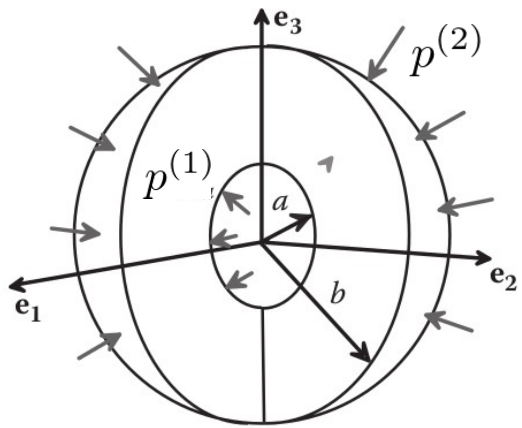
Pressurized hollow sphere



- No body forces act on the sphere
- The inner surface $R{=}a$ is subjected to pressure $p^{(1)}$, which implies

$$\sigma_{R\theta} = \sigma_{R\phi} = 0\,, \quad \sigma_{RR} = -p^{(1)} \quad \text{on } R = a$$

- The outer surface $R{=}b$ is subjected to pressure $p^{(2)}$, which implies,

$$\sigma_{R\theta} = \sigma_{R\phi} = 0\,, \quad \sigma_{RR} = -p^{(2)} \quad \text{on } R = b\,.$$

Let us solve for displacements first.

As the geometry and loading is symmetric, we also assume that the solution displacement field exhibits the spherical symmetry. The displacement components are then,

$$u_R = U(R)\,, \quad u_\theta = u_\phi = 0$$

Symmetry also implies that the problem is independent of and ϕ direction.

Then, the Navier's equation become,

$$(\lambda + 2\mu)\frac{d}{dR}\left[\frac{1}{R^2}\frac{d}{dR}(R^2U)\right] = 0$$

From the integration of the equation twice, we get,

$$U(R) = AR + B/R^2,$$

where A and B are constants of integration.

Now, strain-displacement relations can be used to find the strain components.

$$\varepsilon_{RR} = \frac{\partial U(R)}{\partial R} = \frac{dU(R)}{\partial R}, \quad \varepsilon_{\theta\theta} = \varepsilon_{\phi\phi} = U(R)/R.$$

Using constitutive relations stress components can be found.

$$\sigma_{rr} = (\lambda + 2\mu)\varepsilon_{rr} + \lambda\varepsilon_{\theta\theta} + \lambda\varepsilon_{\phi\phi} = (\lambda + 2\mu)\frac{dU(R)}{dR} + 2\lambda\frac{U(R)}{R}$$

$$\sigma_{rr} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1+\nu)A - 2(1-2\nu)\frac{B}{R^3}\right].$$

By applying boundary conditions, constant A and B can be found .

$$A = \frac{\left(p^{(2)}b^3 - p^{(1)}a^3\right)(1-2\nu)}{\left(a^3 - b^3\right)E}, B = \frac{\left(p^{(2)} - p^{(1)}\right)a^3b^3(1+\nu)}{2E\left(a^3 - b^3\right)}.$$

Finally, stress components are,

$$\sigma_{RR} = \left(\frac{a^3p^{(1)} - b^3p^{(2)}}{b^3 - a^3}\right) - \frac{a^3b^3}{R^3}\left(\frac{p^{(1)} - p^{(2)}}{b^3 - a^3}\right) ,$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = \left(\frac{a^3p^{(1)} - b^3p^{(2)}}{b^3 - a^3}\right) + \frac{a^3b^3}{2R^3}\left(\frac{p^{(1)} - p^{(2)}}{b^3 - a^3}\right) ,$$

$$\sigma_{R\theta} = \sigma_{\theta\phi} = \sigma_{\phi R} = 0 .$$

and displacement components are,

$$u_R = \frac{1}{2E\left(b^3 - a^3\right)R^2}\left[2\left(p^{(1)}a^3 - p^{(2)}b^3\right)(1-2\nu)R^3 + \left(p^{(1)} - p^{(2)}\right)(1+\nu)a^3b^3\right],$$

$$u_\theta = u_\phi = 0.$$

Spacial cases:

