Hyperelasticity

Incompressible hyperelasticity

For imcompressible material volume does not change. Various elastomers can generally sustain high strains without any noticable volume change and behave like an incompressible or nearly-incompressible material. Hence, for incompressible materials a constraint $\operatorname{tr}\boldsymbol{C} = \operatorname{tr}\boldsymbol{b} = 1$ is considered. Thus a suitable strain energy function for incomprssible material is given as,

$$\Psi = \Psi[I_{1C}, I_{2C}] - \frac{p}{2}\left(I_{3C} - 1\right), \qquad \text{..................(11)}$$

where $p/2$ serves as an indeterminate Lagrange multiplier. Now from (7) and (8),

$$\boldsymbol{S} = 2\left[\left(\frac{\partial \Psi}{\partial I_{1C}} + I_{1C}\frac{\partial \Psi}{\partial I_{2C}}\right)\boldsymbol{I} - 2\frac{\partial \Psi}{\partial I_{2C}}\boldsymbol{C}\right] - p\boldsymbol{C}^{-1}. \qquad \text{..................(12)}$$

$$\boldsymbol{\sigma} = 2\left[\left(\frac{\partial \Psi}{\partial I_{1b}} + I_{1b}\frac{\partial \Psi}{\partial I_{2b}}\right)\boldsymbol{b} - \frac{\partial \Psi}{\partial I_{2b}}\boldsymbol{b}^2\right] - p\boldsymbol{I}. \qquad \text{..................(13)}$$

Observe that these stress relations incorporate the unknown scalar p, which must be determined from the equilibrium equations and the boundary conditions.

Example

Consider the biaxial stretching of a thin sheet of incompressible hyperelastic material.

$$\boldsymbol{F} = \sum_{i=1}^{3} \lambda_i \boldsymbol{e}_i \boldsymbol{e}_i, \quad \text{with} \quad J = \lambda_1 \lambda_2 \lambda_3 = 1. \qquad \cdots\cdots\cdots\cdots(14)$$

$$\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^T = \sum_{i=1}^{3} \lambda_i^2 \boldsymbol{e}_i \boldsymbol{e}_i. \qquad \cdots\cdots\cdots\cdots(15)$$

Now from (13),

$$\sigma_\alpha = 2\left[\left(\frac{\partial \Psi}{\partial I_{1b}} + I_{1b}\frac{\partial \Psi}{\partial I_{2b}}\right)\lambda_\alpha^2 - \frac{\partial \Psi}{\partial I_{2b}}\lambda_\alpha^4\right] - p, \quad \alpha = 1, 2, 3. \qquad \cdots\cdots\cdots\cdots(16)$$

If thickness direction is considered to be along $\boldsymbol{e}_3$, then

$$\sigma_3 = 2\left[\left(\frac{\partial \Psi}{\partial I_{1b}} + \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right)\frac{\partial \Psi}{\partial I_{2b}}\right)\lambda_3^2 - \frac{\partial \Psi}{\partial I_{2b}}\lambda_3^4\right] - p = 0$$

$$\Rightarrow p = 2\frac{\partial \Psi}{\partial I_{1b}}\lambda_3^2 + 2\left(\lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2\right)\frac{\partial \Psi}{\partial I_{2b}} = 2\frac{\partial \Psi}{\partial I_{1b}}\lambda_1^{-2}\lambda_2^{-2} + 2\left(\lambda_1^{-2} + \lambda_2^{-2}\right)\frac{\partial \Psi}{\partial I_{2b}} \qquad \cdots\cdots\cdots\cdots(17)$$

Substituting (17) in (16), we get

$$\sigma_1 = 2\frac{\partial\Psi}{\partial I_{1b}}\left(\lambda_1^2 - \lambda_1^{-2}\lambda_2^{-2}\right)+2\frac{\partial\Psi}{\partial I_{2b}}\left(\lambda_1^2\lambda_2^2 - \lambda_1^{-2}\right) = 2\left(\lambda_1^2 - \lambda_1^{-2}\lambda_2^{-2}\right)\left[\frac{\partial\Psi}{\partial I_{1b}} + \lambda_2^2\frac{\partial\Psi}{\partial I_{2b}}\right]$$

$$\sigma_2 = 2\frac{\partial\Psi}{\partial I_{1b}}\left(\lambda_2^2 - \lambda_1^{-2}\lambda_2^{-2}\right)+2\frac{\partial\Psi}{\partial I_{2b}}\left(\lambda_1^2\lambda_2^2 - \lambda_2^{-2}\right) = 2\left(\lambda_2^2 - \lambda_1^{-2}\lambda_2^{-2}\right)\left[\frac{\partial\Psi}{\partial I_{1b}} + \lambda_1^2\frac{\partial\Psi}{\partial I_{2b}}\right]$$

