ME232: Dynamics

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Room # 106

Conservation of energy and momentum

Conservation of energy:

A mass system is said to be conservative if it dissipate energy because of friction or inelasticity.

If no work is done on a conservative system during an interval of motion by external forces (other than gravity or other potential forces), then none of the energy of the system is lost.

For this case, $U'_{1-2} = 0$ and hence, $\Delta T + \Delta V = 0$ or $T_1 + V_1 = T_2 + V_2$.

This is the law of **conservation of dynamical energy**.

The total energy $E = T + V$ is a constant, so that $E_1 = E_2$.

Remember this law holds only in the ideal case where dissipative phenomenon are neglected.

Conservation of momentum:

If, for a certain interval of time, the resultant external force $\Sigma \boldsymbol{F}$ acting on a conservative or nonconservative mass system is zero, it requires that $\dot{\boldsymbol{G}} = 0$, so that during this interval $\boldsymbol{G}_1 = \boldsymbol{G}_2$, which expresses the principle of conservation of linear momentum. Thus, in the absence of an external impulse, the linear momentum of a system remains unchanged.

Similarly, if the resultant moment about a fixed point O or about the mass center G of all external forces on any mass system is zero, then

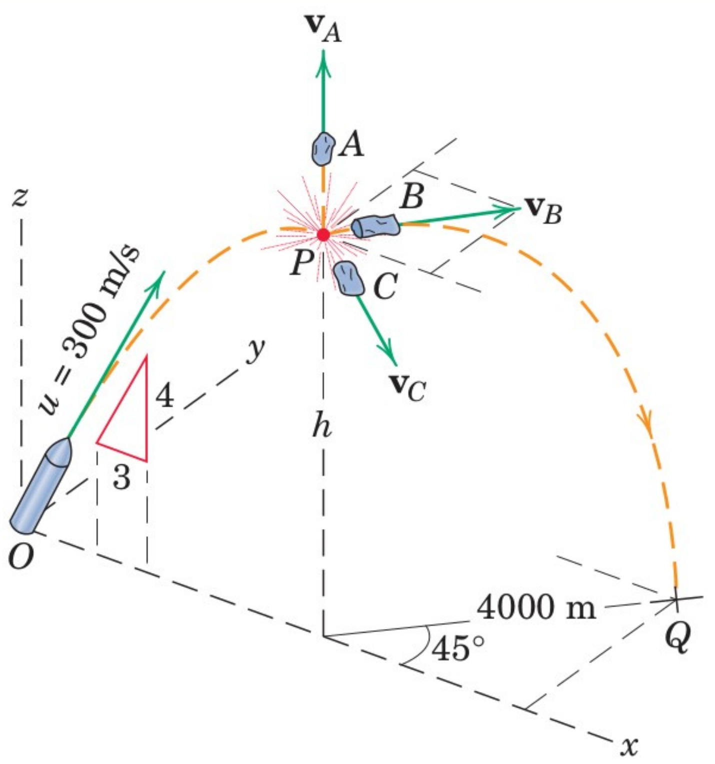
$$(\boldsymbol{H}_O)_1 = (\boldsymbol{H}_O)_2 \qquad \text{or} \qquad (\boldsymbol{H}_G)_1 = (\boldsymbol{H}_{\mathrm{G}})_2.$$

These relations express the principle of conservation of angular momentum for a general mass system in the absence of an angular impulse.

Thus, if there is no angular impulse about a fixed point (or about the mass center), the angular momentum of the system about the fixed point (or about the mass center) remains unchanged.

Example 1

A shell with a mass of 20 kg is fired from point O, with a velocity $u = 300$ m/s in the vertical x-z plane at the inclination shown. When it reaches the top of its trajectory at P, it explodes into three fragments A, B, and C. Immediately after the explosion, fragment A is observed to rise vertically a distance of 500 m above P, and fragment B is seen to have a horizontal velocity $\boldsymbol{v}_B$ and eventually lands at point Q. When recovered, the masses of the fragments A, B, and C are found to be 5, 9, and 6 kg, respectively. Calculate the velocity which fragment C has immediately after the explosion. Neglect atmospheric resistance.



