$$\mathbf{S} = 2\frac{\partial\psi(\mathbf{C})}{\partial \mathbf{C}} = 2\left[\left(\frac{\partial\psi}{\partial I_1} + I_1\frac{\partial\psi}{\partial I_2}\right)\mathbf{I} - \frac{\partial\psi}{\partial I_2}\mathbf{C} + I_3\frac{\partial\psi}{\partial I_3}\mathbf{C}^{-1}\right] \quad \text{—— (13)}$$

where ψ - strain energy function.

$$\boldsymbol{\sigma} = J^{-1}\mathbf{F}\mathbf{S}\mathbf{F}^{T} \quad \text{—— (14)}$$

using (13) & (14)

$$\boldsymbol{\sigma} = 2\left[\left(\frac{\partial\psi}{\partial I_1} + I_1\frac{\partial\psi}{\partial I_2}\right)\mathbf{F}\mathbf{I}\mathbf{F}^{T} - \frac{\partial\psi}{\partial I_2}\mathbf{F}\mathbf{C}\mathbf{F}^{T} + I_3\frac{\partial\psi}{\partial I_3}\mathbf{F}\mathbf{C}^{-1}\mathbf{F}^{T}\right]J^{-1}$$

$$= 2\left[\left(\frac{\partial\psi}{\partial I_1} + I_1\frac{\partial\psi}{\partial I_2}\right)\mathbf{b} - \frac{\partial\psi}{\partial I_2}F(F^{T}F)F^{T} + I_3\frac{\partial\psi}{\partial I_3}F(F^{-1}F^{-T})F^{T}\right]J^{-1}$$

$$\boxed{\boldsymbol{\sigma} = 2\left[\left(\frac{\partial\psi}{\partial I_1} + I_1\frac{\partial\psi}{\partial I_2}\right)\mathbf{b} - \frac{\partial\psi}{\partial I_2}\mathbf{b}^{2} + I_3\frac{\partial\psi}{\partial I_3}\mathbf{I}\right]J^{-1}} \quad \text{—— (15)}$$

Another form.

$\therefore$ $\cancel{\boldsymbol{\sigma} = 2J^{-1}\left[I_2 \partial\psi\right.}$

$$\boxed{\boldsymbol{\sigma} = 2J^{-1}\left[\left(I_2\frac{\partial\psi}{\partial I_2} + I_3\frac{\partial\psi}{\partial I_3}\right)\mathbf{I} + \frac{\partial\psi}{\partial I_1}\mathbf{b} - I_3\frac{\partial\psi}{\partial I_2}\mathbf{b}^{-1}\right]} \quad \text{—— (15a)}$$

(18) is nothing but having a form

$$\boxed{\boldsymbol{\sigma} = 2\frac{\partial\psi(\mathbf{b})}{\partial\mathbf{b}}\mathbf{b}J^{-1}} \quad \text{—— (16)}$$

We know that $\boldsymbol{\nu} = \mathbf{b}^{1/2}$. Using this relation we can write an alternate relation for σ as

$$\boxed{\sigma = J^{-1}\frac{\partial\psi(\boldsymbol{\nu})}{\partial\boldsymbol{\nu}}\boldsymbol{\nu}} \quad \text{—— (17)}$$

Constitutive eq. in terms of principal stretches

$$\Psi = \Psi(\underset{\sim}{C}) = \Psi(\lambda_1, \lambda_2, \lambda_3)$$

For a stress free configuration,

$$\Psi = \Psi(1,1,1) = 0$$

Thus. using a form of Ψ in terms of λ_α $(\alpha = 1,2,3)$ we can write alternate equations for stress. For this purpose we use the form (17) to write

$$\boxed{\sigma_\alpha = J^{-1} \lambda_\alpha \frac{\partial \Psi}{\partial \lambda_\alpha}} \qquad \{\alpha = 1,2,3\} \qquad \text{——— (18)}$$

with $J = \lambda_1 \lambda_2 \lambda_3$ ——— (19)

Other forms are

$$\boxed{\underset{\sim}{P} = \frac{\partial \Psi}{\partial \lambda_\alpha}, \quad \underset{\sim}{S} = \frac{1}{\lambda_\alpha} \frac{\partial \Psi}{\partial \lambda_\alpha}} \qquad \{\alpha = 1,2,3\} \qquad \text{——— (20)}$$

Incompressible Hyperelastic Materials

→ Several polymeric materials can go under finite deformations without noticeable change in the volume.

