ME531: Advanced Mechanics of Solids

Motion, Strain and Stress

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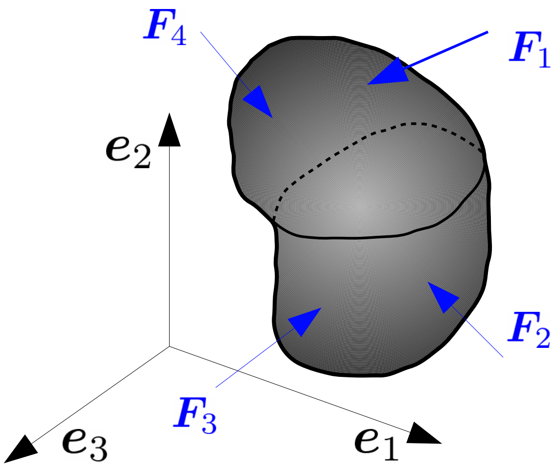
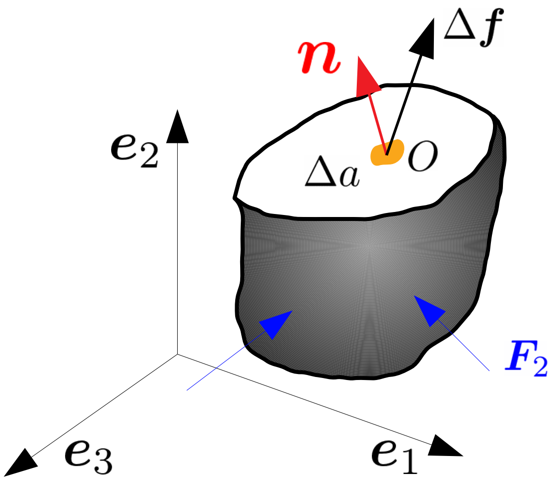
Stress





Consider a body in the deformed configuration acted upon by several external forces. Let the body be cut by an imaginary plane passing though point O.

Now, consider an elemental area ΔA in the neighborhood of point O. The reaction force on the area is $\Delta \boldsymbol{F}$, and normal to the area is $\boldsymbol{n}$.

We define a traction vector $\boldsymbol{t}$ corresponding to normal $\boldsymbol{n}$ and point O is defined as,

$$\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}) = \lim_{\Delta a \to 0} \frac{\Delta \boldsymbol{f}}{\Delta a} = \frac{d\boldsymbol{f}}{da}$$

where $\boldsymbol{t}(\boldsymbol{x},\boldsymbol{n})$ is called *Cauchy-traction vector* (force per unit surface area in the current configuration), exerted on the area da having normal $\boldsymbol{n}$. Traction vectors are also referred to as *surface traction*, *contact forces*, *stress vectors* or *loads*.

Cauchy's stress theorem

There exist a unique second-order tensor fields so that,

$$\boldsymbol{t}(\boldsymbol{x},\boldsymbol{n}) = \boldsymbol{n}\cdot\boldsymbol{\sigma}(\boldsymbol{x}), \quad \text{or} \quad \boldsymbol{t}(\boldsymbol{x},\boldsymbol{n}) = \boldsymbol{\sigma}^T(\boldsymbol{x})\cdot\boldsymbol{n}, \quad \text{or} \quad t_i = \sigma_{ji}n_j.$$

where $\boldsymbol{\sigma}$ is known as *Cauchy stress tensor.* This relation between the stress tensor and the traction vector is known as *Cauchy's stress theorem.* It states that if traction vector $\boldsymbol{t}$ depends upon the outward unit normal $\boldsymbol{n}$ then it must be linear in $\boldsymbol{n}$. It immediately follows that

