ME632: Fracture Mechanics

Timings	
Monday	10:00 to 11:20
Thursday	08:30 to 09:50

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Room No. # 106

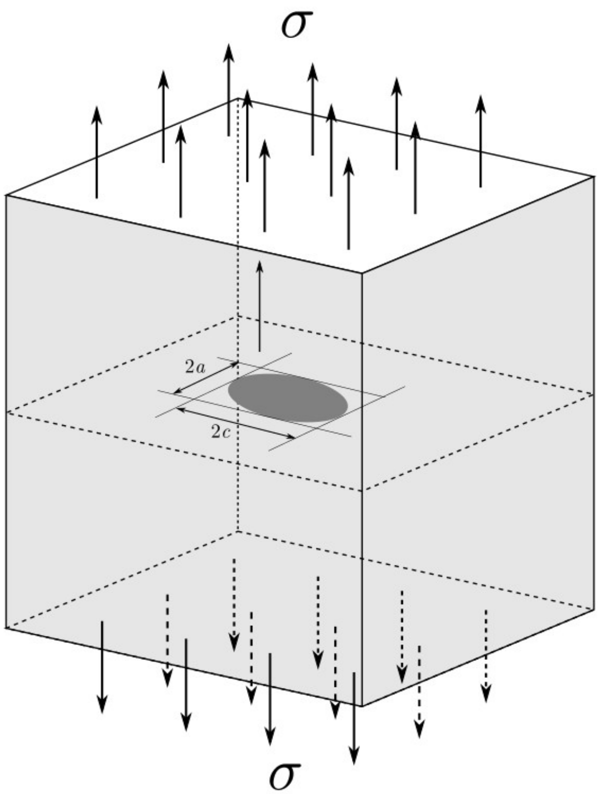
Critical stress intensity factor (SIF)

- So we have seen that the stress intensity factor (SIF) defines the amplitude of singularity at the crack-tip.
- We have also seen that SIF is a function of applied stress (σ) and crack length (a).
- For a given crack length (a) crack starts growing at a critical applied stress (σ_c). The SIF corresponding to (a and σ_c) is critical stress intensity factor K_C.
- Similarly for a given applied stress (σ) crack become unstable after a critical crack length (a_c). Again the stress intensity factor corresponding to (σ and a_c) is critical stress intensity factor K_C.
- Critical stress intensity factor K_{IC}, K_{IIC}, K_{IIIC} is material property.
- We can now define a new fracture based design criteria as $K_{I/II/III} \leq K_{IC/IIC/IIIC}$.
- For the case of a mixed crack-tip loading by K_I , K_{II} , and K_{III}, then a generalized fracture criterion $f(K_{IC}, K_{IIC}, K_{IIIC})=0$ must be defined.
- For such a design criteria to follow, we should be able to calculate SIF corresponding to any configuration.

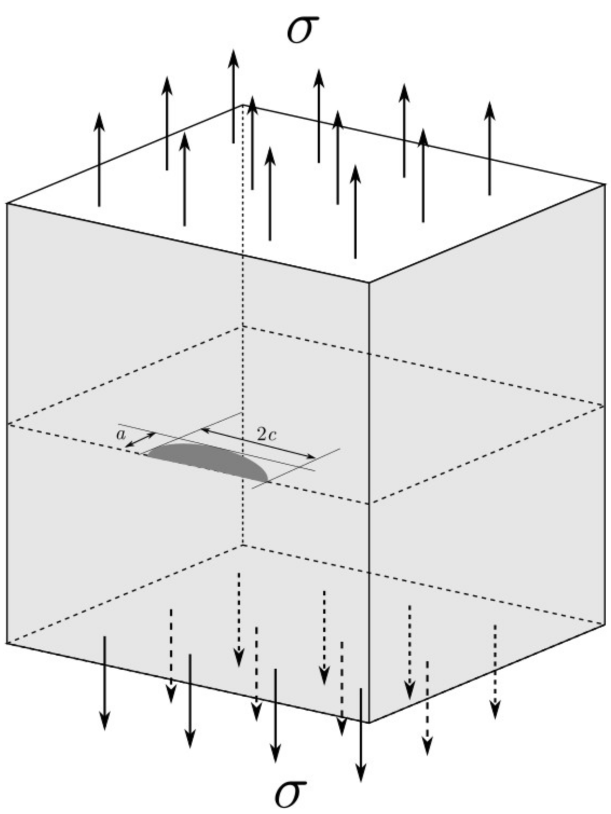
- For thin specimen critical stress intensity factor is a function of thickness. For plane stress case it is higher than the corresponding value in a plane strain. Generally values of K_{IC} reported is the values for plane strain case and the same value is also used for design in case of plane stress case as a conservative approach, unless a very low factor of safety is required.
- Critical SIF of a material also depends on many factors, such as
  - Heat treatment of the material
  - Speed of the crack
  - Temperature
  - Process of manufacturing (e.g., vacuum furnaced or air melted, as cast or rolled).
  - Orientation of the crack with respect to the grains at the crack tip.
  - Test method.

