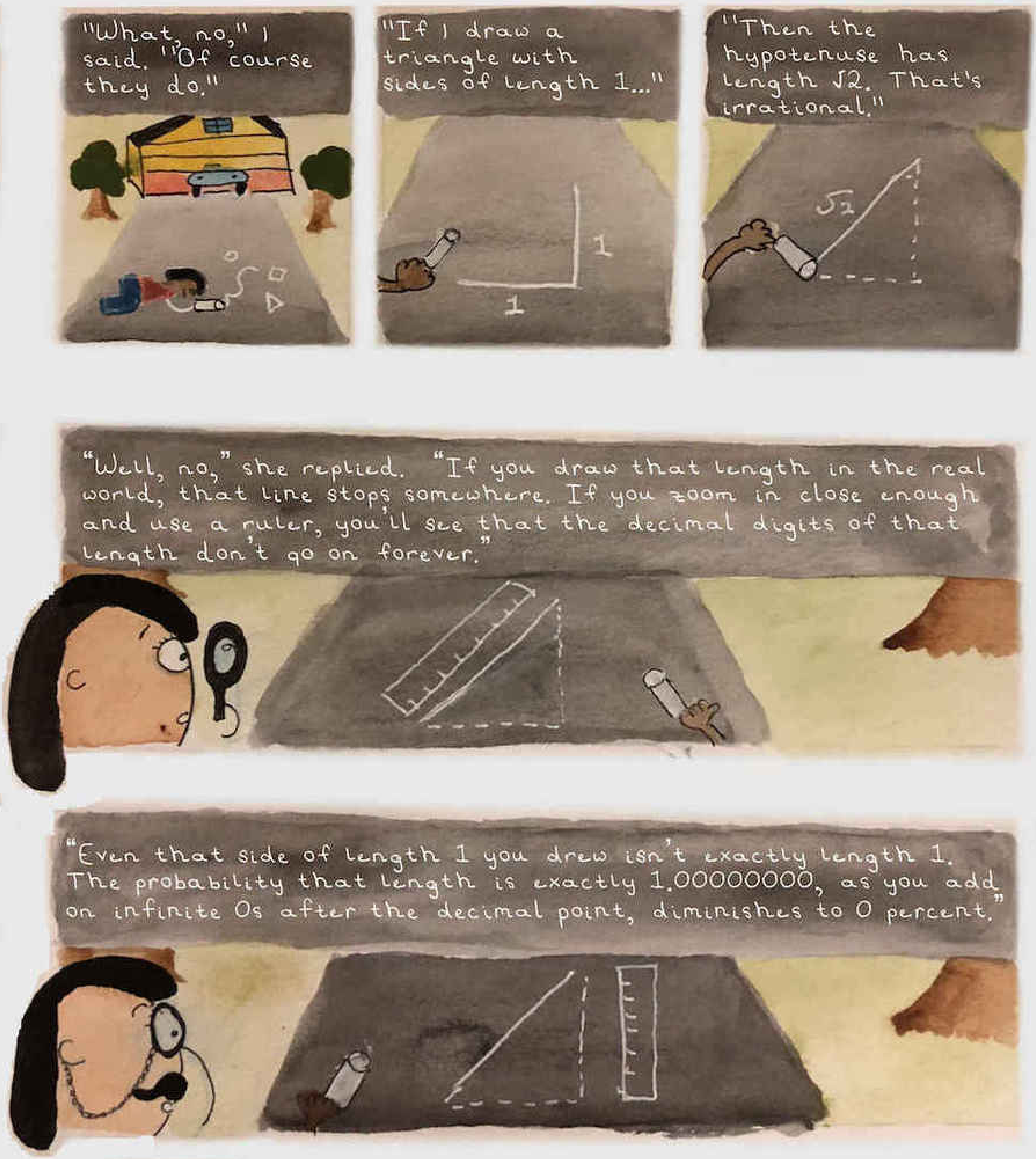
Let's prove q is divisible by two.



* Remember, we chose a p and q that shared no common factors.

9. But do irrational numbers actually exist?

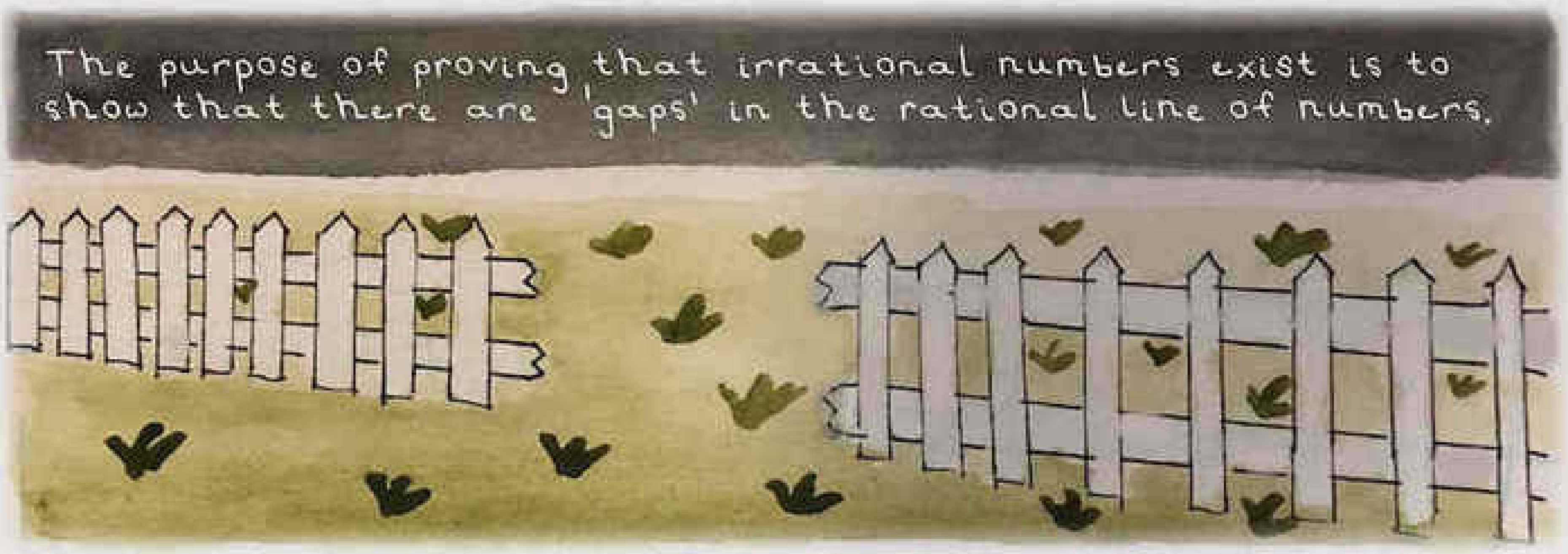
A friend once told me that irrational numbers don't exist in the real world.

