For a spatial field ψ, describing a physical quantity of a particle per unit mass at time t integration over the volume of body Ω will be,

$$\bar{I}(t) = \int_{\Omega} \rho\psi(\boldsymbol{x}, t)dv.$$

Using the Reynolds' transport theorem we can write,

$$\frac{D}{Dt}\int_{\Omega} \rho\psi dv = \int_{\Omega} \left[\dot{\overline{\rho\psi}} + \rho\psi \mathrm{div}\boldsymbol{v}\right] dv, \Rightarrow \int_{\Omega} \left[\dot{\rho}\psi + \rho\dot{\psi} + \rho\psi \mathrm{div}\boldsymbol{v}\right] dv,$$

Now, from the mass continuity equation, we have,

$$\dot{\rho} + \rho \mathrm{div}\boldsymbol{v} = 0, \quad \Rightarrow \dot{\rho} = -\rho \mathrm{div}\boldsymbol{v}.$$

This implies,

$$\frac{D}{Dt}\int_{\Omega} \rho\psi dv = \int_{\Omega} \left[\rho\dot{\psi} - \rho\psi \mathrm{div}\boldsymbol{v} + \rho\psi \mathrm{div}\boldsymbol{v}\right] dv = \int_{\Omega} \rho\dot{\psi} dv,$$

$$\boxed{\frac{D}{Dt}\int_{\Omega} \rho(\boldsymbol{x}, t)\psi(\boldsymbol{x}, t)dv = \int_{\Omega} \rho(\boldsymbol{x}, t)\dot{\psi}(\boldsymbol{x}, t)dv.}$$

Linear and Angular Momentum

The **total linear momentum** of a continuum body (closed system) occupying a region Ω in the space is defined as,

$$\boxed{\boldsymbol{L}(t) = \int_{\Omega} \rho(\boldsymbol{x}, t)\boldsymbol{v}(\boldsymbol{x}, t)dv = \int_{\Omega_0} \rho_0(\boldsymbol{X})\boldsymbol{V}(\boldsymbol{X}, t)dV,}$$

where ρ and ρ_0 are spatial and material density, $\boldsymbol{v}$ and $\boldsymbol{V}$ are spatial and material velocity field.

The *total angular momentum* relative to a fixed point (whose position vector is $\boldsymbol{x}_0$) is defined as,

$$\boxed{\boldsymbol{J}(t) = \int_{\Omega} \boldsymbol{r} \times \rho(\boldsymbol{x}, t)\boldsymbol{v}(\boldsymbol{x}, t)dv = \int_{\Omega_0} \boldsymbol{r} \times \rho_0(\boldsymbol{X})\boldsymbol{V}(\boldsymbol{X}, t)dV,}$$

where $\boldsymbol{r}(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{x}_0$.

Angular momentum is also referred as **moment of momentum** or **the rotational momentum**.

Momentum balance principle

Total time derivative of linear and angular momentum of a continuum body results in following momentum balance principles.

The balance of linear momentum balance is,

$$\boxed{\dot{\boldsymbol{L}}(t) = \frac{D}{Dt}\int_{\Omega} \rho(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)dv = \frac{D}{Dt}\int_{\Omega_0} \rho_0(\boldsymbol{x})\boldsymbol{V}(\boldsymbol{X},t)dV = \boldsymbol{F}(t),}$$

where $\boldsymbol{F}$(t) is the resultant force on the body.

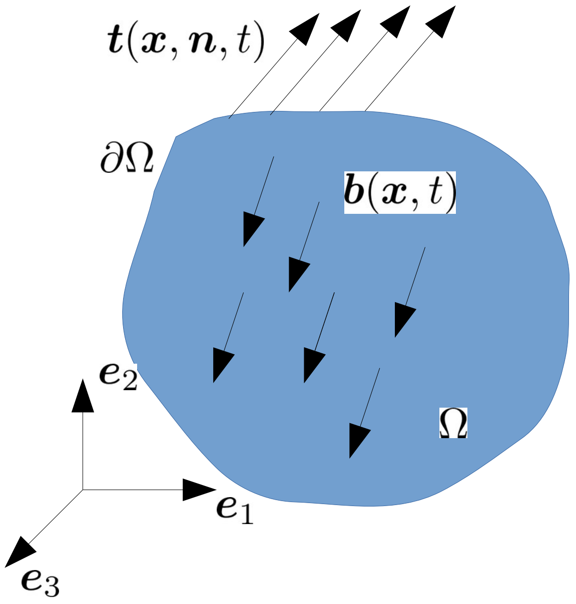
The balance of angular momentum balance is,

