

# Real Analysis

by Anshula Gandhi



## 1. What is this book?

"What is the point of this?"

...is a common question in math classes for a reason.

In math, we are often presented with the tools that answer questions, without being given the questions themselves.

For example, in math class, I learned what a "field" was, but had no idea why we used it, until I realized it helped answer the question "what is a number?"

The purpose of this book is to present the guiding philosophical questions that have led to core mathematical concepts in real analysis.

After all, math is like philosophy, but with answers.

I love real analysis - the math of formalizing calculus. You might or you might not be into that.

Either way, this book is for you, and I hope you'll like analysis by the end of this book. I hope it's something you'd want to read if you're curious about math, even if you don't have to pass a real analysis class. But if you are taking a class, this book should teach what you'd be tested on.

I'll be adding new pages weekly. While pages are being added, I hope you'll feel free to skip around and read whatever looks interesting to you.

Chapter 0

# Introduction

## 2. What is real analysis?

You learned some amount of rubbish about calculus in high school (even if you didn't realize it).

Analysis is about cleaning up that rubbish.



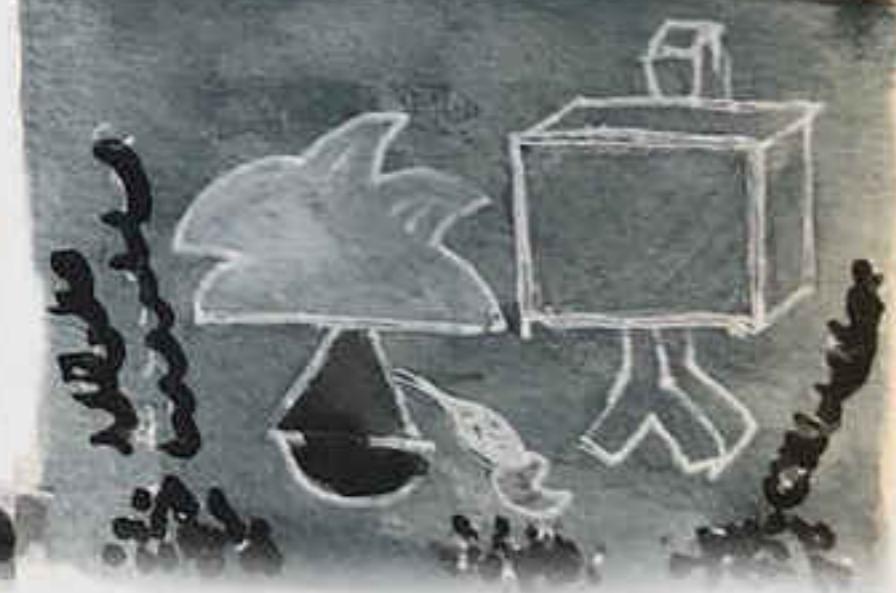
To do analysis...



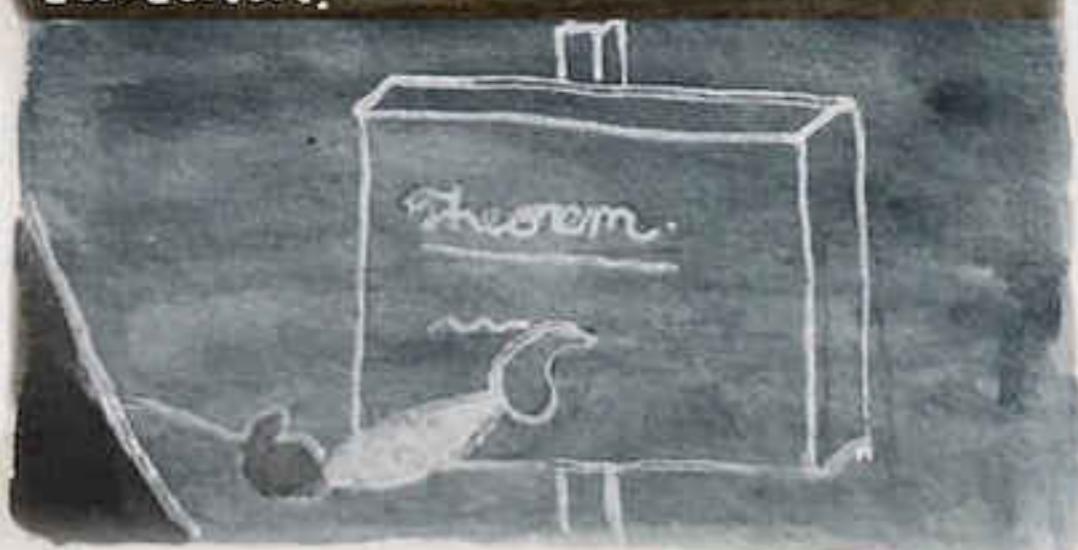
We have to forget everything we know about math.



Then we'll build it back up again, proving each step, making sure it's all true.



Note: Analysis is about proving calculus, not doing calculus.



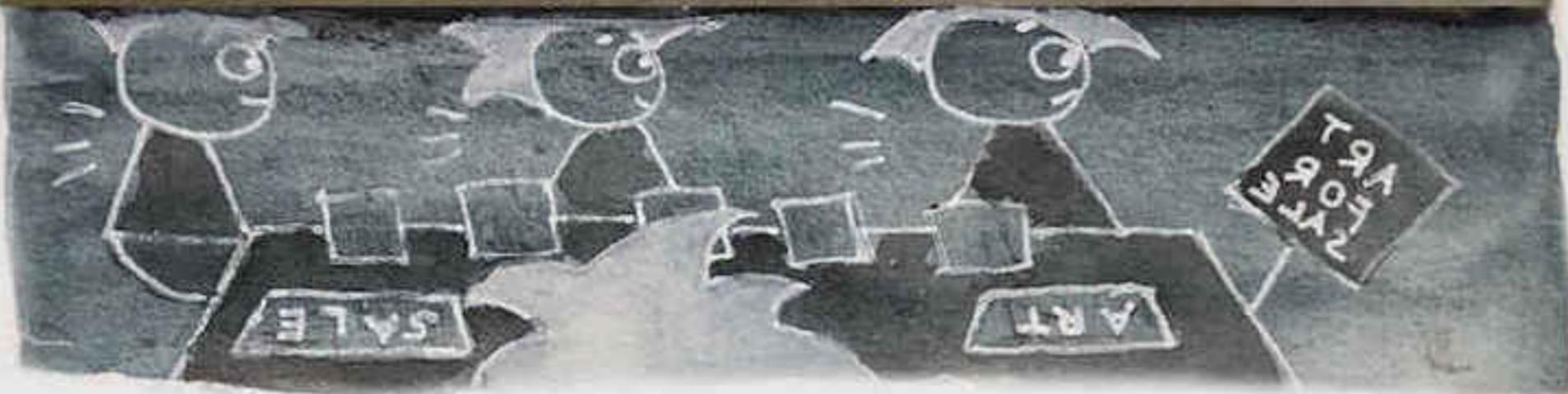
For example, you'll prove that integrals exist, but never calculate a single one.



### 3. Why do analysis?

Disclaimer: Analysis isn't technically "useful," in that:

- 1) You can rarely apply it to the real world, and
  - 2) it probably won't help you land a job.



