ME231: Solid Mechanics-I

Stress, Strain and Temperature relationship

Thermal strains

Temperature may affect the elastic region of a material in two ways: (i) by modifying in the values of the elastic constants, and (ii) second, by directly producing a strain even in the absence of stress.

As a part of this course, we will not be interested in the first effect, as change in the elastic constants for many materials is small for a temperature change of about 100°C.

Second effect is of our interest now. The strain due to temperature change in the absence of stress is called thermal strain and is denoted by the superscript t on the strain symbol. For an isotropic material, using symmetry arguments it can be shown that the thermal strain must be a pure expansion or contraction with no shear-strain components referred to any set of axes.

The thermal strains are not exactly linear with temperature change, but for many engineering material thermal strains can be approximated as linear variation within temperature changes of 30-100°C .

Thus, thermal strains due to a change in temperature from T_0 to T is

$$\begin{aligned} \epsilon_x^t = \epsilon_y^t = \epsilon_z^t &= \alpha(T - T_0), \\ \gamma_{xy}^t = \gamma_{yz}^t = \gamma_{zx}^t &= 0. \end{aligned} \qquad \cdots\cdots\cdots\cdots(20)$$

Here, α is known as the coefficient of linear expansion. It has a unit of 1/temperature.

The total strain at a point in an elastic body is the sum of that due to stress and that due to temperature. Denoting the elastic strain due to stress by superscript e and the thermal part by superscript t , the total strain derived from the displacements is given by

$$\epsilon = \epsilon^e + \epsilon^t \qquad \cdots\cdots\cdots\cdots(21)$$

For example, consider a material rigidly restrained between supports so that no strain is possible, then

$$\epsilon = \epsilon^e + \epsilon^t = 0 \quad \text{or} \quad \epsilon^e = -\epsilon^t, \qquad \cdots\cdots\cdots\cdots(22)$$

i.e., elastic part of the strain will be the negative of the thermal strain.

Complete equations of Elasticity

- The theory of elasticity is the subject dealing with the distribution of stress and strain in elastic bodies subjected to given loads, displacements, and distributions of temperature. Now we are in a position to state completely the foundations of the theory of elasticity.
- The problem is to find distributions of stress and strain which
    - meet the prescribed loads and displacements on the boundary and which
    - satisfy the equilibrium equations, the stress-strain-temperature relations, and the geometrical conditions associated with the definition of strain and the concept of continuous displacements at every point.
- All equations which are required to be satisfied at each point of a nonaccelerating, isotropic, homogeneous, linear-elastic body subject to small strains.

Equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0,$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0,$$
$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + Z = 0.$$

where X, Y, and Z are body forces which are distributed over the volume with intensities X, Y, and Z per unit volume.

............(23)

Geometric compatibility

The displacements must match the geometrical boundary conditions and must be continuous functions of position with which the strain components are associated, as follows:

$$\epsilon_x = \frac{\partial u}{\partial x}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$
$$\epsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}$$
$$\epsilon_z = \frac{\partial w}{\partial z}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

where u, v, and w are the displacement components in the x, y, and z-directions.

............(24)

Stress – Strain – Temperature relationship

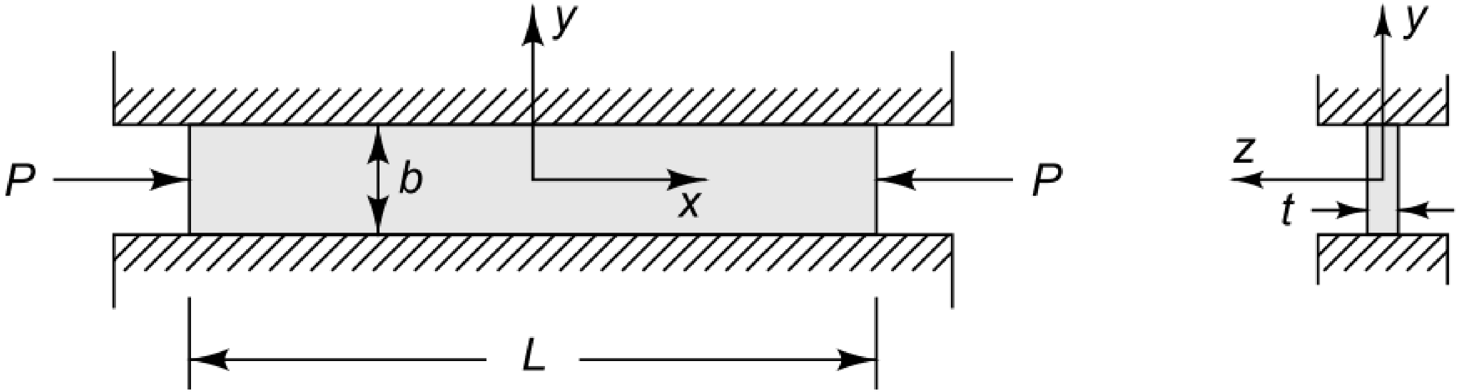
$$
\begin{aligned}
\epsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] + \alpha(T - T_0), \quad & \gamma_{xy} &= \frac{\tau_{xy}}{G}, \\
\epsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})\right] + \alpha(T - T_0), \quad & \gamma_{yz} &= \frac{\tau_{yz}}{G}, \\
\epsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right] + \alpha(T - T_0), \quad & \gamma_{xz} &= \frac{\tau_{xz}}{G}.
\end{aligned}
\qquad \text{............}(25)
$$

