

Now, we define a fundamental quantity at current position \boldsymbol{x} and time t called the **internal energy**, which is **sum of all microscopic forms of energy**. It denoted as $e_c(\boldsymbol{x}, t)$ and is defined an per unit current volume. The internal energy of a body occupying region Ω is,

$$\mathcal{E}(t) = \int_{\Omega} e_c(\boldsymbol{x}, t) dv.$$

Since only mechanical energy is considered, we state that **the rate of work done on the continuum body by internal stresses (i.e., internal power) = the rate of internal energy**, i.e.

$$\mathcal{P}_{\text{int}} = \frac{D}{Dt} \mathcal{E}(t).$$

Now, balance of mechanical energy can be written in term of internal energy as,

$$\frac{D}{Dt} \mathcal{K}(t) + \frac{D}{Dt} \mathcal{E}(t) = \mathcal{P}_{\text{ext}}(t), \quad \text{or}$$

$$\frac{D}{Dt} \int_{\Omega} \left(\frac{1}{2} \rho \boldsymbol{v}^2 dv + e_c \right) dv = \int_{\partial\Omega} \boldsymbol{t} \cdot \boldsymbol{v} ds + \int_{\Omega} \boldsymbol{b} \cdot \boldsymbol{v} dv.$$

The term $\int_{\Omega} \left(\frac{1}{2} \rho \mathbf{v}^2 + e_c \right) dv$ characterizes the total energy, i.e. the sum of the kinetic energy and internal energies.

Note that contribution to the total energy is only due to the external sources, i.e the external mechanical power.

Material description of balance of mechanical energy

Material description of the balance of mechanical energy equations is,

$$\frac{D}{Dt} \int_{\Omega_0} \left(\frac{1}{2} \rho_0 \mathbf{V}^2 + e \right) dV = \int_{\partial\Omega_0} \mathbf{T} \cdot \mathbf{V} ds + \int_{\Omega_0} \mathbf{B} \cdot \mathbf{V} dV.$$

Here, $e=e(\mathbf{X},t)$ is the material description of internal energy, which is defined such that,

$$\mathcal{E}(t) = \int_{\Omega} e_c(\mathbf{x}, t) dv = \int_{\Omega_0} e(\mathbf{X}, t) dV.$$

Internal power is related to e as, $\mathcal{P}_{\text{int}} = \frac{D}{Dt} \int_{\Omega_0} \mathbf{P} : \dot{\mathbf{F}} dV = \frac{D}{Dt} \int_{\Omega_0} e(\mathbf{X}, t) dV.$ 28

Alternate expressions for the stress power

We already derived internal power as,

$$\mathcal{P}_{\text{int}} = \frac{D}{Dt} \int_{\Omega} \boldsymbol{\sigma} : \mathbf{d} dv.$$

It can be shown that internal power can be expressed in following forms also,

$$\mathcal{P}_{\text{int}} = \frac{D}{Dt} \int_{\Omega} \boldsymbol{\sigma} : \mathbf{d} dv. = \frac{D}{Dt} \int_{\Omega_0} \mathbf{P} : \dot{\mathbf{F}} dV = \frac{D}{Dt} \int_{\Omega_0} \mathbf{S} : \dot{\mathbf{E}} dV = \frac{D}{Dt} \int_{\Omega_0} \frac{1}{2} \mathbf{S} : \dot{\mathbf{C}} dV.$$

Definition of different measure of stress (and strain) and their inter-relationship will be useful to derive the above relationship.

Stress definition and corresponding measure of deformation, which define the internal power are terms as **work-conjugate pairs**.

Conservative systems

Consider scalar valued functions,

Π_{ext} – **potential energy of external loads (or external potential energy)** and

Π_{int} – **total strain energy (or internal potential energy)** of the body, respectively.

$\Pi = \Pi_{\text{ext}}(t) + \Pi_{\text{int}}(t)$ – the sum of both the potential energy is known as **total potential energy (or energy functional)** of a mechanical system.