(a) When wall of the spherical shell is very thin, i.e. $t << a$, which implies

$$b^3 - a^3 = (b-a)(b^2 + ab + a^2) \approx 3ta^2 \ ,$$

$$a^3 p^{(1)} - b^3 p^{(2)} \approx a^3 (p^{(1)} - p^{(2)}) \ ,$$

$$b^3 \approx a^3 \ ,$$

$$R^3 \approx a^3 \ ,$$

which leads to

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} \approx \frac{a(p^{(1)} - p^{(2)})}{2t} \quad \text{and } \sigma_{rr} \approx 0,$$

This is the thin-walled pressure vessel approximation commonly studied in strength of materials.

Spacial cases:

(b) In the limit as $b \to \infty$, the problem reduces to that of a pressurized spherical void in an infinite medium under remote hydrostatic pressure. The solution to this problem then is found by taking the solution derived for hollow sphere in the limit as $b \to \infty$, as

$$\sigma_{RR} = -p^{(2)} - (p^{(1)} - p^{(2)})\frac{a^3}{R^3}\,, \qquad \sigma_{\theta\theta} = \sigma_{\phi\phi} = -p^{(2)} + \frac{1}{2}(p^{(1)} - p^{(2)})\frac{a^3}{R^3}\,.$$

Far away from the void, as $R \to \infty$, the state of stress is one of pure hydrostatic pressure:

$$\sigma_{RR} = \sigma_{\theta\theta} = \sigma_{\phi\phi} = -p^{(2)}.$$

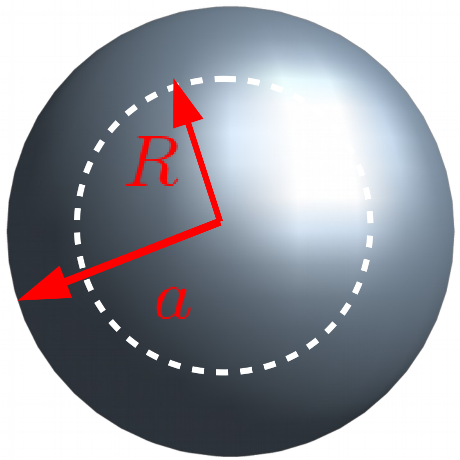
If the void is unpressurized (i.e. $p^{(1)} = 0$), then

$$\sigma_{RR} = -p^{(2)}\left(1 - \frac{a^3}{R^3}\right)\,, \quad \sigma_{\theta\theta} = \sigma_{\phi\phi} = -p^{(2)}\left(1 + \frac{a^3}{2R^3}\right)\,.$$

At $R = a$, $\sigma_{RR} = 0$ and $\sigma_{\theta\theta} = \sigma_{\phi\phi} = -3p^{(2)}/2$. Far away from the void, the maximum absolute value of the principal stresses is $|p^{(2)}|$, whereas at $R = a$, it is $3|p^{(2)}|/2$. The ratio of the later over the former is the *stress concentration factor*: s.c.f $= 3/2$,

for an unpressurized spherical void in an infinite medium.

Gravitating sphere



A planet under its own gravitational attraction may be idealized (rather crudely) as a solid sphere with radius a. The solid is subjected to the following loading:

- A body force $\boldsymbol{b} = -(gR/a)\boldsymbol{e}_{\mathrm{R}}$ per unit mass, where g is the acceleration attributable to gravity at the surface of the sphere
- A traction-free surface at $R{=}a$

We start with Navier's equation,

$$(\lambda + 2\mu)\frac{d}{dR}\left[\frac{1}{R^2}\frac{d}{dR}(R^2U)\right] - \rho_0\frac{gR}{a} = 0,$$

or

$$\frac{d}{dR}\left[\frac{1}{R^2}\frac{d}{dR}(R^2U)\right] = \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\rho_0\frac{gR}{a}.$$

Similar to the previous problem, integrate the equation twice to get the displacement,

$$U(R) = \frac{(1+\nu)(1-2\nu)}{E(1-\nu)} \frac{\rho_0 g R^3}{10a} + AR + B/R^2,$$

where A and B are constants of integration which can be found by applying boundary conditions.

