ME232: Dynamics

Plane Kinetics of Rigid Bodies

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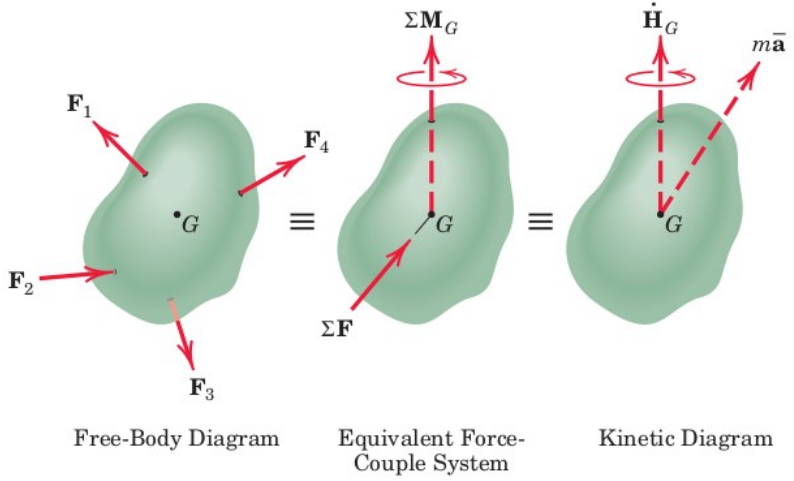
General equations of motion

- For a general rigid body in three dimensions, the following equations can be written,

$$\sum \boldsymbol{F} = m\overline{\boldsymbol{a}} \qquad \cdots\cdots\cdots\cdots(1)$$

$$\sum \boldsymbol{M}_G = \dot{\boldsymbol{H}}_G$$



- For plane motion,

$$\boldsymbol{H}_G = \sum \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i = \sum \boldsymbol{\rho}_i \times m_i(\boldsymbol{\omega} \times \boldsymbol{\rho}_i)$$

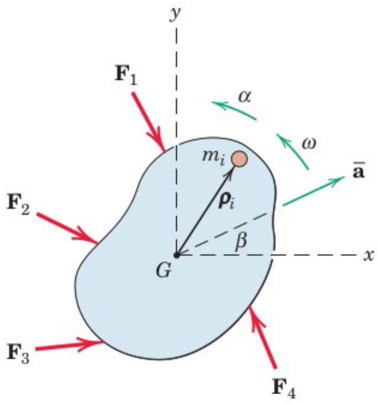
where, magnitude of $H_G = \sum m_i \rho_i^2 \omega$.

Here, $\sum m_i \rho_i^2 = \int \rho_i^2 dm = \overline{I}$ is the mass moment of inertia of the body about the z-axis through G. Thus, $H_G = \overline{I}\omega$.

Hence, for a rigid body in plane motion general equations of motion become,

$$\sum \boldsymbol{F} = m\overline{\boldsymbol{a}} \qquad \cdots\cdots\cdots\cdots(1a)$$

$$\sum \boldsymbol{M}_G = \overline{I}\alpha$$



Alternate moment equations

Moment about any arbitrary point P can be written as (already derived),

$$\sum \boldsymbol{M}_P = \dot{\boldsymbol{H}}_G + \overline{\boldsymbol{\rho}} \times m\overline{\boldsymbol{a}} \qquad \text{...........(2)}$$

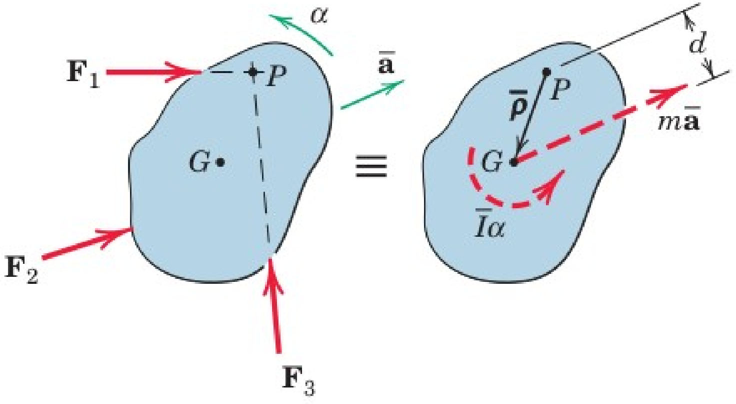
For a two-dimensional body,

$$\sum M_P = \overline{I}\alpha + m\overline{a}d$$

Alternatively, we can also write (already derived),

$$\sum \boldsymbol{M}_P = (\dot{\boldsymbol{H}}_P)_{\text{rel}} + \overline{\boldsymbol{\rho}} \times m\overline{\boldsymbol{a}}_P \qquad \text{...........(2a)}$$

where, $(\dot{\boldsymbol{H}}_P)_{\text{rel}} = I_P\boldsymbol{\alpha}$.



System of interconnected bodies

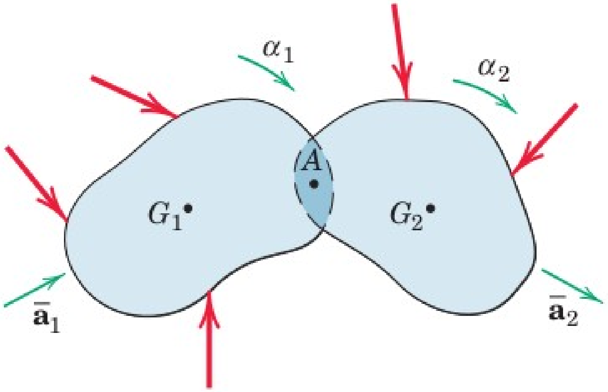
Many times it is required to deal with two or more connected rigid bodies whose motions are related kinematically. It is convenient to analyze the bodies as an entire system.

Two rigid bodies, which are hinged at A, and are subjected to external forces are shown.

For the two bodies we can write,

$$\sum \boldsymbol{F} = \sum m\bar{\boldsymbol{a}}$$

$$\sum M_P = \sum \bar{I}\alpha + \sum m\bar{a}d \qquad \cdots\cdots\cdots\cdots(3)$$



Free-Body Diagram of System

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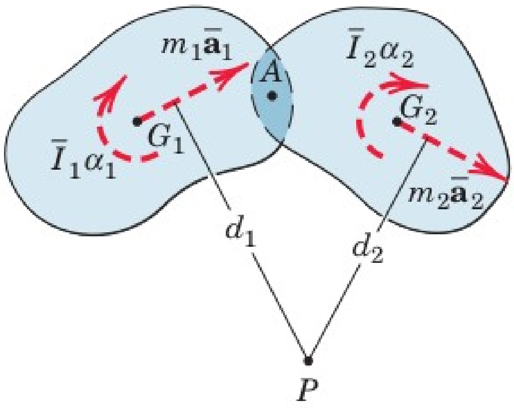


Kinetic Diagram of System

Translation

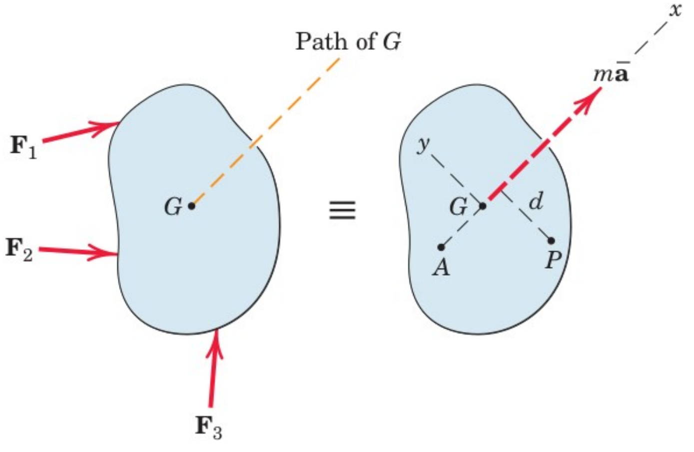
We already studied kinematics of rigid body translation in plane motion.

During translation (rectilinear or curvilinear) all points of the body moves either in straight lines or on congruent curved paths. In either case there is no angular motion of the body, so that both ω and α are zero.

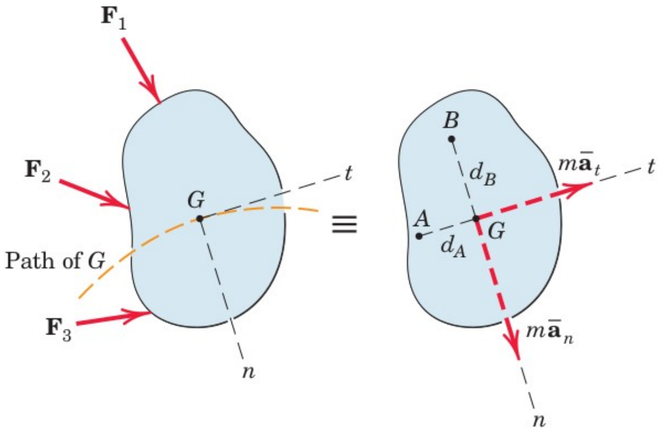
Hence, from (1a), general equations for rigid bodies in plane motions are

$$\sum \boldsymbol{F} = m\bar{\boldsymbol{a}} \qquad \cdots\cdots\cdots\cdots(4)$$

$$\sum \boldsymbol{M}_G = \bar{I}\alpha = 0$$



(*a*) Rectilinear Translation
($\alpha = 0,\ \omega = 0$)



(*b*) Curvilinear Translation
($\alpha = 0,\ \omega = 0$)

Fixed-axis rotation

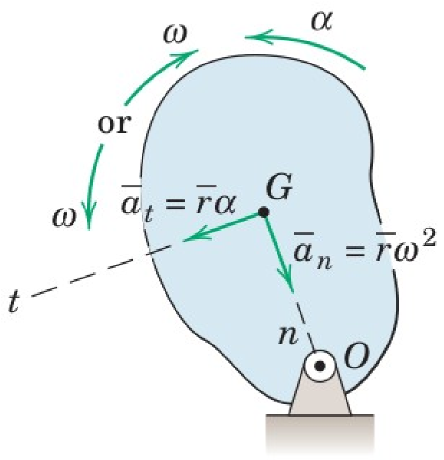
For rigid body rotation about fixed axis we have already seen that all points of the body rotates in a circle about the rotation axis, and all lines of the body have same angular velocity ω and acceleration α. Thus, the general equation for plane motions are directly applicable.