But despite analysis being close to useless in a job application, it is useful in that:

- 1) The hand-waving of calculus can lead to paradoxes. Analysis fixes that by proving calculus rigorously.



- 2) Analysis presents an opportunity to think and prove things in an entirely new way.



- 3) Analysis offers escape from reality - a chance to philosophize about problems that have nothing to do with your everyday life.



And sure, analysis does have some practical applications. But who cares, anyway? We do it because it's fun.



## DIGRESSION

### 4. Why do analysis books rarely use pictures?

Analysis came about as part of a movement in the 1700s. Some mathematicians wanted math to be more "pure" and abstract, and rely less on the "crutches" of figures and diagrams.

It was something of a challenge, perhaps, to define math without relying on figures

And that's probably why most analysis texts shy away from visuals: because a big point of analysis was to not need visuals anymore.

Chapter 1

# Real and Complex Numbers

## 5. How do you prove that something doesn't exist?

Proving something doesn't exist can be a lot harder than proving something does exist.

A sociologist, Jerry Lembcke, ran into that problem.

He'd heard stories about antiwar protesters spitting on Vietnam War soldiers as they returned home to America.

But as Lembcke looked into the phenomenon, he could find no single instance of this spitting.



But he couldn't say for sure that "the spitting never happened."



If he only had to prove the spitting happened, he would just have to find one account of it.



But how could he prove that nobody ever spat on a war veteran?

It would be impossible to interview every single Vietnam War veteran - dead and alive.



The sociologist admitted that he couldn't say the "spitting protestor" phenomenon was untrue. He could only say he found no evidence it was true.

## 6. How do you prove non-existence in math?

So, people say "you can't prove a negative statement." Not true. It's hard. But in math, it's possible with a "proof by contradiction."

We start by assuming something does exist, and reason through it.



If we find a contradiction within our reasoning, we must conclude that thing does not exist.



But why?

When two people contradict each other, you know one of them has got to be wrong.



Similarly, in math, if two of the lines in your proof contradict each other, then there's a lie in there somewhere.



And if every step you took in the proof was correct, then the lie can't be in any of the steps you took. The lie must be way back, in the very first assumption you made.

## 7. How do you prove irrationality?

When an ancient Greek discovered that some numbers are irrational\*...

He was exiled.  
(It was heresy to say numbers were so disorderly.)



But how could he have proven that irrational numbers exist?



\*Irrational numbers are those that can't be written as a ratio of integers (e.g.  $\pi$ ).

Did he line up all the infinite fractions in the world, and compare each to a number he thought was irrational?



He couldn't have. He'd have to have checked all infinite fractions in the world before ensuring none of them was his guy.



That method of proof would be impossible.

Instead, we could use proof by contradiction.



## 8. Is the square root of two irrational?

First, let's declare that  $\sqrt{2}$  exists.  
(We should be explicit about the rules we're going by.)

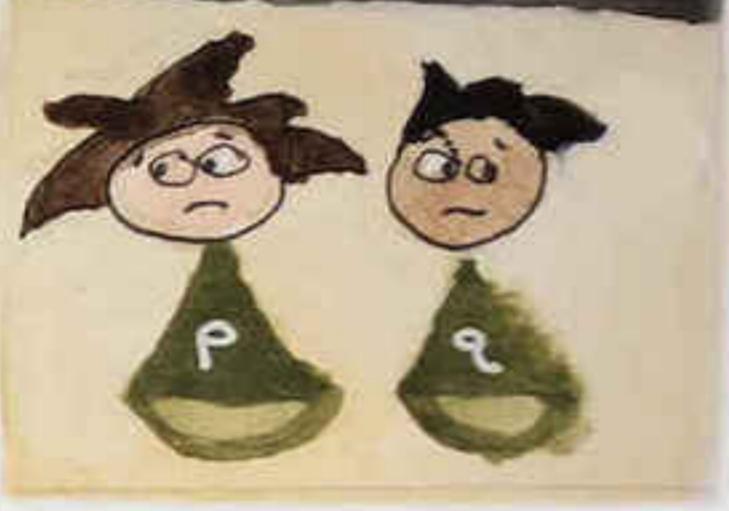
Let's assume, for the purpose of contradiction, that  $\sqrt{2}$  is rational.  
(It's not.)



Then there must exist a fraction  $\frac{p}{q}$  that equals  $\sqrt{2}$ .



Let's use a  $p$  and  $q$  that have no common factors (so that  $\frac{p}{q}$  is simplified.)



Now we'll show the contradiction: even though we chose  $p$  and  $q$  to share no common factors, they always end up sharing a factor of two.

Let's interrogate  $p$  and  $q$  separately.



Let's prove  $p$  is divisible by two.

$$\sqrt{2} = \frac{p}{q}$$

square



$$2 = \frac{p^2}{q^2}$$

move  $q^2$

$$p^2 = 2q^2$$

logic

$p^2$  is even

$p$  is even

more logic

Let's prove  $q$  is divisible by two.

$$\sqrt{2} = \frac{p}{q}$$

since  $p$  is even, we can rewrite  $p$  as  $2k$   
(where  $k$  is some integer)

$$\sqrt{2} = \frac{(2k)}{q}$$

square



$$2 = \frac{4k^2}{q^2}$$

move  
 $q^2$

$$q^2 = 2k^2$$

logic

$q^2$  is even

$q$  is even

more logic

So, p and q share  
a factor of two.  
Contradiction!



\* Remember, we chose a p  
and q that shared no  
common factors.

But all  
the  
steps we  
took  
were  
correct...

