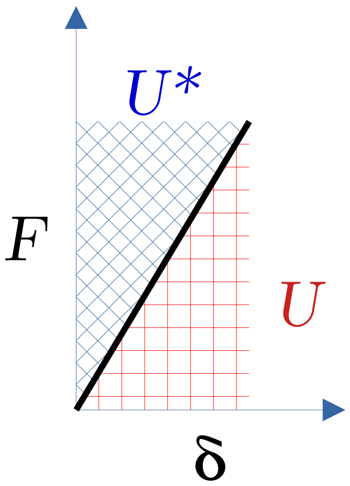
Deflections due to bending

For linear systems, when the force-displacement relation is linear then $U = U^*$.



This means that for linear systems it is not essential to make a distinction between complementary energy and potential energy. Thus for a linear elastic system Castigliano's them can be written as,

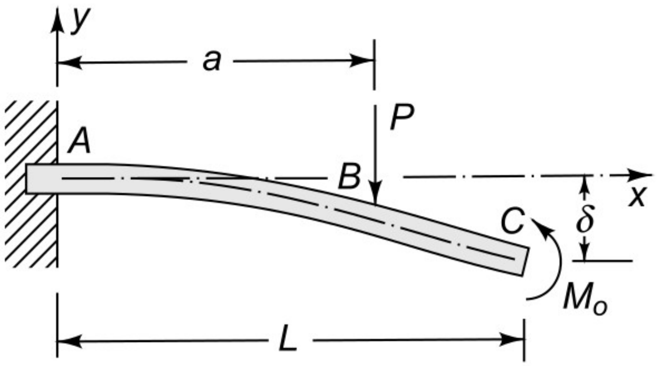
$$\delta_i = \frac{\partial U^*}{\partial P_i} \quad \text{and} \quad \phi_i = \frac{\partial U^*}{\partial M_i}. \qquad \cdots\cdots\cdots\cdots(13)$$

Note that deflection δ_i or rotation ϕ_i is associated with the load P_i or moment M_i acting at the point. If it is required to determine the displacement or rotation at any point where no load is acting, then it can be done by introducing a fictitious force Q at the desired point. Then the elastic strain energy is determined in terms of P_i and Q, the desired displacement is then calculated as,

$$\delta = \left(\frac{\partial U}{\partial Q}\right)_{Q=0}. \qquad \cdots\cdots\cdots\cdots(14)$$

Example 6

A cantilever beam carries a concentrated load P and an end moment M_O. It is desired to predict the deflection δ and slope at the free end C in terms of the constant bending modulus EI and the dimensions shown.



$$M_b = -P(L-x) - M$$

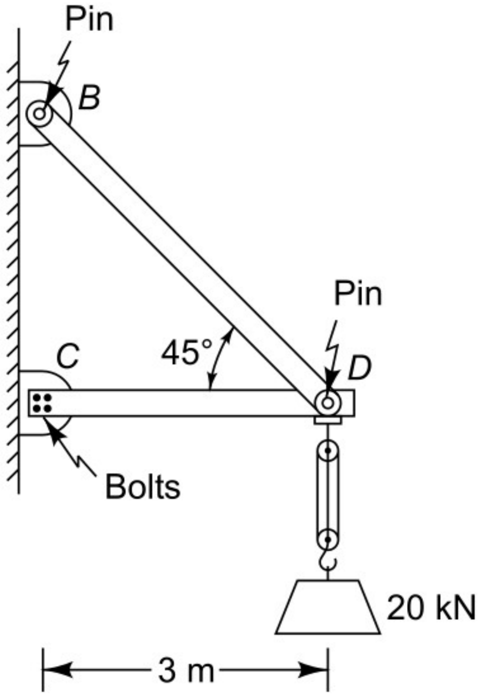
$$U = \int_0^L \frac{M_b^2}{2EI}dx = \int_0^L \frac{\left[P(L-x)+M\right]^2}{2EI}dx$$

