ME632: Fracture Mechanics

Timings	
Monday	10:00 to 11:20
Thursday	08:30 to 09:50

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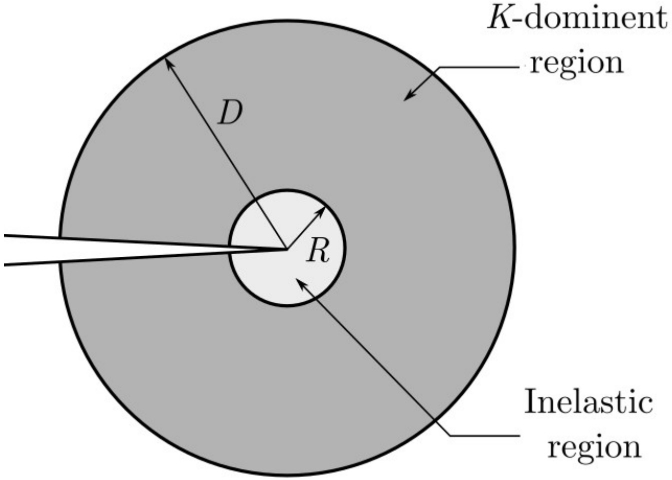
Room No. # 106

Elasto-Plastic fracture mechanics

Till now we discussed about the stress-field in the vicinity of a crack-tip for linear elastic materials. However, assumption of linear elasticity is not true with most engineering material. Engineering materials go under plastic deformation at high stresses. This is also valid for the high stresses near the crack-tip. The material flow after yielding makes the crack tip blunt, which in turn reduces the magnitude of stress components there. Thus, many potential catastrophic failures are avoided just by the local plastic deformation at the crack tip. A proper analysis of plastic deformation near the crack-tip allow accurate determination the stresses and deformation there.

Small-scale yielding

We also saw that the applicability of singular stress fields is confined to a very small region around the crack-tip. Let the singular solution dominate inside a circle of radius D surrounding the crack tip. Consider also that the region of inelastic deformation attending the crack tip is represented by R. When R is sufficiently small compared to D and any other characteristic geometric dimension such as notch radius, plate thickness, crack ligament, etc., the singular stress field governed by the stress intensity factors forms a useful approximation to the elastic field in the ring enclosed by radii R and D. This situation is called "small-scale yielding".



Approximate determination of crack-tip plastic zone

Strictly speaking, the plastic zone should be determined from an elastic-plastic analysis of the stress field around the crack tip. However, such analysis is quite complex and involved. Hence, we can obtain some useful results regarding the shape of the plastic zone from the approximate calculation.

A first estimate of the extent of the plastic zone attending the crack tip can be obtained by determining the locus of points where the elastic stress field satisfies the yield criterion. This calculation is very approximate, since yielding leads to stress redistribution and modifies the size and shape of the plastic zone. To apply the yield criterion let us first determine principle stresses. Stresses at the crack-tip are

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right),$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right),$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}} \sin\theta\sin\frac{3\theta}{2}\cos\frac{3\theta}{2},$$

$$\sigma_z = 0. \quad \text{for plane stress,}$$

$$\sigma_z = \nu(\sigma_x + \sigma_y) \quad \text{for plane strain.}$$

$$\ldots\ldots\ldots\ldots\ldots(1)$$

Principal stresses are

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \sigma_{xy}^2} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\right]$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \sigma_{xy}^2} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\right]$$

$$\sigma_3 = 0 \quad \text{(for plane stress)} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(2)$$

$$\sigma_3 = 2\nu\frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \quad \text{(for plane strain)}$$

By applying the Von-Mises criterion for yielding,

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \geq 2\sigma_Y^2 \qquad \cdots\cdots\cdots\cdots\cdots\cdots(3)$$

Substituting (2) in (3) and simplifying we get,

$$\frac{K_I^2}{2\pi r}\left(1+\cos\theta+\frac{3}{2}\sin^2\theta\right) \geq 2\sigma_Y^2 \qquad \text{(for plane stress)}$$

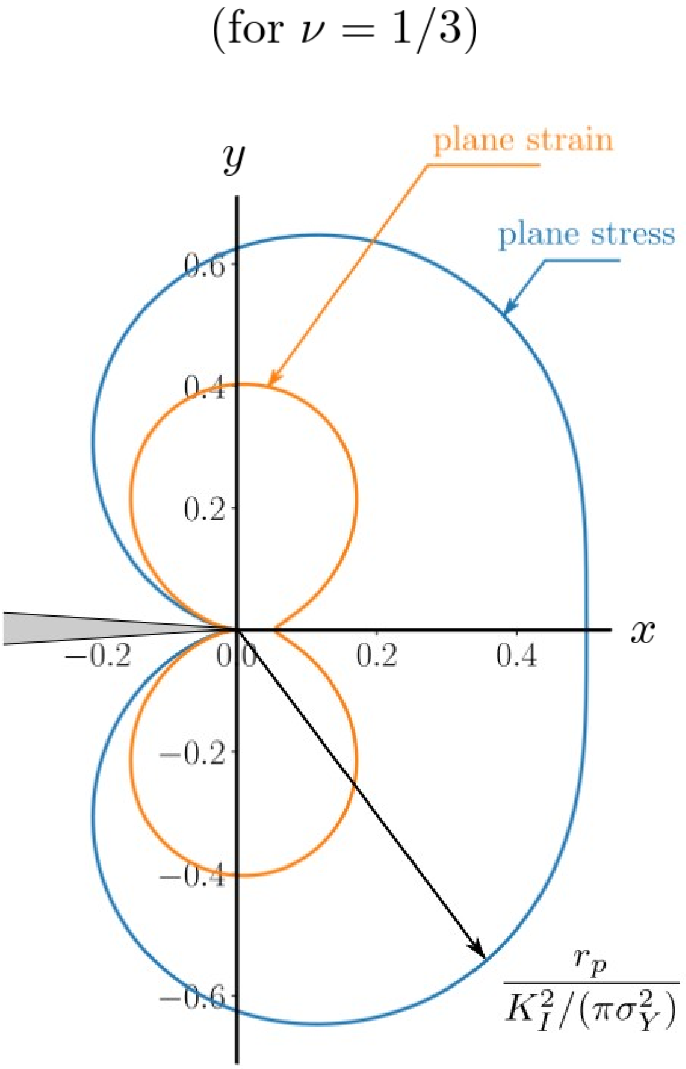
$$\frac{K_I^2}{2\pi r}\left[(1-2\nu)^2(1+\cos\theta)+\frac{3}{2}\sin^2\theta\right] \geq 2\sigma_Y^2 \qquad \text{(for plane strain)}$$

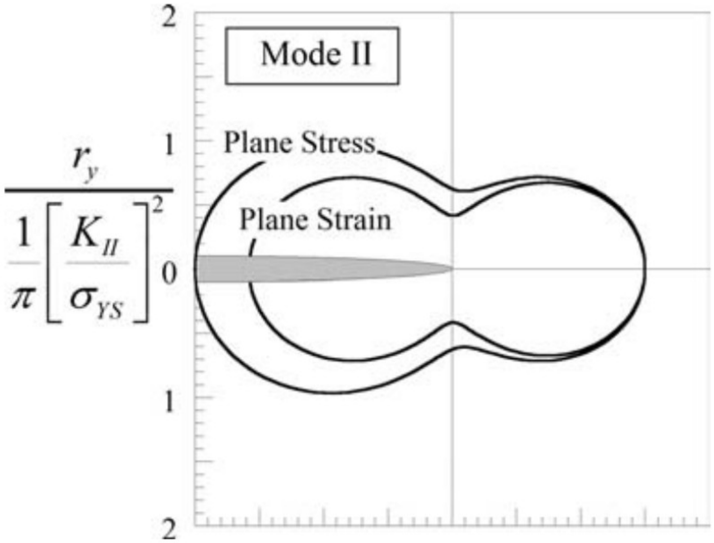
Thus the boundary of elastic plastic interface can be obtained as

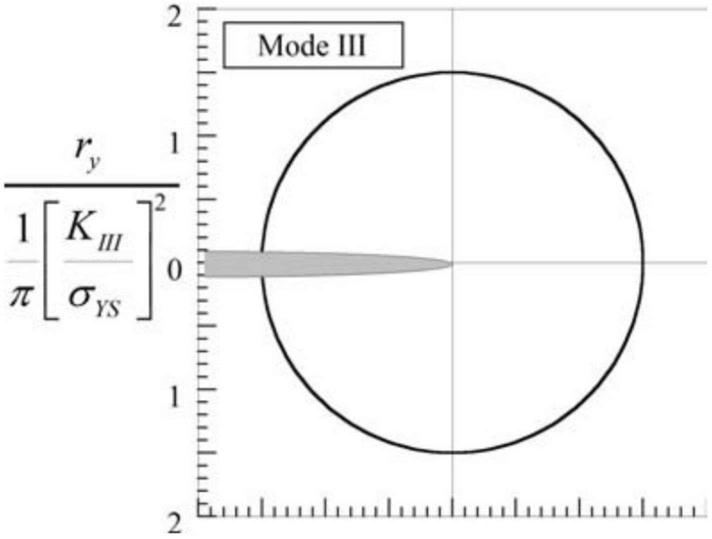
$$r_p(\theta) = \frac{K_I^2}{4\pi\sigma_Y^2}\left(1+\cos\theta+\frac{3}{2}\sin^2\theta\right) \qquad \text{(for plane stress)}$$

$$r_p(\theta) = \frac{K_I^2}{4\pi\sigma_Y^2}\left[(1-2\nu)^2(1+\cos\theta)+\frac{3}{2}\sin^2\theta\right] \qquad \text{(for plane strain)}$$

..................(4)

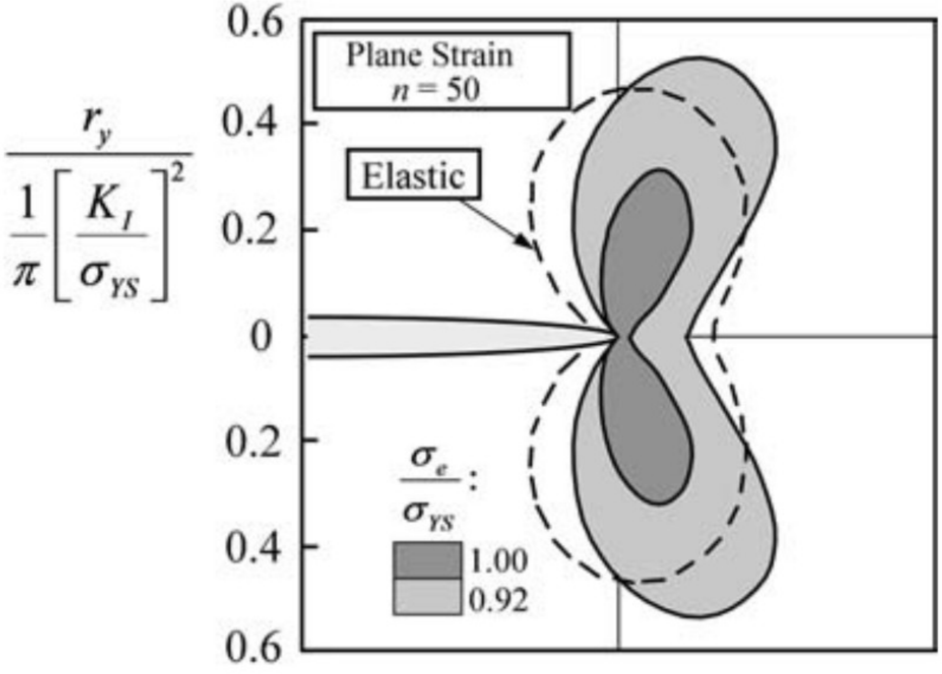






Plastic zone through finite element simulation, using the following stress-strain relation:

$$\frac{\varepsilon}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \alpha\left(\frac{\sigma}{\sigma_0}\right)^n$$



- Observe that the plane stress zone is much larger than the plane strain zone because of the higher constraint for plane strain.
- Plastic zone is proportional to the square of SIF and inversely proportional to the square of the yield stress.
- Whenever yield stress of a metal is increased, say by an appropriate heat treatment, its plastic zone size decreases considerably; this in turn makes the material more prone to crack growth.
- The increase in yield stress may please a conventional designer, because he usually designs structural components based on a yield criterion. But as far as the toughness of a material is concerned, the designer is left with an inferior material. The designer msut explore a satisfactory compromise between yield stress and toughness, while choosing a material and its heat treatment.
- Let us try to understand the yielding behaviour. Tresca criteria helps us in understanding it in a simple manner.