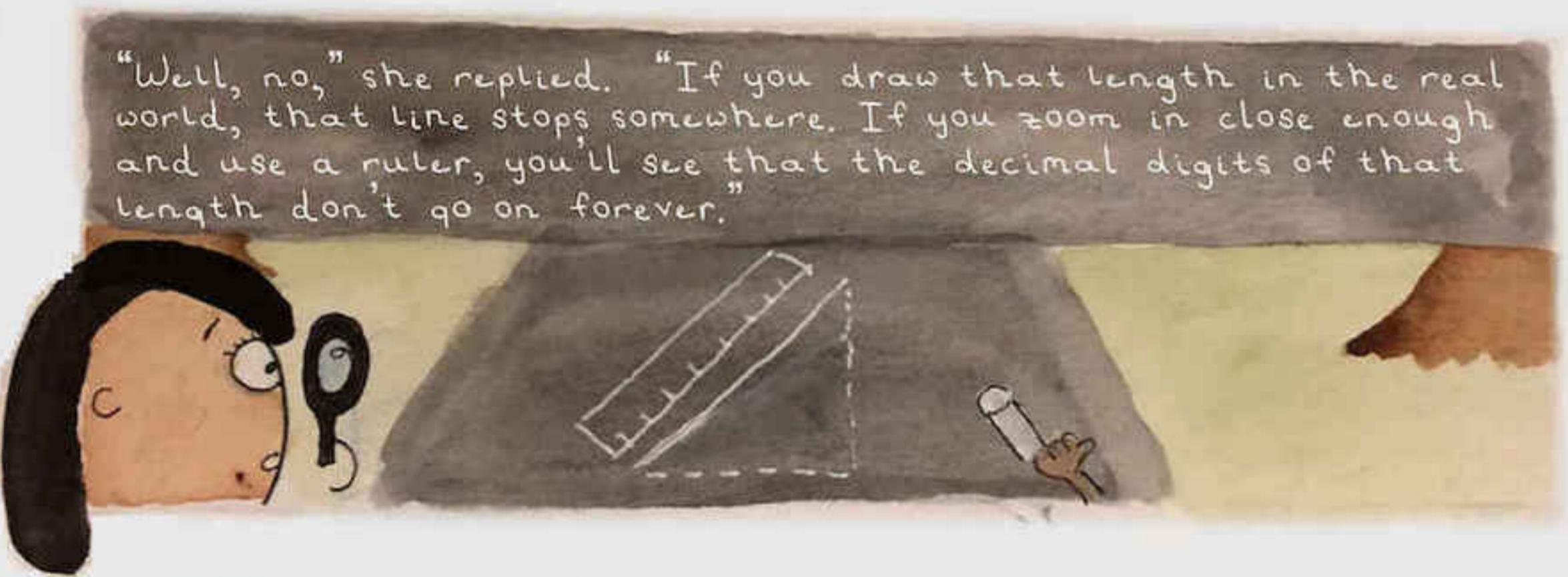
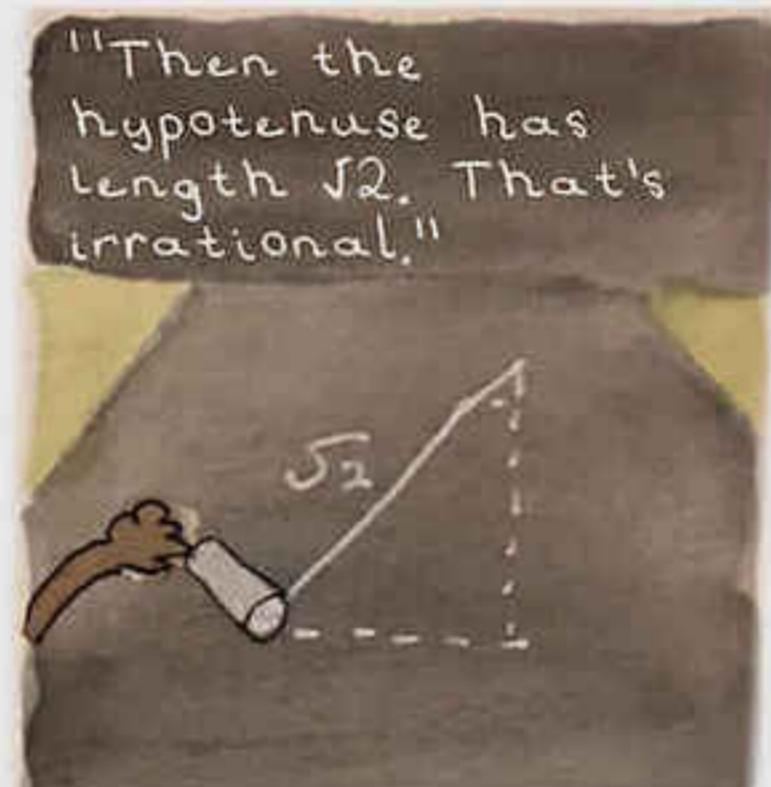
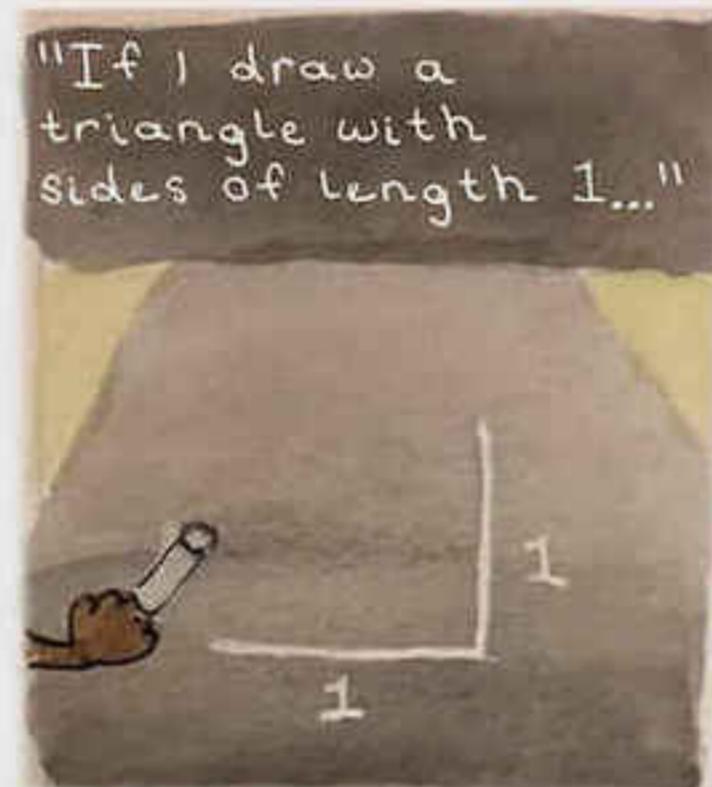
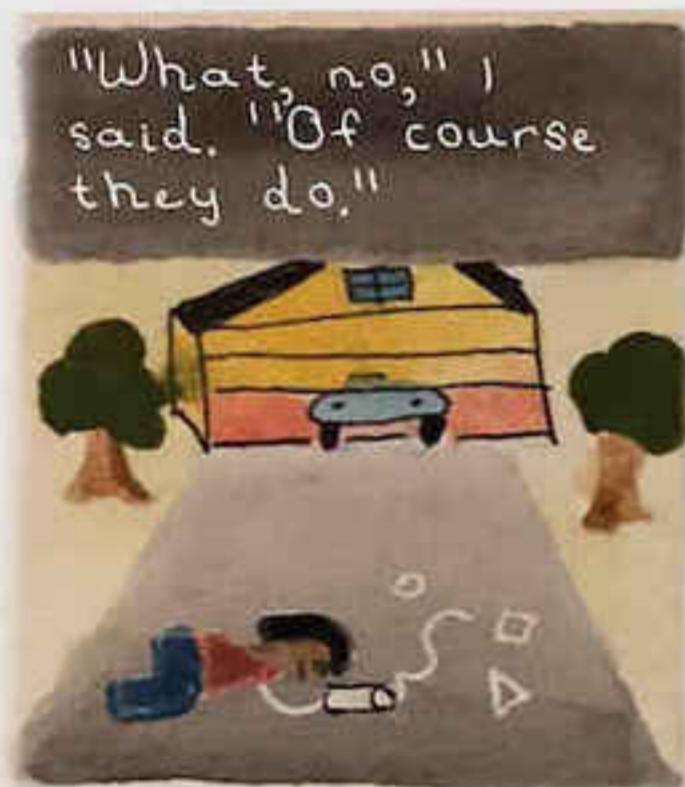
So the flaw must be in  
our assumption, and the  
only thing we assumed  
was that  $\sqrt{2}$  is rational.

So,  $\sqrt{2}$  can't be rational.