....................(18)

Equations (18) are the stresses in 1 and 2 directions as a function of stretches both the directions.

Some forms of strain energy functions

Ogden Model for incompressible materials:

The strain energy function due to Ogden is a function of principal stretches in all directions. The function is as follows,

$$\Psi = \Psi(\lambda_1, \lambda_2, \lambda_3) = \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p} \left(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p} - 3 \right), \qquad \text{......................(19)}$$

where μ_p and a_p are material constants.

On comparison with the linear theory we obtain the (consistency) condition as

$$\mu = \frac{1}{2} \sum_{p=1}^{n} \mu_p \alpha_p > 0, \qquad \text{......................(20)}$$

where μ denotes the classical shear modulus in reference configuration for linear elasticity theory.

Mooney-Rivlin Model for incompressible materials:

Mooney-Rivlin strain energy function for incompressible hyperelastic materials is as follows,

$$\Psi = c_1(I_1 - 3) + c_2(I_2 - 3), \qquad \cdots\cdots\cdots\cdots(21)$$

where c_1 and c_2 are material constants with the shear modulus

$$\mu = 2(c_1 + c_2).$$

neo-Hookean Model for incompressible materials:

neo-Hookean strain energy function is given as,

$$\Psi = c_1(I_1 - 3), \qquad \cdots\cdots\cdots\cdots(22)$$

where c_1 is a material constant, and the shear modulus $\mu = 2c_1$.

This strain-energy function involves a single parameter only and provides a mathematically simple and reliable constitutive model for the nonlinear deformation behavior of isotropic rubber-like materials. It relies on phenomenological considerations and includes typical effects known from nonlinear elasticity within the small strain domain.

Yeoh Model for incompressible materials:

- Largely engineering elastomers contain reinforcing fillers such as carbon black (in natural rubber vulcanizate) or silica (in silicone rubber). These finely distributed fillers form physical and chemical bonds with the polymer chains. The fine filler particles are added to the elastomers in order to improve their physical properties which are mainly tensile and tear strength, or abrasion resistance.

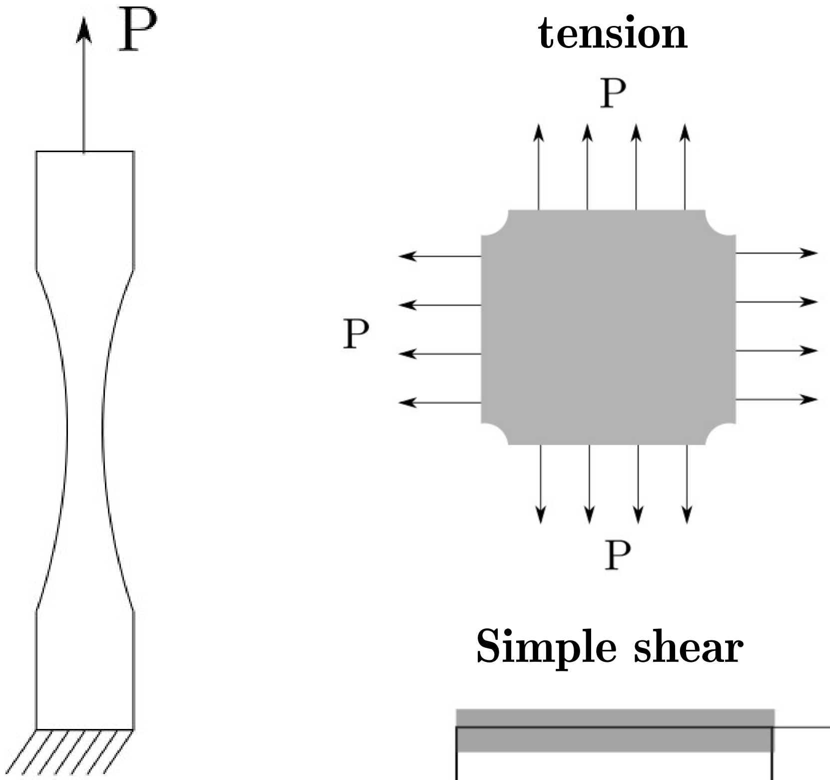
- Carbon-black filled rubbers have important applications in the manufacture of automotive tyres and many other engineering components.

