ME632: Fracture Mechanics

Timings	
Monday	10:00 to 11:20
Thursday	08:30 to 09:50

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Room No. # 106

Solution of Mode-I crack

For mode-I crack Westergaard suggested Φ in the form of a complex function Z_I as,

$$\Phi = \mathrm{Re}\,\overline{\overline{Z}}_I + x_2\,\mathrm{Im}\,\overline{Z}_I, \qquad \cdots\cdots\cdots\cdots\cdots(80)$$

where $Z_I = \dfrac{\overline{Z}_I}{dz}$, and $\overline{Z}_I = \dfrac{\overline{\overline{Z}}_I}{dz}$.

Check that function Φ satisfies the Biharmonic equation.

Using (9) we can show that,

$$\begin{aligned}\sigma_{11} &= \mathrm{Re}\,Z_I - x_2\,\mathrm{Im}\,Z_I', \\ \sigma_{22} &= \mathrm{Re}\,Z_I + x_2\,\mathrm{Im}\,Z_I', \\ \sigma_{12} &= -x_2\,\mathrm{Re}\,Z_I'.\end{aligned} \qquad \cdots\cdots\cdots\cdots\cdots(81)$$

To solve a given problem, the proper form of the Westergaard function $Z_I(z)$ is chosen such that the stress components, determined through (81), satisfy all the boundary conditions. Once such a function is obtained, the stress field in the vicinity of the crack tip can be easily obtained. The Westergaard function does not solve a problem completely; it only aids in solving a problem. We still have to guess the form of the complex function Z_I in a specific problem.

To determine the displacement field u_1 and u_2 we first write strains in the form of complex functions using stress-strain relations (5) and (6). For plane stress case,

$$\begin{aligned}
\varepsilon_{11} &= \frac{\partial u_1}{\partial x_1} = \frac{1}{E}\left[(\operatorname{Re} Z_I - x_2 \operatorname{Im} Z_I') - \nu\,(\operatorname{Re} Z_I + x_2 \operatorname{Im} Z_I')\right] = \frac{1}{2\mu}\left[\left(\frac{1-\nu}{1+\nu}\right)\operatorname{Re} Z_I - x_2 \operatorname{Im} Z_I'\right], \\
\varepsilon_{22} &= \frac{\partial u_2}{\partial x_2} = \frac{1}{E}\left[(\operatorname{Re} Z_I + x_2 \operatorname{Im} Z_I') - \nu\,(\operatorname{Re} Z_I - x_2 \operatorname{Im} Z_I')\right] = \frac{1}{2\mu}\left[\left(\frac{1-\nu}{1+\nu}\right)\operatorname{Re} Z_I + x_2 \operatorname{Im} Z_I'\right], \\
\varepsilon_{12} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = -\frac{x_2}{2\mu}\operatorname{Re} Z_I'. \qquad \cdots\cdots\cdots\cdots\cdots(82)
\end{aligned}$$

Now first and second equations of (82) can be integrated to determine u_1 and u_2 as

$$\begin{aligned}
u_1 &= \frac{1}{2\mu}\left[\left(\frac{1-\nu}{1+\nu}\right)\operatorname{Re} \overline{Z}_I - x_2 \operatorname{Im} Z_I'\right] + f(x_2), \\
u_2 &= \frac{1}{2\mu}\left[\left(\frac{2}{1+\nu}\right)\operatorname{Im} \overline{Z}_I - x_2 \operatorname{Re} Z_I'\right] + g(x_1),
\end{aligned}$$

It can be shown that in (83) functions $f(x_2)$ and $g(x_1)$ contribute to rigid body displacement and rotation. Since they do not contribute to any strain or stress, they can be discarded safely.

Thus, for plane stress,

$$
\begin{aligned}
u_1 &= \frac{1}{2\mu}\left[\left(\frac{1-\nu}{1+\nu}\right)\mathrm{Re}\,\overline{Z}_I - x_2\,\mathrm{Im}\,Z_I'\right], \\
u_2 &= \frac{1}{2\mu}\left[\left(\frac{2}{1+\nu}\right)\mathrm{Im}\,\overline{Z}_I - x_2\,\mathrm{Re}\,Z_I'\right].
\end{aligned}
\qquad \cdots\cdots\cdots\cdots(83)
$$

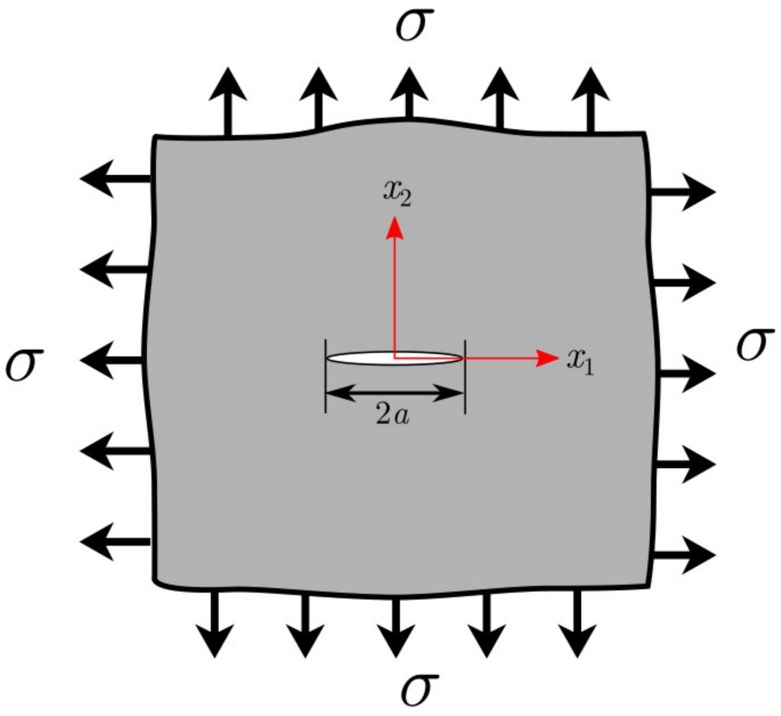
For plane strain,

$$
\begin{aligned}
u_1 &= \frac{1}{2\mu}\left[(1-2\nu)\,\mathrm{Re}\,\overline{Z}_I - x_2\,\mathrm{Im}\,Z_I'\right], \\
u_2 &= \frac{1}{2\mu}\left[2\,(1-\nu)\,\mathrm{Im}\,\overline{Z}_I - x_2\,\mathrm{Re}\,Z_I'\right].
\end{aligned}
\qquad \cdots\cdots\cdots\cdots(83a)
$$

So we have stress and displacement in terms of Westergaard Function Z_I, which is yet to be determined.

Now let us consider the case of infinite plate with through-thickness crack of length $2a$ under a biaxial field of stress σ. By infinite plate we mean that the exterior dimensions of the plate are much larger than the crack length.

The boundary conditions that should be satisfied by the Westergaard function are following:



(i) At the crack tip, (i.e., $x_1 = \pm a$)

$$\sigma_{22} = \infty.$$

(ii) On the crack surface, (i.e., $x_2 = 0, -a < x_1 < a$)

$$\sigma_{22} = \sigma_{12} = 0.$$

(iii) From away from the crack tip, (i.e., $|z| \to \infty$)

$$\sigma_{11} = \sigma_{22} = \sigma \text{ and } \sigma_{12} = 0.$$

Now while guessing the function Z_I above boundary conditions should be taken care by keeping in mind equations (81). Thus the following function is suitable,

$$Z_I(z) = \frac{\sigma z}{\sqrt{(z-a)(z+a)}} = \frac{\sigma z}{\sqrt{(z^2-a^2)}}. \quad \cdots\cdots\cdots\cdots\cdots(84)$$

Now, let us determine the stress field near the crack tip. To do that it is convenient to transform the origin from the center of the crack-tip by substituting $z = a + z_0$, where z_0 is measured from the crack-tip. Thus Z_I becomes,

$$Z_I(z_0) = \frac{\sigma(z_0 + a)}{\sqrt{z_0(z_0 + 2a)}} = \frac{\sigma a(1 + z_0/a)}{\sqrt{2az_0}\sqrt{1 + z_0/2a}} = \frac{\sigma\sqrt{a}(1 + z_0/a)}{\sqrt{2z_0}\sqrt{1 + z_0/2a}}. \quad \text{...............(85)}$$

In the vicinity of the crack-tip $|z_0| \ll a$, then

$$Z_I(z_0) \simeq \frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi z_0}}. \quad \text{...............(86)}$$

Similarly from (81), we can also determine Z_I' in the vicinity of the crack-tip,

$$Z_I'(z_0) \simeq -\frac{\sigma a^2}{(2az_0)^{3/2}}. \quad \text{...............(87)}$$

Writing z_0 in polar coordinate as $z_0 = r(\cos\theta + i\sin\theta)$, we get,

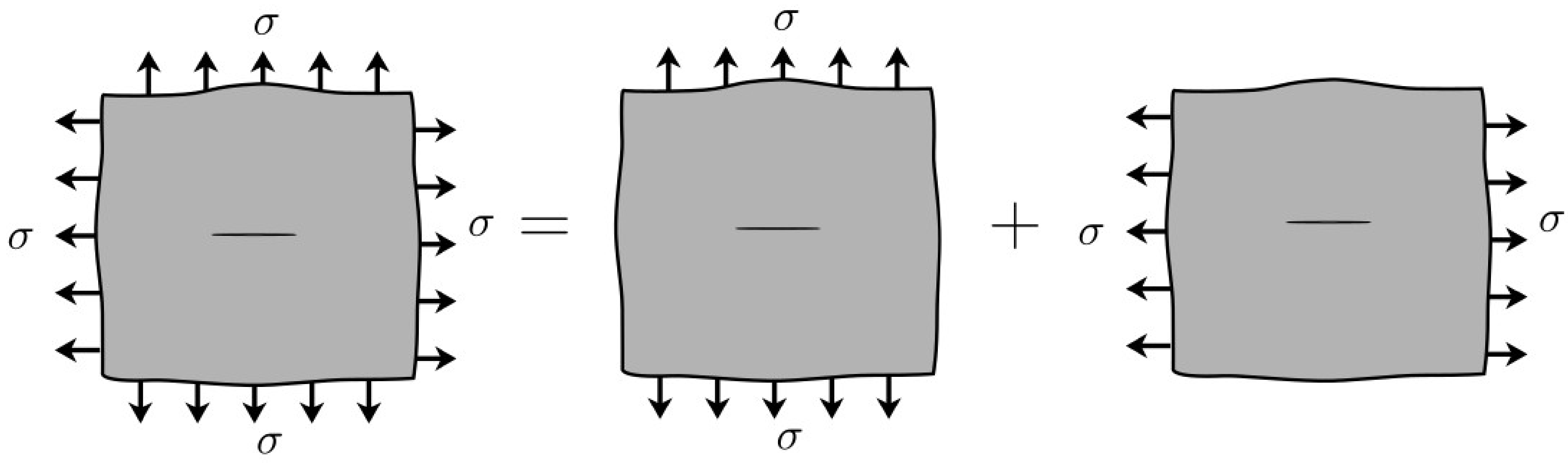
$$Z_I = \frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi r}}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right), \qquad Z_I' = \frac{\sigma\sqrt{\pi a}}{2\sqrt{2\pi}r^{3/2}}\left(\cos\frac{3\theta}{2} - i\sin\frac{3\theta}{2}\right). \quad \text{...............(88)}$$

Using (88) in (81), we can now determine the stress components,

$$\begin{aligned}
\sigma_{11} &= \frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right), \\
\sigma_{22} &= \frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right), \\
\sigma_{12} &= \frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi r}} \sin\theta \sin\frac{3\theta}{2}\cos\frac{3\theta}{2}.
\end{aligned} \qquad \cdots\cdots\cdots\cdots\cdots(89)$$

Comparing (89) will (49) it can be observed that, for an infinite body having through-thickness crack with far-field stress σ stress intensity factor is $K_I = \sigma\sqrt{\pi a}$.

