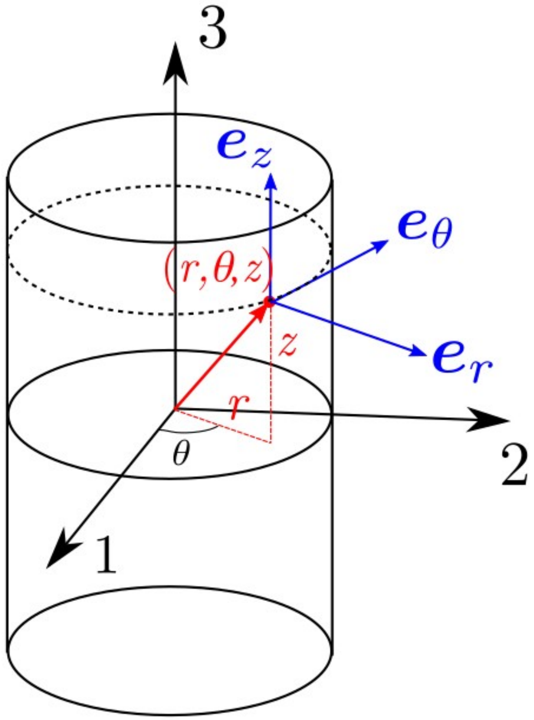
Cylindrical Coordinate System

Selection of appropriate coordinate system can facilitate easier solution to many problems. We will rewrite some important relation in Cylindrical coordinate system.

Position vector of a point, its coordinates, and basis vectors in Cartesian coordinate system and Cylindrical coordinate system are related in the following manner.



$$x_1 = r\cos\theta$$
$$x_2 = r\sin\theta$$
$$x_3 = z$$
$$r = \sqrt{x_1^2 + x_2^2}$$
$$\theta = \tan^{-1}(x_2/x_1)$$

$$\boldsymbol{e}_r = \boldsymbol{e}_1\cos\theta + \boldsymbol{e}_2\sin\theta$$
$$\boldsymbol{e}_\theta = -\boldsymbol{e}_1\sin\theta + \boldsymbol{e}_2\cos\theta$$
$$\boldsymbol{e}_z = \boldsymbol{e}_3$$

$$\boldsymbol{r} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 = r\boldsymbol{e}_r + z\boldsymbol{e}_z$$

It can be observed that the basis vector $\boldsymbol{e}_r$ lies along the radial direction, and $\boldsymbol{e}_\theta$ is always perpendicular to $\boldsymbol{e}_r$. Thus $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ change with change in θ. Thus, it can be shown that,

$$\frac{\partial \boldsymbol{e}_r}{\partial \theta} = \boldsymbol{e}_\theta, \quad \frac{\partial \boldsymbol{e}_\theta}{\partial \theta} = -\boldsymbol{e}_r.$$

Derivative with respect to r and θ can be changed to the derivative w.r.t. x_1 and x_2 using chain rule of differentiation as,

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial r} + \frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial r} = \frac{\partial}{\partial x_1}\cos\theta + \frac{\partial}{\partial x_2}\sin\theta,$$
$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial \theta} + \frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial \theta} = -r\sin\theta\frac{\partial}{\partial x_1} + r\cos\theta\frac{\partial}{\partial x_2},$$

or vice-versa as,

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial}{\partial \theta}\sin\theta,$$
$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial}{\partial \theta}\cos\theta.$$

Using the definition of Gradient operator in the Cylindrical coordinates can be written as,

$$\boldsymbol{\nabla} = \frac{\partial}{\partial x_1}\boldsymbol{e}_1 + \frac{\partial}{\partial x_2}\boldsymbol{e}_2 + \frac{\partial}{\partial x_3}\boldsymbol{e}_3,$$

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial}{\partial \theta}\sin\theta\right)\boldsymbol{e}_1 + \left(\frac{\partial}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial}{\partial \theta}\cos\theta\right)\boldsymbol{e}_2 + \frac{\partial}{\partial z}\boldsymbol{e}_z,$$

$$\boldsymbol{\nabla} = \frac{\partial}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\boldsymbol{e}_\theta + \frac{\partial}{\partial z}\boldsymbol{e}_z.$$

$$\boldsymbol{\nabla}(\bullet) = \frac{\partial(\bullet)}{\partial x_i}\boldsymbol{e}_i = \frac{\partial(\bullet)}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial(\bullet)}{\partial \theta}\boldsymbol{e}_\theta + \frac{\partial(\bullet)}{\partial z}\boldsymbol{e}_z.$$