$$\sum \boldsymbol{F} = m\bar{\boldsymbol{a}}$$

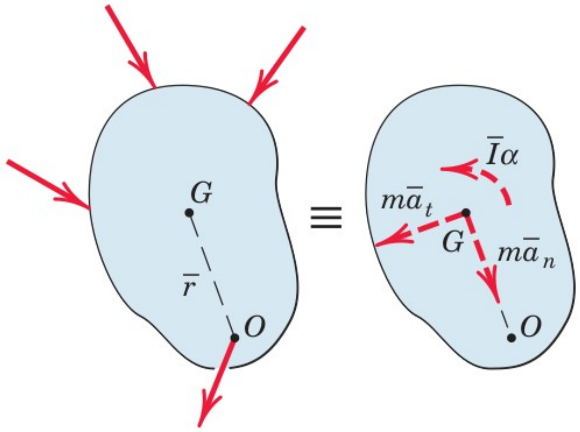
$$\sum \boldsymbol{M}_G = \bar{I}\alpha$$

For fixed-axis roation, it is generally useful to apply a moment equation direction about the rotation axis O, which yield the following equation,

$$\sum M_O = I_O\alpha.$$



Fixed-Axis Rotation
(*a*)



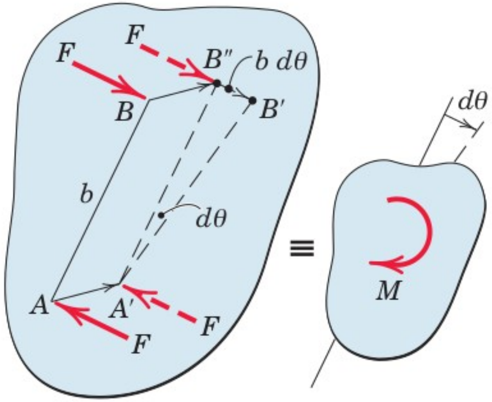
Free-Body Diagram
(*b*)

Kinetic Diagram
(*c*)

Work and Energy

- We have already developed the principles of work and energy and their application to the motion of a particle or system of particles.
- For finite displacements, the work-energy method eliminates the necessity for determining the acceleration and integrating it over the interval to obtain the velocity change.
- These same advantages are realized after extending the work-energy principles to describe rigid-body motion.

Work of forces and couples



The work done by a force $\boldsymbol{F}$ for a displacement of d$\boldsymbol{r}$ at the point of application of load $\boldsymbol{F}$ is given by

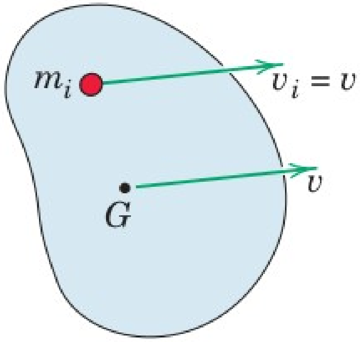
$$U = \int \boldsymbol{F} \cdot d\boldsymbol{r}.$$

Figure shows a couple $\boldsymbol{M} = \boldsymbol{Fb}$ acting on a rigid body which moves in the plane of the couple. Work done by $\boldsymbol{M}$ is determined as follows.

During time dt the body rotates through an angle $d\theta$, and line AB moves to $A'B'$. We may consider this motion in two parts, first a translation to $A'B''$ and then a rotation d about A'. It can be seen that **during the translation the work done by one of the forces cancels that done by the other force**, so that the net work done is $dU = F(b\ d\theta\ ) = M\ d\theta$ due to the rotational part of the motion.

If the couple acts in the sense opposite to the rotation, the work done is negative. During a finite rotation, the work done by a couple M whose plane is parallel to the plane of motion is, therefore, $U = \int M \cdot d\theta.$

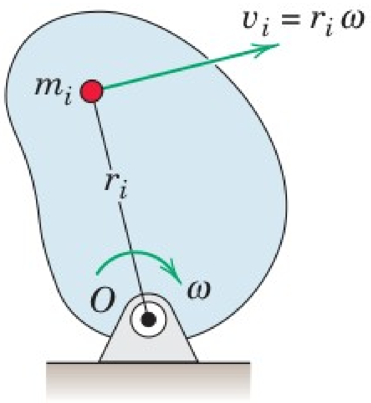
Kinetic energy



(a) Translation

The kinetic energy of a rigid body under pure translation is

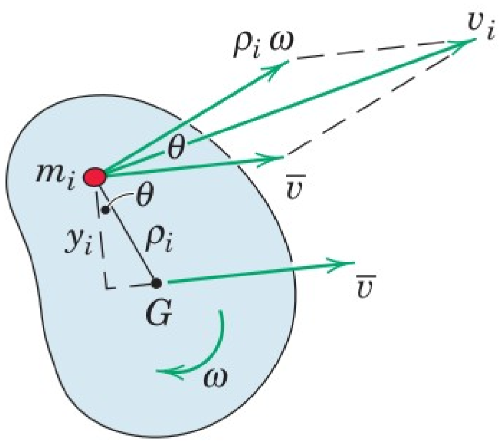
$$T = \sum \tfrac{1}{2} m_i v^2 = \tfrac{1}{2}\left(\sum m_i\right) v^2 = \tfrac{1}{2} M v^2.$$



(b) Fixed-Axis Rotation

The kinetic energy of a rigid body under rotation about a fixed-axis through O is

$$T = \sum \tfrac{1}{2} m_i r_i^2 \omega^2 = \tfrac{1}{2}\left(\sum m_i r_i^2\right) \omega^2 = \tfrac{1}{2} I_O \omega^2.$$



(c) General Plane Motion

For general plane motion, kinetic energy of the body is

$$T = \sum \tfrac{1}{2} m_i v_i^2,$$

$$T = \sum \tfrac{1}{2} m_i (\overline{v}^2 + \rho_i^2 \omega^2 + 2\overline{v}\rho_i \omega \cos\theta),$$

It can be shown that the third term is zero. Thus,

$$T = \tfrac{1}{2} m \overline{v}^2 + \tfrac{1}{2} \overline{I} \omega^2.$$

The kinetic energy of plane motion may also be expressed in terms of the rotational velocity about the instantaneous center C of zero velocity. Thus,

$$T = \tfrac{1}{2} I_C \omega^2.$$

Potential energy and the work-energy equation

The work-energy relation studied earlier is also applicable to rigid bodies in motion, as

$$T_1 + U_{1\text{-}2} = T_2, \text{ or}$$

$$T_1 + V_1 + U'_{1\text{-}2} = T_2 + V_2,$$

where, T is the kinetic energy of the system, $V = V_e + V_g$ is the total potential potential energy, and $U'_{1\text{-}2}$ is the work of all forces external to the system.

Power:

Power is the time rate at which work is performed. If the force $\boldsymbol{F}$ and the couple M is acting on a body and v is the velocity of the point of application of force $\boldsymbol{F}$ and ω is the angular velocity of the body, then, the total instantaneous power is

$$P = \boldsymbol{F} \cdot \boldsymbol{V} + M\omega.$$

From work-energy relation, it can be derived that,

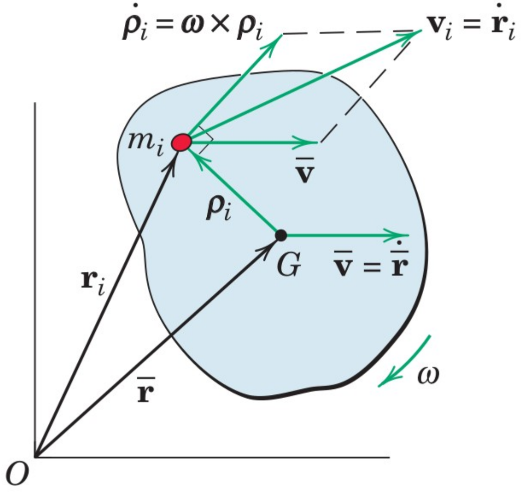
$$P = \frac{dU}{dt} = \dot{T}.$$

Thus, the power developed by the active forces and couples equals the rate of change of kinetic energy of the body or system of bodies.

$$\begin{aligned}
\dot{T} &= \frac{d}{dt}\left(\frac{1}{2}m\overline{\boldsymbol{v}}\cdot\overline{\boldsymbol{v}} + \frac{1}{2}\overline{I}\omega^2\right) \\
&= \frac{1}{2}m(\overline{\boldsymbol{a}}\cdot\overline{\boldsymbol{v}} + \overline{\boldsymbol{v}}\cdot\overline{\boldsymbol{a}}) + \overline{I}\omega\dot{\omega} \\
&= m\overline{\boldsymbol{a}}\cdot\overline{\boldsymbol{v}} + \overline{I}\alpha\omega = \boldsymbol{R}\cdot\overline{\boldsymbol{v}} + \overline{M}\omega
\end{aligned}$$

where $\boldsymbol{R}$ is the resultant of all forces acting on the body and $\overline{M}$ is the resultant moment about the mass center G of all forces.

Impulse and momentum



Expressions of impulse-momentum principle can be extended to motion of bodies in two-dimensions.

Linear momentum:

Linear momentum of a body under general plane motion can be defined as,

$$\boldsymbol{G} = m\overline{\boldsymbol{v}}$$

where m is the overall mass of the body, and $\overline{\boldsymbol{v}}$ is the velocity of mass center.

Impulse-Linear Momentum principle:

From Newton's generalized second law we get the impulse-linear momentum principle as

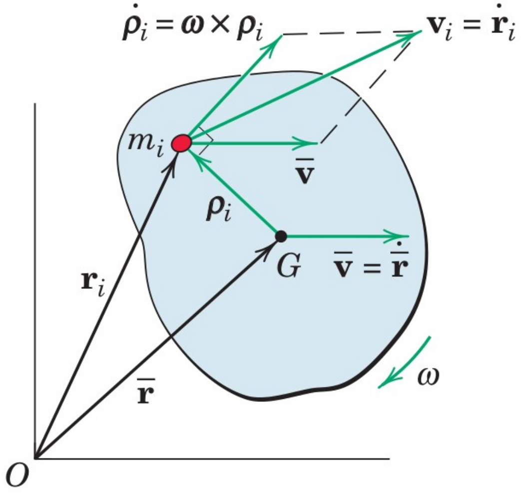
$$\boldsymbol{G}_1 + \int_{t_1}^{t_2} \sum \boldsymbol{F} dt = \boldsymbol{G}_2$$

or in component form

$$(G_x)_1 + \int_{t_1}^{t_2} \sum F_x dt = (G_x)_2$$

$$(G_y)_1 + \int_{t_1}^{t_2} \sum F_y dt = (G_y)_2$$

Angular Momentum:



Angular momentum is defined as the moment of linear momentum.