It may be argued that comparing (89) and (49) is not valid, as both are not the same configuration. However, we will justify the comparison by showing that the solution of biaxial loading is not different from the mode-I loading.



Since we are working with small deformations and linear elasticity, principal of superposition is applicable. The biaxial loading can be assumed as the superposition of two configurations,

(i) loading perpendicular to the crack,

(ii) loading parallel to the crack.

Observe that the configuration (i) is nothing but mode-I which tries to open the crack, whereas configuration (ii) does not try to open the crack. Though it modifies the stress-field near the crack-tip to a certain extent. However, the solution of configuration (ii) is not simple and we usually neglect its effect in engineering solutions to fracture mechanics.

Therefore, for most of the practical purposes, the solution of a biaxially loaded plate is employed for both uniaxially and biaxially loaded problems.

Solution for the displacement can be obtained by calculating $\overline{Z_I}$ from (86) and substituting (83) as,

$$\text{(for plane stress)} \quad \begin{aligned} u_1 &= \frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}\left(\frac{1-\nu}{1+\nu}+\sin^2\frac{\theta}{2}\right), \\ u_2 &= \frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left(\frac{2}{1+\nu}-\cos^2\frac{\theta}{2}\right). \end{aligned} \qquad \cdots\cdots\cdots\cdots(90)$$

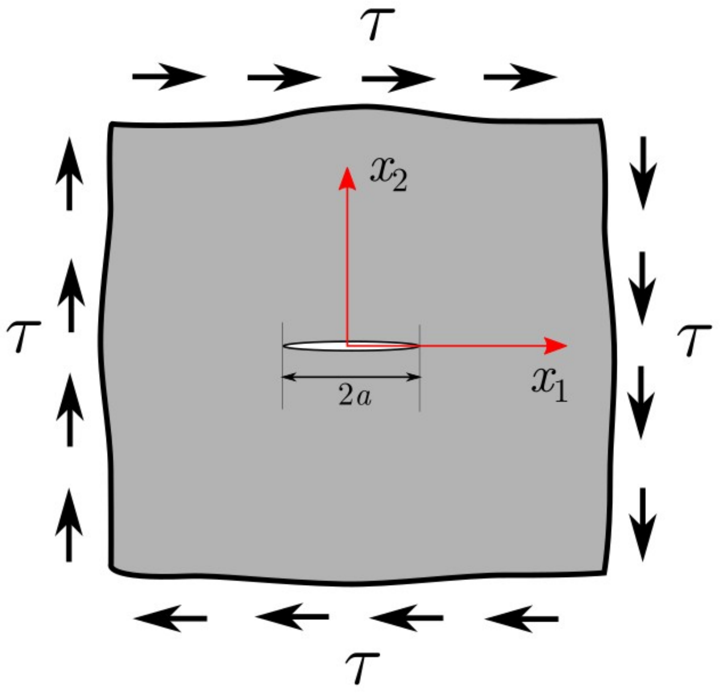
$$\text{(for plane strain)} \quad \begin{aligned} u_1 &= \frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}\left(1-2\nu+\sin^2\frac{\theta}{2}\right), \\ u_2 &= \frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left(2-2\nu-\cos^2\frac{\theta}{2}\right). \end{aligned} \qquad \cdots\cdots\cdots\cdots(90a)$$

The distance between the two crack faces is as the Crack Opening Displacement (COD) and it is useful from experimental point of view. Thus for plane stress equations (83),

$$\mathrm{COD} = 2\times u_2|_{x_2=0} = \frac{1}{\mu}\frac{2}{1-\nu}(\mathrm{Im}\, Z_I')_{x_2=0}. \qquad \cdots\cdots\cdots\cdots(91)$$

Using (84), we get, $\mathrm{COD} = \dfrac{4\sigma}{E}\sqrt{a^2-x_1^2}$, with $(\mathrm{COD})_{\max} = \dfrac{4\sigma a}{E}$, at $x_1 = 0$. $\cdots\cdots\cdots\cdots(92)$

For Mode-II crack



For an infinite plate with center-crack subjected to Mode II loading the following expression of the Airy Stress Function Φ is suggested:

$$\Phi = -x_2 \operatorname{Re} \overline{Z}_{II}. \qquad \cdots\cdots\cdots\cdots\cdots(93)$$

Stress field is given as,

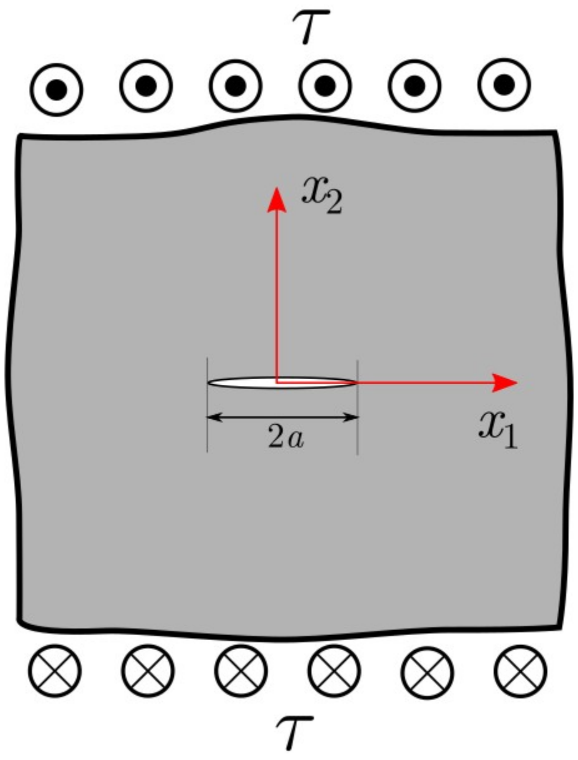
$$\begin{aligned} \sigma_{11} &= 2 \operatorname{Im} Z_{II} + x_2 \operatorname{Re} Z'_{II}, \\ \sigma_{22} &= -x_2 \operatorname{Re} Z'_{II}, \\ \sigma_{12} &= \operatorname{Re} Z_{II} - x_2 \operatorname{Im} Z'_{II}. \end{aligned} \qquad \cdots\cdots\cdots\cdots\cdots(94)$$

Displacement field for plane stress and plane strain can be derived.

Westergaard function for Mode-II is

$$Z_{II} = \frac{\tau z}{\sqrt{z^2 - a^2}}. \qquad \cdots\cdots\cdots\cdots\cdots(95)$$

For Mode-III crack



For mode-III crack, out-of-plane displacement u_z is defined in terms of complex functions. For an infinite plate with center-crack subjected to Mode III loading the following expression of u_z is suggested:

$$u_z = \frac{1}{\mu}\,\mathrm{Im}\,Z_{III}. \qquad \cdots\cdots\cdots\cdots\cdots(96)$$

Non-zero stress components are given as,

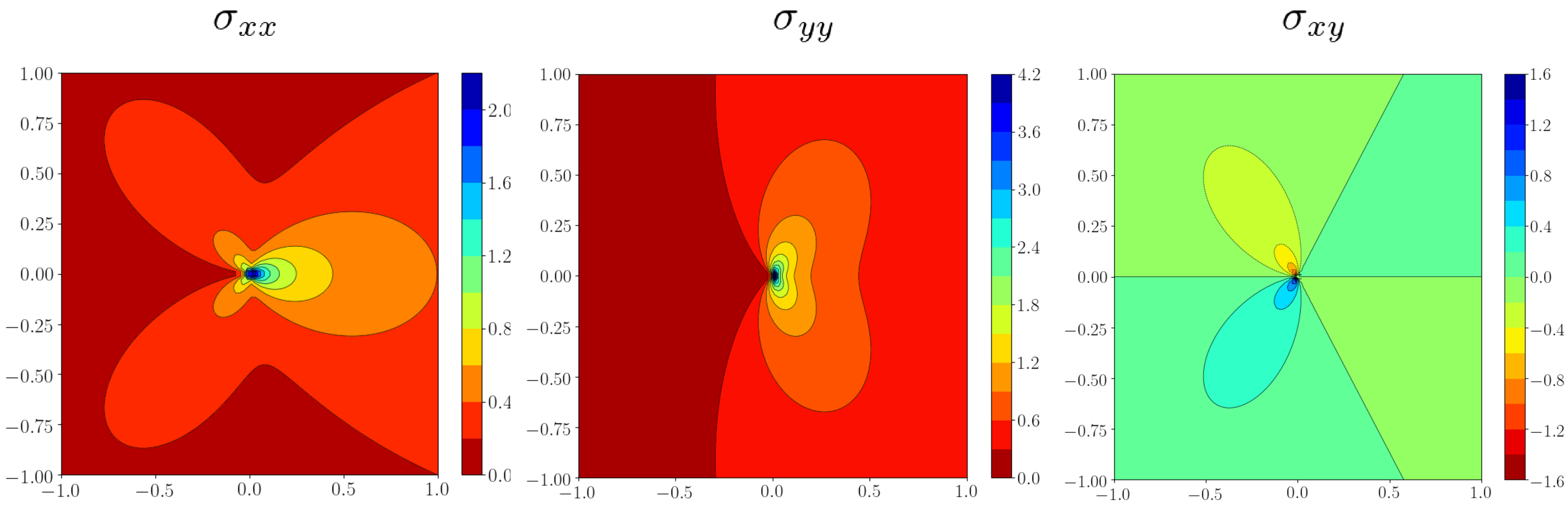
$$\begin{aligned} \sigma_{13} &= \mathrm{Im}\,Z'_{III}, \\ \sigma_{23} &= \mathrm{Re}\,Z'_{III}. \end{aligned} \qquad \cdots\cdots\cdots\cdots\cdots(97)$$

Subsequently displacement field can be derived.