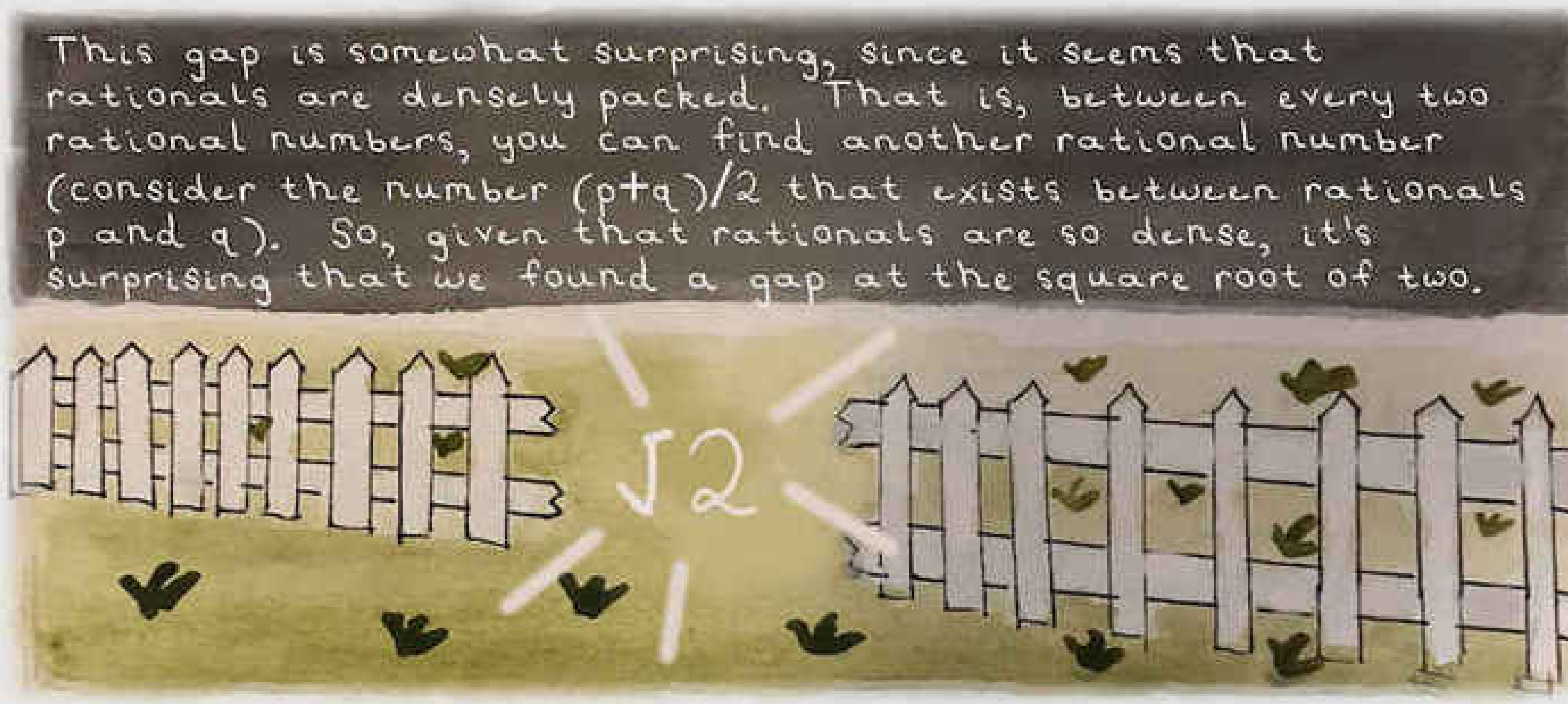


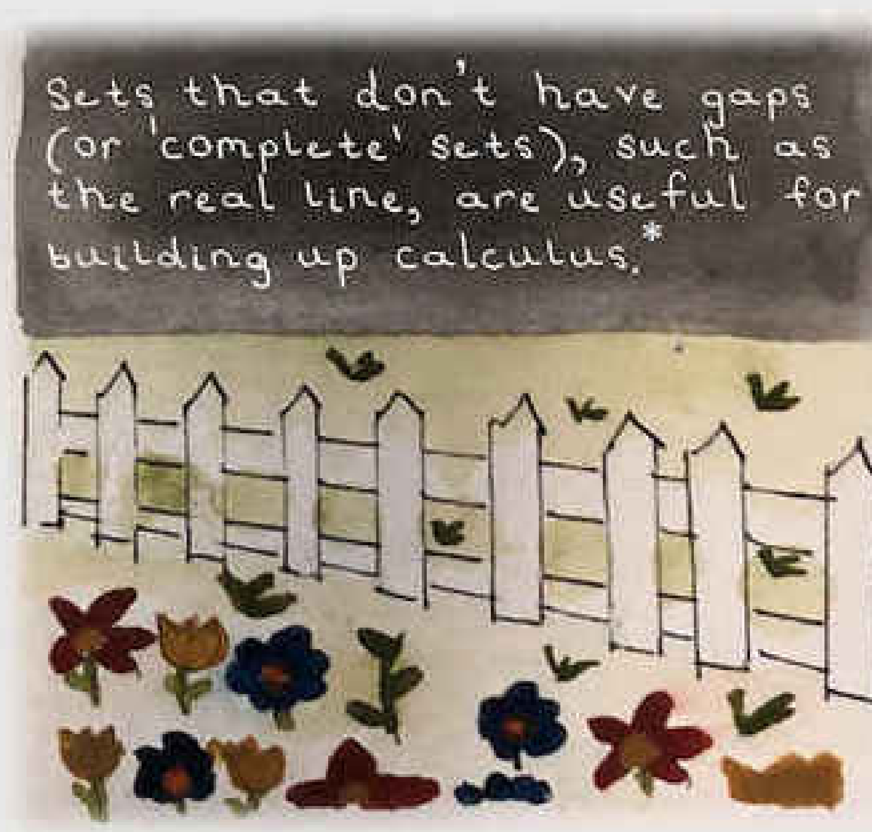

She's right.
Like a lot of analysis concepts, infinite decimals exist perfectly only in our heads.
So why bother with all of this stuff if it all only exists in our heads?
"Well, why do we do philosophy?" asked a friend.
"It's not always useful. And it's sometimes ridiculous."
"But it forces you to view things from a different perspective, and get a fuller understanding of your assumptions."
And that's exactly what makes analysis so cool. It's an entirely different way of thinking.

10. How does proving irrationality relate to analysis?

So what does proving that irrational numbers exist have to do with real analysis – the building of calculus?