A first order tensor $\boldsymbol{u}$ and a second order tensor $\boldsymbol{\sigma}$ in Cylindrical coordinate system is given as,

$$\boldsymbol{u} = u_r\boldsymbol{e}_r + u_\theta\boldsymbol{e}_\theta + u_z\boldsymbol{e}_z$$

$$\begin{aligned}\boldsymbol{\sigma} =& \sigma_{rr}\boldsymbol{e}_r\boldsymbol{e}_r + \sigma_{r\theta}\boldsymbol{e}_r\boldsymbol{e}_\theta + \sigma_{rz}\boldsymbol{e}_r\boldsymbol{e}_z + \sigma_{\theta r}\boldsymbol{e}_\theta\boldsymbol{e}_r + \sigma_{\theta\theta}\boldsymbol{e}_\theta\boldsymbol{e}_\theta + \sigma_{\theta z}\boldsymbol{e}_\theta\boldsymbol{e}_z + \\ & \sigma_{zr}\boldsymbol{e}_z\boldsymbol{e}_r + \sigma_{z\theta}\boldsymbol{e}_z\boldsymbol{e}_\theta + \sigma_{zz}\boldsymbol{e}_z\boldsymbol{e}_z.\end{aligned}$$

Gradient of vector $\boldsymbol{u}$ is now defined as,

$$\boldsymbol{\nabla u} = \frac{\partial \boldsymbol{u}}{\partial x_i}\boldsymbol{e}_i = \frac{\partial \boldsymbol{u}}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial \boldsymbol{u}}{\partial \theta}\boldsymbol{e}_\theta + \frac{\partial \boldsymbol{u}}{\partial z}\boldsymbol{e}_z.$$

$$\begin{aligned}\boldsymbol{\nabla u} =& \frac{\partial u_r}{\partial r}\boldsymbol{e}_r\boldsymbol{e}_r + \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right)\boldsymbol{e}_r\boldsymbol{e}_\theta + \frac{\partial u_r}{\partial z}\boldsymbol{e}_r\boldsymbol{e}_z + \\ & \frac{\partial u_\theta}{\partial r}\boldsymbol{e}_\theta\boldsymbol{e}_r + \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right)\boldsymbol{e}_\theta\boldsymbol{e}_\theta + \frac{\partial u_\theta}{\partial z}\boldsymbol{e}_\theta\boldsymbol{e}_z + \\ & \frac{\partial u_z}{\partial r}\boldsymbol{e}_z\boldsymbol{e}_r + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\boldsymbol{e}_z\boldsymbol{e}_\theta + \frac{\partial u_z}{\partial z}\boldsymbol{e}_z\boldsymbol{e}_z\end{aligned}$$

Using the above expression, small strain components in Cylindrical coordinates are given as,

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right), \quad \varepsilon_z = \frac{\partial u_z}{\partial z}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right), \quad \varepsilon_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right), \quad \varepsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right).$$

Divergence of second order tensor $\boldsymbol{\sigma}$ is

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}=\frac{\partial\boldsymbol{\sigma}}{\partial r}\cdot\boldsymbol{e}_r+\frac{1}{r}\frac{\partial\boldsymbol{\sigma}}{\partial\theta}\cdot\boldsymbol{e}_\theta+\frac{\partial\boldsymbol{\sigma}}{\partial z}\cdot\boldsymbol{e}_z.$$

Let us first derive the second term (i.e. $\partial\boldsymbol{\sigma}/\partial\theta$)