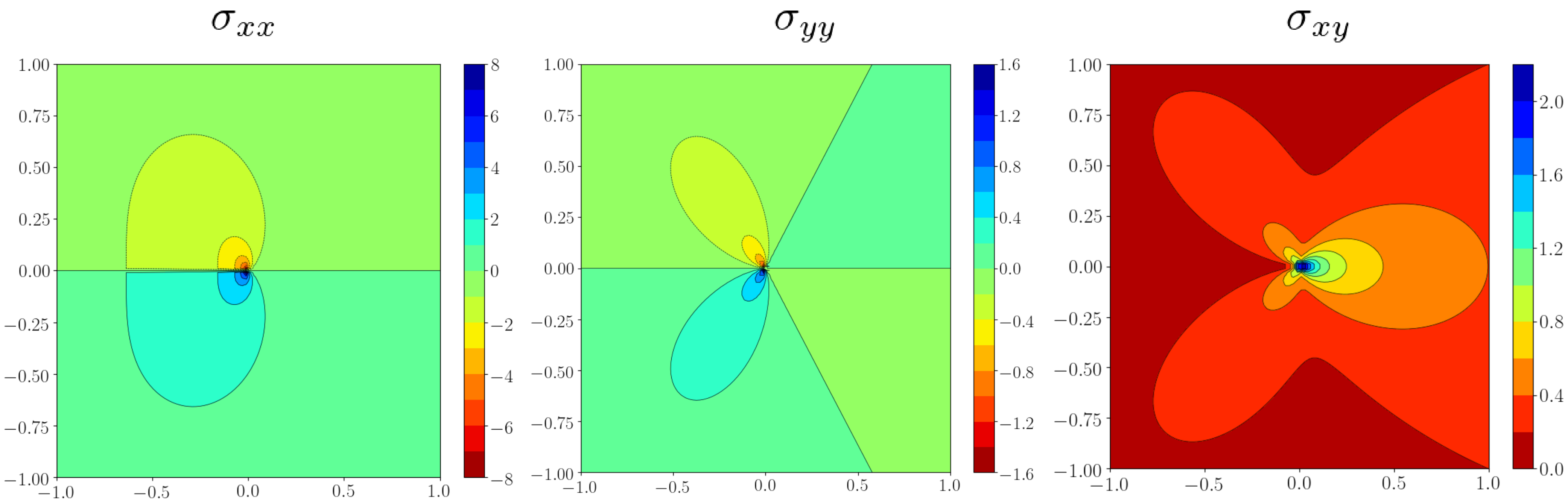
Westergaard function for Mode-III is

$$Z'_{III} = \frac{\tau z}{\sqrt{z^2 - a^2}}. \qquad \cdots\cdots\cdots\cdots\cdots(98)$$

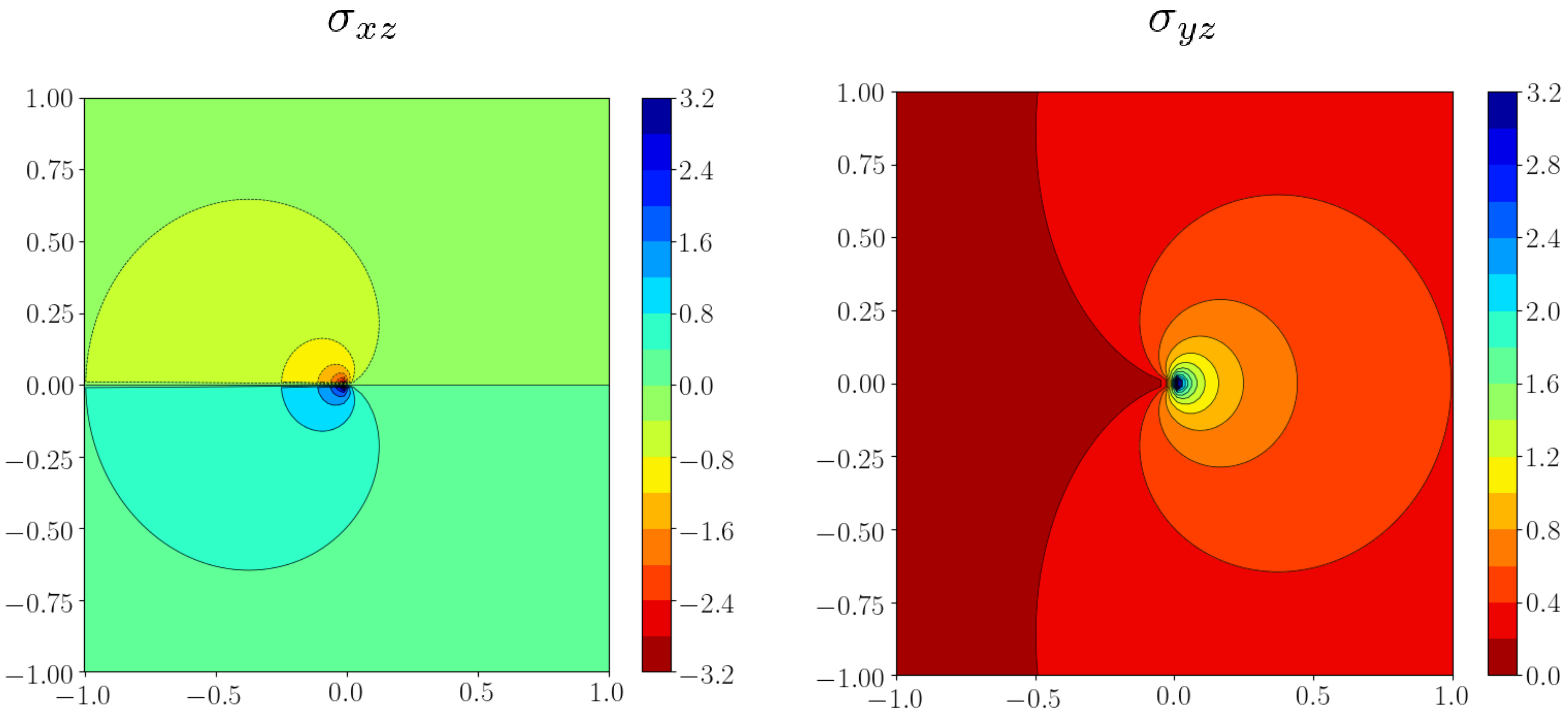
Mode-I stress fields



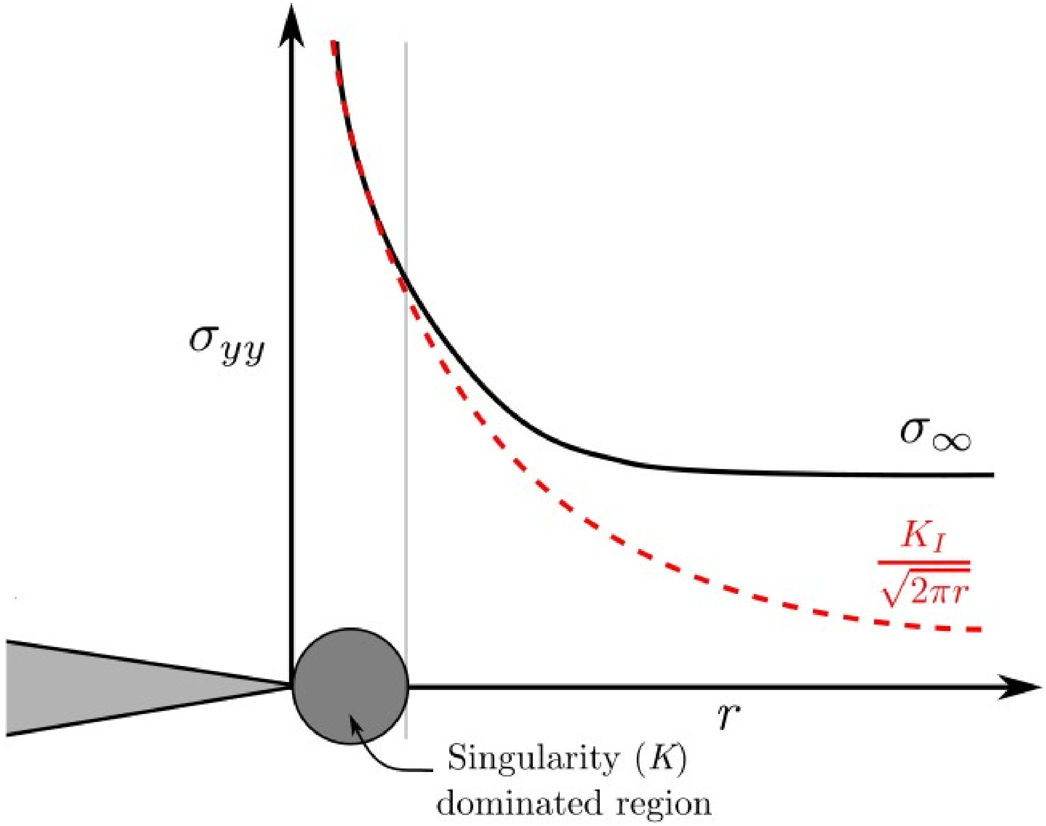
Mode-II stress fields



Mode-III stress fields



Singularity or K-dominant region



Critical stress intensity factor (SIF)

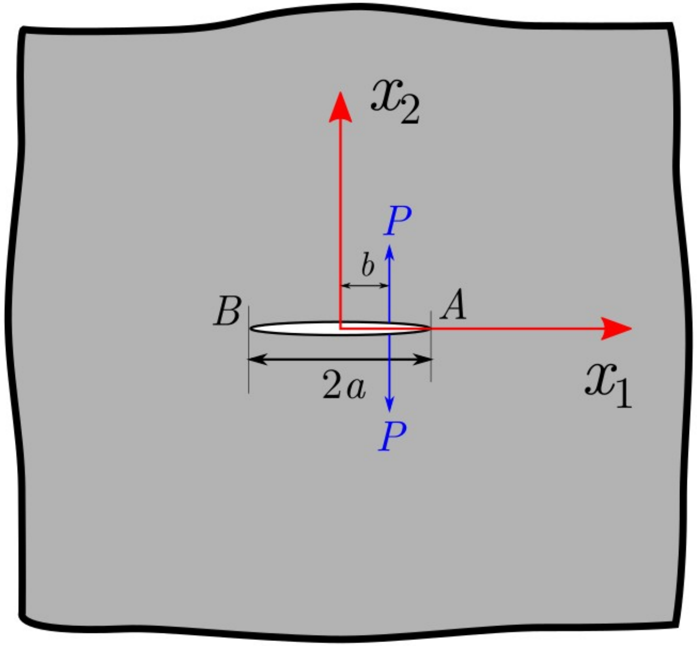
- So we have seen that the stress intensity factor (SIF) defines the amplitude of singularity at the crack-tip.
- We have also seen that SIF is a function of applied stress (σ) and crack length (a).
- For a given crack length (a) crack starts growing at a critical applied stress (σ_c). The SIF corresponding to (a and σ_c) is critical stress intensity factor K_C.
- Similarly for a given applied stress (σ) crack become unstable after a critical crack length (a_c). Again the stress intensity factor corresponding to (σ and a_c) is critical stress intensity factor K_C.
- Critical stress intensity factor K_{IC}, K_{IIC}, K_{IIIC} is material property.
- We can now define a new fracture based design criteria as $K_{I/II/III} \leq K_{IC/IIC/IIIC}$.
- For the case of a mixed crack-tip loading by K_I , K_{II} , and K_{III}, then a generalized fracture criterion $f(K_{IC}, K_{IIC}, K_{IIIC})=0$ must be defined.
- For such a design criteria to follow, we should be able to calculate SIF corresponding to any configuration.

- For thin specimen critical stress intensity factor is a function of thickness. For plane stress case it is higher than the corresponding value in a plane strain. Generally values of K_{IC} reported is the values for plane strain case and the same value is also used for design in case of plane stress case as a conservative approach, unless a very low factor of safety is required.
- Critical SIF of a material also depends on many factors, such as
    - Heat treatment of the material
    - Speed of the crack
    - Temperature
    - Process of manufacturing (e.g., vacuum furnaced or air melted, as cast or rolled).
    - Orientation of the crack with respect to the grains at the crack tip.
    - Test method.

Material	Yield Stress (MPa)	K_{IC} (MPa$\sqrt{m}$)
Mild Steel	240	Very high (~220)
Medium Carbon Steel	260	54
	1070	77
	1515	60
	1850	47
Rotor Steel	626	50
Nuclear Reactor Steel	350	190
Maraging Steel	1770	93
	2000	47
	2240	38
Aluminum		
2014-T4	460	29
2014-T651	455	24
7075-T651	495	24
7178-T651	570	23
Titanium (Ti-6Al-4V)	910	55
Perspex (PMMA)		1.6
PVC		3.5
Nylon		3

Westergaard function for few more cases

Westergaard function for an infinite plate with a crack of length $2a$ subjected to a pair of forces at $x_1 = b$.

