ME531: Advanced Mechanics of Solids

Motion, Strain and Stress

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Measure of Deformation is solids

From our knowledge of Strength of materials (UG course), we know that in linear stress-strain analysis the deformation of a continuum body is measured in terms of small strains.

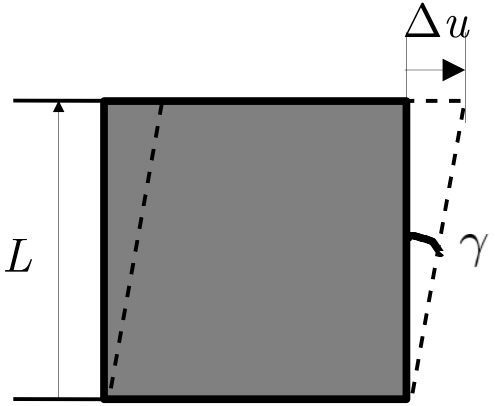
For instance, small strain in axial loading is defined as

$$\epsilon = \frac{\Delta l}{L},$$

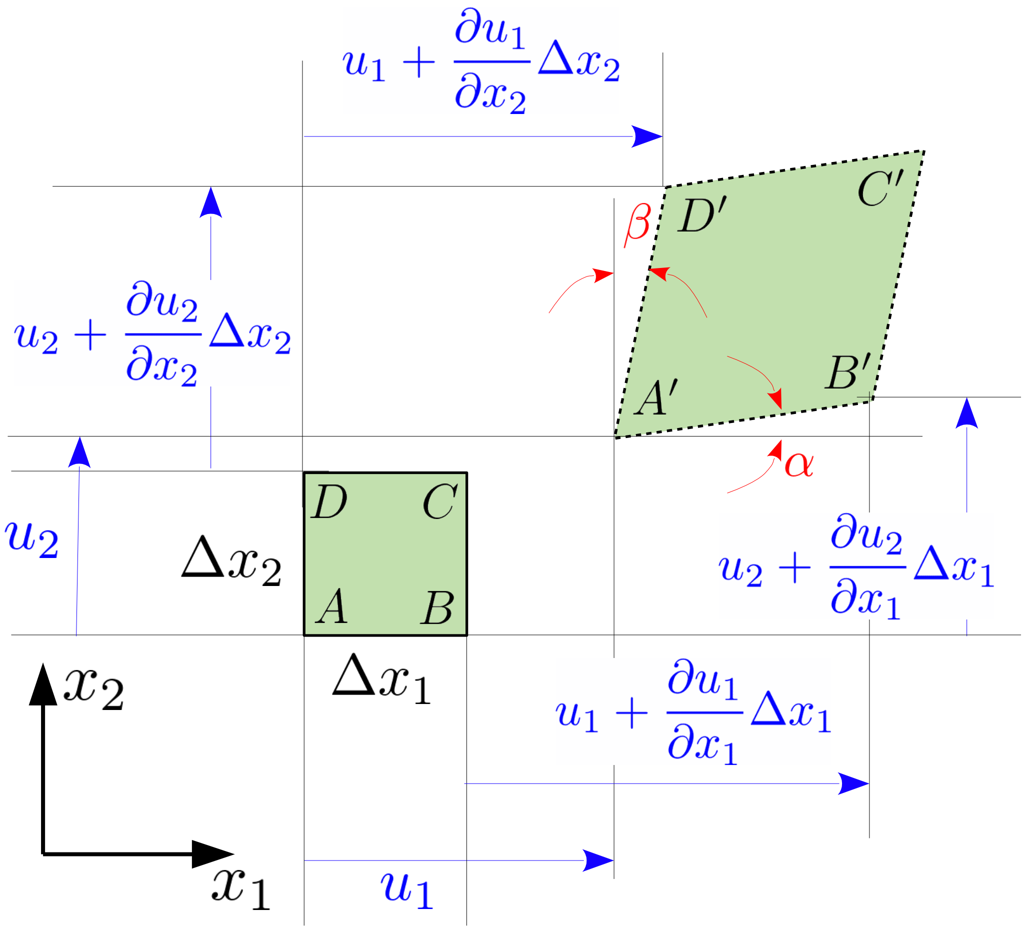
where L is the initial length and Δl is the elongation.

