ME231: Solid Mechanics-I

Stress, Strain and Temperature relationship

Thermal strains

Temperature may affect the elastic region of a material in two ways: (i) by modifying in the values of the elastic constants, and (ii) second, by directly producing a strain even in the absence of stress.

As a part of this course, we will not be interested in the first effect, as change in the elastic constants for many materials is small for a temperature change of about 100°C.

Second effect is of our interest now. The strain due to temperature change in the absence of stress is called **thermal strain** and is denoted by the superscript t on the strain symbol. For an isotropic material, using symmetry arguments it can be shown that the **thermal strain must be a pure expansion or contraction with no shear strain components** referred to any set of axes.

The thermal strains are not exactly linear with temperature change, but for many engineering material thermal strains can be approximated as linear variation within temperature changes of 30-100°C .

Thus, thermal strains due to a change in temperature from T_0 to T is

$$\begin{aligned} \epsilon_x^t &= \epsilon_y^t = \epsilon_z^t = \alpha(T - T_0), \\ \gamma_{xy}^t &= \gamma_{yz}^t = \gamma_{zx}^t = 0. \end{aligned} \qquad \cdots\cdots\cdots\cdots(20)$$

Here, α is known as the **coefficient of linear expansion**. It has a unit of **1/temperature**.

The total strain at a point in an elastic body is the sum of that due to stress and that due to temperature. Denoting the elastic strain due to stress by superscript e and the thermal part by superscript t, the total strain derived from the displacements is given by

$$\epsilon = \epsilon^e + \epsilon^t \qquad \cdots\cdots\cdots\cdots(21)$$

For example, consider a material rigidly restrained between supports so that no strain is possible, then

$$\epsilon = \epsilon^e + \epsilon^t = 0 \quad \text{or} \quad \epsilon^e = -\epsilon^t, \qquad \cdots\cdots\cdots\cdots(22)$$

i.e., elastic part of the strain will be the negative of the thermal strain.

Complete equations of Elasticity

- The theory of elasticity is the subject dealing with the distribution of stress and strain in elastic bodies subjected to given loads, displacements, and distributions of temperature. Now we are in a position to state completely the foundations of the theory of elasticity.
- The problem is to find distributions of stress and strain which
  - meet the prescribed loads and displacements on the boundary and which
  - satisfy the equilibrium equations, the stress-strain-temperature relations, and the geometrical conditions associated with the definition of strain and the concept of continuous displacements at every point.
- All equations which are required to be satisfied at each point of a nonaccelerating, isotropic, homogeneous, linear-elastic body subject to small strains.

Equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0,$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0,$$
$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + Z = 0.$$

where X, Y, and Z are body forces which are distributed over the volume with intensities X, Y, and Z per unit volume.

............(23)

Geometric compatibility

The displacements must match the geometrical boundary conditions and must be continuous functions of position with which the strain components are associated, as follows:

$$\epsilon_x = \frac{\partial u}{\partial x}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$
$$\epsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}$$
$$\epsilon_z = \frac{\partial w}{\partial z}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

where u, v, and w are the displacement components in the x, y, and z-directions.

............(24)