For plane strain (considering $\nu = 1/3$)

$$\sigma_{\max} = \sigma_1 = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\right]$$

$$\text{for } \theta \leq 38.9^\circ,\ \sigma_{\min} = \sigma_3 = 2\nu \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}$$

$$\text{for } \theta \geq 38.9^\circ,\ \sigma_{\min} = \sigma_2 = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\right]$$

Thus,

$$\text{for } \theta \leq 38.9^\circ,\quad \tau_{\max} = \frac{\sigma_{\max} - \sigma_{\min}}{2} = \frac{1}{2}\frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - 2\nu + \sin\frac{\theta}{2}\right]$$

$$\text{for } \theta \geq 38.9^\circ,\quad \tau_{\max} = \frac{\sigma_{\max} - \sigma_{\min}}{2} = \frac{1}{2}\frac{K_I}{\sqrt{2\pi r}} \sin\theta$$

Hence, from Tresca criteria, ...(5)

$$\text{for } \theta \leq 38.9^\circ,\quad r_p = \frac{K_I^2}{2\pi\sigma_Y^2} \cos^2\frac{\theta}{2}\left[1 - 2\nu + \sin\frac{\theta}{2}\right]^2, \quad \text{and} \quad \text{for } \theta \leq 38.9^\circ,\quad r_p = \frac{K_I^2}{2\pi\sigma_Y^2} \sin^2\theta.$$

For plane stress,

$$\sigma_{\max} = \sigma_1 = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\right]$$

$$\sigma_{\min} = \sigma_3 = 0$$

Now, as per Tresca criteria for yielding

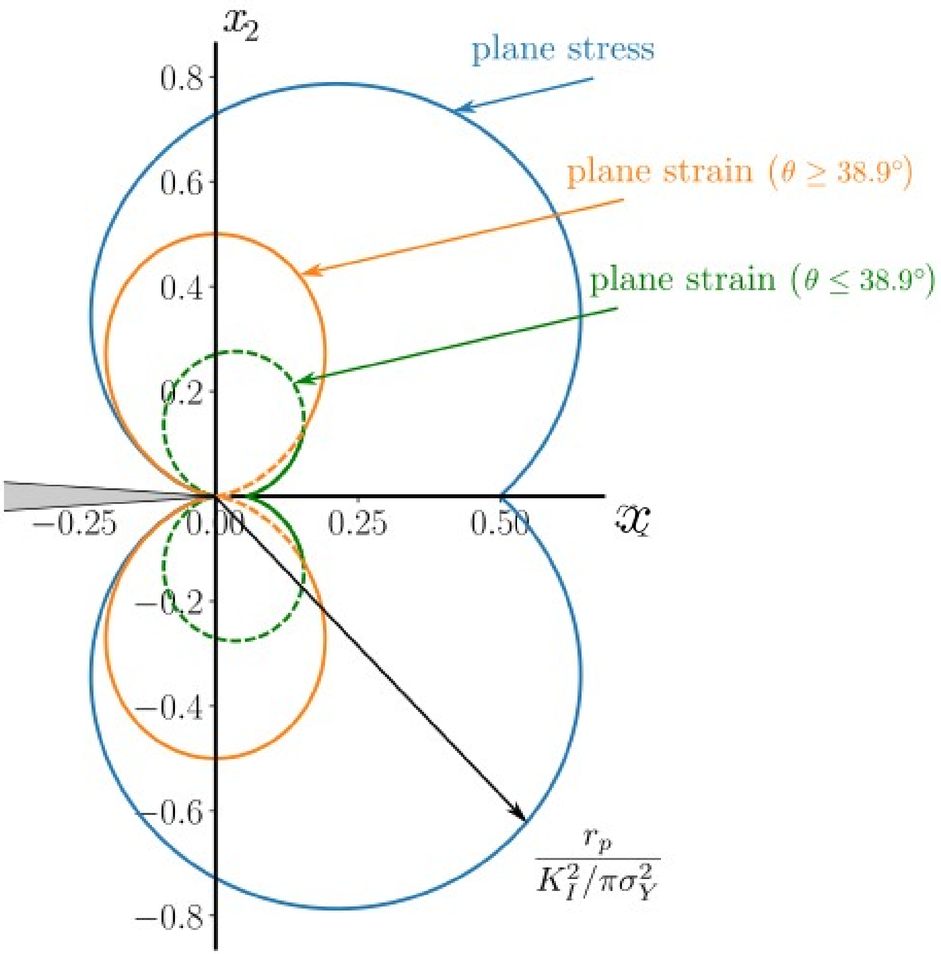
$$\tau_{\max} = \frac{\sigma_{\max} - \sigma_{\min}}{2} \geq \frac{\sigma_Y}{2},$$

i.e.,

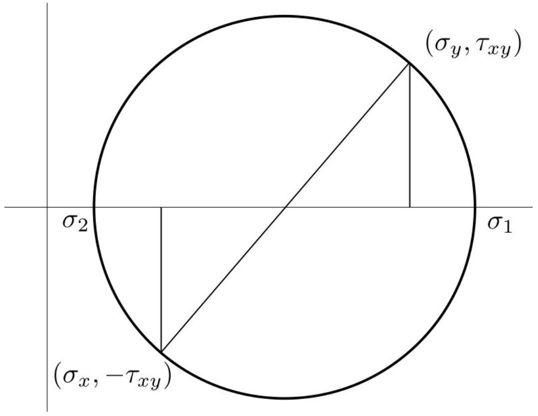
$$\frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\right] \geq \sigma_Y .$$

Thus,

$$r_p(\theta) = \frac{K_I^2}{2\pi\sigma_Y^2} \cos^2\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\right]^2 . \qquad \cdots\cdots\cdots\cdots(6)$$



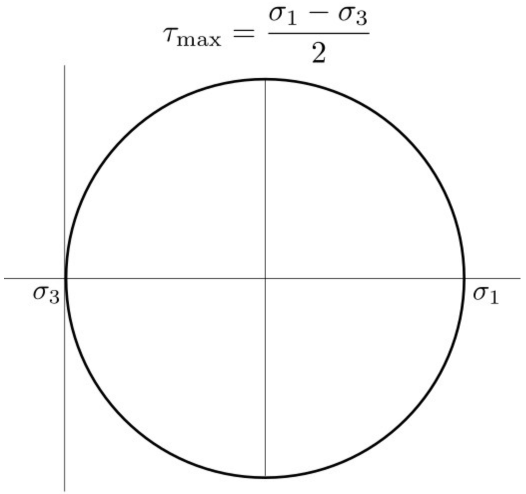
Let us now use Mohr's circle to visualize the planes of yielding in plane stress and plane strain cases under Mode-I loading. During the plastic deformation there is no change in volume and hence, $\nu = 0.5$ will be considered here.



The state of stress for an element at $\theta = 0°$ plane is,

$$\sigma_x = \sigma_y = \frac{K_I}{\sqrt{2\pi r}} \quad \text{and} \quad \sigma_{xy} = 0.$$

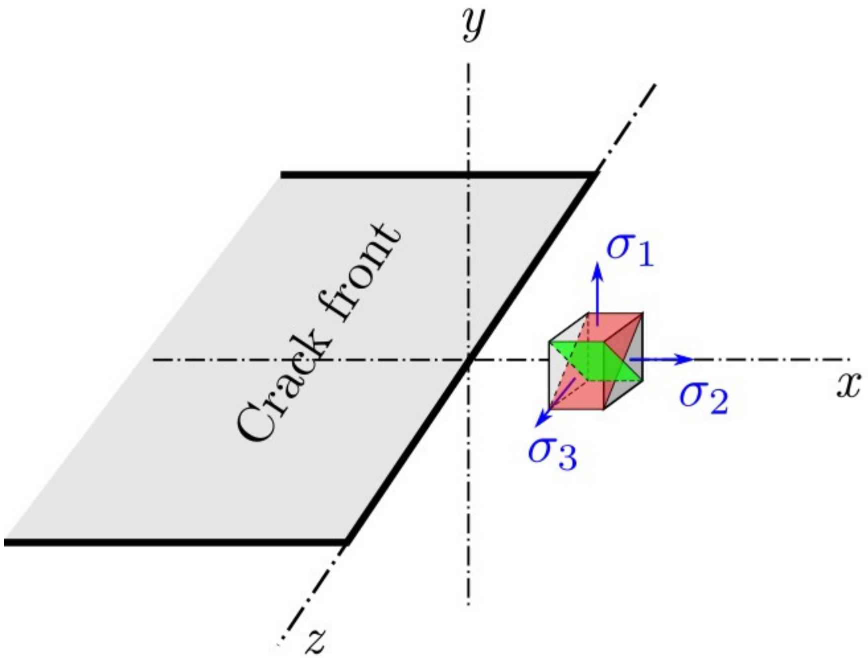
In this case, stress in x- and y- direction are equal. However, if we consider a small deviation from $\theta = 0°$ then σ_x and σ_y will be different and $\sigma_y > \sigma_x$. In this case the maximum principle stress σ_1 is in y- direction and the minimum principal stress is in the z- direction which is equal to zero.