From our knowledge of projectile motion, the time required for the shell to reach P and its vertical rise are

$$t = u_z / g = 24.5 \text{ s}$$

$$h = u_z^2 / 2g = 2940 \text{ m}$$

The velocity of A has the magnitude

$$v_A = (2gh)^{1/2} = 99 \text{ m/s}$$

With no z-component of velocity initially, fragment B requires 24.5 s to return to the ground. Thus, its horizontal velocity, which remains constant, is

$$v_B = s/t = 163.5 \text{ m/s.}$$

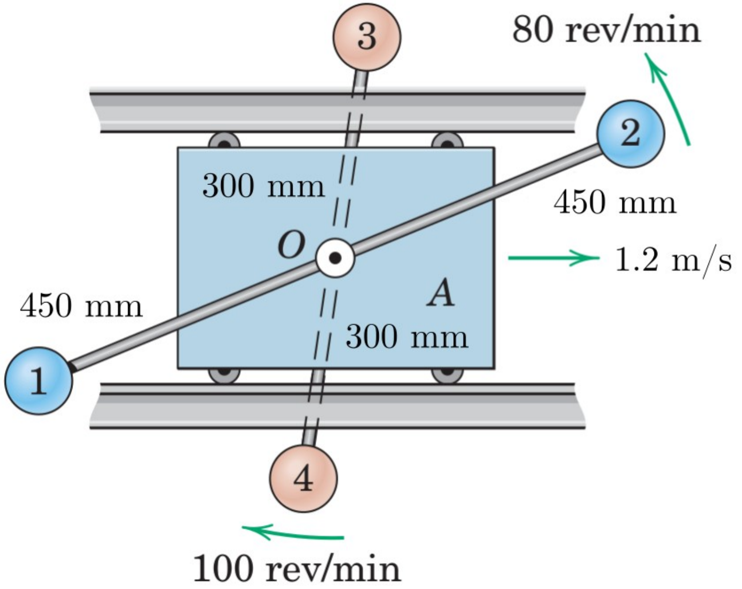
Since the force of the explosion is internal to the system of the shell and its three fragments, the linear momentum of the system remains unchanged during the explosion. Thus,

$$\boldsymbol{G}_1 = \boldsymbol{G}_2 \qquad \text{or} \qquad m\boldsymbol{v} = m_A \boldsymbol{v}_A + m_B \boldsymbol{v}_B + m_C \boldsymbol{v}_C.$$

$\boldsymbol{v}_c$ can now be obtained.

Example 2

The 16-kg carriage A moves horizontally in its guide with a speed of 1.2 m/sec and carries two assemblies of balls and light rods which rotate about a shaft at O in the carriage. Each of the four balls weighs 1.6-kg The assembly on the front face rotates counterclockwise at a speed of 80 rev/min, and the assembly on the back side rotates clockwise at a speed of 100 rev/min. For the entire system, calculate (a) the kinetic energy T, (b) the magnitude $\boldsymbol{G}$ of the linear momentum, and (c) the magnitude $\boldsymbol{H}_O$ of the angular momentum about point O.



The velocities of the balls with respect to O are

$(v_{\text{rel}})_{1,2} = (0.450) \cdot 80\ (2\pi/60) = 3.77$ m/s, $(v_{\text{rel}})_{3,4} = (0.3) \cdot 100\ (2\pi/60) = 3.14$ m/s

Transnational part of the KE of the the system is,

$$\frac{1}{2}m\bar{v}^2 = \frac{1}{2}\left[16 + 4(1.6)\right](1.2)^2 = 16.13 \text{ J}$$

Rotational part of the KE of the the system is,

$$\sum \frac{1}{2}m_i \left|\dot{\boldsymbol{\rho}}\right|^2 = 38.5 \text{ J}$$

Total kinetic energy is $16.13 + 38.5 = 54.7$ J

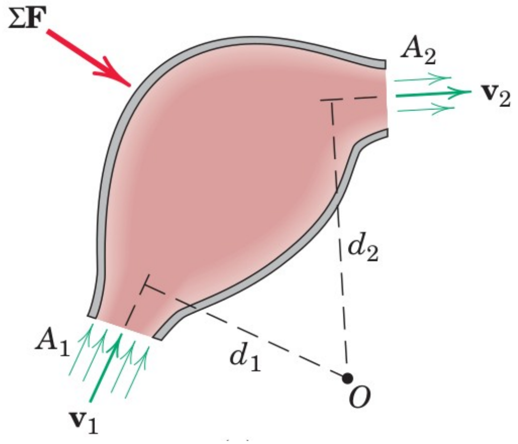
Linear momentum $G = m\bar{\boldsymbol{v}}$ =26.9 kg.m/s