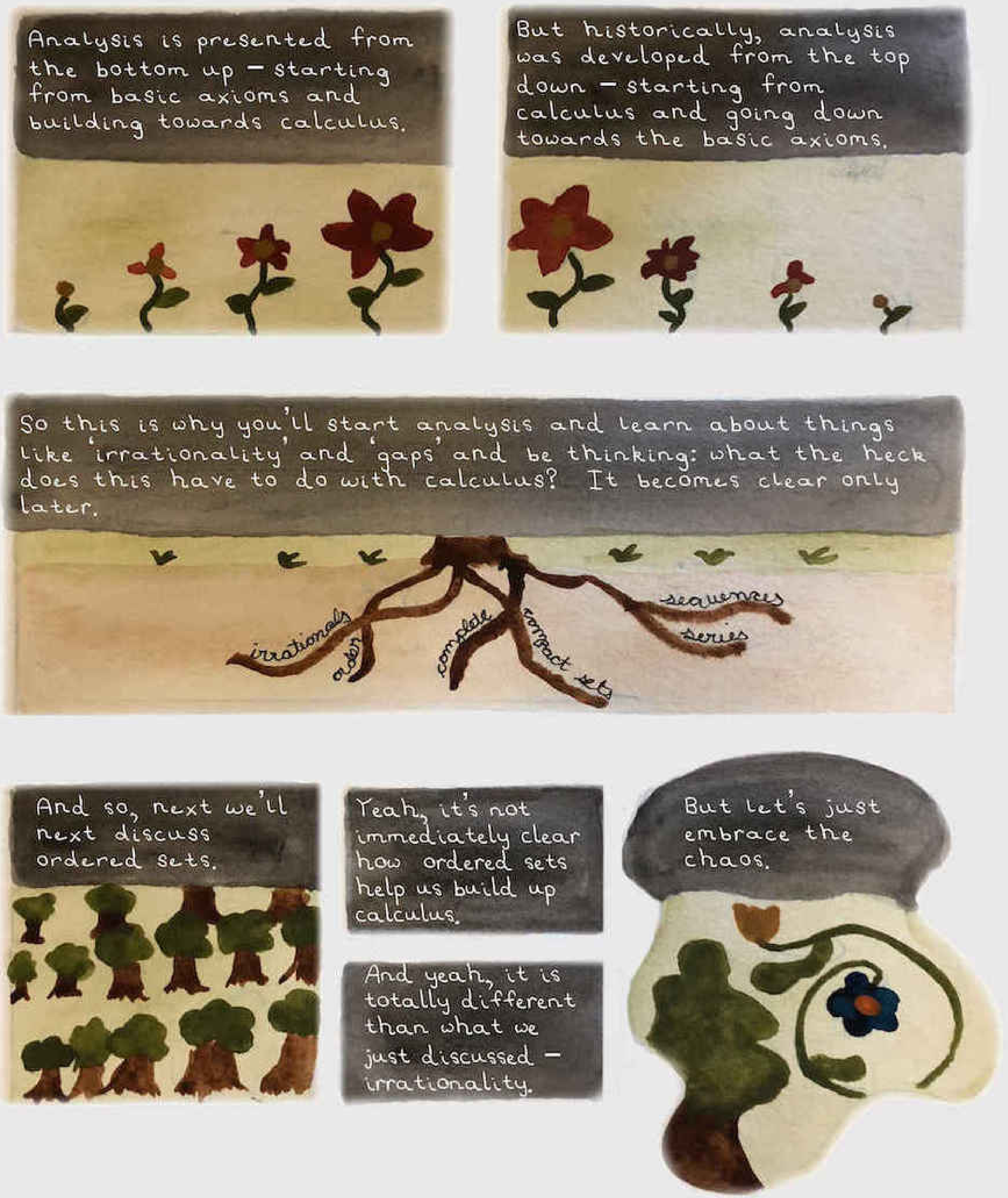




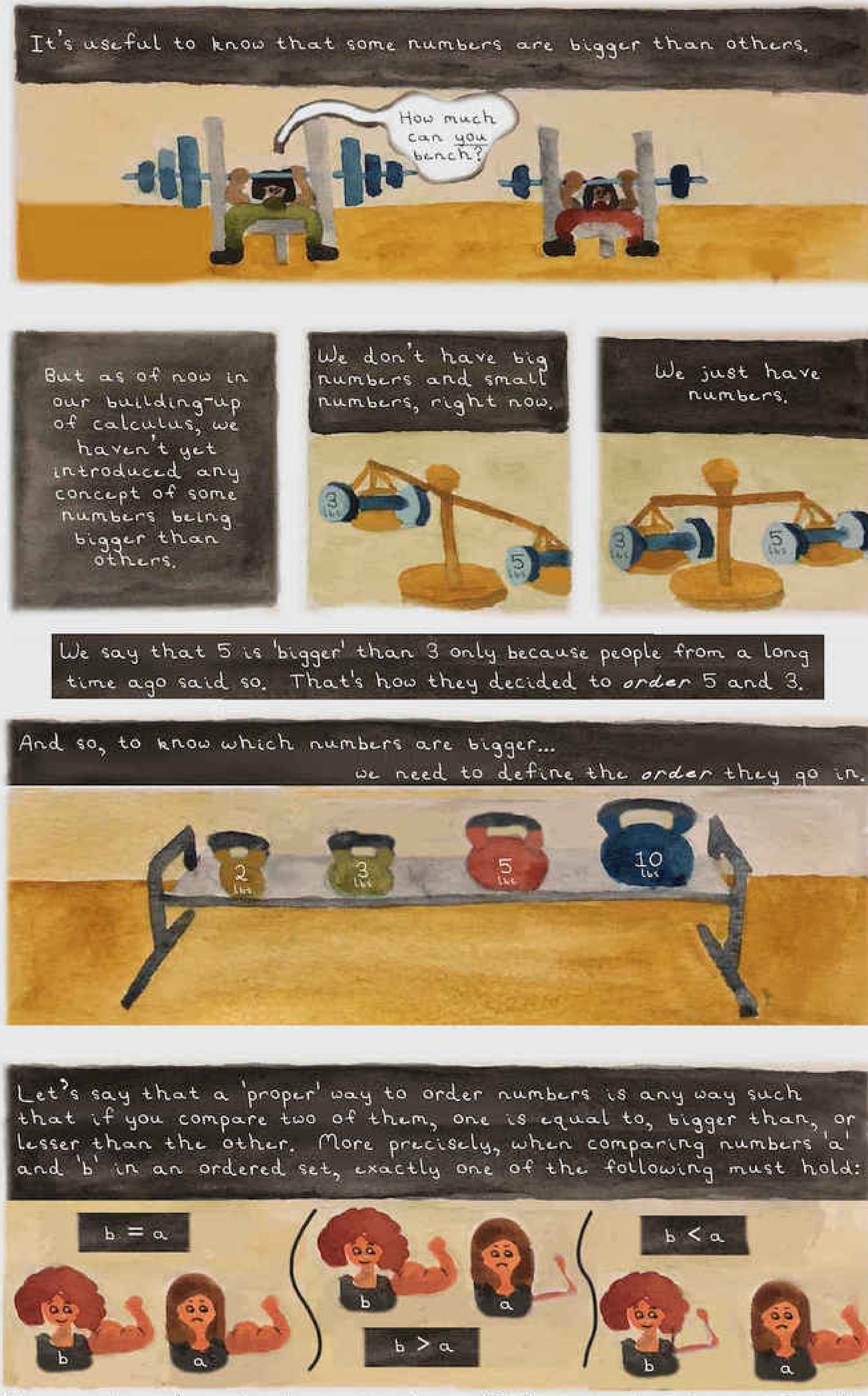


*For example, we know that limits are a foundational concept in calculus. But a sequence might not have a limit in an incomplete set. For example, consider the sequence of rational numbers that slowly approaches pi: 3, 3.1, 3.14, and so on. It will have no limit in the rationals (because its limit is pi).

11. Why does analysis feel so unrelated to calculus?



12. How do we know where to put numbers on a number line?



*For example, a 'proper' ordering for the set {3,5} *can* be that 3 *equals* 5 (it's not the ordering that most mathematicians go by, but it does satisfy our definition of a proper ordering).
But an order *can't* be that 3 is *both* greater than and less than 5. And it *can't* be that 3 has *no* relation to 4. If these last two were orderings, it would be impossible to answer the question: which is bigger: 3 or 4?

First, what exactly do I mean by *order*? I mean that given two numbers, the *order* should tell us *which number is bigger*. For example, we can ask...

How should we order the set $\{1,2,3\}$?

We could order it like: $1 < 2 < 3$.

Or, we could order it like: $1 > 2 > 3$.

(This is a 'proper' ordering, according to our definition. It is just not the one that most mathematicians use.)

Now for a tougher ordering problem. How should we order the rationals?

The ordering can't just say that the bigger number is bigger.

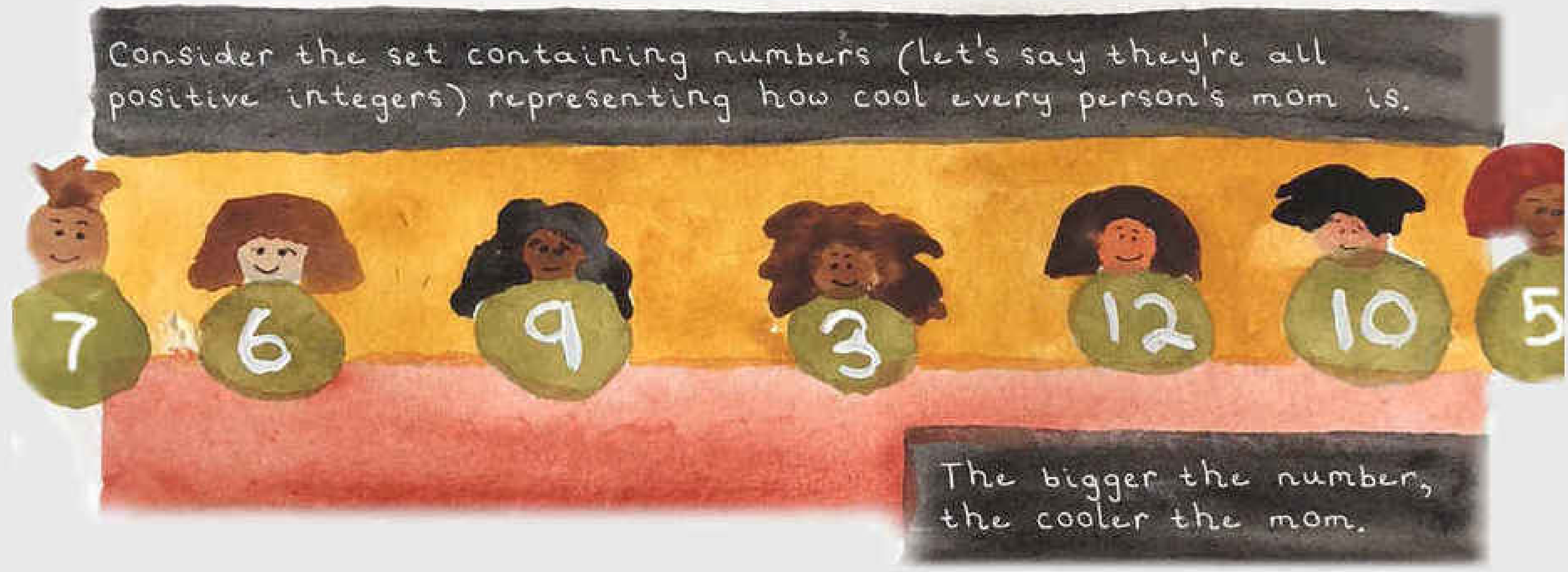
The purpose of ordering is to *define* the word 'bigger.'

