ME531: Advanced Mechanics of Solids

Motion, Strain and Stress

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Maximum and minimum values of shear stresses:

Next we determine the direction n at point x that gives maximum or minimum values of shear stresses.

We start with the representation of stress tensor in eigenvector basis as,

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \sum_{i=1}^{3} \sigma_i \boldsymbol{n}_i \otimes \boldsymbol{n}_i.$$

Then traction vector along a vector $\boldsymbol{N} = N_1\boldsymbol{n}_1 + N_2\boldsymbol{n}_2 + N_3\boldsymbol{n}_3$, which is a unit normal vector to an arbitrary oriented plane, will be

$$\boldsymbol{t} = \boldsymbol{N} \cdot \boldsymbol{\sigma}(\boldsymbol{x}) = \sigma_1 N_1 \boldsymbol{n}_1 + \sigma_2 N_2 \boldsymbol{n}_2 + \sigma_3 N_3 \boldsymbol{n}_3,$$

Now the magnitude of normal and shear stress on the plane will be,

$$\sigma = \boldsymbol{N} \cdot \boldsymbol{t} = \sigma_1 N_1^2 + \sigma_2 N_2^2 + \sigma_3 N_3^2, \text{ and}$$

$$\tau^2 = |\boldsymbol{t}|^2 - \sigma^2 = \sigma_1^2 N_1^2 + \sigma_2^2 N_2^2 + \sigma_3^2 N_3^2 - (\sigma_1 N_1^2 + \sigma_2 N_2^2 + \sigma_3 N_3^2)^2.$$

Now we use the constraint $N_1^2 + N_2^2 + N_3^2 = 1$ to replace N_3 with N_1 and N_2 in the previous equation, so that τ^2 become a function of N_1 and N_2 only. To identify the extremal values of τ^2 the expression is partially differentiated w.r.t N_1 and N_2 and equated to zero as,

$$\frac{\partial \tau^2}{\partial N_1} = 2N_1(\sigma_1 - \sigma_3)\left[\sigma_1 - \sigma_3 - 2\left\{(\sigma_1 - \sigma_3)N_1^2 + (\sigma_2 - \sigma_3)N_2^2\right\}\right] = 0,$$

$$\frac{\partial \tau^2}{\partial N_2} = 2N_2(\sigma_2 - \sigma_3)\left[\sigma_2 - \sigma_3 - 2\left\{(\sigma_1 - \sigma_3)N_1^2 + (\sigma_2 - \sigma_3)N_2^2\right\}\right] = 0.$$

To solve for the direction of extremal shear stress the very first choice can be N_1=N_2=0, with N_3=±1. Similar if we choose N_1=N_3=0, corresponding component is N_2=±1 and for N_2=N_3=0, corresponding component is N_1=±1. Thus we obtain three unit vector which are same as eigenvectors. Basically these vector corresponds to the minimum solution as in the principal direction magnitude of shear stress is zero.

Another solution is obtained by setting N_1=0, which results N_2=$\pm 1/\sqrt{2}$ from the last set of equations and $N_3 = \pm 1/\sqrt{2}$ by the condition that $|\boldsymbol{N}| = 1$.

Similarly by setting N_2=0, we obtain $N_1 = N_3 = \pm 1/\sqrt{2}$ and, by setting N_3=0, $N_1 = N_2 = \pm 1/\sqrt{2}$. These vector putting back in the equation for τ^2 results in the following extremal values.

$$\boldsymbol{N} = \pm\frac{1}{\sqrt{2}}\boldsymbol{n}_2 \pm \frac{1}{\sqrt{2}}\boldsymbol{n}_3, \quad \tau^2 = \frac{1}{4}\left(\sigma_2 - \sigma_3\right)^2,$$

$$\boldsymbol{N} = \pm\frac{1}{\sqrt{2}}\boldsymbol{n}_1 \pm \frac{1}{\sqrt{2}}\boldsymbol{n}_2, \quad \tau^2 = \frac{1}{4}\left(\sigma_1 - \sigma_2\right)^2,$$

$$\boldsymbol{N} = \pm\frac{1}{\sqrt{2}}\boldsymbol{n}_1 \pm \frac{1}{\sqrt{2}}\boldsymbol{n}_3, \quad \tau^2 = \frac{1}{4}\left(\sigma_1 - \sigma_3\right)^2.$$

Consequently maximum value of shear stress is given by maximum of above three, i.e., $\tau_{\max} = \frac{1}{2}|\sigma_{\max} - \sigma_{\min}|$, where $\sigma_{\max}$ and $\sigma_{\min}$ indicate the maximum and minimum values of principal stresses.

It is important to note that the maximum shear stress acts on a plane that is shifted about an angle of $\pm 45°$ to the principal plane in which the maximum and minimum principal stresses act.