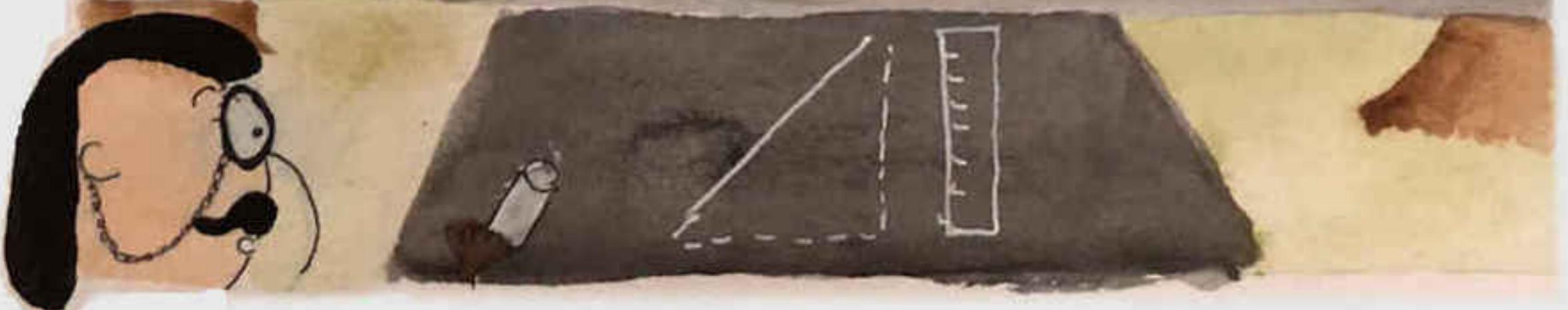


## 9. But do irrational numbers actually exist?

A friend once told me that irrational numbers don't exist in the real world.

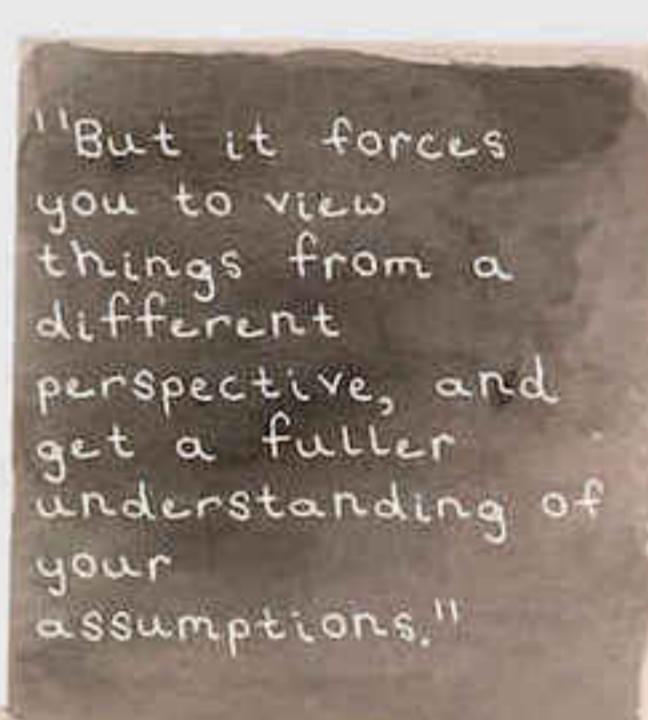
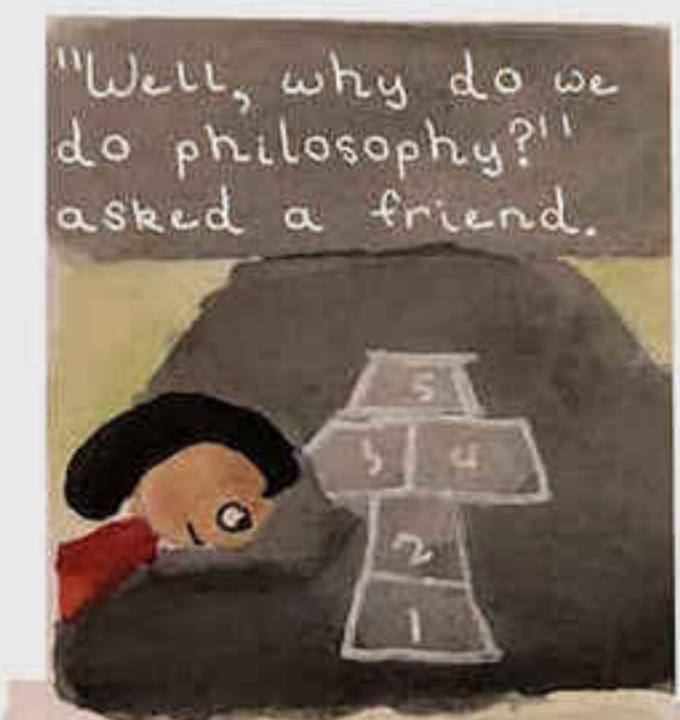


"Well, no," she replied. "If you draw that length in the real world, that line stops somewhere. If you zoom in close enough and use a ruler, you'll see that the decimal digits of that length don't go on forever."





So why bother with all of this stuff if it all only exists in our heads?



And that's exactly what makes analysis so cool. It's an entirely different way of thinking.



## 10. How does proving irrationality relate to analysis?

So what does proving that irrational numbers exist have to do with real analysis – the building of calculus?

The purpose of proving that irrational numbers exist is to show that there are 'gaps' in the rational line of numbers.



This gap is somewhat surprising, since it seems that rationals are densely packed. That is, between every two rational numbers, you can find another rational number (consider the number  $(p+q)/2$  that exists between rationals  $p$  and  $q$ ). So, given that rationals are so dense, it's surprising that we found a gap at the square root of two.



It's not only surprising, but also somewhat inconvenient that rationals have gaps.



Sets that don't have gaps (or 'complete' sets), such as the real line, are useful for building up calculus.\*



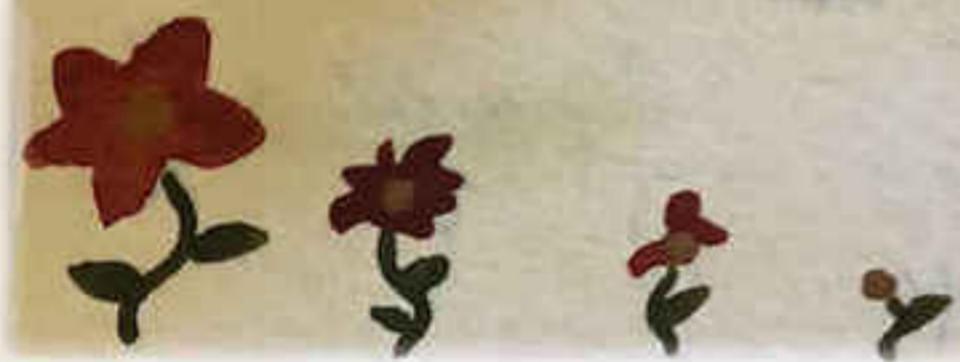
\*For example, we know that limits are a foundational concept in calculus. But a sequence might not have a limit in an incomplete set. For example, consider the sequence of rational numbers that slowly approaches pi: 3, 3.1, 3.14, and so on. It will have no limit in the rationals (because its limit is pi).

## 11. Why does analysis feel so unrelated to calculus?

Analysis is presented from the bottom up - starting from basic axioms and building towards calculus.