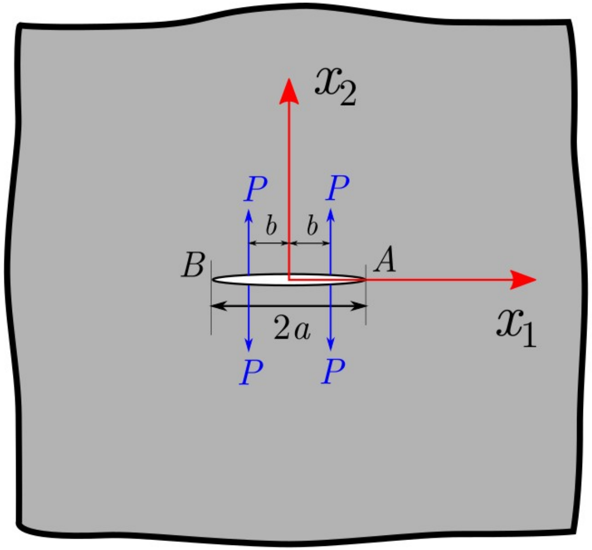


$$Z_I = \frac{P}{\pi(z-b)}\sqrt{\frac{a^2-b^2}{z^2-a^2}} \quad \cdots\cdots\cdots\cdots(99)$$

Using this function it can be shown that the SIF at crack tips A and B are

$$K_I^A = \frac{P}{\sqrt{\pi a}}\sqrt{\frac{a+b}{a-b}}$$

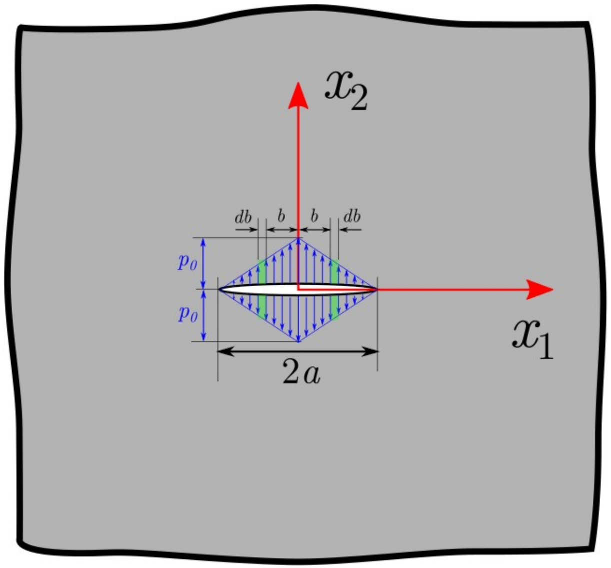
$$K_I^B = \frac{P}{\sqrt{\pi a}}\sqrt{\frac{a-b}{a+b}} \quad \cdots\cdots\cdots\cdots(100)$$



Using the SIF for previous case i.e., equation (100) and principal of superposition, SIF for the case shown is

$$K_I^A = K_I^B = \frac{P}{\sqrt{\pi a}}\sqrt{\frac{a+b}{a-b}} + \frac{P}{\sqrt{\pi a}}\sqrt{\frac{a-b}{a+b}}$$

$$K_I^A = K_I^B = \frac{2P}{\pi}\sqrt{\frac{\pi a}{a^2-b^2}} \qquad \ldots\ldots\ldots\ldots\ldots(101)$$



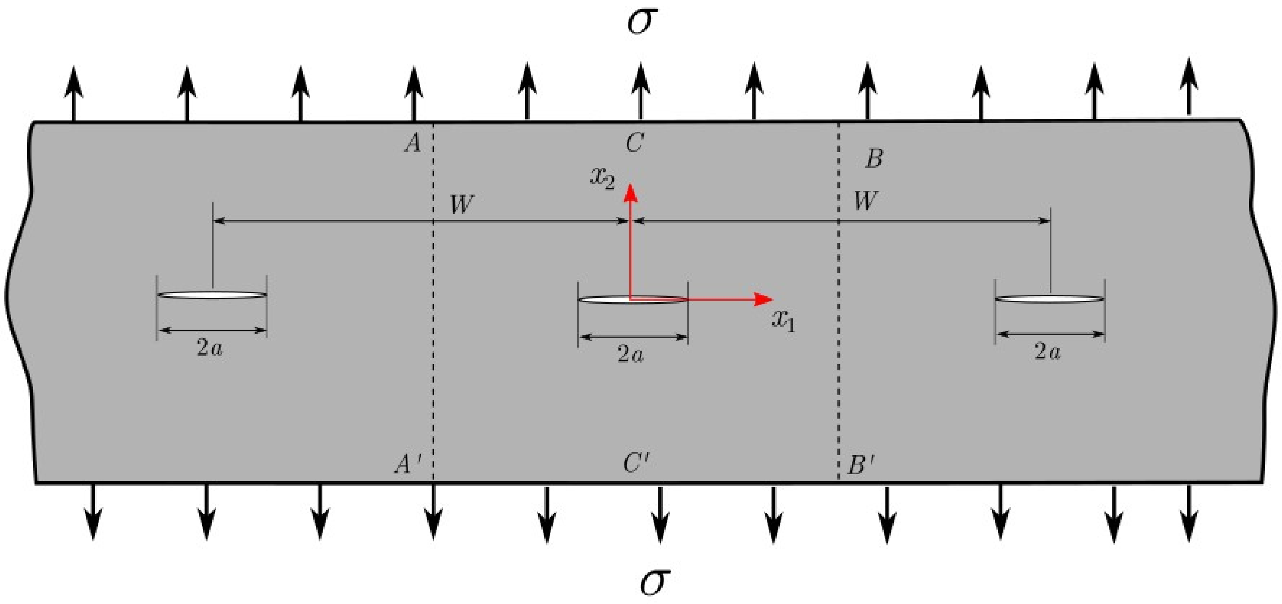
In fact, equation (100) or (101) can be used to determined the SIF for any type of load distribution on the crack face. For e.g., for triangular pressure distribution as shown.

$$dK_I = \frac{2dP}{\pi}\sqrt{\frac{\pi a}{a^2-b^2}} \quad \text{where} \quad dp = p_0 db\frac{(a-b)}{a}.$$

Hence,

$$K_I = \int_0^a dK_I = \int_0^a \frac{p_0}{\pi a}(a-b)\sqrt{\frac{\pi a}{a^2-b^2}}db$$

$$K_I = \left(1-\frac{2}{\pi}\right)p_0\sqrt{\pi a}. \qquad \ldots\ldots\ldots\ldots\ldots(102)$$



Collinear cracks in an infinitely long strip is a classical problem in fracture mechanics. Figure shows identical cracks, each of length $2a$, are separated by a distance W.

As an immediate thought this problem may not seem to be practical. However, this is not the case, because the problem acts as a stepping stone to several real life problems dealing with finite size plates.

Subsequently appropriate portions will be cut out from this strip to solve problems encountered in several engineering applications of importance.

The Westergaard function for this problem is

$$Z_I = \frac{\sigma \sin(\pi z/W)}{\sqrt{\sin^2(\pi z/W) - \sin^2(\pi a/W)}}. \qquad \text{..............(103)}$$

Transforming axis to the crack-tip by using $z = z_0 + a$ leads to

$$\sin\frac{\pi z}{W} = \sin\frac{\pi z_0}{W}\cos\frac{\pi a}{W} + \cos\frac{\pi z_0}{W}\sin\frac{\pi a}{W}. \qquad \text{..............(104)}$$

In the vicinity of the crack-tip, $|z| << a$, hence we can use

$$\sin\frac{\pi z_0}{W} \approx \frac{\pi z_0}{W}, \quad \text{and} \quad \cos\frac{\pi z_0}{W} \approx 1. \qquad \text{..............(105)}$$

Using (104) and (105), (103) can be written as.

$$Z_I = \frac{\sigma\left[\frac{\pi z_0}{W}\cos\frac{\pi a}{W} + \sin\frac{\pi a}{W}\right]}{\left[\left(\frac{\pi z_0}{W}\right)^2\cos^2\frac{\pi a}{W} + \frac{2\pi z_0}{W}\cos\frac{\pi a}{W}\sin\frac{\pi a}{W}\right]^{1/2}}$$

Since z_0/W is a small number, we can approximate Z_I as $Z_I = \dfrac{\sigma\left[\sin\frac{\pi a}{W}\right]^{1/2}}{\left[\frac{2\pi z_0}{W}\cos\frac{\pi a}{W}\right]^{1/2}}$ ....(106)

From (86), one can see another way to determine SIF as

$$K_I = \lim_{z_0 \to 0} \sqrt{2\pi z_0} Z_I(z_0).$$

Thus, K_I from the present case is

$$K_I = \frac{\sigma \left[\sin \frac{\pi a}{W}\right]^{1/2}}{\left[\frac{1}{W} \cos \frac{\pi a}{W}\right]^{1/2}} \qquad \text{or} \quad K_I = \sigma\sqrt{\pi a}\sqrt{\frac{\tan(\pi a/W)}{(\pi a/W)}}. \qquad \text{...............}(107)$$

Crack in a plate with finite dimensions

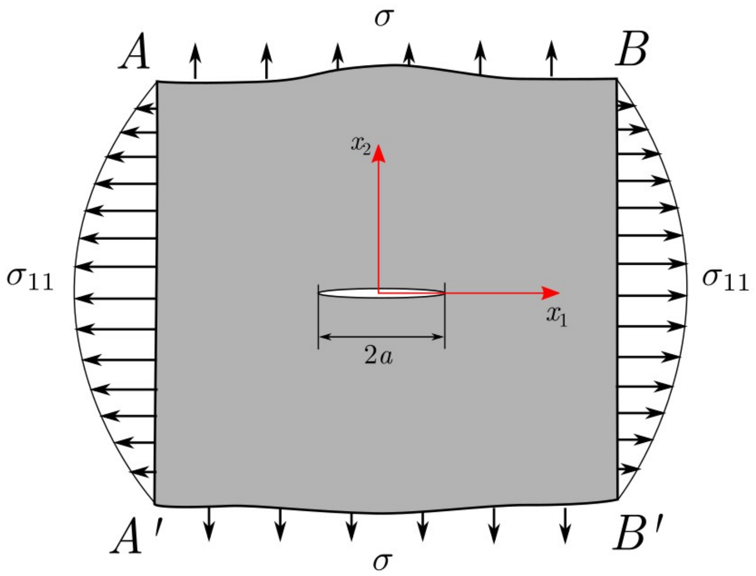
In practical applications edge of a component may be close to the crack tip. Since the edge is traction free, it disturbs the stress field around the crack tip. The edge may have a considerable influence on the stress field in the vicinity of the crack tip and on the SIF and accurate determination of SIF is then required. If the distance of the edge from the crack tip is less than the crack length, or of the order of the crack length, the component is of finite dimensions.

Even under the laboratory conditions meeting the conditions of infinite plate is difficult because of practical difficulties. For e.g., to save money on the material of the specimen, to reduce machining charges, to use a test-machine of low load capacity and to minimize material handling problems.