Material	Yield Stress (MPa)	K_{IC} (MPa$\sqrt{m}$)
Mild Steel	240	Very high (~220)
Medium Carbon Steel	260	54
	1070	77
	1515	60
	1850	47
Rotor Steel	626	50
Nuclear Reactor Steel	350	190
Maraging Steel	1770	93
	2000	47
	2240	38
Aluminum		
2014-T4	460	29
2014-T651	455	24
7075-T651	495	24
7178-T651	570	23
Titanium (Ti-6Al-4V)	910	55
Perspex (PMMA)		1.6
PVC		3.5
Nylon		3

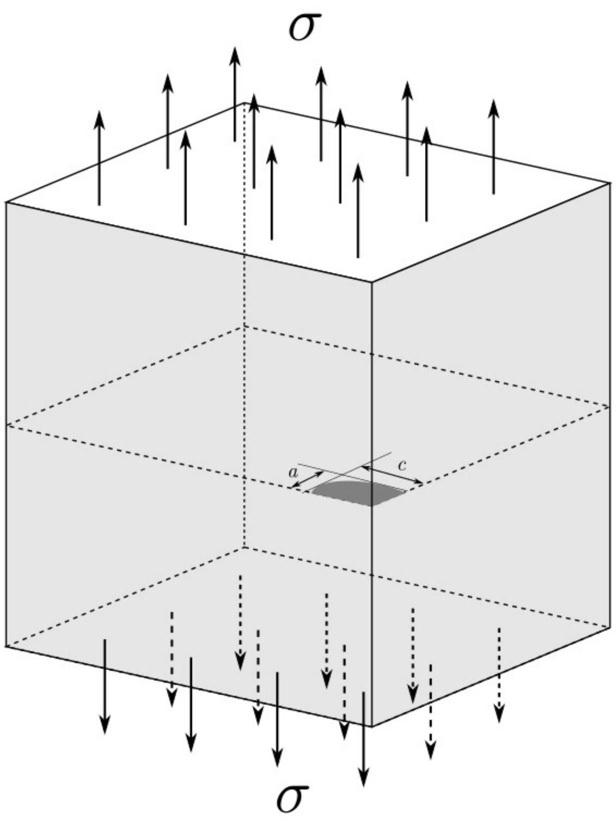
Embedded cracks



A fully embedded elliptical crack

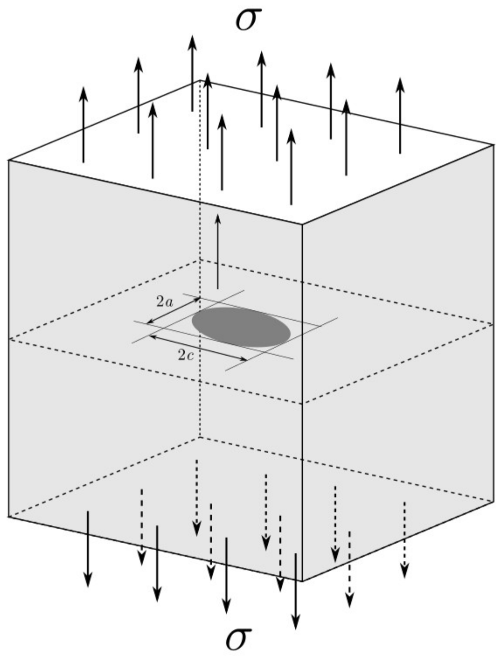


A semi-elliptical crack

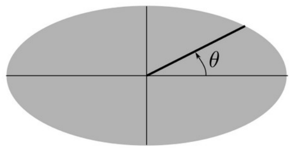


A quarter-elliptical crack

Elliptical cracks

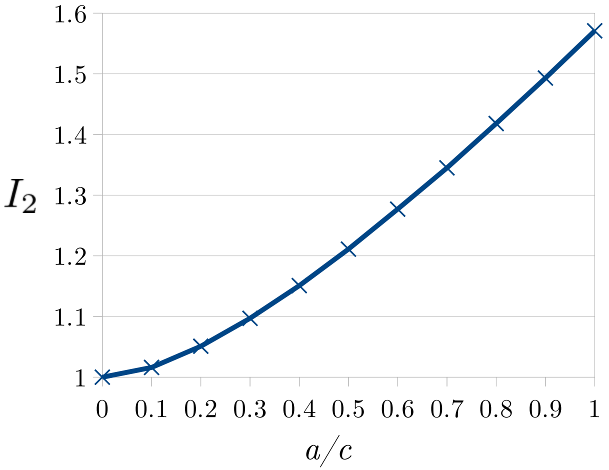


Theoretical solution for an elliptical crack embedded in an infinite medium, subjected to load σ at the far field is given by Irwin in 1962. The stress intensity factor is varies at every point of the crack front and is a function of angle θ.



$$K_I = \frac{\sigma\sqrt{\pi a}}{I_2}\left[\sin^2\theta + \left(\frac{a}{c}\right)^2\cos^2\theta\right]^{1/4}, \quad \cdots\cdots(111)$$

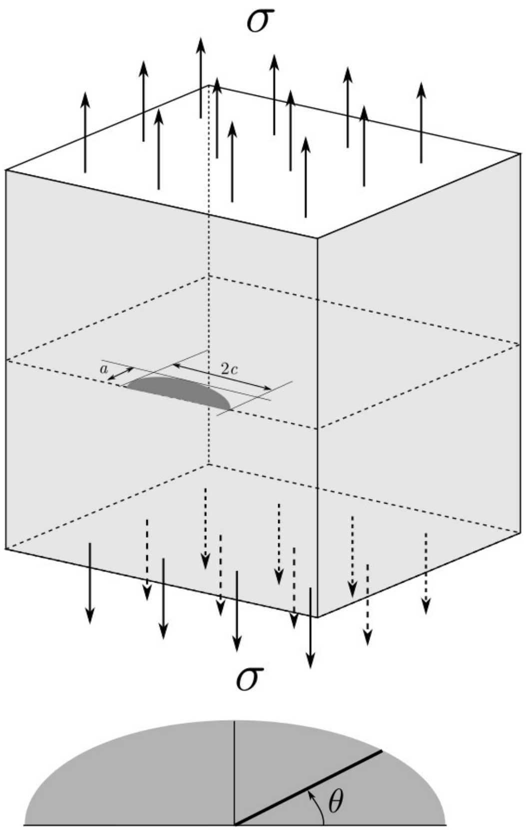
$$\text{where, } I_2 = \int_0^{\pi/2}\left(1 - \frac{c^2 - a^2}{c^2}\sin^2\alpha\right)^{1/2} d\alpha \quad \cdots\cdots(112)$$



Integral I_2 can be evaluated and its variation with a/c ratio is shown.