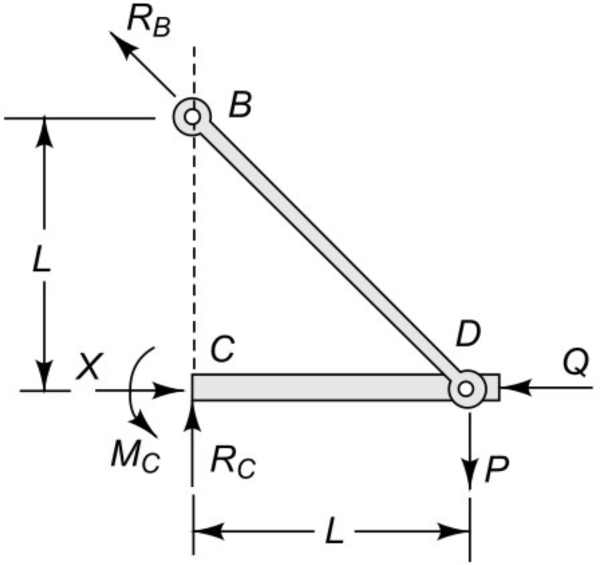
$$\delta_C = \frac{\partial U}{\partial P} = \int_0^L \frac{P(L-x)^2 + M(L-x)}{EI}dx = \frac{PL^3}{3EI} + \frac{ML^2}{2EI}$$

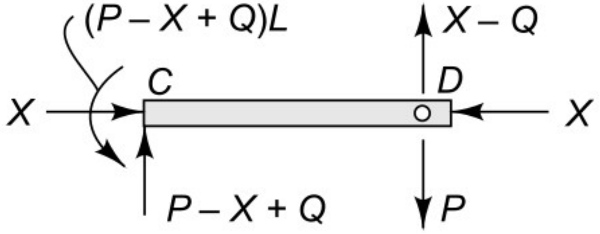
$$\phi_C = \frac{\partial U}{\partial M} = \int_0^L \frac{P(L-x)+M}{EI}dx = \frac{PL^2}{2EI} + \frac{ML}{EI}$$

Example 7

The simple triangular frame shown in figure is used to support a small chain hoist. The chain hoist is supporting its rated capacity of 20 kN. The rod BD is pinned at its ends. The member CD is pinned at D and secured with four bolts at C. Estimate the displacement at the point D.





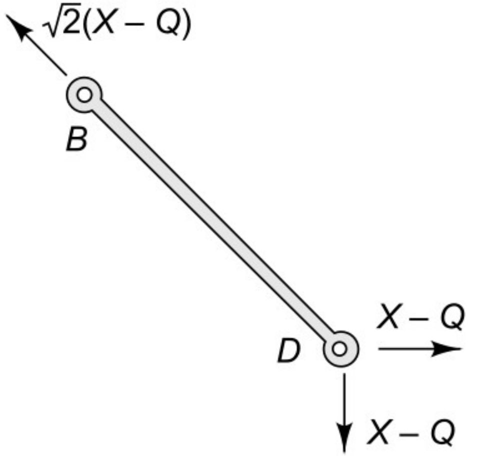


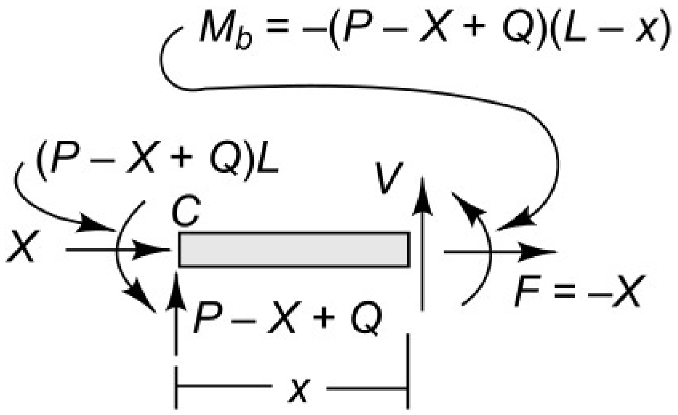
Lets convert the problem into a determinate problem by assuming that the horizontal reaction force X is an applied load and we express all other reaction forces in terms of X by using equations of equilibrium for the whole system as well as for individual members. Following relations are obtained.

$$R_C = P + Q - X$$

$$R_B = \sqrt{2}(X - Q)$$

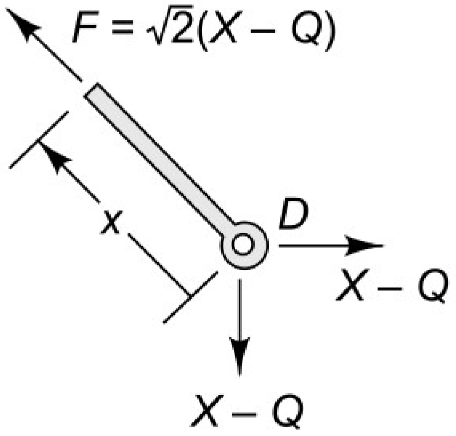
$$M_C = (P + Q - X)L$$





Now we determine the strain energy of the links. Strain energy in CD will be because of bending and axial load. Stress energy because of shear can also be considered; however since we assume that the contribution of shear stresses to the beam deflection is negligible, we do not consider it.