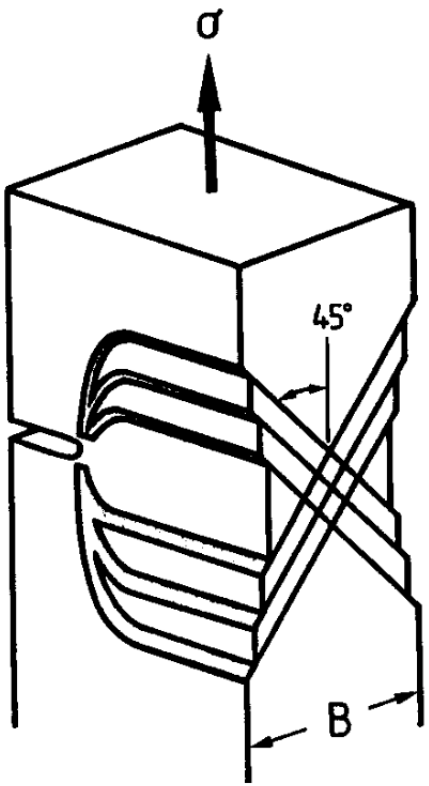
$$\sigma_{\max} = \sigma_1 = \sigma_y, \quad \sigma_{\min} = \sigma_3 = \sigma_z = 0,$$

Thus, the plane of maximum shear stress at planes parallel to x-axis and inclined at an angle of $\pm 45°$ from the direction of σ_1 (i.e., y-axis)

Planes having maximum shear stress under mode-I loading for plane stress case

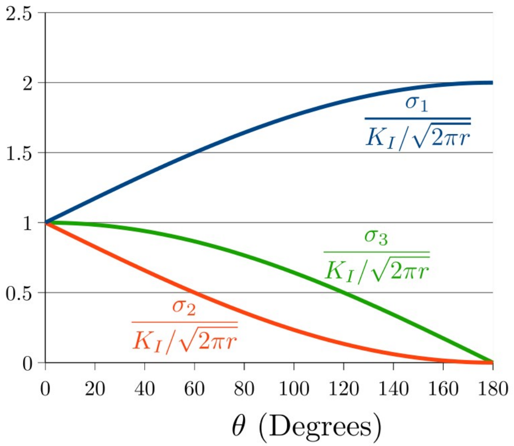


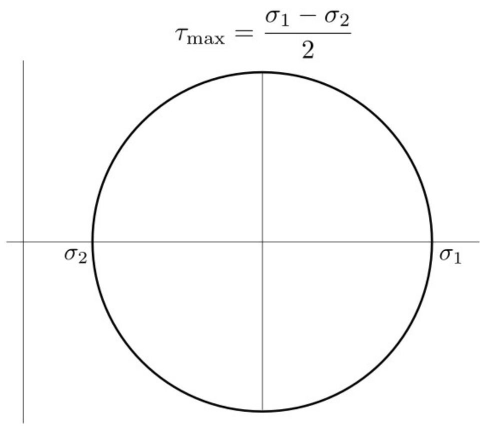
Slip-planes around a mode-I crack for plane stress case



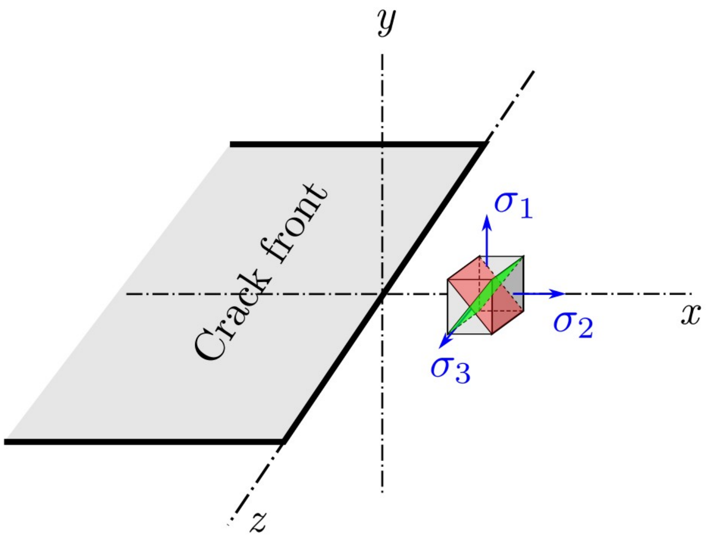
Source: Fracture Mechanics, E. E. Gdoutos

For plane strain case (considering $\nu = 0.5$) the maximum principal stress is σ_1 and minimum principal stress is always σ_2 hence the maximum shear stress will be occurs in planes which are parallel to z-axis and inclined at an angle of $\pm 45°$ from the direction of σ_1 (i.e., y-axis)

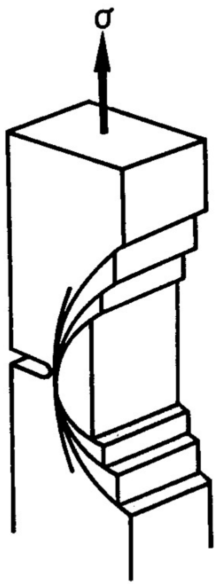




Planes having maximum shear stress under mode-I loading for plane stain case

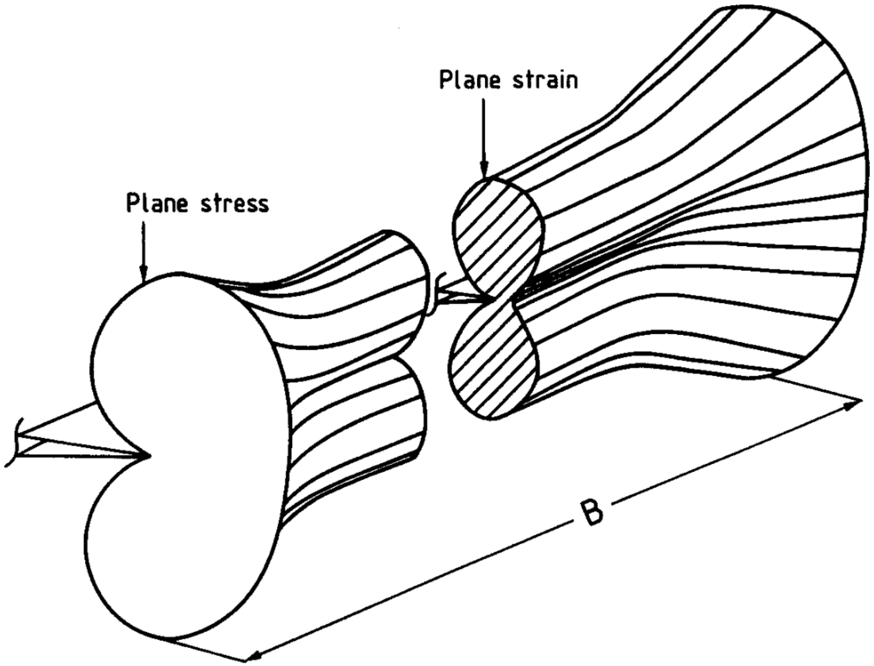


Slip-planes around a mode-I crack for plane stress case



Source: Fracture Mechanics, E. E. Gdoutos

In cracked plates, conditions of plane stress dominate at the traction-free surfaces, while plane strain prevails in the interior. This results in a variation of the plastic zone size through the plate thickness, which decreases from the surface to the interior of the plate.

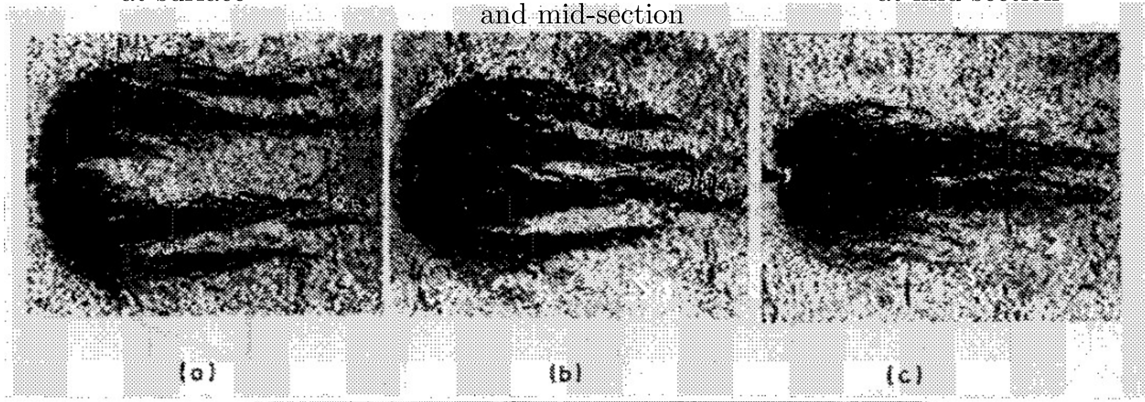


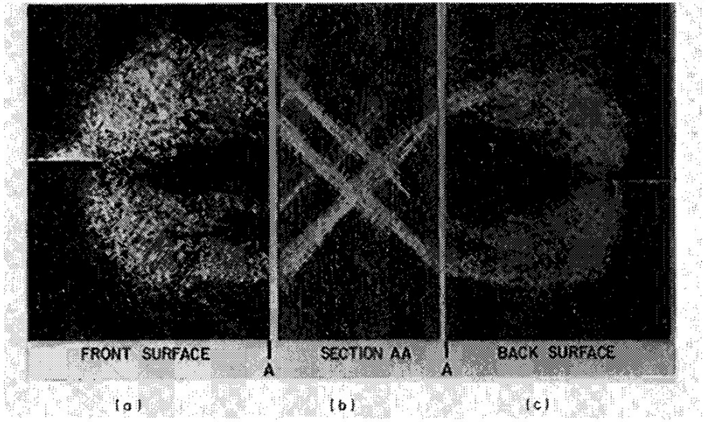
Source: Fracture Mechanics, E. E. Gdoutos

at surface

Halfway between surface and mid-section

at mid-section

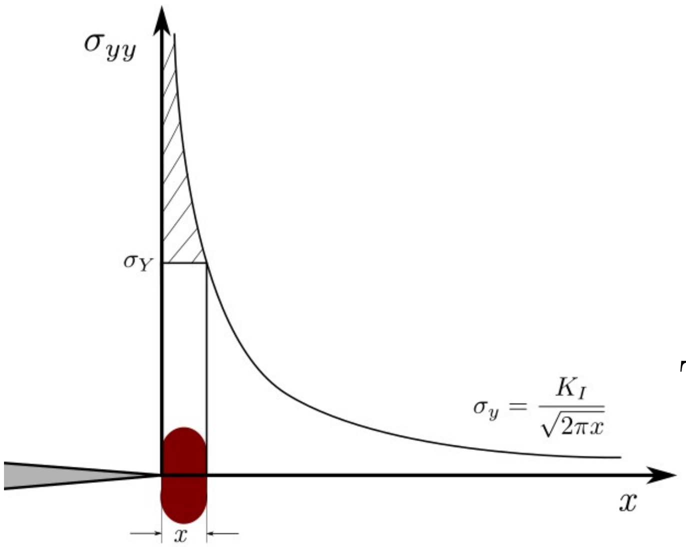




Source: Fracture Mechanics, E. E. Gdoutos

Approximate method for plastic zone calculation

In a very approximate manner the size of plastic zone can be determined by identifying the distance ahead of the crack-tip at which vertical stress reaches yield stress. We first use the Von-Mises yield criteria to determine the the amount of stress in y-direction during yield. Principal stresses ahead of the crack-tip at $\theta = 0°$is



Thus the distance at which yield stress reaches,

For plane stress

$$\sigma_1 = \sigma_{xx} (= \sigma_{yy})$$
$$\sigma_2 = \sigma_{yy}$$
$$\sigma_3 = 0.$$