Semi-elliptical cracks

A surface crack is modeled as a half ellipse with its minor axis into the thickness direction. The SIF of the surface crack is then 12% higher over the corresponding SIF of the elliptical crack. Thus, the SIF at a point of the crack front of a semi-elliptical crack is



$$K_I = \frac{1.12\sigma\sqrt{\pi a}}{I_2}\left[\sin^2\theta + \left(\frac{a}{c}\right)^2\cos^2\theta\right]^{1/4}, \qquad \cdots\cdots(113)$$

Note that the SIF at the extreme end of the minor axis (θ=90°) is

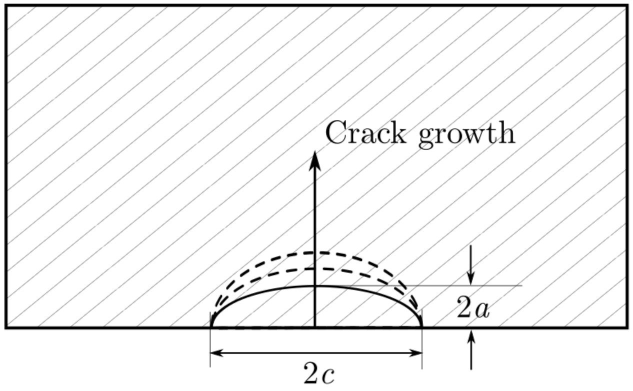
$$K_I^{90} = \frac{1.12\sigma\sqrt{\pi a}}{I_2}, \qquad \cdots\cdots(114)$$

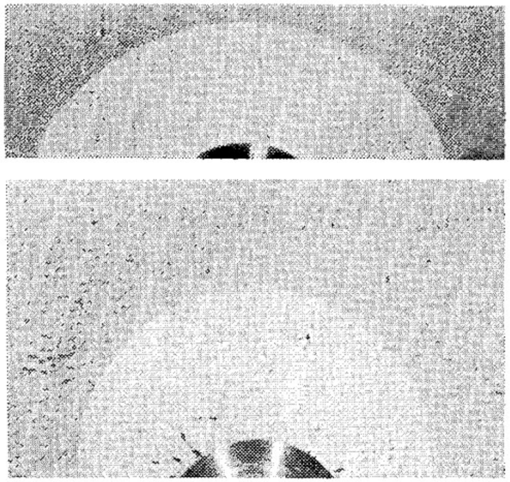
and SIF at extreme ends of major axis is,

$$K_I^{0/180} = \frac{1.12\sigma\sqrt{\pi a}}{I_2}\left(\frac{a}{c}\right)^2. \qquad \cdots\cdots(115)$$

So the segment of the crack which is inside the material is having higher SIF.

This different in SIF causes the crack to grow in a particular manner as shown.



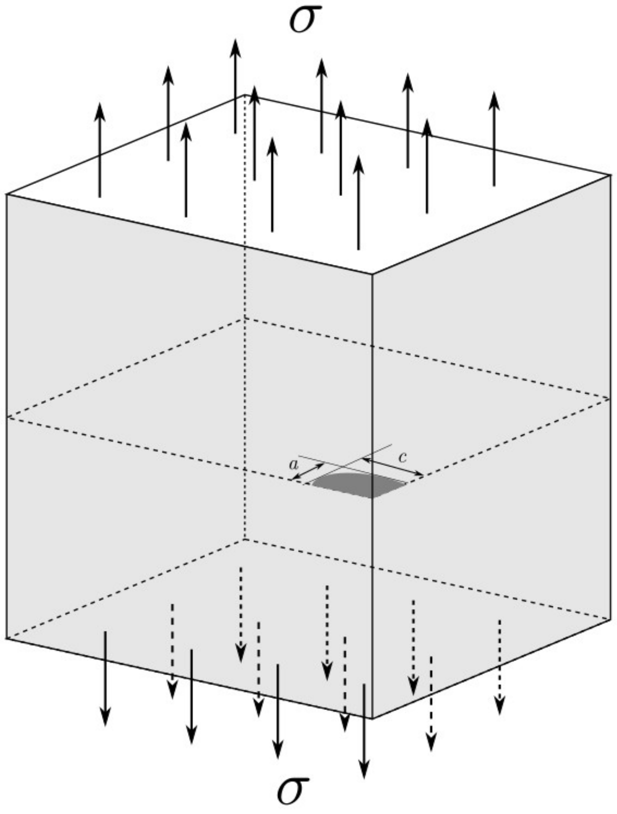


Till the time $a/c < 1$, $K_I^{90} > K_I^{0/180}$ and crack grows in the direction of minor axis.

When a=c, i.e., when crack becomes semi-circular, $K_I^{90} = K_I^{0/180}$, and then crack grows in both the direction of major as well as minor axis as semi-circular crack.

Also, note that for a very shallow crack ($a << c$), a/c is close to unity and the SIF $K_I^{90} \approx 1.12\sigma\sqrt{\pi a}$, which is the same as the result of through-the-thickness edge crack of length a. In this case, the dimension of the major axis is no longer relevant. Therefore, a shallow crack is equivalent to a through-the-thickness edge crack of length a.