A mechanical system is called **conservative**, if the mechanical external power and internal power can be expressed as,

$$\mathcal{P}_{\text{ext}}(t) = -\frac{D}{Dt}\Pi_{\text{ext}}(t) = -\dot{\Pi}_{\text{ext}}(t), \quad \text{and}$$
$$\mathcal{P}_{\text{int}}(t) = \frac{D}{Dt}\Pi_{\text{int}}(t) = \dot{\Pi}_{\text{int}}(t), \quad \text{with } \Pi_{\text{int}}(t) = \int_{\Omega_0} \Psi dV,$$

where, Ψ is the **strain energy function** (or **strain energy** or **stored energy**) defined per unit reference volume. Strain energy is also denoted in W .

With the definitions and assumptions in the previous slide, balance of mechanical energy become,

$$\frac{D}{Dt}\mathcal{K}(t) + \frac{D}{Dt}\Pi_{\text{int}}(t) + \frac{D}{Dt}\Pi_{\text{ext}}(t) = 0, \text{ or ,}$$

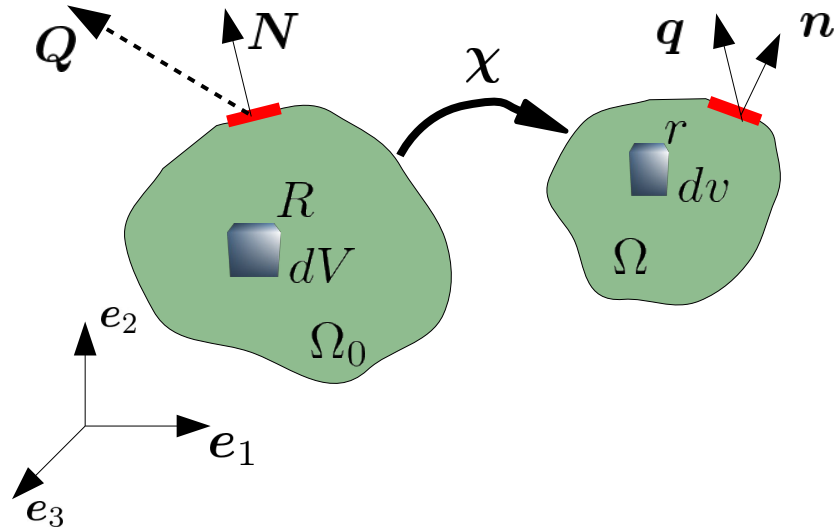
$$\mathcal{K}(t) + \Pi_{\text{int}}(t) + \Pi_{\text{ext}}(t) = \text{const.},$$

which implies that the sum of potential energy and kinetic energy is conserved during a dynamic process.

It should be noted that most surface traction and all problems associated with dissipative energy (such as external or internal friction, viscous or plastic effects) lead to non-mechanical energy which are **non-conservative** in the sense that **they can not be derived from a potential**.

Balance of energy in continuum thermodynamics

- In this section we will consider both the **mechanical** and **thermal energy** of the system, which are essential in many problems of physics and engineering.
- In a thermodynamic context, conservation of mass and momentum balance principles are supplemented by **balance of energy and entropy inequality** laws.
- A continuum with has both mechanical and thermal energy is known as **thermodynamic continuum**.
- Thermodynamic state of a system is known if all quantities throughout the system is known. All quantities characterizing a system at certain state are called **thermodynamic state variables**. The function that describe a certain state variable is called **thermodynamic state function**. Any function that interrelates state variables is known as **equation of state** or **constitutive relation**.



Let the thermal power or rate of thermal work be denoted by \mathcal{Q} and defined as,

$$\mathcal{Q}(t) = \int_{\partial\Omega} q_n ds + \int_{\Omega} r dv = \int_{\partial\Omega_0} Q_N dS + \int_{\Omega_0} R dV,$$

in the spatial and material coordinates, respectively.

Functions q_n and Q_n – The heat fluxes, i.e., heat per unit time per unit current (ds) and reference (dS) surface area, respectively.

$\int_{\partial\Omega} q_n ds$ and $\int_{\partial\Omega_0} Q_n dS$ – The total heat flux. It measures the rate at which heat enters (inward normal flux) the body across the current and reference surface area, respectively.

Scalar fields $r(\boldsymbol{x},t)$ and $R(\boldsymbol{X},t)$ – The heat source per unit time and per unit current or reference volume, respectively.

