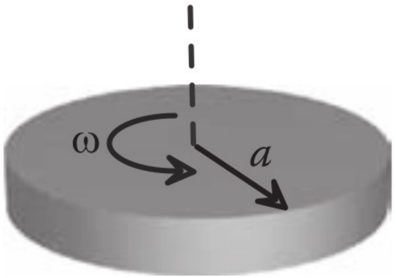
Spinning circular plate



Consider a thin solid plate with radius a that spins with angular speed ω about its axis. Following are our assumptions:

- No body forces act on the disk.
- The disk has constant angular velocity.
- The outer surface $r = a$ and the top and bottom faces of the disk are free of traction.
- The disk is sufficiently thin to ensure a state of plane stress in the disk.

It can be easily observed that geometry is symmetric; hence the solution also exhibits the cylindrical symmetry. It also implies that the problem is independent of and ϕ direction.

Then, the Navier's equation become,

$$(\lambda + 2\mu)\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru)\right] = -\rho_0\omega^2 r$$

The equation can be integrated to see that

$$u = Ar + \frac{B}{r} - \frac{\left(1 - \nu^2\right)}{8E} \rho_0 \omega^2 r^3 .$$

The radial stress follows as

$$\sigma_{rr} = \frac{E}{1 - \nu^2}\left(\frac{du}{dr} + \nu \frac{u}{r}\right) = \frac{E}{1 - \nu^2}\left\{(1 + \nu)A - (1 - \nu)\frac{B}{r^2} - \frac{\left(1 - \nu^2\right)\rho_0 \omega^2}{8E}(3 + \nu)r^2\right\}.$$

The radial stress must be bounded at $r = 0$, which is only possible if $B = 0$. In addition, the radial stress must be zero at $r = a$, which requires that

$$A = \frac{\rho_0 \omega^2}{8E}\frac{(3 + \nu)}{(1 + \nu)} a^2 .$$

The solution is:

$$\mathbf{u} = (1-\nu)\frac{\rho_0\omega^2}{8E}\left\{(3+\nu)a^2 r - (1+\nu)r^3\right\}\mathbf{e}_r + z\varepsilon_{zz}\mathbf{e}_z$$

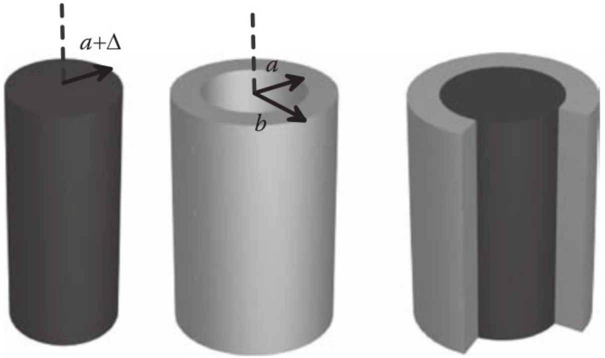
$$\varepsilon_{rr} = (1-\nu)\frac{\rho_0\omega^2}{8E}\left\{(3+\nu)a^2 - 3(1+\nu)r^2\right\} \quad \varepsilon_{\theta\theta} = (1-\nu)\frac{\rho_0\omega^2}{8E}\left\{(3+\nu)a^2 - (1+\nu)r^2\right\}$$

$$\varepsilon_{zz} = -\nu\frac{\rho_0\omega^2}{8E}\left\{2(3+\nu)a^2 - (3\nu+2)r^2\right\}$$

$$\sigma_{rr} = (3+\nu)\frac{\rho_0\omega^2}{8}\left\{a^2 - r^2\right\}$$

$$\sigma_{\theta\theta} = \frac{\rho_0\omega^2}{8}\left\{(3+\nu)a^2 - (3\nu+1)r^2\right\}.$$

Stresses induced by an interference fit between two cylinders



Interference fits are often used to secure a bushing or a bearing housing to a shaft. In this problem, we calculate the stress induced by such an interference fit.