$$U_{CD} = \int_0^L \frac{X^2}{2A_{CD}E}dx + \int_0^L \frac{(P - X + Q)^2(L - X)^2}{2EI}dx$$



Strain energy in BD is only because of axial force. Thus,

$$U_{BD} = \int_0^{\sqrt{2}L} \frac{(X - Q)^2}{2A_{BD}E}dx$$

The total strain energy of the structure is now,

$$U = U_{CD} + U_{BD} = \int_0^L \frac{X^2}{2A_{CD}E} dx + \int_0^L \frac{(P - X + Q)^2 (L - X)^2}{2EI} dx + \int_0^{\sqrt{2}L} \frac{(X - Q)^2}{2A_{BD}E} dx$$

X can be determined by applying the condition that the deflection at point C is zero, i.e.,

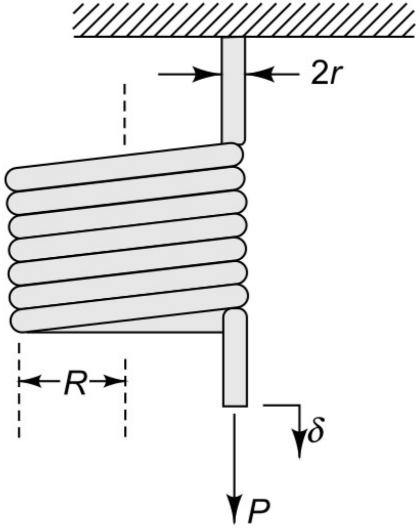
$$\delta_{hC} = \left. \frac{\partial U}{\partial X} \right|_{Q=0} = 0$$

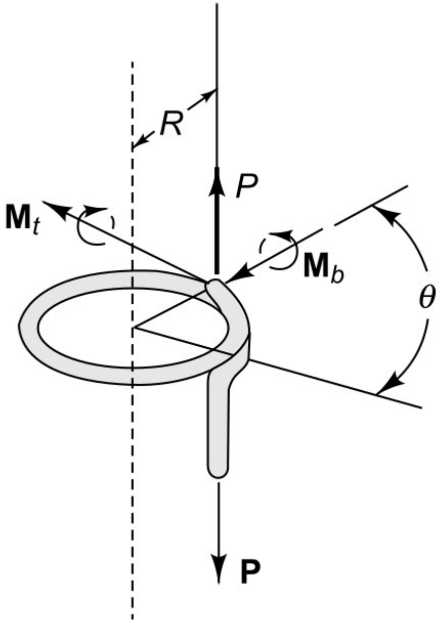
Horizontal and vertical displacements at point D can be obtained as

$$\delta_{hD} = \left. \frac{\partial U}{\partial Q} \right|_{Q=0} \quad \text{and} \quad \delta_{vD} = \left. \frac{\partial U}{\partial P} \right|_{Q=0} .$$

Example 8

Calculate the deflection δ of the tightly coiled spring in . The wire has radius r and is formed into n complete turns of radius R. The ends of the spring are not brought into the center of the coil but extend directly from the rim of the coil. We will see that this "small" difference has an important effect on the stiffness of the spring.





$$M_t = PR(1 - \cos\theta)$$

$$M_b = PR\sin\theta$$

The total strain energy due to bending and torsion is

$$U = \int_0^{2\pi n} \frac{P^2 R^2 (1-\cos\theta)^2}{2GI_x} R d\theta + \int_0^{2\pi n} \frac{P^2 R^2 \sin^2\theta}{2EI} R d\theta$$

$$U = \frac{P^2 R^3}{2GI_x} 3\pi n + \frac{P^2 R^3}{2EI} \pi n$$

The deflection is now obtained as,

$$\delta = \frac{\partial U}{\partial P}.$$

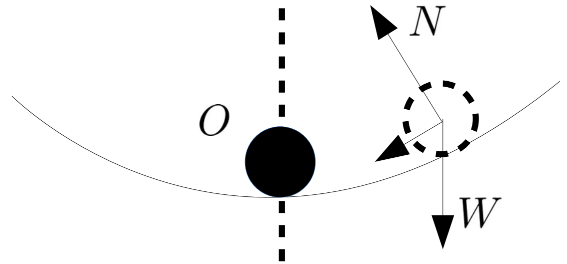
Calculate the deflection and make yourself convinced that this spring is less stiffer than the spring with the centered end. Under the same load this spring deflects nearly twice as much as the spring with centered ends.