$$\begin{aligned}\frac{\partial\boldsymbol{\sigma}}{\partial\theta}=&\frac{\partial\sigma_{rr}}{\partial\theta}\boldsymbol{e}_r\boldsymbol{e}_r+\sigma_{rr}\frac{\partial\boldsymbol{e}_r}{\partial\theta}\boldsymbol{e}_r+\sigma_{rr}\boldsymbol{e}_r\frac{\partial\boldsymbol{e}_r}{\partial\theta}+\frac{\partial\sigma_{r\theta}}{\partial\theta}\boldsymbol{e}_r\boldsymbol{e}_\theta+\sigma_{r\theta}\frac{\partial\boldsymbol{e}_r}{\partial\theta}\boldsymbol{e}_\theta+\sigma_{r\theta}\boldsymbol{e}_r\frac{\partial\boldsymbol{e}_\theta}{\partial\theta}+\\&\frac{\partial\sigma_{rz}}{\partial\theta}\boldsymbol{e}_r\boldsymbol{e}_z+\sigma_{rz}\frac{\partial\boldsymbol{e}_r}{\partial\theta}\boldsymbol{e}_z+\sigma_{rz}\boldsymbol{e}_r\frac{\partial\boldsymbol{e}_z}{\partial\theta}+\frac{\partial\sigma_{\theta r}}{\partial\theta}\boldsymbol{e}_\theta\boldsymbol{e}_r+\sigma_{\theta r}\frac{\partial\boldsymbol{e}_\theta}{\partial\theta}\boldsymbol{e}_r+\sigma_{\theta r}\boldsymbol{e}_\theta\frac{\partial\boldsymbol{e}_r}{\partial\theta}+\\&\frac{\partial\sigma_{\theta\theta}}{\partial\theta}\boldsymbol{e}_\theta\boldsymbol{e}_\theta+\sigma_{\theta\theta}\frac{\partial\boldsymbol{e}_\theta}{\partial\theta}\boldsymbol{e}_\theta+\sigma_{\theta\theta}\boldsymbol{e}_\theta\frac{\partial\boldsymbol{e}_\theta}{\partial\theta}+\frac{\partial\sigma_{\theta z}}{\partial\theta}\boldsymbol{e}_\theta\boldsymbol{e}_z+\sigma_{\theta z}\frac{\partial\boldsymbol{e}_\theta}{\partial\theta}\boldsymbol{e}_z+\sigma_{\theta z}\boldsymbol{e}_\theta\frac{\partial\boldsymbol{e}_z}{\partial\theta}+\\&\frac{\partial\sigma_{zr}}{\partial\theta}\boldsymbol{e}_z\boldsymbol{e}_r+\sigma_{zr}\frac{\partial\boldsymbol{e}_z}{\partial\theta}\boldsymbol{e}_r+\sigma_{zr}\boldsymbol{e}_z\frac{\partial\boldsymbol{e}_r}{\partial\theta}+\frac{\partial\sigma_{z\theta}}{\partial\theta}\boldsymbol{e}_z\boldsymbol{e}_\theta+\sigma_{z\theta}\frac{\partial\boldsymbol{e}_z}{\partial\theta}\boldsymbol{e}_\theta+\sigma_{z\theta}\boldsymbol{e}_z\frac{\partial\boldsymbol{e}_\theta}{\partial\theta}+\\&\frac{\partial\sigma_{zz}}{\partial\theta}\boldsymbol{e}_z\boldsymbol{e}_z+\sigma_{zz}\frac{\partial\boldsymbol{e}_z}{\partial\theta}\boldsymbol{e}_z+\sigma_{zz}\boldsymbol{e}_z\frac{\partial\boldsymbol{e}_z}{\partial\theta}\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \boldsymbol{\sigma}}{\partial \theta} =& \frac{\partial \sigma_{rr}}{\partial \theta}\boldsymbol{e}_r\boldsymbol{e}_r + \sigma_{rr}\boldsymbol{e}_\theta\boldsymbol{e}_r + \sigma_{rr}\boldsymbol{e}_r\boldsymbol{e}_\theta + \frac{\partial \sigma_{r\theta}}{\partial \theta}\boldsymbol{e}_r\boldsymbol{e}_\theta + \sigma_{r\theta}\boldsymbol{e}_\theta\boldsymbol{e}_\theta - \sigma_{r\theta}\boldsymbol{e}_r\boldsymbol{e}_r + \\