But historically, analysis was developed from the top down - starting from calculus and going down towards the basic axioms.



So this is why you'll start analysis and learn about things like 'irrationality' and 'gaps' and be thinking: what the heck does this have to do with calculus? It becomes clear only later.



And so, next we'll next discuss ordered sets.



Yeah, it's not immediately clear how ordered sets help us build up calculus.

And yeah, it is totally different than what we just discussed - irrationality.

But let's just embrace the chaos.



## 12. How do we know where to put numbers on a number line?

It's useful to know that some numbers are bigger than others.



But as of now in our building-up of calculus, we haven't yet introduced any concept of some numbers being bigger than others.

We don't have big numbers and small numbers, right now.

We just have numbers.



We say that 5 is 'bigger' than 3 only because people from a long time ago said so. That's how they decided to order 5 and 3.

And so, to know which numbers are bigger...  
we need to define the order they go in.



Let's say that a 'proper' way to order numbers is any way such that if you compare two of them, one is equal to, bigger than, or lesser than the other. More precisely, when comparing numbers ' $a$ ' and ' $b$ ' in an ordered set, exactly one of the following must hold:



\*For example, a 'proper' ordering for the set  $\{3, 5\}$  can be that 3 equals 5 (it's not the ordering that most mathematicians go by, but it does satisfy our definition of a proper ordering).

But an order can't be that 3 is both greater than and less than 5. And it can't be that 3 has no relation to 4. If these last two were orderings, it would be impossible to answer the question: which is bigger: 3 or 4?

First, what exactly do I mean by *order*? I mean that given two numbers, the *order* should tell us which number is bigger.  
For example, we can ask...

How should we order the set  $\{1, 2, 3\}$ ?

We could order it like:

$$1 < 2 < 3.$$



Or, we could order it like:

$$1 > 2 > 3.$$



(This is a 'proper' ordering, according to our definition. It is just not the one that most mathematicians use.)

Now for a tougher ordering problem.  
How should we order the rationals?

The ordering can't just say that the bigger number is bigger.

The purpose of ordering is to define the word 'bigger.'

Instead, let's say our ordering is that the rational  $s$  is bigger than  $r$  when  $s - r$  is positive\*.



\*Sounds good right? But we haven't defined positive. Now, in your head you might be thinking that it's obvious how to define positive: a number is positive when it's bigger than zero. But then, our definition of positive relies on bigger (positive means a number is bigger than zero) and our definition of bigger relies on positive (a number is bigger than another when  $s - r$  is positive). So for now, I'll trust we all know the difference between a positive and negative number. But we will formally define what positive and negative mean later.

Ok. Now we know how to order sets. So, we can figure out which numbers are bigger than others.

13. Does every set have  
a biggest number?

Enough of  
smaller and bigger.



Let's talk  
smallest and biggest.



Consider the set containing numbers (let's say they're all positive integers) representing how cool every person's mom is.



The bigger the number,  
the cooler the mom.

That set does have a biggest number -  
the coolness of my mom.

59371143...

But do you think every set of items has a biggest number?

Sure, sets with finite elements (like our mom-coolness set) contains a biggest number.

But imagine a world with infinite moms, each mom cooler than the last. (That is, their coolnesses make up the set  $\{1, 2, 3, 4, 5, \dots\}$  and so on).



That set (the set of positive integers) has no biggest number.

If I asked you to prove why, what would you say? Well, whenever a mom of coolness  $x$  is in the set, so too is the mom with coolness  $x+1$ , so  $x$  cannot be the biggest number in the set. That holds for every single  $x$  in the set.





So, some sets don't have  
a biggest number.

Big whoop. If numbers can  
get as big as you want, then  
sure there is no biggest  
number in the set.



So, let's make this  
more interesting...

But first, a note.

14. Isn't infinity the biggest element in the set  $\{1, 2, 3 \dots\}$ ?

Why isn't infinity the biggest element in the set of positive integers  $\{1, 2, 3 \dots\}$ ?



Well, infinity isn't an integer. So it can't be in the set of positive integers at all.



Yes, numbers in the set get as high as you'd like them to get. But they don't end - not at infinity or anywhere else.



## 15. How do you prove infinity isn't an integer?

Sure, infinity seems like an integer at first...

...until we take a closer look under its disguise.



So let's prove that infinity isn't an integer.



First, let's officially define infinity. One agreeable definition for infinity is that it's bigger than any other integer.



So let's suppose (for the purposes of contradiction) that infinity is an integer.

Then, what is infinity + 1?



Note that infinity + 1 must be an integer, since an integer plus an integer should equal an integer.

If you say  
infinity + 1 = infinity...

...then, when you subtract infinity from both sides, you get  $0=1$ . That can't be right.



If you say  
infinity + 1 > infinity....

...then that contradicts the fact that infinity was supposed to be larger than every other integer.



If you say  
infinity + 1 < infinity...

...then that's just silly. Adding one to a number should make it bigger, not smaller.



So infinity + 1 can't be equal to infinity, greater than infinity, or less than infinity. So infinity must not be an integer at all.



## 16. Does every upper bounded set have a biggest element?

Oops. I got a little sidetracked by these infinity shenanigans.  
Let's rewind.

I was saying that sometimes sets are unbounded (like  $\{1, 2, 3, \dots\}$ ), and therefore have no biggest element.



Well, fine. But what if the set is bounded?



That is, what if every number in the set is less than or equal to some upper bound we call "alpha"?

Then the set has to have a biggest element, right?



Oddly enough, no. You would think that because the elements of the set are upper bounded, there has to be a biggest element. But consider the set  $\{0.9, 0.99, 0.999, \dots\}$ .



The elements in the set are all upper-bounded by 1, but there is no biggest element.

And I'll even prove that the set has no biggest element.

Tell me a number in the set that you claim is the biggest...



...and I'll add a 9 to the end, and come up with an even bigger number in the set.



Therefore, any number that you claimed is the biggest cannot be. So we can agree that the set has no biggest number.



17. What's the most important upper bound of a set?

What's the upper bound of our set  $\{0.9, 0.99, 0.999\dots\}$ ?



I think you'd say the upper bound is 1.



But 1 isn't the only upper bound of that set.

1...

5...

... and 1 billion are all upper bounds of that set.



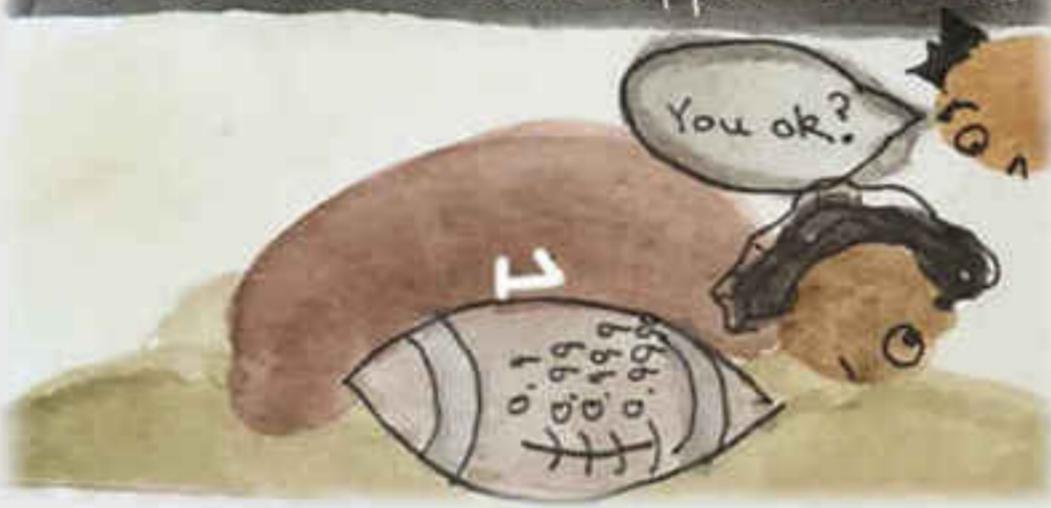
They are upper bounds because they are all numbers bigger than every number in the set.



