ME232: Dynamics

3D dynamics of rigid bodies

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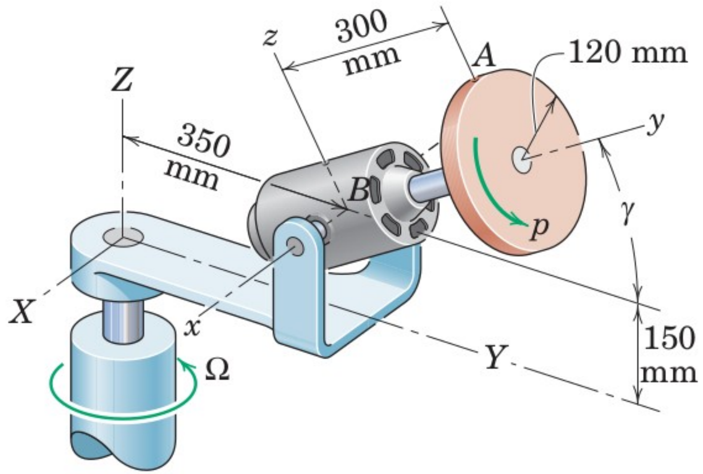
Example 3

The motor housing and its bracket rotate about the Z-axis at the constant rate $\Omega = 3$ rad/s. The motor shaft and disk have a constant angular velocity of spin $p = 8$ rad/s with respect to the motor housing in the direction shown. If γ is constant at 30°, determine the velocity and acceleration of point A at the top of the disk and the angular acceleration α of the disk.



The rotating reference axes x-y-z are attached to the motor housing, and the rotating base for the motor has the momentary orientation shown with respect to the fixed axes X-Y-Z. We will use both X-Y-Z components with unit vectors $\boldsymbol{I}$, $\boldsymbol{J}$, $\boldsymbol{K}$ and x-y-z components with unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$. The angular velocity of the x-y-z axes becomes $\boldsymbol{\Omega} = \Omega\boldsymbol{K} = 3\boldsymbol{K}$ rad/s.

The velocity of A is given by

$$\boldsymbol{v}_A = \boldsymbol{v}_B + \boldsymbol{\Omega} \times \boldsymbol{r}_{A/B} + \boldsymbol{v}_{\text{rel}}$$

$$\boldsymbol{v}_B = \boldsymbol{\Omega} \times \boldsymbol{r}_B = 3\boldsymbol{K} \times 0.350\boldsymbol{J} = -1.05\boldsymbol{I} = -1.05\boldsymbol{i} \text{ m/s}$$

$$\boldsymbol{\Omega} \times \boldsymbol{r}_{A/B} = 3\boldsymbol{K} \times (0.300\boldsymbol{j} + 0.120\boldsymbol{k}) = -0.599\boldsymbol{i} \text{ m/s}$$

$$\boldsymbol{v}_{\text{rel}} = \boldsymbol{p} \times \boldsymbol{r}_{A/B} = 8\boldsymbol{j} \times (0.300\boldsymbol{j} + 0.120\boldsymbol{k}) = 0.960\boldsymbol{i} \text{ m/s}$$

Thus, $\boldsymbol{v}_A = -1.05\boldsymbol{i} - 0.599\boldsymbol{i} + 0.960\boldsymbol{i} = -0.689\boldsymbol{i}$ m/s.

The acceleration of A is given by

$$\boldsymbol{a}_A = \boldsymbol{a}_B + \dot{\boldsymbol{\Omega}} \times \boldsymbol{r}_{A/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_{A/B}) + 2\boldsymbol{\Omega} \times \boldsymbol{v}_{\text{rel}} + \boldsymbol{a}_{\text{rel}}$$

where,

$$\boldsymbol{a}_B = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_b) = -3.15\boldsymbol{J} = -2.73\boldsymbol{j} + 1.575\boldsymbol{k} \text{ m/s}^2$$

$$\dot{\boldsymbol{\Omega}} = \boldsymbol{0}$$

$$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_{A/B}) = -1.557\boldsymbol{j} + 0.899\boldsymbol{k} \text{ m/s}^2$$

$$2\boldsymbol{\Omega} \times \boldsymbol{r}_{\text{rel}} = -4.99\boldsymbol{j} - 2.88\boldsymbol{k} \text{ m/s}^2$$

$$\boldsymbol{a}_{\text{rel}} = \boldsymbol{p} \times (\boldsymbol{p} \times \boldsymbol{r}_{A/B}) = 8\boldsymbol{j} \times [8\boldsymbol{j} \times (0.3\boldsymbol{j} + 0.120\boldsymbol{k})] = -7.68\boldsymbol{k}$$

Substituting in expression of $\boldsymbol{a}_A$ we get, $\boldsymbol{a}_A = 0.703\ \boldsymbol{j}$ -8.09 $\boldsymbol{k}$ m/s^2.

Angular acceleration $\boldsymbol{\alpha}$ is calculated as,

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \boldsymbol{\Omega} \times \boldsymbol{\omega} = -20.8\boldsymbol{i} \text{ rad/s}^2$$

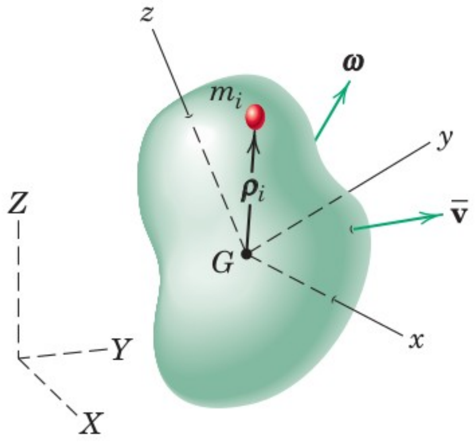
Kinetics

The force equation for a rigid or nonrigid in three-dimensional motion is the simple extension of Newton's second law.

The moment equation for three-dimensional motion is not as simple as for plane motion since the change of angular momentum has a number of additional components which are absent in plane motion.