$$\boxed{\dot{\boldsymbol{J}}(t) = \frac{D}{Dt}\int_{\Omega} \boldsymbol{r}\times\rho(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)dv = \frac{D}{Dt}\int_{\Omega_0} \boldsymbol{r}\times\rho_0(\boldsymbol{X})\boldsymbol{V}(\boldsymbol{X},t)dV = \boldsymbol{M}(t),}$$

where $\boldsymbol{M}(t)$ is the resultant moment about $\boldsymbol{x}_0$.

Using Reynolds' transport theorem, alternate form of the momentum balance principles can be written as follows,

$$\boxed{\begin{aligned}\dot{\boldsymbol{L}}(t) &= \int_{\Omega} \rho \dot{\boldsymbol{v}} dv = \int_{\Omega_0} \rho_0 \dot{\boldsymbol{V}} dV = \boldsymbol{F}(t), \\ \dot{\boldsymbol{J}}(t) &= \int_{\Omega} \boldsymbol{r} \times \rho \dot{\boldsymbol{v}} dv = \int_{\Omega_0} \boldsymbol{r} \times \rho_0 \dot{\boldsymbol{V}} dV = \boldsymbol{M}(t).\end{aligned}}$$



Consider a body in the current configuration with a volume Ω and the surface area as $\partial\Omega$. $\boldsymbol{t}(\boldsymbol{x},\boldsymbol{n},t)$ is the Cauchy traction vector and $\boldsymbol{b}(\boldsymbol{x},t)$ is a spatial vector field called body force.

Now, resultant force $\boldsymbol{F}(t)$ and the moment $\boldsymbol{M}(t)$ on the body will be given as,

$$\boldsymbol{F}(t) = \int_{\partial\Omega} \boldsymbol{t} ds + \int_{\Omega} \boldsymbol{b} dv, \text{ and}$$

$$\boldsymbol{M}(t) = \int_{\partial\Omega} \boldsymbol{r} \times \boldsymbol{t} ds + \int_{\Omega} \boldsymbol{r} \times \boldsymbol{b} dv.$$

Now, the global form of linear and angular momentum balance in spatial description can be written as,

$$\boxed{\begin{aligned} \dot{\boldsymbol{L}} &= \frac{D}{Dt}\int_{\Omega} \rho \boldsymbol{v} dv = \int_{\Omega} \rho \dot{\boldsymbol{v}} dv = \int_{\partial\Omega} \boldsymbol{t} ds + \int_{\Omega} \boldsymbol{b} dv, \\ \dot{\boldsymbol{J}} &= \frac{D}{Dt}\int_{\Omega} \boldsymbol{r} \times \rho \boldsymbol{v} dv = \int_{\Omega} \boldsymbol{r} \times \rho \dot{\boldsymbol{v}} dv = \int_{\partial\Omega} \boldsymbol{r} \times \boldsymbol{t} ds + \int_{\Omega} \boldsymbol{r} \times \boldsymbol{b} dv. \end{aligned}}$$

These are fundamental equations in the continuum mechanics.

Note that for balance of angular momentum we have assumed that the distributed resultant couples are neglected.

To obtain the expression for material description of momentum balance, we define reference body forces $\boldsymbol{B}(\boldsymbol{X},t)$ and its related with the body force $\boldsymbol{b}(\boldsymbol{x},t)$ in the following manner,

$$\int_{\Omega} \boldsymbol{b}(\boldsymbol{x},t)dv = \int_{\Omega_0} \boldsymbol{b}(\boldsymbol{\chi}(\boldsymbol{X},t),t)J(\boldsymbol{X},t)dV = \int_{\Omega_0} \boldsymbol{B}(\boldsymbol{X},t)dV,$$

or in the local form,

$$\boldsymbol{B}(\boldsymbol{X},t) = \boldsymbol{b}(\boldsymbol{x},t)J(\boldsymbol{X},t), \quad \text{or} \quad B_i = Jb_i.$$

