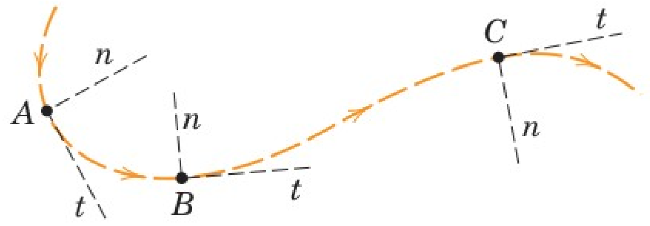
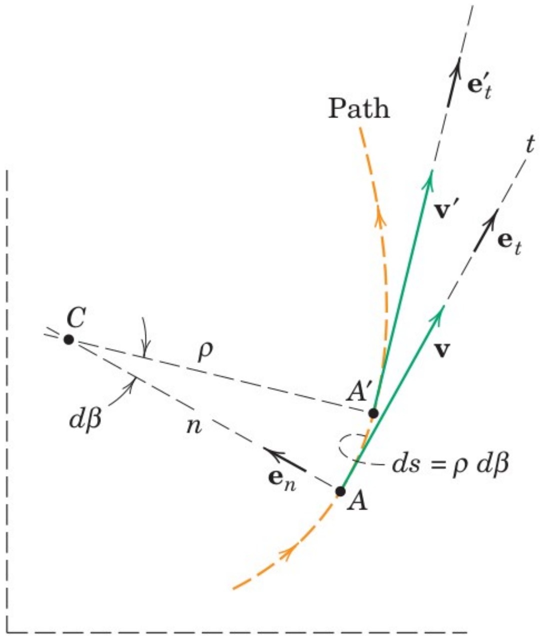
ME232: Dynamics

Anshul Faye
afaye@iitbhilai.ac.in
Room # 106

Normal and tangential coordinates (n-t)





We now use the coordinates $\boldsymbol{n}$ and $\boldsymbol{t}$ to describe the velocity $\boldsymbol{v}$ and acceleration $\boldsymbol{a}$.

Unit vectors $\boldsymbol{e}_n$ and $\boldsymbol{e}_t$ are defined in the n-and t-direction, respectively at point A.

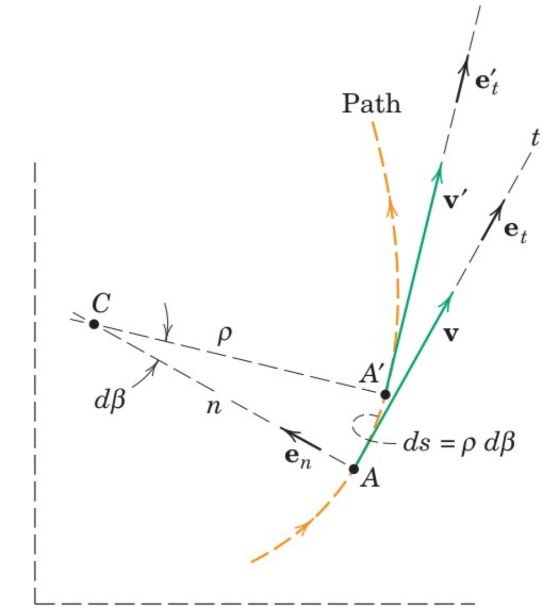
During a differential increment of time dt, the particle moves a differential distance ds along the curve from A to A'.