Stress – Strain – Temperature relationship

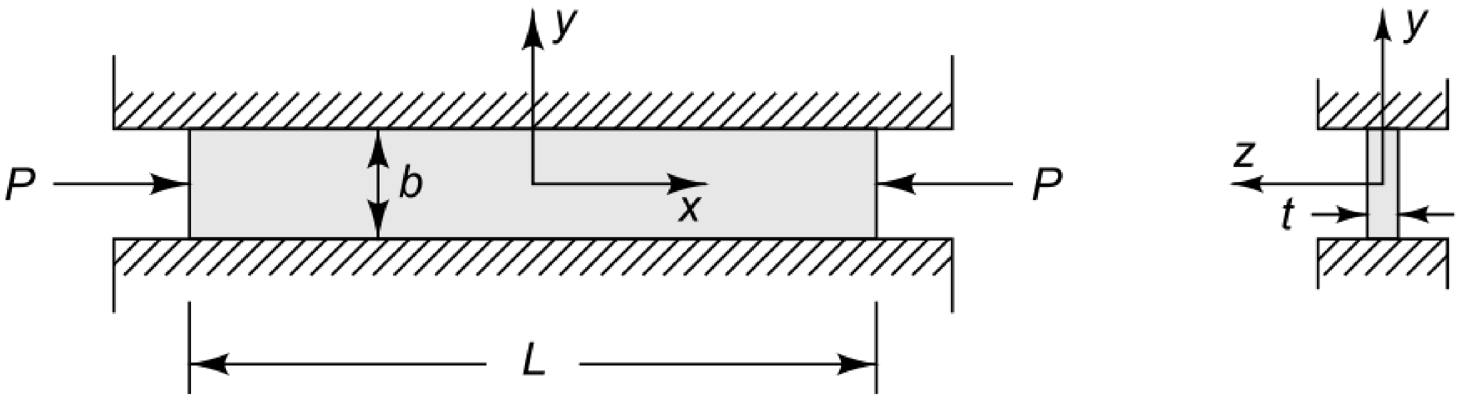
$$\epsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] + \alpha(T - T_0), \qquad \gamma_{xy} = \frac{\tau_{xy}}{G},$$
$$\epsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})\right] + \alpha(T - T_0), \qquad \gamma_{yz} = \frac{\tau_{yz}}{G},$$
$$\epsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right] + \alpha(T - T_0), \qquad \gamma_{xz} = \frac{\tau_{xz}}{G}. \qquad \text{............(25)}$$

Equations (23) – (25) are total 15 equations for the six components of stress, the six components of strain, and the three components of displacement. These 15 equations are the foundation for what is commonly called linear elasticity theory.

However, in this course we will be solving simplified problem, which will evolve only few of these equations.

Example 1

A long, thin plate of width b, thickness t, and length L is placed between two rigid walls a distance b apart and is acted on by an axial force P, as shown in Figure. We wish to find the deflection of the plate parallel to the force P.



Idealization:

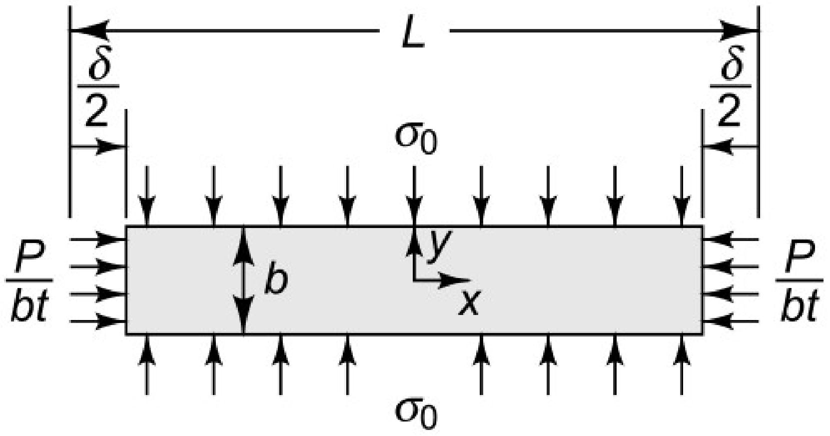
1. Uniform axial stress in x-direction because of applied load P

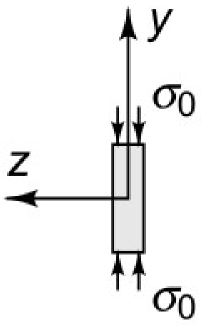
2. No normal stress in z-direction, as plate is very thin in that direction (plane stress in xy-plane)

3. No deformation in y-direction. (Plane strain in yz-plane)

4. Uniform distribution of contact stress between plate and rigid walls.

5. Frictionless contact





Equilibrium:

Equilibrium with external load will be satisfied when stresses are

$$\sigma_{xx} = -\frac{P}{bt}, \quad \sigma_{yy} = -\sigma_0, \quad \sigma_{zz} = 0,$$
$$\tau_{xy} = \tau_{xz} = \tau_{yz} = 0. \qquad \text{................(1.a)}$$

It can be verified that these stresses satisfies equilibrium equations.