Equations (23) – (25) are total 15 equations for the six components of stress, the six components of strain, and the three components of displacement. These 15 equations are the foundation for what is commonly called linear elasticity theory.

However, in this course we will be solving simplified problem, which will evolve only few of these equations.

Example 1

A long, thin plate of width b, thickness t, and length L is placed between two rigid walls a distance b apart and is acted on by an axial force P, as shown in Figure. We wish to find the deflection of the plate parallel to the force P.



Idealization:

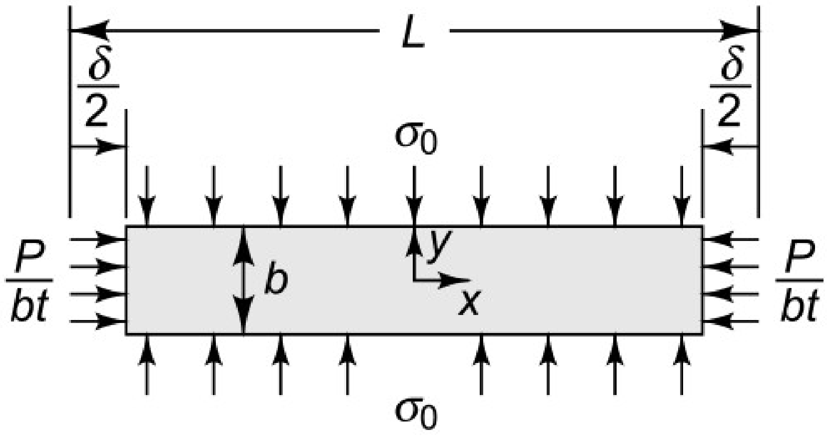
1. Uniform axial stress in x-direction because of applied load P

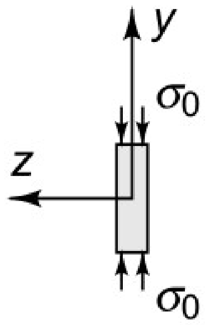
2. No normal stress in z-direction, as plate is very thin in that direction (plane stress in xy-plane)

3. No deformation in y-direction. (Plane strain in yz-plane)

4. Uniform distribution of contact stress between plate and rigid walls.

5. Frictionless contact





Equilibrium:

Equilibrium with external load will be satisfied when stresses are

$$\sigma_{xx} = -\frac{P}{bt}, \quad \sigma_{yy} = -\sigma_0, \quad \sigma_{zz} = 0,$$

$$\tau_{xy} = \tau_{xz} = \tau_{yz} = 0. \qquad \text{................(1.a)}$$

It can be verified that these stresses satisfies equilibrium equations.

Geometric compatibility:

Because of rigid walls, $\epsilon_{yy} = 0$, and $\epsilon_{xx} = -\dfrac{\delta}{L}$ ....(1.b)

Stress-strain relations:

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$$\epsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}), \quad \gamma_{xy} = 0,$$

$$\epsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx}), \quad \gamma_{yz} = 0, \qquad \text{................(1.c)}$$

$$\epsilon_{zz} = \frac{-\nu}{E}(\sigma_{xx} + \sigma_{yy}), \quad \gamma_{xz} = 0.$$

$$-\frac{\delta}{L} = \frac{1}{E}\left(-\frac{P}{bt} + \nu\sigma_0\right), \quad \text{................(1.d)}$$

$$0 = \frac{1}{E}\left(-\sigma_0 + \nu\frac{P}{bt}\right), \Rightarrow \sigma_0 = \frac{\nu P}{bt} \quad \text{................(1.e)}$$

$$\epsilon_{zz} = \frac{-\nu}{E}\left(-\frac{P}{bt} - \sigma_0\right). \quad \text{................(1.f)}$$

Using results from (1.e) to (1.d) and (1.f), we get

$$\delta = (1-\nu^2)\frac{PL}{Ebt} \quad \text{................(1.g)}$$

$$\epsilon_{zz} = \nu(1+\nu)\frac{P}{Ebt} = \frac{\nu}{1-\nu}\frac{\delta}{L}. \quad \text{................(1.h)}$$

From strain-displacement relations, we can find

$$u = -\frac{\delta}{L}x, \quad v = 0, \quad w = -\frac{\nu}{1-\nu}\frac{\delta}{L}z.$$