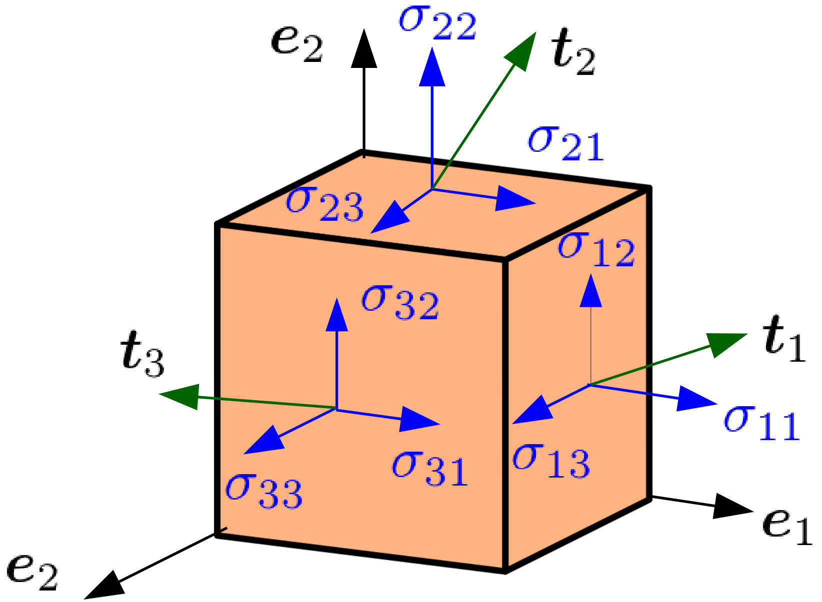
$$\boldsymbol{t}(\boldsymbol{x},\boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{x},-\boldsymbol{n}),$$

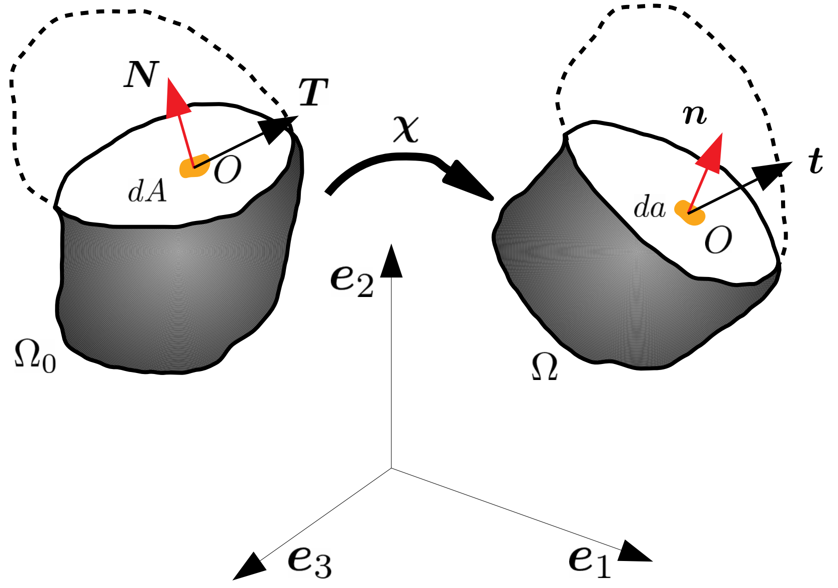
for all unit normal vectors n. This is known as *Newton's (third) law of action and reaction.*

Now, we find the traction vectors at planes having basis vectors $\{\boldsymbol{e}_i\}$ as normal,

$$\begin{aligned}
\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{e}_1) &= \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{e}_1 = \sigma_{11}\boldsymbol{e}_1 + \sigma_{12}\boldsymbol{e}_2 + \sigma_{13}\boldsymbol{e}_3, \\
\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{e}_2) &= \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{e}_2 = \sigma_{21}\boldsymbol{e}_1 + \sigma_{22}\boldsymbol{e}_2 + \sigma_{23}\boldsymbol{e}_3, \\
\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{e}_3) &= \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{e}_3 = \sigma_{31}\boldsymbol{e}_1 + \sigma_{32}\boldsymbol{e}_2 + \sigma_{33}\boldsymbol{e}_3.
\end{aligned}$$

State of stress at a point can be represented by considering an infinitesimal cubic element surrounding it as,





Now, consider a body under stress in its reference and current configuration. Traction vector in the current configuration and the corresponding normal is $\boldsymbol{t}$ and $\boldsymbol{n}$, respectively. Corresponding normal and (pseudo) traction vector is the reference configurations are $\boldsymbol{T}$ and $\boldsymbol{N}$ respectively.