Angular momentum:

Consider a rigid body moving with any general motion in space. Axes x-y-z are attached to the body with origin at the mass center G. Thus, the angular velocity $\boldsymbol{\omega}$ of the body becomes the angular velocity of the x-y-z axes as observed from the fixed reference axes X-Y-Z. The absolute angular momentum $\boldsymbol{H}_G$ of the body about its mass center G is the sum of the moments about G of the linear momenta of all elements of the body as,



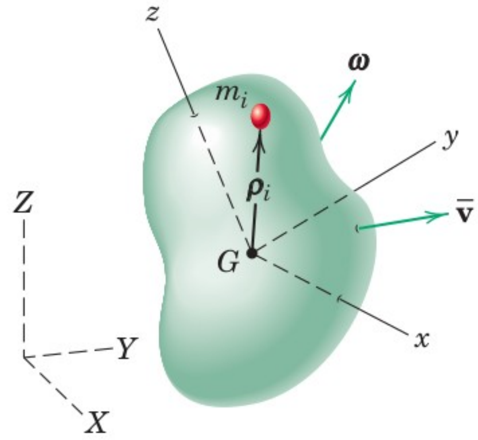
$$\boldsymbol{H}_G = \sum(\boldsymbol{\rho}_i \times m_i \boldsymbol{v}_i).$$

For a rigid body,

$$\boldsymbol{v}_i = \overline{\boldsymbol{v}} + \boldsymbol{\omega} \times \boldsymbol{\rho}_i$$

and thus,

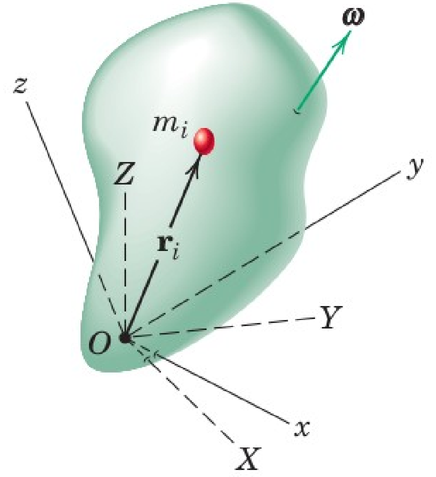
$$\boldsymbol{H}_G = -\overline{\boldsymbol{v}} \times \sum m_i \boldsymbol{\rho}_i + \sum \left[\boldsymbol{\rho}_i \times m_i(\boldsymbol{\omega} \times \boldsymbol{\rho}_i)\right].$$



Above equations can further be written (by considering mass $\boldsymbol{m}_i$ to be elemental mass dm) as,

$$\boldsymbol{H}_G = \int \boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) dm. \qquad \cdots\cdots\cdots\cdots\cdots(6)$$

Also consider the case of a rigid body rotating about a fixed point O, the x-y-z axes are attached to the body, and both body and axes have an angular velocity $\boldsymbol{\omega}$. The angular momentum about O can be written as,



$$\boldsymbol{H}_O = \int \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r}) dm. \qquad \cdots\cdots\cdots\cdots\cdots(7)$$

Moments and Product of inertia:

Observe that in (6) and (7), the position vectors $\boldsymbol{\rho}$ and $\boldsymbol{r}$ are given by the same expression $x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. Thus, they are identical in form.

Let us expand the integrand in the two expressions for angular momentum, recognizing that the components of $\boldsymbol{\omega}$ are invariant with respect to the integrals over the body and thus become constant multipliers of the integrals, and we get,

$$
\begin{aligned}
d\boldsymbol{H} = &\boldsymbol{i}\left[(y^2+z^2)\omega_x \quad - xy\omega_y \quad - xz\omega_z\right] dm \\
&\boldsymbol{j}\left[-xy\omega_x \quad + (z^2+x^2)\omega_y \quad - yz\omega_z\right] dm \\
&\boldsymbol{k}\left[-zx\omega_x \quad - yz\omega_y \quad + (x^2+y^2)\omega_z\right] dm
\end{aligned}
\quad
\begin{aligned}
\Rightarrow \boldsymbol{H} = &(I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\,\boldsymbol{i} \\
&+ (-I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z)\,\boldsymbol{j} \\
&+ (-I_{zx}\omega_x - I_{yz}\omega_y + I_{zz}\omega_z)\,\boldsymbol{k}
\end{aligned}
\qquad \ldots\ldots\ldots\ldots\ldots(8)
$$

where, $$I_{xx} = \int (y^2+z^2)dm, \quad I_{xy} = \int xy\, dm, \quad I_{yy} = \int (z^2+x^2)dm,$$

$$I_{xz} = \int xz\, dm, \quad I_{zz} = \int (x^2+y^2)dm, \quad I_{yz} = \int yz\, dm. \qquad \ldots\ldots\ldots\ldots\ldots(9)$$

The quantities I_{xx}, I_{yy}, I_{zz} are called the moments of inertia of the body about the respective axes, and I_{xy}, I_{xz}, I_{yz} are the products of inertia with respect to the coordinate axes.

Observe that $I_{xy} = I_{yx}$, $I_{xz} = I_{zx}$, and $I_{yz} = I_{zy}$ and the components of $\boldsymbol{H}$ are

$$\begin{aligned} H_x &= I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\ H_y &= -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z \\ H_z &= -I_{zx}\omega_x - I_{yz}\omega_y + I_{zz}\omega_z. \end{aligned} \qquad \cdots\cdots\cdots\cdots\cdots(10)$$

(8) represents the general expression for the angular momentum either about the mass center G or about a fixed point O for a rigid body rotating with an instantaneous angular velocity $\boldsymbol{\omega}$.

In each of the two cases represented, the reference axes x-y-z are attached to the rigid body, which makes **the moment-of-inertia integrals and the product-of-inertia integrals invariant with time.**

If the x-y-z axes were to rotate with respect to an irregular body, then these inertia integrals would be functions of the time, which would introduce an undesirable complexity into the angular-momentum relations.

Principal axis:

The array of moments and products of inertia $\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{zy} & I_{zz} \end{bmatrix}$

is called the **inertia matrix or inertia tensor**.

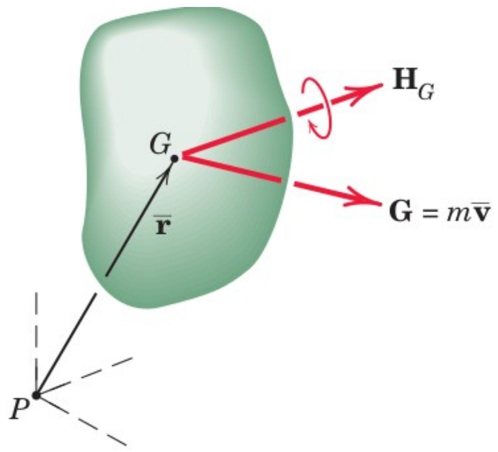
With the change of the orientation of the axes relative to the body, the moments and products of inertia will also change in value. It can be shown that there is one unique orientation of axes x-y-z for a given origin for which **the products of inertia vanish** and the moments of inertia I_{xx} , I_{yy} , I_{zz} take on stationary values. For this orientation, the inertia matrix takes the form

$$\begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

and is said to be diagonalized. The axes x-y-z for which the products of inertia vanish are called **the principal axes of inertia**, and I_{xx} , I_{yy} , and I_{zz} are called the **principal moments of inertia**. The principal moments of inertia for a given origin represent the maximum, the minimum, and an intermediate value of the moments of inertia.