Geometric compatibility:

Because of rigid walls, $\epsilon_{yy} = 0$, and $\epsilon_{xx} = -\frac{\delta}{L}$ ....(1.b)

Stress-strain relations:

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$$\epsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}), \quad \gamma_{xy} = 0,$$
$$\epsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx}), \quad \gamma_{yz} = 0,$$
$$\epsilon_{zz} = \frac{-\nu}{E}(\sigma_{xx} + \sigma_{yy}), \quad \gamma_{xz} = 0. \qquad \text{................(1.c)}$$

$$-\frac{\delta}{L} = \frac{1}{E}\left(-\frac{P}{bt} + \nu\sigma_0\right), \qquad \cdots\cdots\cdots\cdots\cdots\cdots(1.d)$$

$$0 = \frac{1}{E}\left(-\sigma_0 + \nu\frac{P}{bt}\right), \Rightarrow \sigma_0 = \frac{\nu P}{bt} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(1.e)$$

$$\epsilon_{zz} = \frac{-\nu}{E}\left(-\frac{P}{bt} - \sigma_0\right). \qquad \cdots\cdots\cdots\cdots\cdots\cdots(1.f)$$

Using results from (1.e) to (1.d) and (1.f), we get

$$\delta = (1-\nu^2)\frac{PL}{Ebt} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(1.g)$$

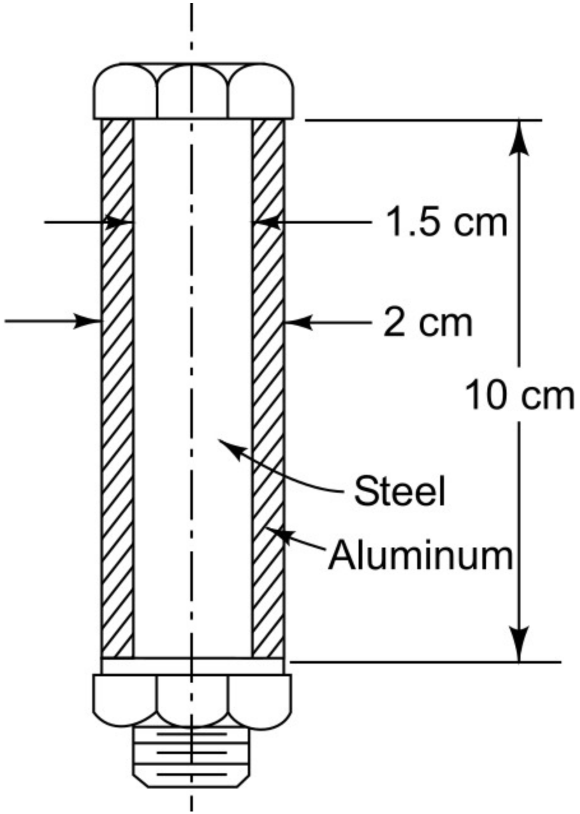
$$\epsilon_{zz} = \nu(1+\nu)\frac{P}{Ebt} = \frac{\nu}{1-\nu}\frac{\delta}{L}. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(1.h)$$

From strain-displacement relations, we can find

$$u = -\frac{\delta}{L}x, \quad v = 0, \quad w = -\frac{\nu}{1-\nu}\frac{\delta}{L}z.$$

Example 2

A steel bolt and nut and an aluminum sleeve is shown. The bolt has 6 threads per cm and, when the material is at 60°F, the nut is tightened one-quarter turn. The temperature is then raised from 60 to 100°F. Determine the stresses in both bolt and sleeve.



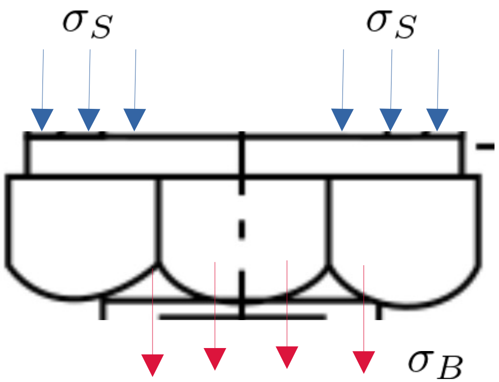
When the nut is turned on the bolt against the sleeve, a tensile force acts on the bolt through nut. Now, if tensile stress in the bolt is σ_B, then strain in the bolt is,

$$\varepsilon_B = \sigma_{MB}/E_B.$$

Resultant deformation in the sleeve consists of the following:
(I) Because of compression applied by the nut.
(II) Because of extension of the bolt.