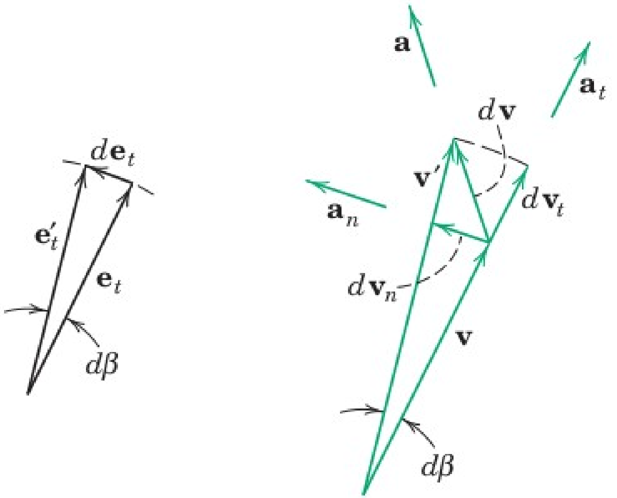
$\boldsymbol{\rho}$ being the radius of curvature of the path at A, $ds = \boldsymbol{\rho}\ d\beta$, where β is in radians. (change in $\boldsymbol{\rho}$ is neglected)

Thus, the magnitude of the velocity can be written

$v = ds\ /dt = \boldsymbol{\rho}\ d\beta/dt$, and the velocity vector is

$$\boldsymbol{v} = v\boldsymbol{e}_t = \rho\dot{\beta}\boldsymbol{e}_t. \qquad \text{.................}(16)$$





The acceleration $\boldsymbol{a}$ of the particle $\boldsymbol{a} = d\boldsymbol{v} / dt$, which is obtained as,

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{d(v\boldsymbol{e}_t)}{dt} = v\dot{\boldsymbol{e}}_t + \dot{v}\boldsymbol{e}_t, \qquad \cdots\cdots\cdots\cdots(17)$$

Note that $\dot{\boldsymbol{e}}_t \neq 0$ because it changes the direction.

Change is et during small time increment dt is $d\boldsymbol{e}_t$ in the direction of $\boldsymbol{e}_n$. Thus, we can write

$$|d\boldsymbol{e}_t| = |\boldsymbol{e}_t| d\beta = d\beta.$$

$$d\boldsymbol{e}_t = |d\boldsymbol{e}_t|\boldsymbol{e}_n = d\beta\boldsymbol{e}_n \Rightarrow \dot{\boldsymbol{e}}_t = \dot{\beta}\boldsymbol{e}_n. \qquad \cdots\cdots\cdots\cdots(18)$$

Substituting (18) in (17), the acceleration becomes,

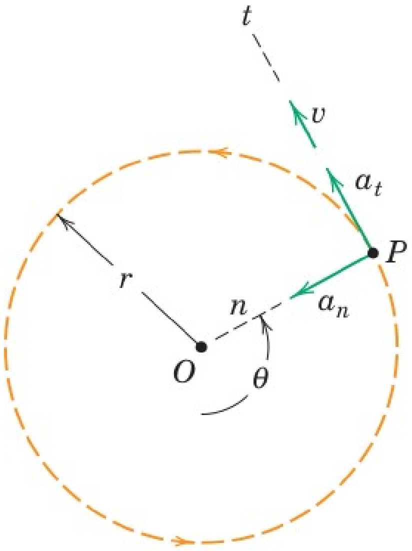
$$\boldsymbol{a} = \frac{v^2}{\rho}\boldsymbol{e}_n + \dot{v}\boldsymbol{e}_t = a_n\boldsymbol{e}_n + a_t\boldsymbol{e}_t. \qquad \cdots\cdots\cdots\cdots(19)$$

where,

$$a_n = v^2/\rho = \rho\dot{\beta}^2 = v\dot{\beta}, \quad a_t = \dot{v} = \ddot{s}. \qquad \cdots\cdots\cdots\cdots(20)$$

Circular motion

Circular motion is an important special case of plane curvilinear motion where the radius of curvature ρ becomes the constant radius r of the circle and the angle β is replaced by the angle measured from any convenient radial reference to OP. The velocity and the acceleration components for the circular motion of the particle P become