Transfer principle for angular momentum:

The momentum properties of a rigid body may be represented by the resultant linear-momentum vector $\boldsymbol{G} = m\overline{\boldsymbol{v}}$ through the mass center and the resultant angular-momentum vector $\boldsymbol{H}_G$ about the mass center.



These vectors have properties analogous to those of a force and a couple. Thus, the angular momentum about any point P equals the free vector $\boldsymbol{H}_G$ plus the moment of the linear-momentum vector G about P. Therefore, we may write

$$\boldsymbol{H}_P = \boldsymbol{H}_G + \overline{\boldsymbol{r}} \times \boldsymbol{G}.$$

Kinetic energy

We already developed the expression for the kinetic energy T of any general system of mass, rigid or nonrigid, and expressed as,

$$T = \frac{1}{2} m \overline{v}^2 + \sum \frac{1}{2} m_i |\boldsymbol{\rho}_i|^2$$

where $\overline{v}$ is the velocity of the mass center and $\boldsymbol{\rho}_i$ is the position vector of a representative element of mass m_i with respect to the mass center. The first term is the kinetic energy due to the translation of the system and the second term is the kinetic energy associated with the motion relative to the mass center. The translational term may be written as

$$\frac{1}{2} m \overline{v}^2 = \frac{1}{2} m \dot{\overline{\boldsymbol{r}}} \cdot \dot{\overline{\boldsymbol{r}}} = \frac{1}{2} \overline{\boldsymbol{v}} \cdot \boldsymbol{G}$$

where $\dot{\overline{\boldsymbol{r}}}$ is the velocity $\overline{\boldsymbol{v}}$ of the mass center and $\boldsymbol{G}$ is the linear momentum of the body.

For a rigid body, the relative term becomes the kinetic energy due to rotation about the mass center. Because $\dot{\boldsymbol{\rho}}_i$ is the velocity of the representative particle with respect to the mass center, then for the rigid body we may write it as $\dot{\boldsymbol{\rho}}_i = \boldsymbol{\omega} \times \boldsymbol{\rho}_i$, where $\boldsymbol{\omega}$ is the angular velocity of the body. With this substitution, the relative term in the kinetic energy expression becomes

$$\sum \tfrac{1}{2} m_i |\boldsymbol{\rho}_i|^2 = \sum \tfrac{1}{2} m_i (\boldsymbol{\omega} \times \boldsymbol{\rho}_i) \cdot (\boldsymbol{\omega} \times \boldsymbol{\rho}_i) = \tfrac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{H}_G$$

Thus, the general expression for the kinetic energy of a rigid body moving with mass-center velocity $\overline{v}$ and angular velocity $\boldsymbol{\omega}$ is

$$T = \tfrac{1}{2} \overline{\boldsymbol{v}} \cdot \boldsymbol{G} + \tfrac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{H}_G. \qquad \text{...............(11)}$$

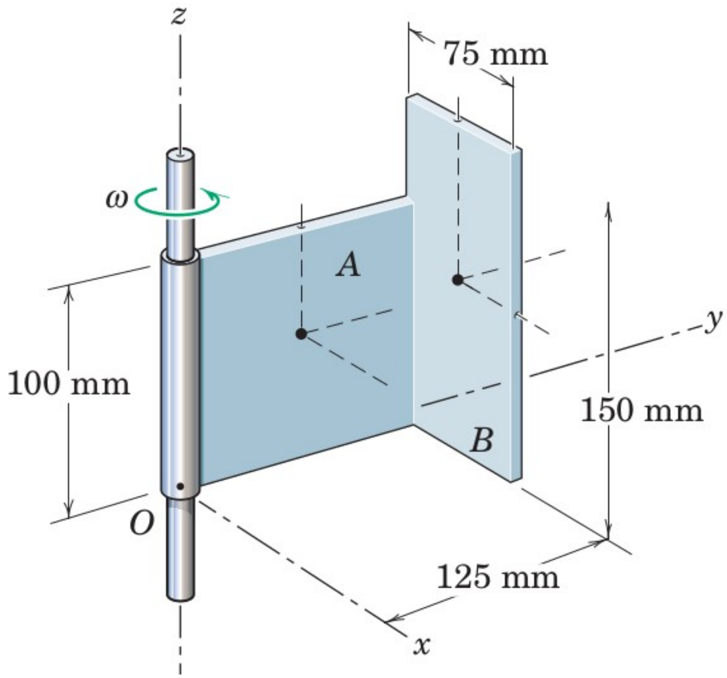
When a rigid body is pivoted about a fixed point O or when there is a point O in the body which momentarily has zero velocity, the kinetic energy is

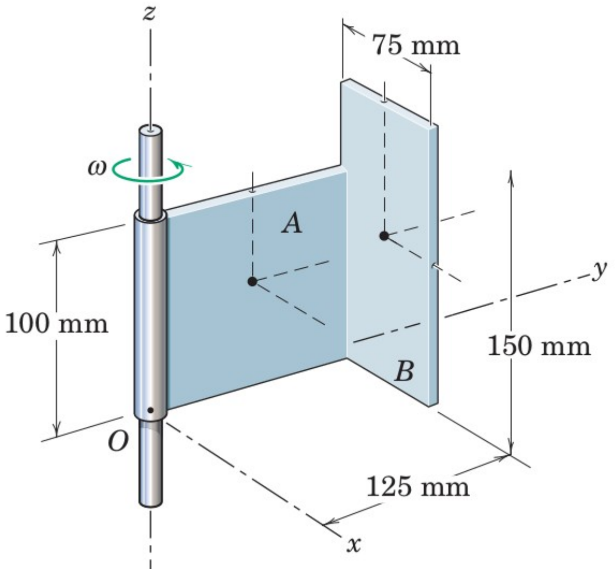
$$T = \tfrac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{H}_O, \qquad \text{...............(11a)}$$

where $\boldsymbol{H}_O$ is the angular momentum about O.

Example 4

The bent plate has a mass of 70 kg/m^2 of surface area and revolves about the z-axis at the rate $\omega = 30$ rad/s. Determine (a) the angular momentum $\boldsymbol{H}$ of the plate about point O and (b) the kinetic energy T of the plate. Neglect the mass of the hub and the thickness of the plate compared with its surface dimensions.





The masses of the parts are

$m_A = (0.100)(0.125)(70) = 0.875$ kg, and

$m_B = (0.075)(0.150)(70) = 0.788$ kg.

Moments and products of inertia can be calculated w.r.t. x-y-z axis.