But some of those upper bounds seem sort of useless and redundant, like 1 billion.

So let's define the supremum to be the *least* upper bound.

(Which in a sense, is the most important upper bound.)



For example, the supremum of the set  $\{0.9, 0.99, 0.999, \dots\}$  is 1. Although we can't know for sure, because we haven't proved it, yet. Just take my word for it, for now.



## 18. Does every upper-bounded set contain its supremum?

Does every upper-bounded set have a supremum?

Yes.



Does every upper-bounded set contain its supremum?

No.

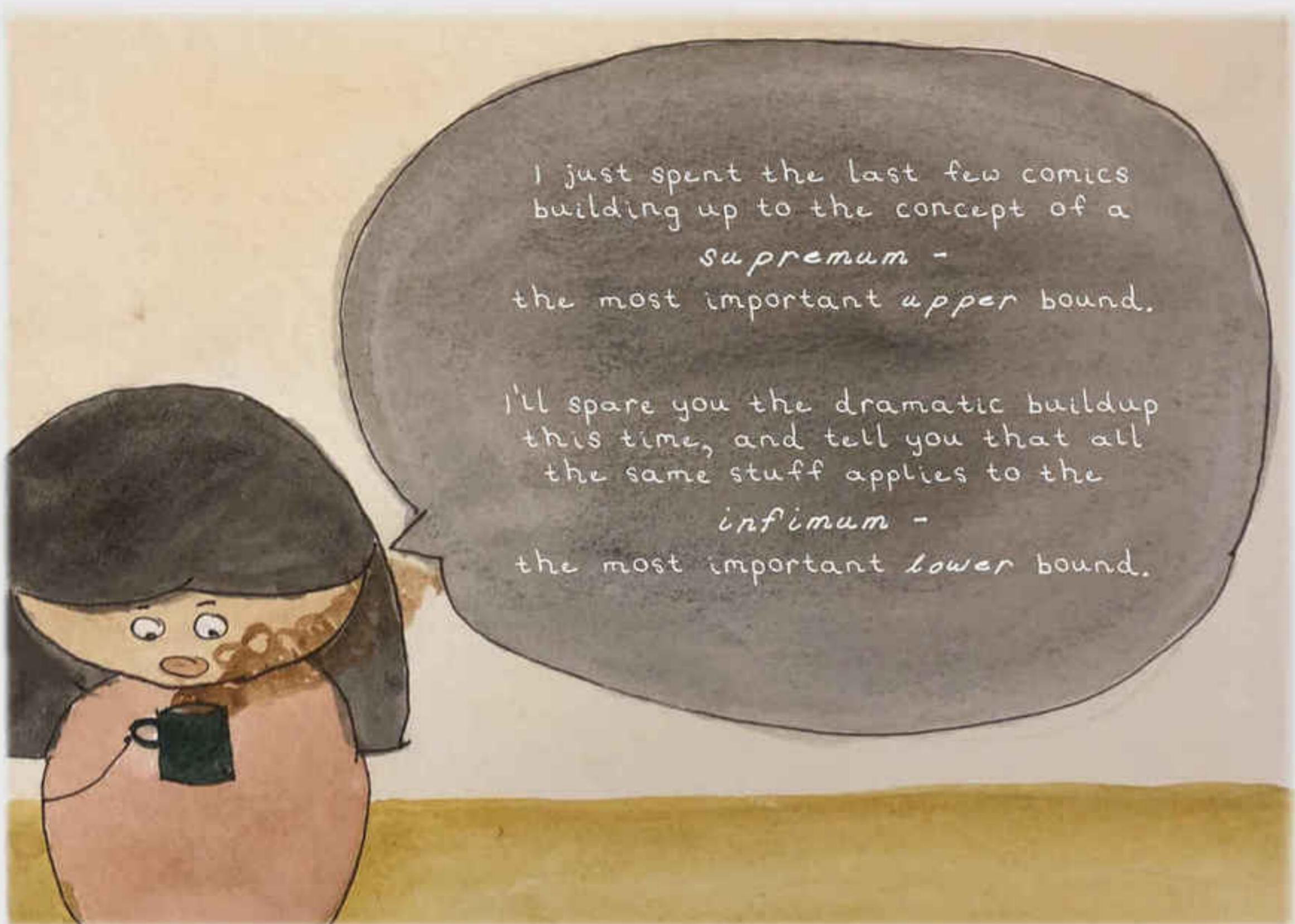


For example, the set  $\{0.9, 0.99, 0.999, \dots\}$  has a supremum of 1, but never actually reaches 1 in the set, and therefore doesn't contain 1.\*



\*You might say the set does contain 1, since the set goes on forever, and 0.999 repeating forever equals 1. But while the numbers in the set get arbitrarily close to "0.999 repeating forever" (aka 1), they never do reach it.

19. What's the most important lower bound of a set?



## 20. Does every lower-bounded set contain its infimum?

Ok, so maybe I won't completely spare you.

Does every lower-bounded set have an infimum?

Yes.

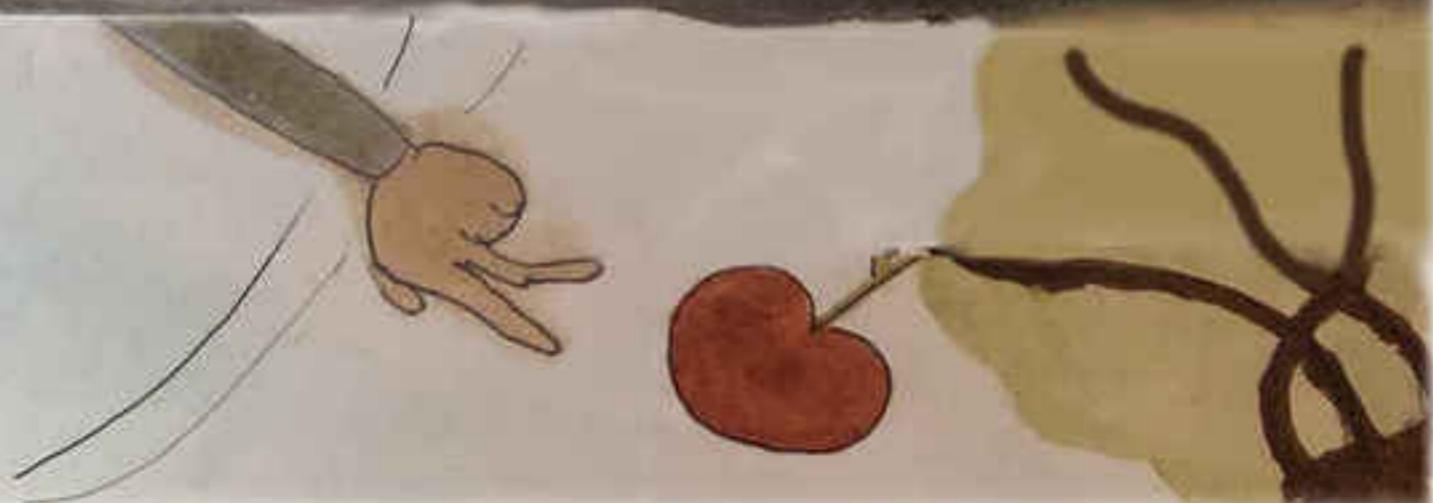


Does every lower-bounded set contain its infimum?