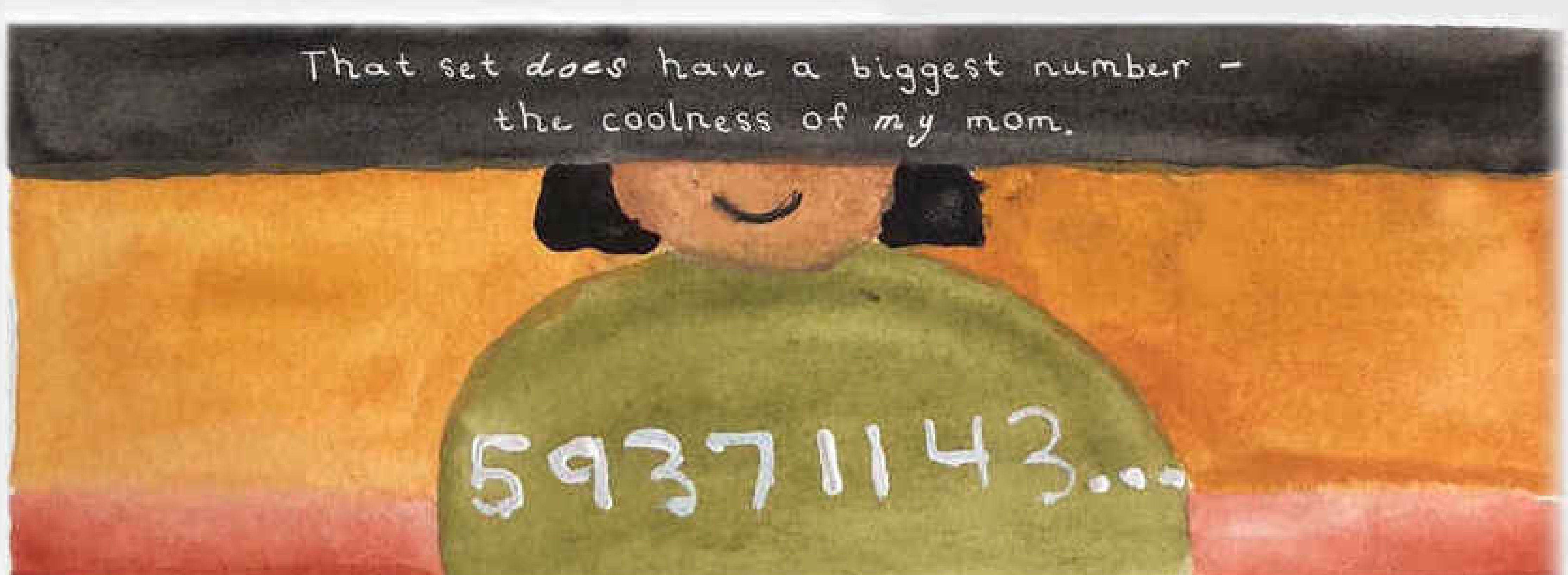
Instead, let's say our ordering is that the rational s is bigger than r when s−r is positive*.

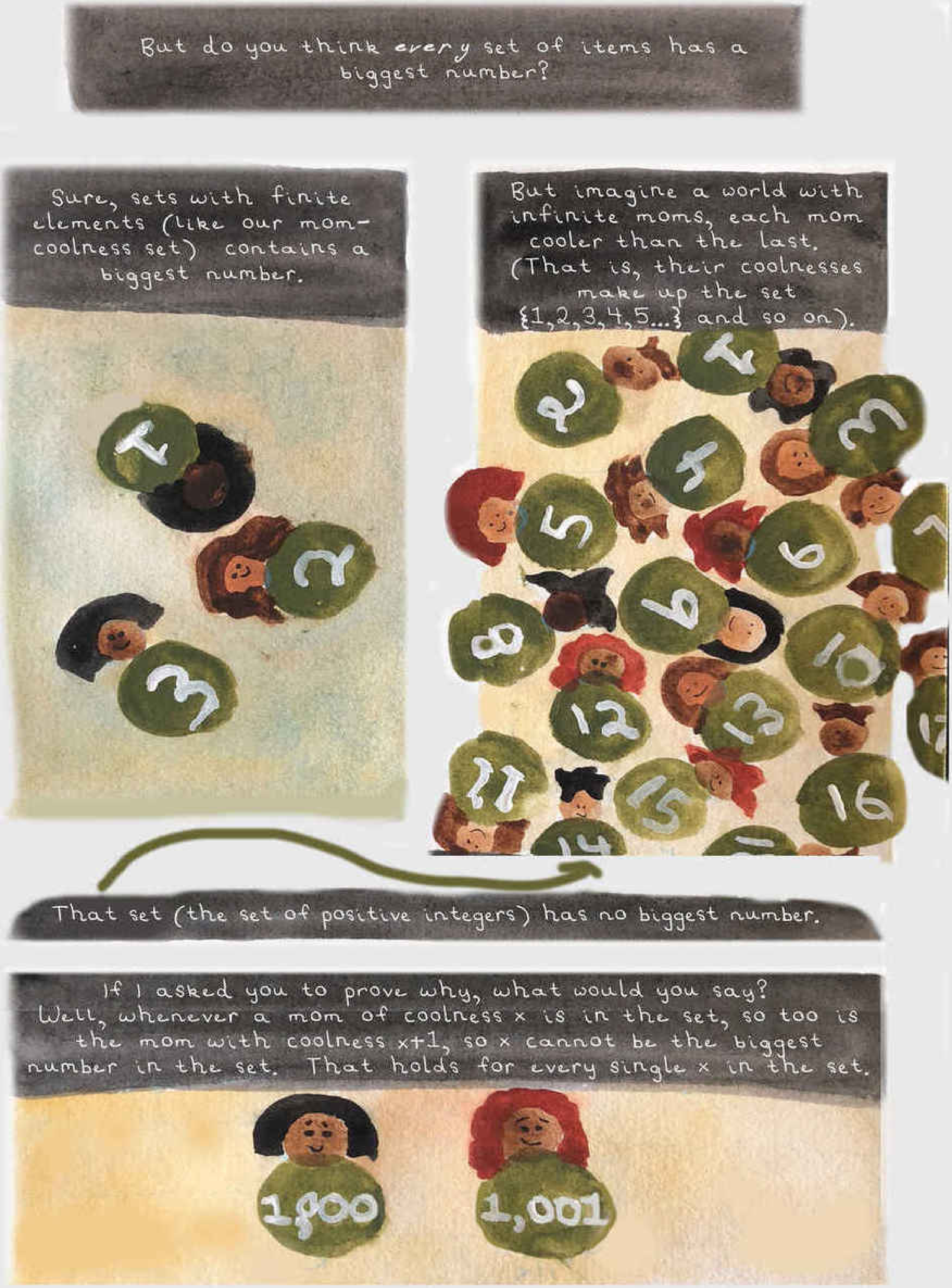
*Sounds good right? But we haven't defined positive. Now, in your head you might be thinking that it's *obvious* how to define positive: a number is positive when it's bigger than zero. But then, our definition of *positive* relies on *bigger* (*positive* means a number is *bigger* than zero) and our definition of *bigger* relies on *positive* (a number is *bigger* than another when s−r is *positive*). So for now, I'll trust we all know the difference between a positive and negative number. But we will formally define what positive and negative mean later.

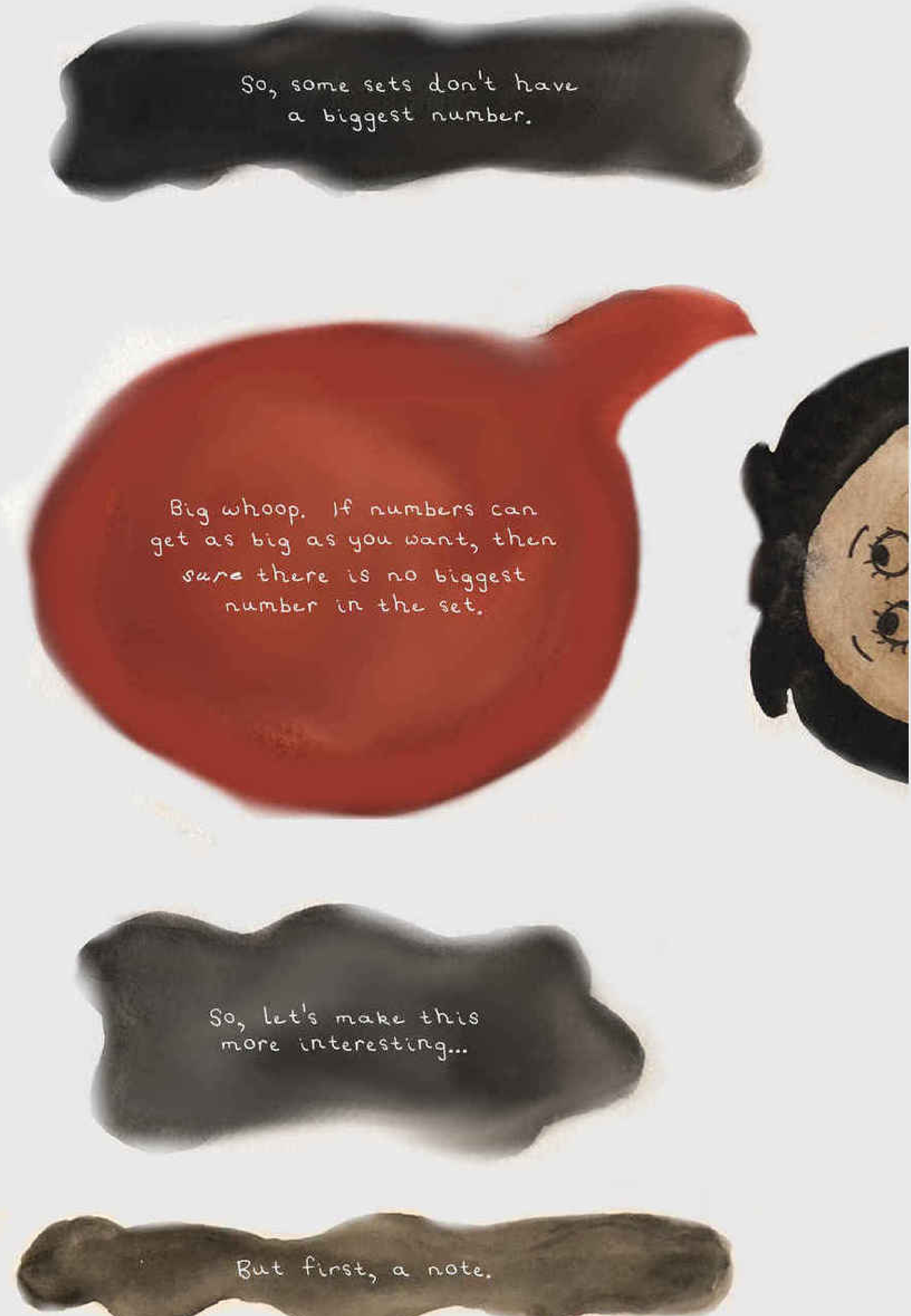
Ok. Now we know how to order sets. So, we can figure out which numbers are bigger than others.

