ME232: Dynamics

Anshul Faye
afaye@iitbhilai.ac.in
Room # 106

The shape of the path followed by m may be obtained by solving the first of (55), and (56) and eliminating the time t.

To that end substitute $r = 1/u$. Thus, $\dot{r} = -(1/u^2)\dot{u}$, which from (56) becomes $\dot{r} = -h(\dot{u}/\dot{\theta})$ or $\dot{r} = -h(du/d\theta)$. The second time derivative is $\ddot{r} = -h(d^2u/d\theta^2)\dot{\theta}$, which by combining with (56), become $\ddot{r} = -h^2u^2(d^2u/d\theta^2)$.

Substitution into the first of (55) now gives,

$$-Gm_0u^2 = -h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}h^2u^4, \quad \text{or} \quad \frac{d^2u}{d\theta^2} + u = \frac{Gm_0}{h^2}, \qquad \cdots\cdots\cdots(57)$$

which is a nonhomogeneous linear differential equation.

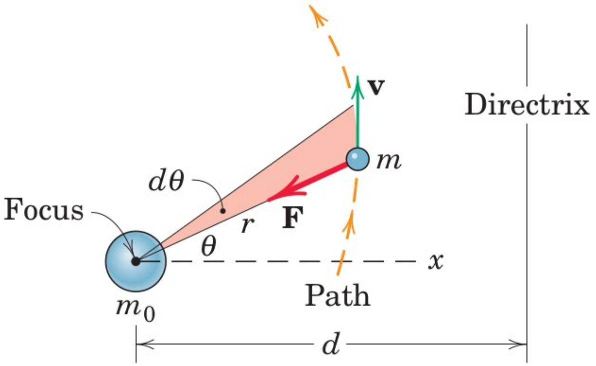
The solution of (57) may be verified by direct substitution and is

$$u = \frac{1}{r} = C\cos(\theta + \delta) + \frac{Gm_0}{h^2}, \qquad \cdots\cdots\cdots(58)$$

where C and δ are the two integration constants. The phase angle δ may be eliminated by choosing the x-axis so that r is a minimum when $\theta = 0$. Thus,

$$\frac{1}{r} = C\cos\theta + \frac{Gm_0}{h^2}. \qquad \cdots\cdots\cdots(59)$$

The interpretation of (59) requires a knowledge of the equations for conic sections. A conic section is formed by the locus of a point which moves so that the ratio e of its distance from a point (focus) to a line (directrix) is constant. Thus, $e = r/(d - r\cos\theta)$, which may be rewritten as

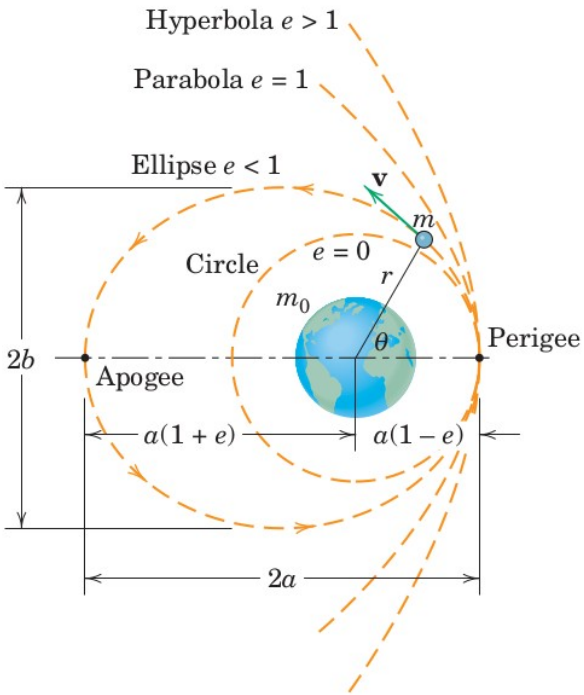


$$\frac{1}{r} = \frac{1}{d}\cos\theta + \frac{1}{ed}. \qquad \cdots\cdots\cdots(60)$$

It can be observed that (59) and (60) have same forms. Thus, we see that the motion of m is along a conic section with $d = 1/C$ and $ed = h^2/(Gm_0)$, or

$$e = h^2C/Gm_0. \qquad \cdots\cdots\cdots(61)$$

The three cases to be investigated correspond to $e < 1$ (ellipse), $e = 1$ (parabola), and $e > 1$ (hyperbola). The trajectory for each of these cases is shown in figure.



Case 1: ellipse ($e < 1$):

From (60) we deduce that r is a minimum when $\theta = 0$ and is a maximum when $\theta = \pi$. Thus,

$$2a = r_{\min} + r_{\max} = \frac{ed}{1+e} + \frac{ed}{1-e} \quad \text{or} \quad a = \frac{ed}{1-e^2}.$$

With the distance d expressed in terms of a, (60) and the maximum and minimum values of r may be written as

$$\frac{1}{r} = \frac{1+e\cos\theta}{a(1-e^2)}, \quad r_{\min} = a(1-e), \quad r_{\max} = a(1+e). \qquad \cdots\cdots\cdots(62)$$

Equation (62) is an expression of **Kepler's first law**, which says that **the planets move in elliptical orbits around the sun as a focus**. The period τ for the elliptical orbit is the total area A of the ellipse divided by the constant rate $\dot{A}$ at which the area is swept through.

Thus, from (56),

$$\tau = \frac{A}{\dot{A}} = \frac{\pi ab}{\frac{1}{2}r^2\dot{\theta}} \quad \text{or} \quad \tau = \frac{2\pi ab}{h}.$$

Eliminate $\dot{\theta}$ or h in the expression for τ by substituting (61), the identity $d = 1/C$, the geometric relationships $a = ed/(1-e^2)$ and $b = a(1-e^2)^{1/2}$ for the ellipse, and the equivalence $Gm_0 = gR^2$. The result after simplification is

$$\tau = 2\pi \frac{a^{3/2}}{R\sqrt{g}}. \qquad \cdots\cdots\cdots(63)$$

In this equation note that R is the mean radius of the central attracting body and g is the absolute value of the acceleration due to gravity at the surface of the attracting body. Equation (63) expresses **Kepler's third law** of planetary motion which states that **the square of the period of motion is proportional to the cube of the semi-major axis of the orbit**.

Case 2: parabola ($e = 1$):

Equations (60) and (61) become,

$$\frac{1}{r} = \frac{1}{d}(1 + \cos\theta) \quad \text{and} \quad h^2 C = Gm_0. \qquad \cdots\cdots\cdots(64)$$

The radius vector becomes infinite as θ approaches π, so the dimension a is infinite.

Case 3: hyperbola ($e > 1$):

From (60) we see that the radial distance r becomes infinite for the two values of the polar angle θ_1 and $-\theta_1$ defined by $\cos\theta_1 = -1/e$. Only branch I corresponding to $-\theta_1 < \theta < \theta_1$ represents a physically possible motion.