Thus from Von-mises criteria,

$$2\sigma_{yy}^2 = 2\sigma_Y^2$$
$$\Rightarrow \sigma_{yy} = \sigma_Y.$$

$$x = \frac{K_I^2}{2\pi\sigma_Y^2}$$

For plane strain (for $\nu = 1/3$)

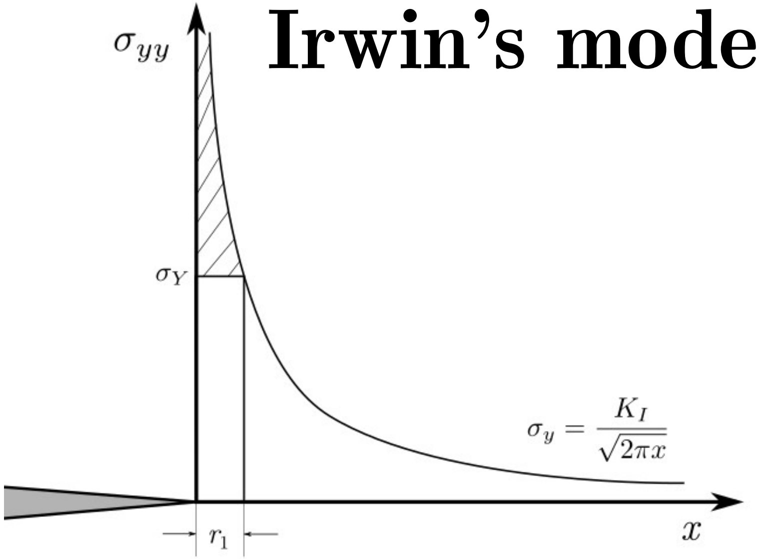
$$\sigma_1 = \sigma_{xx} (= \sigma_{yy})$$
$$\sigma_2 = \sigma_{yy}$$
$$\sigma_3 = 2\nu(\sigma_1 + \sigma_2) = \frac{4}{3}\sigma_{yy}$$

Thus from Von-mises criteria,

$$2\left(\frac{4}{3}\sigma_{yy} - \sigma_{yy}\right)^2 = 2\sigma_Y^2$$
$$\Rightarrow \sigma_{yy} = 3\sigma_Y.$$

$$x = \frac{K_I^2}{18\pi\sigma_Y^2}$$

Irwin's model for plastic zone calculation



Irwin presented a simplified model for the determination of the plastic zone at the crack tip under small-scale yielding.

His focus was only on the extent along the crack axis and not on the shape of the plastic zone, for an elastic-perfectly plastic material.

Consider the elastic distribution of the σ_y along the crack axis for plane stress case. The extent of the plastic zone in front of the crack can be determined using (4a) as,

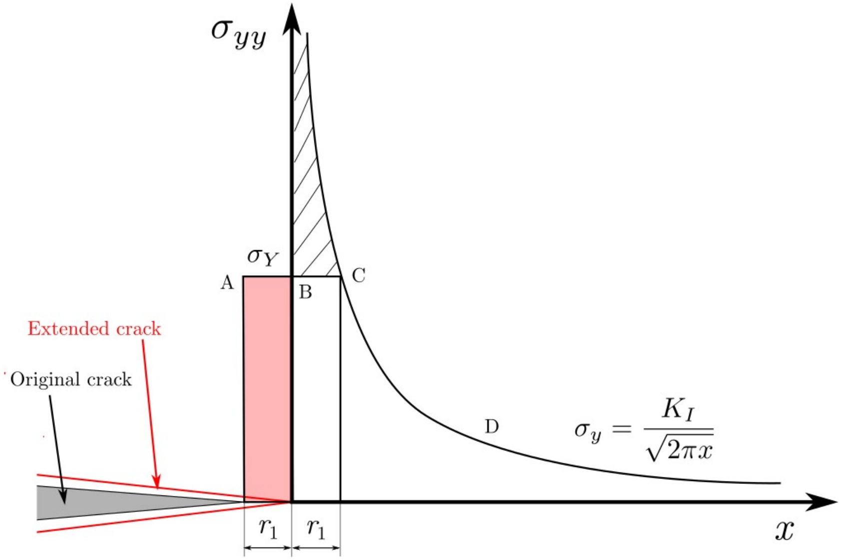
$$r_1 = \frac{K_I^2}{2\pi\sigma_Y^2}. \qquad \cdots\cdots\cdots\cdots(7)$$

Let us calculate the load equivalent to of the elastic stress distribution ahead of the crack-tip till distance r_1, which is

$$P = \int_0^{r_1} B\sigma_{yy}dx = \int_0^{r_1} B\frac{K_I}{\sqrt{2\pi x}}dx = B\frac{K_I^2}{\pi\sigma_Y} \qquad \cdots\cdots\cdots\cdots(8)$$

Now, If the stress upto a distance of r_1 is limited to constant yield stress of σ_Y then the equilibrium condition along the y-direction is violated and to maintain this equilibrium the load (8) must be maintained. We can rewrite (8) as,

$$P = B\frac{K_I^2}{\pi\sigma_Y} = B\left(\frac{K_I^2}{2\pi\sigma_Y^2}\right)(2\sigma_Y) = B(2\sigma_Y r_1). \qquad \text{(from 7)}$$



The result suggests that to satisfy equilibrium along the y-direction the original crack should be extended by a length r_1, so that, the σ_{yy} distribution is now represented by the curve ABCD, and the area under this curve is equal to the area underneath the σ_{yy} curve for earlier case. Thus, the length of the plastic zone c in front of the crack is

$$c = 2r_1 = \frac{K_I^2}{\pi\sigma_Y^2}. \qquad \text{..................(9)}$$

These observations led Irwin to propose that the effect of plasticity makes the plate behave as if it had a crack longer than the actual crack size and the fictitious crack length is $(a+r_1)$.

Irwin's correction to the plane strain case is useful to determine the plastic zone size. Due to the plastic deformation the crack tip becomes rounded. Since the rounded tip acts as a free surface, σ_{xx} is released to zero, which also affect σ_{xx} stress upto some distance on the x-axis beyond the crack tip. Irwin found that σ_{yy} stress is no longer 3 times the yield stress (σ_Y) it is closer to $2\sqrt{2} \approx \sqrt{3}$ times, thus the plastic zone size

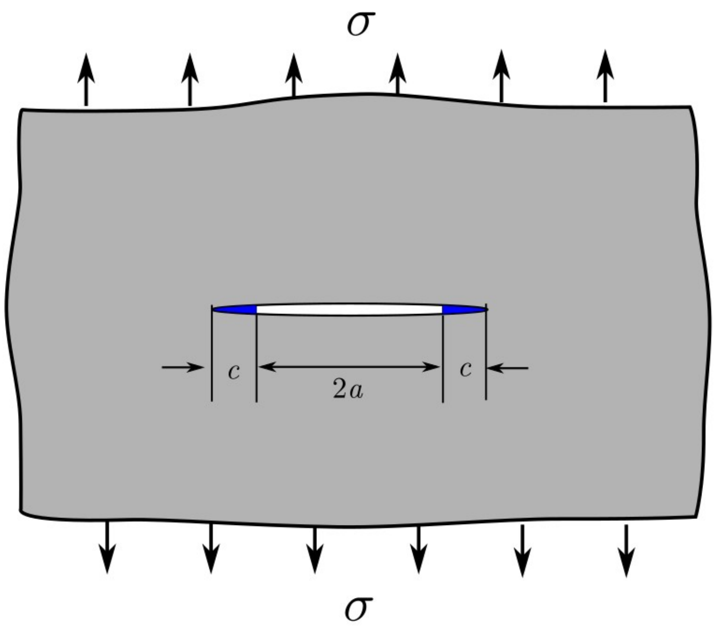
$$c = \frac{K_I^2}{3\pi\sigma_Y^2}. \qquad \text{.................}(10)$$

For an experimental determination of K_{IC} of material, plane strain conditions are assured by taking plate thickness to be much thicker than the plastic zone size.

According to the ASTM Standard E399 the stress condition is characterized as plane stress when $c = B$ and as plane strain when $c < B/25$, where B is the thickness of the plate. Therefore from (10) we obtain for plane strain case,

$$B > \frac{25K_I^2}{3\pi\sigma_Y^2} \approx \frac{2.5K_I^2}{\sigma_Y^2}. \qquad \text{.................}(11)$$

Dugdale's model



Dugdale (1960) and Barenblatt (1962) individually proposed a simplified method to determine the size of the plastic zone ahead of crack-tip. Following are the assumption.

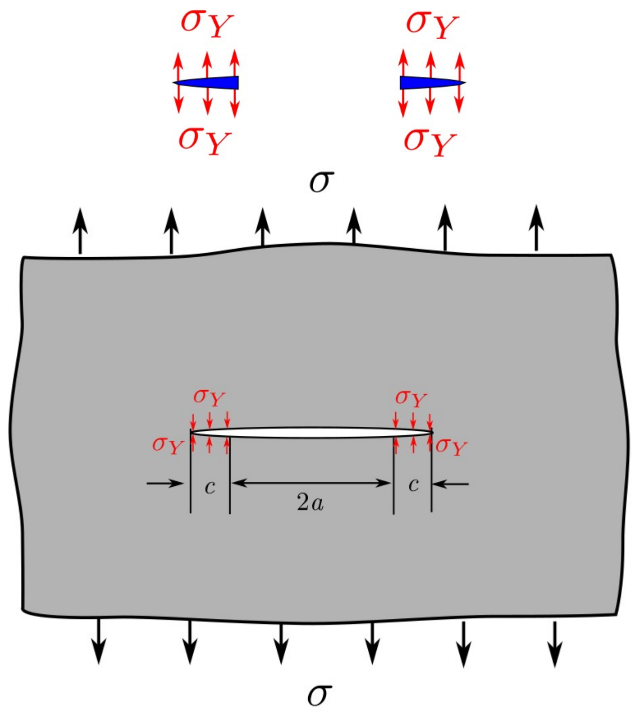
(i) Applicable to very thin plates (i.e., plane stress conditions).

(ii) Infinite plate having crack of length $2a$, and subjected to uniform tensile load σ.

(ii) Material is elastic-perfectly plastic and it obeys the Tresca yield criterion.

(iii) All plastic deformation concentrates in a line in front of the crack.

(iv) The crack has an effective length of $2(a+c)$, which exceeds that of the physical crack ($2a$) by the length of the plastic zone.



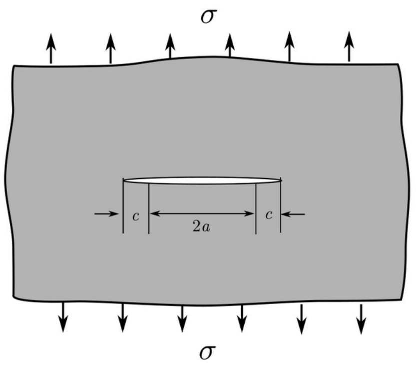
Isolate the strips of material which has gone under the plastic deformation froσm the plate. Stresses on these strips are the yield stress.

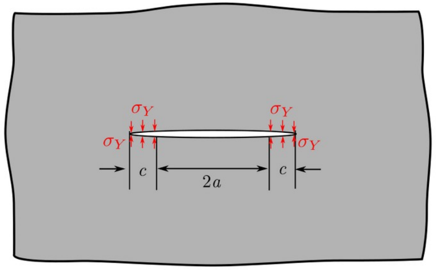
Equal and opposite pressure distribution will work on the plate from where the stripes have been isolated.