Quarter-elliptical cracks

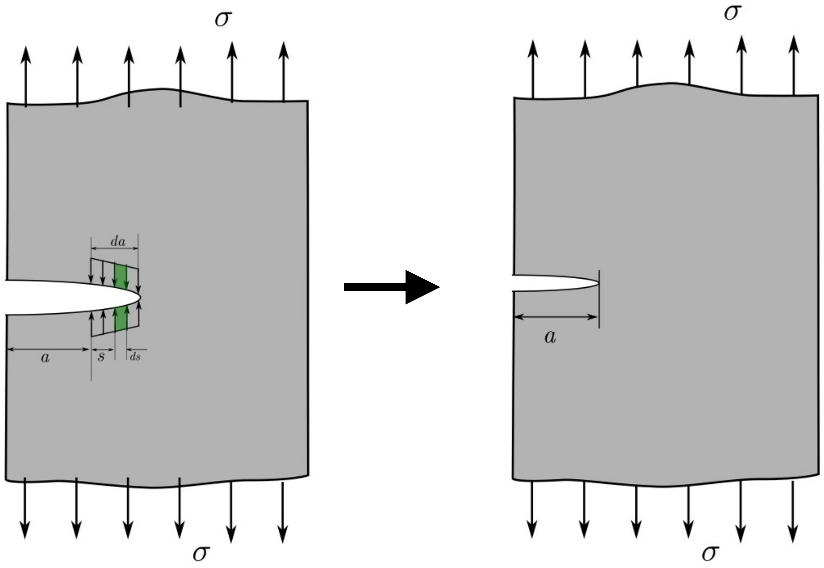


A corner crack under tensile load is exposed to two free surfaces, each one applying an additional correction factor on the SIF. The overall correction factor of such cracks in thick plates has found to be about 20% higher i.e., the highest SIF on the crack front is

$$K_I^{90} = 1.2\sigma\sqrt{\pi a}. \qquad \cdots\cdots(116)$$

The relationship between G and K

- Energy release rate G is a global parameter and deals with energy.
- On the other hand, stress intensity factor K is a local parameter which deals with displacement and stress fields in the vicinity of the crack.
- Although the approaches are entirely different, the goal is same, i.e., to characterize a crack. Therefore there should be a relationship between G and K.
- The relation was obtained by Irwin

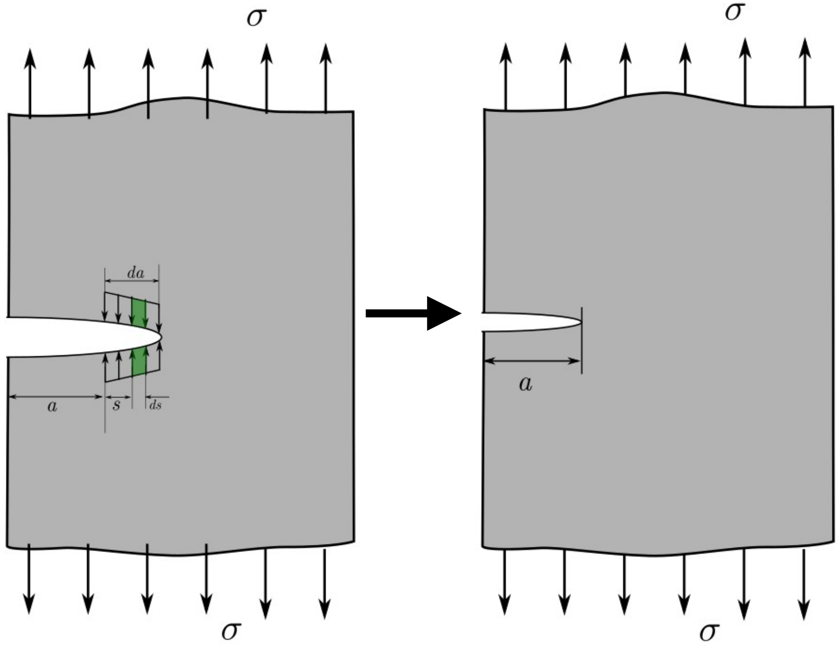


Consider a crack of length a which is extended by an small incremental length Δa..

For the case of extended crack, let us find the the displacement of the crack face at a distance of (Δa-s) from the crack-tip.

For plane stress case, we use (90) with θ=180°, which gives

$$u_2(s) = \frac{K_I'}{\mu}\sqrt{\frac{\Delta a - s}{2\pi}}\frac{2}{1-\nu}. \qquad \cdots\cdots\cdot(117)$$



Now think of a hypothetical experiment. The extended length Δa of the crack is closed by applying stress $\sigma_{22}(s)$ on crack faces. σ_{22} is evaluated from the stress field of the initial (unextended) crack of length a, and therefore at a distance s, it is given by

$$\sigma_{22}(s) = \frac{K_I}{\sqrt{2\pi s}}. \qquad \cdots\cdots(118)$$

Every point of the extended crack need to be moved by a displacement $u_2(s)$ to close it.

Thus, the crack is closed by length Δa. Irwin argued that the total elastic work required by σ_{22} in closing the crack is equal to the energy released during the extension of crack by Δa length. Balancing the two energies, we have

$$G_I B \Delta a = 2B \int_0^{\Delta a} \frac{1}{2} \sigma_{22}(s) u_2(s) ds. \qquad \cdots\cdots(119)$$

Substituting (117) and (118) in (119), we get,

$$G_I = \lim_{\Delta a \to 0} \frac{1}{\Delta a} \int_0^{\Delta a} \frac{K_1}{\sqrt{2\pi s}} \frac{K_I'}{\mu} \sqrt{\frac{\Delta a - s}{2\pi}} \frac{2}{1-\nu} ds, \quad \text{where} K_I' = K_I + \Delta K,$$

$$G_I = \lim_{\Delta a \to 0} \frac{2}{2\pi\mu(1-\nu)} \frac{1}{\Delta a} \int_0^{\Delta a} K_I(K_I + \Delta K_I) \frac{\sqrt{\Delta a - s}}{\sqrt{s}} ds.$$

For sufficiently small Δa, $K_I + \Delta K_I \approx K_I$. Thus,

$$G_I = \lim_{\Delta a \to 0} \frac{K_I^2}{\pi\mu(1-\nu)} \frac{1}{\Delta a} \int_0^{\Delta a} \frac{\sqrt{\Delta a - s}}{\sqrt{s}} ds.$$

This equation can be solved by substituting $s = \Delta a \sin^2 a$ and changing the limits accordingly.

$$G_I = \frac{K_I^2}{2\mu(1-\nu)} = \frac{K_I^2}{E}. \qquad \cdots\cdots\cdots\cdots(120)$$

The relation is simple, but is rigorous only for brittle materials in which the components remain elastic.