For part A:

$$I_{xx} = \overline{I}_{xx} + m_A d^2 = 0.00747 \text{ kg} \cdot \text{m}^2$$

$$I_{yy} = \tfrac{1}{3} m_A l^2 = 0.00292 \text{ kg} \cdot \text{m}^2$$

$$I_{zz} = \tfrac{1}{3} m_A l^2 = 0.00456 \text{ kg} \cdot \text{m}^2$$

$$I_{xy} = I_{xz} = 0$$

$$I_{yz} = \overline{I}_{yz} + m_A d_y d_z = 0.00273 \text{ kg} \cdot \text{m}^2$$

For part B:

$$I_{xx} = \overline{I}_{xx} + m_B d^2 = 0.01821 \text{ kg} \cdot \text{m}^2$$

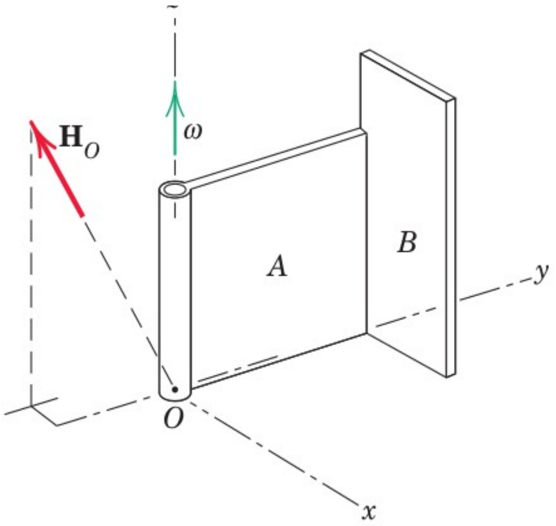
$$I_{yy} = \overline{I}_{yy} + m_B d^2 = 0.00738 \text{ kg} \cdot \text{m}^2$$

$$I_{zz} = \overline{I}_{zz} + m_B d^2 = 0.01378 \text{ kg} \cdot \text{m}^2$$

$$I_{xy} = \overline{I}_{xy} + m_A d_x d_y = 0.00369 \text{ kg} \cdot \text{m}^2$$

$$I_{xz} = \overline{I}_{xz} + m_A d_x d_z = 0.00221 \text{ kg} \cdot \text{m}^2$$

$$I_{yz} = \overline{I}_{yz} + m_A d_y d_z = 0.00738 \text{ kg} \cdot \text{m}^2$$



Angular momentum of the body is

$$\boldsymbol{H}_O = (I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\,\boldsymbol{i} + (-I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z)\,\boldsymbol{j} + (-I_{zx}\omega_x - I_{yz}\omega_y + I_{zz}\omega_z)\,\boldsymbol{k}$$

$\omega_x = \omega_y = 0$ and $\omega_z = \omega.$

Thus, H_O can be calculated.

The kinetic energy can be calculated as $\mathrm{T} = \frac{1}{2}\,\boldsymbol{\omega}\cdot\boldsymbol{H}_O = 8.25$ J.

Momentum and energy equations of motion

Momentum equations:

The general linear- and angular-momentum equations for a system of constant mass are

$$\sum \boldsymbol{F} = \dot{\boldsymbol{G}} \quad \text{and} \quad \sum \boldsymbol{M} = \dot{\boldsymbol{H}}.$$

In the derivation of the moment principle, the derivative of $\boldsymbol{H}$ was taken with respect to an absolute coordinate system. When $\boldsymbol{H}$ is expressed in terms of components measured relative to a moving coordinate system x-y-z which has an angular velocity $\boldsymbol{\Omega}$, then we can write,

$$\begin{aligned}\sum M &= \left(\frac{d\boldsymbol{H}}{dt}\right)_{xyz} + \boldsymbol{\Omega} \times \boldsymbol{H} \\ &= \left(\dot{H}_x\boldsymbol{i} + \dot{H}_y\boldsymbol{j} + \dot{H}_z\boldsymbol{k}\right) + \boldsymbol{\Omega} \times \boldsymbol{H}\end{aligned}$$

$$\begin{aligned}\Rightarrow \sum \boldsymbol{M} =& (\dot{H}_x - H_y\Omega_z + H_z\Omega_y)\boldsymbol{i} \\ &+ (\dot{H}_y - H_z\Omega_x + H_x\Omega_z)\boldsymbol{j} \\ &+ (\dot{H}_z - H_x\Omega_y + H_y\Omega_x)\boldsymbol{k}\end{aligned}$$

...............(12)

(12) is the most general form of the moment equation about a fixed point O or about the mass center G.

For a rigid body where the coordinate axes are attached to the body, *the moments and products of inertia are invariant with time*, (when expressed in x-y-z coordinates) and $\boldsymbol{\Omega} = \boldsymbol{\omega}$. Thus, for axes attached to the body, (12) becomes

$$\begin{aligned} \sum \boldsymbol{M} =& (\dot{H}_x - H_y\omega_z + H_z\omega_y)\boldsymbol{i} \\ &+ (\dot{H}_y - H_z\omega_x + H_x\omega_z)\boldsymbol{j} \\ &+ (\dot{H}_z - H_x\omega_y + H_y\omega_x)\boldsymbol{k} \end{aligned} \qquad \cdots\cdots\cdots\cdots\cdots(13)$$

(13) is the most general moment equations for rigid-body motion with axes attached to the body. They hold w.r.t. axes through a fixed point O or through the mass center G.

Energy equations:

The resultant of all external forces acting on a rigid body may be replaced by the resultant force $\Sigma \boldsymbol{F}$ acting through the mass center and a resultant couple $\Sigma \boldsymbol{M}_G$ acting about the mass center. Work is done by the resultant force and the resultant couple at the respective rates $\sum \boldsymbol{F} \cdot \overline{\boldsymbol{v}}$ and $\sum \boldsymbol{M}_G \cdot \boldsymbol{\omega}$, where $\overline{\boldsymbol{v}}$ is the linear velocity of the mass center and $\boldsymbol{\omega}$ is the angular velocity of the body. Integration over the time from condition 1 to condition 2 gives the total work done during the time interval. Equating the work done to the respective changes in the kinetic energy as (11) gives

$$\int_{t_1}^{t_2} \sum \boldsymbol{F} \cdot \overline{\boldsymbol{v}}\, dt = \frac{1}{2} \overline{\boldsymbol{v}} \cdot \boldsymbol{G} \Big|_1^2, \quad \int_{t_1}^{t_2} \sum \boldsymbol{M}_G \cdot \overline{\boldsymbol{\omega}}\, dt = \frac{1}{2} \overline{\boldsymbol{\omega}} \cdot \boldsymbol{H}_G \Big|_1^2, \qquad \cdots\cdots\cdots\cdots\cdots (14)$$

These equations express the change in translational and rotational kinetic energy, respectively, for the interval during which $\Sigma \boldsymbol{F}$ or $\Sigma \boldsymbol{M}_G$ acts, and the sum of two expressions equals ΔT.