13. Does every set have a biggest number?

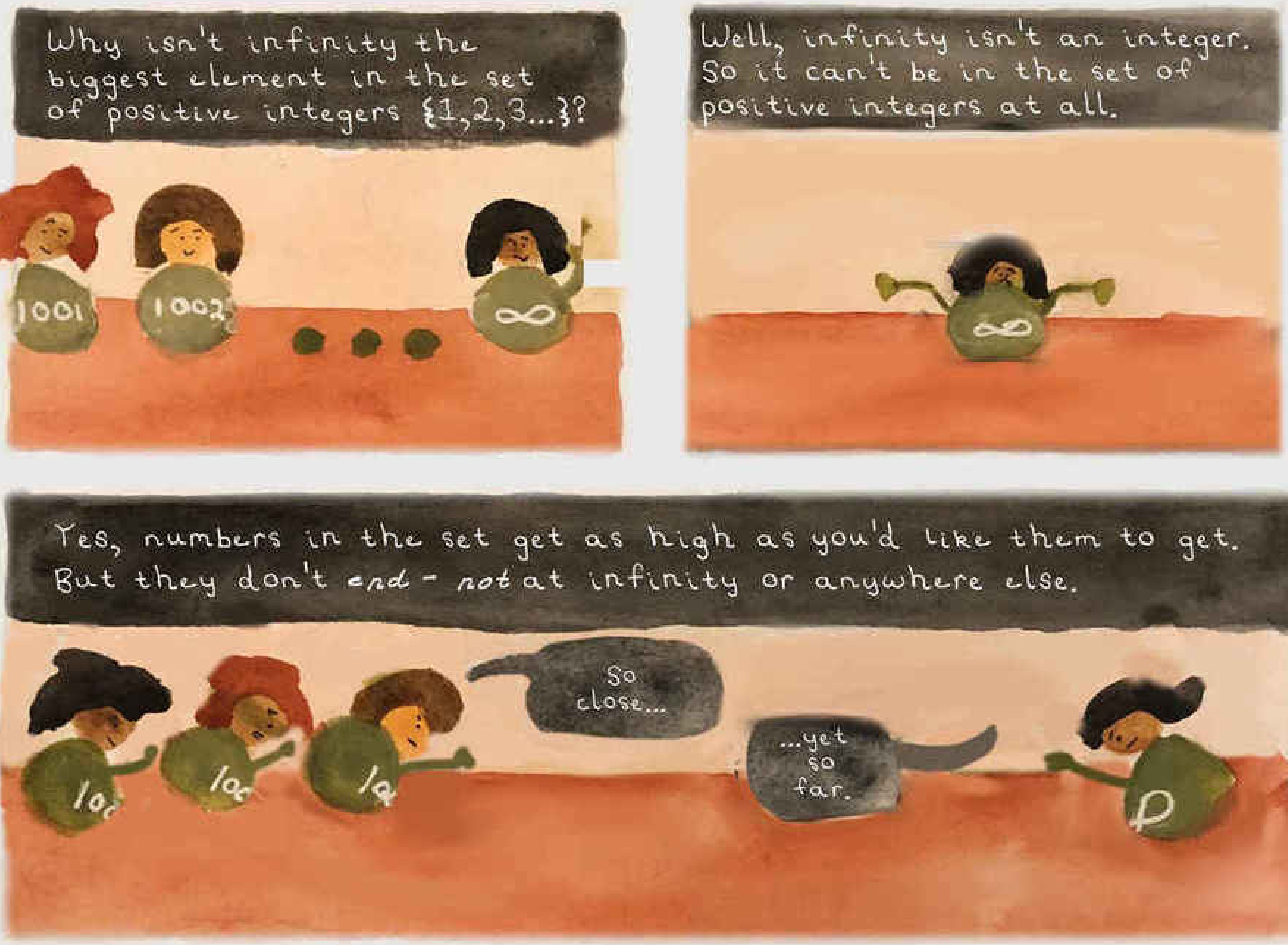




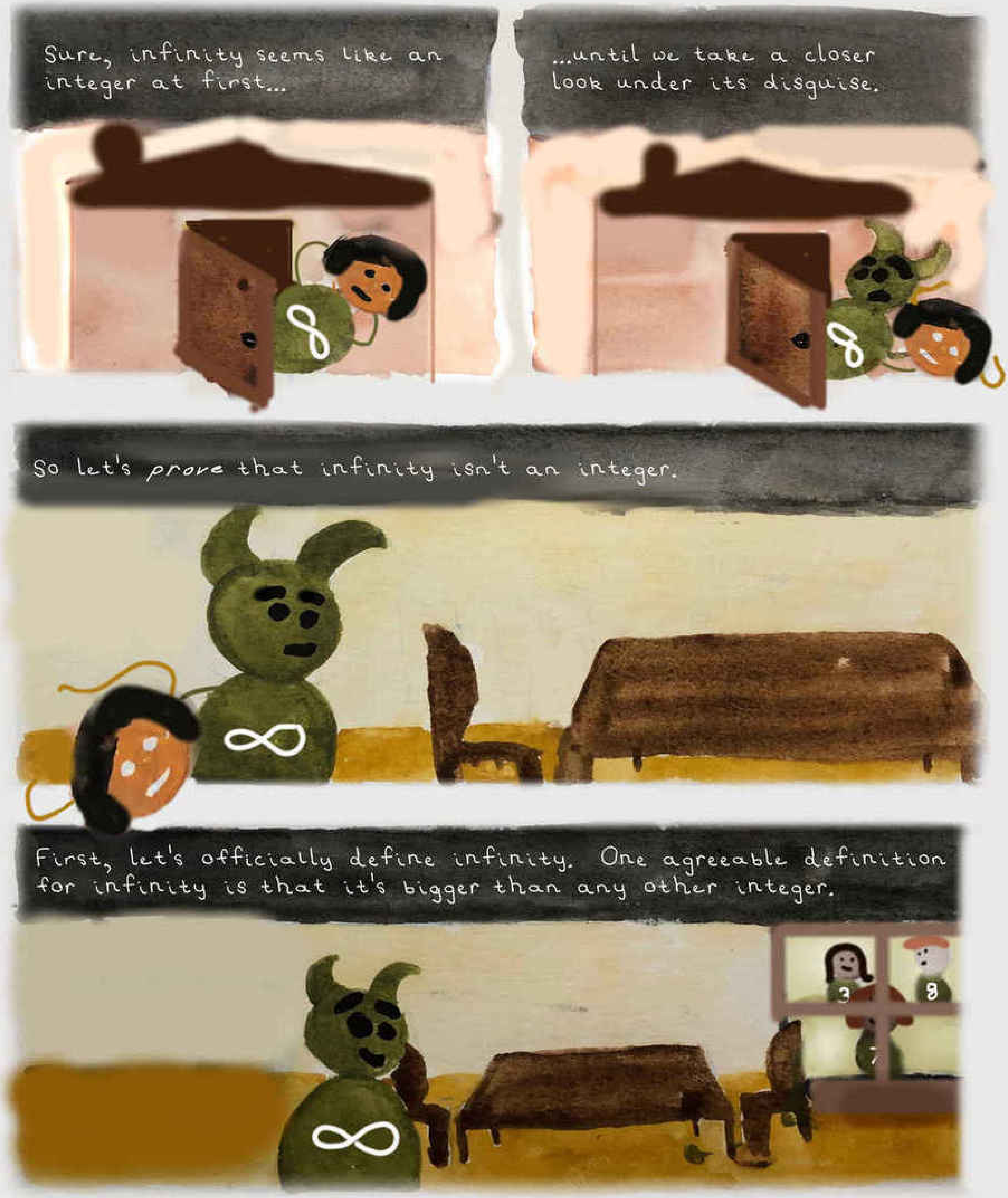

But do you think *every* set of items has a biggest number?
Sure, sets with finite elements (like our mom-coolness set) contains a biggest number.
1
2
3
But imagine a world with infinite moms, each mom cooler than the last. (That is, their coolnesses make up the set {1,2,3,4,5...} and so on).
1
2
3
4
5
6
9
8
10
12
13
11
15
16
That set (the set of positive integers) has no biggest number.
If I asked you to prove why, what would you say? Well, whenever a mom of coolness x is in the set, so too is the mom with coolness x+1, so x cannot be the biggest number in the set. That holds for every single x in the set.
1,000
1,001


So, some sets don't have a biggest number.
Big whoop. If numbers can get as big as you want, then sure there is no biggest number in the set.
So, let's make this more interesting...
But first, a note.