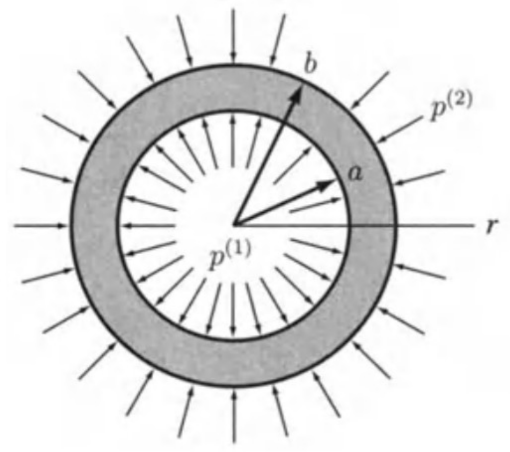
Expression for radial stress can be derived by first using strain-displacement relations, and then the constitutive relations (similar to the previous problem) as,

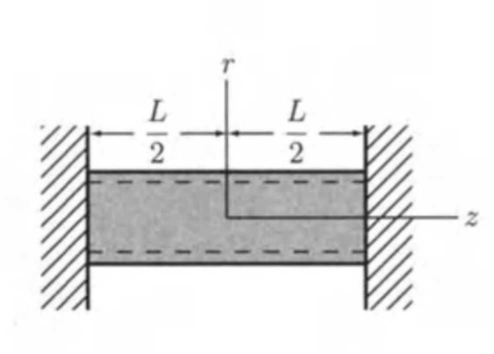
$$\sigma_{rr} = \frac{\rho_0 g(3-\nu)R^2}{10a(1-\nu)} + \frac{E}{(1+\nu)(1-2\nu)} \left[(1+\nu)A - 2(1-2\nu)\frac{B}{R^3} \right].$$

Constants A and B can be determined by using following:
(i) the stress must be finite at $R \to 0$, which is only possible if $B = 0$;
(ii) the surface of the sphere is traction free, which requires $\sigma_{RR} = 0$ at $R = a$, which implies that

$$A = -\frac{(1-2\nu)(3-\nu)\rho_0 g a}{10E(1-\nu)}.$$

Pressurized long cylinder





- Position vector: $\boldsymbol{x} = r\boldsymbol{e}_r + z\boldsymbol{e}_z$
- Displacement vector: $\boldsymbol{u} = u(r)\boldsymbol{e}_r$

Assume that body forces are negligible. For cross sections sufficiently far from the ends, it is clear that $u_z = 0$ and that u_r and u are independent of z. Also, $u = 0$.

- The inner surface $R{=}a$ is subjected to pressure $p^{(1)}$, which implies

$$\sigma_{rr} = -p^{(1)}\ , \quad \sigma_{r\theta} = 0 \quad \text{on } r = a\ .$$

- The outer surface $R{=}b$ is subjected to pressure $p^{(2)}$, which implies,

$$\sigma_{rr} = -p^{(2)}\ , \quad \sigma_{r\theta} = 0 \quad \text{on } r = b\ .$$

This problem can be solved starting from Navier's equation, which will be very similar to that method we used for the pressurized hollow sphere. The only different here will be that we will be using equations corresponding to cylindrical coordinate system.