Parallel-plane motion:

When all particles of a rigid body moves in planes which are parallel to a fixed plane, the body has a general form of plane motion. Every line is such a body which is normal to the fixed plane remains parallel to itself at all times. Considering G as the origin of coordinates x-y-z which are attached to the body, with the x-y plane coinciding with the plane of motion P. The components of the angular velocity of both the body and he attached axes become $\omega_x=\omega_y=0$, $\omega_z\neq0$. For this case, the angular-momentum components are

$$H_x = -I_{xz}\omega_z \qquad H_y = -I_{yz}\omega_z \qquad H_z = I_{zz}\omega_z.$$

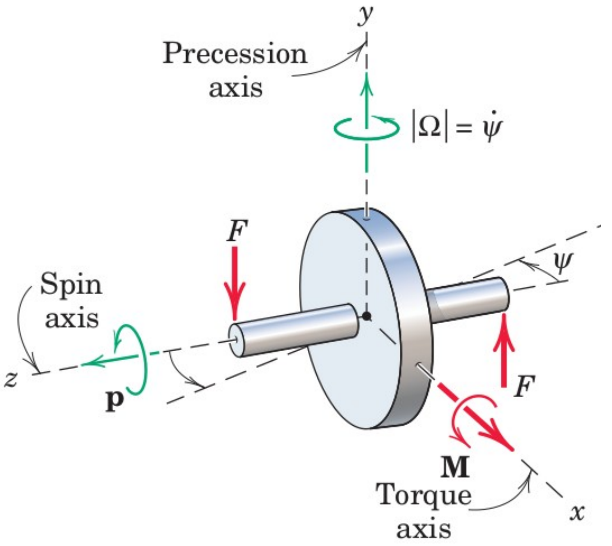
and the moment relations reduce to

$$\sum M_x = -\, I_{xz}\dot{\omega}_z + I_{yz}\omega_z^2, \quad \sum M_y = \quad -I_{yz}\dot{\omega}_z + I_{xz}\omega_z^2, \quad \sum M_z = -\, I_{zz}\dot{\omega}_z.$$

Gyroscopic motion: Steady precession

- Gyroscopic motion is one of the most interesting of all problems in dynamics.
- This motion occurs whenever the axis about which a body is spinning is itself rotating about another axis.
- Although the complete description of this motion involves considerable complexity, we will focus on a most common and useful examples of gyroscopic motion occur when the axis of a rotor spinning at constant speed turns (precesses) about another axis at a steady rate.
- Let us first understand the dynamics of this motion. Then we will look at the engineering applications of gyroscopic motion (or gyroscope)

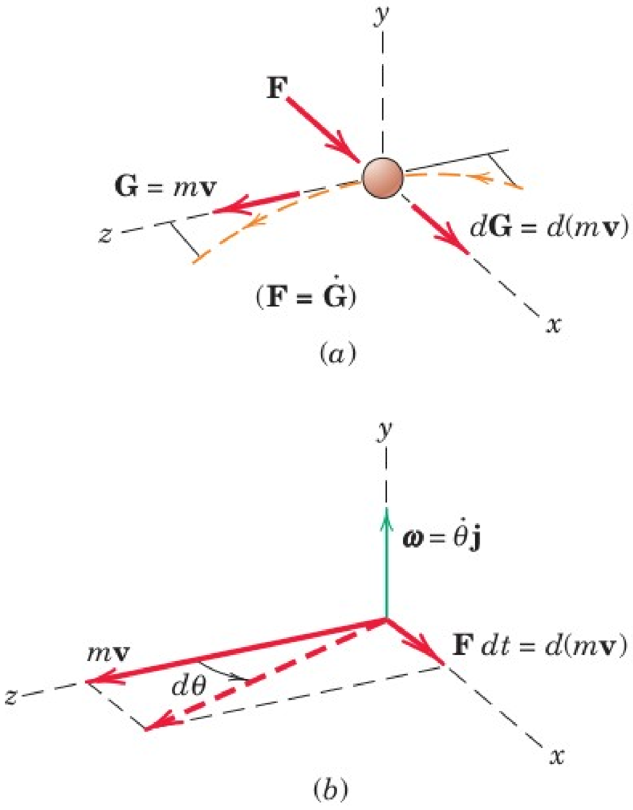
Figure shows a symmetrical rotor spinning about the z-axis with a large angular velocity $\boldsymbol{p}$, known as the **spin velocity**. If we apply two forces $\boldsymbol{F}$ to the rotor axle to form a couple $\boldsymbol{M}$ whose vector is directed along the x-axis, we will find that the rotor shaft rotates in the x-z plane about the y-axis in the sense indicated, with a relatively slow angular velocity Ω known as the **precession velocity**. Thus, we identify the spin axis ($\boldsymbol{p}$), the torque axis ($\boldsymbol{M}$), and the precession axis ($\boldsymbol{\Omega}$), where the usual right-hand rule identifies the sense of the rotation vectors.



Note that, the rotor shaft does not turn about the x-axis in the sense of $\boldsymbol{M}$, as it would if the rotor were not spinning.

For understanding this phenomenon, a direct analogy may be made between the rotation vectors and the familiar vectors describing the curvilinear motion of a particle

Figure shows a particle of mass m moving in the x-z plane with constant speed v. The application of a force $\boldsymbol{F}$ normal to its linear momentum $\boldsymbol{G} = m\boldsymbol{v}$ causes a change $d\boldsymbol{G} = d(m\boldsymbol{v})$ in its momentum. We see that $d\boldsymbol{G}$, and thus d$\boldsymbol{v}$, is a vector in the direction of the normal force $\boldsymbol{F}$ according to Newton's second law $\boldsymbol{F} = \dot{\boldsymbol{G}}$, which may be written as $\boldsymbol{F}dt = d\boldsymbol{G}$.



We see that, in the limit, $\tan d\theta = d\theta = Fdt/mv$ or $F = mv\dot{\theta}$.

In vector notation with $\boldsymbol{\omega} = \dot{\theta}\boldsymbol{j}$, the force becomes

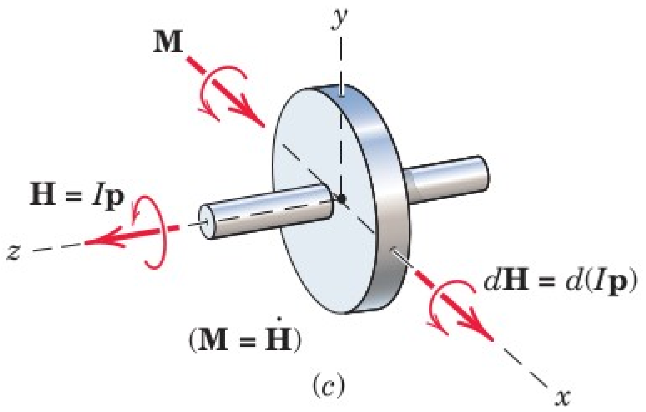
$$\boldsymbol{F} = m\boldsymbol{\omega} \times \boldsymbol{v} \quad \cdots\cdots\cdots\cdots\cdots(15)$$

which is the vector equivalent of scalar relation $F_n = ma_n$ for the normal force on the particle.