Thus we convert the elastic-plastic crack problem to a elastic problem having crack-length $2(a+c)$. The SIF for the problem will be the superpostion of the SIFs for the following problems.

(i) SIF for infinite plate having crack length $2(a+c)$ and subjected to tensile load σ (K_σ).

(ii) SIF for Infinite plate having crack length $2(a+c)$ and subjected with pressure σ_Y on the crack faces from length $\pm a$ to $\pm(a+c)$ ($K_{\sigma Y}$).





Thus, $$K = K_\sigma + K_{\sigma_Y}$$

$$K = \sigma\sqrt{\pi(a+c)} - \frac{2\sigma_Y\sqrt{\pi(a+c)}}{\pi}\cos^{-1}\frac{a}{a+c}$$

The effect of plasticity ahead of the crack-tip is to kill the singularity there. Hence, the K for the problem will be zero.

$$K = \sigma\sqrt{\pi(a+c)} - \frac{2\sigma_Y\sqrt{\pi(a+c)}}{\pi}\cos^{-1}\frac{a}{a+c} = 0$$

$$\Rightarrow \frac{a}{a+c} = \cos\frac{\pi\sigma}{2\sigma_Y}$$

$$\Rightarrow c = a\left(\sec\frac{\pi\sigma}{2\sigma_Y} - 1\right) \qquad \text{..................(12)}$$

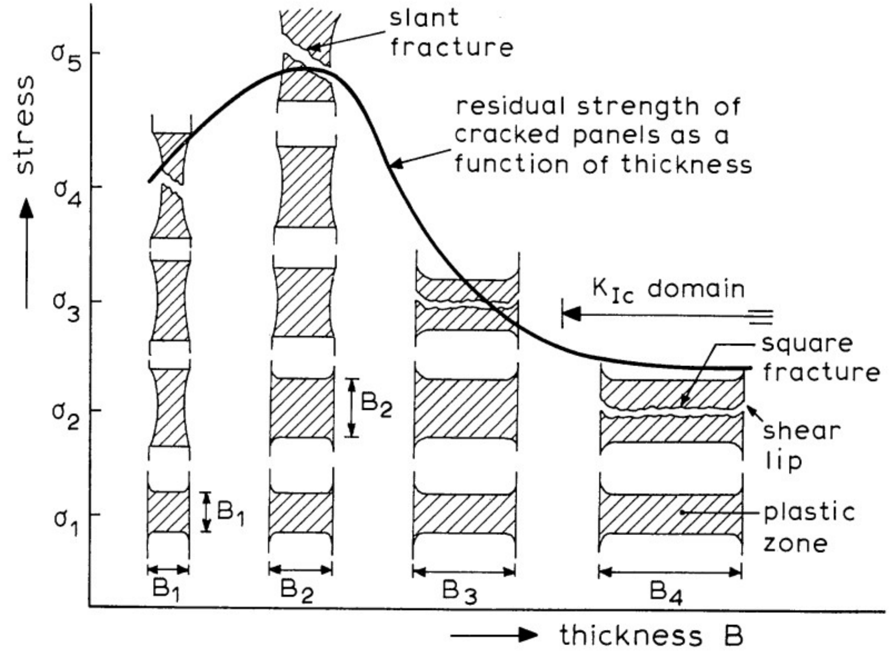
Equation (12) can be simplified to determine the size of plastic zone required to remove the singularity at the crack-tip. When $\sigma << \sigma_Y$ and $c << a$, then the above relation is approximated to

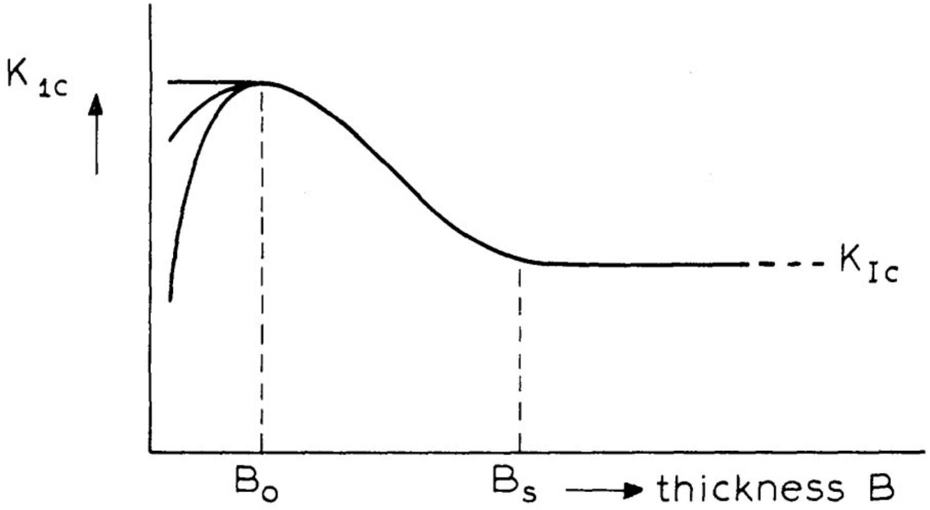
$$1 - \frac{c}{a} = 1 - \frac{\pi^2\sigma^2}{8\sigma_Y^2}$$

$$\Rightarrow c = \frac{\pi^2\sigma^2 a}{8\sigma_Y^2} = \frac{\pi K_I^2}{8\sigma_Y^2} \qquad \text{..................(13)}$$

Equation (13) overestimates the plastic zone size by about 20%, compared to Irwin model for $\sigma << \sigma_Y$.

Effect of plate thickness

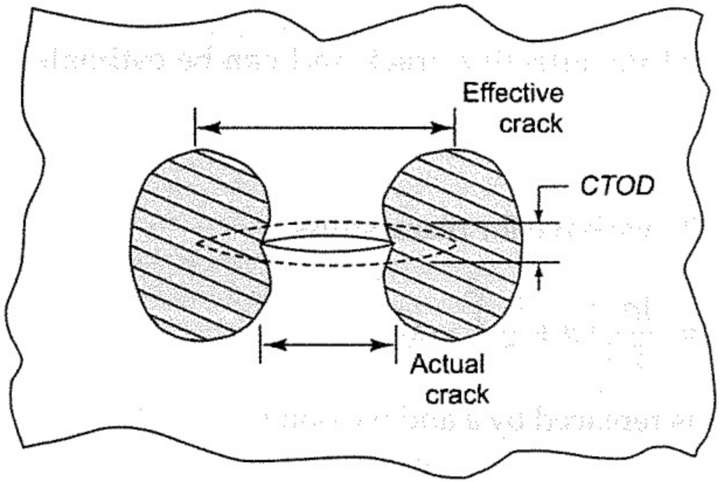


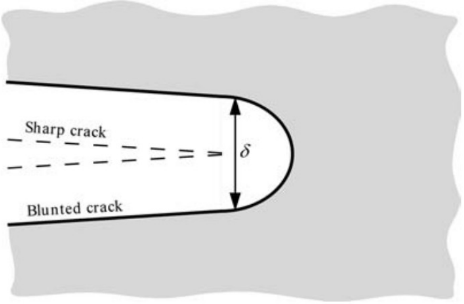


Source: Elementary Engineering Fracture Mechanics, David Broek

Crack tip opening displacement (CTOD)

Crack-tip opening displacement (CTOD) is another parameter suitable to characterize a crack. It can be used for both linear elastic fracture mechanics (LEFM) and elastic-plastic fracture mechanics (EPFM).





We have already seen that in the presence of plasticity the structure becomes more complaint and this effect can be simulated by considering an effective crack-length which is longer than the actual crack length.
With this assumption it can be observed that the tip of the actual crack in inside the effective crack and there is some opening at the actual crack-tip. The displacement of effective crack faces at the localtion of actual crack tip is called the crack-tip opening displacement (CTOD).
In reality, there is hardly any opening of the crack tip; only the tip may become more rounded as the plastic deformation increases, which is called the blunting of the crack-tip.

We have already derived the expression for COD for a center crack of length $2a$ under Mode-I loading as,

$$\text{COD} = \frac{4\sigma}{E}\sqrt{a^2 - x_1^2}, \qquad \text{(for plane stress)}$$

For effective crack length a_{eff}, COD at a distance of a from the center from the crack-tip is

$$\begin{aligned}\text{CTOD} &= \frac{4\sigma}{E}\sqrt{a_{\text{eff}}^2 - a^2} = \frac{4\sigma}{E}\sqrt{\left(a + \frac{r_p}{2}\right)^2 - a^2}, \\ &= \frac{4\sigma}{E}\sqrt{\frac{r_p^2}{4} + ar_p} \\ &\approx \frac{4\sigma}{E}\sqrt{ar_p} \quad \text{(for small } r_p\text{)} \qquad \cdots\cdots\cdots\cdots(14)\end{aligned}$$

From (9), length of plastic zone, $r_p = \dfrac{K_I^2}{\pi\sigma_Y^2} = \dfrac{\sigma^2\pi a}{\pi\sigma_Y^2} = \dfrac{\sigma^2 a}{\sigma_Y^2}$. (for plane stress)

Thus,

$$\text{CTOD} = \frac{4\sigma^2 a}{\sigma_Y E} = \frac{4K_I^2}{\pi\sigma_Y E} \qquad \text{(for plane stress)} \qquad \cdots\cdots\cdots\cdots(15)$$

If Dugdale model is used then,

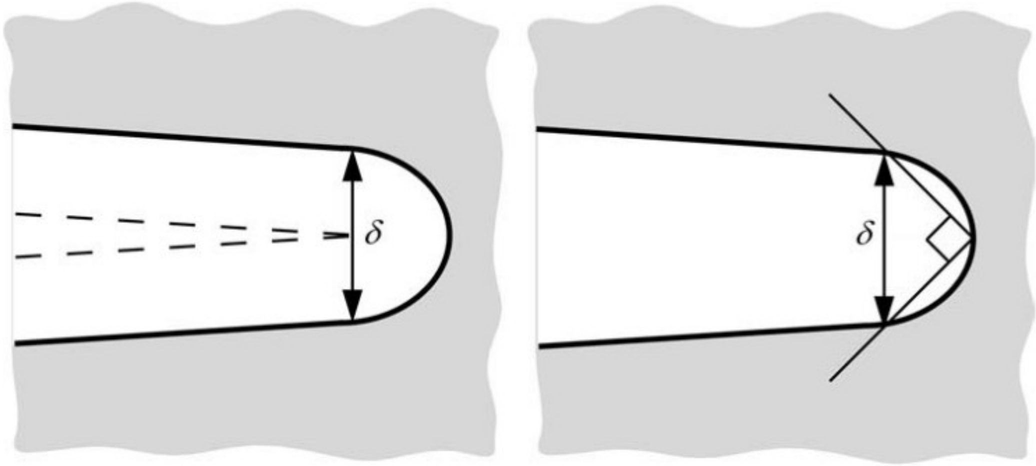
$$\text{CTOD} = \frac{4\sigma^2 a}{\sigma_Y E} = \frac{K_I^2}{\sigma_Y E} = \frac{G_I}{\sigma_Y}. \qquad \cdots\cdots\cdots\cdots(16)$$

Thus in general CTOD can be written as,

$$\text{CTOD} = \frac{K_I^2}{\lambda \sigma_Y E}, \qquad \cdots\cdots\cdots\cdots\cdots\cdots(17)$$

where value of λ depends upon the type of model used for finding the plastic zone size. From the direct experimental method it has been found that λ is close to unity.