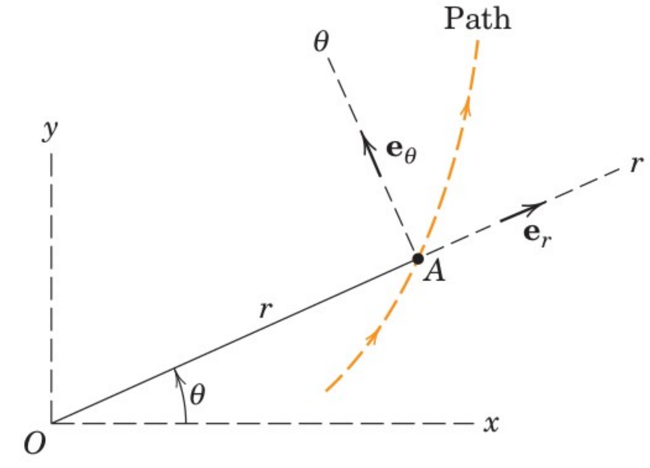


$$v = r\dot{\theta}$$

$$a_n = v^2/r = r\dot{\theta}^2 = v\dot{\theta} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(21)$$

$$a_t = r\ddot{\theta}$$

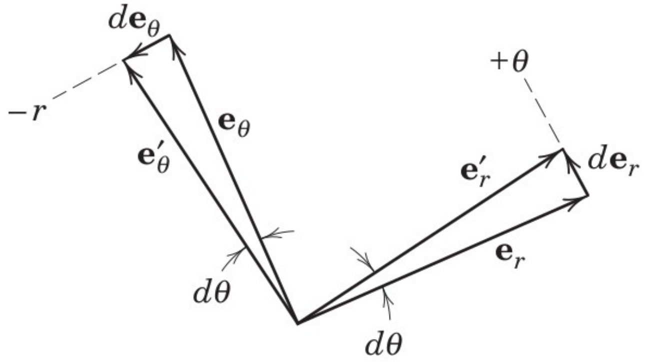
Polar coordinates (r-θ)



Polar coordinates are particularly useful when a motion is constrained through the control of a radial distance and an angular position or when an unconstrained motion is observed by measurements of a radial distance and an angular position.

$\boldsymbol{r}$ is the position vector to the particle at A, which is expressed as

$$\boldsymbol{r} = r\boldsymbol{e}_r. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(22)$$



During time dt the coordinate directions as well as unit vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ rotate through an angle $d\theta$ to $\boldsymbol{e}'_r$ and $\boldsymbol{e}'_\theta$. We can write

$$\frac{d\boldsymbol{e}_r}{d\theta} = \boldsymbol{e}_\theta, \quad \text{and} \quad \frac{d\boldsymbol{e}_\theta}{d\theta} = -\boldsymbol{e}_r, \text{ or,} \quad \cdots\cdots\cdots\cdots\cdots\cdots(23)$$

$$\dot{\boldsymbol{e}}_r = \dot{\theta}\boldsymbol{e}_\theta, \quad \text{and} \quad \dot{\boldsymbol{e}}_\theta = -\dot{\theta}\boldsymbol{e}_r. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(24)$$

Now, the velocity

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{r}\boldsymbol{e}_r + r\dot{\boldsymbol{e}}_r = \dot{\boldsymbol{r}} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta, \qquad \text{[From (24)]} \qquad \cdots\cdots\cdots\cdots(25)$$

where, $v_r = \dot{r}, v_\theta = r\dot{\theta}$ and $v = \sqrt{v_r^2 + v_\theta^2}$.

The r-component of $\boldsymbol{v}$ is the rate at which the vector $\boldsymbol{r}$ stretches. The θ-component of $\boldsymbol{v}$ is due to the rotation of $\boldsymbol{r}$.