Similarly, shear strain is defined as

$$\tan\gamma \approx \gamma = \frac{\Delta u}{L}.$$



Small strains in 2D



$$\varepsilon_{11} = \lim_{\Delta x_1 \to 0} \frac{A'B' - AB}{AB} = \lim_{\Delta x_1 \to 0} \frac{\Delta x_1 + \frac{\partial u}{\partial x_1}\Delta x_1 - \Delta x_1}{\Delta x_1},$$

$$\Rightarrow \varepsilon_{11} = \frac{\partial u_1}{\partial x_1},$$

$$\varepsilon_{22} = \lim_{\Delta x_2 \to 0} \frac{A'D' - AD}{AD} = \lim_{\Delta x_2 \to 0} \frac{\Delta x_2 + \frac{\partial u_2}{\partial x_2}\Delta x_2 - \Delta x_2}{\Delta x_2},$$

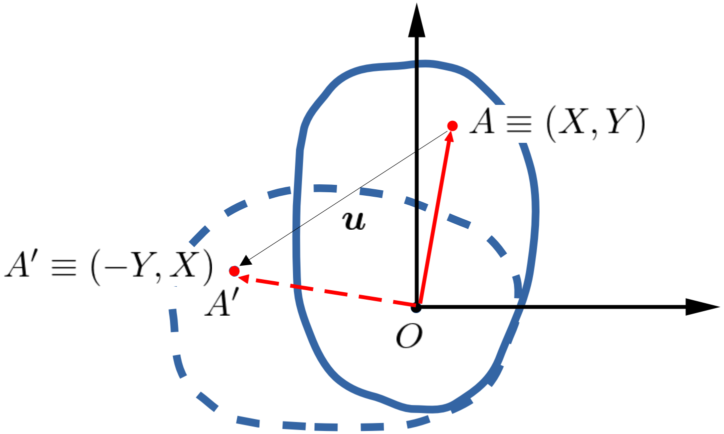
$$\Rightarrow \varepsilon_{22} = \frac{\partial u_2}{\partial x_2},$$

$$\gamma_{12} = \lim_{\substack{\Delta x_1 \to 0 \\ \Delta x_2 \to 0}} \frac{\pi}{2} - \angle B'A'D' = \alpha + \beta$$

$$\Rightarrow \gamma_{12} = \lim_{\substack{\Delta x_1 \to 0 \\ \Delta x_2 \to 0}} \frac{\frac{\partial u_2}{\partial x_1}\Delta x_1}{\Delta x_1 + \frac{\partial u_1}{\partial x_1}\Delta x_1} + \frac{\frac{\partial u_1}{\partial x_2}\Delta x_2}{\Delta x_2 + \frac{\partial u_2}{\partial x_2}\Delta x_2}$$

$$\Rightarrow \gamma_{12} \approx \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}.$$

What if displacements are large?



Consider a body which goes under a rotation of 90° in the anti-clockwise direction. During the rotation, the point A which had its coordinates as (X, Y) before the rotation, take a new position A' with its coordinates as (-Y,X) after the rotation.