$$u_r = \frac{r}{2(\mu+\lambda)}\left(\frac{a^2p^{(1)} - b^2p^{(2)}}{b^2-a^2}\right) + \frac{a^2b^2}{2\mu r}\left(\frac{p^{(1)} - p^{(2)}}{b^2-a^2}\right) ,$$

$$u_\theta = 0$$

$$\sigma_{rr} = \left(\frac{a^2p^{(1)} - b^2p^{(2)}}{b^2-a^2}\right) - \frac{a^2b^2}{r^2}\left(\frac{p^{(1)} - p^{(2)}}{b^2-a^2}\right) ,$$

$$\sigma_{\theta\theta} = \left(\frac{a^2p^{(1)} - b^2p^{(2)}}{b^2-a^2}\right) + \frac{a^2b^2}{r^2}\left(\frac{p^{(1)} - p^{(2)}}{b^2-a^2}\right) ,$$

$$\sigma_{zz} = 2\nu\left(\frac{a^2p^{(1)} - b^2p^{(2)}}{b^2-a^2}\right) ,$$

$$\sigma_{r\theta} = 0 .$$

Remember that this solution is only valid for cross sections sufficiently far from the ends of the pressure vessel.

This solution can be generalized by superposing a uniaxial stress field. Consider a problem where, instead of rigid supports at $z = \pm L/2$, there are end caps, such as on an actual cylindrical pressure vessel. The internal pressure has a resultant force on each end cap of $F^{(1)} = \pi a^2 p^{(1)}$ along the z-direction and the external pressure has a resultant force of $F^{(2)} = \pi b^2 p^{(2)}$. The cross-sectional area of the vessel wall is $A = \pi(b^2 - a^2)$. For equilibrium with the pressure on the end caps, therefore, the axial stress σ_{zz} should be

$$\sigma_{zz} = \frac{F^{(1)} - F^{(2)}}{A} = \frac{a^2 p^{(1)} - b^2 p^{(2)}}{b^2 - a^2} . \tag{7.2.34}$$

To meet this condition, a uniaxial stress field given by

$$\sigma_{zz} = (1 - 2\nu)\frac{a^2 p^{(1)} - b^2 p^{(2)}}{b^2 - a^2} , \tag{7.2.35}$$

along with the corresponding components of strain and displacement, can be superposed on the solution (7.2.33).

If the cylindrical vessel is thin-walled with thickness $t \equiv b - a$, $t \ll a$, then

$$\begin{aligned} b^2 - a^2 = (b-a)(b+a) &\approx 2ta \ , \\ a^2 p^{(1)} - b^2 p^{(2)} &\approx a^2 (p^{(1)} - p^{(2)}) \ , \\ b^2 &\approx a^2 \ , \\ r^2 &\approx a^2 \end{aligned} \tag{7.2.36}$$

and it follows from (7.2.33) that

$$\sigma_{\theta\theta} \approx \frac{a(p^{(1)} - p^{(2)})}{t} \ . \tag{7.2.37}$$

This is the thin-walled cylindrical pressure vessel approximation from mechanics of materials.

In the limit as $b \to \infty$, the problem reduces to that of a pressurized cylindrical hole in an infinite body under remote "plane strain pressure" (as $r \to \infty$, $\sigma_{rr} = \sigma_{\theta\theta} = -p^{(2)}$ and $\sigma_{zz} = -2\nu p^{(2)}$). Letting $b \to \infty$ in (7.2.33),

$$\begin{aligned}
\sigma_{rr} &= -p^{(2)} - (p^{(1)} - p^{(2)})\frac{a^2}{r^2} \ , \\
\sigma_{\theta\theta} &= -p^{(2)} + (p^{(1)} - p^{(2)})\frac{a^2}{r^2} \ , \\
\sigma_{zz} &= -2\nu p^{(2)} \ .
\end{aligned} \tag{7.2.38}$$

If the hole is unpressurized ($p^{(1)} = 0$), then

$$\begin{aligned}
\sigma_{rr} &= -p^{(2)}\left(1 - \frac{a^2}{r^2}\right) \ , \\
\sigma_{\theta\theta} &= -p^{(2)}\left(1 + \frac{a^2}{r^2}\right) \ , \\
\sigma_{zz} &= -2\nu p^{(2)} \ .
\end{aligned} \tag{7.2.39}$$

Far from the hole ($r \to \infty$), the greatest principal stress magnitude is $|p^{(2)}|$, whereas at the hole ($r = a$), the greatest principal stress magnitude is $2|p^{(2)}|$. Thus, a cylindrical hole in an infinite body under far-field plane strain pressure has a stress concentration factor of 2.