Consider a hollow cylindrical bushing, with outer radius b and inner radius a. Suppose that a solid shaft with radius $a + \Delta$, with $\Delta/a << 1$, is inserted into the cylinder (in practice, this is done by heating the cylinder or cooling the shaft until they fit and then letting the system return to thermal equilibrium).

Our assumptions are:

- No body forces act on the solids.
- The cylinders have uniform temperature.
- The ends of the cylinder are free of force.
- Both the shaft and cylinder have the same Young's modulus E and Poisson's ratio ν.
- The cylinder and shaft are sufficiently long to ensure that a state of generalized plane strain can be developed in each solid.

The solution of the problem can be found using the solution to a pressurized cylinder. After the shaft is inserted into the tube, a pressure p acts to compress the shaft , and the same pressure pushes outward to expand the cylinder. Suppose that this pressure induces a radial displacement $u^S(r)$ in the solid cylinder and a radial displacement $u^C(r)$ in the hollow tube. To accommodate the interference, the displacements must satisfy

$$u^C(r=a) - u^S(r=a) = \Delta.$$

Evaluating the relevant displacements using the formulas in Section 4.1.9 gives

$$u^S(r=a) = -\frac{(1-2\nu)(1+\nu)}{E} pa - \frac{2\nu^2}{E} pa$$

$$u^C(r=a) = \frac{(1+\nu)a^2b^2}{E\left(b^2-a^2\right)}\left\{\frac{p}{a} + (1-2\nu)\frac{pa}{b^2}\right\} + \frac{2\nu^2 pa^3}{E\left(b^2-a^2\right)}.$$

Using above equations, pressure at the interface can be calculated as,

$$p = \frac{E\Delta\left(b^2-a^2\right)}{2ab^2}.$$

This pressure can then be substituted back into the formulas in Section 4.1.9 to evaluate the stresses.

The displacements, strains, and stresses in the solid shaft ($r < a$) are

$$\mathbf{u} = -\frac{(1+\nu)(1-2\nu)\Delta\left(b^2 - a^2\right)}{2ab^2} r\mathbf{e}_r - 2\nu^2 \frac{\Delta\left(b^2 - a^2\right)}{2ab^2} r\mathbf{e}_r + 2\nu \frac{\Delta\left(b^2 - a^2\right)}{2ab^2} z\mathbf{e}_z$$

$$\varepsilon_{rr} = \varepsilon_{\theta\theta} = -\frac{(1+\nu)(1-2\nu)\Delta\left(b^2 - a^2\right)}{2ab^2} - 2\nu^2 \frac{\Delta\left(b^2 - a^2\right)}{2ab^2}$$

$$\sigma_{rr} = \sigma_{\theta\theta} = -\frac{E\Delta\left(b^2 - a^2\right)}{2ab^2} \qquad \sigma_{zz} = 0.$$

In the hollow cylinder, they are

$$\mathbf{u} = \frac{(1+\nu)a}{r}\frac{\Delta}{2}\left\{1 + (1-2\nu)\frac{r^2}{b^2}\right\}\mathbf{e}_r - \nu^2 \frac{\Delta a}{b^2} r\mathbf{e}_r + 2\nu^2 \frac{\Delta a}{b^2} z\mathbf{e}_z$$

$$\varepsilon_{rr} = \frac{(1+\nu)a}{r^2}\frac{\Delta}{2}\left\{-1 + (1-2\nu)\frac{r^2}{b^2}\right\} - \nu^2 \frac{\Delta a}{b^2}$$

$$\varepsilon_{\theta\theta} = \frac{(1+\nu)a}{r^2}\frac{\Delta}{2}\left\{1 + (1-2\nu)\frac{r^2}{b^2}\right\} - \nu^2 \frac{\Delta a}{b^2}$$

$$\sigma_{rr} = \frac{E\Delta a}{2b^2}\left\{1 - \frac{b^2}{r^2}\right\} \qquad \sigma_{\theta\theta} = \frac{E\Delta a}{2b^2}\left\{1 + \frac{b^2}{r^2}\right\} \qquad \sigma_{zz} = 0.$$