→ Such materials are treated as incompressible materials and for them only isochoric motions are possible

→ Thus for incompressible materials we have

$$\boxed{J=1} \qquad \text{——— (21)}$$

→ Incompressibility introduces a new constraint.

Recall the expression for internal dissipation

$$D_{int} = P:\dot{F} - \dot{\psi} = \left(P - \frac{\partial \psi}{\partial \underset{\sim}{F}}\right) : \dot{\underset{\sim}{F}} = 0 \qquad \text{——— (22)}$$

→ For hyperelastic materials we derived an expression for stress by using the argument that $\underset{\sim}{F}$ and $\dot{\underset{\sim}{F}}$ are independent and hence $\dot{\underset{\sim}{F}}$ is arbitrary.

→ However when incompressibility enters then we have

$$J=1$$

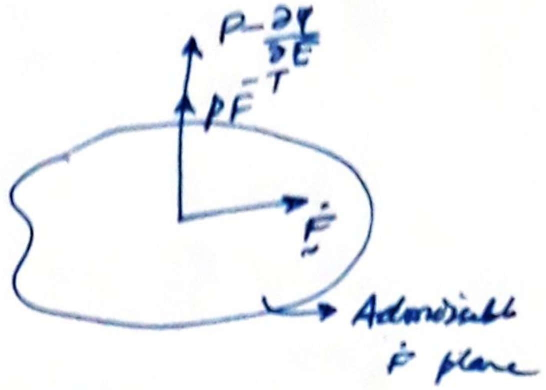
$$\therefore \dot{\underset{\sim}{J}} = 0 = \underset{\sim}{F}^{-T} : \dot{\underset{\sim}{F}}, \qquad \text{——— (23)}$$

which suggest that $\underset{\sim}{F}$ and $\dot{\underset{\sim}{F}}$ are now not independent.

Hence, the argument given earlier is not valid now.

Hence, (22) & (23) combined gives us the condition that

$$\left(P - \frac{\partial \psi}{\partial F}\right) = -pF^{-T}$$



Hence,

$$\boxed{P = \frac{\partial \psi}{\partial F} - pF^{-T}} \quad (24)$$

{ p is a scalar constant }

Other definitions of stresses can be derived using (24) as

$$\boxed{S = 2\frac{\partial \psi(C)}{\partial C} - pC^{-1}J} \quad (25)$$

{ $J = 1$ }

$$\boxed{\sigma = F\left(\frac{\partial \psi(F)}{\partial F}\right)^{T} - pI} \quad (26)$$

From (26), we can identify that the term involving the constant p is the hydrostatic part of the stress, hence p is nothing but the pressure term.

Hence, the other part is the deviatoric part of the stress

$$\therefore \boxed{S' = 2\frac{\partial \psi(C)}{\partial C} \quad \text{or} \quad \sigma' = F\left[\frac{\partial \psi(F)}{\partial F}\right]^{T} \quad \text{or} \quad P' = \frac{\partial \psi}{\partial F}} \quad (27)$$

* Constant p is determined from boundary conditions in a given problem.

Considering above constraints, we can write a suitable form of strain energy function ψ as

$$\boxed{\begin{aligned}\psi &= \psi[I_1(\underset{\sim}{C}), I_2(\underset{\sim}{C})] - \frac{1}{2}p(I_3 - 1)\\ \psi &= \psi[I_1(\underset{\sim}{b}), I_2(\underset{\sim}{b})] - \frac{1}{2}p(I_3 - 1)\end{aligned}}$$

Some forms of strain-energy function

Ogden model for rubber-like materials ...

$$\psi = \psi(\lambda_1, \lambda_2, \lambda_3) = \sum_{i=1}^{N} \frac{\mu_i}{\alpha_i}\left(\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3\right)$$

Mooney Rivlin model $\{N=2, \alpha_1 = 2, \alpha_2 = -2\}$

$$\psi = C_1\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right) + C_2\left(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3\right)$$