With these relations in mind, now we discuss the problem of rotation. Recall now the analogous equation $\boldsymbol{M} = \dot{\boldsymbol{H}}$ which we developed for any prescribed mass system, rigid or nonrigid, referred to its mass center or to a fixed point O.

Now apply this relation to symmetrical rotor. For a high rate of spin $\boldsymbol{p}$ and a low precession rate $\boldsymbol{\Omega}$ about the y-axis, the angular momentum is represented by the vector $\boldsymbol{H} = I\boldsymbol{p}$, where $I = I_{zz}$ is the moment of inertia of the rotor about the spin axis.

Initially, we neglect the small component of angular momentum about the y-axis which accompanies the slow precession. The application of the couple $\boldsymbol{M}$ normal to $\boldsymbol{H}$ causes a change $d\boldsymbol{H} = d(I\boldsymbol{p})$ in the angular momentum. We see that $d\boldsymbol{H}$, and thus $d\boldsymbol{p}$, is a vector in the direction of the couple $\boldsymbol{M}$ since $\boldsymbol{M} = \dot{\boldsymbol{H}}$, which may also be written $\boldsymbol{M}dt = d\boldsymbol{H}$. Just as the change in the linear-momentum vector of the particle is in the direction of the applied force, so is the change in the angular-momentum vector of the gyro in the direction of the couple.



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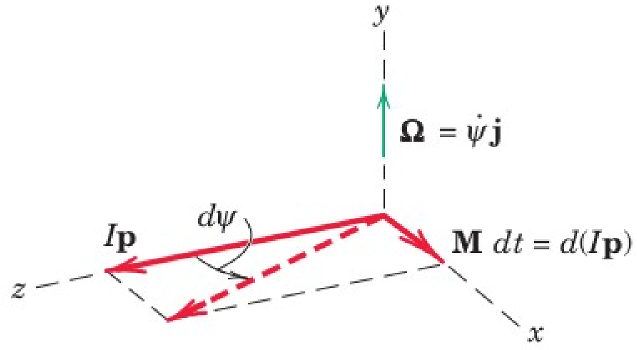
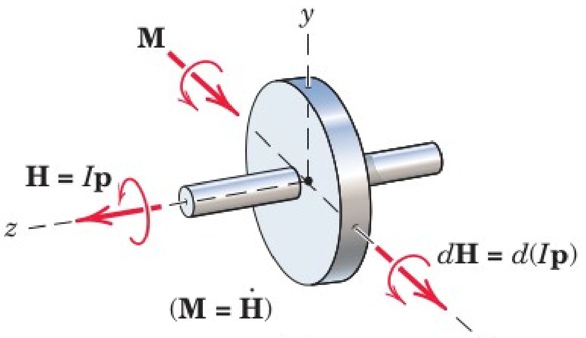


(d)

Thus, we see that the vectors $\boldsymbol{M}$, $\boldsymbol{H}$, and $d\boldsymbol{H}$ are analogous to the vectors $\boldsymbol{F}$, $\boldsymbol{G}$, and $d\boldsymbol{G}$. With this insight, it is no longer strange to see the rotation vector undergo a change which is in the direction of $\boldsymbol{M}$, thereby causing the axis of the rotor to precess about the y-axis.

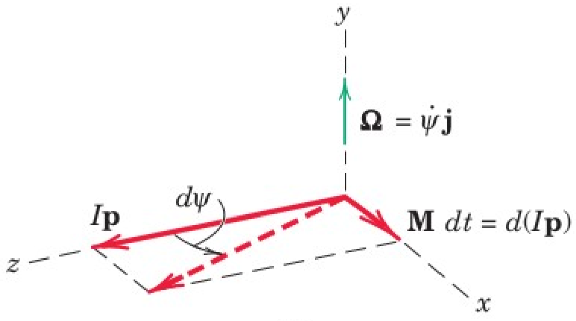


(c)

It can be seen that during time dt the angular-momentum vector $\boldsymbol{Ip}$ has swung through the angle $d\psi$, so that in the limit with

$\tan d\psi = d\psi$, we have

$$d\psi = \frac{M\,dt}{IP} \quad \text{or} \quad M = I\frac{d\psi}{dt} = I\Omega p. \qquad \text{.............(16)}$$



(d)

Note that $\boldsymbol{M}$, $\boldsymbol{\Omega}$, and $\boldsymbol{p}$ are mutually perpendicular vectors, and it can be written as

$$\boldsymbol{M} = \boldsymbol{I\Omega} \times \boldsymbol{p}, \qquad \text{.............(16a)}$$

which is analogous to the relation (15) applicable for curvilinear motion of particle.

The correct spatial relationship among the three vectors may be remembered from the fact that $d\boldsymbol{H}$, and thus $d\boldsymbol{p}$, is in the direction of $\boldsymbol{M}$, which establishes the correct sense for the precession $\boldsymbol{\Omega}$. Therefore, the spin vector $\boldsymbol{p}$ always tends to rotate toward the torque vector $\boldsymbol{M}$.

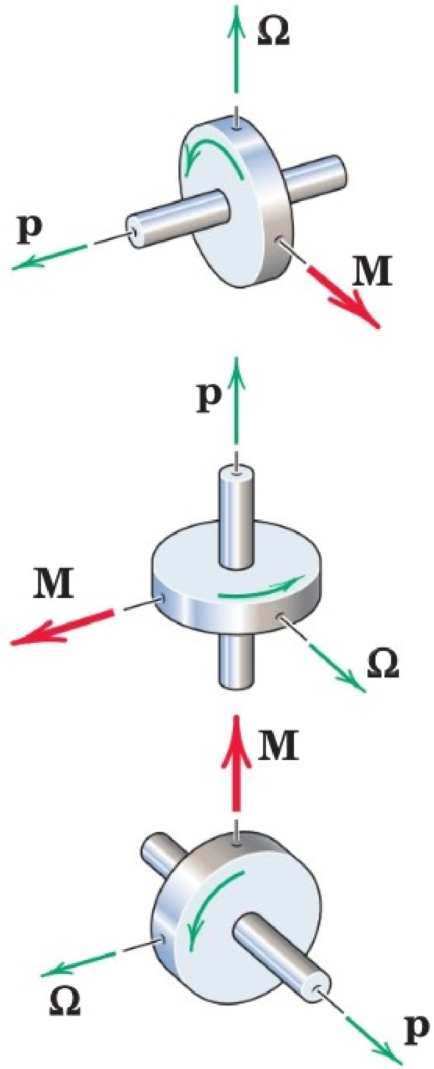


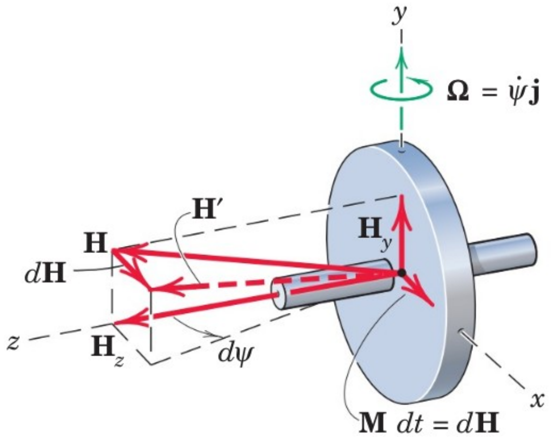
Figure shows three orientations of the three vectors which are consistent with their correct order.