14. Isn't infinity the biggest element in the set {1,2,3...}?

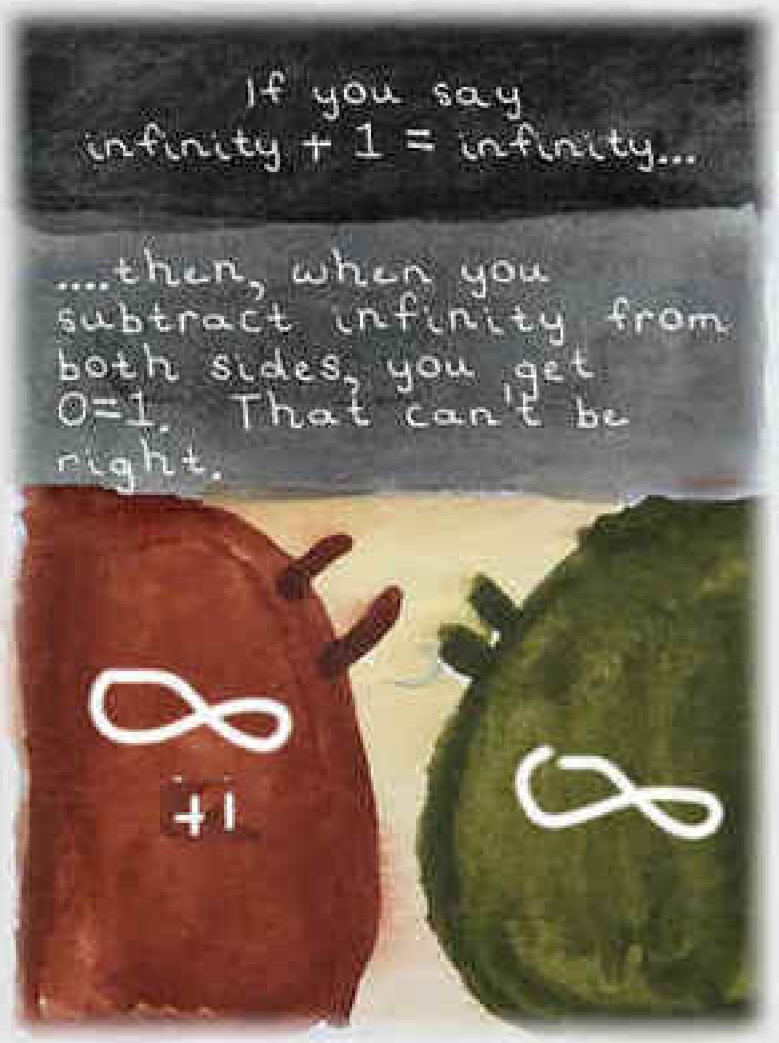


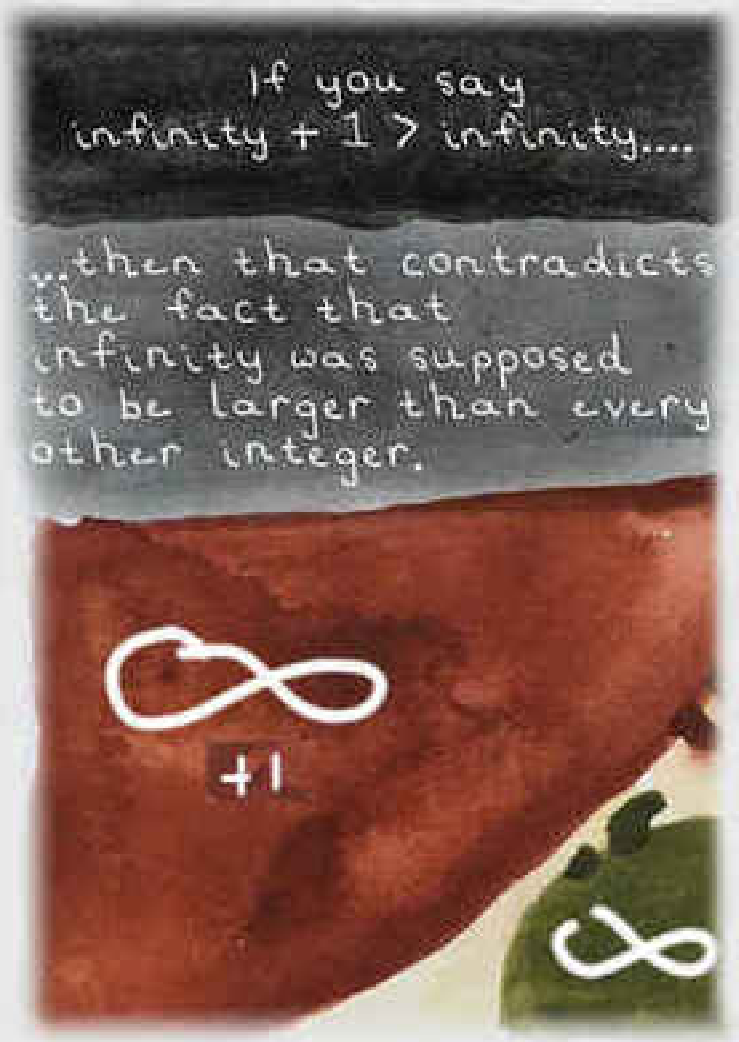
15. How do you prove infinity isn't an integer?