The angular momentum about the mass center of any prescribed system of mass as

$$\boldsymbol{H}_G = \sum \boldsymbol{\rho}_i \times m_i \boldsymbol{v}_i = \sum \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i$$

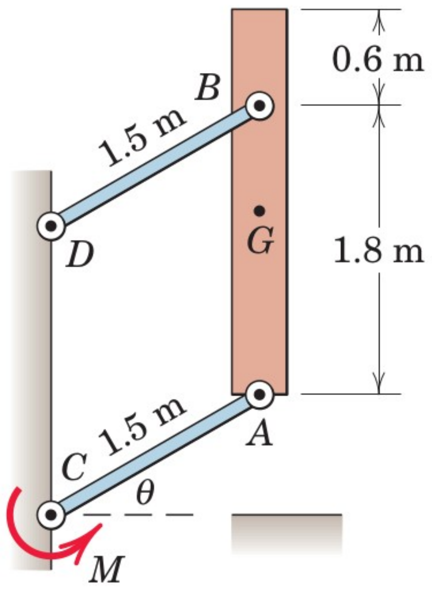
$$\boldsymbol{H}_G = \sum m_i \rho_i^2 \omega \boldsymbol{k} = \overline{I} \omega \boldsymbol{k}$$

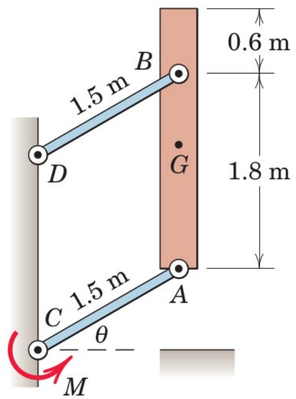
where $\overline{I} = \sum m_i \rho_i^2$ is the mass moment of inertia of the body about its mass center. Again from Newton's generalized second law, we get

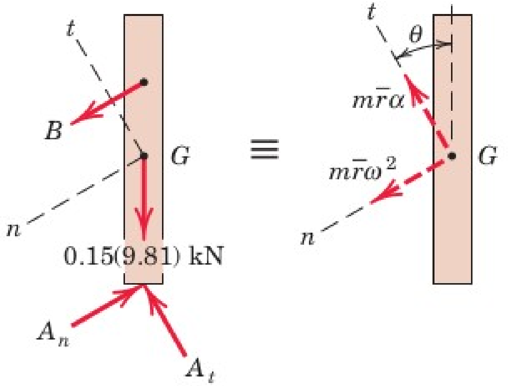
$$\sum M_G = \dot{H}_G \qquad \text{or} \qquad (H_G)_1 + \int_{t_1}^{t_2} \sum M_G dt = (H_G)_2$$

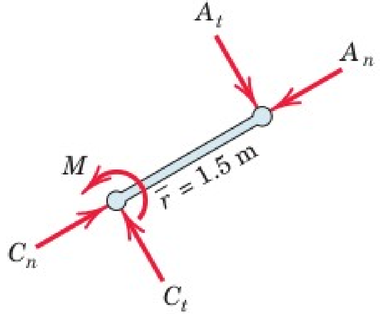
Example 1

The vertical bar AB has a mass of 150 kg with center of mass G midway between the ends. The bar is elevated from rest at $\theta = 0°$ by means of the parallel links of negligible mass, with a constant couple $M = 5$ kN·m applied to the lower link at C. Determine the angular acceleration of the links as a function of θ and find the force B in the link DB at the instant when $\theta = 30°$.









The bar translates on curvilinear path. With the circular motion of the mass center G, let us choose n- and t-coordinates for description of motion.

Considering negligible mass of the links, component A_t can be obtained from the analysis of link AC ($\Sigma M_C=0$).

Now applying kinetic equations to link AB.

$\sum F_t = m\bar{a}_t$ gives relation between α and θ.

With α as a function of θ, angular velocity ω as a function of can be obtained as follows,

$$\int_0^\omega \omega d\omega = \int_0^\theta \alpha(\theta)d\theta.$$

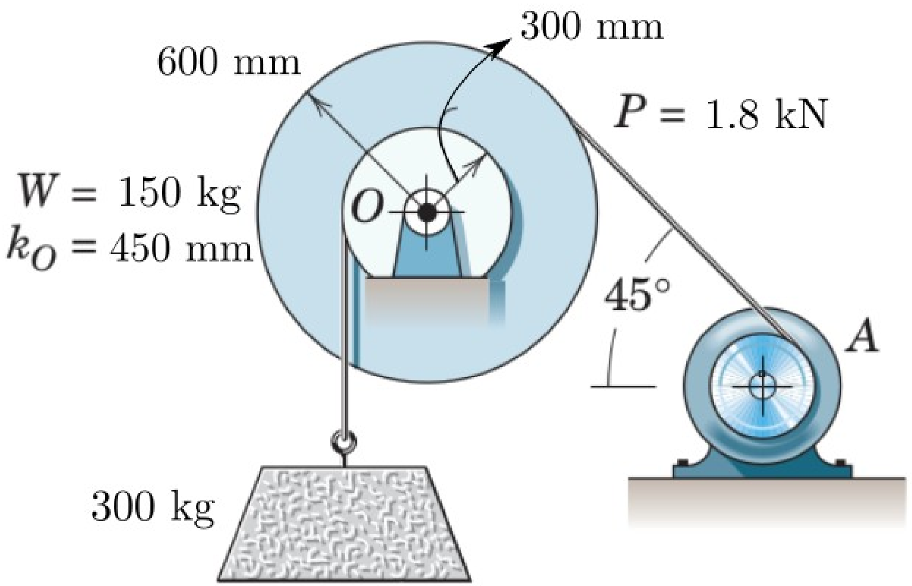
Now, force B at θ=30° can be obtained by taking moment about point A, as

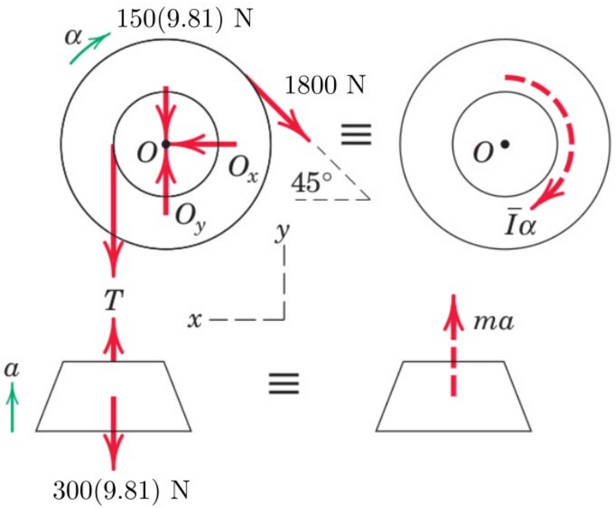
$$\sum M_A = m\bar{a}d$$

$$\Rightarrow 1.8\cos 30° B = m\bar{r}\omega^2(1.2\cos 30°) + m\bar{r}\alpha(1.2\sin 30°)$$

Example 2

The concrete block weighing 300 kg is elevated by the hoisting mechanism shown, where the cables are securely wrapped around the respective drums. The drums, which are fastened together and turn as a single unit about their mass center at O, have a combined weight of 150 kg and a radius of gyration about O of 450 mm. If a constant tension $P = 1.8$ kN is maintained by the power unit at A, determine the vertical acceleration of the block and the resultant force on the bearing at O.





The free-body and kinetic diagrams components showing all forces which act. The resultant of the force system on the drums for centroidal rotation is the couple

$\bar{I}\alpha = I_O\alpha$, where $\bar{I} = I_O = (0.45)^2 150 = 30.4$ kg·m^2

Now apply kinetic equations for the pulley and the block,

$$\sum M_G = \bar{I}\alpha \Rightarrow 1800(0.6) - T(0.3) = 30.4\alpha$$

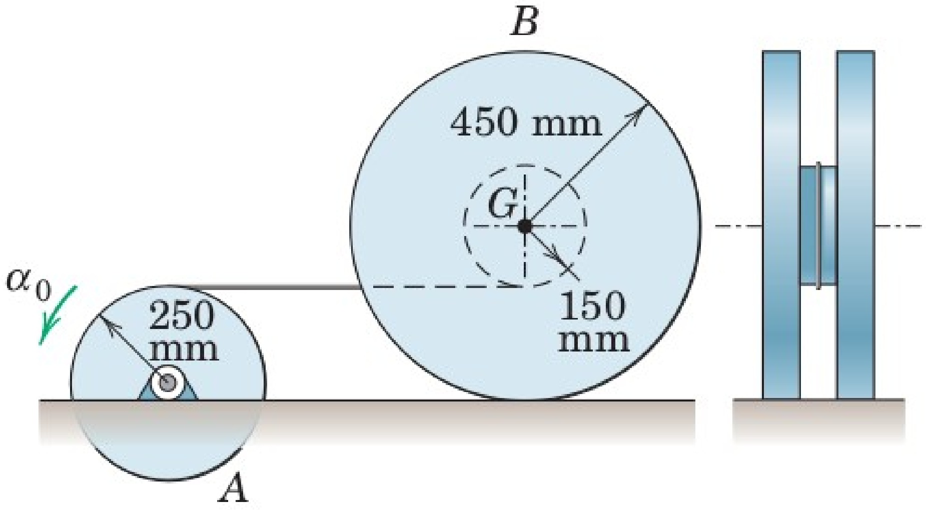
$$\sum F_y = ma_y \Rightarrow T - 300(9.81) = 300a$$

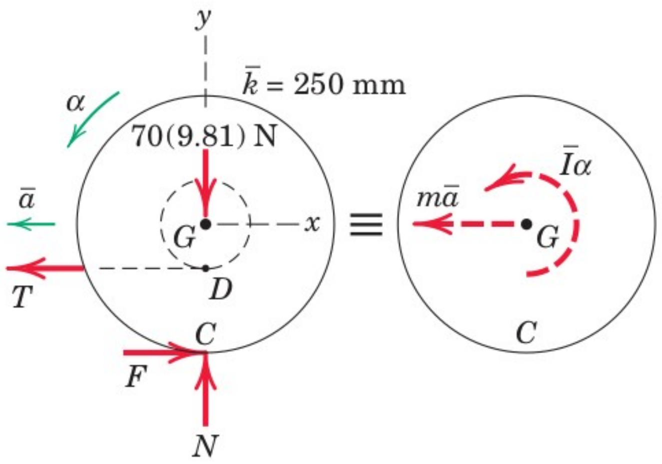
Note that, $a = r\alpha = 0.3\alpha$.