For plane strain case,

$$G_I = \frac{K_I^2}{E'} = (1-\nu^2)\frac{K_I^2}{E}. \qquad \cdots\cdots\cdots\cdots(120a)$$

Similarly, we can also establish relations for mode II and mode III as,

$$G_{II} = \frac{K_{II}^2}{E} \text{ (for plane stress)} \quad \text{and} \quad G_{II} = (1-\nu^2)\frac{K_{II}^2}{E} \text{ (for plane strain)},$$

and

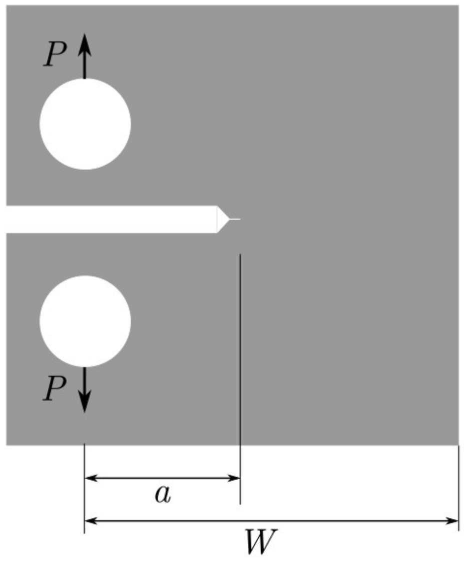
$$\cdots\cdots\cdots\cdots(121)$$

$$G_{III} = \frac{K_{III}^2}{2\mu}. \qquad \cdots\cdots\cdots\cdots(122)$$

In a general case where all three modes can be present the total energy release rate is,

$$G = G_I + G_{II} + G_{III}.$$

Compact Tension (CT) Specimen

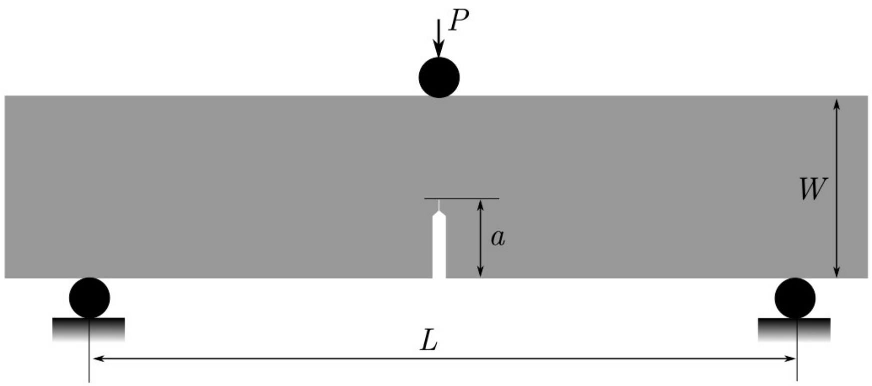


$$K_I = \frac{P}{B\sqrt{W}} f(\alpha)$$

$$\alpha = a/W$$

$$f(\alpha) = \frac{3\sqrt{\alpha}\left[1.99 - \alpha(1-\alpha)(2.15 - 3.93\alpha + 2.7\alpha^2\right]}{2(1+2\alpha)(1-\alpha)^{3/2}}$$

Single-Edge-Notch-Bend (SENB) Specimen

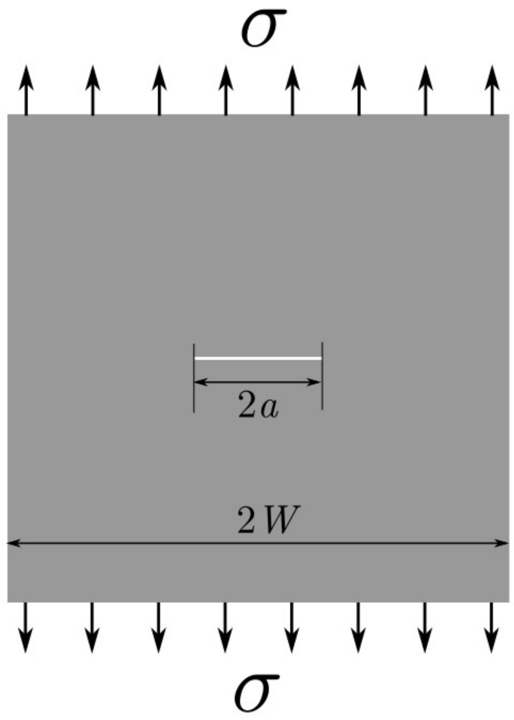


$$K_I = \frac{PS}{BW^{3/2}} f(\alpha)$$

$$\alpha = a/W$$

$$f(\alpha) = \frac{3\sqrt{\alpha}\left[1.99 - \alpha(1-\alpha)(2.15 - 3.93\alpha + 2.7\alpha^2\right]}{2(1+2\alpha)(1-\alpha)^{3/2}}$$