Stability of equilibrium: Buckling

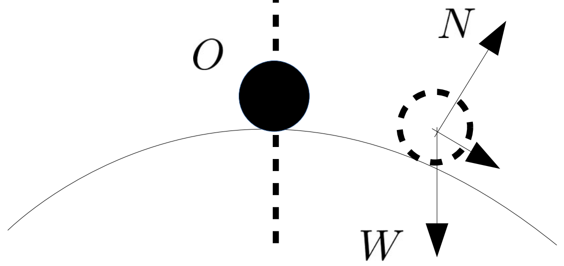
Stability of Equilibrium

- Till now we have investigated the systems and configurations which were in equilibrium by applying the conditions of force and moment equilibrium.
- Now we investigate the behaviour of systems when they are disturbed slightly from their equilibrium position.
- When disturbed from the equilibrium position, forces (and/or moments) in the system are no longer in equilibrium. Hence, their will be acceleration and its presence may result in complicated motions.
- In this course, we will not be investigating the motion in detail. Our scope in the this course will be limited to investigate that *when a system is slightly disturbed from equilibrium does the system tend to return to its equilibrium position or does it tend to depart from the equilibrium position.*

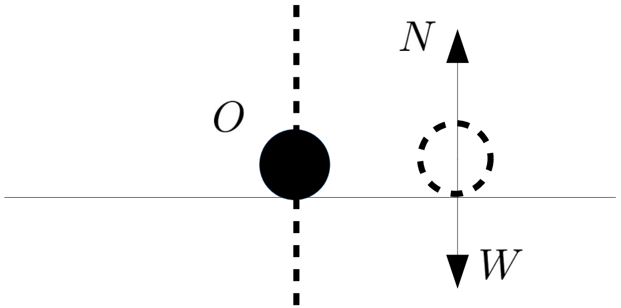
Consider a particle on a frictionless surface. Forces on the particle (weight and reaction) are balance when the surface is horizontal.



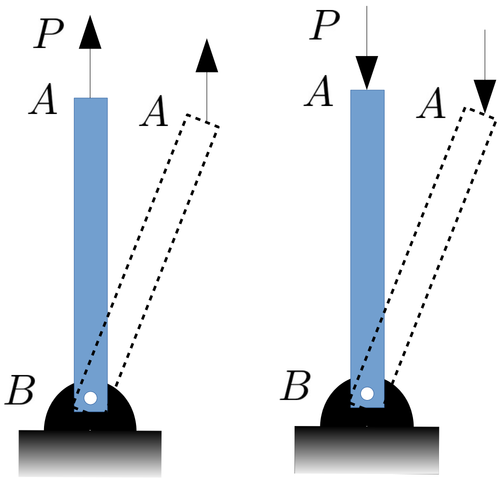
Point O shows the equilibrium position for the particle. When equilibrium position is disturbed, then the forces are no longer balanced, but the unbalanced force is an *restoring force*, i.e., it accelerate the particle towards the equilibrium position. Such an equilibrium is called *stable equilibrium*.



Now consider an opposite situation. At the disturbed position, unbalance force is an upsetting force, i.e., it accelerates the particle away from the equilibrium position. Such an equilibrium is called *unstable*.