Remember that, $\boldsymbol{M} = I\boldsymbol{\alpha}$, is an equation of motion, where $\boldsymbol{M}$ represents the couple due to all forces acting on the rotor, as disclosed by a correct free-body diagram of the rotor.

When a rotor is forced to precess, as occurs with the turbine in a ship which is executing a turn, the **motion will generate a gyroscopic couple $\boldsymbol{M}$** which obeys (16a) in both magnitude and sense.

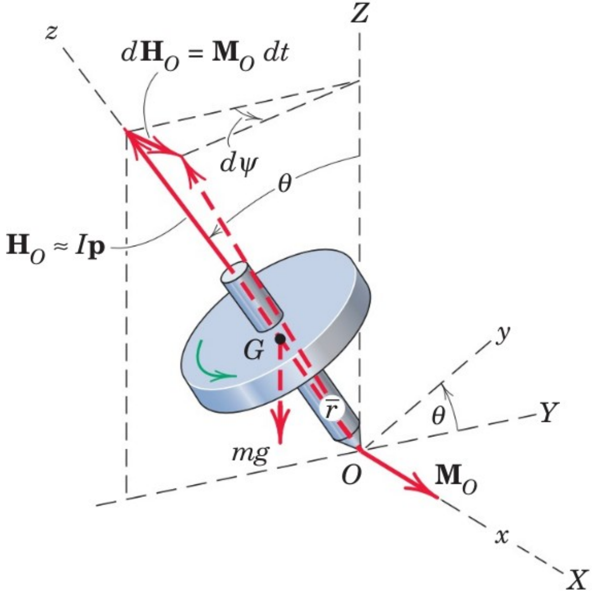
In the foregoing discussion of gyroscopic motion, it was assumed that **the spin was large and the precession was small.** Although we can see from (16) that for given values of I and $\boldsymbol{M}$, the precession $\boldsymbol{\Omega}$ must be small if $\boldsymbol{p}$ is large, let us now examine the influence of $\boldsymbol{\Omega}$ on the momentum relations.

We restrict our attention to steady precession, where $\boldsymbol{\Omega}$ has a constant magnitude. Figure shows the same rotor again. Because it has a moment of inertia about the y-axis and an angular velocity of precession about this axis, there will be an additional component of angular momentum about the y-axis. Thus, we have the two components $H_z = Ip$ and $H_y = I_O \Omega$, where $I_O = I_{yy}$ and $I = I_{zz}$. The total angular momentum is $\boldsymbol{H}$ as shown.



The change in $\boldsymbol{H}$ remains $d\boldsymbol{H} = \boldsymbol{M}\,dt$ as previously, and the precession during time dt is the angle $d\psi = Mdt/H_z = M\,dt/(Ip)$ as before. Thus, (16) is still valid and for steady precession is an exact description of the motion as long as the spin axis is perpendicular to the axis around which precession occurs.

Consider now the steady precession of a symmetrical top spinning about its axis with a high angular velocity $\boldsymbol{p}$ and supported at its point O. Here the spin axis makes an angle θ with the vertical Z-axis around which precession occurs. Again, we will neglect the small angular-momentum component due to the precession and consider $\boldsymbol{H} = I\boldsymbol{p}$, the angular momentum about the axis of the top associated with the spin only. The moment about O is due to the weight and is $mgr\sin\theta$, where r is the distance from O to the mass center G. From the diagram, we see that the angular-momentum vector $\boldsymbol{H}_O$ has a change $d\boldsymbol{H}_O = \boldsymbol{M}_O\, dt$ in the direction of M_O during time dt and that θ is unchanged. The increment in precessional angle around the Z-axis is



$$d\psi = \frac{M_O dt}{Ip\sin\theta}.$$

Substituting, $M_O = mg\bar{r}\sin\theta$ and $\Omega = d\psi/dt$, we get, $mg\bar{r}\sin\theta = I\Omega p\sin\theta$ or $mg\bar{r} = I\Omega p$, which is independent of θ. Introducing the radius of gyration so that $I = mk^2$ and solving for the precessional velocity give

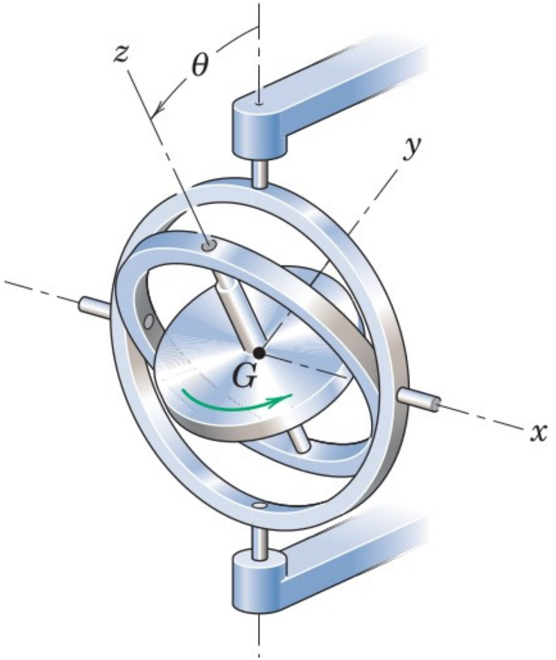
$$\Omega = \frac{g\bar{r}}{k^2 p}. \qquad \cdots\cdots\cdots\cdots\cdots(17)$$

(17) is an approximation based on the assumption that the angular momentum associated with $\boldsymbol{\Omega}$ is negligible compared with that associated with $\boldsymbol{p}$. On the basis of our analysis, the top will have a steady precession at the constant angle only if it is set in motion with a value of $\boldsymbol{\Omega}$ which satisfies (17). When these conditions are not met, the precession becomes unsteady, and may oscillate with an amplitude which increases as the spin velocity decreases. The corresponding rise and fall of the rotation axis is called **nutation**.

Applications of Gyroscope

The gyroscope has important engineering applications.