Angular momentum about $O = H_O = \Sigma \mid \boldsymbol{r}_i \text{ x } m\boldsymbol{v}_i \mid = 2.41$ kg.m^2/s

Steady mass flow

- The momentum relation developed for a general system of mass enable us to analyze the action of mass flow. This is a situation where a change in momentum occurs.
- The dynamics of mass flow is of importance in the description of fluid machinery of all types including turbines, pumps, nozzles, air-breathing jet engines, and rockets.
- An important cases of mass flow is the steady-flow where the **rate at which mass enters** a given volume **equals the rate at which mass leaves** the same volume. The volume may be enclosed by a rigid container, fixed or moving, such as the nozzle of a jet aircraft or rocket, the space between blades in a gas turbine, the volume within the casing of a centrifugal pump, or the volume within the bend of a pipe through which a fluid is flowing at a steady rate.
- The design of such fluid machines depends on the analysis of the forces and moments associated with the corresponding momentum changes of the flowing mass.

Analysis of flow through a rigid container:

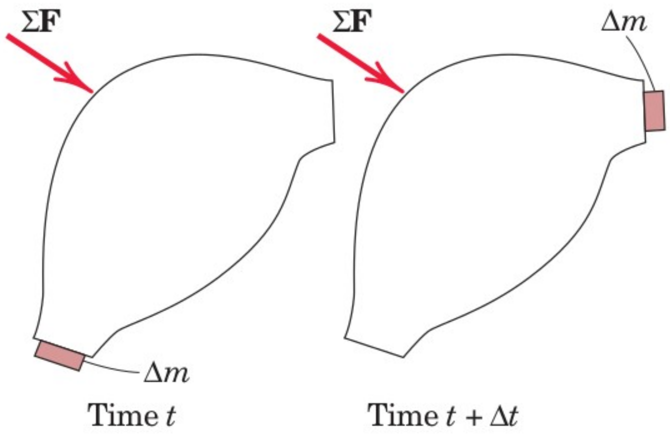


Consider a rigid container into which mass flows in a steady stream at the rate m' through the entrance section of area A_1. Mass leaves the container through the exit section of area A_2 at the same rate. This means that there is no accumulation or depletion of the total mass within the container during the period of observation.

The velocity of the entering stream is v_1 normal to A_1 and that of the leaving stream is v_2 normal to A_2. If 1 and 2 are the respective densities of the two streams, conservation of mass requires that

$$\rho_1 v_1 A_1 = \rho_2 v_2 A_2 = m'. \quad \text{..........(22)}$$

- We isolate the mass of fluid within the container if the forces between the container and the fluid were to be described.
- The entire container and the fluid within it is isolated when the forces external to the container are desired.



Following the later approach, isolation of the system is described by a free-body diagram of the mass within a closed volume defined by the exterior surface of the container and the entrance and exit surfaces. All forces applied externally to this system are shown as the vector sum $\Sigma \boldsymbol{F}$, which includes:

1. the forces exerted on the container at points of its attachment to other structures, including attachments at A_1 and A_2 , if present,

2. the forces acting on the fluid within the container at A_1 and A_2 due to any static pressure which may exist in the fluid at these positions, and

3. the weight of the fluid and structure if appreciable.

The resultant $\Sigma \boldsymbol{F}$ of all of these external forces must equal $\dot{\boldsymbol{G}}$, the time rate of change of the linear momentum of the isolated system. The expression for $\dot{\boldsymbol{G}}$ may be obtained by an incremental analysis.

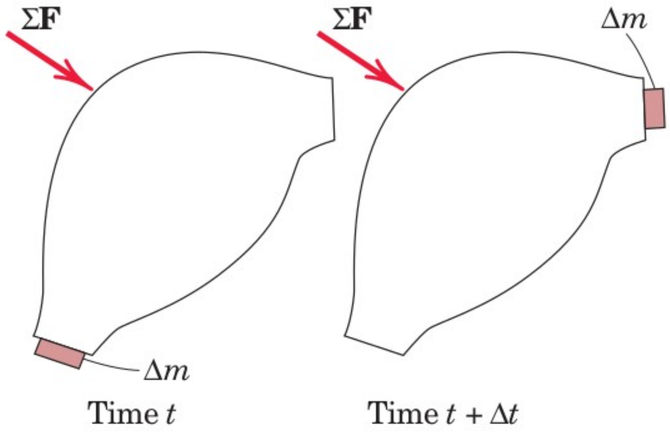


Figure shows the system at time t when the system mass is that of the container, the mass within it, and an increment Δm about to enter during time Δt.