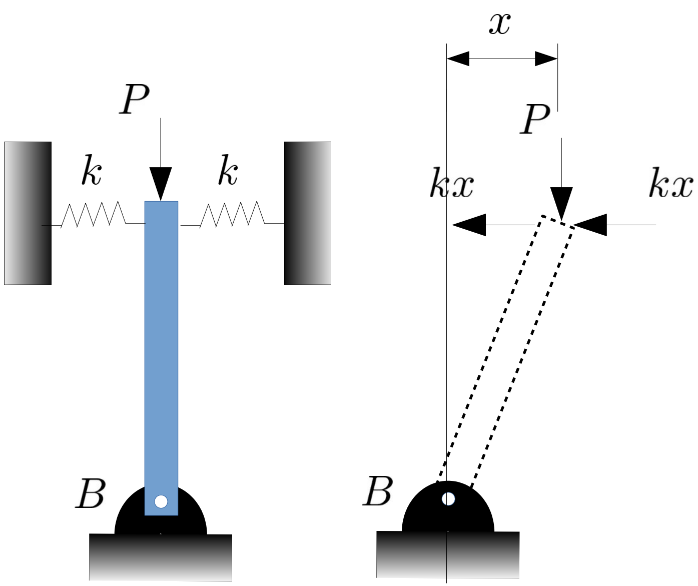
Now consider a particle laying on a horizontal plane. At the disturbed position also particle remain in balance and their is no unbalanced force. Particle neither tend to go toward or go away from the initial position. Such an equilibrium is called *neutral equilibrium*.



(a) (b)

Now consider a weightless bar AB in the vertical position. When a vertical load P is applied at point A, bar will be in equilibrium whether P is in upward (case (a)) or downward (case (b)) position.

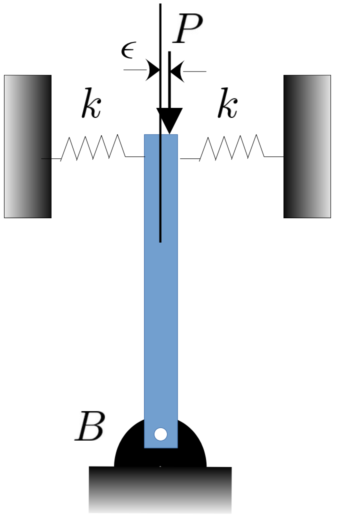
If a small rotation is given to the bar and load P remain vertical then it gives rise to a restoring force in case (a), whereas in case (b) it gives rise to a upsetting force. Thus bar is unstable in case (b).



Unstable equilibrium of the bar can be stablized by addition two springs as shown in the figure. Rotation of the bar will cause spring forces (kx)to act on the bar and create a torque in the opposite direction to torque due to load P. If horizontal deflection of point A is x then the follwing are the conditions for stable and unstable equilibrium of the bar,

$Px < 2kxL$ (stable), $Px > 2kxL$ (unstable).

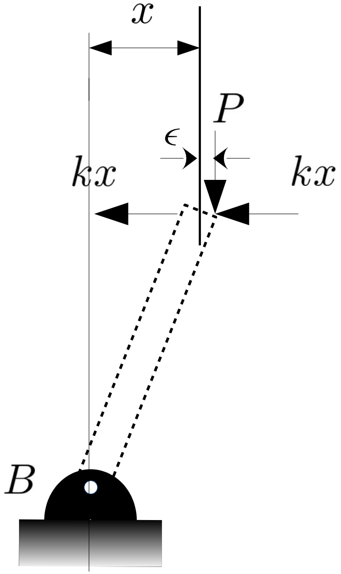
The load $P = 2kL$ is the *critical load.*

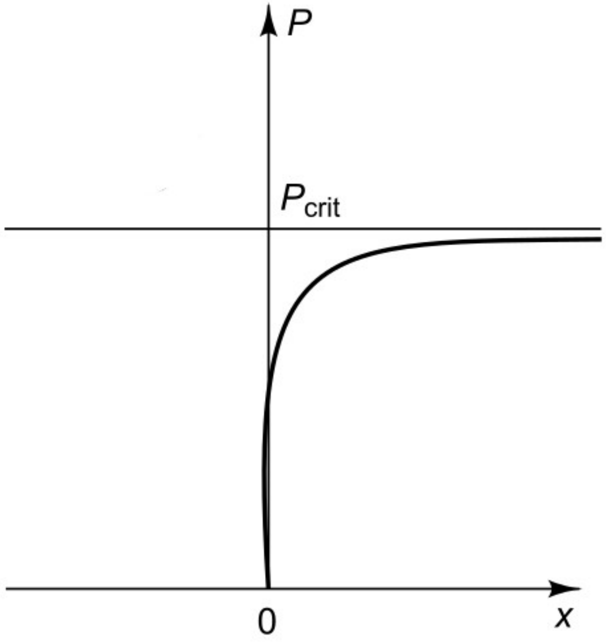


Now consider another situation. The force P is applied slightly off-center. The small distance ϵ is called eccentricity of the load Now we solve the problem for the transverse displacement x in the equilibrium position by balancing torques about B.