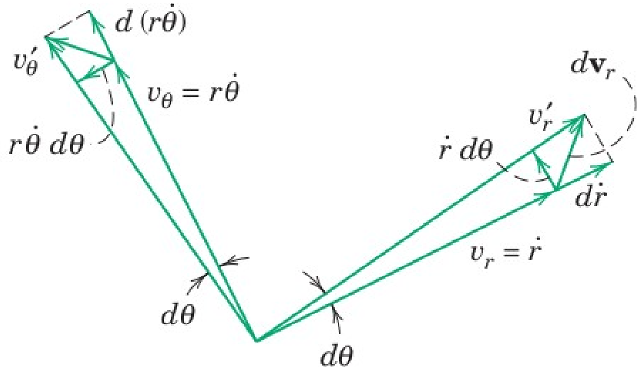
To obtain acceleration, we differentiate (25), so,

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = (\ddot{r}\boldsymbol{e}_r + \dot{r}\dot{\boldsymbol{e}}_r) + (\dot{r}\dot{\theta}\boldsymbol{e}_\theta + r\ddot{\theta}\boldsymbol{e}_\theta + r\dot{\theta}\dot{\boldsymbol{e}}_\theta).$$

Substituting derivatives of unit vectors from (24),

$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e}_\theta, \qquad \cdots\cdots\cdots\cdots(26)$$

where $a_r = (\ddot{r} - r\dot{\theta}^2), a_\theta = (r\ddot{\theta} + 2\dot{r}\dot{\theta})$, and $a = \sqrt{a_r^2 + a_\theta^2}$.



Following changes are shown in the figure:

(a) Magnitude change of $\boldsymbol{v}_r$: This change is the increase in length of v_r or $dv_r = d\dot{r}$, and the corresponding acceleration term is $d\dot{r}/dt = \ddot{r}$ in the positive r-direction.

(b) Direction change of $\boldsymbol{v}_r$: The magnitude of this change is $v_r d\theta = \dot{r} d\theta,$ and its contribution to the acceleration becomes $\dot{r} d\theta/dt = \dot{r}\dot{\theta},$ which is in the positive θ-direction.

(c) Magnitude change of $\boldsymbol{v}_\theta$: This term is the change in length of v_θ or $d(r\dot{\theta})$, and its contribution to the acceleration is $d(r\dot{\theta})/dt = r\ddot{\theta} + \dot{r}\dot{\theta}$, and is in the positive θ-direction.

(d) Direction Change of $\boldsymbol{v}_\theta$: The magnitude of this change is $v_\theta d\theta = r\dot{\theta} d\theta,$ and the corresponding acceleration term is observed to be $r\dot{\theta}(d\theta/dt) = r\dot{\theta}^2$, in the negative r-direction.

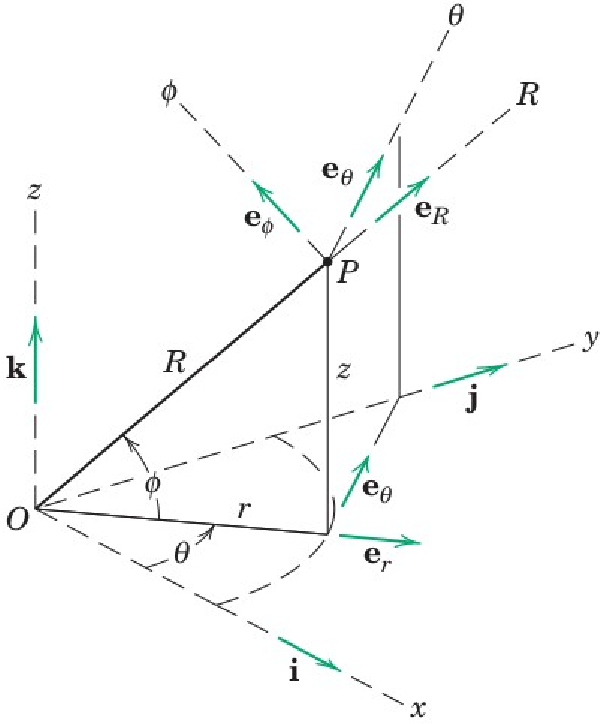
Circular motion

For motion in a circular path with r constant, the components of velocity and acceleration become,