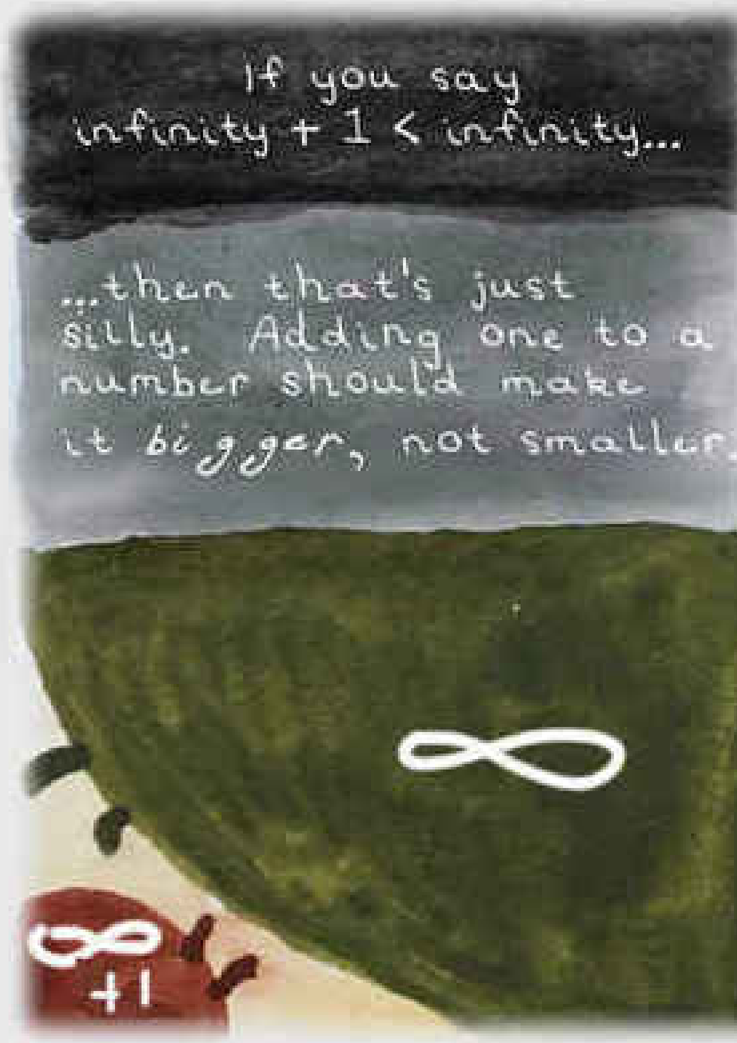


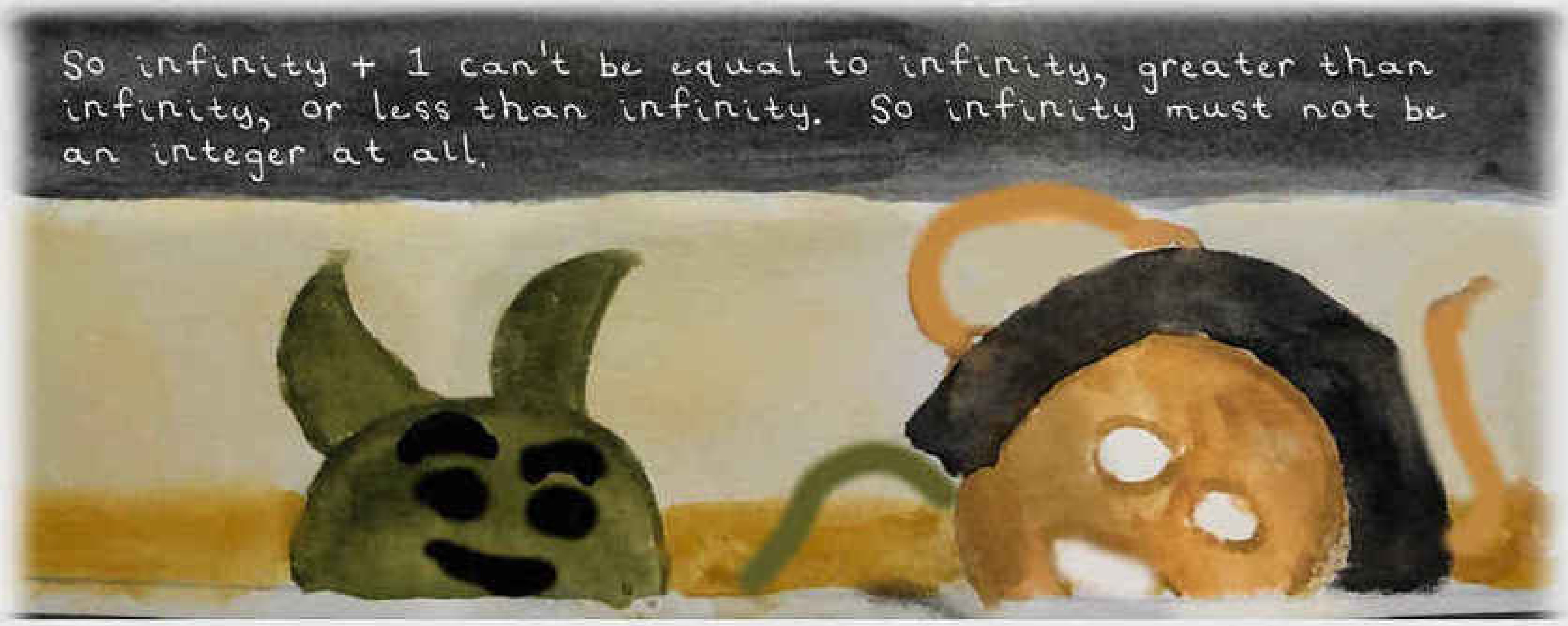
So let's suppose (for the purposes of contradiction) that infinity is an integer.

Note that infinity + 1 must be an integer, since an integer plus an integer should equal an integer.

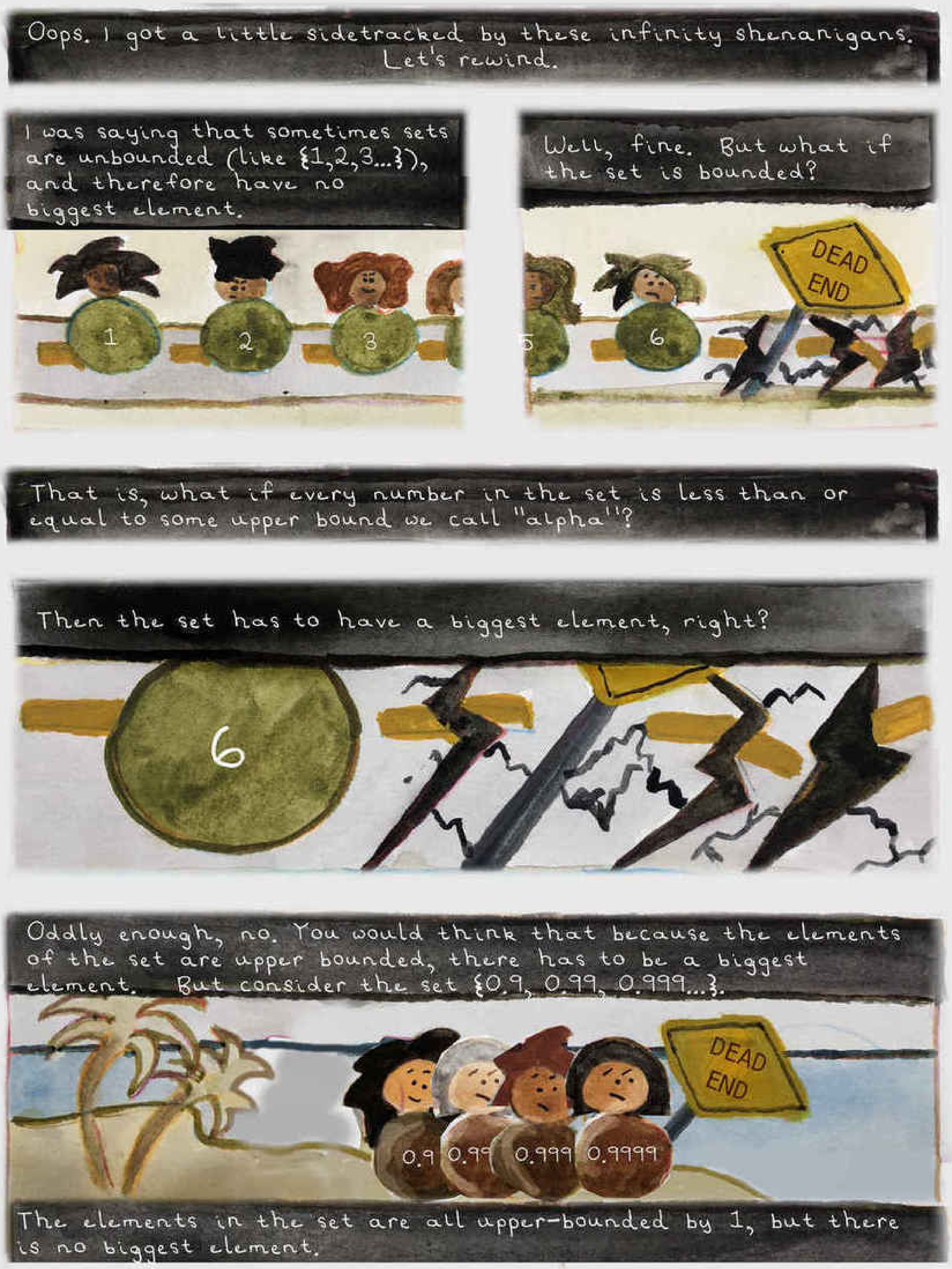








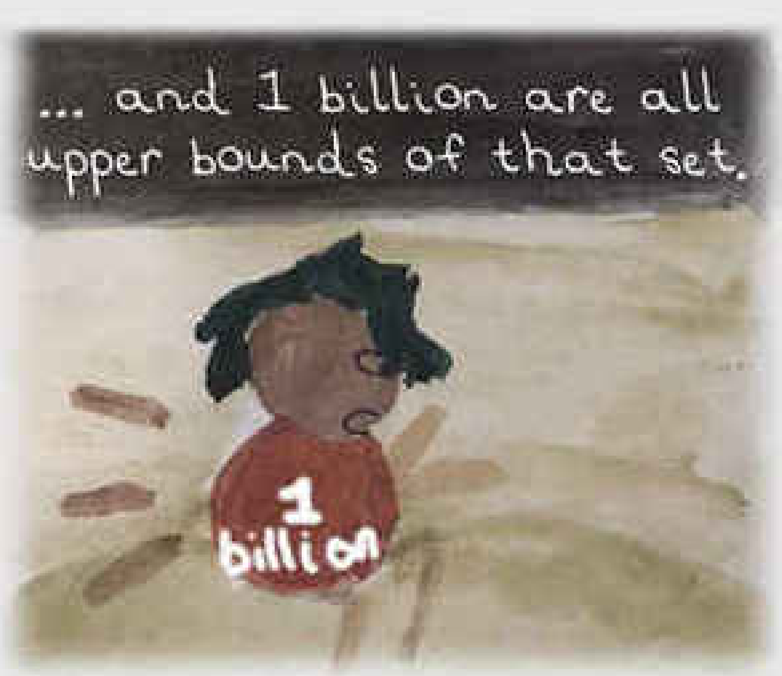
16. Does every upper bounded set have a biggest element?