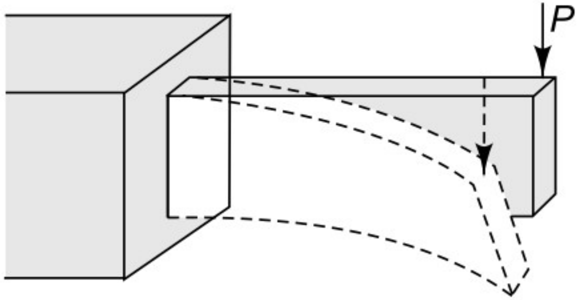
$$P(x+\epsilon) = 2kxL$$

$$x = \epsilon \frac{P}{2kL - P} \qquad \cdots\cdots\cdots\cdots\cdots(1)$$

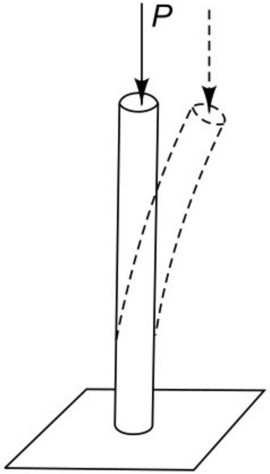




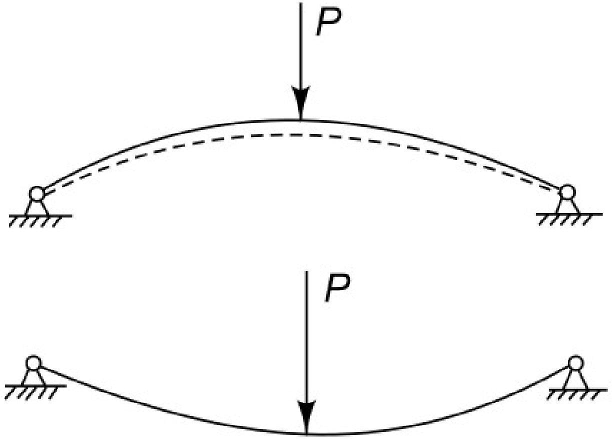
Examples of instability



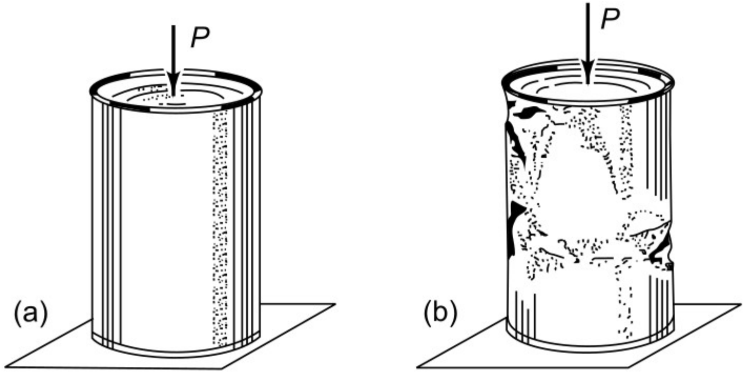
Twist-bend buckling of a deep narrow beam



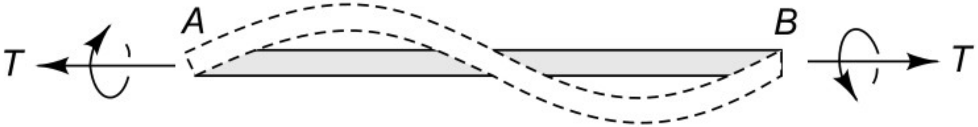
Buckling of columns under compression



Snap-through buckling of shallow curved member



Buckling and crumpling of can walls under compression



Twist-bend buckling of a shaft in tension

Elastic instability of flexible columns

In this course, we will discuss the buckling of columns in details.

We first derive the equations governing the bending of a beam subjected to longitudinal as well as transverse loads.