For specimen of finite dimension SIF is expressed in the following form,

$$K_I = \sigma\sqrt{\pi a} f(a/W),$$

where, W is the width of plate and the function f depends on a/W. For most cases, $f(a/W)$ is written as a series of ratio a/W. For most of the laboratory specimen this function f is obtained from numerical simulations



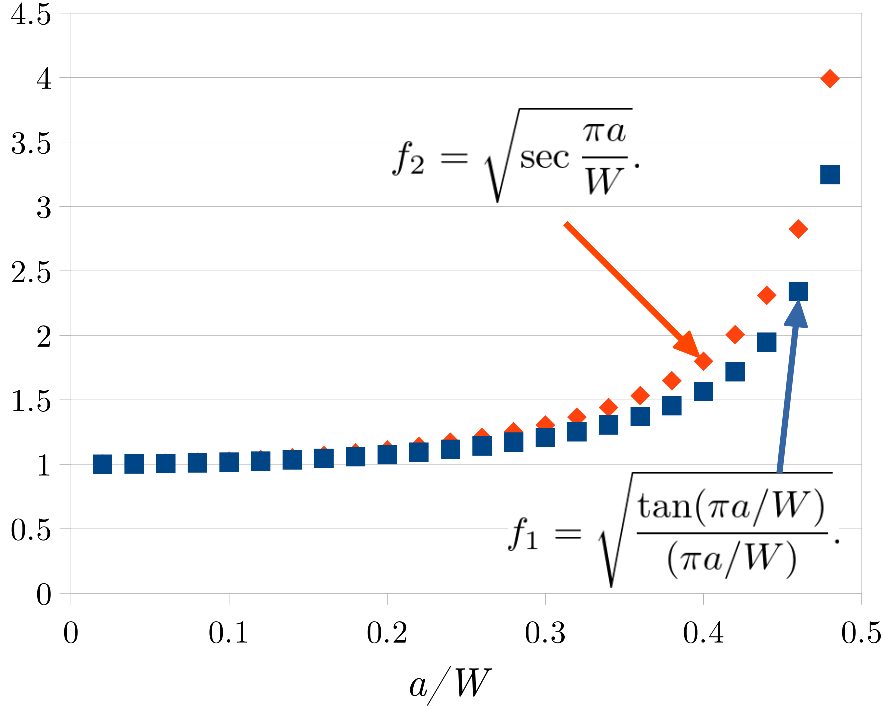
For a center-cracked plate of finite dimensions under Mode-I loading, SIF is estimated using the results of collinear cracks in an infinite strip. A portion is cut from the strip at AA' and BB', which leads to traction at both the faces. Since AA' and BB' are the planes of symmetry, shear stress on them is zero. The stress component σ_{11} has some distribution, shown qualitatively in the figure.

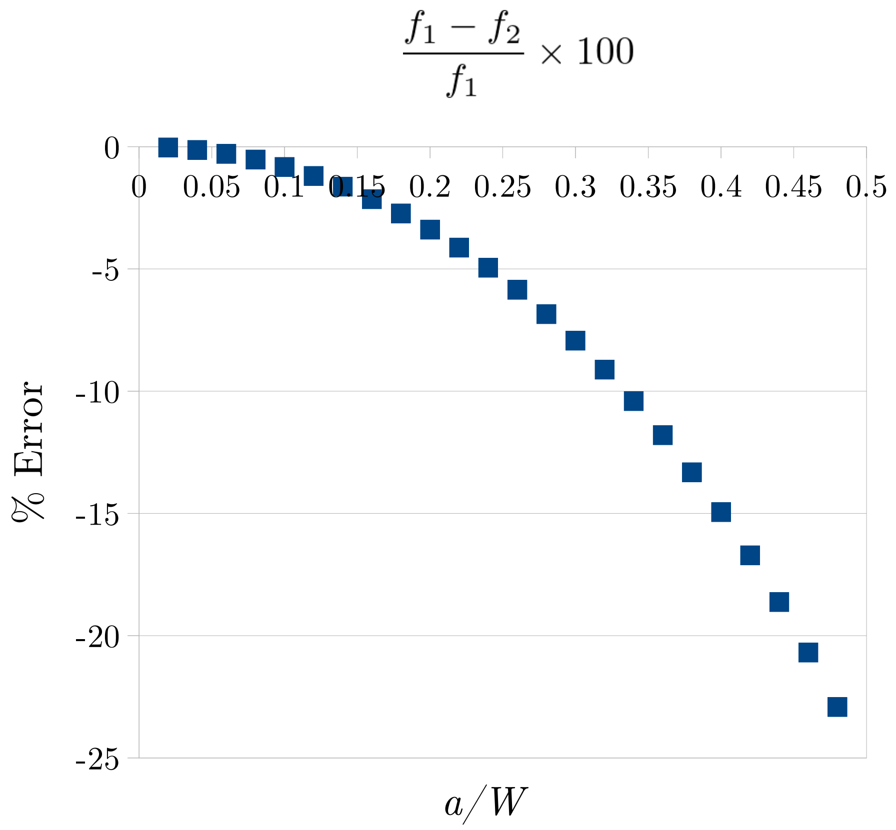
In fact, the problem is somewhat similar to the biaxial loading, where it was argued that σ_{11} does not change the SIF significantly. On the same lines, the effect of σ_{11} on cut faces may be ignored. Then, the SIF of a plate with a center-crack is approximated to be same as that of the case of collinear cracks in an infinite strip, i.e.,

$$K_I = \sigma\sqrt{\pi a}\sqrt{\frac{\tan(\pi a/W)}{(\pi a/W)}}. \qquad \cdots\cdots\cdots\cdots\cdots(108)$$

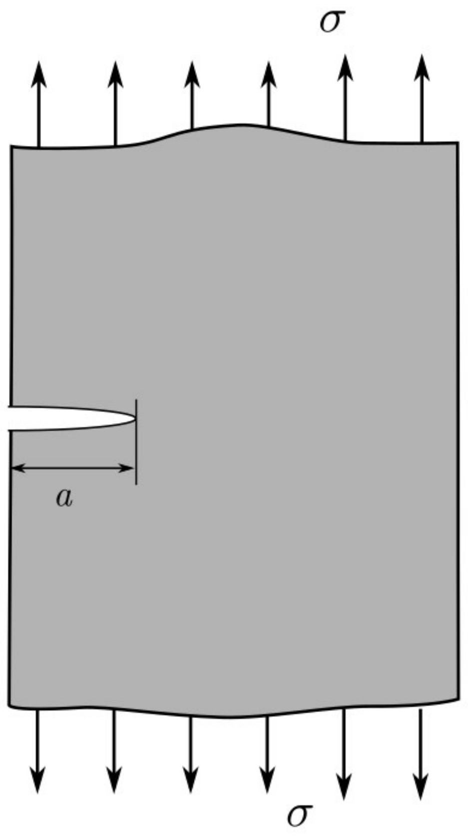
However, exact solution of this problem obtained through advanced mathematical method and numerical solutions is more close to

$$K_I = \sigma\sqrt{\pi a}\sqrt{\sec\frac{\pi a}{W}}. \qquad \cdots\cdots\cdots\cdots\cdots(109)$$


4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
a/W
$f_2 = \sqrt{\sec \frac{\pi a}{W}}$.
$f_1 = \sqrt{\frac{\tan(\pi a/W)}{(\pi a/W)}}$.


$\frac{f_1 - f_2}{f_1} \times 100$
0
-5
-10
-15
-20
-25
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
% Error
a/W

Edge crack



We have already discussed that a free edge close to the crack influences the stress field near the crack tip. In the case of an edge

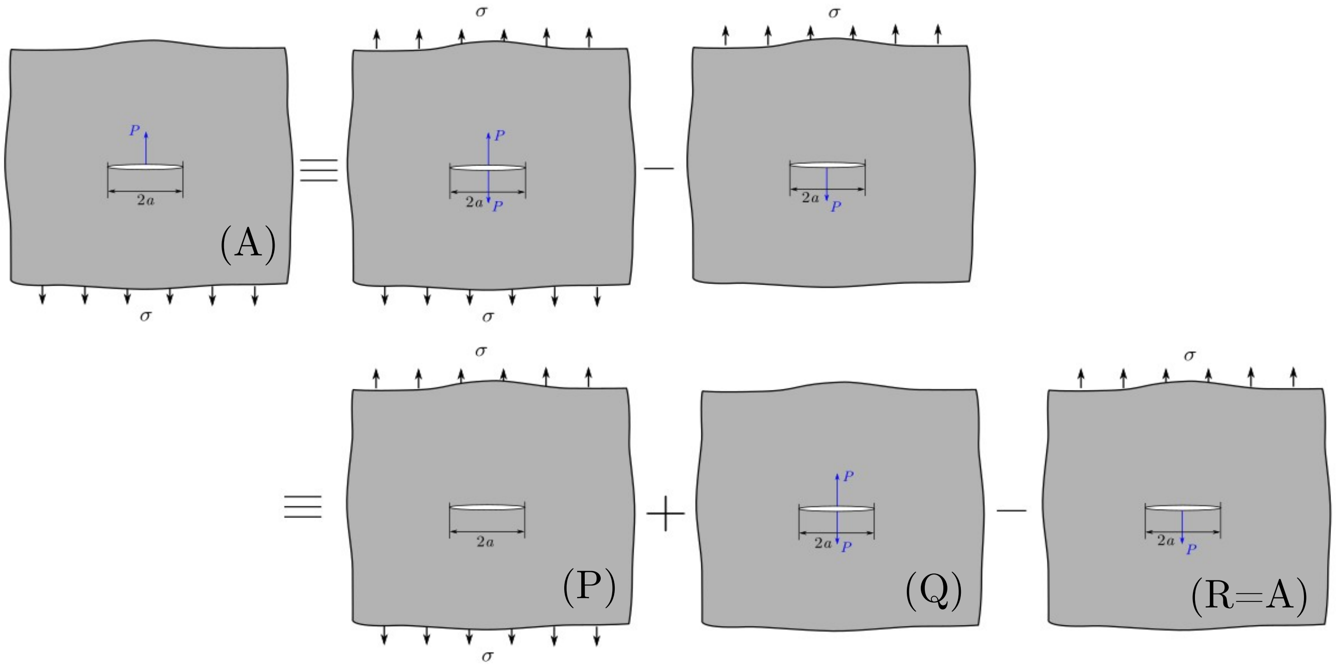
crack, the free edge is not only close to the crack, but it intersects the crack. Edge cracks are very commonly encountered in day-to-day life. Since they are more dangerous, special attention is required to deal with them.

Consider an edge crack in a semi-infinite plate which is loaded by a far field stress. The stress intensity factor for this case is known $1.12\sigma\sqrt{(\pi a)}$. When the edge crack is compared with one half of the overall length of an interior crack, the value of the SIF is about 12% more, as the ends of cracked faces at the free edge tend to open up more easily.

The problem of edge crack can be solved by separating a portion $CBB'C'$ from the strip of collinear cracks and invoking the principle of superposition to make the traction zero on section. However, the solution is quite complex for the same.