Thus from above equations, T, a and α can be determined. Bearing reaction can be computed by applying equations.
$\Sigma F_x = 0$ and $\Sigma F_y = 0$.

Example 3

The drum A is given a constant angular acceleration α_0 of 3 rad/s and causes the 70 kg spool B to roll on the horizontal surface by means of the connecting cable, which wraps around the inner hub of the spool. The radius of gyration k of the spool about its mass center G is 250 mm, and the coefficient of static friction between the spool and the horizontal surface is 0.25. Determine the tension T in the cable and the friction force F exerted by the horizontal surface on the spool.





The free-body diagram and the kinetic diagram of the spool are shown. The correct direction of the friction force may be assigned in this problem by observing from both diagrams that with counterclockwise angular acceleration, a moment sum about point G must be counterclockwise.

A point on the connecting cable has an acceleration $a_t = r\alpha = 0.25(3) = 0.75\ \text{m/s}^2$, which is also the horizontal component of the acceleration of point D on the spool.

It will be assumed initially that the spool rolls without slipping, in which case it has a counterclockwise angular acceleration $\alpha = (a_D)_x / DC = 0.75/0.30 = 2.5\ \text{rad/s}^2$.

The acceleration of the mass center G is, therefore, $a = r\alpha = 0.45(2.5) = 1.125\ \text{m/s}^2$.

Apply equations of motion as

$$\sum F_x = m\bar{a}_x, \quad \sum F_y = m\bar{a}_y, \quad \sum M_G = \bar{I}\alpha.$$

By solving above equations we obtain, $F = 75.8$ N, $T = 154.6$ N, and $N = 687$ N.

To establish the validity of our assumption of no slipping, we see that the surfaces are capable of supporting a maximum friction force $F_{max} = \mu_s N = 0.25(687) = 171.7$ N. Since only 75.8 N of friction force is required, we conclude that our assumption of rolling without slipping is valid.

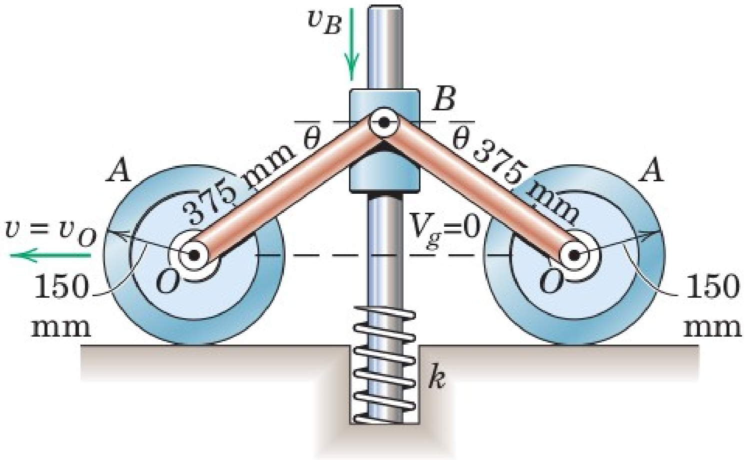
If the coefficient of static friction had been 0.10, for example, then the friction force would have been limited to $0.10(687) = 68.7$ N, which is less than 75.8 N, and the spool would slip. In this event, the kinematic relation a $= r\alpha$ would no longer hold. With $(a_D)_x$ known, the angular acceleration would be .

$$\alpha = [\overline{a} - (a_D)_x]/GD$$

Using this relation along with $F = \mu_k N = 68.7$ N, we would then re-solve the three equations of motion for the unknowns T, a, and a.

Example 4

In the mechanism shown, each of the two wheels has a mass of 30 kg and a centroidal radius of gyration of 100 mm. Each link OB has a mass of 10 kg and may be treated as a slender bar. The 7-kg collar at B slides on the fixed vertical shaft with negligible friction. The spring has a stiffness $k = 30$ kN/m and is contacted by the bottom of the collar when the links reach the horizontal position. If the collar is released from rest at the position θ= 45° and if friction is sufficient to prevent the wheels from slipping, determine (a) the velocity v_B of the collar as it first strikes the spring and (b) the maximum deformation x of the spring.



The mechanism executes plane motion and is conservative with the neglect of kinetic friction losses. Three states are defined as, 1, 2 and 3 at θ= 45°, 0°, and maximum spring deflection, respectively. The datum for zero gravitational potential energy V_g is conveniently taken through O as shown.

(a) Note that for the interval from θ= 45° to θ= 0°, the initial and final kinetic energies of the wheels are zero since each wheel starts from rest and momentarily comes to rest at θ= 0°. Also, at position 2, each link is only rotating about its point O so that

$$T_2 = \left[2 \times \frac{1}{2} I_O \omega^2\right]_{\text{links}} + \left[\frac{1}{2} m v^2\right]_{\text{collar}} \qquad (I_O = \tfrac{1}{3} m L^2)$$

$\omega = 0.375\ v_B$, hence, $T_2 = 6.83 v_B^2$.

Reg. Potential energy, the collar at B travels a distance $0.375/\sqrt{2} = 0.265$ m so that

$V_1 = (2\times10\text{g}\times0.265/2) + (0.265\times7\text{g}) = 44.2$ J, $\quad V_2 = 0.$

Also $U'_{1\text{-}2} = 0.$

Hence, $V_1 + T_1 + U'_{1\text{-}2} = V_2 + T_2$, and we get v_B as 2.54 m/s.

(b) At the condition of maximum deformation x of the spring, all parts are momentarily at rest, which makes $T_3 = 0$. Thus,

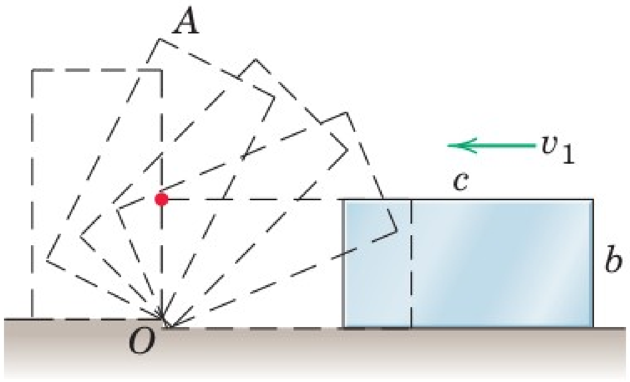
$$V_1 + T_1 + U'_{1\text{-}3} = V_3 + T_3,$$

$$0 + (2\times 10\text{g}\times 0.265/2) + (0.265\times 7\text{g}) = 0 - (2\times 10\text{g}\times x/2) - 7\text{g}\times x + 1/2\times(30\text{e}3)\times x^2$$

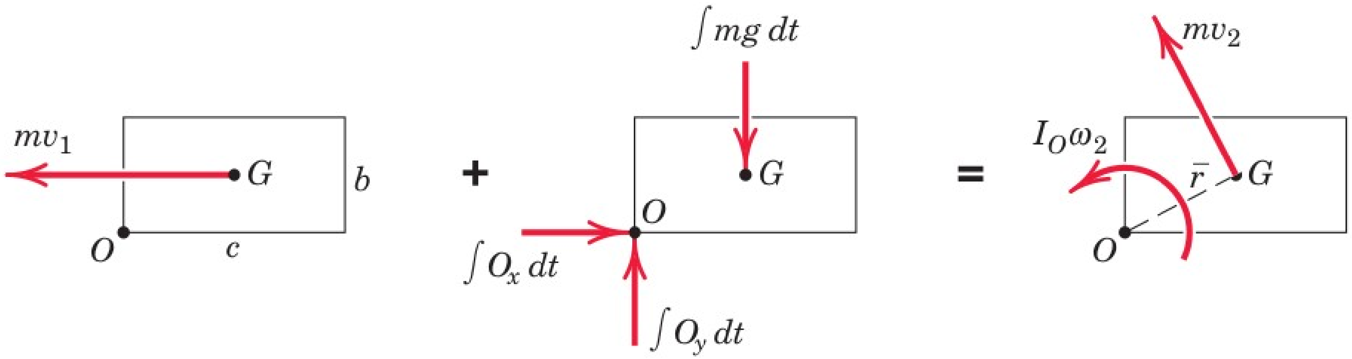
Solution for the positive value of x gives $x = 60.1$ mm.

Example 5

The uniform rectangular block of dimensions shown is sliding to the left on the horizontal surface with a velocity v_1 when it strikes the small step at O. Assume negligible rebound at the step and compute the minimum value of v_1 which will permit the block to pivot freely about O and just reach the standing position A with no velocity.



We break the overall process into two subevents:
(i) the collision, (ii) and the subsequent rotation.



I. Collision. With the assumption that the weight mg is nonimpulsive, angular momentum about O is conserved. The initial angular momentum of the block about O just before impact is the moment about O of its linear momentum and is $(H_O)_1 = mv_1(b/2)$. The angular momentum about O just after impact when the block is starting its rotation about O is

$$(H_O)_2 = I_O\omega_2 = \frac{m}{3}(b^2 + c^2)\omega_2$$

From conservation of angular momentum i.e., $(H_O)_1 = (H_O)_2$ we get, $\quad \omega_2 = \dfrac{3v_1 b}{2(b^2 + c^2)}$.