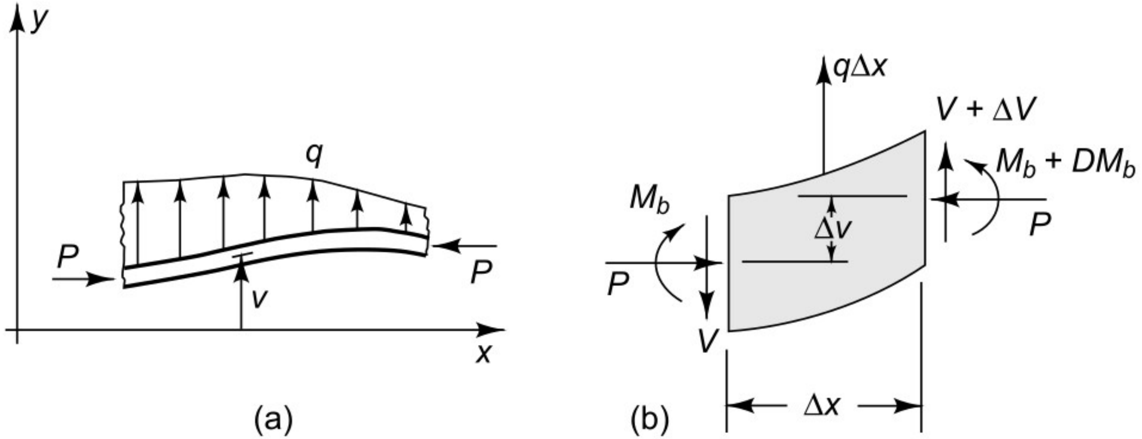
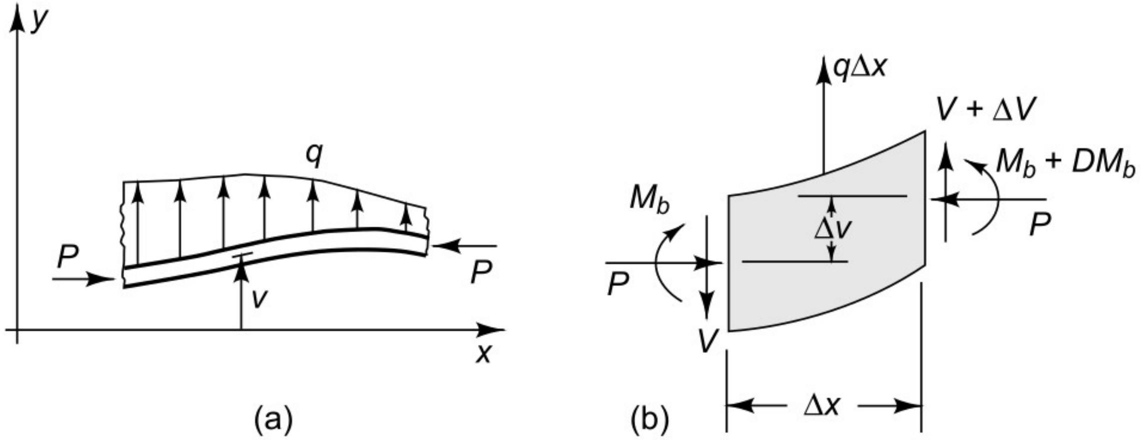


Figure shows a portion of such beam and an enlarged element isolated from the rest of the beam.



For this element to be in equilibrium, we must have

$$V + \Delta V - V + q\Delta x = 0$$

$$M_b + \Delta M_b - M_b + V\frac{\Delta x}{2} + (V + \Delta V)\frac{\Delta x}{2} + P\Delta v = 0$$

In limit $\Delta x \to 0$, and neglecting higher order terms, we get

$$\frac{dV}{dx} + q = 0$$

$$\frac{dM_b}{dx} + V + P\frac{dv}{dx} = 0 \qquad \text{................(2)}$$

Note that in the absence of longitudinal load P, equations (2) are same as the differential relationship derived earlier.

Also note that dv/dx is the slope of the beam, and thus we can relate moment curvature relationship to equations (2). Moment-curvature relations is

$$EI\frac{d^2v}{dx^2} = M_b. \qquad \text{................(3)}$$

Eliminating M_b and V from (2) and (3), we get,

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) + \frac{d}{dx}\left(P\frac{dv}{dx}\right) = q. \qquad \text{................(4)}$$

Equation (3) is applicable for a beam subjected to transverse load $q(x)$ per unit length and axial compressive force $P(x)$. For a constant load P and constant flextural modulus (EI) equation (4) become,

$$EI\frac{d^4v}{dx^4} + P\frac{d^2v}{dx^2} = q. \qquad \text{................(4a)}$$