Displacements of point A along X and Y directions are,

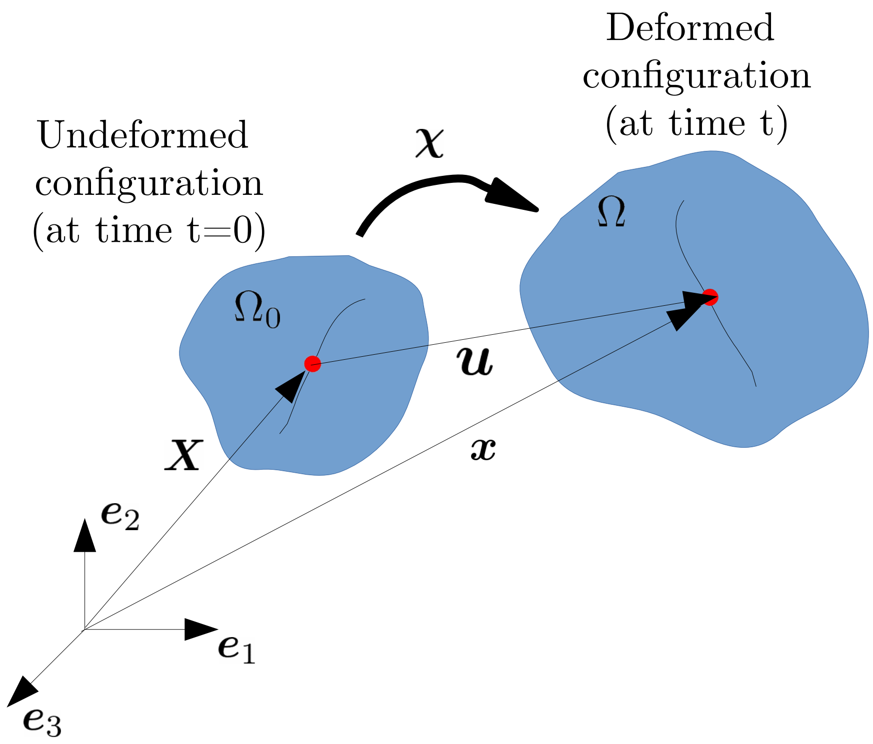
$$u_x = -Y - X \quad \text{and} \quad u_y = X - Y.$$

Let us determine the components of strain for this case as,

$$\varepsilon_x = \frac{\partial u_x}{\partial X} = -1, \quad \varepsilon_Y = \frac{\partial u_y}{\partial Y} = -1, \quad \text{and} \quad \gamma_{xy} = \frac{\partial u_x}{\partial Y} + \frac{\partial u_y}{\partial X} = 0.$$

Are this strains correct?

Motion and Deformation



Consider a body in its *undeformd* (or *initial*) configuration with a volume Ω_0. After time t, the body moves to a new position occupying a volume Ω in the space. The configuration at time t is called *deformed* (or *current*) configuration.

A function $\boldsymbol{\chi}$, which maps a point $\boldsymbol{X}$ in Ω_0 to another point $\boldsymbol{x}$ in Ω is called *motion* and the motion is assumed to be uniquely *invertible*, i.e.

$$\boldsymbol{x} = \chi(\boldsymbol{X}, t) \text{ and } \boldsymbol{X} = \chi^{-1}(\boldsymbol{x}, t).$$

It can be observed that, $\boldsymbol{x}(\boldsymbol{X}, t) = \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X}, t)$, where $\boldsymbol{u}$ is the *displacment vector*.

Lagrangian and Eulerian Description

Charecterization of motion (or any other quantity) with respect to material (or referential) coordinates X_i is called *material* or *Lagrangian* descrition. *Spatial* or *Eulerian* description is the characterized with respect to spatial (or current) coordinates x_i.

E.g. Displacement field in the Lagrantial form is $\boldsymbol{U}(\boldsymbol{X},t) = \boldsymbol{x}(\boldsymbol{X},t) - \boldsymbol{X}$, and Eulerian form of the displacement is $\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{x} - \boldsymbol{X}(\boldsymbol{x},t)$.