$\boldsymbol{T}$ is known as *first Piola-Kirchhoff traction vector* and works in the same directions as $\boldsymbol{t}$. It should be noted that $\boldsymbol{T}$ does not describe the actual intensity as it works on the reference volume Ω_0 and is a function of reference position $\boldsymbol{X}$. Cauchy's stress theorem is also applicable for traction vector $\boldsymbol{T}(\boldsymbol{X},\boldsymbol{N})$ and normal vector $\boldsymbol{N}$, which is as follows,

$$\boldsymbol{T}(\boldsymbol{X},\boldsymbol{N}) = \boldsymbol{N} \cdot \boldsymbol{P}(\boldsymbol{X}), \quad \text{or} \quad T_i = P_{ji} N_j,$$

Here, $\boldsymbol{P}$ is a second order tensor and known as *first-Piola Kirchhoff stress tensor*.[37]

It should be noted that the force on the element area can be expressed as,

$$\mathrm{d}\boldsymbol{f} = \boldsymbol{T}\mathrm{d}A = \boldsymbol{t}\,\mathrm{d}a.$$

Above relation can be used to establish a relation between the Cauchy stress tensor, and the first Piola-Kirchhoff stress tensor. We use the Cauchy's stress theorem and write the previous equation as,

$$\boldsymbol{N}\cdot\boldsymbol{P}(\boldsymbol{X})\mathrm{d}A = \boldsymbol{n}\cdot\boldsymbol{\sigma}(\boldsymbol{x})\mathrm{d}a.$$

Using Nanson's relations, we write,

$$\boldsymbol{P}(\boldsymbol{X}) = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}(\boldsymbol{x}), \qquad P_{ij} = JF_{ik}^{-1}\sigma_{kj}.$$

We can also write,

$$\boldsymbol{\sigma} = J^{-1}\boldsymbol{F}\boldsymbol{P}, \qquad \sigma_{ij} = J^{-1}F_{ik}P_{kj}.$$

Later, we will prove that the Cauchy stress tensor is symmetric under the assumption of zero resultant couples. i.e.

$$\boldsymbol{\sigma} = J^{-1}\boldsymbol{F}\boldsymbol{P} = \boldsymbol{\sigma}^T, \qquad \sigma_{ij} = J^{-1}P_{ij} = F_{ik}\sigma_{kj},$$

which implies that $\boldsymbol{F}\boldsymbol{P} = \boldsymbol{P}^T\boldsymbol{F}^T$, which also suggests that the tensor $\boldsymbol{P}$ in general is not symmetric.

Example

Deformation of a body is described by,

$$x_1 = \text{-}6X_2,\ x_2 = 0.5X_1, \qquad x_3 = 1/3\ X_3.$$

The Cauchy stress tensor for certain part of the body is given by the matrix representation as,

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathrm{kg/cm^2}$$

Determine the Cauchy traction vector $\boldsymbol{t}$ and the first Piola-Kirchhoff traction vector $\boldsymbol{T}$ acting on a plane, which is characterized by the outward unit normal $\boldsymbol{n} = \boldsymbol{e}_2$ in the current configuration.

For the given deformation,

$$[\boldsymbol{F}] = \begin{bmatrix} 0 & -6 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}, \quad [\boldsymbol{F}]^{-1} = \begin{bmatrix} 0 & 2 & 0 \\ -1/6 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad \text{and } \det \boldsymbol{F} = 1.$$

The components for first Piola-Kirchoff stress tensor will be

$$[\boldsymbol{P}] = J[\boldsymbol{F}]^{-1}[\boldsymbol{\sigma}] = \begin{bmatrix} 0 & 100 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{kN/cm}^2.$$

Outward unit normal $\boldsymbol{N}$ in the reference configuration will be related to $\boldsymbol{n}$ by Nanson's formula as,

Thus, for $\boldsymbol{n} = \boldsymbol{e}_2$, $\boldsymbol{N} dS = J^{-1} \boldsymbol{F}^T \boldsymbol{n} ds \Rightarrow \boldsymbol{N} dS = \dfrac{\boldsymbol{e}_1}{2} ds.$

Hence, $\boldsymbol{N} = \boldsymbol{e}_1$ and $dS = ds/2$.

Finally, using Cauchy's stress theorem,

$$\{\boldsymbol{t}\} = [\boldsymbol{\sigma}]^T[\boldsymbol{n}] = \begin{bmatrix} 0 \\ 50 \\ 0 \end{bmatrix} \text{kN/cm}^2, \ \{\boldsymbol{T}\} = [\boldsymbol{P}]^T[\boldsymbol{N}] = \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix} \text{kN/cm}^2.$$

i.e. $\boldsymbol{t} = 50\boldsymbol{e}_2$ and $\boldsymbol{T} = 100\boldsymbol{e}_2$.

It can be observed that both $\boldsymbol{t}$ and $\boldsymbol{T}$ have same direction, but the magnitudes of $\boldsymbol{T}$ is twice that of $\boldsymbol{t}$, as deformed area is half the undeformed area.