At time $t + \Delta t$ the same total mass is that of the container, the mass within it, and an equal increment Δm which leaves the container in time Δt.

The linear momentum of the container and mass within it between the two sections A_1 and A_2 remains unchanged during Δt.

So the change in momentum of the system in time Δt is only because of mass Δm as

$$\Delta \boldsymbol{G} = (\Delta m)\boldsymbol{v}_2 - (\Delta m)\boldsymbol{v}_1 = \Delta m\,(\boldsymbol{v}_2 - \boldsymbol{v}_1)$$

Division by Δt and taking the limit yield

$$\dot{\boldsymbol{G}} = m'\Delta \boldsymbol{v},$$

where

$$m' = \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} = \frac{dm}{dt}.$$

Thus from the expression for linear momentum,

$$\Sigma \boldsymbol{F} = m'\Delta \boldsymbol{v}. \qquad \cdots\cdots\cdots(23)$$

Equation (23) establishes the relation between the resultant force on a steady-flow system and the corresponding mass flow rate and vector velocity increment.

We can now see one of the powerful applications of our general force-momentum equation which we derived for any mass system.

The system includes a body which is rigid (the structural container for the mass stream) and particles which are in motion (the flow of mass).

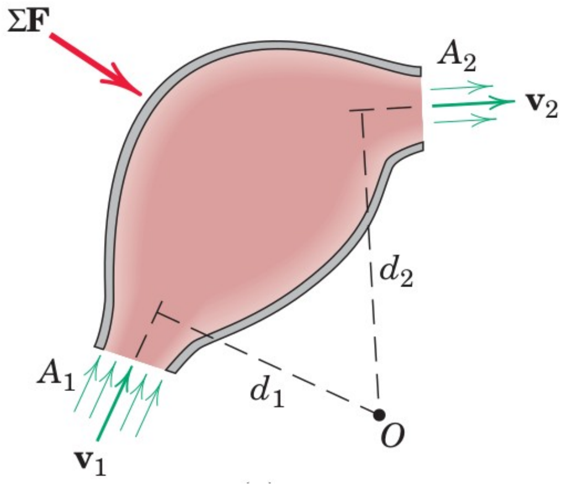
By defining the boundary of the system, the mass within which is constant for steady-flow conditions, we are able to utilize the generality of Newton's second law.

However, care must be taken while accounting for all external forces acting on the system, and they become clear if our free-body diagram is correct.

The blades of a helicopter impart downward momentum to a column of air, thereby creating the forces necessary for hovering and maneuvering.

Analysis of flow through a rigid container:

A similar formulation is obtained for the case of angular momentum in steady-flow systems. The resultant moment of all external forces about some fixed point O on or off the system, equals the time rate of change of angular momentum of the system about O (Equation 13).



For the case of steady flow in a single plane, becomes

$$\Sigma M_O = m' \,(v_2 \, d_2 - v_1 \, d_1) \qquad \text{..........(24)}$$

When the velocities of the incoming and outgoing flows are not in the same plane, the equation may be written in vector form as

$$\Sigma \boldsymbol{M}_O = m' \,(\boldsymbol{d}_2 \times \boldsymbol{v}_2 - \boldsymbol{d}_1 \times \boldsymbol{v}_1) \qquad \text{..........(24a)}$$

where $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are the position vectors to the centers of A_1 and A_2 from the fixed reference O. As discussed earlier, the mass center G may be used alternatively as a moment center.

Equations (23) and (24a) are very simple relations having important use in describing relatively complex fluid actions. Note that these equations relate external forces to the resultant changes in momentum and are independent of the flow path and momentum changes internal to the system.

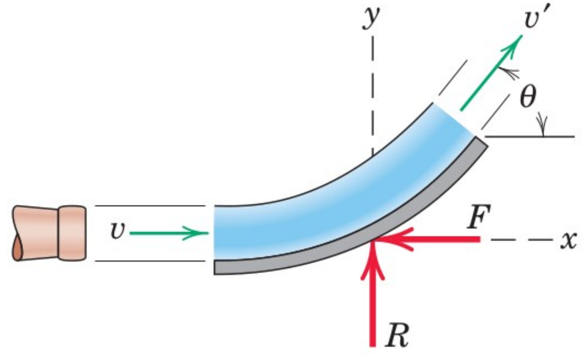
Above analysis may also be applied to systems which move with constant velocity (i.e. for inertial frame). The only restriction is that the mass within the system remain constant with respect to time.