$\int_{\Omega} r dv$ and $\int_{\Omega_0} R dV$ – The total heat source. It measure the total heat generated into a certain region of a body.

Stokes' heat flux theorem in thermodynamics states that function q_n and Q_n are linear functions of outward unit normal vectors, such that

$$\begin{aligned} q_n &= -\mathbf{q}(\mathbf{x}, t) \cdot \mathbf{n} \quad \text{or} \quad q_n = -q_i n_i, \\ Q_N &= -\mathbf{Q}(\mathbf{X}, t) \cdot \mathbf{N} \quad \text{or} \quad Q_N = -Q_i N_i. \end{aligned}$$

The time dependent vector field \mathbf{q} are called **Cauchy heat flux (or true heat flux)** define per unit surface area, and \mathbf{n} is the outward normal vector to an infinitesimal spatial surface element at the current position \mathbf{x} .

\mathbf{Q} is known as **Piola-Kirchoff heat flux** per unit surface area in material configuration, and \mathbf{N} is the outward normal vector to an infinitesimal material surface element.

Negative signs are necessary because, normals are outward normals but we claim that the heat enters inside the body (i.e. inward normal flux).

$$\mathbf{Q} = J\mathbf{F}^{-1}\mathbf{q} \quad \text{or} \quad Q_i = JF_{ij}^{-1}q_j.$$

\mathbf{Q} and \mathbf{q} are related by **Piola transformation** as,

First law of thermodynamics (spatial description)

Consider a case when thermal power is added to the body. Thus, the rate of work done on the continuum body, i.e. the sum of the rate of internal mechanical work \mathcal{P} and the rate of thermal work \mathcal{Q} , equals the rate of internal energy \mathcal{E} , i.e.,

$$\mathcal{P}_{\text{int}}(t) + \mathcal{Q}(t) = \frac{D}{Dt}\mathcal{E}(t),$$

which is referred as the **balance of thermal energy**. From the balance of mechanical energy, we can write,

$$\mathcal{P}_{\text{ext}}(t) = \mathcal{P}_{\text{int}}(t) + \frac{D}{Dt}\mathcal{K}(t) \Rightarrow \mathcal{P}_{\text{ext}}(t) = \frac{D}{Dt}\mathcal{E}(t) - \mathcal{Q}(t) + \frac{D}{Dt}\mathcal{K}(t),$$

$$\Rightarrow \frac{D}{Dt}\mathcal{E}(t) + \frac{D}{Dt}\mathcal{K}(t) = \mathcal{P}_{\text{ext}}(t) + \mathcal{Q}(t).$$

Substituting the expressions for individual terms in the above equations,

It is the thermodynamic extension of balance of mechanical energy and postulates the **balance of energy (mechanical and thermal)**, a.k.a **first law of thermodynamics** in spatial description.

$$\frac{D}{Dt} \int_{\Omega} \left(e_c + \frac{1}{2} \rho \mathbf{v}^2 \right) dv = \int_{\partial\Omega} (\mathbf{t} \cdot \mathbf{v} + q_n) ds + \int_{\Omega} (\mathbf{b} \cdot \mathbf{v} + r) dv.$$

First law of thermodynamics states that

the **rate of change of total energy (internal energy and kinetic energy)** of a thermodynamic system =

the rate at which external mechanical work is done the system by surface tractions and body forces +

the rate at which thermal work is done by the heat fluxes and heat sources.

It governs the transformation from one type of energy involved in a thermodynamic process into another, but it never governs the direction of that energy transfer. 37

With the help of Stokes' heat flux theorem and Gauss divergence theorem, expression for the thermal work can be written as,

$$\mathcal{Q}(t) = \int_{\Omega} (-\operatorname{div} \mathbf{q} + r) dv.$$

Recall the balance of mechanical energy,

$$\frac{D}{Dt} \int_{\Omega} \frac{1}{2} \rho \mathbf{v}^2 dv + \int_{\Omega} \boldsymbol{\sigma} : \mathbf{d} dv = \int_{\partial\Omega} \mathbf{t} \cdot \mathbf{v} ds + \int_{\Omega} \mathbf{b} \cdot \mathbf{v} dv.$$