$$v_r = 0, \quad v_\theta = r\dot{\theta},$$

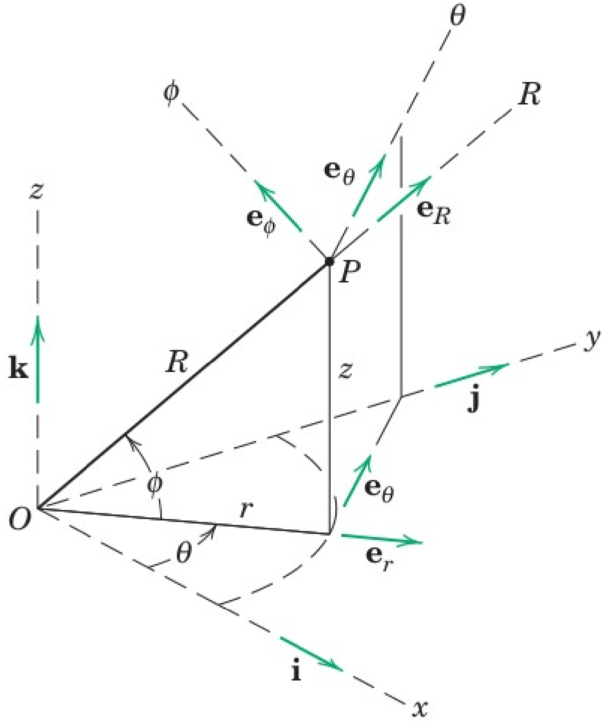
$$a_r = -r\dot{\theta}^2, \quad a_\theta = r\ddot{\theta}.$$

Space curvilinear motion



For the general case of three-dimensional motion of a particle along a space curve three coordinate systems, rectangular (x-y-z), cylindrical (r--z), and spherical (R--ϕ), are commonly used to describe this motion. These systems are indicated in Figure, which also shows the unit vectors for the three coordinate systems.

After understanding physical meaning of different terms in velocity and acceleration for plane motion, we simply extend the concept for the space curvilinear motion and just write the mathematical expressions.



Rectangular coordinates (x-y-z):

$$\boldsymbol{R} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$
$$\dot{\boldsymbol{R}} = \boldsymbol{v} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k} \qquad \cdots\cdots\cdots\cdots(27)$$
$$\ddot{\boldsymbol{R}} = \dot{\boldsymbol{v}} = \boldsymbol{a} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{z}\boldsymbol{k}$$

Cylindrical coordinates (r-θ-z):

$$\boldsymbol{R} = r\boldsymbol{e}_r + z\boldsymbol{k}$$
$$\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta + \dot{z}\boldsymbol{k} \qquad \cdots\cdots\cdots\cdots(28)$$
$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e}_\theta + \ddot{z}\boldsymbol{k}$$

Spherical coordinates (R-θ-ϕ):

$$\boldsymbol{R} = R\boldsymbol{e}_R$$
$$\boldsymbol{v} = v_R\boldsymbol{e}_R + v_\theta\boldsymbol{e}_\theta + v_\phi\boldsymbol{e}_\phi$$
$$\boldsymbol{a} = a_R\boldsymbol{e}_R + a_\theta\boldsymbol{e}_\theta + a_\phi\boldsymbol{e}_\phi$$

$$v_R = \dot{R}, \quad v_\theta = R\dot{\theta}\cos\phi, \quad v_\phi = R\dot{\phi},$$
$$a_R = \ddot{R} - R\dot{\phi}^2 - R\dot{\theta}^2\cos^2\phi, \qquad \cdots\cdots\cdots\cdots(29)$$
$$a_\theta = \frac{\cos\phi}{R}\frac{d}{dt}(R^2\dot{\theta}) - 2R\dot{\theta}\dot{\phi}\sin\phi,$$
$$a_\phi = \frac{1}{R}\frac{d}{dt}(R^2\dot{\phi}) + R\dot{\theta}^2\sin\phi\cos\phi.$$