& \frac{\partial \sigma_{rz}}{\partial \theta}\boldsymbol{e}_r\boldsymbol{e}_z + \sigma_{rz}\boldsymbol{e}_\theta\boldsymbol{e}_z + \sigma_{rz}\boldsymbol{e}_r \cdot 0 + \frac{\partial \sigma_{\theta r}}{\partial \theta}\boldsymbol{e}_\theta\boldsymbol{e}_r - \sigma_{\theta r}\boldsymbol{e}_r\boldsymbol{e}_r + \sigma_{\theta r}\boldsymbol{e}_\theta\boldsymbol{e}_\theta + \\
& \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\boldsymbol{e}_\theta\boldsymbol{e}_\theta - \sigma_{\theta\theta}\boldsymbol{e}_r\boldsymbol{e}_\theta - \sigma_{\theta\theta}\boldsymbol{e}_\theta\boldsymbol{e}_r + \frac{\partial \sigma_{\theta z}}{\partial \theta}\boldsymbol{e}_\theta\boldsymbol{e}_z - \sigma_{\theta z}\boldsymbol{e}_r\boldsymbol{e}_z + \sigma_{\theta z}\boldsymbol{e}_\theta \cdot 0 + \\
& \frac{\partial \sigma_{zr}}{\partial \theta}\boldsymbol{e}_z\boldsymbol{e}_r + \sigma_{zr} \cdot 0\boldsymbol{e}_r + \sigma_{zr}\boldsymbol{e}_z\boldsymbol{e}_\theta + \frac{\partial \sigma_{z\theta}}{\partial \theta}\boldsymbol{e}_z\boldsymbol{e}_\theta + \sigma_{z\theta} \cdot 0\boldsymbol{e}_\theta - \sigma_{z\theta}\boldsymbol{e}_z\boldsymbol{e}_r + \\
& \frac{\partial \sigma_{zz}}{\partial \theta}\boldsymbol{e}_z\boldsymbol{e}_z + \sigma_{zz} \cdot 0\boldsymbol{e}_z + \sigma_{zz}\boldsymbol{e}_z \cdot 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \boldsymbol{\sigma}}{\partial \theta} \cdot \boldsymbol{e}_\theta = & \Bigg(\frac{\partial \sigma_{rr}}{\partial \theta}\boldsymbol{e}_r\boldsymbol{e}_r + \sigma_{rr}\boldsymbol{e}_\theta\boldsymbol{e}_r + \sigma_{rr}\boldsymbol{e}_r\boldsymbol{e}_\theta + \frac{\partial \sigma_{r\theta}}{\partial \theta}\boldsymbol{e}_r\boldsymbol{e}_\theta + \sigma_{r\theta}\boldsymbol{e}_\theta\boldsymbol{e}_\theta - \sigma_{r\theta}\boldsymbol{e}_r\boldsymbol{e}_r + \\
& \frac{\partial \sigma_{rz}}{\partial \theta}\boldsymbol{e}_r\boldsymbol{e}_z + \sigma_{rz}\boldsymbol{e}_\theta\boldsymbol{e}_z + \frac{\partial \sigma_{\theta r}}{\partial \theta}\boldsymbol{e}_\theta\boldsymbol{e}_r - \sigma_{\theta r}\boldsymbol{e}_r\boldsymbol{e}_r + \sigma_{\theta r}\boldsymbol{e}_\theta\boldsymbol{e}_\theta + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\boldsymbol{e}_\theta\boldsymbol{e}_\theta - \\
& \sigma_{\theta\theta}\boldsymbol{e}_r\boldsymbol{e}_\theta - \sigma_{\theta\theta}\boldsymbol{e}_\theta\boldsymbol{e}_r + \frac{\partial \sigma_{\theta z}}{\partial \theta}\boldsymbol{e}_\theta\boldsymbol{e}_z - \sigma_{\theta z}\boldsymbol{e}_r\boldsymbol{e}_z + \frac{\partial \sigma_{zr}}{\partial \theta}\boldsymbol{e}_z\boldsymbol{e}_r + \sigma_{zr}\boldsymbol{e}_z\boldsymbol{e}_\theta + \\
& \frac{\partial \sigma_{z\theta}}{\partial \theta}\boldsymbol{e}_z\boldsymbol{e}_\theta\boldsymbol{e}_\theta - \sigma_{z\theta}\boldsymbol{e}_z\boldsymbol{e}_r + \frac{\partial \sigma_{zz}}{\partial \theta}\boldsymbol{e}_z\boldsymbol{e}_z \Bigg) \cdot \boldsymbol{e}_\theta
\end{aligned}
$$