For a very simple motion given as,

$$
\begin{aligned}
x_1(X_1, X_2, X_3, t) &= X_1 + \gamma X_2,\\
x_2(X_1, X_2, X_3, t) &= X_2,\\
x_3(X_1, X_2, X_3, t) &= X_3,
\end{aligned}
$$

Material description of displacement is

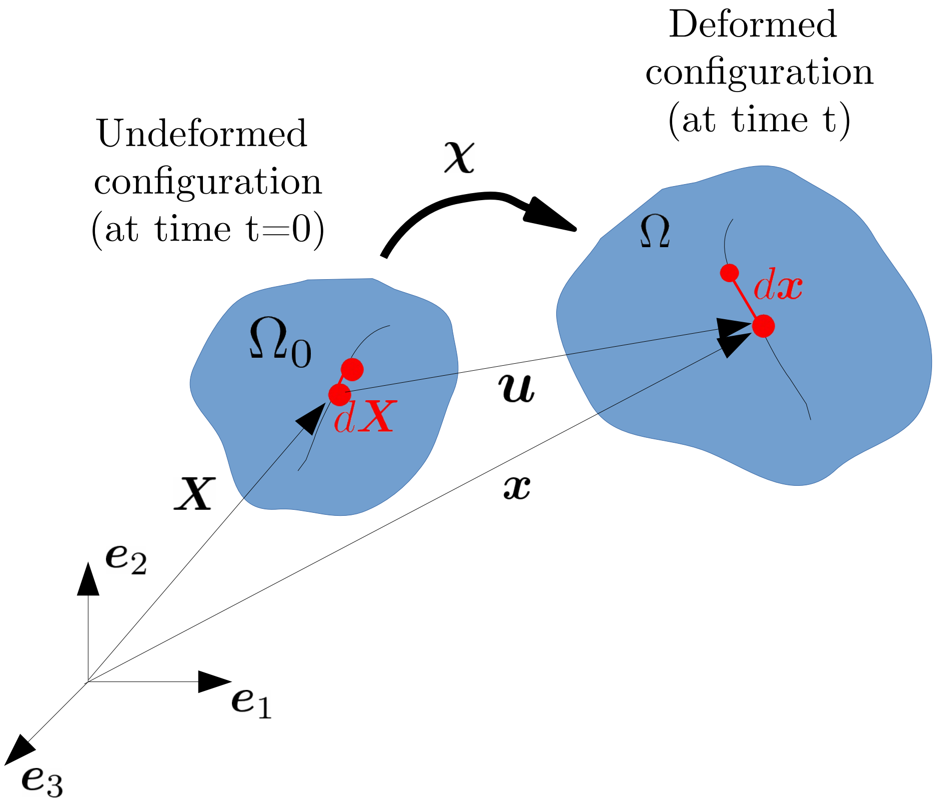
$$
\begin{aligned}
U_1(X_1, X_2, X_3, t) &= x_1 - X_1 = \gamma X_2,\\
U_2(X_1, X_2, X_3, t) &= 0,\\
U_3(X_1, X_2, X_3, t) &= 0,
\end{aligned}
$$

and special description of is

$$
\begin{aligned}
u_1(x_1, x_2, x_3, t) &= x_1 - X_1 = \gamma X_2 = \gamma x_2,\\
u_2(x_1, x_2, x_3, t) &= 0,\\
u_3(x_1, x_2, x_3, t) &= 0.
\end{aligned}
$$

It should be noticed that $\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{U}(\boldsymbol{X},t)$.

Deformation gradient



Consider a line element $d\boldsymbol{X}$ in Ω_0 which deforms to a line element $d\boldsymbol{x}$. *Deformation gradient* tensor maps the line element $d\boldsymbol{X}$ to $d\boldsymbol{x}$ as,

$$d\boldsymbol{x} = \boldsymbol{F} d\boldsymbol{X} \text{ or } dx_i = F_{iJ} dX_J,$$

$$\text{where, } \boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} \text{ or } F_{iJ} = \frac{\partial x_i}{\partial X_J},$$

$$\text{or } \boldsymbol{F} = \text{Grad } \boldsymbol{x} = \boldsymbol{\nabla}_X \boldsymbol{x}$$

is an invertible tensor. $\boldsymbol{F}^{-1}$ is the *inverse deformation gradient* tensor, defined as

$$\boldsymbol{F}^{-1} = \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}} \text{ or } F^{-1}_{Ij} = \frac{\partial X_I}{\partial x_j}.$$

$$\text{or } \boldsymbol{F}^{-1} = \text{grad } \boldsymbol{X} = \boldsymbol{\nabla} \boldsymbol{X}$$