The two most common method of measuring CTOD are shown, are the displacement at the original crack tip and the 90° intercept. Note that these two definitions are equivalent if the crack blunts in a semicircle.



Path independent integrals

Consider a elastic (linear or nonlinear), homogeneous, anisotropic solid body which is in a state of static equilibrium under the action of surface tractions. The body occupies a region Ω in the space which is bounded by surface $\partial\Omega$. For simplification we assume that the deformations remain small.

For elastic body the stress tensor can be obtained from the elastic strain energy ω as,

$$\sigma_{ij} = \frac{\partial w}{\partial \varepsilon_{ij}}, \quad w(0) = 0, \qquad \text{.................}(17)$$

where ε_{ij} are the components of small strain tensor. The strain energy density of the solid is

$$w = \int_0^{\varepsilon_{kl}} \sigma_{ij} d\varepsilon_{ij}. \qquad \text{.................}(18)$$

Note that this integral is path independent since the material is elastic.

In the absence of any body forces equilibrium equation and traction vector on surface are

$$\sigma_{ij,i} = \frac{\partial \sigma_{ij}}{\partial x_i} = 0, \quad t_j = \sigma_{ij} n_i, \qquad \text{.................}(19)$$

where n_i is the normal to the surface.

Now consider the surface integrals

$$Q_j = \int_{\partial\Omega} (w n_j - t_k u_{k,j})\, dS, \quad j = 1, 2, 3. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(20)$$

where $\partial\Omega$ is a closed surface bounding a region Ω which is assumed to be free of singularities. Substituting (19) in (20),

$$Q_j = \int_{\partial\Omega} (w n_j - \sigma_{lk} n_l u_{k,j})\, dS = \int_{\partial\Omega} (w\delta_{jl} - \sigma_{lk} u_{k,j})\, n_l dS. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(21)$$

Applying Gauss divergence theorem,

$$Q_j = \int_{\Omega} \frac{\partial}{\partial x_l} (w\delta_{jl} - \sigma_{lk} u_{k,j})\, dV = \int_{\Omega} (w\delta_{jl} - \sigma_{lk} u_{k,j})_{,l}\, dV. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(22)$$

We look at the integrand,

$$\begin{aligned}(w\delta_{jl} - \sigma_{lk} u_{k,j})_{,l} &= w_{,l}\delta_{jl} - \cancelto{0}{\sigma_{lk,l} u_{k,j}} - \sigma_{lk} u_{k,jl} = w_{,j} - \sigma_{lk} u_{k,jl} \\ &= \frac{\partial w}{\partial \varepsilon_{lk}} \varepsilon_{lk,j} - \sigma_{lk} u_{k,jl} = \sigma_{lk}\varepsilon_{lk,j} - \sigma_{lk} u_{k,jl} \\ &= \sigma_{lk} (\varepsilon_{lk,j} - u_{k,jl}) = \sigma_{lk} (\varepsilon_{lk} - u_{k,l})_{,j}\end{aligned}$$

$$
\begin{aligned}
(w\delta_{jl} - \sigma_{lk}u_{k,j})_{,l} &= \sigma_{lk}\left[\frac{1}{2}(u_{l,k} + u_{k,l}) - u_{k,l}\right]_{,j} \\
&= \sigma_{lk}\frac{1}{2}(u_{l,k} - u_{k,l})_{,j} = \sigma_{lk}\omega_{lk,j} = 0
\end{aligned}
$$

(Since, $\boldsymbol{\omega}$ is an antisymmetric tensor and contraction of a symmetric and anti-symmetric tensor is zero).

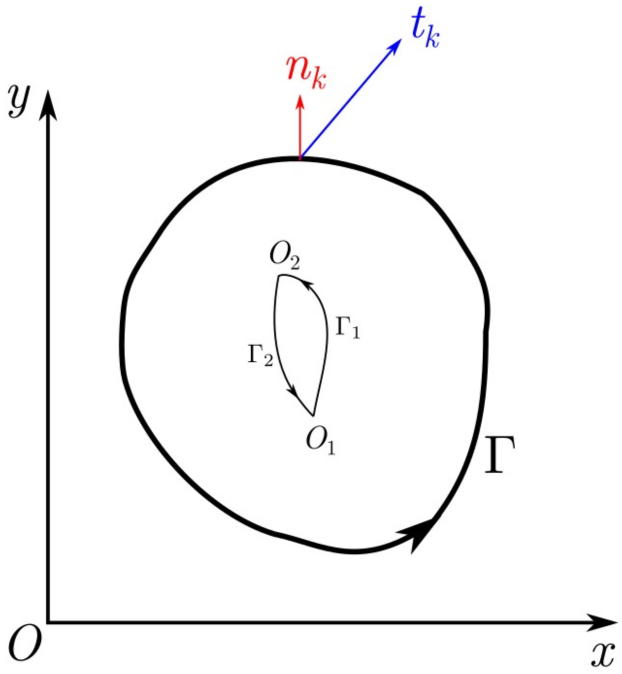
Thus,

$$
Q_j = 0. \qquad \text{.................}(23)
$$

i.e., the integral (20) is equal to zero for any surface $\partial\Omega$.

J-integral

For the particular case of the two-dimensional plane elastic problem, consider the integral



$$J = Q_1 = \int_\Gamma \left(w n_1 - t_k \frac{\partial u_k}{\partial x} \right) dS, \quad \text{where} \quad n_1 = dy/dS,$$

Thus,

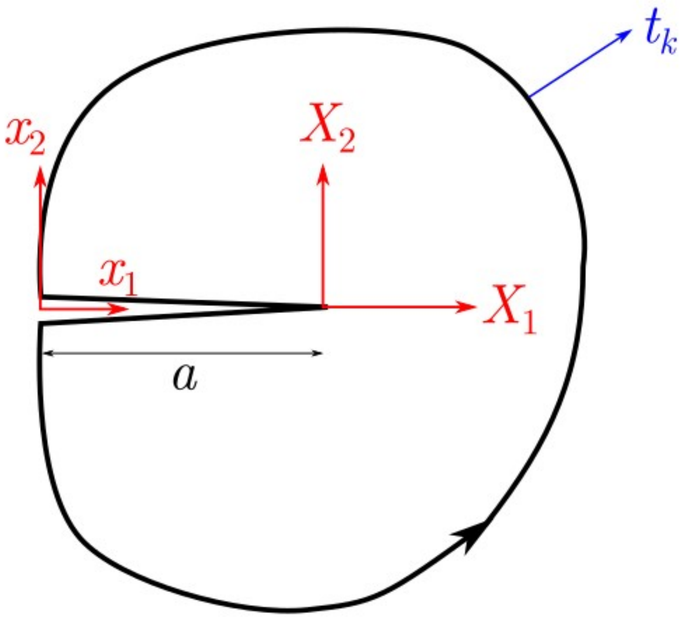
$$J = \int_\Gamma w dy - t_k \frac{\partial u_k}{\partial x} dS, \quad (k = 1, 2) \quad \cdots\cdots\cdots\cdots\cdots\cdots (24)$$

Equation (24) defines the J-integral along a closed contour in the two-dimensional space. From (23) it follows that $J = 0$.

J=0 for any other closed integral which does not include singularity within. Hence,

$$J = J_{\Gamma_1} + J_{\Gamma_2} = 0 \Rightarrow J_{\Gamma_1} = -J_{\Gamma_2}.$$

Relationship between J and G



Consider a linear or nonlinear elastic plane body, with a crack of length a subjected to prescribed tractions and displacements along parts of its boundary. Tractions and displacements are assumed to be independent of crack length. The body is referred to a fixed system of Cartesian coordinates x_1-x_2 with the x_1-axis parallel to the crack faces. It is also assumed that the crack extends in a self-similar manner.

The potential energy of the body is

$$\Pi(a) = \int_A w dA - \int_\Gamma t_k u_k ds,$$

where A is the area of the region and Γ is its boundary.

$$\frac{d\Pi(a)}{da} = \int_A \frac{dw}{da} dA - \int_\Gamma t_k \frac{du_k}{da} ds, \qquad \text{.................}(25)$$

We now use a new coordinate system X_1-X_2 attached to the crack tip is introduced,

$$X_1 = x_1 - a,\; X_2 = x_2.$$

Any function which depends upon a and X_1,

$$\frac{d}{da} = \frac{\partial}{\partial a} + \frac{\partial}{\partial X_1}\frac{\partial X_1}{\partial a} = \frac{\partial}{\partial a} - \frac{\partial}{\partial x_1} \qquad \because \left(\frac{\partial}{\partial X_1} = \frac{\partial}{\partial x_1} \quad \text{and} \quad \frac{\partial X_1}{\partial a} = -1\right)$$

Thus from (25),

$$\frac{d\Pi(a)}{da} = \int_A \left(\frac{\partial w}{\partial a} - \frac{\partial w}{\partial x_1}\right) dA - \int_\Gamma t_k \left(\frac{\partial u_k}{\partial a} - \frac{\partial u_k}{\partial x_1}\right) ds, \qquad \cdots\cdots\cdots\cdots(26)$$

Look at the term,

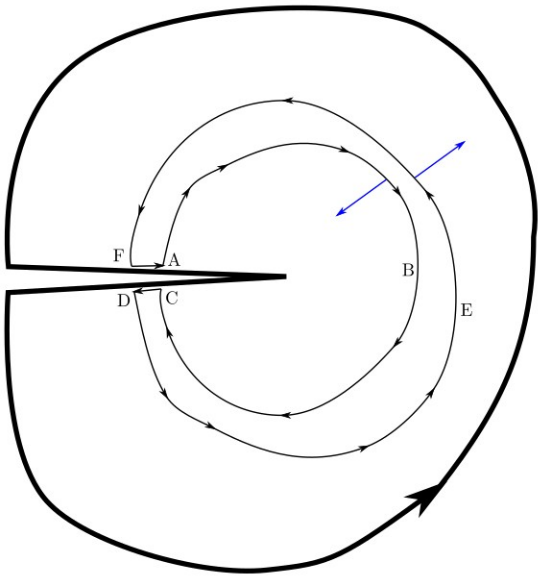
$$\begin{aligned}\int_A \frac{\partial w}{\partial a} dA &= \int_A \frac{\partial w}{\partial \varepsilon_{ij}}\frac{\partial \varepsilon_{ij}}{\partial a} dA = \int_A \sigma_{ij}\frac{\partial \varepsilon_{ij}}{\partial a} dA = \frac{1}{2}\int_A \left(\sigma_{ij}\frac{\partial u_i}{\partial x_j \partial a} + \sigma_{ij}\frac{\partial u_j}{\partial x_i \partial a}\right) dA \\ &= \frac{1}{2}\int_A \left(\sigma_{ij}\frac{\partial u_i}{\partial x_j \partial a} + \sigma_{ji}\frac{\partial u_j}{\partial x_i \partial a}\right) dA = \int_A \sigma_{ij}\frac{\partial u_i}{\partial x_j \partial a} dA \\ &= \int_A \frac{\partial}{\partial x_j}\left(\sigma_{ij}\frac{\partial u_i}{\partial a}\right) - \cancelto{0}{\frac{\partial \sigma_{ij}}{\partial x_j}}\frac{\partial u_i}{\partial a} dA = \int_\Gamma \sigma_{ij}\frac{\partial u_i}{\partial a} n_j ds = \int_\Gamma t_i \frac{\partial u_i}{\partial a} ds\end{aligned}$$