$$\frac{\partial \boldsymbol{\sigma}}{\partial \theta} \cdot \boldsymbol{e}_\theta = \left(\sigma_{rr} - \sigma_{\theta\theta} + \frac{\partial \sigma_{r\theta}}{\partial \theta}\right)\boldsymbol{e}_r + \left(\sigma_{r\theta} + \sigma_{\theta r} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\right)\boldsymbol{e}_\theta + \left(\sigma_{zr} + \frac{\partial \sigma_{z\theta}}{\partial \theta}\right)\boldsymbol{e}_z$$

Other two terms are given as,

$$\frac{\partial \boldsymbol{\sigma}}{\partial r} \cdot \boldsymbol{e}_r = \frac{\partial \sigma_{rr}}{\partial r}\boldsymbol{e}_r + \frac{\partial \sigma_{\theta r}}{\partial r}\boldsymbol{e}_\theta + \frac{\partial \sigma_{zr}}{\partial r}\boldsymbol{e}_z$$

$$\frac{\partial \boldsymbol{\sigma}}{\partial z} \cdot \boldsymbol{e}_z = \frac{\partial \sigma_{rz}}{\partial z}\boldsymbol{e}_r + \frac{\partial \sigma_{\theta z}}{\partial z}\boldsymbol{e}_\theta + \frac{\partial \sigma_{zz}}{\partial z}\boldsymbol{e}_z$$

Substitute back into the original expression,

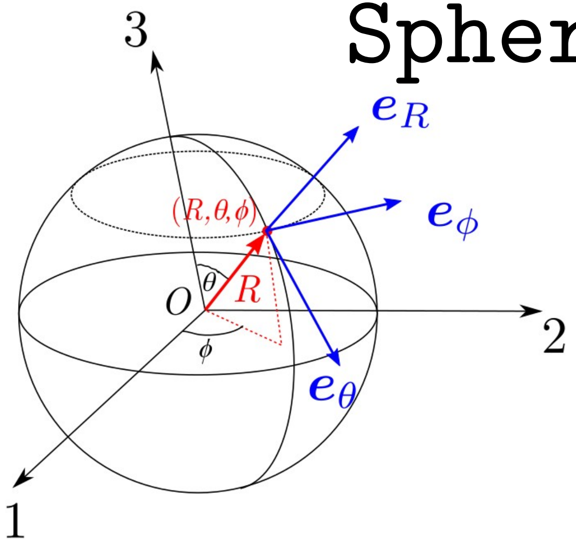
$$\begin{aligned}\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = &\left(\frac{\partial \sigma_{rr}}{\partial r}\boldsymbol{e}_r + \frac{\partial \sigma_{\theta r}}{\partial r}\boldsymbol{e}_\theta + \frac{\partial \sigma_{zr}}{\partial r}\boldsymbol{e}_z\right) + \\ &\frac{1}{r}\left(\sigma_{rr} - \sigma_{\theta\theta} + \frac{\partial \sigma_{r\theta}}{\partial \theta}\right)\boldsymbol{e}_r + \frac{1}{r}\left(\sigma_{r\theta} + \sigma_{\theta r} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\right)\boldsymbol{e}_\theta + \frac{1}{r}\left(\sigma_{zr} + \frac{\partial \sigma_{z\theta}}{\partial \theta}\right)\boldsymbol{e}_z + \\ &\left(\frac{\partial \sigma_{rz}}{\partial z}\boldsymbol{e}_r + \frac{\partial \sigma_{\theta z}}{\partial z}\boldsymbol{e}_\theta + \frac{\partial \sigma_{zz}}{\partial z}\boldsymbol{e}_z\right)\end{aligned}$$

Rearranging the terms will give the following expression for divergence of $\boldsymbol{\sigma}$,

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = & \left[\frac{\partial\sigma_{rr}}{\partial r}+\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{\partial\sigma_{rz}}{\partial z}+\frac{1}{r}\left(\sigma_{rr}-\sigma_{\theta\theta}\right)\right]\boldsymbol{e}_r+\\
& \left[\frac{\partial\sigma_{\theta r}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{\partial\sigma_{\theta z}}{\partial z}+\frac{1}{r}\left(\sigma_{r\theta}+\sigma_{\theta r}\right)\right]\boldsymbol{e}_\theta+\\
& \left[\frac{\partial\sigma_{zr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{z\theta}}{\partial\theta}+\frac{\partial\sigma_{zz}}{\partial z}+\frac{1}{r}\sigma_{zr}\right]\boldsymbol{e}_z.
\end{aligned}
$$

Above relation will be useful in writing the Cauchy's equations of motion in Cylindrical coordinate system.