- The phenomenological material model by YEOH is motivated in order to simulate the mechanical behavior of carbon-black filled rubber vulcanizates with the typical stiffening effect in the large strain domain. The corresponding strain energy function is as follows,

$$\Psi = c_1(I_1 - 3) + c_2(I_1 - 3)^2 + c_3(I_1 - 3)^3, \quad \cdots\cdots\cdots\cdots\cdots\cdots(23)$$

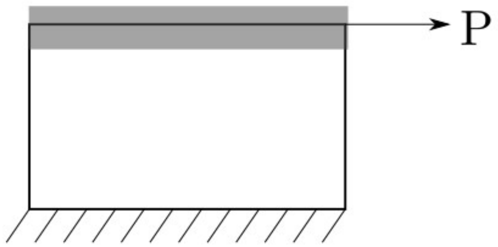
  where c_1, c_2 and c_3 are material parameters.

Determining material parameters

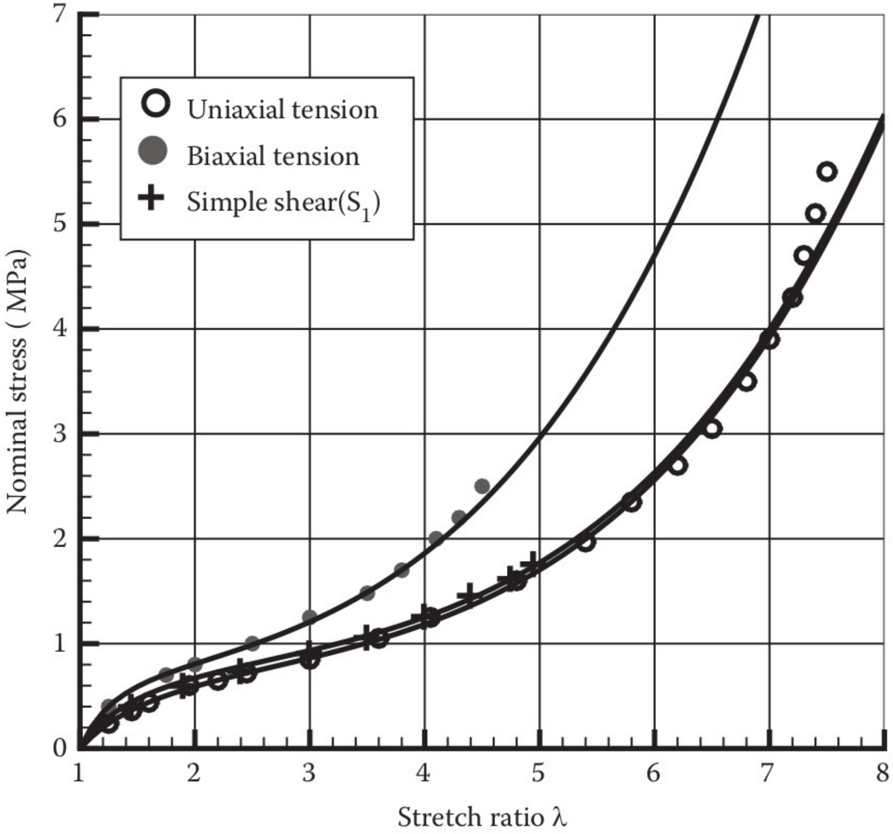


Uniaxial
tension

Simple shear



Experimental data for vulcanized rubber and Corresponding fit with Ogden model

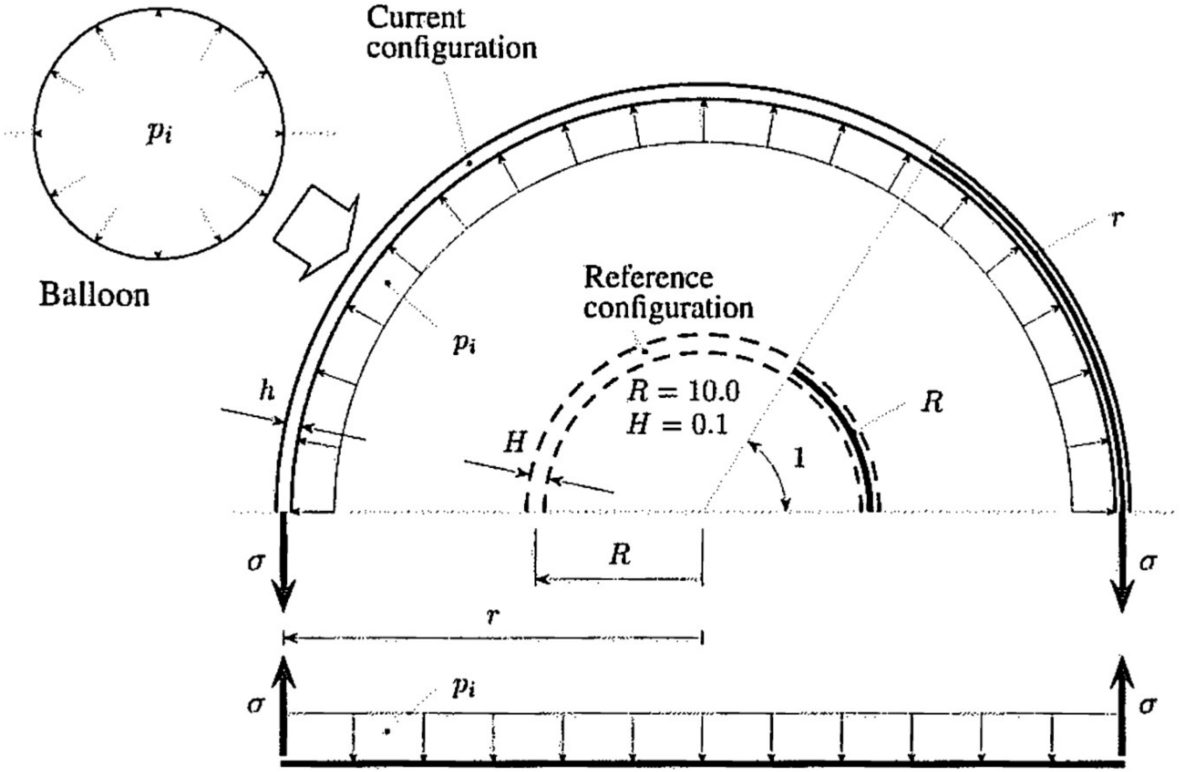


Example

Consider the inflation of a spherical balloon made of an incompressible material. During the analysis we are particularly interested in computing the inflation pressure p_i, i.e. the internal pressure in the balloon, and the circumferential Cauchy stress σ as a function of the circumferential stretch λ of the balloon.

Let the initial (zero-pressure) radius of the rubber balloon be R=10 and the initial thickness of the wall be $H = 0.1$.

We compare the behaviour of inflation for three different material models.



For the purpose we use following material parameters which are obtained by fitting the experimental data for vulcanized rubber to different material models.

Ogden:

$$\alpha_1 = 1.3, \quad \mu_1 = 0.63 \text{ MPa}$$

$$\alpha_2 = 5.0, \quad \mu_2 = 0.0012 \text{ MPa}$$

$$\alpha_3 = -2.0, \quad \mu_3 = -0.01 \text{ MPa},$$

which gives the shear modulus $\mu = 0.4225$ MPa.

Mooney-Rivlin:

$$c_1 = 0.4375\mu, \quad c_2 = 0.0625\mu.$$

neo-Hookean:

$$c_1 = 0.5\mu.$$

During the inflation every point on the balloon goes under equi-biaxial stretching, and the state of stress at every point is given as,

$$\sigma_1 = \sigma_2 = \sigma \text{ and } \sigma_3 = 0, \qquad \cdots\cdots\cdots(24)$$

for a thin spherical shell, where 1 and 2 are the hoop directions and 3 is the thickness direction.