No.



For example, the set  $\{0.1, 0.01, 0.001, \dots\}$  has an infimum of 0, but doesn't contain 0 in the set.



Alright. That's all the repetitiveness for now, I promise.

## 21. What's the point of formal definitions?

We previously said that:

The supremum is the smallest upper bound.

and

The infimum is the biggest lower bound.



But more formally we can say that:

An upper bound ' $u$ ' of set ' $S$ ' is called a supremum of  $S$  if for all upper bounds  $b$  of  $S$ ,  $u \leq b$ .

and

A lower bound ' $l$ ' of set ' $S$ ' is called an infimum of  $S$  if for all lower bounds  $b$  of  $S$ ,  $l \geq b$ .



No kidding? We can define a supremum using a bunch of math words and symbols? Well, isn't that wonderful.

So what's the point of this formal definition?

Sometimes, there is no point (just math people using the weird language they've always used).

But sometimes, formal definitions come in handy during proofs.

For example, say I wanted to convince someone that the infimum of  $\{1, .01, .001, \dots\}$  is 0.

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What if that someone insisted the infimum is 0.0036?

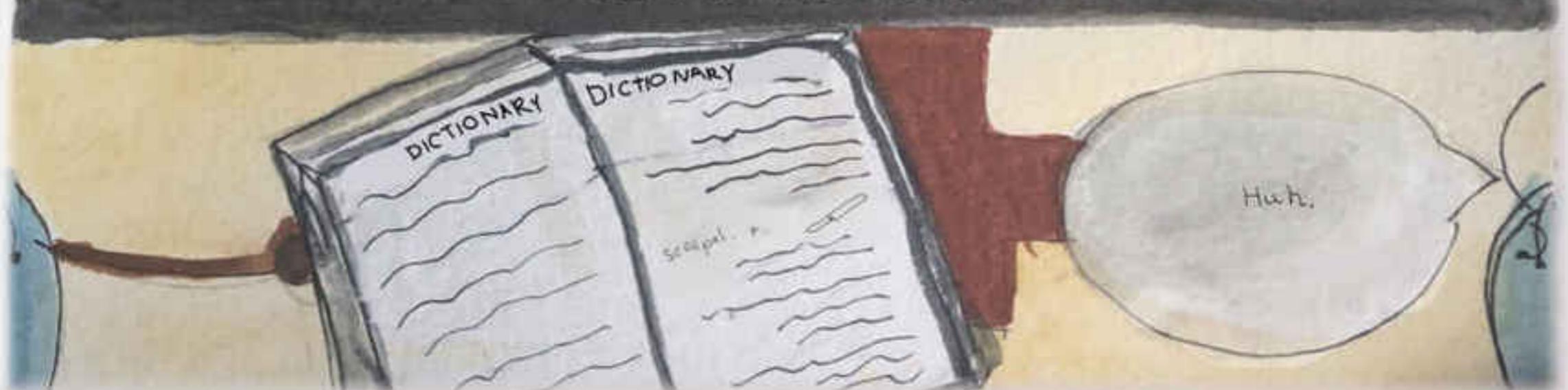
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I'd never convince them if my only reason for why 0 is the greatest lower bound is: "Uhh...it's definitely a lower bound...and it's definitely the biggest one..believe me."



In order to actually convince someone, I'd have to go through a proof by contradiction, and you'd need to use the formal definition. Let's do that next.



## 22. How can we prove something is an infimum?

Let's prove 0 is the greatest lower bound of the set  $\{0.1, 0.01, 0.001\ldots\}$ .

First, let's prove that 0 is a lower bound.

Well, yes, 0 is smaller than any element of the set.



Finally, let's prove that 0 is the greatest lower bound.

Let's do so by contradiction.

Let's assume that there was a number strictly bigger than 0 (call it alpha) that was the actual greatest lower bound.



If we replace the two most significant digits of alpha with "01" and truncate everything after that...then we get an element in the set that is definitely smaller than alpha.

$\alpha = 0.01$

So alpha can't be a lower bound of the set (because we found something in the set smaller than it)..

Thus, 0 must be the greatest lower bound.



23. What's the point of formal proofs when we have intuition?

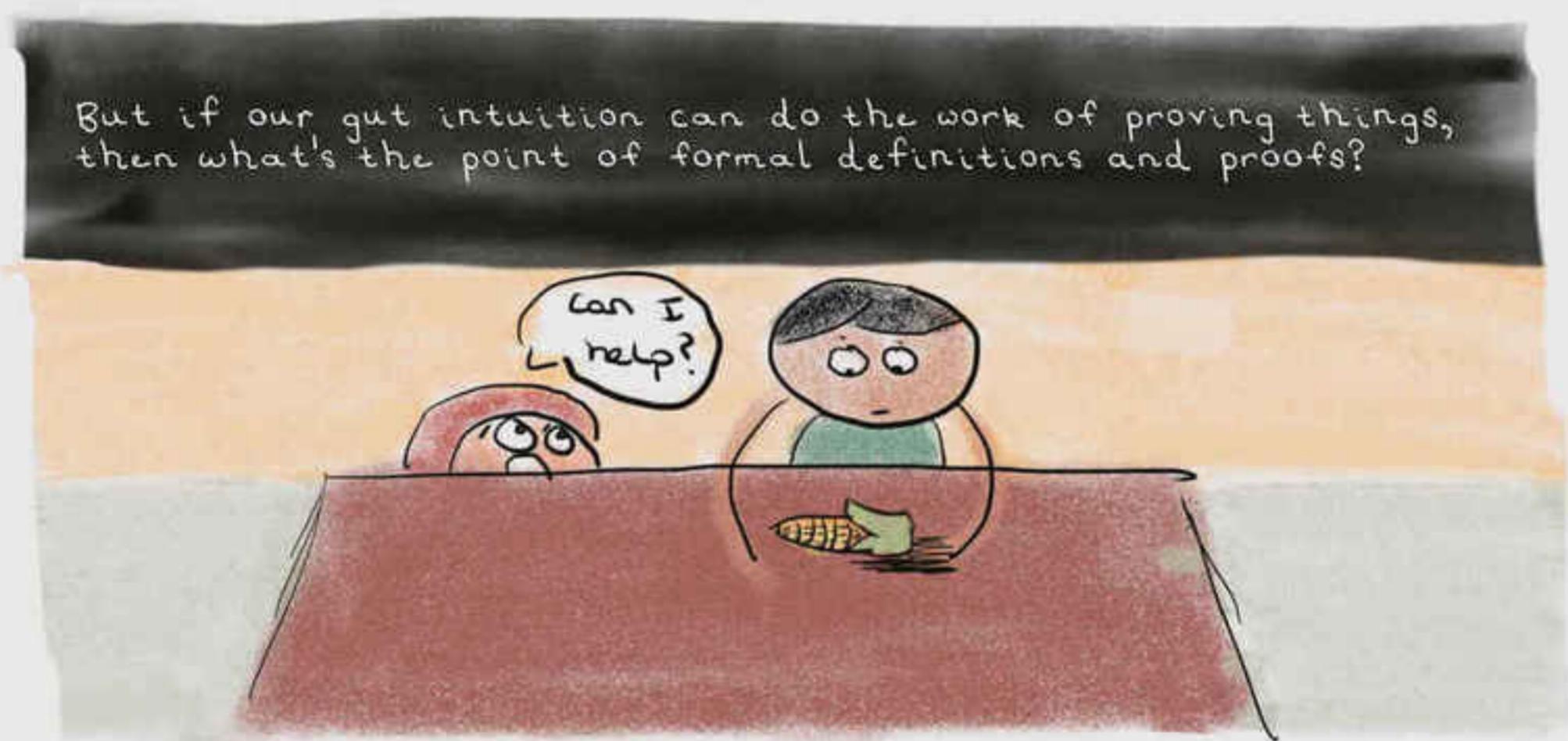
Our intuition would tell us that the greatest lower bound of  $\{0.1, 0.01, 0.001, \dots\}$  is 0.