Spherical coordinate system



Spherical coordinate variables (R, θ, ϕ) are related to Cartesian coordinate variables (x_1, x_2, x_3) as,

$$R = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad \theta = \tan^{-1} \frac{\sqrt{x_1^2 + x_2^2}}{x_3}, \quad \phi = \tan^{-1} \frac{x_2}{x_1},$$

and

$$x_1 = R \sin\theta \cos\phi, \quad x_2 = R \sin\theta \sin\phi, \quad x_3 = R\cos\theta.$$

An orthonormal basis can be defined in a spherical coordinate system with base vectors at a point in the space that are tangent to the curvilinear coordinate axes (i.e., lines along which two of the three coordinate variables are constant). Thus, base vector $\boldsymbol{e}_R$ is a unit vector that points in the direction of increasing R when θ and ϕ are held constant; similarly for $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\phi$. The base vectors are related to the base vectors in Cartesian coordinate system

$$\boldsymbol{e}_R = \sin\theta\cos\phi \boldsymbol{e}_1 + \sin\theta\sin\phi \boldsymbol{e}_2 + \cos\theta \boldsymbol{e}_3,$$

$$\boldsymbol{e}_\theta = \cos\theta\cos\phi \boldsymbol{e}_1 + \cos\theta\sin\phi \boldsymbol{e}_2 - \sin\theta \boldsymbol{e}_3,$$

$$\boldsymbol{e}_\phi = -\sin\phi \boldsymbol{e}_1 + \cos\phi \boldsymbol{e}_2$$

It can be observed that the basis vector changes with the change in angle θ and ϕ. Thus, similar to Cylindrical coordinate system, differentiation of basis vector w.r.t θ and ϕ will be considered. It can be shown that,

$$\frac{\partial \boldsymbol{e}_R}{\partial \theta} = \boldsymbol{e}_\theta, \quad \frac{\partial \boldsymbol{e}_\theta}{\partial \theta} = -\boldsymbol{e}_R,$$
$$\frac{\partial \boldsymbol{e}_R}{\partial \phi} = \sin\theta \boldsymbol{e}_\phi, \quad \frac{\partial \boldsymbol{e}_\theta}{\partial \phi} = -\cos\theta \boldsymbol{e}_\phi, \quad \frac{\partial \boldsymbol{e}_\phi}{\partial \phi} = -\sin\theta \boldsymbol{e}_R - \cos\theta \boldsymbol{e}_\theta.$$

It can be shown that small strain components in Spherical coordinates are,

$$\varepsilon_{RR} = \frac{\partial u_R}{\partial R}, \quad \varepsilon_{\theta\theta} = \frac{1}{R}\left(u_R + \frac{\partial u_\theta}{\partial \theta}\right), \quad \varepsilon_{\phi\phi} = \frac{1}{R\sin\theta}\left(\frac{\partial u_\phi}{\partial \phi} + u_R \sin\theta + u_\theta \cos\theta\right)$$
$$\varepsilon_{R\theta} = \frac{1}{2}\left(\frac{1}{R}\frac{\partial u_R}{\partial \theta} + \frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R}\right), \quad \varepsilon_{\theta\phi} = \frac{1}{2R}\left(\frac{1}{\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{\partial u_\phi}{\partial \theta} - u_\phi \cot\theta\right),$$
$$\varepsilon_{\phi r} = \frac{1}{2}\left(\frac{1}{R\sin\theta}\frac{\partial u_R}{\partial \phi} + \frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R}\right).$$

One can also derive the following Cauchy's equation of motion in Spherical coordinate system.

$$
\begin{aligned}
&\frac{\partial \sigma_{RR}}{\partial R} + \frac{1}{R}\frac{\partial \sigma_{R\theta}}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial \sigma_{R\phi}}{\partial \phi} + \\
&\qquad \frac{1}{R}\left(2\sigma_{RR} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{R\theta}\cot\theta\right) + b_r = \rho a_r \\
&\frac{\partial \sigma_{\theta R}}{\partial R} + \frac{1}{R}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial \sigma_{\theta\phi}}{\partial \phi} + \\
&\qquad \frac{1}{R}\left[(\sigma_{RR} - \sigma_{\phi\phi})\cot\theta + 2\sigma_{\theta R} + \sigma_{R\theta}\right] + b_\theta = \rho a_\theta \\
&\frac{\partial \sigma_{\phi R}}{\partial R} + \frac{1}{R}\frac{\partial \sigma_{\phi\theta}}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \\
&\qquad \frac{1}{R}\left[(\sigma_{\theta\phi} + \sigma_{\phi\theta})\cot\theta + 2\sigma_{\phi R} + \sigma_{R\phi}\right] + b_\phi = \rho a_\phi
\end{aligned}
$$