Thus total strain in sleeve is $$\varepsilon_S = -\frac{\frac{1}{6}\cdot\frac{1}{4}}{10} + \frac{\sigma_{MB}}{E_B},$$

Hence, stress in the sleeve is, $$\sigma_{MS} = E_S\varepsilon_S = E_S\left(-\frac{\frac{1}{6}\cdot\frac{1}{4}}{10} + \frac{\sigma_{MB}}{E_B}\right) \qquad \text{................(2.a)}$$



From equilibrium of nut,

$$\sigma_{MS}A_S + \sigma_{MB}A_B = 0 \qquad \text{................(2.b)}$$

Solving (2.a) and (2.b) will give stress because of mechanical deformation only.

Now, we calculate the stress only because of change in temperature.
When the temperature is increased by ΔT amount, thermal strains are generated in both the bolt and the sleeve and the resultant strain is,

$$\varepsilon_{BT} = \frac{\sigma_{BT}}{E_B} + \alpha_B \Delta T \qquad \cdots\cdots\cdots\cdots\cdots(2.c)$$

$$\varepsilon_{ST} = \frac{\sigma_{ST}}{E_S} + \alpha_S \Delta T.$$

Due to constraint of the nut,

$$\Delta_{BT} = \Delta_{ST} \text{ or } \varepsilon_{BT} = \varepsilon_{ST} \qquad (\because L_B = LS). \qquad \cdots\cdots\cdots\cdots\cdots(2.d)$$

Again from equilibrium of the nut, we can write,

$$\sigma_{ST} A_S + \sigma_{BT} A_B = 0 \qquad \cdots\cdots\cdots\cdots\cdots(2.e)$$

Solving (2.c) to (2.e) gives stress because of change in temperature.

Now total stress is bolt and sleeve is

$$\sigma_B = \sigma_{MB} + \sigma_{TB} \quad \text{and} \quad \sigma_S = \sigma_{MS} + \sigma_{TS}. \qquad \cdots\cdots\cdots\cdots\cdots(2.f)$$

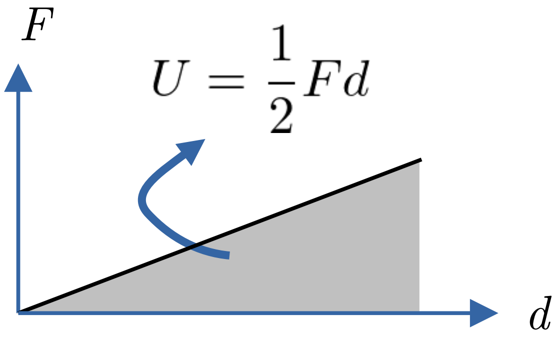
Strain energy in an elastic body

We will understand the strain energy in linear elastic bodies subjected to small deformation.

The elastic energy U stored in a linear spring is given in three forms:

- in terms of the deflection d,
- in terms of the force F, or
- in terms of the deflection d and the force F.

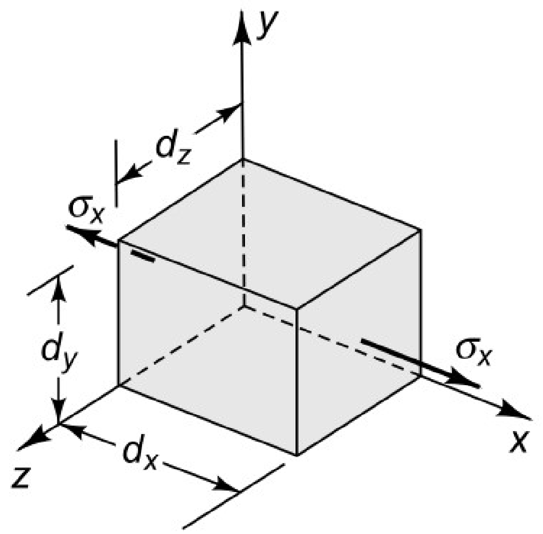
Because of the linearity, force and deflection grow in proportion during the loading process, and thus the total work done is just one-half the product of the final force and the final deflection.



We apply this concept to an infinitesimal element of a linear elastic body. Figure shows a uniaxial stress component σ_x and corresponding deformation for a cuboidal element. The elastic energy stored in such an element is commonly called strain energy.

In this case the force $\sigma_x\, dy\, dz$ acting on the positive x face does work as the element undergoes the elongation $\varepsilon_x\, dx$.