Now, the linear and angular momentum balance principle in material co-ordinates can be written as,

$$\boxed{\begin{gathered} \frac{D}{Dt}\int_{\Omega_0} \rho_0 \boldsymbol{V}\, dV = \int_{\Omega_0} \rho_0 \dot{\boldsymbol{V}}\, dV = \int_{\partial\Omega_0} \boldsymbol{T} dS + \int_{\Omega_0} \boldsymbol{B} dV, \\ \frac{D}{Dt}\int_{\Omega_0} \boldsymbol{r} \times \rho_0 \boldsymbol{V}\, dV = \int_{\Omega_0} \boldsymbol{r} \times \rho_0 \dot{\boldsymbol{V}}\, dV = \int_{\partial\Omega_0} \boldsymbol{r} \times \boldsymbol{T} dS + \int_{\Omega_0} \boldsymbol{r} \times \boldsymbol{B} dV. \end{gathered}}$$

where, $\boldsymbol{T}(\boldsymbol{X},\boldsymbol{N},t)$ is the Piola-Kirchoff traction vector.

Equation of motion

Consider the spatial form of linear momentum balance equation,

$$\int_{\Omega} \rho \dot{\boldsymbol{v}} dv = \int_{\partial\Omega} \boldsymbol{t} ds + \int_{\Omega} \boldsymbol{b} dv.$$

By using Cauchy's stress theorem, we can write,

$$\int_{\Omega} \rho \dot{\boldsymbol{v}} dv = \int_{\partial\Omega} \boldsymbol{n} \cdot \boldsymbol{\sigma} ds + \int_{\Omega} \boldsymbol{b} dv, \quad \text{or} \quad \int_{\Omega} \rho \dot{v}_i dv = \int_{\partial\Omega} \sigma_{ji} n_j ds + \int_{\Omega} b_i dv.$$

Applying Gauss-divergence theorem,

$$\int_{\Omega} \rho \dot{\boldsymbol{v}} dv = \int_{\Omega} (\mathrm{div}\boldsymbol{\sigma} + \boldsymbol{b})\, dv, \quad \text{or} \quad \int_{\Omega} \rho \dot{v}_i dv = \int_{\Omega} (\sigma_{ji,j} + b_i)\, dv.$$

$$\Rightarrow \boxed{\int_{\Omega} (\mathrm{div}\boldsymbol{\sigma} + \boldsymbol{b} - \rho \dot{\boldsymbol{v}})\, dv = 0, \quad \text{or} \quad \int_{\Omega} (\sigma_{ji,j} + b_i - \rho \dot{v}_i)\, dv = 0.}$$

Above equation is known as **Cauchy's equation (or first equation) of motion** in the global form.

By applying the localization theorem as v is an arbitrary volume in region Ω local form is

$$\boxed{\mathrm{div}\boldsymbol{\sigma} + \boldsymbol{b} = \rho\dot{\boldsymbol{v}}, \quad \text{or} \quad \sigma_{ji,j} + b_i = \rho\dot{v}_i,}$$

for each point $\boldsymbol{x}$ of v at all time t.

If acceleration if assumed to be zero (i.e. constant velocity) then,

$$\boxed{\mathrm{div}\boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{O}, \quad \text{or} \quad \sigma_{ji,j} + b_i = 0,}$$

which is known as **Cauchy's equation of equilibrium.**

In the absence of body forces equilibrium equation become,

$$\mathrm{div}\boldsymbol{\sigma} = 0, \quad \text{or} \quad \sigma_{ji,j} = 0.$$

A spatial stress field satisfying the above equation is said to be **self-equilibrated.**

For solid bodies it is more convenient to work in material coordinates. Hence, material description of Cauchy's equation of motion is,

$$\boxed{\int_{\Omega_0} \left(\mathrm{div}\boldsymbol{P} + \boldsymbol{B} - \rho_0 \dot{\boldsymbol{V}}\right) dV = 0, \quad \text{or} \quad \int_{\Omega_0} \left(P_{ji,j} + B_i - \rho_0 \dot{V}_i\right) dV = 0.}$$

Local form of the above equation is,

$$\boxed{\mathrm{div}\boldsymbol{P} + \boldsymbol{B} = \rho_0 \dot{\boldsymbol{V}}, \quad \text{or} \quad P_{ji,j} + B_i = \rho_0 \dot{V}_i.}$$

For motions having zero acceleration,

$$\mathrm{div}\boldsymbol{P} + \boldsymbol{B} = \boldsymbol{O}, \quad \text{or} \quad P_{ji,j} + B_i = 0.$$