Principal of superposition for determining SIF

In many cases, principal of superposition can be exploited to find SIF for complex loading by considering it as a combination of simple loading cases.



$$K_I^A = K_I^P + K_I^Q - K_I^R$$
$$= K_I^P + K_I^Q - K_I^A$$

or $\quad K_I^A = \frac{1}{2}\left(K_I^P + K_I^Q\right)$

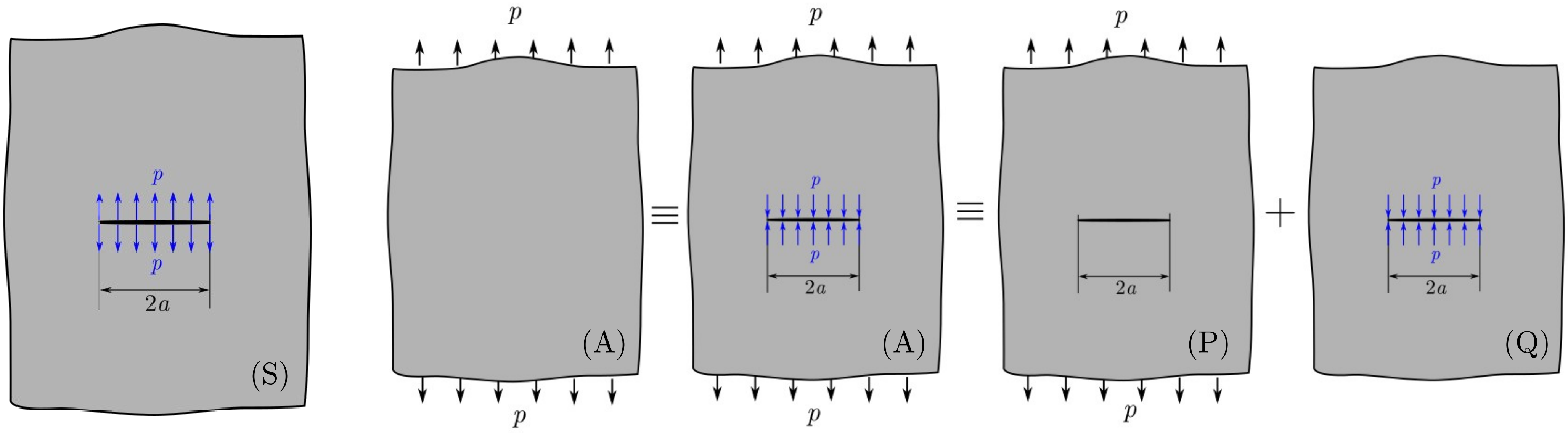
Thus,

$$K_I^A = \frac{1}{2}\left[\sigma\sqrt{\pi a} + \frac{P}{\sqrt{\pi a}}\right]$$

If width of plate is W, then from equilibrium of configuration (A) we can write $P = \sigma W$ and thus,

$$K_I^A = \frac{1}{2}\left[\sigma\sqrt{\pi a} + \frac{\sigma W}{\sqrt{\pi a}}\right]$$

Crack with internal pressure

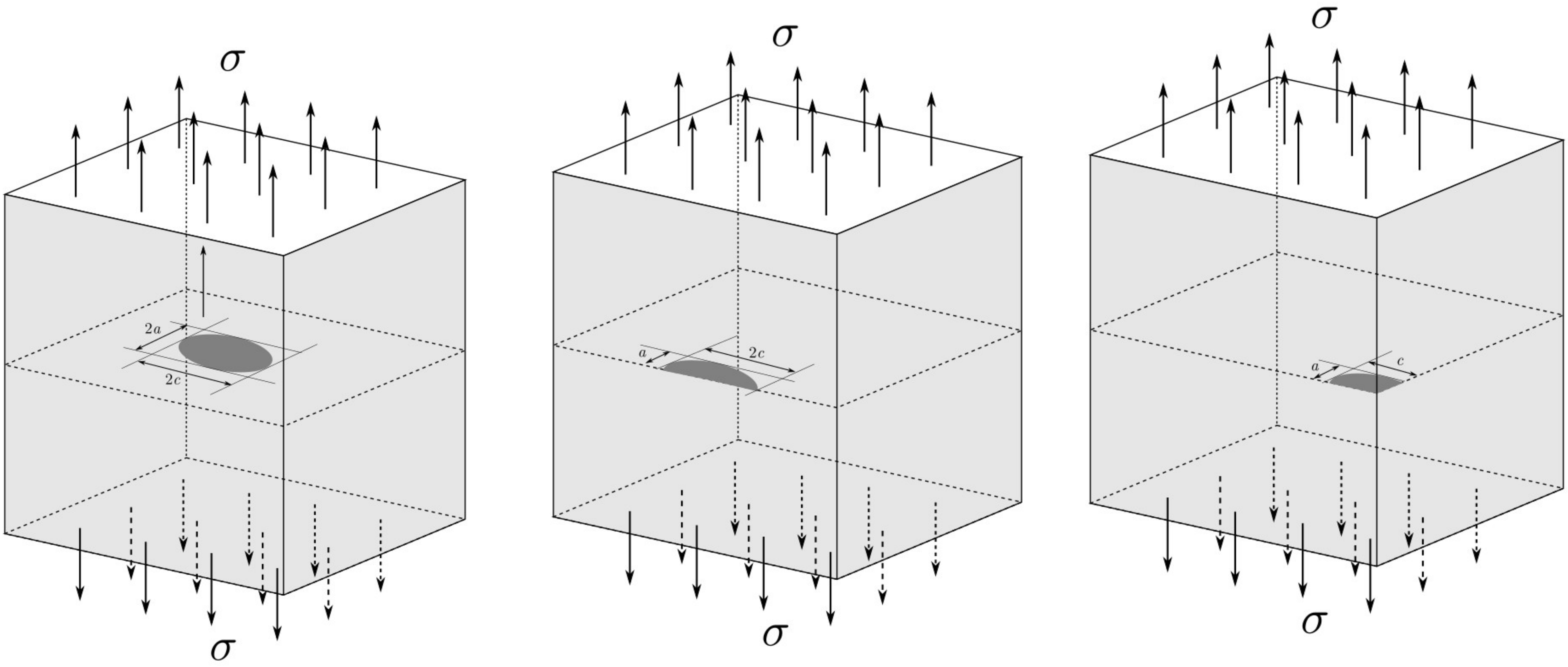


The problem of a plate with center crack can be solved using the Green's function as discussed earlier. However, we will also see an application of the method of superposition to determine the SIF for this case.

$$K_I^A = K_I^P + K_I^Q = K_I^P - K_I^S = 0$$

Thus, $\qquad K_I^S = K_I^P = p\sqrt{\pi a}$ ....(110)

Embedded cracks

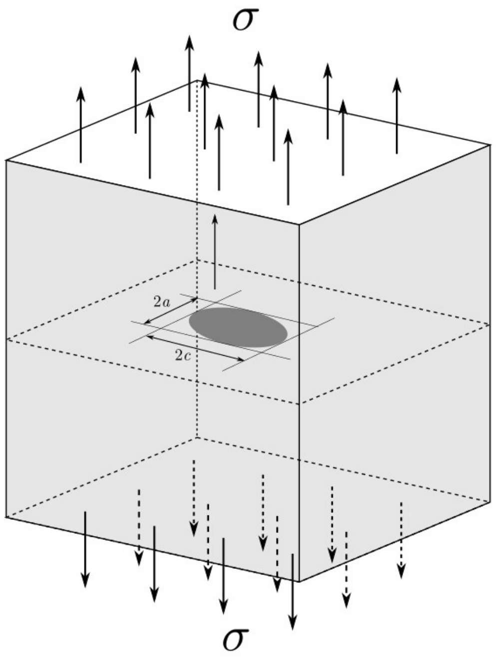


A fully embedded elliptical crack

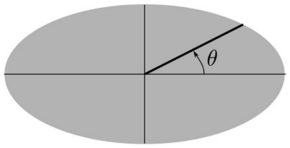
A semi-elliptical crack

A quarter-elliptical crack

Elliptical cracks

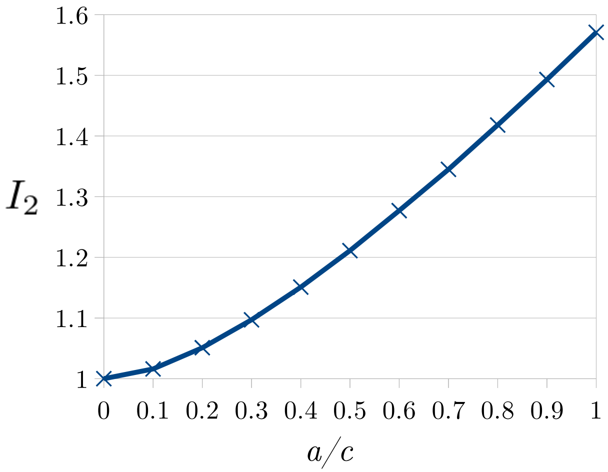


Theoretical solution for an elliptical crack embedded in an infinite medium, subjected to load σ at the far field is given by Irwin in 1962. The stress intensity factor is varies at every point of the crack front and is a function of angle θ.



$$K_I = \frac{\sigma\sqrt{\pi a}}{I_2}\left[\sin^2\theta + \left(\frac{a}{c}\right)^2\cos^2\theta\right]^{1/4}, \quad \cdots\cdots(111)$$

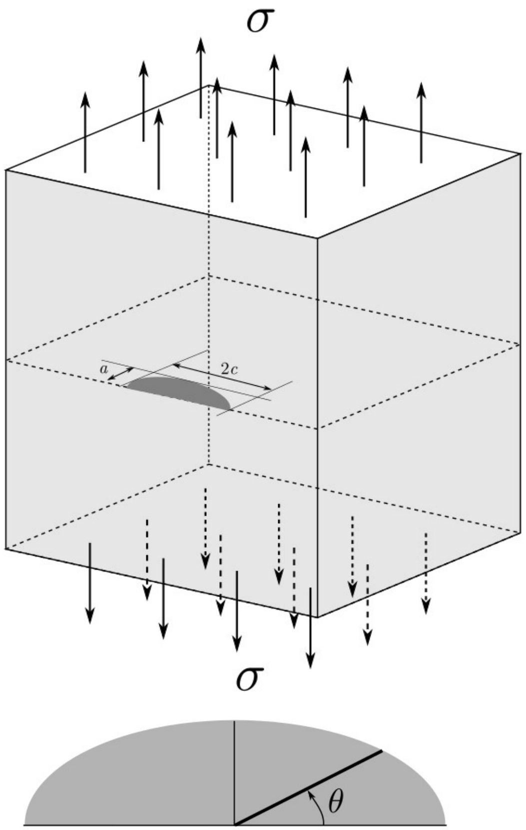
$$\text{where, } I_2 = \int_0^{\pi/2}\left(1 - \frac{c^2 - a^2}{c^2}\sin^2\alpha\right)^{1/2} d\alpha \quad \cdots\cdots(112)$$



Integral I_2 can be evaluated and its variation with a/c ratio is shown.

Semi-elliptical cracks

A surface crack is modeled as a half ellipse with its minor axis into the thickness direction. The SIF of the surface crack is then 12% higher over the corresponding SIF of the elliptical crack. Thus, the SIF at a point of the crack front of a semi-elliptical crack is



$$K_I = \frac{1.12\sigma\sqrt{\pi a}}{I_2}\left[\sin^2\theta + \left(\frac{a}{c}\right)^2\cos^2\theta\right]^{1/4}, \qquad \cdots\cdots(113)$$

Note that the SIF at the extreme end of the minor axis (θ=90°) is

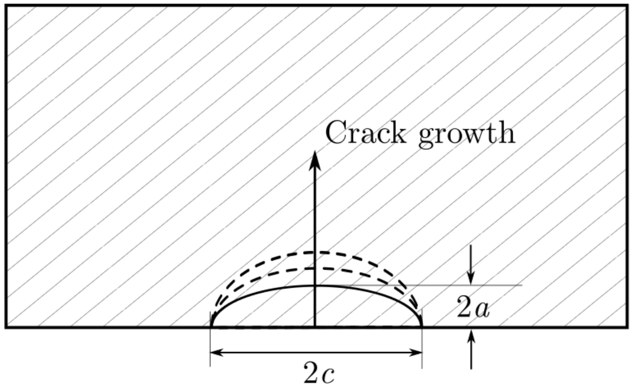
$$K_I^{90} = \frac{1.12\sigma\sqrt{\pi a}}{I_2}, \qquad \cdots\cdots(114)$$

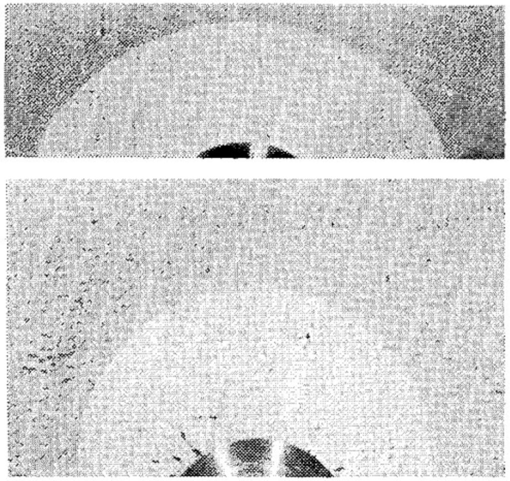
and SIF at extreme ends of major axis is,

$$K_I^{0/180} = \frac{1.12\sigma\sqrt{\pi a}}{I_2}\left(\frac{a}{c}\right)^2. \qquad \cdots\cdots(115)$$

So the segment of the crack which is inside the material is having higher SIF.