By substituting the it can be shown that the normal stress associated with $\tau_{\max}$ has the value

$$\sigma = \frac{1}{2}|\sigma_{\max} + \sigma_{\min}|.$$

Stress invariants

Following are the invariant of stress tensor.

$$I_1 = \mathrm{tr}\boldsymbol{\sigma} = \sigma_{ii}$$
$$I_2 = \frac{1}{2}\left[(\mathrm{tr}\boldsymbol{\sigma})^2 - \mathrm{tr}\boldsymbol{\sigma}^2\right] = \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji})$$
$$I_3 = \det \boldsymbol{\sigma}$$

Hydrostatic and Deviatoric stress

Stress being a second order tensor can be splitted in to a spherical and deviatoric part as,

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + \boldsymbol{\sigma}' \text{ or } \sigma_{ij} = -p\delta_{ij} + \sigma'_{ij},$$

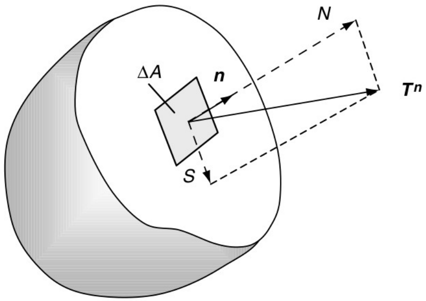
where $p =-\mathrm{tr}\boldsymbol{\sigma}$, is the hydrostatic stress component and $\boldsymbol{\sigma}'$ is called the deviatoric stress. Hydrostatic stress is the volume changing and shape conserving part, whereas deviatoric stress is volume conserving and shape changing part of the stress.

Problem Set

Problem 1: For a pure expansion the deformation gradient is $\boldsymbol{F} = a\boldsymbol{I}$, where a is a scalar. Show that the rate of deformation is

$$\dot{\boldsymbol{d}} = \frac{\dot{\alpha}}{\alpha}\boldsymbol{I}.$$

Problem 2: Show that shear stress S acting on a plane defined by the unit normal vector $\boldsymbol{n}$ can be written as,



$$S = \left[n_1^2 n_2^2(\sigma_1 - \sigma_2)^2 + n_2^2 n_3^2(\sigma_2 - \sigma_3)^2 + n_3^2 n_1^2(\sigma_3 - \sigma_1)^2\right].$$

Problem 3: Show that the principal directions of the deviatoric stress tensor coincide with the principal directions of the stress tensor. Also determine the principal values of the deviatoric stress in terms of principal values of the total stress.

Problem 4:

A circular cylinder in its reference configuration has radius A and its axis lies along the X_3–axis. It undergoes the following deformation,

$$
\begin{aligned}
x_1 &= \mu[X_1 \cos(\psi X_3) + X_2 \sin(\psi X_3)],\\
x_2 &= \mu[-X_1 \sin(\psi X_3) + X_2 \cos(\psi X_3)],\\
x_3 &= \lambda X_3.
\end{aligned}
$$

(a) Find condition(s) in terms of constants λ, μ and ψ, if cylinder's volume remain constant (incompressible material).

(b) A line drawn on the surface of the cylinder has unit length and is parallel to the axis of the cylinder in the reference configuration. Find its length after the deformation.

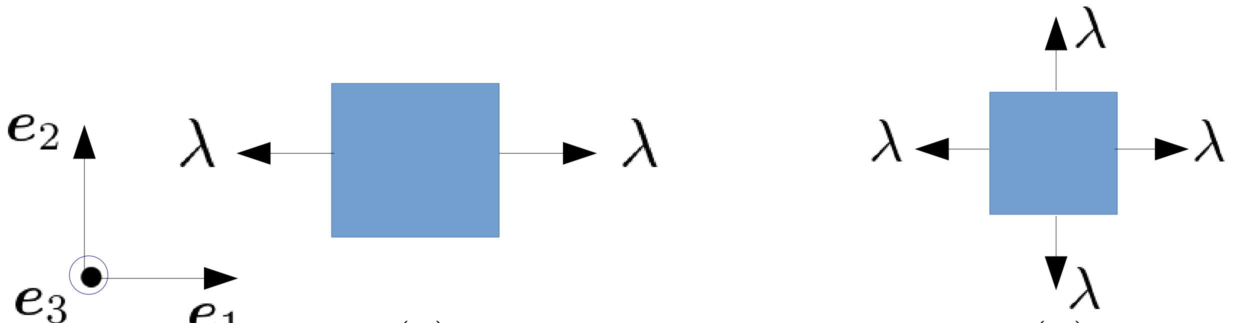
(c) Find also the initial length of a line on the surface which has unit length and is parallel to the axis after the deformation.

Problem 5:

For the uni-axial strain case, find the Engineering, Green's, and Almansi strain tensor in terms of the stretch λ. Using these expressions show that when the Engineering strain is small, all three strain measures converge to the same value.

Problem 6:

Consider a solid for which volume remains constant during the deformation. Write deformation gradient tensor,
(a) for uniaxial stretch case in terms of the stretch λ (see figure).
(b) for equal biaxial stretch case in terms of λ (see figure).



(a) (b)

Problem 7:

At a point the stress matrix, referred to xyz-axes, has the component shown below. Find the following.

(a) the three rectangular components of the traction vector acting on a plane through a point with unit normal (2/3, -2/3, 1/3),

(b) the magnitude of the traction vector of (a),

(c) its component in the direction of the normal,

(d) the angle between the traction vector and the normal.

$$\begin{bmatrix} 36 & 27 & 0 \\ 27 & -36 & 0 \\ 0 & 0 & 18 \end{bmatrix} \text{ MPa}$$

Problem 8:

Prove the following: $\dot{J} = J \mathrm{div} \boldsymbol{v}$