Hence from (26),

$$-\frac{d\Pi(a)}{da}=\int_A \frac{\partial w}{\partial x_1}dA-\int_\Gamma t_k\frac{\partial u_k}{\partial x_1}ds=\int_\Gamma\left(wn_1-t_k\frac{\partial u_k}{\partial x_1}\right)ds=J \quad \text{from (24).}$$

Thus, for any contour surrounding the crack-tip

$$J=-\frac{d\Pi(a)}{da}=G. \qquad \text{.................(27)}$$



Now consider another contour ABCDEF as shown in the figure. This contour is draws such that it does not include the crack-tip (hence, no singularities inside the contour). For such case we have already derived that $J=0$. Thus,

$$J_{\text{ABCDEF}}=J_{\text{ABC}}+J_{\text{CD}}+J_{\text{DEF}}+J_{\text{FA}}=0.$$

Note that $J_{\text{FA}}=-\ J_{\text{DC}}$.
Hence,

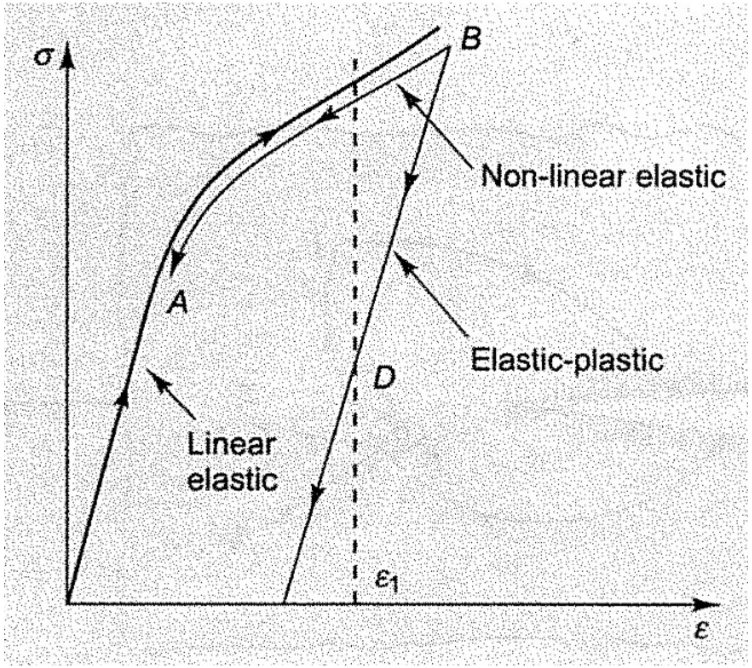
$$J_{\text{ABC}}+J_{\text{DEF}}=0 \text{ or } J_{\text{ABC}}=J_{\text{FED}}.$$

The J-integral is independent of the contour, and it can be calculated for any contour taken inside the material.

Since J is equal to G, hence J is very popular candidate for fracture criteria. For crack initiation $J \geq J_C$ $(= G_C)$.

Note that the J-integral is derived for elastic (linear or nonlinear) material response, for which strain energy function w is defined.

Attempts have been made to extend the realm of applicability of the J-integral fracture criterion to ductile fracture where extensive plastic deformation and possibly stable crack growth precede fracture instability.

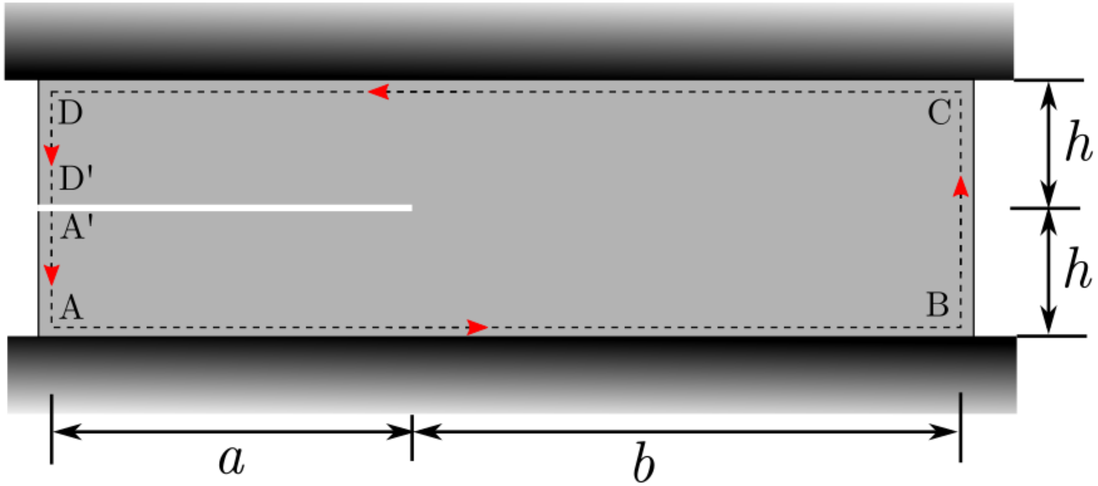


In fact, the presence of plastic zone nullifies the path independence property of the J-integral. For any closed path surrounding the crack tip and taken entirely within the plastic zone or within the elastic zone, the necessary requirements for path independence are not satisfied.

In an effort to establish path independence for the J-integral the deformation theory of plasticity has been invoked. This theory is a nonlinear elasticity theory, and no unloading is permitted. Any solution based on the this theory remains valid under proportional loading (the stress components change in fixed proportion to one another).

No unloading is permitted at any point of the plastic zone. Although, strictly speaking, the condition of proportional loading is not satisfied in practice.

It is argued that in a number of stationary problems, under a single monotonically applied load, the loading condition is close to proportionality. This argument has been supported by finite element simulations.



Consider an infinite strip of height $2h$ with a semi-infinite crack is rigidly clamped along its upper and lower faces at $y = \pm h$. The upper and lower faces are moved in the positive and negative y-direction over distances u_0, respectively.

Let us determine the value of the J-integral and the stress intensity factor.

Consider the path A$'$ABCDD$'$ along the upper and lower surfaces of the strip up to infinity and traversing the strip perpendicularly to the crack. J is calculated from

$$J_{\mathrm{A'ABCDD'}} = J_{\mathrm{A'A}} + J_{\mathrm{AB}} + J_{\mathrm{BC}} + J_{\mathrm{CD}} + J_{\mathrm{DD'}}$$

Remember the expression for J

$$J = \int_{\Gamma} w dy - t_1 \frac{\partial u_1}{\partial x} - t_2 \frac{\partial u_2}{\partial x} dS.$$

Let us evaluate J for every segment of this contour.

For segment AB and CD, we have

$$dy = 0 \quad \text{and } du_{1,2}/dx = 0,$$

Hence, $J_{\text{AB}} = J_{\text{CD}} = 0$.

For segment DD' and AA',

$$w = 0 \quad \text{and } du_{1,2}/dx = 0,$$

Hence, $J_{\text{DD'}} = J_{\text{AA'}} = 0$.

For segment BC,

$$\varepsilon_y = u_0/h, \sigma_y = \beta E\epsilon_y = \beta E u_0/h,$$

$$u_{1,2}/dx = 0,$$

Hence,

$$J = \int_{-h}^{h} w|_{x\to\infty} dy = \int_{-h}^{h} \frac{1}{2}\sigma_y \varepsilon_y dy.$$

where, $\beta = 1/(1-\nu^2)$ (for plane stress), and $\beta = (1-\nu)/[(1+\nu)(1-2\nu)]$ (for plane strain).

Thus, $J = \dfrac{\beta E u_0^2}{h}$ $K_I = \sqrt{\dfrac{JE}{\eta}}$. ($\eta = 1$ for plane stress and $\eta = (1-\nu^2)$ for plane strain)

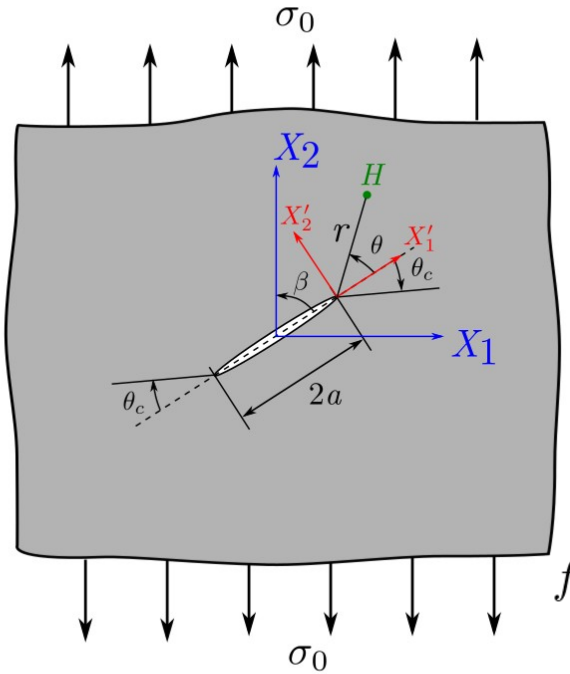
Mixed-mode crack growth

When direction of load is not aligned with the orientation of the crack, the crack-tip stress fields are governed by a combination of stress intensity factors K_{I}, K_{II} and K_{III}. The direction of crack initiation also depends on a failure criterion which is a function of K_{I}, K_{II} and K_{III}, resulting in a curved crack path. We will discuss the following criteria:

(a) Maximum Tangential Stress (MTS) criterion

(b) Strain Energy Density (SED) criterion

MTS criterion



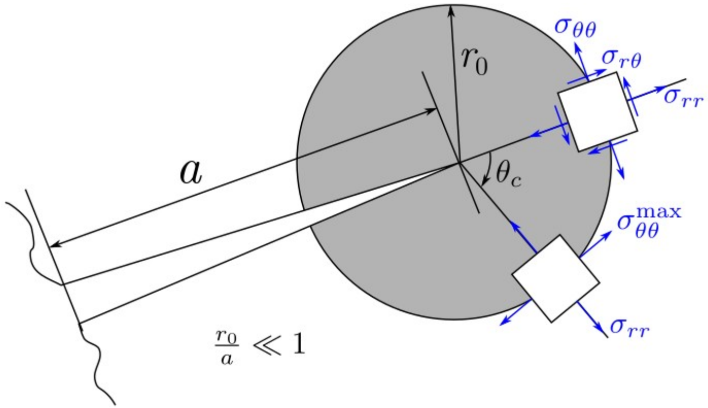
This criterion was proposed by Erdogan and Sih in 1963.

$$\sigma_{rr} = K_{\mathrm{I}} f_{11}(r,\theta) + K_{\mathrm{II}} f_{12}(r,\theta)$$
$$\sigma_{\theta\theta} = K_{\mathrm{I}} f_{21}(r,\theta) + K_{\mathrm{II}} f_{22}(r,\theta)$$
$$\sigma_{r\theta} = K_{\mathrm{I}} f_{31}(r,\theta) + K_{\mathrm{II}} f_{32}(r,\theta)$$

$$f_{11} = \frac{1}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right), \quad f_{22} = \frac{1}{\sqrt{2\pi r}}\left(-\frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2}\right)$$
$$f_{12} = \frac{1}{\sqrt{2\pi r}}\left(-\frac{5}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right), \quad f_{31} = \frac{1}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right)$$
$$f_{21} = \frac{1}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right), \quad f_{32} = \frac{1}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right)$$

According to MTS criterion, crack extension will occur in the direction where tangential stress component $\sigma_{\theta\theta}$ at an infinitesimal radial distance r_0 from the crack tip is maximum and the extension will take place when the maximum tangential stress reaches a critical value which is a material dependent parameter.