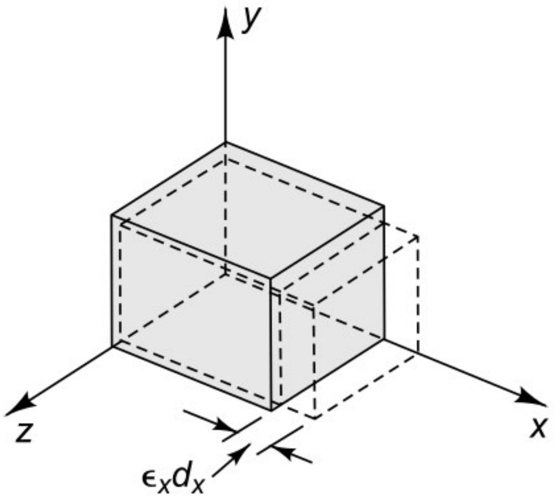
As strain is proportional to the stress, the strain energy dU stored in the element, is



$$dU = \frac{1}{2}(\sigma_x dydz)(\varepsilon_x dx) = \frac{1}{2}\sigma_x \epsilon_x dV.$$

For a linear uniaxial member stress and strain can be taken to be uniform through the volume of the member.

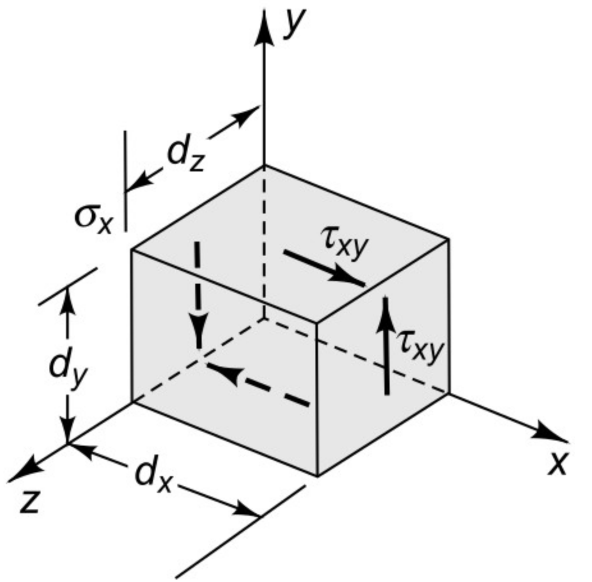
Thus, $\sigma_x = P/A$ and $\varepsilon_x = \delta/L$, then we have the total strain energy as,



$$U = \int dU = \int_V \frac{1}{2}\sigma_x \epsilon_x dV = \sigma_x \epsilon_x V = \frac{P}{A}\frac{\delta}{L}(AL) = \frac{1}{2}P\delta,$$

which is same as the relation for the stored energy in a linear elastic spring.

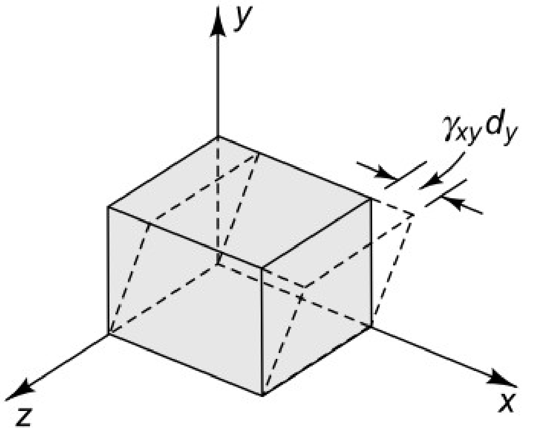
Now, consider the shear-stress component τ_{xy} acting on the infinitesimal element. The corresponding deformation due to the shear-strain component γ_{xy} . In this case the force τ_{xy} dx dz acting on the positive y face does work as that face translates through the distance γ_{xy} dy. As τ_{xy} and γ_{xy} are proportional to each other, the strain energy stored in the element is,



$$dU = \frac{1}{2}(\tau_{xy} dx dz)(\gamma_{xy} dy) = \frac{1}{2}\tau_{xy}\gamma_{xy} dV.$$

Similar expressions will be valid for other components of stresses and strains.

Finally, for an element with all components of stress and stress present working on it, we can write the strain energy stored in the element as,



$$dU = \frac{1}{2}\left[\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right] dV.$$

The total energy is then $U = \int_V dU.$