With a mounting in gimbal rings, the gyro is free from external moments, and its axis will retain a fixed direction in space regardless of the rotation of the structure to which it is attached. In this way, the gyro is used for inertial guidance systems and other directional control devices.

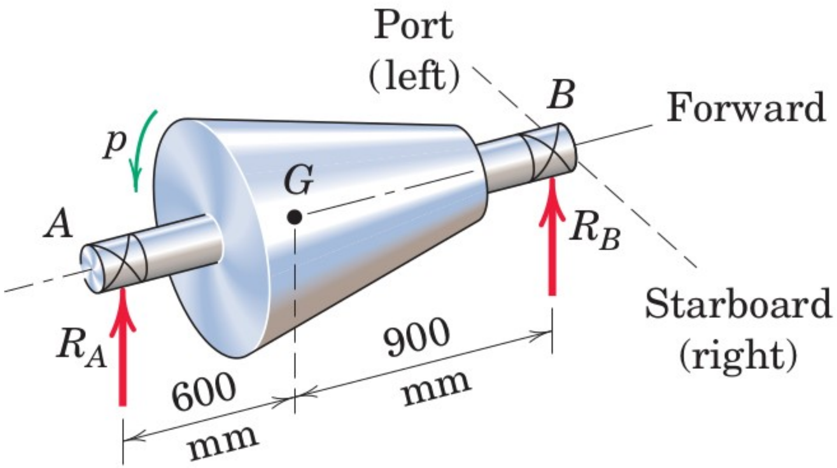


The gyroscope has also found important use as a stabilizing device. The controlled precession of a large gyro mounted in a ship is used to produce a gyroscopic moment to counteract the rolling of a ship at sea.

The gyroscopic effect is also an extremely important consideration in the design of bearings for the shafts of rotors which are subjected to forced precessions.

Example 5

The turbine rotor in a ship's power plant has a mass of 1000 kg, with center of mass at G and a radius of gyration of 200 mm. The rotor shaft is mounted in bearings A and B with its axis in the horizontal fore-and-aft direction and turns counterclockwise at a speed of 5000 rev/min when viewed from the stern. Determine the vertical components of the bearing reactions at A and B if the ship is making a turn to port (left) of 400-m radius at a speed of 25 knots (1 knot $=$ 0.514 m/s). Does the bow of the ship tend to rise or fall because of the gyroscopic action?



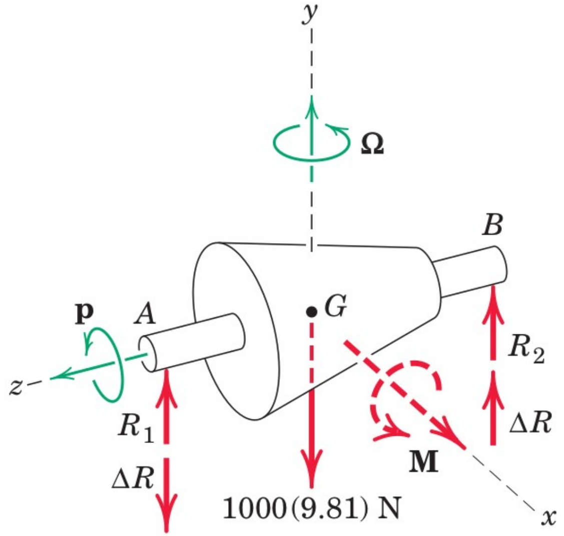
The vertical component of the bearing reactions will equal the static reactions R_1 and R_2 due to the weight of the rotor, plus or minus the increment ΔR due to the gyroscopic effect.

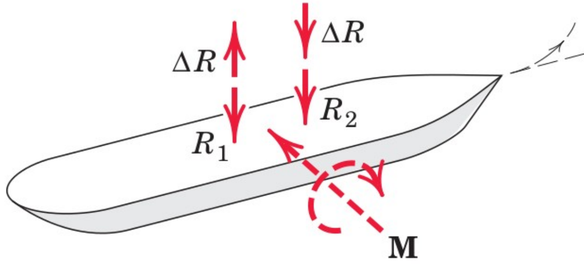
R_1 and R_2 can be easily calculated from the principles of statics (moment principle) as $R_1 = 5890$ N and $R_2 = 3920$ N.

The given directions of the spin velocity $\boldsymbol{p}$ and the precession velocity $\boldsymbol{\Omega}$ are shown with the free-body diagram of the rotor. Because the spin axis always tends to rotate toward the torque axis, the torque axis $\boldsymbol{M}$ points in the starboard direction.

The sense of the ΔR's is, therefore, up at B and down at A to produce the couple M. Thus, the bearing reactions at A and B are

$$R_A = R_1 - \Delta R \qquad \text{and} \qquad R_B = R_2 + \Delta R.$$





The precession velocity Ω is the speed of the ship divided by the radius of its turn, i.e.,

$$\Omega = v/\rho = 25(0.514)/400 = 0.321 \text{ rad/s}$$

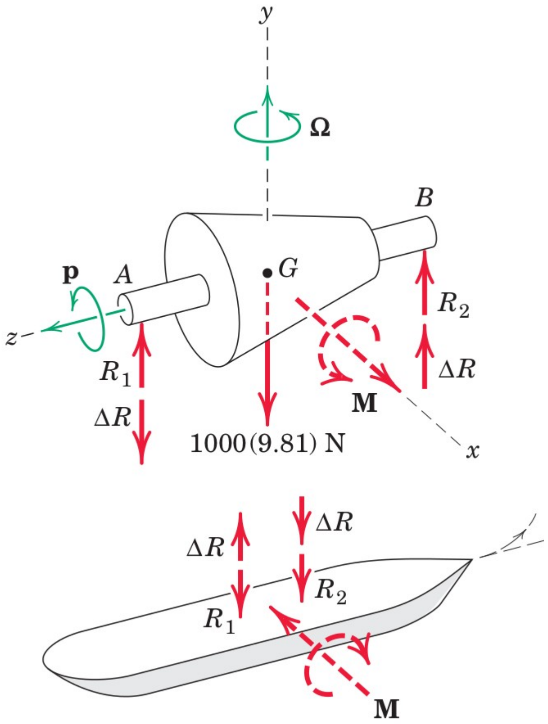
From the relation between M, Ω, and p, which is (about G)

$$M = I\,\Omega\,p$$

$1.5(\Delta R) = 1000(0.2)^2(0.0321)(2\pi \cdot 5000/60)$,

which gives

$\Delta R = 449$ N.



Now observe that the forces we just computed are exerted on the rotor shaft by the structure of the ship.

Consequently, from the principle of action and reaction, the equal and opposite forces are applied to the ship by the rotor shaft. Therefore, the effect of the gyroscopic couple is to generate the increments ΔR shown, and the bow will tend to fall and the stern to rise (but only slightly).