It relates line elements as,

$$d\boldsymbol{X} = \boldsymbol{F}^{-1} d\boldsymbol{x} \text{ or } dX_I = F^{-1}_{Ij} dx_j.$$

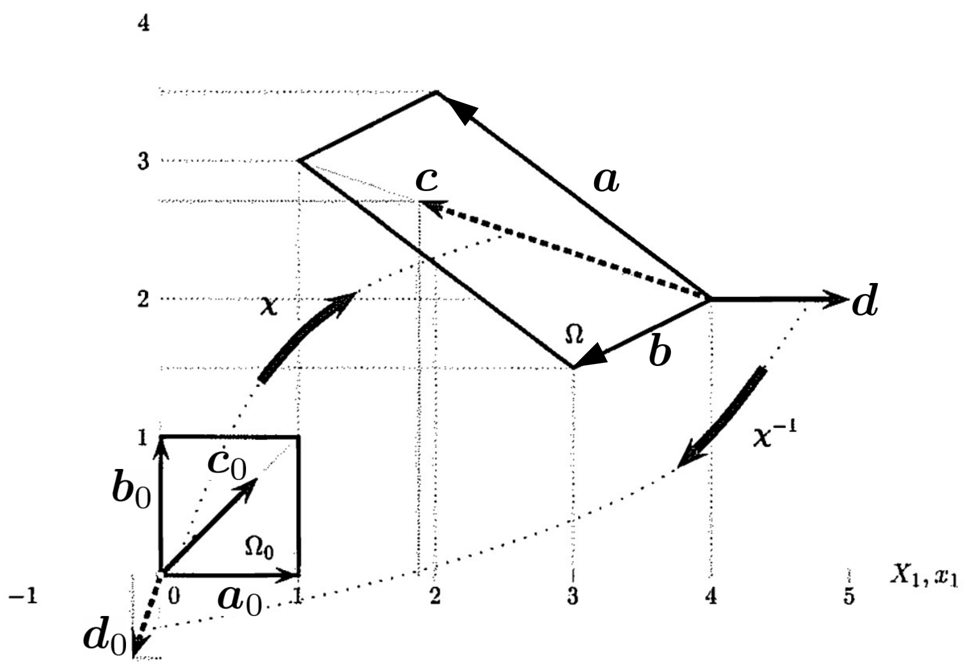
Example

Consider a two dimensional motion given by two equations as

$$x_1 = 4 - 2X_1 - X_2$$
$$x_2 = 2 + 1.5X_1 - 0.5X_2$$

Deformation gradient for the given motion is calculated as,

$$[\boldsymbol{F}] = \begin{bmatrix} -2 & -1 \\ 1.5 & -0.5 \end{bmatrix}, \text{ and } [\boldsymbol{F}]^{-1} = \frac{1}{5}\begin{bmatrix} -1 & 2 \\ -3 & -4 \end{bmatrix}$$



Now consider following vectors $\boldsymbol{a}_0 = [1,0]$, $\boldsymbol{b}_0=[0,1]$ and $\boldsymbol{c}_0=[0.707, 0.707]$ in the reference configuration. Current or deformed vectors can be obtained as $\boldsymbol{a} = \boldsymbol{F}\boldsymbol{a}_0$, $\boldsymbol{b} = \boldsymbol{F}\boldsymbol{b}_0$ and $\boldsymbol{c} = \boldsymbol{F}\boldsymbol{c}_0$.

Consider another vector $\boldsymbol{d}=[1,0]$ in the current configuration. Reference configuration $\boldsymbol{d}_0$ for vector d can be obtained as $\boldsymbol{d}_0 = \boldsymbol{F}^{1}\boldsymbol{d}$.