This different in SIF causes the crack to grow in a particular manner as shown.



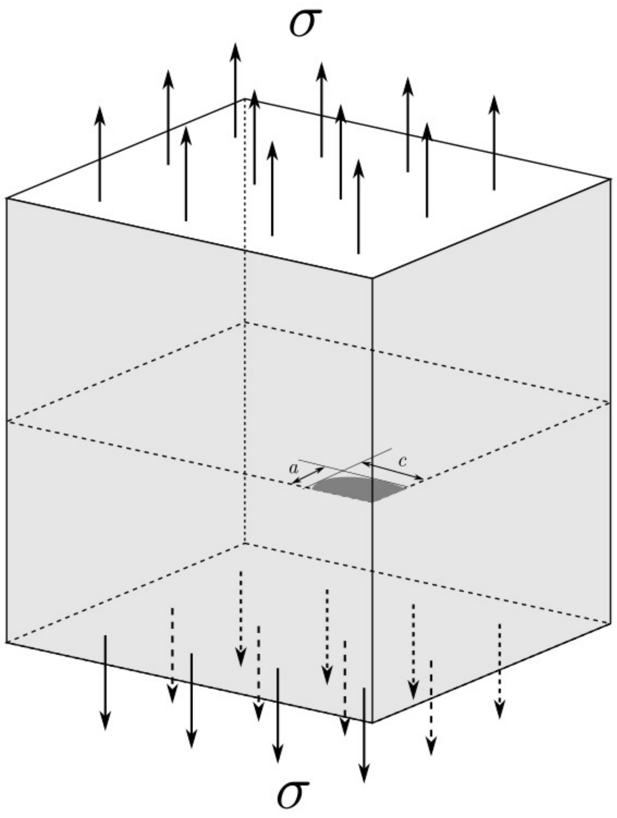


Till the time $a/c < 1$, $K_I^{90} > K_I^{0/180}$ and crack grows in the direction of minor axis.

When $a{=}c$, i.e., when crack becomes semi-circular, $K_I^{90} = K_I^{0/180}$, and then crack grows in both the direction of major as well as minor axis as semi-circular crack.

Also, note that for a very shallow crack ($a << c$), a/c is close to unity and the SIF $K_I^{90} \approx 1.12\sigma\sqrt{\pi a}$, which is the same as the result of through-the-thickness edge crack of length a. In this case, the dimension of the major axis is no longer relevant. Therefore, a shallow crack is equivalent to a through-the-thickness edge crack of length a.

Quarter-elliptical cracks

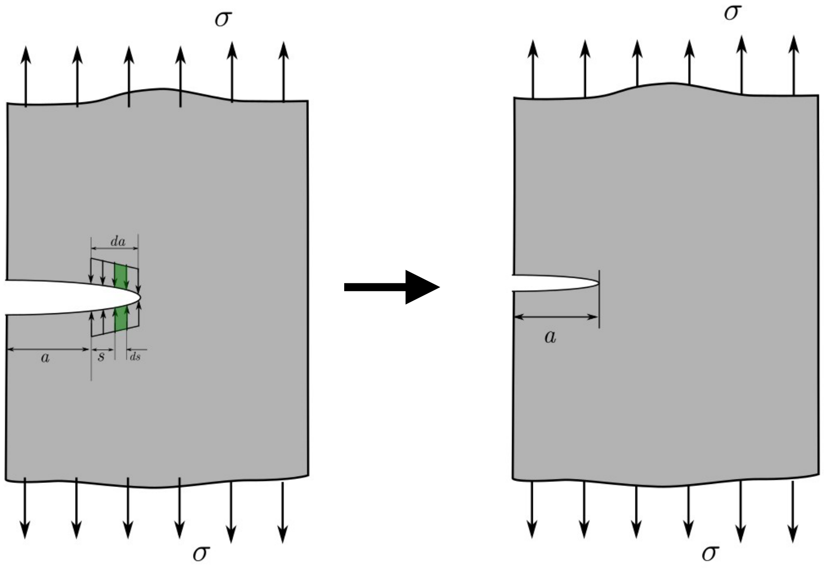


A corner crack under tensile load is exposed to two free surfaces, each one applying an additional correction factor on the SIF. The overall correction factor of such cracks in thick plates has found to be about 20% higher i.e., the highest SIF on the crack front is

$$K_I^{90} = 1.2\sigma\sqrt{\pi a}. \qquad \cdots\cdots(116)$$

The relationship between G and K

- Energy release rate G is a global parameter and deals with energy.
- On the other hand, stress intensity factor K is a local parameter which deals with displacement and stress fields in the vicinity of the crack.
- Although the approaches are entirely different, the goal is same, i.e., to characterize a crack. Therefore there should be a relationship between G and K.
- The relation was obtained by Irwin

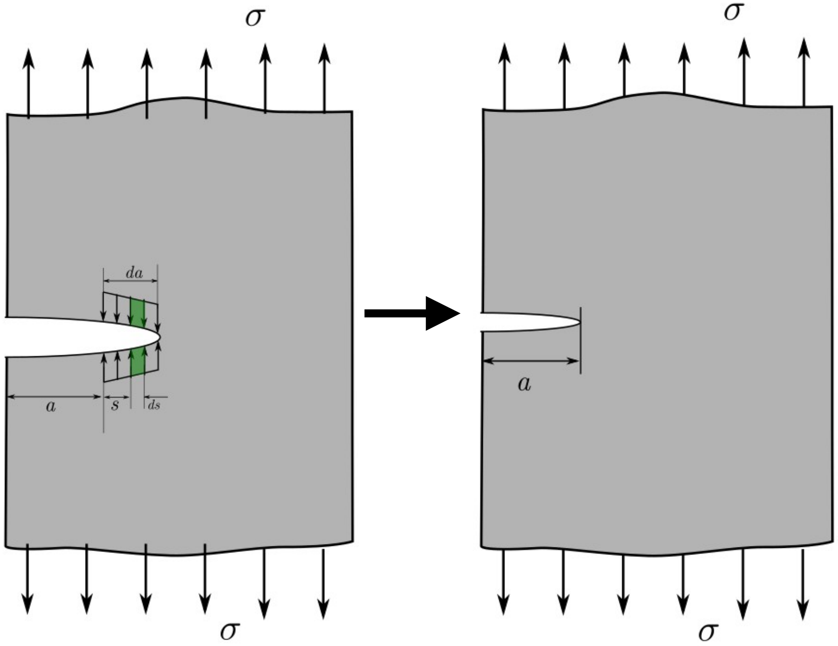


Consider a crack of length a which is extended by an small incremental length Δa..

For the case of extended crack, let us find the the displacement of the crack face at a distance of (Δa-s) from the crack-tip.

For plane stress case, we use (90) with θ=180°, which gives

$$u_2(s) = \frac{K_I'}{\mu}\sqrt{\frac{\Delta a - s}{2\pi}}\frac{2}{1-\nu}. \qquad \cdots\cdots(117)$$



Now think of a hypothetical experiment. The extended length Δa of the crack is closed by applying stress $\sigma_{22}(s)$ on crack faces. σ_{22} is evaluated from the stress field of the initial (unextended) crack of length a, and therefore at a distance s, it is given by

$$\sigma_{22}(s) = \frac{K_I}{\sqrt{2\pi s}}. \qquad \cdots\cdots\cdot(118)$$

Every point of the extended crack need to be moved by a displacement $u_2(s)$ to close it.

Thus, the crack is closed by length Δa. Irwin argued that the total elastic work required by σ_{22} in closing the crack is equal to the energy released during the extension of crack by Δa length. Balancing the two energies, we have

$$G_I B \Delta a = 2B \int_0^{\Delta a} \frac{1}{2}\sigma_{22}(s)u_2(s)ds. \qquad \cdots\cdots\cdot(119)$$

Substituting (117) and (118) in (119), we get,

$$G_I = \lim_{\Delta a \to 0} \frac{1}{\Delta a} \int_0^{\Delta a} \frac{K_1}{\sqrt{2\pi s}} \frac{K_I'}{\mu} \sqrt{\frac{\Delta a - s}{2\pi}} \frac{2}{1-\nu} ds, \quad \text{where} K_I' = K_I + \Delta K,$$

$$G_I = \lim_{\Delta a \to 0} \frac{2}{2\pi\mu(1-\nu)} \frac{1}{\Delta a} \int_0^{\Delta a} K_I(K_I + \Delta K_I) \frac{\sqrt{\Delta a - s}}{\sqrt{s}} ds.$$

For sufficiently small Δa, $K_I + \Delta K_I \approx K_I$. Thus,

$$G_I = \lim_{\Delta a \to 0} \frac{K_I^2}{\pi\mu(1-\nu)} \frac{1}{\Delta a} \int_0^{\Delta a} \frac{\sqrt{\Delta a - s}}{\sqrt{s}} ds.$$

This equation can be solved by substituting $s = \Delta a \sin^2 a$ and changing the limits accordingly.

$$G_I = \frac{K_I^2}{2\mu(1-\nu)} = \frac{K_I^2}{E}. \qquad \cdots\cdots\cdots\cdots(120)$$

The relation is simple, but is rigorous only for brittle materials in which the components remain elastic.