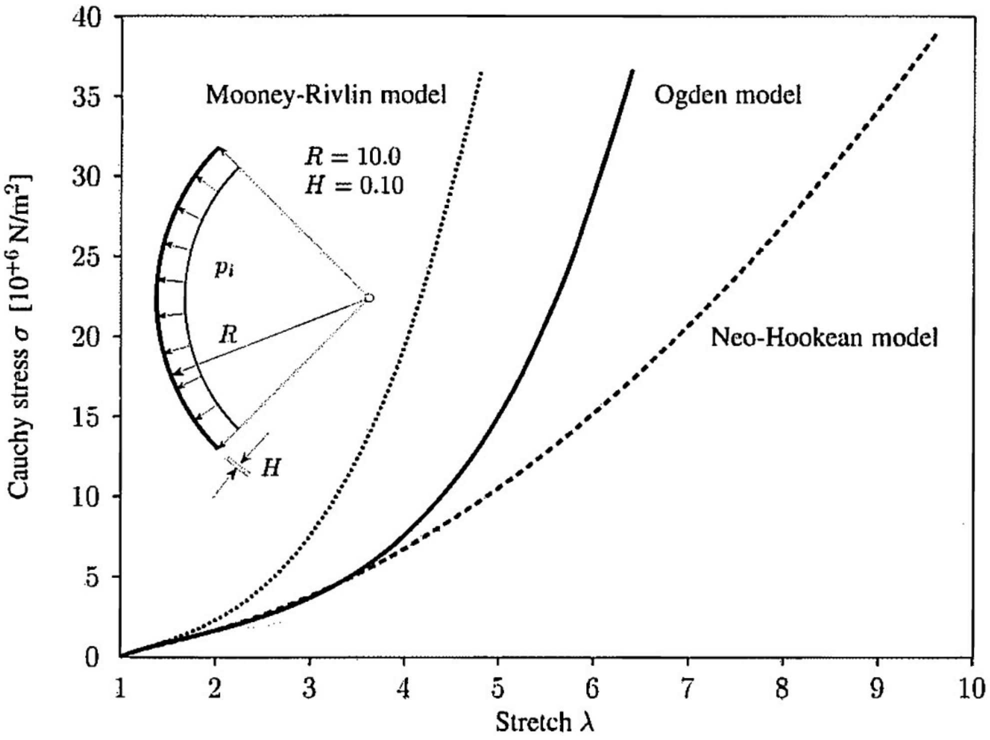
By considering the equilibrium of balloon, a relationship between the internal pressure and the stress can be established as,

$$p_i \cdot \pi r^2 = \sigma \cdot 2\pi r h \Rightarrow p_i = 2\sigma h/r. \qquad \cdots\cdots\cdots(25)$$

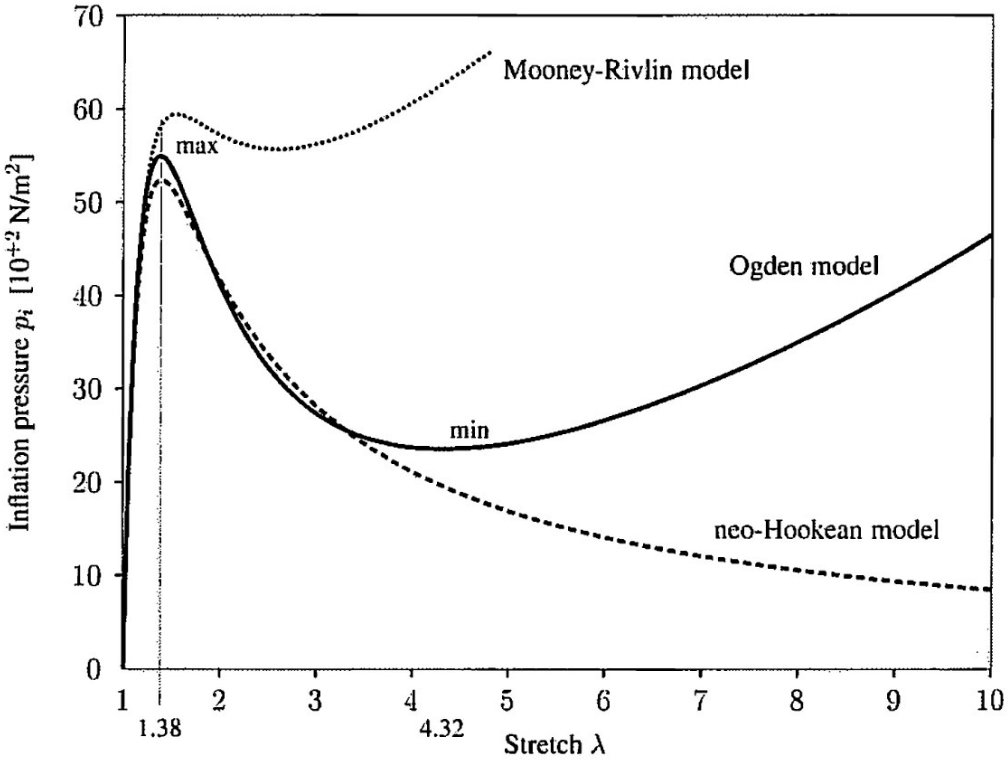
Stretch at any point of the balloon is defined as $r/R = \lambda$, and the stretch in the thickness direction is given as $h/H = \lambda_3 = 1/\lambda^2$, which can be obtained by incompressibility condition, i.e.,