II. Rotation about *O*: With the assumptions that the rotation is like that about a fixed frictionless pivot and that the location of the effective pivot O is at ground level, mechanical energy is conserved during the rotation according to

$$T_2 + V_2 = T_3 + V_3$$

$$\frac{1}{2}I_O\omega_2^2 + 0 = 0 + mg\left[\sqrt{(b/2)^2 - (c/2)^2} - b/2\right]$$

Substituting ω_2 in terms of v_1 from part I, v_1 can now be calculated.

ME232: Dynamics

3D dynamics of rigid bodies

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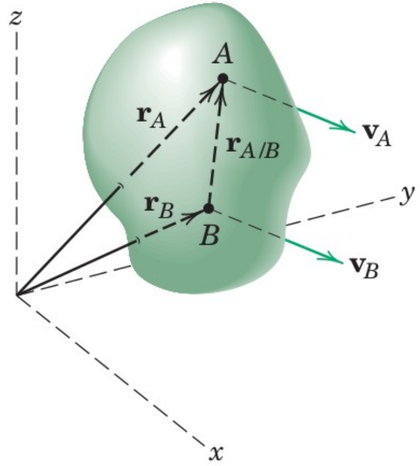
Introduction

- Although a large percentage of dynamics problems in engineering can be solved by the principles of plane motion, modern developments have focused increasing attention on problems which call for the analysis of motion in three dimensions.
- Inclusion of the third dimension **adds considerable complexity** to the kinematic and kinetic relationships.
- Introduction of third dimension not only add a third component to vectors such as force, linear velocity, linear acceleration, and linear momentum, but it also adds the **possibility of two additional components for vectors representing angular quantities** including moments of forces, angular velocity, angular acceleration, and angular momentum. Thus in three-dimensional motion application of vector analysis helps a lot.

Kinematics

Translation

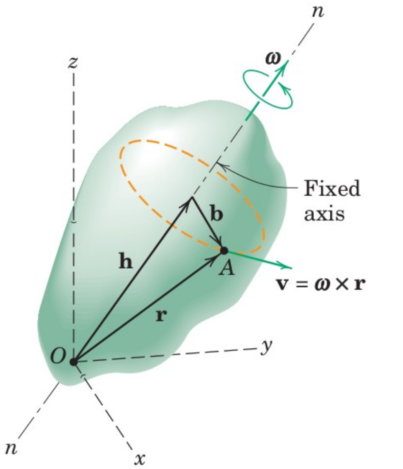


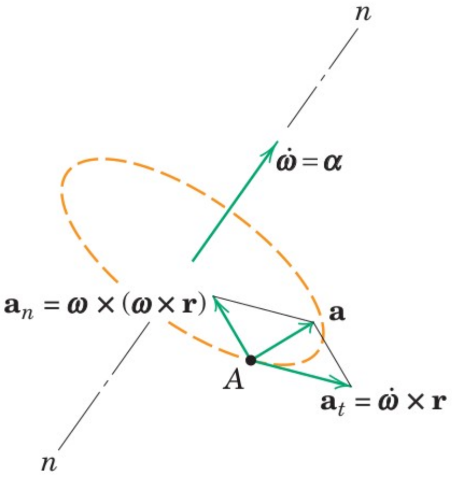
$$\boldsymbol{r}_A = \boldsymbol{r}_B + \boldsymbol{r}_{A/B}$$
$$\boldsymbol{v}_A = \boldsymbol{v}_B$$
$$\boldsymbol{a}_A = \boldsymbol{a}_B$$

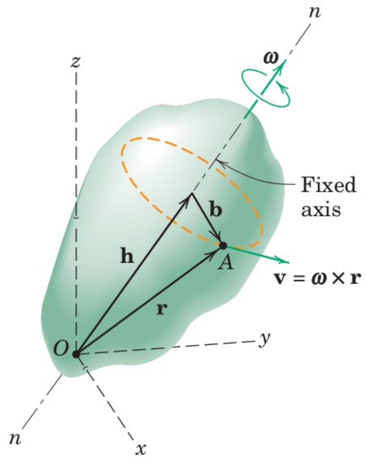
Fixed-axis rotation

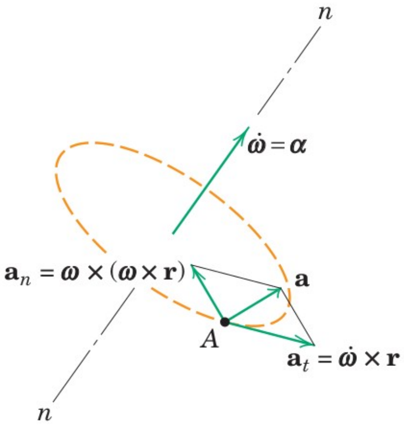
Consider the rotation of a rigid body about a fixed axis n-n in space with an angular velocity ω. $\boldsymbol{\omega}$ is a vector in the direction of the rotation axis with a sense established by the right-hand rule. For fixed-axis rotation, $\boldsymbol{\omega}$ does not change its direction since it lies along the axis. We choose the origin O of the fixed coordinate system on the rotation axis for convenience. Any point such as A which is not on the axis moves in a circular arc in a plane normal to the axis and has a velocity

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} = \boldsymbol{\omega} \times (\boldsymbol{b} + \boldsymbol{h}) = \boldsymbol{\omega} \times \boldsymbol{b} \quad \text{...............(1)}$$









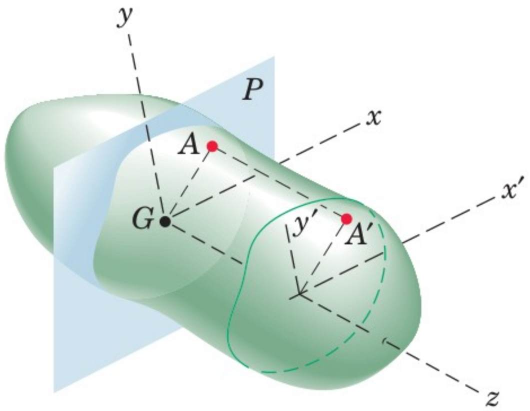
Acceleration of A is given by the time derivative of (1) as,

$$\boldsymbol{a} = \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \quad \text{..............}(2)$$

Normal and tangential component of acceleration are shown in the figure.

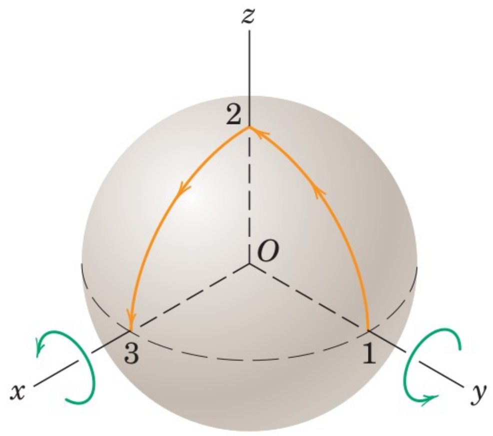
Parallel-plane motion

When all points in a rigid body moves in a plane P, we have a general plane motion. The reference plane is customarily taken through the mass center G and is call the plane of motion. Because each point in the body has a motion identical with the motion of corresponding point in the plane P, it follows that the kinematics of plane motion completely describes the motion of body.



Rotation about a fixed point

When a body rotates about a fixed point, the angular-velocity vector no longer remains fixed in direction, and this change calls for a more general concept of rotation.

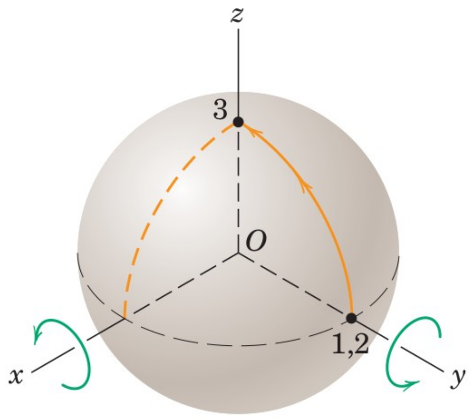


θ_x then θ_y

Rotation and Proper vectors:

Consider a solid sphere which is cut from a rigid body confined to rotate about a fixed point O. The x-y-z axes here are taken as fixed in space and do not rotate with the body.

Now, consider two successive 90° rotations of the sphere, first about the x-axis and, second about the y-axis, which result in the motion of point 1 to point 2 and then to point 3, successively.

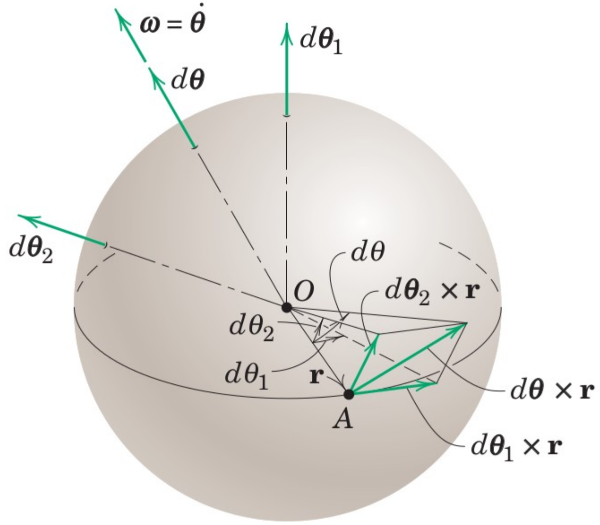


θ_y then θ_x

Now reverse the order of rotation, i.e. first rotate about y-axis and then about x-axis. We see that for the two cases do not produce the same final position.

The previous example shows that **finite rotations do not generally obey the parallelogram law of vector addition and are not commutative**. Thus, finite rotations may not be treated as vectors.