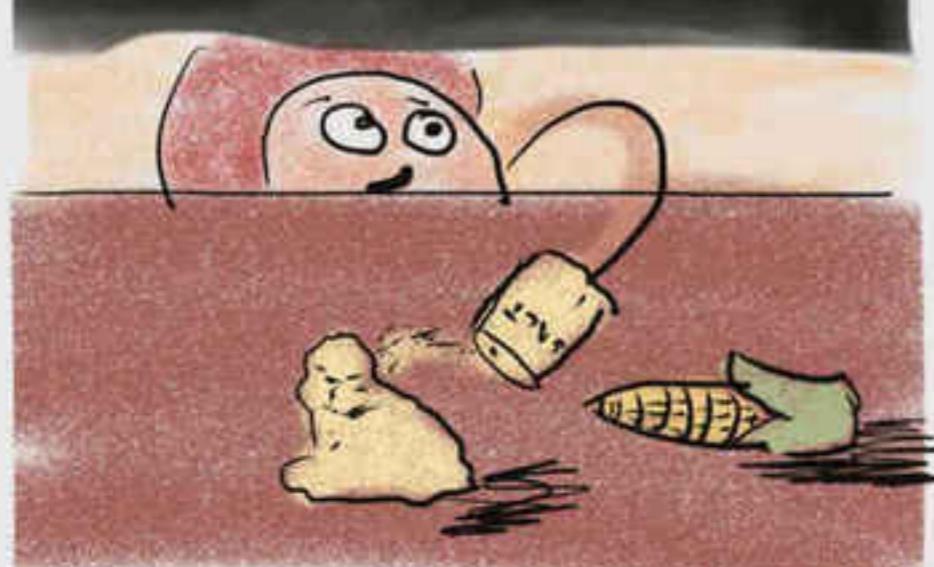
And it would be right.



But if our gut intuition can do the work of proving things, then what's the point of formal definitions and proofs?



Well, it might be useful if mathematicians we worked with came up with the same results we did.



Despite different intuitions, we want consistent results.



24. Is  $\inf(A) = -\sup(-A)$ ?

Now, for an even more challenging proof.

Say set  $A$  is any set of real numbers with a lower bound.  
Say set  $-A$  is the set of the negatives of those numbers.  
Let's prove  $\inf(A) = -\sup(-A)$ .



First, let's consider an example.

Say  $A = \{1, 2, 3, \dots\}$ .  
Then  $\inf(A) = 1$ .

Now let's flip it.  
Then  $-A = \{-1, -2, -3, \dots\}$ .  
And so  $-\sup(-A) = -(-1) = 1$ .

So the equation checks out, at least for this particular set  $A$ .

Now, let's prove the equation holds for every set ever.  
We'll do so in two steps...

In this step, we will prove  $\inf(A) \leq -\sup(-A)$ .

In the next step, we will prove  $\inf(A) \geq -\sup(-A)$ .

If both are true, that must mean  $\inf(A) = -\sup(-A)$ .

Step 1: Prove that  $\inf(A) \leq -\sup(-A)$ .

We know  $A$  has an **infimum**, since it's bounded below.

And by the definition of infimums:

$\inf(A) \leq a$  for any  $a$  in  $A$ .



Let's flip the inequality, multiplying both sides by  $-1$ :

So  $-\inf(A) \geq -a$ .

But guess what? The element " $-a$ " could be any element in  $-A$ .

So  $-\inf(A)$  upper bounds  $-A$ .



So since  $-\inf(A)$  is an upper bound of  $-A$ ...

it has to be bigger than the least upper bound of  $-A$ :

$-\inf(A) \geq \sup(-A)$



So  $\inf(A) \leq -\sup(-A)$ .

Step 2: Prove that  $\inf(A) \geq -\sup(-A)$ .

We know  $-A$  has a supremum, since it's bounded above.

And by the definition of supremums:

$$\sup(-A) \geq -a \text{ for any } -a \text{ in } -A.$$

Let's flip the inequality, multiplying both sides by  $-1$ :

$$-\sup(-A) \leq a.$$

But guess what? The element "a" could be any element in A.

So  $-\sup(-A)$  lower bounds A.

So since  $-\sup(-A)$  is a lower bound of A...  
it has to be smaller than the greatest lower bound of A:

$$-\sup(-A) \leq \inf(A).$$

$$\text{So } \inf(A) \geq -\sup(-A).$$

In step 1, we proved  $\inf(A) \leq -\sup(-A)$ .

In step 2, we proved  $\inf(A) \geq -\sup(-A)$ .

Since both are true, that must mean  $\inf(A) = -\sup(-A)$ .

## Bibliography

(By Chapter)

1 I owe my comparison of math and philosophy to Amanda Gefter, who once told me that "physics is like philosophy, but with answers."

3 For more on the origins of abstract mathematics, see the book *Duel at Dawn: Heroes, Martyrs, and the Rise of Modern Mathematics* by Amir Alexander.

5 Lembeck's book on the Vietnam War spitting story is called *The Spitting Image: Myth, Memory and the Legacy of Vietnam*. I owe this reference to Thalia Ruiz at MIT.

7 For more on the Greek mathematician exiled for asserting that some numbers are irrational, see the fascinating Ted Ed video: "A Brief History of Banned Numbers."

8 The proof that the square root of two is irrational was adapted from page 2 of *Principles of Mathematical Analysis* by Walter Rudin.

9 Thank you to Amanda Sobel at MIT for the wonderful comparison of analysis to philosophy.

11 For more on the contrast between the top-down historical development of analysis and the bottom-up teaching of analysis, see page 37 of the book *How to Think about Analysis* by Lara Alcock.