In the absence of body forces, $\mathrm{div}\boldsymbol{P} = \boldsymbol{O}$, or $P_{ji,j} = 0$, which is known as **Piola Identity.**

Symmetry of Cauchy's stress tensor

Start with the spatial form of angular momentum balance equation,

$$\int_\Omega \boldsymbol{r}\times\rho\dot{\boldsymbol{v}}dv = \int_{\partial\Omega}\boldsymbol{r}\times\boldsymbol{t}ds + \int_\Omega \boldsymbol{r}\times\boldsymbol{b}dv, \text{ or } \int_\Omega e_{ijk}\rho r_i\dot{v}_j dv = \int_{\partial\Omega} e_{ijk}r_i t_j ds + \int_\Omega e_{ijk}r_i b_j dv,$$

With the application of Cauchy's stress theorem and then Gauss-divergence theorem,

$$\int_\Omega e_{ijk}\rho r_i\dot{v}_j dv = \int_\Omega \left[e_{ijk}(r_i\sigma_{pj})_{,p} + e_{ijk}r_i b_j\right] dv, \qquad (r_i = x_i - x_{0i})$$

$$\Rightarrow \int_\Omega e_{ijk}\rho r_i\dot{v}_j dv = \int_\Omega \left[e_{ijk}\left(r_i\sigma_{pj,p} + r_{i,p}\sigma_{pj}\right) + e_{ijk}r_i b_j\right] dv,$$

$$\Rightarrow \int_\Omega e_{ijk}\rho r_i\dot{v}_j dv = \int_\Omega \left[e_{ijk}\left(r_i\sigma_{pj,p} + \sigma_{ij}\right) + e_{ijk}r_i b_j\right] dv, \qquad (r_{i,p} = x_{i,p} = \delta_{ip})$$

$$\Rightarrow \int_\Omega e_{ijk}r_i\left(\sigma_{pj,p} + b_j - \rho\dot{v}_j\right) + e_{ijk}\sigma_{ij} dv = 0,$$

From the balance of linear momentum,

$$\sigma_{pj,p} + b_j - \rho \dot{v}_j = 0,$$

which implies

$$\int_{\Omega} e_{ijk}\sigma_{ij} dv = 0,$$

where v is an arbitrary volume, hence, $e_{ijk}\sigma_{ij} = 0$, which results in following equations,

$$\sigma_{12} - \sigma_{21} = 0,\ \sigma_{23} - \sigma_{32} = 0,\ \text{and}\ \sigma_{13} - \sigma_{31} = 0.$$

This is only possible when, $\boxed{\boldsymbol{\sigma} = \boldsymbol{\sigma}^T,\ \text{or}\ \sigma_{ij} = \sigma_{ji}.}$

This is an important result from the local form of balance of angular momentum, often refereed as **Cauchy's second equation of motion.**

Symmetry of Cauchy stress implied the symmetry of Kirchoff stress and Second Piola-Kirchoff stress; whereas First Piola-Kirchoff is not symmetric.

Note that symmetric property of Cauchy stress tensor does not hold if distributed resultant couples are not neglected while writing balance of angular momentum.

Balance of mechanical energy

We will be considering only the balance of mechanical energy. Other forms of energy are neglected in the present context.

The **external mechanical power** or the **rate of external work** is defined as power input on a region Ω at time t done by the system of forces $(\boldsymbol{t}, \boldsymbol{b})$, i.e.,

$$\mathcal{P}_{\text{ext}} = \int_{\partial\Omega} \boldsymbol{t} \cdot \boldsymbol{v} ds + \int_{\Omega} \boldsymbol{b} \cdot \boldsymbol{v} dv, \quad \text{or} \quad \mathcal{P}_{\text{ext}} = \int_{\partial\Omega} t_i v_i ds + \int_{\Omega} b_i v_i dv.$$

Here $\boldsymbol{v}$ is the spatial velocity field. The scalar quantities $t_i v_i$ and $b_i v_i$ give the external mechanical power per unit current surface s and, current volume v, respectively.