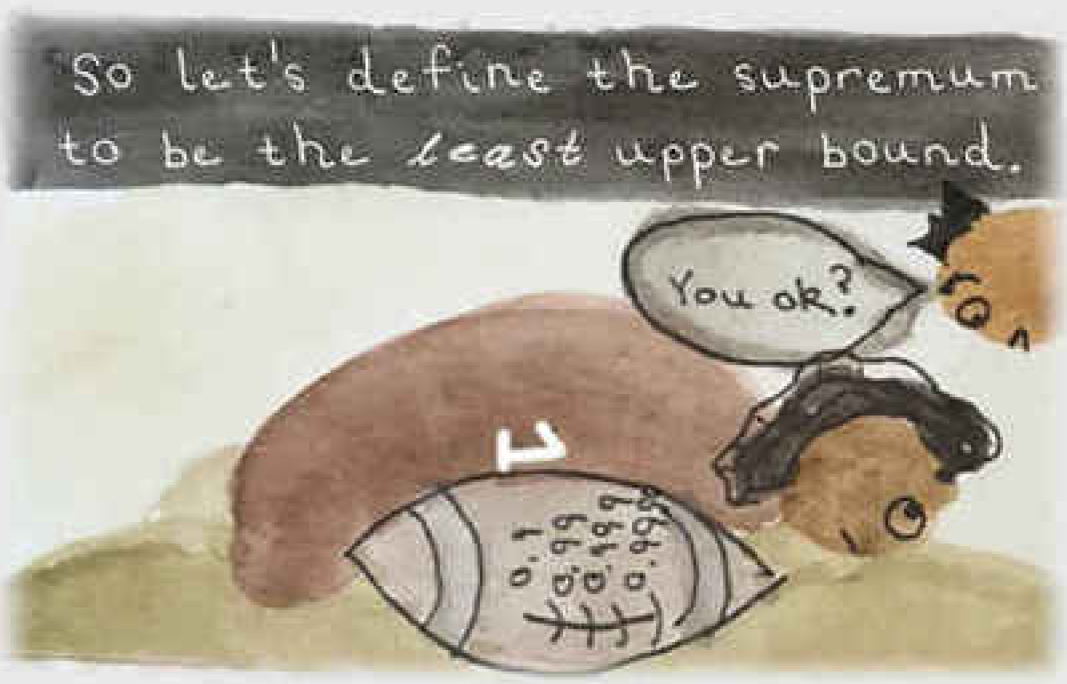

And I'll even prove that the set has no biggest element.
Tell me a number in the set that you claim is the biggest...
0.99
...and I'll add a 9 to the end, and come up with an even bigger number in the set.
0.999
Therefore, any number that you claimed is the biggest cannot be. So we can agree that the set has no biggest number.

17. What's the most important upper bound of a set?

But 1 isn't the only upper bound of that set.

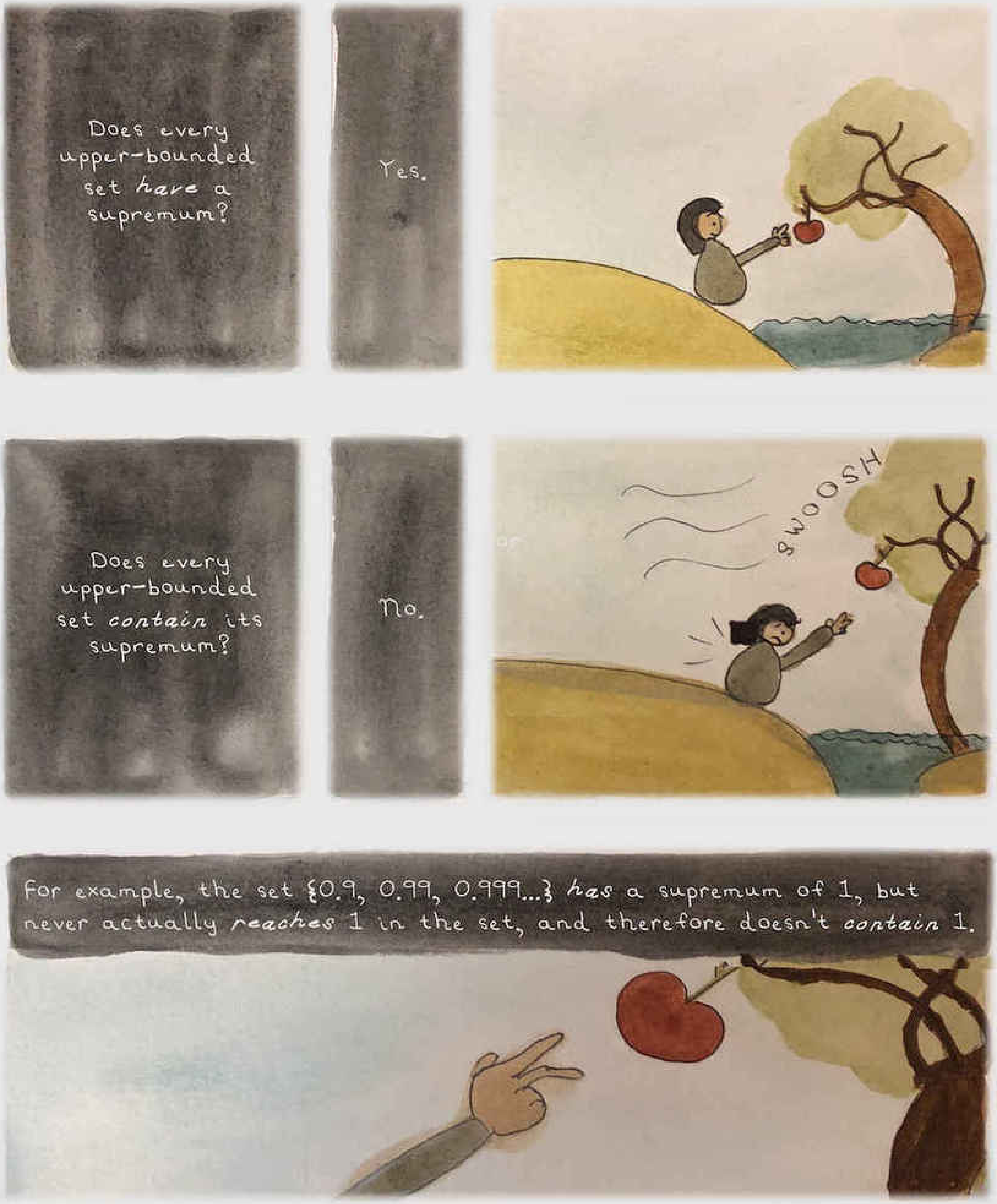


But some of those upper bounds seem sort of useless and redundant, like 1 billion.



For example, the supremum of the set $\{0.9, 0.99, 0.999, \ldots\}$ is 1.
Although we can't know for sure, because we haven't proved it, yet.
Just take my word for it, for now.

Does every upper-bounded set contain its supremum?



You might say the set does contain 1, since the set goes on forever, and 0.999 repeating forever equals 1. But even infinite sets don't contain an "infinitieth" element. The numbers in the set get arbitrarily *close* to 1, but they never do *reach* 1.

Bibliography

(By Chapter)

1 I owe my comparison of math and philosophy to Amanda Gefter, who once told me that "physics is like philosophy, but with answers."

3 For more on the origins of abstract mathematics, see the book *Duel at Dawn: Heroes, Martyrs, and the Rise of Modern Mathematics* by Amir Alexander.

5 Lembke's book on the Vietnam War spitting story is called *The Spitting Image: Myth, Memory and the Legacy of Vietnam*. I owe this reference to Thalia Rubio at MIT.

7 For more on the Greek mathematician exiled for asserting that some numbers are irrational, see the fascinating Ted Ed video: "A Brief History of Banned Numbers."

8 The proof that the square root of two is irrational was adapted from page 2 of *Principles of Mathematical Analysis* by Walter Rudin.

9 Thank you to Amanda Sobel at MIT for the wonderful comparison of analysis to philosophy.

11 For more on the contrast between the top-down historical development of analysis and the bottom-up teaching of analysis, see page 37 of the book *How to Think about Analysis* by Lara Alcock.