Centre-Cracked Plate under Uniform Tension

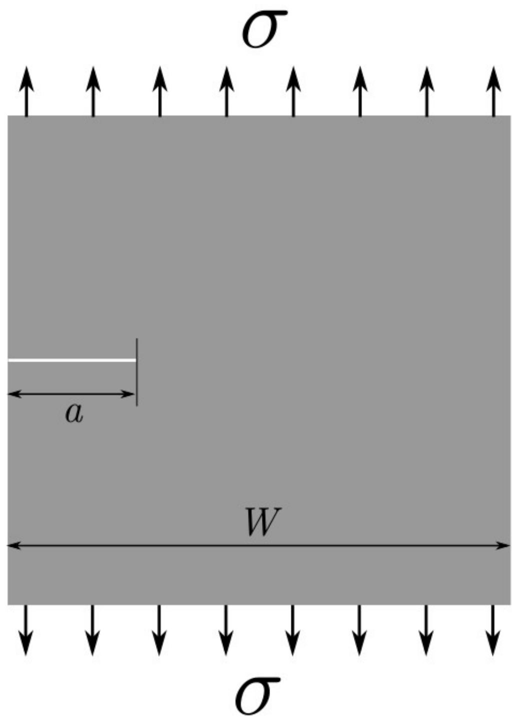


$$K_I = \sigma\sqrt{\pi a} f(\alpha)$$

$$\alpha = a/W \text{ for } 0 < \alpha < 0.7$$

$$f(\alpha) = 1.0 + 0.128\alpha - 0.288\alpha^2 + 1.53\alpha^3$$

Single-Edge-Cracked Plate under Uniform Tension

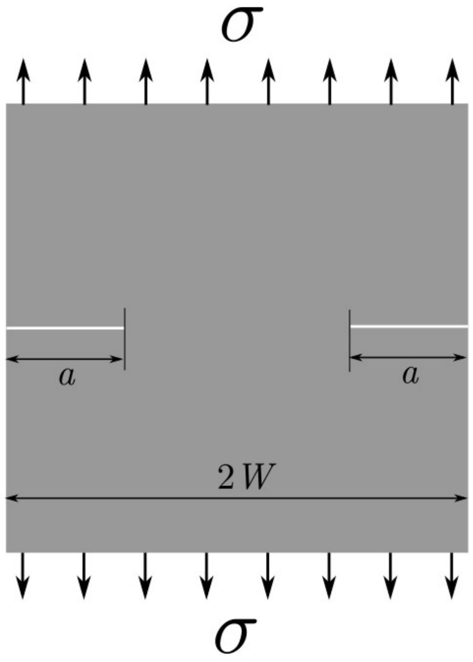


$$K_I = \sigma\sqrt{\pi a} f(\alpha)$$

$$\alpha = a/W \text{ for } 0 < \alpha < 0.6$$

$$f(\alpha) = 1.12 - 0.23\alpha + 10.55\alpha^2 - 21.72\alpha^3 + 30.39\alpha^4$$

Double-Edge-Cracked Plate under Uniform Tension

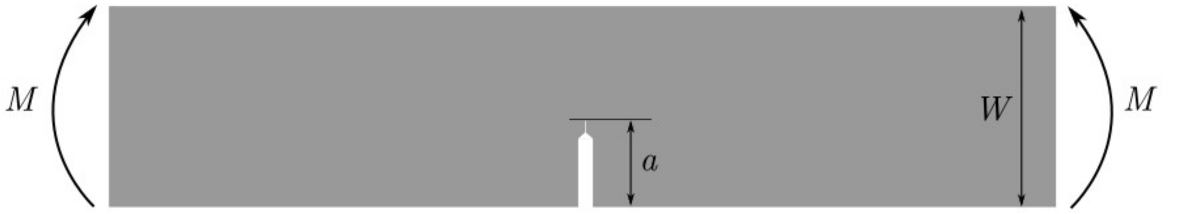


$$K_I = \sigma\sqrt{\pi a}f(\alpha)$$

$$\alpha = a/W \text{ for } 0 < \alpha < 0.7$$

$$f(\alpha) = 1.12 - 0.20\alpha - 1.20\alpha^2 + 1.93\alpha^3$$

Strip with Edge-Crack under bending

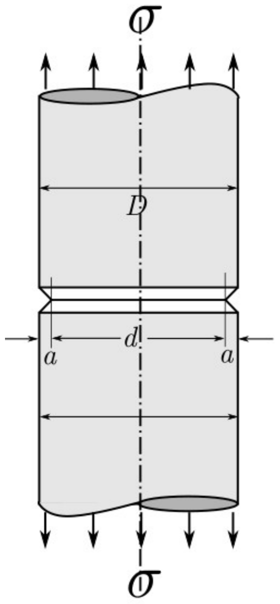


$$K_I = \frac{6M}{BW^2}\sqrt{\pi a}f(\alpha)$$

$$\alpha = a/W$$

$$f(\alpha) = 1.12 - 1.40\alpha + 7.33\alpha^2 - 13.083\alpha^3 + 14\alpha^4$$

Circumferentially Cracked Round Bar under Tension

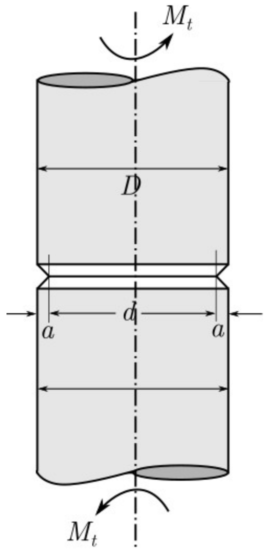


$$K_I = \sqrt{\pi a} f(\beta)$$

$$\beta = d/B$$

$$f(\beta) = \frac{1}{2\sqrt{\beta}}\left(\frac{1}{\beta} + \frac{1}{2} + \frac{3}{8}\beta - 0.316\beta^2 + 7.33\beta^3\right).$$

Circumferentially Cracked Round Bar under Torsion

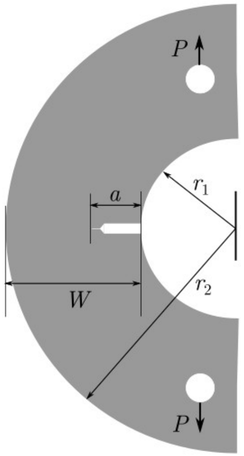


$$K_I = \frac{16M}{\pi D^3}\sqrt{\pi a} f(\beta)$$

$$\beta = d/B$$

$$f(\beta) = \frac{3}{8\sqrt{\beta}}\left(\frac{1}{\beta^2} + \frac{0.5}{\beta} + \frac{3}{8} + \frac{5}{16}\beta + \frac{35}{128}\beta^2 + 0.21\beta^3\right).$$