$$4\pi R^2 H = 4\pi r^2 h. \qquad \cdots\cdots\cdots(26)$$

Now using (18) with different energy function given by (19)-(22), expression for σ in terms of stretch λ can be derived. With the help of (25) relationship between pi and stretch λ can also be obtained.



Cauchy stress σ [10^+6 N/m^2]
Mooney-Rivlin model
Ogden model
Neo-Hookean model
R = 10.0
H = 0.10
p_i
R
H
Stretch λ



Inflation pressure p_i [10^+2 N/m^2]
Mooney-Rivlin model
max
Ogden model
min
neo-Hookean model
1.38
4.32
Stretch λ

Problem Set

Problem 1: Consider a thin sheet of incompressible hyperelastic material ($I_3 = 1$).

(a) Consider a simple tension for which $\lambda = \lambda_1$. Then, obeying incompressibility constraint $\lambda_1\lambda_2\lambda_3 = 1$, the equal stretch ratios in the transverse directions are, by symmetry, $\lambda_2 = \lambda_3 = \lambda^{-1/2}$. Show that for this mode of deformation the homogeneous stress state reduces to $\sigma_1 = \sigma$, $\sigma_2 = \sigma_3 = 0$, with

$$\sigma = 2\left(\lambda^2 - \frac{1}{\lambda}\right)\left(\frac{\partial \Psi}{\partial I_1} + \frac{1}{\lambda}\frac{\partial \Psi}{\partial I_2}\right)$$

(b) Consider a case of equibiaxial tension in 1-2 plane and show that

$$\sigma_1 = \sigma_2 = \sigma = 2\left(\lambda^2 - \frac{1}{\lambda^4}\right)\left(\frac{\partial \Psi}{\partial I_1} + \lambda^2\frac{\partial \Psi}{\partial I_2}\right)$$

(c) Consider a case of pure shear in 1-2 plane and show that nonzero Cauchy stress components are

$$\sigma_1 = 2\left(\lambda^2 - \frac{1}{\lambda^2}\right)\left(\frac{\partial \Psi}{\partial I_1} + \frac{\partial \Psi}{\partial I_2}\right), \quad \sigma_2 = 2\left(1 - \frac{1}{\lambda^2}\right)\left(\frac{\partial \Psi}{\partial I_1} + \lambda^2\frac{\partial \Psi}{\partial I_2}\right),$$

Problem 2:

For all the cases (a)-(c) considered in problem 1, derive the stresses for the following material models.
(a) Neo-hookean
(b) Mooney-Rivlin
(c) Yeoh
(d) Ogden

Problem 3:

Compare the bheaviour of different models by plotting equations obtained in problem 2. User the properties given in slide 17.

Problem 4:

Show that in the limit of small stretches equivalent linear elastic shear modulus for Ogden model is given as follows:

$$\mu = \frac{1}{2}\sum_{p=1}^{n} \mu_p \alpha_p.$$