For plane strain case,

$$G_I = \frac{K_I^2}{E'} = (1-\nu^2)\frac{K_I^2}{E}. \qquad \cdots\cdots\cdots\cdots(120a)$$

Similarly, we can also establish relations for mode II and mode III as,

$$G_{II} = \frac{K_{II}^2}{E} \text{ (for plane stress)} \quad \text{and} \quad G_{II} = (1-\nu^2)\frac{K_{II}^2}{E} \text{ (for plane strain)},$$

and

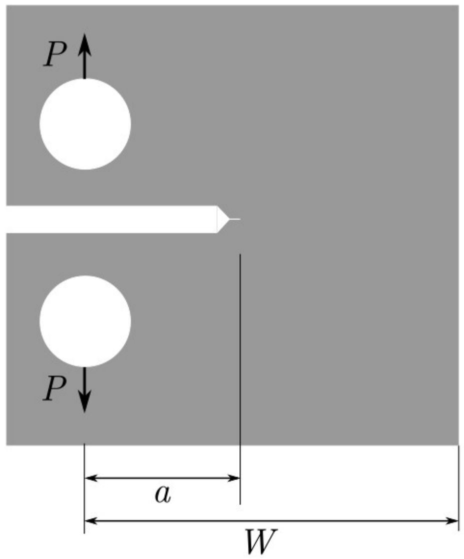
$$\cdots\cdots\cdots\cdots(121)$$

$$G_{III} = \frac{K_{III}^2}{2\mu}. \qquad \cdots\cdots\cdots\cdots(122)$$

In a general case where all three modes can be present the total energy release rate is,

$$G = G_I + G_{II} + G_{III}.$$

Compact Tension (CT) Specimen

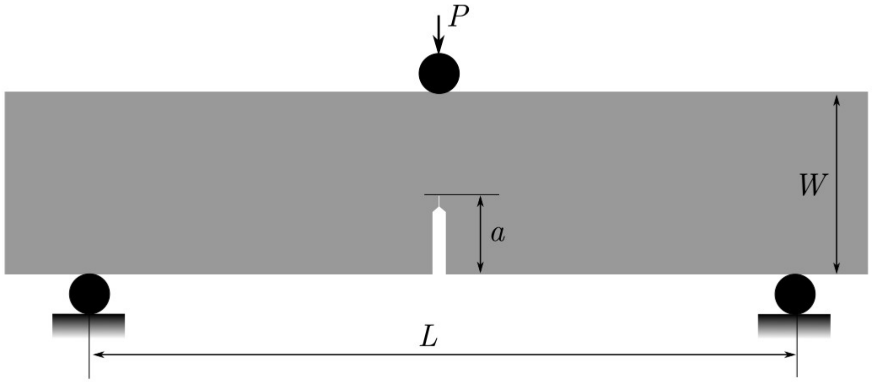


$$K_I = \frac{P}{B\sqrt{W}} f(\alpha)$$

$$\alpha = a/W$$

$$f(\alpha) = \frac{3\sqrt{\alpha}\left[1.99 - \alpha(1-\alpha)(2.15 - 3.93\alpha + 2.7\alpha^2\right]}{2(1+2\alpha)(1-\alpha)^{3/2}}$$

Single-Edge-Notch-Bend (SENB) Specimen

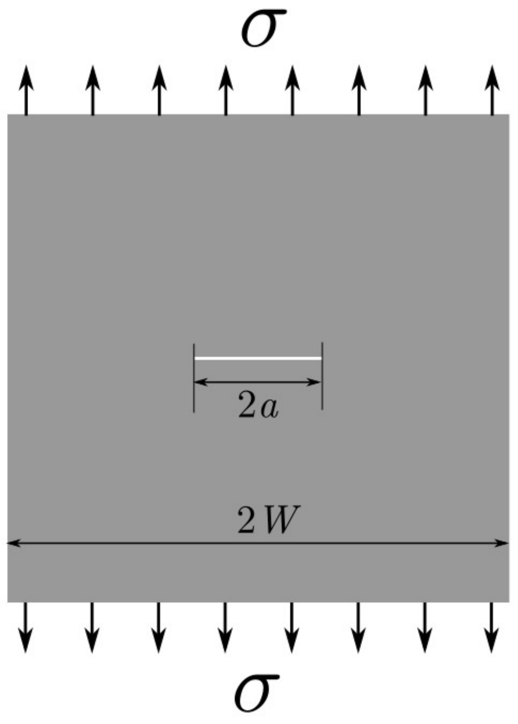


$$K_I = \frac{PS}{BW^{3/2}} f(\alpha)$$

$$\alpha = a/W$$

$$f(\alpha) = \frac{3\sqrt{\alpha}\left[1.99 - \alpha(1-\alpha)(2.15 - 3.93\alpha + 2.7\alpha^2\right]}{2(1+2\alpha)(1-\alpha)^{3/2}}$$

Centre-Cracked Plate under Uniform Tension

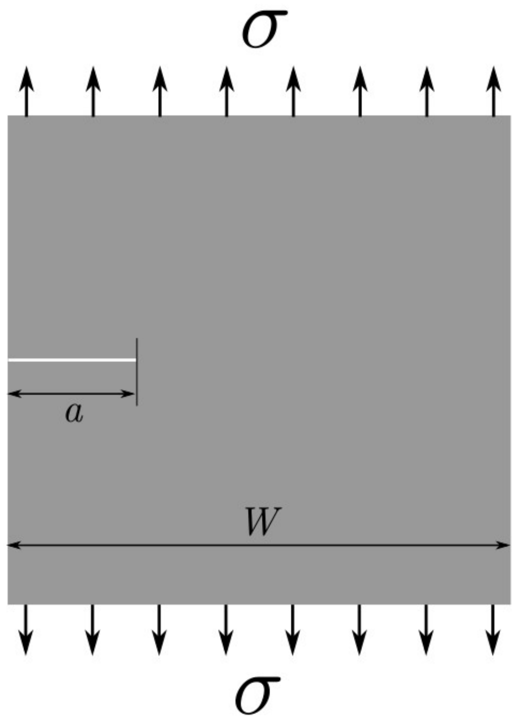


$$K_I = \sigma\sqrt{\pi a}f(\alpha)$$

$$\alpha = a/W \text{ for } 0 < \alpha < 0.7$$

$$f(\alpha) = 1.0 + 0.128\alpha - 0.288\alpha^2 + 1.53\alpha^3$$

Single-Edge-Cracked Plate under Uniform Tension

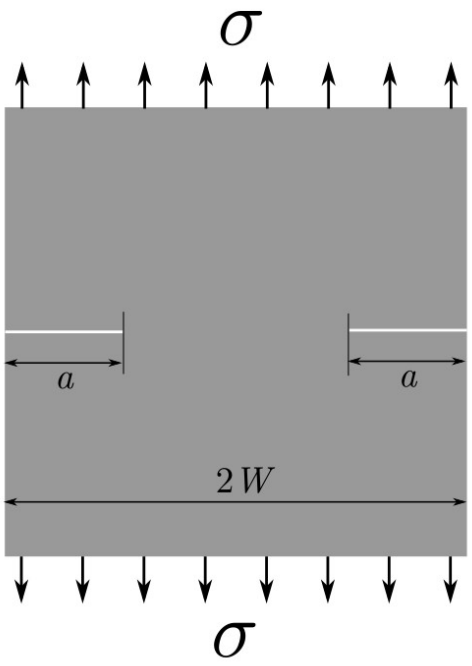


$$K_I = \sigma\sqrt{\pi a}f(\alpha)$$

$$\alpha = a/W \text{ for } 0 < \alpha < 0.6$$

$$f(\alpha) = 1.12 - 0.23\alpha + 10.55\alpha^2 - 21.72\alpha^3 + 30.39\alpha^4$$

Double-Edge-Cracked Plate under Uniform Tension

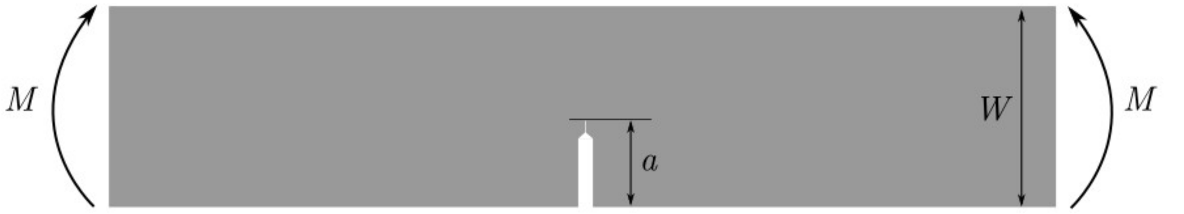


$$K_I = \sigma\sqrt{\pi a}f(\alpha)$$

$$\alpha = a/W \text{ for } 0 < \alpha < 0.7$$

$$f(\alpha) = 1.12 - 0.20\alpha - 1.20\alpha^2 + 1.93\alpha^3$$

Strip with Edge-Crack under bending

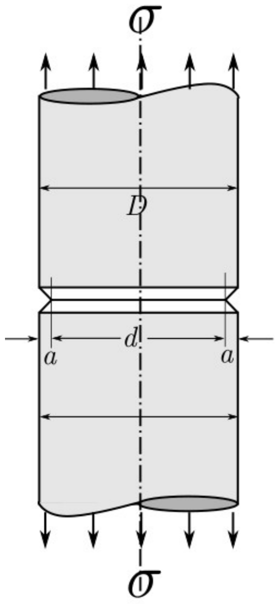


$$K_I = \frac{6M}{BW^2}\sqrt{\pi a}f(\alpha)$$

$$\alpha = a/W$$

$$f(\alpha) = 1.12 - 1.40\alpha + 7.33\alpha^2 - 13.083\alpha^3 + 14\alpha^4$$

Circumferentially Cracked Round Bar under Tension

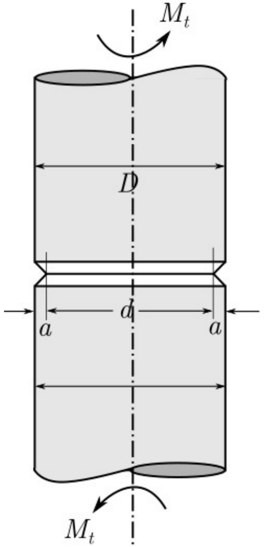


$$K_I = \sqrt{\pi a} f(\beta)$$

$$\beta = d/B$$

$$f(\beta) = \frac{1}{2\sqrt{\beta}}\left(\frac{1}{\beta} + \frac{1}{2} + \frac{3}{8}\beta - 0.316\beta^2 + 7.33\beta^3\right).$$

Circumferentially Cracked Round Bar under Torsion

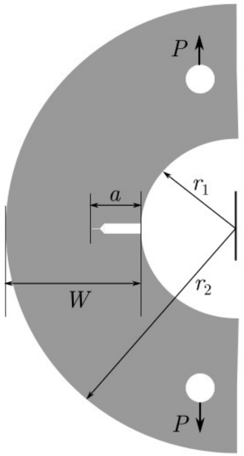


$$K_I = \frac{16M}{\pi D^3}\sqrt{\pi a} f(\beta)$$

$$\beta = d/B$$

$$f(\beta) = \frac{3}{8\sqrt{\beta}}\left(\frac{1}{\beta^2} + \frac{0.5}{\beta} + \frac{3}{8} + \frac{5}{16}\beta + \frac{35}{128}\beta^2 + 0.21\beta^3\right).$$

Arc-Shaped Tension (AT) Specimen

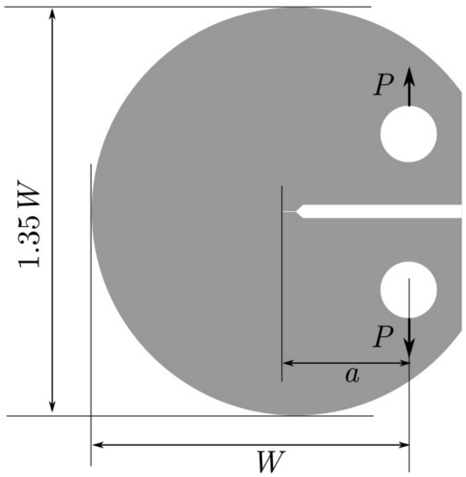


$$K_I = \frac{P}{B\sqrt{W}} f(\alpha)$$

$$\alpha = a/W$$

$$f(\alpha) = \left[\frac{3X}{W} + 1.9 + 1.1\alpha\right]\left[1 + 0.25(1-\alpha)^2\left(1 - \frac{r_1}{r_2}\right)\right]$$
$$\left[\frac{\alpha^{1/2}}{(1-\alpha)^{3/2}}\right]\left(3.74 - 6.3\alpha + 6.32\alpha^2 - 2.43\alpha^3\right).$$

Disc-Shaped Compact Tension (DCT) Specimen



$$K_I = \frac{P}{B\sqrt{W}} f(\alpha)$$

$$\alpha = a/W$$

$$\begin{aligned} f(\alpha) =& (2+\alpha)(0.76 + 4.8\alpha - 11.85\alpha^2 \\ &+ 11.43\alpha^3 - 4.08\alpha^4)(1-\alpha)^{-3/2} \end{aligned}$$