Arc-Shaped Tension (AT) Specimen

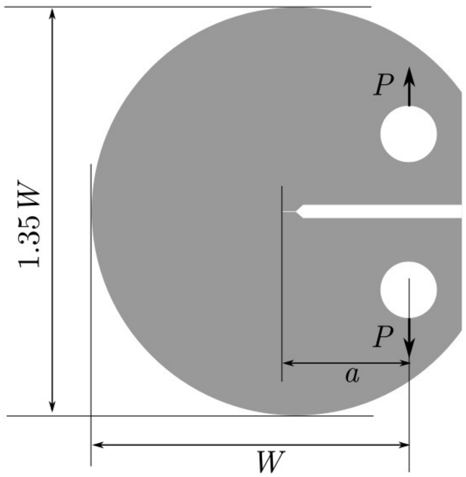


$$K_I = \frac{P}{B\sqrt{W}} f(\alpha)$$

$$\alpha = a/W$$

$$f(\alpha) = \left[\frac{3X}{W} + 1.9 + 1.1\alpha\right]\left[1 + 0.25(1-\alpha)^2\left(1 - \frac{r_1}{r_2}\right)\right]$$
$$\left[\frac{\alpha^{1/2}}{(1-\alpha)^{3/2}}\right]\left(3.74 - 6.3\alpha + 6.32\alpha^2 - 2.43\alpha^3\right).$$

Disc-Shaped Compact Tension (DCT) Specimen



$$K_I = \frac{P}{B\sqrt{W}} f(\alpha)$$

$$\alpha = a/W$$

$$f(\alpha) = (2+\alpha)(0.76 + 4.8\alpha - 11.85\alpha^2 + 11.43\alpha^3 - 4.08\alpha^4)(1-\alpha)^{-3/2}$$

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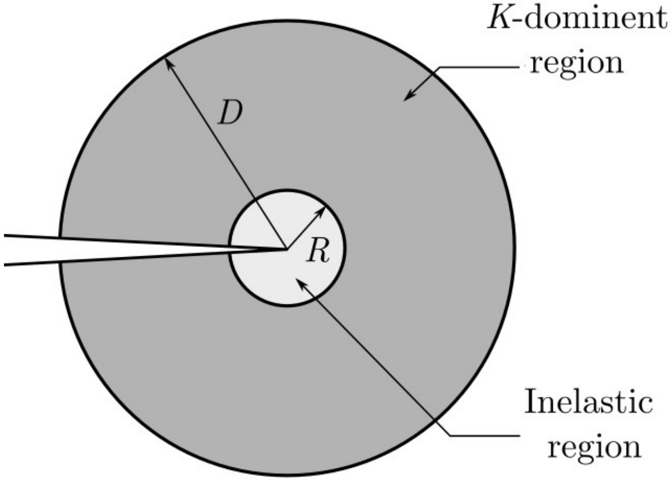
Elasto-Plastic fracture mechanics

Till now we discussed about the stress-field in the vicinity of a crack-tip for linear elastic materials. However, assumption of linear elasticity is not true with most engineering material. Engineering materials go under plastic deformation at high stresses. This is also valid for the high stresses near the crack-tip. The material flow after yielding makes the crack tip blunt, which in turn reduces the magnitude of stress components there. Thus, many potential catastrophic failures are avoided just by the local plastic deformation at the crack tip. A proper analysis of plastic deformation near the crack-tip allow accurate determination the stresses and deformation there.

Small-scale yielding

We also saw that the applicability of singular stress fields is confined to a very small region around the crack-tip. Let the singular solution dominate inside a circle of radius D surrounding the crack tip. Consider also that the region of inelastic deformation attending the crack tip is represented by R. When R is sufficiently small compared to D and any other characteristic geometric dimension such as notch radius, plate thickness, crack ligament, etc., the singular stress field governed by the stress intensity factors forms a useful approximation to the elastic field in the ring enclosed by radii R and D. This situation is called “small-scale yielding”.



Approximate determination of crack-tip plastic zone

Strictly speaking, the plastic zone should be determined from an elastic-plastic analysis of the stress field around the crack tip. However, such analysis is quite complex and involved. Hence, we can obtain some useful results regarding the shape of the plastic zone from the approximate calculation.

A first estimate of the extent of the plastic zone attending the crack tip can be obtained by determining the locus of points where the elastic stress field satisfies the yield criterion. This calculation is very approximate, since yielding leads to stress redistribution and modifies the size and shape of the plastic zone. To apply the yield criterion let us first determine principle stresses. Stresses at the crack-tip are

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right),$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right),$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}} \sin\theta \sin\frac{3\theta}{2}\cos\frac{3\theta}{2},$$

$$\sigma_z = 0. \quad \text{for plane stress,}$$

$$\sigma_z = \nu(\sigma_x + \sigma_y) \quad \text{for plane strain.}$$

$$\ldots\ldots\ldots\ldots\ldots(1)$$

Principal stresses are

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \sigma_{xy}^2} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\right]$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \sigma_{xy}^2} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\right]$$

$$\sigma_3 = 0 \quad \text{(for plane stress)} \qquad \cdots\cdots\cdots\cdots(2)$$

$$\sigma_3 = 2\nu \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \quad \text{(for plane strain)}$$

By applying the Von-Mises criterion for yielding,

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \geq 2\sigma_Y^2 \qquad \cdots\cdots\cdots\cdots(3)$$