Using above expression, first law of thermodynamics can be written as,

$$\frac{D}{Dt} \int_{\Omega} \left(e_c + \frac{1}{2} \rho \mathbf{v}^2 \right) dv = \frac{D}{Dt} \int_{\Omega} \frac{1}{2} \rho \mathbf{v}^2 dv + \int_{\Omega} (\boldsymbol{\sigma} : \mathbf{d} - \operatorname{div} \mathbf{q} + r) dv, \quad \text{or}$$

$$\boxed{\frac{D}{Dt} \int_{\Omega} e_c dv = \int_{\Omega} (\boldsymbol{\sigma} : \mathbf{d} - \operatorname{div} \mathbf{q} + r) dv.}$$

which is **reduced global form of first law of thermodynamics** in spatial description.

First law of thermodynamics (material description)

Material description of global form of the first law of thermodynamics is,

$$\frac{D}{Dt} \int_{\Omega} \left(e + \frac{1}{2} \rho_0 \mathbf{V}^2 \right) dv = \int_{\partial\Omega_0} (\mathbf{T} \cdot \mathbf{V} + Q_N) ds + \int_{\Omega_0} (\mathbf{B} \cdot \mathbf{V} + R) dV.$$

The reduced global form is given as,

$$\frac{D}{Dt} \int_{\Omega_0} e dV = \int_{\Omega_0} \left(\mathbf{P} : \dot{\mathbf{F}} - \text{Div} \mathbf{Q} + R \right) dV.$$

Noting that the volume V is arbitrary, the local form of the balance of energy in material description can be written as,

$$\dot{e} = \mathbf{P} : \dot{\mathbf{F}} - \text{Div} \mathbf{Q} + R \quad \text{or} \quad \dot{e} = P_{ij} : \dot{F}_{ij} - \frac{\partial Q_i}{\partial X_i} + R.$$

Entropy inequality principle

After the first law of thermodynamics, we now derive another principle which governs the direction of energy transfer; this is known as **the second law of thermodynamics**.

Here, we introduce a fundamental state variable called **entropy**. It can be viewed as the qualitative measure of **microscopic randomness and disorder**. $\eta_c(\mathbf{x}, t)$ and $\eta(\mathbf{X}, t)$ denote entropy per unit current and reference volume, respectively at certain point at time t . The total entropy of a system is denoted as,

$$\mathcal{S} = \int_{\Omega} \eta_c(\mathbf{x}, t) dv = \int_{\Omega_0} \eta(\mathbf{X}, t) dV, \quad \text{where,} \quad \eta(\mathbf{X}, t) = J(\mathbf{X}, t) \eta_c(\mathbf{x}, t).$$

Let the rate of entropy input ($\tilde{\mathcal{Q}}$) to a certain region of a continuum body consist of the value of **entropy transferred across its boundary surface** and the **entropy generated (or destroyed) inside that region**, which is given as,

$$\tilde{\mathcal{Q}}(t) = - \int_{\partial\Omega} \mathbf{h} \cdot \mathbf{n} ds + \int_{\Omega} \tilde{r} dv = - \int_{\partial\Omega_0} \mathbf{H} \cdot \mathbf{N} dS + \int_{\Omega_0} \tilde{R} dV, \quad 40$$

where, $\tilde{r}(\mathbf{x}, t)$ and $\tilde{R}(\mathbf{X}, t)$ – entropy sources per unit time and per unit current and reference volume, and

$\mathbf{h}(\mathbf{x}, t)$ and $\mathbf{H}(\mathbf{X}, t)$ – Cauchy entropy flux per unit current surface area of Piola-Kirchoff entropy flux per unit reference surface area.

Now, **The total entropy production per unit time (Γ) =**
the **difference between the rate of change of entropy $\dot{\mathcal{S}}$ and the rate of entropy input \tilde{Q}**

We postulate that the total entropy production for all thermodynamic processes is never negative, i.e.

$$\Gamma(t) = \frac{D}{Dt} \mathcal{S}(t) - \tilde{Q}(t) \geq 0,$$

This is known as the **second law of thermodynamics**. It is an inequality; hence referred as **entropy inequality principle**.