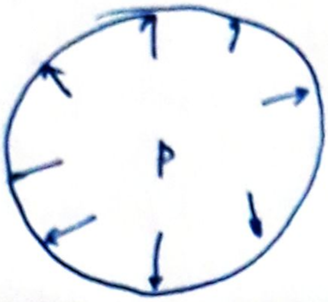
$$\psi = C_1(I_1 - 3) + C_2(I_2 - 3)$$

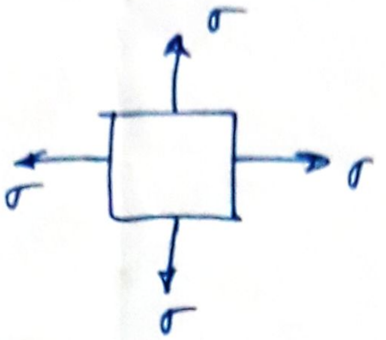
Neo-hookean model $\{N=1, \alpha_1 = 2\}$

$$\psi = C_1\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right) = C_1(I_1 - 3)$$

and several others - ——

Example - Inflation of a spherical balloon





$\sigma_1 = \sigma_2 = \sigma$

$\sigma_3 = 0$

{plane stress}

Let us derive expression of stresses for Ogden model

$$\Psi = \sum_{i=1}^{N} \frac{\mu_i}{\lambda_i}\left(\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3\right) - \frac{p}{2}(I_3 - 1)$$

We start with the form (159),

$$\underset{\sim}{\sigma} = 2J^{-1}\left[\left(I_2 \frac{\partial \Psi}{\partial I_2} + I_3 \frac{\partial \Psi}{\partial I_3}\right)\underset{\sim}{I} + \frac{\partial \Psi}{\partial I_1}\underset{\sim}{b} - I_3 \frac{\partial \Psi}{\partial I_2}\underset{\sim}{b}^{-1}\right]$$

$$\frac{\partial \Psi}{\partial I_3} = -\frac{p}{2} \quad , \quad I_3 = 1 \ \& \ \sigma = 1$$

$$\therefore \underset{\sim}{\sigma} = 2J^{-1}\left[\left(I_2 \frac{\partial \Psi}{\partial I_2} - \frac{p}{2}\right)\underset{\sim}{I} + \frac{\partial \Psi}{\partial I_1}\underset{\sim}{b} - \frac{\partial \Psi}{\partial I_2}\underset{\sim}{b}^{-1}\right]$$

$$= -p\underset{\sim}{I} + 2\frac{\partial \Psi}{\partial I_1}\underset{\sim}{b} + 2I_2\frac{\partial \Psi}{\partial I_2}\underset{\sim}{I} - 2\frac{\partial \Psi}{\partial I_2} b^{-1}$$

or
$$\underset{\sim}{\sigma} = \underbrace{\left\{2I_2\frac{\partial \Psi}{\partial I_2} - p\right\}}_{-p'}\underset{\sim}{I} + 2\frac{\partial \Psi}{\partial I_1}\underset{\sim}{b} - 2\frac{\partial \Psi}{\partial I_2} b^{-1}$$

or
$$\boxed{\underset{\sim}{\sigma} = -p'\underset{\sim}{I} + 2\frac{\partial \Psi}{\partial I_1}\underset{\sim}{b} - 2\frac{\partial \Psi}{\partial I_2} b^{-1}}$$

It can be shown using chain rule of differentiation that in terms of $\lambda_1, \lambda_2, \lambda_3$ above relation can be written as

$$\sigma_\alpha = -p + \lambda_\alpha \frac{\partial \psi}{\partial \lambda_\alpha}$$

Now, using this equation with Ogden energy function we get

$$\sigma_a = -p + \sum_{i=1}^{N} \mu_i \lambda_a^{\alpha_i} \qquad \{a = 1, 2, 3\}$$

as discussed constant p will be determined from B.C

In this case we have $\sigma_3 = 0$.

$$\therefore \quad \sigma_3 = 0 = -p + \sum_{i=1}^{N} \mu_i \lambda_3^{\alpha_i}$$

$$\therefore \quad p = \sum_{i=1}^{N} \mu_i \lambda_3^{\alpha_i}$$

$\therefore$ we get

$$\sigma_1 = \sum_{i=1}^{N} \mu_i \left[\lambda_1^{\alpha_i} - (\lambda_1 \lambda_2)^{-\alpha_i} \right]$$

$$\sigma_2 = \sum_{i=1}^{N} \mu_i \left[\lambda_2^{\alpha_i} - (\lambda_1 \lambda_2)^{-\alpha_i} \right]$$

for our case $\lambda_1 = \lambda_2 = \lambda$, hence

$$\boxed{\sigma_1 = \sigma_2 = \sigma = \sum_{i=1}^{N} \left(\lambda^{\alpha_i} - \lambda^{-2\alpha_i} \right)}$$