Substituting (2) in (3) and simplifying we get,

$$\frac{K_I^2}{2\pi r}\left(1+\cos\theta+\frac{3}{2}\sin^2\theta\right)\geq 2\sigma_Y^2 \qquad \text{(for plane stress)}$$

$$\frac{K_I^2}{2\pi r}\left[(1-2\nu)^2\,(1+\cos\theta)+\frac{3}{2}\sin^2\theta\right]\geq 2\sigma_Y^2 \qquad \text{(for plane strain)}$$

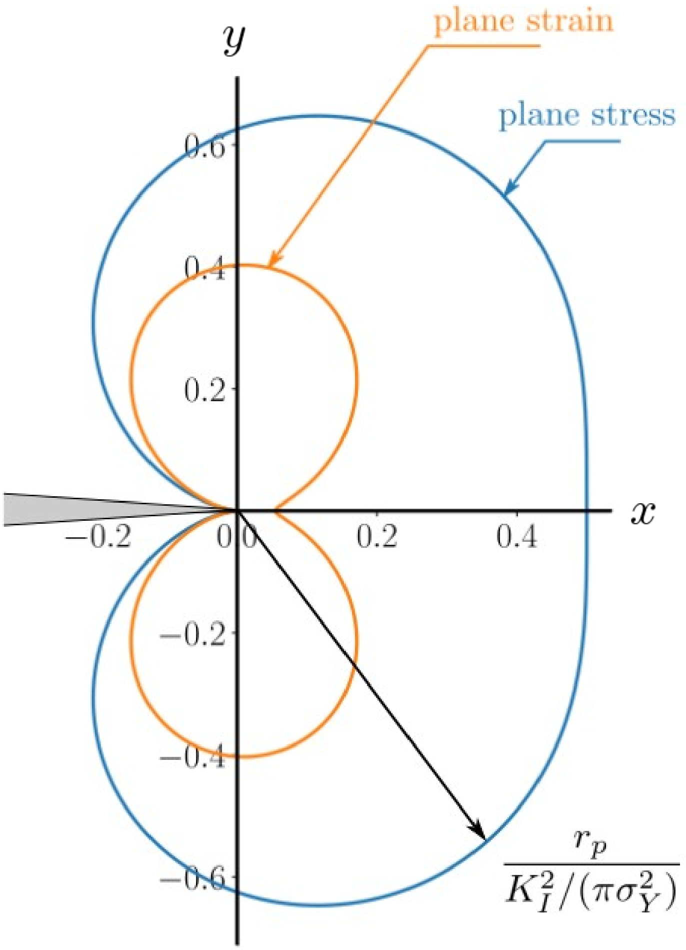
Thus the boundary of elastic plastic interface can be obtained as

$$r_p(\theta)=\frac{K_I^2}{4\pi\sigma_Y^2}\left(1+\cos\theta+\frac{3}{2}\sin^2\theta\right) \qquad \text{(for plane stress)}$$

$$r_p(\theta)=\frac{K_I^2}{4\pi\sigma_Y^2}\left[(1-2\nu)^2\,(1+\cos\theta)+\frac{3}{2}\sin^2\theta\right] \qquad \text{(for plane strain)}$$

..................(4)

(for $\nu = 1/3$)



- Observe that the plane stress zone is much larger than the plane strain zone because of the higher constraint for plane strain.
- Plastic zone is proportional to the square of SIF and inversely proportional to the square of the yield stress.
- Whenever yield stress of a metal is increased, say by an appropriate heat treatment, its plastic zone size decreases considerably; this in turn makes the material more prone to crack growth.
- The increase in yield stress may please a conventional designer, because he usually designs structural components based on a yield criterion. But as far as the toughness of a material is concerned, the designer is left with an inferior material. The designer msut explore a satisfactory compromise between yield stress and toughness, while choosing a material and its heat treatment.
- Let us try to understand the yielding behaviour. Tresca criteria helps us in understanding it in a simple manner.

For plane strain (considering $\nu = 1/3$)

$$\sigma_{\max} = \sigma_1 = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\right]$$

$$\text{for } \theta \leq 38.9^\circ,\ \sigma_{\min} = \sigma_3 = 2\nu \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}$$

$$\text{for } \theta \geq 38.9^\circ,\ \sigma_{\min} = \sigma_2 = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\right]$$

Thus,

$$\text{for } \theta \leq 38.9^\circ,\quad \tau_{\max} = \frac{\sigma_{\max} - \sigma_{\min}}{2} = \frac{1}{2}\frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - 2\nu + \sin\frac{\theta}{2}\right]$$

$$\text{for } \theta \geq 38.9^\circ,\quad \tau_{\max} = \frac{\sigma_{\max} - \sigma_{\min}}{2} = \frac{1}{2}\frac{K_I}{\sqrt{2\pi r}} \sin\theta$$

Hence, from Tresca criteria, ...(5)

$$\text{for } \theta \leq 38.9^\circ,\quad r_p = \frac{K_I^2}{2\pi\sigma_Y^2} \cos^2\frac{\theta}{2}\left[1 - 2\nu + \sin\frac{\theta}{2}\right]^2, \quad \text{and} \quad \text{for } \theta \leq 38.9^\circ,\quad r_p = \frac{K_I^2}{2\pi\sigma_Y^2} \sin^2\theta.$$

For plane stress,

$$\sigma_{\max} = \sigma_1 = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\right]$$

$$\sigma_{\min} = \sigma_3 = 0$$

Now, as per Tresca criteria for yielding

$$\tau_{\max} = \frac{\sigma_{\max} - \sigma_{\min}}{2} \geq \frac{\sigma_Y}{2},$$

i.e.,

$$\frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\right] \geq \sigma_Y.$$

Thus,

$$r_p(\theta) = \frac{K_I^2}{2\pi\sigma_Y^2} \cos^2\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\right]^2. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(6)$$