Kinetic energy of the body is defined as,

$$\mathcal{K}(t) = \int_{\Omega} \frac{1}{2} \rho \boldsymbol{v}^2 dv = \int_{\Omega} \frac{1}{2} \rho \boldsymbol{v} \cdot \boldsymbol{v} dv, \quad \text{or} \quad \mathcal{K}(t) = \int_{\Omega} \frac{1}{2} \rho v_i v_i dv.$$

The **stress power** or the **rate of internal mechanical work** by the stress field is defined as,

$$\mathcal{P}_{\text{int}} = \int_{\partial\Omega} \boldsymbol{\sigma} : \boldsymbol{d} dv \quad \text{or} \quad \mathcal{P}_{\text{int}} = \int_{\partial\Omega} \sigma_{ij} d_{ij} dv.$$

For the **rigid body motion stress power is zero**, since the rate of deformation tensor vanishes.

Balance of mechanical energy (or theorem of power expended) states,

$$\frac{D}{Dt}\mathcal{K}(t) + \mathcal{P}_{\text{int}}(t) = \mathcal{P}_{\text{ext}}(t), \quad \text{or}$$

$$\frac{D}{Dt}\int_{\Omega} \frac{1}{2}\rho \boldsymbol{v}^2 dv + \int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{d} dv = \int_{\partial\Omega} \boldsymbol{t} \cdot \boldsymbol{v} ds + \int_{\Omega} \boldsymbol{b} \cdot \boldsymbol{v} dv.$$

the rate of change of kinetic energy of a mechanical system +

the rate of internal mechanical work (stress-power) done by internal stresses =

the rate of external mechanical work done on the system by surface traction and body forces.

Proof:

We starts with the term, $\int_{\partial\Omega} \boldsymbol{t} \cdot \boldsymbol{v} ds$ or $\int_{\partial\Omega} t_i v_i ds.$

Using Cauchy's equation and applying Gauss-divergence theorem we can write,

$$\int_{\partial\Omega} t_i v_i ds = \int_{\partial\Omega} \sigma_{ij} n_j v_i ds = \int_{\Omega} \left(\sigma_{ij} v_i\right)_{,j} dv$$

$$\Rightarrow \int_{\Omega} \left(\sigma_{ij} v_i\right)_{,j} dv = \int_{\Omega} \sigma_{ij,j} v_i + \sigma_{ij} v_{i,j} dv.$$

Now, external power can be written as,

$$\mathcal{P}_{\text{ext}} = \int_{\partial\Omega} t_i v_i ds + \int_{\Omega} b_i v_i dv = \int_{\Omega} \left(\sigma_{ij,j} v_i + \sigma_{ij} v_{i,j} + b_i v_i\right) dv,$$

$$\Rightarrow \int_{\Omega} \left[\left(\sigma_{ij,j} + b_i\right) v_i + \sigma_{ij} v_{i,j}\right] dv,$$

$$\Rightarrow \int_{\Omega} \left[\rho \dot{v}_i v_i + \sigma_{ij} v_{i,j}\right] dv, \qquad \text{(From linear momentum balance)}$$

$$\mathcal{P}_{\text{ext}} = \int_\Omega \left[\rho \dot{v}_i v_i + \sigma_{ij} v_{i,j}\right] dv = \int_\Omega \left[\rho \dot{v}_i v_i + \sigma_{ij} l_{ij}\right] dv = \int_\Omega \left[\rho \dot{v}_i v_i + \sigma_{ij}\left(d_{ij} + w_{ij}\right)\right] dv,$$

$\boldsymbol{\sigma}$ being a symmetric tensor and $\boldsymbol{w}$ being an anti-symmetric tensor, $\sigma_{ij} w_{ij} = 0$. Thus,

$$\mathcal{P}_{\text{ext}} = \int_\Omega \left[\rho \dot{v}_i v_i + \sigma_{ij} d_{ij}\right] dv,$$

which means that spin tensor $\boldsymbol{w}$ does not contribute to the rate of work. Above can be written in the form,

$$\mathcal{P}_{\text{ext}} = \frac{D}{Dt}\int_\Omega \frac{\rho}{2} v_i v_i dv + \int_\Omega \sigma_{ij} d_{ij} dv = \frac{D}{Dt}\underbrace{\boxed{\int_\Omega \frac{\rho}{2} \boldsymbol{v}\cdot\boldsymbol{v} dv}}_{\mathcal{K}(t)} + \underbrace{\boxed{\int_\Omega \boldsymbol{\sigma} : \boldsymbol{d} dv,}}_{\mathcal{P}_{\text{int}}}$$

which is the RHS of the theorem.