Crack extension direction:



$$\sigma_{\theta\theta} = \frac{K_{\mathrm{I}}}{4\sqrt{2\pi r}}\left(3\cos\frac{\theta}{2} + \cos\frac{3\theta}{2}\right) - \frac{3K_{\mathrm{II}}}{4\sqrt{2\pi r}}\left(\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right)$$

$$\frac{\partial\sigma_{\theta\theta}}{\partial\theta} = 0$$

$$\frac{\partial^2\sigma_{\theta\theta}}{\partial\theta^2} < 0$$

$$K_{\mathrm{I}}\sin\theta_c + K_{\mathrm{II}}\left(3\cos\theta_c - 1\right) = 0$$

Note that at $\theta = \theta_c$ shear stress $\tau_{r\theta}$ become zero; hence σ_{rr} and $\sigma_{\theta\theta}$ become principle stresses in that direction.

Critical condition:

$$\sigma_{\theta\theta}^{\max} = (\sigma_{\theta\theta})_{\theta=\theta_c}$$

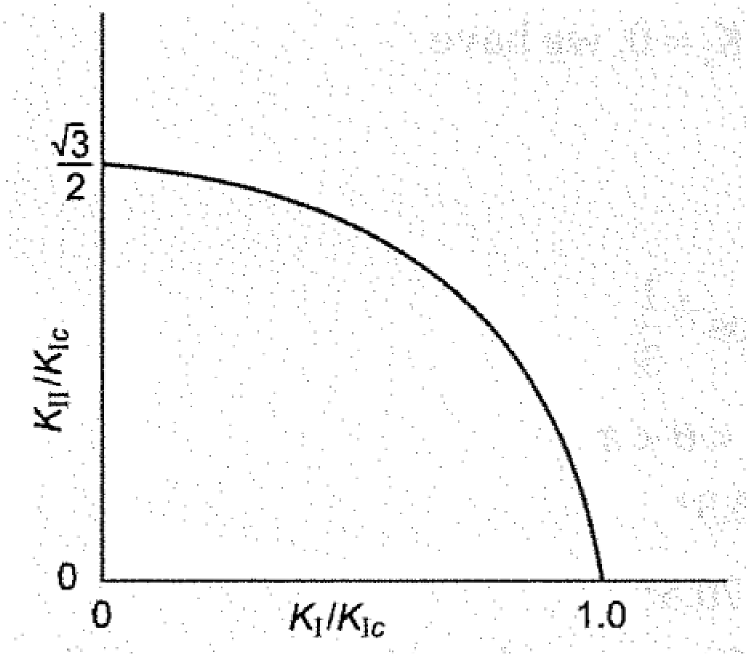
$$\sigma_{\theta\theta}^{\max} = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r_0}} \cos^3\frac{\theta_c}{2} - \frac{3K_{\mathrm{II}}}{2\sqrt{2\pi r_0}} \cos\frac{\theta_c}{2} \sin\theta_c$$

Crack extension occurs when $\sigma_{\theta\theta}^{\max}$ reaches a critical value σ_c which is a material property; σ_c is usually obtained from pure Mode I loading where $\theta_c = 0$ and $K_{\mathrm{I}} = K_{\mathrm{Ic}}$, that is,

$$\sigma_c = \frac{K_{\mathrm{Ic}}}{\sqrt{2\pi r_0}}$$

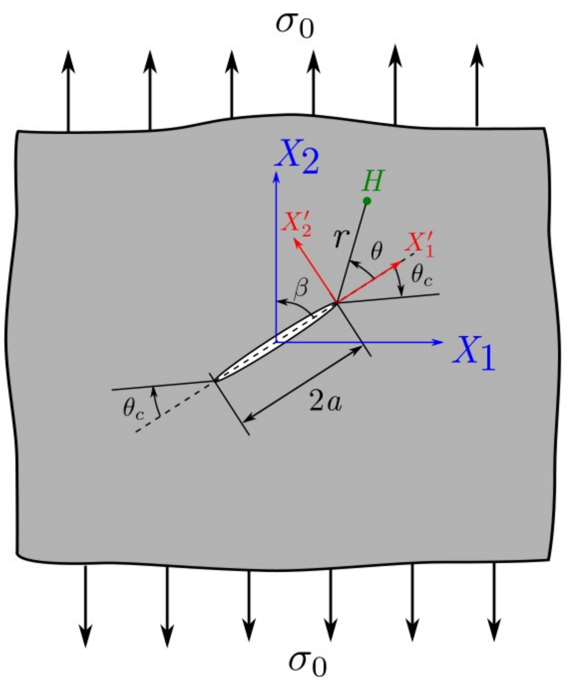
Failure will occur when, $\sigma_{\theta\theta}^{\max} = \sigma_c$, which gives the equation of failure surface as,

$$K_{\mathrm{Ic}} = K_{\mathrm{I}} \cos^3\frac{\theta_c}{2} - \frac{3}{2}K_{\mathrm{II}} \cos\frac{\theta_c}{2} \sin\theta_c$$



SED criterion

Based on energy principles, Sih in 1973 proposed Strain Energy Density (SED) criterion. Consider a crack subjected to Modes I and II loading. Strain energy density is defined as the strain energy per unit volume at a given point in solid, which can be obtained from stress and strain field as:



$$W = \frac{\partial U}{\partial V} = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij}.$$

For plane linear elasticity problems W can be written in the following form in terms of stress components

$$W = \frac{1+\nu}{2E}\left[\frac{\kappa+1}{4}\left(\sigma_{11}+\sigma_{22}\right)^2 - 2\left(\sigma_{11}\sigma_{22}-\sigma_{12}\right)^2\right]$$

where,

$$\begin{aligned} \kappa &= 3-4\nu && \text{(for plane strain)} \\ &= (3-\nu)/(1+\nu) && \text{(for plane stress)} \end{aligned}$$

The Cartesian stress components in the vicinity of crack tip in terms of polar coordinate system are given as:

$$\sigma_{11} = K_{\mathrm{I}} f_{11}(r,\theta) + K_{\mathrm{II}} f_{12}(r,\theta)$$
$$\sigma_{22} = K_{\mathrm{I}} f_{21}(r,\theta) + K_{\mathrm{II}} f_{22}(r,\theta)$$
$$\sigma_{12} = K_{\mathrm{I}} f_{31}(r,\theta) + K_{\mathrm{II}} f_{32}(r,\theta)$$

$$f_{11} = \frac{1}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right), \qquad f_{22} = \frac{1}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2},$$
$$f_{12} = -\frac{1}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\left(2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right), \qquad f_{31} = \frac{1}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2},$$
$$f_{21} = \frac{1}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right), \qquad f_{32} = \frac{1}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right).$$

Strain energy density function can be written in terms of stress field as,

$$W = \frac{1}{\pi r}\left[g_{11}K_{\mathrm{I}}^2 + 2g_{12}K_{\mathrm{I}}K_{\mathrm{II}} + g_{22}K_{\mathrm{II}}^2\right]$$

$$g_{11} = \frac{1}{16\mu}(1+\cos\theta)(\kappa - \cos\theta),$$
$$g_{12} = \frac{1}{16\mu}\sin\theta\left[2\cos\theta - (\kappa - 1)\right],$$
$$g_{22} = \frac{1}{16\mu}\left[(\kappa+1)(1-\cos\theta) + (1+\cos\theta)(3\cos\theta - 1)\right],$$

Strain energy density function poses singularity of order one at the crack tip. Sih proposed a strain energy density factor S in a quadratic form which is independent of the coordinate r and it is defined as:

$$S(\theta) = \frac{1}{\pi}\left(g_{11}K_{\mathrm{I}}^2 + 2g_{12}K_{\mathrm{I}}K_{\mathrm{II}} + g_{22}K_{\mathrm{II}}^2\right)$$

According to SED criterion, crack extension will occur in the direction of minimum strain energy density $S(\theta)$ and the extension will occur when the $S(\theta)$ reaches a critical value S, which is a material property.

Crack extension direction:

$$\left(\frac{\partial W}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial^2 W}{\partial \theta^2} > 0\right) \qquad \text{or} \qquad \left(\frac{\partial S}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial^2 S}{\partial \theta^2} > 0\right)$$

These conditions leads to the following conditions which yields the direction of crack initiation,

$$[2\cos\theta - (\kappa - 1)]\sin\theta\; K_{\mathrm{I}}^2 + 2\left[2\cos 2\theta - (\kappa - 1)\cos\theta\right] K_{\mathrm{I}}K_{\mathrm{II}} + \left[(\kappa - 1 - 6\cos\theta)\sin\theta\right] K_{\mathrm{II}}^2 = 0,$$
$$[2\cos 2\theta - (\kappa - 1)\cos\theta]\;\; K_{\mathrm{I}}^2 + 2\left[(\kappa - 1)\sin\theta - 4\sin 2\theta\right] K_{\mathrm{I}}K_{\mathrm{II}} + \left[(\kappa - 1)\cos\theta - 6\cos 2\theta\right] K_{\mathrm{II}}^2 > 0.$$

Critical condition:

Crack extension will occur when minimum value of strain energy density function (S_{min}) reaches a critical value of strain energy density factor S_c Thus, the condition is expressed as:

$$(S_{\min}) \geq S_c$$

S_c is usually obtained from pure Mode I loading where $\theta_c = 0$ and $K_{\mathrm{I}} = K_{\mathrm{Ic}}$, that is,

$$S_c = \frac{(1+\nu)(\kappa-1)}{4\pi E} K_{\mathrm{Ic}}^2$$

Exercise:

1. Find out the stress intensity factor at crack initiation and direction of crack initiation for pure mode-I and pure mode-II loading

(a) Using MTS criterion

(b) Using SED criterion

2. Consider an infinite plate with a crack of length $2a = 80$ mm, inclined at angle β with the applied tensile stress σ_0. K_{Ic} of the material is known to be 40 MPa.√m, its elastic constants are $E = 200$ GPa and $\nu = 0.3$, and the plate is subjected to plane strain.

(i) Determine initial crack extension direction using MTS and SED fracture criteria for $\beta = 60°$,
(ii) find the applied stress σ_0 corresponding to the crack initiation using MTS and SED fracture criteria for $\beta = 60°$ and,

(iii) determine relations θ_c vs. β and critical σ_0 vs. β for both fracture criteria for β varying between 10° and 90°.

Crack propagation