Example 3

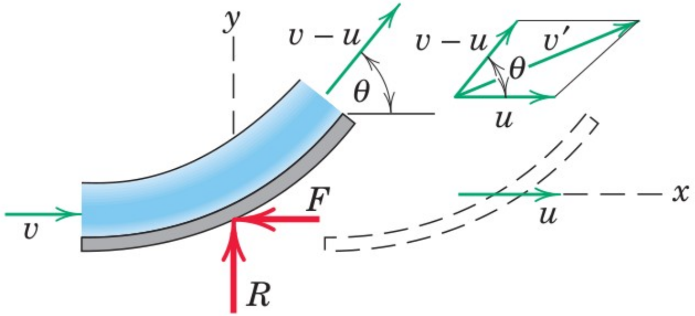
The smooth vane shown diverts the open stream of fluid of cross-sectional area A, mass density ρ, and velocity v.

(a) Determine the force components R and F required to hold the vane in a fixed position.

(b) Find the forces when the vane is given a constant velocity u less than v and in the direction of v.



Fixed vane

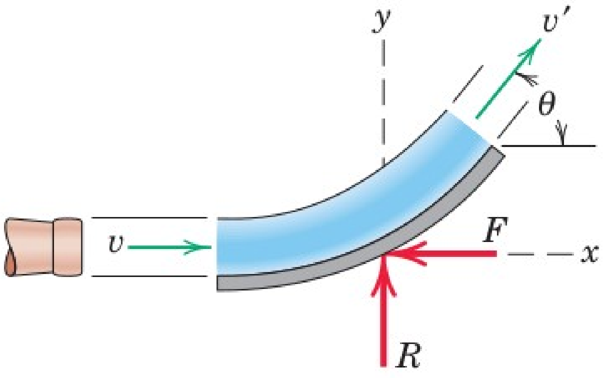


Moving vane

Part a:

The momentum equation may be applied to the isolated system for the change in motion in both the x- and y-directions.

With the vane stationary, the magnitude of the exit velocity v' equals that of the entering velocity v with fluid friction neglected.



Fixed vane

The changes in the velocity components are then

$$\Delta v_x = v'\cos\theta - v = -v\,(1 - \cos\theta\,) \text{ and } \Delta v_y = v'\sin\theta - 0 = v\sin\theta.$$

The mass rate of flow is $m' = Av$, thus,

$\Sigma F_x = m'\Delta v_x$, $\quad -F = \rho A v\,[-v\,(1\text{-}\cos\theta\,)]$ $\quad$ or $\quad F = \rho A v^2\,(1\text{-}\cos\theta\,)$.

$\Sigma F_y = m'\Delta v_y$, $\quad R = \rho A v\,[v\sin\theta\,]$ $\quad$ or $\quad R = \rho A v^2 \sin\theta$.

Part b:

In the case of the moving vane, the magnitude of velocity of the fluid during entry and exit is the velocity of the fluid relative to the vane v - u.

Thus, the change in x-velocity of the stream is

$$\Delta v_x = (v - u)\cos\theta - (v - u) = -(v - u)(1 - \cos\theta)$$

The y-component of v' is $(v - u)\sin\theta$, so that the change in the y - velocity of the stream is

$$\Delta v_y = (v - u)\sin\theta\ .$$

The mass rate of flow m' is the mass undergoing momentum change per unit of time. This rate is the mass flowing over the vane per unit time and not the rate of issuance from the nozzle. Thus, $m' = \rho A(v - u)$.

The mass rate of flow is $m' = A(v\text{-}u)$, thus,

$\Sigma F_x = m'\Delta v_x$, $\quad -F = \rho A\ (v\text{-}u)\ [-(v\text{-}u)\ (1\text{-}\cos\theta\ )]$ or $F = \rho A(v\text{-}u)^2 (1\text{-}\cos\theta\ )$.

$\Sigma F_y = m'\Delta v_y$, $\quad R = \rho A(v\text{-}u)^2 \sin\theta$.