Cauchy-elastic materials

Cauchy-elastic material

- A material is called Cauchy-elastic or elastic if
  - the stress field at time t depends only on the state of deformation (and the state of temperature) at this time t and not on the deformation history (and temperature history).
- Hence, the stress field of a Cauchy-elastic material is independent of the deformation path (independent of the time).
- A constitutive equation (or equation of state) represents the intrinsic physical properties of a continuum body. It determines generally the state of stress at any point of that body to any arbitrary motion at time t.
- A constitutive equation is either regarded as mathematically generalized (axiomatic) or is based upon experimental data (empirical).
- The constitutive equation of an isothermal, homogeneous elastic body relates the Cauchy stress tensor $\boldsymbol{\sigma}(\boldsymbol{x},t)$ at each place $\boldsymbol{x}\,(\boldsymbol{X},\ t)$ with the deformation gradient $\boldsymbol{F}(\boldsymbol{X},\ t)$. A general form of the constitutive equation is

$$\boldsymbol{\sigma} = \mathfrak{g}(\boldsymbol{F}),$$

where $\mathfrak{g}$ is referred to as the response function associated with the Cauchy stress tensor.

Let us check the objectivity of the constitutive relation for elastic material. For the purpose consider a superimposed rigid body motion $\boldsymbol{x}^*$, i.e.,

$$\boldsymbol{\sigma}^* = \mathfrak{g}(\boldsymbol{F}^*).$$

Note that function $\mathfrak{g}$ remain same because elastic material remain same for deformed as well as superimposed rotated configuration.

Thus,

$$\boldsymbol{\sigma}^* = \mathfrak{g}(\boldsymbol{F}^*) \Rightarrow \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T = \mathfrak{g}(\boldsymbol{Q}\boldsymbol{F})$$

$$\Rightarrow \boxed{\boldsymbol{Q}\mathfrak{g}(\boldsymbol{F})\boldsymbol{Q}^T = \mathfrak{g}(\boldsymbol{Q}\boldsymbol{F}).} \qquad \cdots\cdots\cdots(1)$$

Above equation is a restriction on $\mathfrak{g}$ for every nonsingular $\boldsymbol{F}$ and orthogonal $\boldsymbol{Q}$ for constitutive equation to be independent of the observer.

Using the right polar decomposition $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}$ on the right-hand side of (1),

$$\boldsymbol{Q}\mathfrak{g}(\boldsymbol{F})\boldsymbol{Q}^T = \mathfrak{g}(\boldsymbol{Q}\boldsymbol{R}\boldsymbol{U}).$$

Since (1) holds for all proper orthogonal tensors $\boldsymbol{Q}$, it also holds for the special choice $\boldsymbol{Q} = \boldsymbol{R}^T$. Hence, using the orthogonality condition $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}$, we obtain a corresponding reduced form of (1) as,

$$\boldsymbol{R}^T\mathfrak{g}(\boldsymbol{F})\boldsymbol{R} = \mathfrak{g}(\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{U}) \Rightarrow \mathfrak{g}(\boldsymbol{F}) = \boldsymbol{R}\mathfrak{g}(\boldsymbol{U})\boldsymbol{R}^T,$$

For all $\boldsymbol{F}$ and $\boldsymbol{R}$. Hence the associated constitutive relation become

$$\boxed{\boldsymbol{\sigma} = \boldsymbol{R}\mathfrak{g}(\boldsymbol{U})\boldsymbol{R}^T.} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(2)$$

It shows that the properties of an elastic material are independent of the rotational part of $\boldsymbol{F}$.

Exercise: Check that the constitutive equation (2) is compatible with the principle of material objectivity.
Hint: First show that $\boldsymbol{U}^* = \boldsymbol{U}$.