However, **infinitesimal rotations do obey the parallelogram law of vector addition.** Figure shows the combined effect of two infinitesimal rotations $d\theta_1$ and $d\theta_2$ of a rigid body about the respective axes through the fixed point O. As a result either order of infinitesimal rotations clearly produce the same resultant displacement, which is $d\theta_1 \times \boldsymbol{r} + d\theta_2 \times \boldsymbol{r}$. Thus, the two rotations are equivalent to the single rotation $d\theta = d\theta_1 + d\theta_2$. It follows that the angular velocities $\boldsymbol{\omega}_1 = \dot{\boldsymbol{\theta}}_1$ and $\boldsymbol{\omega}_2 = \dot{\boldsymbol{\theta}}_2$ may be added vectorially to give

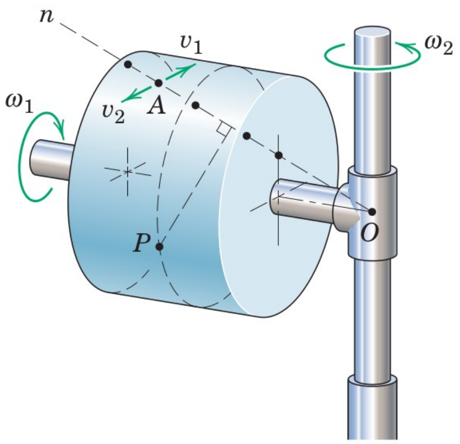


$$\boldsymbol{\omega} = \dot{\boldsymbol{\theta}} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2.$$

Therefore it concludes that **at any instant of time a body with one fixed point is rotating instantaneously about a particular axis passing through the fixed point.**

Instantaneous axis of rotation

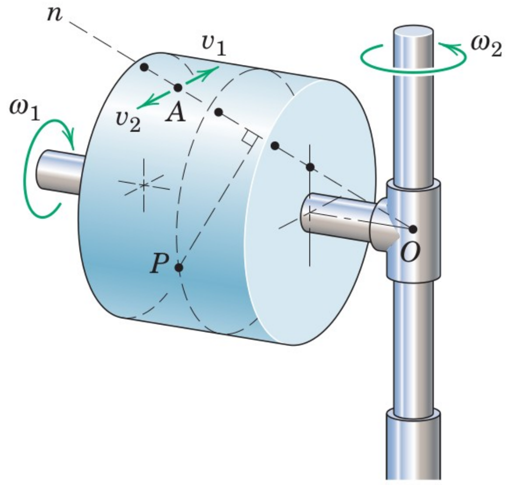
Consider a solid cylindrical rotor made of clear plastic containing many black particles embedded in the plastic. The rotor is spinning about its shaft axis at constant angular velocity ω_1, and its shaft, in turn, is rotating about the fixed vertical axis at with constant angular velocity ω_2.



If the rotor is photographed at a certain instant during its motion, the resulting picture would show **one line of black dots sharply defined**, indicating that, momentarily, **their velocity was zero**. This line of points with no velocity defines the instantaneous position of the axis of rotation O-n. Any dot on this line, such as A, would have **equal and opposite velocity components, v_1 due to ω_1 and v_2 due to ω_2**.

All other dots, such as the one at P, would appear blurred, and their movements would show as short streaks in the form of small circular arcs in planes normal to the axis O-n. Thus, all such particles of the body, are **momentarily rotating in circular arcs about the instantaneous axis of rotation**.

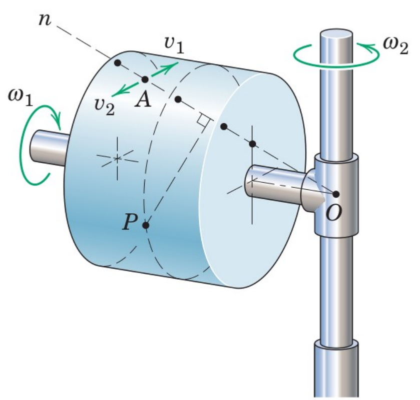
If a succession of photographs were taken, we would observe in each photograph that the rotation axis would be defined by a new series of sharply-defined dots and that the axis would change position both in space and relative to the body. For rotation of a rigid body about a fixed point, then, it is seen that the rotation axis is, in general, not a line fixed in the body.

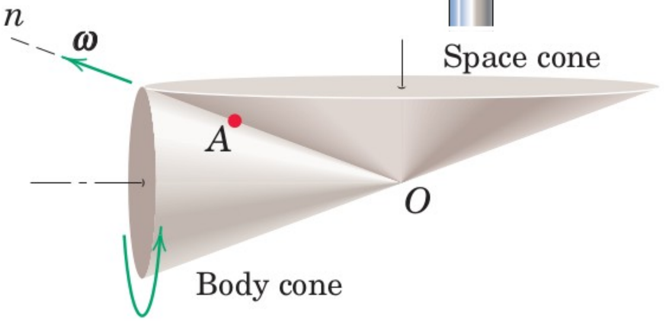


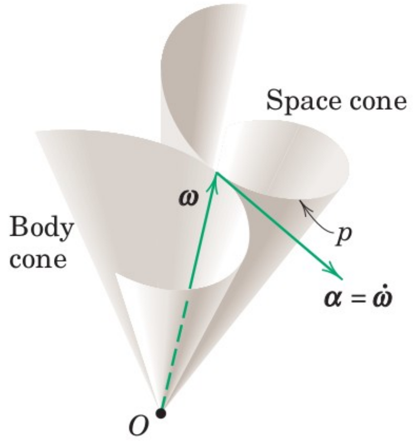
Body and space cones:

Relative to the plastic cylinder, the instantaneous axis of rotation O-A-n generates a right-circular cone about the cylinder axis called the **body cone**. As the two rotations continue and the cylinder swings around the vertical axis, the instantaneous axis of rotation also generates a right-circular cone about the vertical axis called the **space cone.** These cones are shown for this particular example.

We see that the body cone rolls on the space cone and that the **angular velocity $\boldsymbol{\omega}$ of the body is a vector which lies along the common element of the two cones.** For a more general case where the rotations are not steady, the space and body cones are not right-circular cones but the body cone still rolls on the space cone.





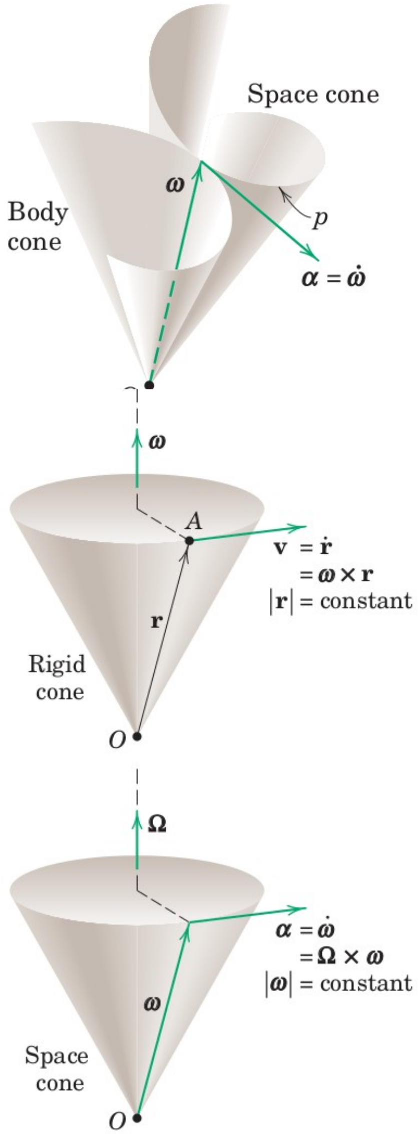


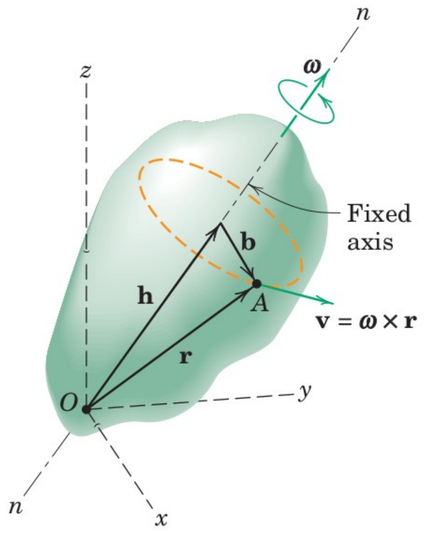
Angular accelerations:

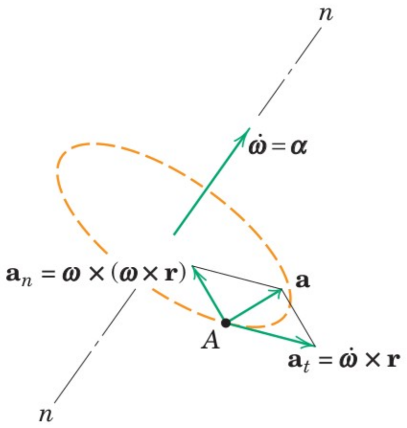
The angular acceleration $\boldsymbol{\alpha}$ of a rigid body in three-dimensional motion is the time derivative of its angular velocity, $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$. In three-dimensional motion **$\boldsymbol{\alpha}$ reflects the change in direction of $\boldsymbol{\omega}$ as well as its change in magnitude**. Thus as the angular velocity vector $\boldsymbol{\omega}$ follows the space curve p and changes in both magnitude and direction, the angular acceleration $\boldsymbol{\alpha}$ becomes a vector tangent to this curve in the direction of the change in $\boldsymbol{\omega}$.

When the magnitude of $\boldsymbol{\omega}$ remains constant, the angular acceleration $\boldsymbol{\alpha}$ is normal to $\boldsymbol{\omega}$. For this case, if we let $\boldsymbol{\Omega}$ stand for the angular velocity with which the vector $\boldsymbol{\omega}$ itself rotates (*precesses*) as it forms the space cone, the angular acceleration may be written

$$\boldsymbol{\alpha} = \boldsymbol{\Omega} \times \boldsymbol{\omega}.$$







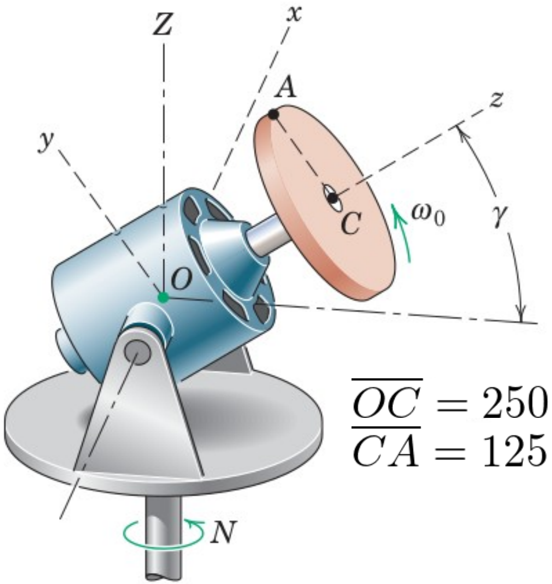
If the figure represent a rigid body rotating about a fixed point O with the instantaneous axis of rotation n-n, we see that the velocity $\boldsymbol{v}$ and acceleration $\boldsymbol{a} = \dot{\boldsymbol{v}}$ of any point A in the body are given by the same expressions as apply to the case in which the axis is fixed.

The one difference between the case of rotation about a fixed axis and rotation about a fixed point lies in the fact that for rotation about a fixed point, the angular acceleration $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$ will have **a component normal to $\boldsymbol{\omega}$ due to the change in direction of $\boldsymbol{\omega}$**, as well as **a component in the direction of $\boldsymbol{\omega}$ to reflect any change in the magnitude of $\boldsymbol{\omega}$.** Although any point on the rotation axis n-n momentarily will have zero velocity, it will not have zero acceleration as long as $\boldsymbol{\omega}$ is changing its direction.

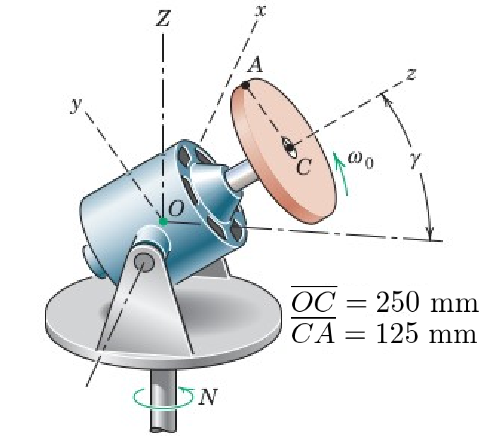
On the other hand, for rotation about a fixed axis, $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$ has **only the one component along the fixed axis to reflect the change in the magnitude of $\boldsymbol{\omega}$.** Furthermore, points which lie on the fixed rotation axis clearly have no velocity or acceleration.

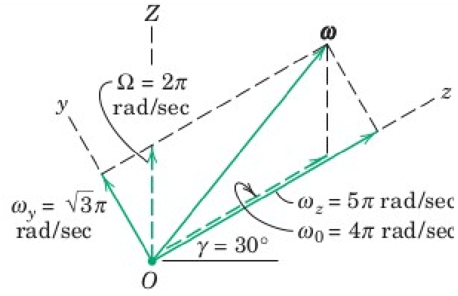
Example 1

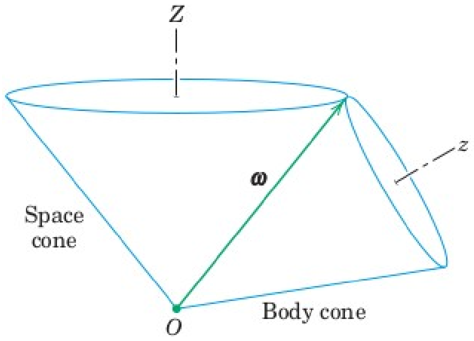
The electric motor with an attached disk is running at a constant low speed of 120 rpm in the direction shown. Its housing and mounting base are initially at rest. The entire assembly is next set in rotation about the vertical Z-axis at the constant rate $N = 60$ rpm with a fixed angle γ of 30°. Determine (a) the angular velocity and angular acceleration of the disk, (b) the space and body cones, and (c) the velocity and acceleration of point A at the top of the disk for the instant shown.



$\overline{OC} = 250$ mm
$\overline{CA} = 125$ mm







The axes x-y-z with unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are attached to the motor frame, with the z-axis coinciding with the rotor axis and the x-axis coinciding with the horizontal axis through O about which the motor tilts. The Z-axis is vertical and carries the unit vector $\boldsymbol{K} = \boldsymbol{j}\cos\gamma + \boldsymbol{k}\sin\gamma$.

(a) The rotor and disk have two components of angular velocity: $\omega_0 = 120(2\pi)/60 = 4\pi$ rad/sec about the z-axis and $\Omega = 60(2\pi)/60 = 2\pi$ rad/sec about the Z-axis. Thus, the angular velocity becomes

$$\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \boldsymbol{\Omega} = \omega_0\boldsymbol{k} + \Omega\boldsymbol{K}$$

$$\boldsymbol{\omega} = \omega_0\boldsymbol{k} + \Omega(\boldsymbol{j}\cos\gamma + \boldsymbol{k}\sin\gamma) = \pi(\sqrt{3}\boldsymbol{j} + 5\boldsymbol{k})\ \mathrm{rad/s}$$

The angular acceleration of the disk

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \boldsymbol{\Omega} \times \boldsymbol{\omega} = 68.4\boldsymbol{i}\ \mathrm{rad/s^2}$$

(c) The position vector of point A for the instant considered is

$$\boldsymbol{r} = 0.125\boldsymbol{j} + 0.250\boldsymbol{k} \text{ m}$$

Velocity of point A,

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} = -0.1920\pi\boldsymbol{i} \text{ m/s}$$

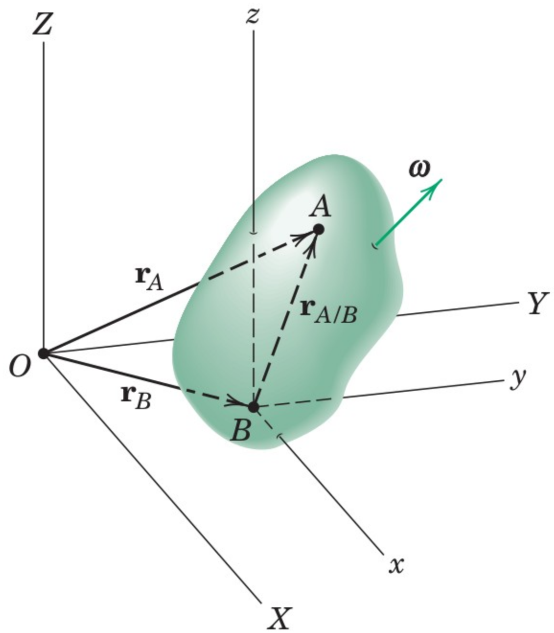
Acceleration of point A,

$$\boldsymbol{a} = \boldsymbol{\alpha} \times \boldsymbol{r} + \boldsymbol{\omega} \times \boldsymbol{v} = -26.6\boldsymbol{j} + 11.83\boldsymbol{k} \text{ m/s}^2$$

General motion

Translating reference axis:

Figure shows a rigid body which has an angular velocity $\boldsymbol{\omega}$. We choose point B as the origin of a translating reference system x-y-z. The velocity $\boldsymbol{v}$ and acceleration $\boldsymbol{a}$ of any other point A in the body are given by the relative-velocity and relative-acceleration expressions, which were developed for plane motion of rigid bodies.



$$\boldsymbol{v}_A = \boldsymbol{v}_B + \boldsymbol{v}_{A/B}$$
$$\boldsymbol{a}_A = \boldsymbol{a}_B + \boldsymbol{a}_{A/B}$$

As already discussed earlier that for rigid-body motion in space the distance AB remains constant. Thus, from an observer's position on x-y-z, the body appears to rotate about the point B and point A appears to lie on a spherical surface with B as the center. Consequently, we may view the general motion as a translation of the body with the motion of B plus a rotation of the body about B.

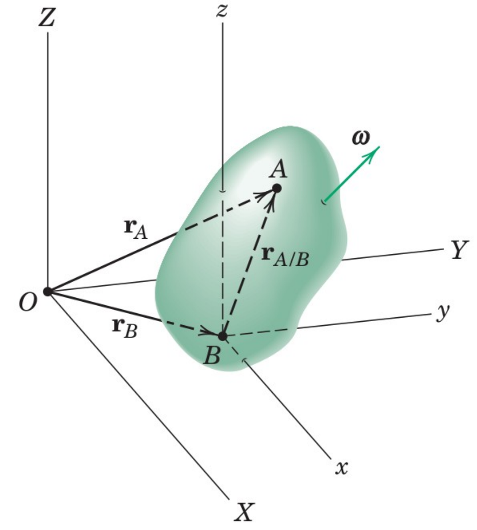
The relative-motion terms represent the effect of the rotation about B and are identical to the velocity and acceleration expressions discussed for rotation of a rigid body about a fixed point. Therefore, the relative-velocity and relative-acceleration equations may be written

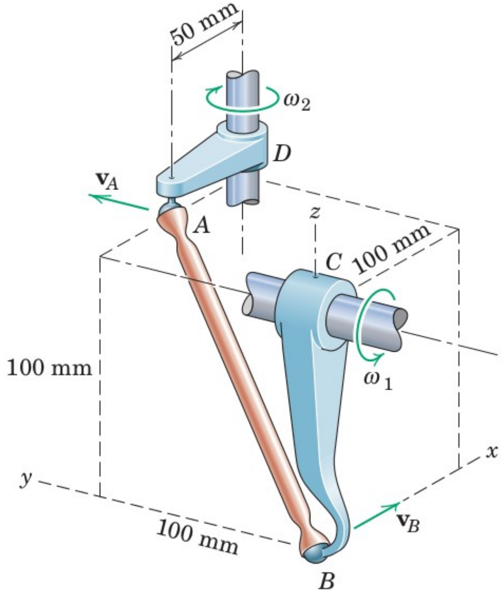
$$\begin{aligned} \boldsymbol{v}_A &= \boldsymbol{v}_B + \boldsymbol{\omega} \times \boldsymbol{r}_{A/B} \\ \boldsymbol{a}_A &= \boldsymbol{a}_B + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{A/B} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{A/B}) \end{aligned} \qquad \cdots\cdots\cdots\cdots\cdots(3)$$

where $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\omega}}$ are the instantaneous angular velocity and angular acceleration of the body, respectively.

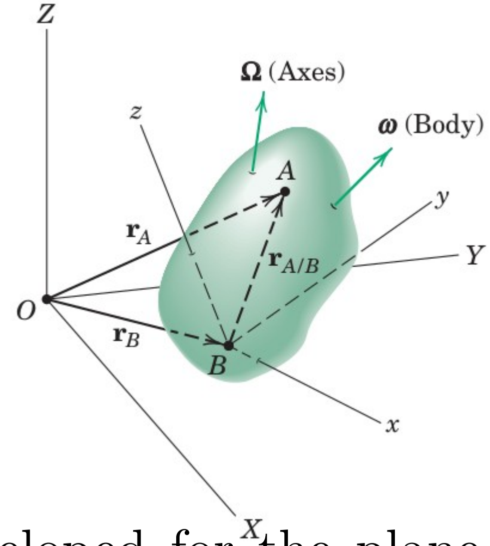
If points A and B represent the ends of a rigid control link in a spatial mechanism where the end connections act as ball-and-socket joints, it is necessary to impose certain kinematic requirements.

Any rotation of the link about its own axis AB does not affect the action of the link. Thus, the angular velocity $\boldsymbol{\omega}_n$ whose vector is normal to the link describes its action. It is necessary, therefore, that $\boldsymbol{\omega}_n$ and $\boldsymbol{r}_{A/B}$ be at right angles, and this condition is satisfied if $\boldsymbol{\omega}_n \cdot \boldsymbol{r}_{A/B} = 0$. Similarly, it is only the component $\boldsymbol{\alpha}_n$ of the angular acceleration of the link normal to AB which affects its action, so that $\boldsymbol{\alpha}_n \cdot \boldsymbol{r}_{A/B} = 0$ must also hold.





Rotating reference axis:



A more general formulation of the motion of a rigid body in space calls for the use of reference axes which rotate as well as translate. The reference axes whose origin is attached to the reference point B rotate with an absolute angular velocity $\boldsymbol{\Omega}$ which may be different from the absolute angular velocity $\boldsymbol{\omega}$ of the body.

Motion of the body can be described using the expressions developed for the plane motion of a rigid body with the use of rotating axes. The extension of these relations from two to three dimensions is easily accomplished by merely including the z-component of the vectors. The expressions for the velocity and acceleration of point A become

$$\boldsymbol{v}_A = \boldsymbol{v}_B + \boldsymbol{\Omega} \times \boldsymbol{r}_{A/B} + \boldsymbol{v}_{\text{rel}}$$

$$\boldsymbol{a}_A = \boldsymbol{a}_B + \dot{\boldsymbol{\Omega}} \times \boldsymbol{r}_{A/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_{A/B}) + 2\boldsymbol{\Omega} \times \boldsymbol{v}_{\text{rel}} + \boldsymbol{a}_{\text{rel}} \qquad \text{...............(4)}$$

where $\boldsymbol{v}_{\text{rel}} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k}$ and $\boldsymbol{a}_{\text{rel}} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{z}\boldsymbol{k}$ are, respectively, the velocity and acceleration of point A measured relative to x-y-z by an observer attached to x-y-z.

Note that $\boldsymbol{r}_{A/B}$ remains constant in magnitude for points A and B fixed to a rigid body, but it will change direction with respect to x-y-z when the angular velocity $\boldsymbol{\Omega}$ of the axes is different from the angular velocity $\boldsymbol{\omega}$ of the body. Further, if x-y-z are rigidly attached to the body, $\boldsymbol{\Omega} = \boldsymbol{\omega}$ and $\boldsymbol{v}_{\text{rel}}$ and $\boldsymbol{a}_{\text{rel}}$ are both zero, which makes the equations (4) identical to (3).

The relationship developed earlier between the time derivative of a vector $\boldsymbol{V}$ as measured in the fixed X-Y system and the time derivative of $\boldsymbol{V}$ as measured relative to the rotating x-y system. For our three-dimensional case, this relation becomes

$$\left(\frac{d\boldsymbol{V}}{dt}\right)_{XYZ} = \left(\frac{d\boldsymbol{V}}{dt}\right)_{xyz} + \boldsymbol{\Omega} \times \boldsymbol{V}. \qquad \cdots\cdots\cdots\cdots\cdots(5)$$

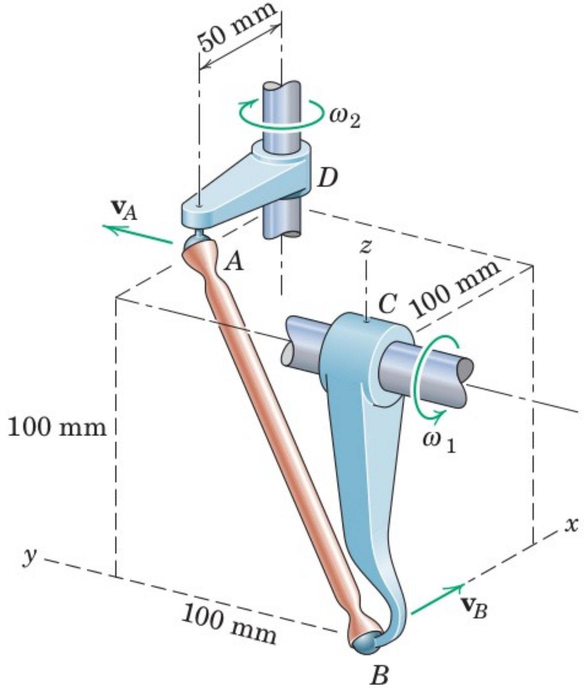
Using the above transformation to the expression of relative position vector, and velocity vector expressions (4) can be derived.

Example 2

Crank CB rotates about the horizontal axis with an angular velocity $\omega_1 = 6$ rad/s which is constant for a short interval of motion which includes the position shown. The link AB has a ball-and-socket fitting on each end and connects crank DA with CB. For the instant shown,

(a) determine the angular velocity ω_2 of crank DA and the angular velocity $\boldsymbol{\omega}_n$ of link AB.

(b) determine the angular acceleration $\dot{\omega}_2$ of crank AD. Also find the angular acceleration $\dot{\boldsymbol{\omega}}_n$ of link AB.



The relative-velocity relation will be solved first using translating reference axes attached to B. The equation is

$$\boldsymbol{v}_A = \boldsymbol{v}_B + \boldsymbol{\omega}_n \times \boldsymbol{r}_{A/B} \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\text{(a)}$$

where $\boldsymbol{\omega}_n$ is the angular velocity of link AB taken normal to AB. The velocities of A and B are

$$\boldsymbol{v}_A = 50\omega_2\,\boldsymbol{j}, \qquad \boldsymbol{v}_B = 100(6)\ \boldsymbol{i} = 600\ \boldsymbol{i}\ \text{mm/s}.$$

Also $\boldsymbol{r}_{A/B} = 50\,\boldsymbol{i} + 100\,\boldsymbol{j} + 100\,\boldsymbol{k}$ mm.

After substituting in (a) we get,

$$50\omega_2\,\boldsymbol{j} = 600\ \boldsymbol{i} + (\omega_{nx}\,\boldsymbol{i} + \omega_{ny}\,\boldsymbol{j} + \omega_{nz}\,\boldsymbol{k}\,) \times (50\,\boldsymbol{i} + 100\,\boldsymbol{j} + 100\,\boldsymbol{k}\,)$$

Comparing both side, we get following equations

$$-6 = \omega_{ny} - \omega_{nz}$$

$$\omega_2 = -2\omega_{nx} + \omega_{nz} \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\text{(b)}$$

$$0 = -2\omega_{nx} - \omega_{ny}$$

Above equations can be solved to get $\omega_2 = 6$ rad/s.

To determine ω_n, we incorporate additional condition, i.e. $\boldsymbol{\omega}_n\cdot\boldsymbol{r}_{A/B}=0$.

Using this condition with (b), $\boldsymbol{\omega}_n$ can be determined as, $\boldsymbol{\omega}_n = 3\ (-2\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k})$ rad/s.

The accelerations of the links may be found from the following equation

$$\boldsymbol{a}_A = \boldsymbol{a}_B + \dot{\boldsymbol{\omega}}_n \times \boldsymbol{r}_{A/B} + \boldsymbol{\omega}_n \times (\boldsymbol{\omega}_n \times \boldsymbol{r}_{A/B}) \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(c)$$

the accelerations of A and B are

$$\begin{aligned} \boldsymbol{a}_A &= 50\omega_2^2\,\boldsymbol{i} + 50\dot{\omega}_2\,\boldsymbol{j} = 1800\,\boldsymbol{i} + \dot{\omega}_2\,\boldsymbol{j} \text{ mm/s}^2 \\ \boldsymbol{a}_B &= 100\omega_1^2\,\boldsymbol{k} = 3600\,\boldsymbol{k} \text{ mm/s}^2 \end{aligned} \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(d)$$

Also,

$$\begin{aligned} \boldsymbol{\omega}_n \times (\boldsymbol{\omega}_n \times \boldsymbol{r}_{A/B}) &= 20(50\,\boldsymbol{i} + 100\,\boldsymbol{j} + 100\,\boldsymbol{k}) \text{ mm/s}^2 \\ \dot{\boldsymbol{\omega}}_n \times \boldsymbol{r}_{A/B} &= (100\dot{\omega}_{ny} - 100\dot{\omega}_{nz})\boldsymbol{i} + (50\dot{\omega}_{nz} - 100\dot{\omega}_{nx})\boldsymbol{j} + (100\dot{\omega}_{nx} - 50\dot{\omega}_{ny})\boldsymbol{k} \end{aligned} \qquad \cdots\cdots(e)$$

Substituting (e) and (d) in (c), we get following equations,

$$\begin{aligned} 28 &= \dot{\omega}_{ny} - \dot{\omega}_{nz} \\ \dot{\omega}_2 + 40 &= 2\dot{\omega}_{nx} + \dot{\omega}_{nz} \\ -32 &= 2\dot{\omega}_{nx} - \dot{\omega}_{ny} \end{aligned} \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(f)$$

From above equations we can determine, $\dot{\omega}_2 = -36$ rad/s^2.

The vector $\dot{\boldsymbol{\omega}}_n$ is normal to $\boldsymbol{r}_{A/B}$ but is not normal to $\boldsymbol{v}_{A/B}$, as was the case with $\boldsymbol{\omega}_n$.
Imposing additional condition, i.e. $\dot{\boldsymbol{\omega}}_n \cdot \boldsymbol{r}_{A/B} = 0$.
Using this condition along with (f), we can determine,

$$\dot{\boldsymbol{\omega}}_n = 4(2\boldsymbol{i} + 4\boldsymbol{j} - 3\boldsymbol{k}) \text{ rad/s}^2.$$