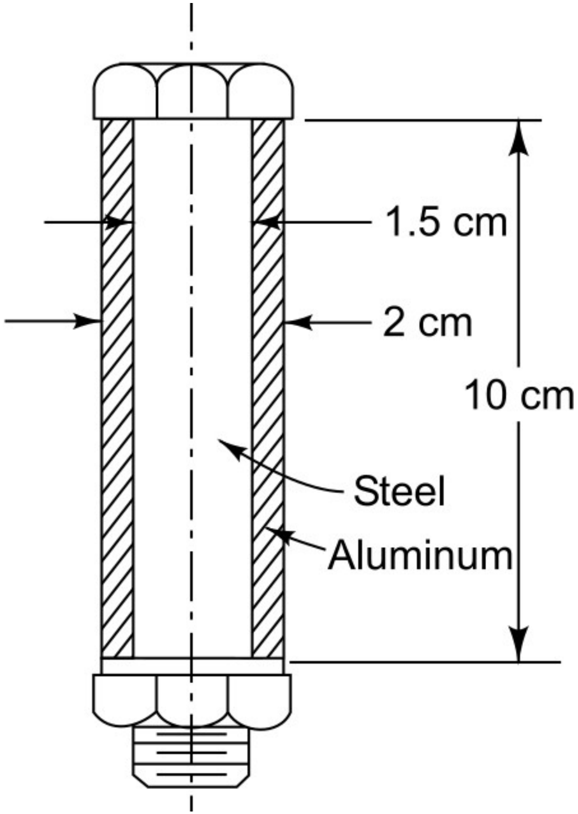
Stress, Strain and Temperature relationship

Example 2

A steel bolt and nut and an aluminum sleeve is shown. The bolt has 6 threads per cm and, when the material is at 60°F, the nut is tightened one-quarter turn. The temperature is then raised from 60 to 100°F. Determine the stresses in both bolt and sleeve.



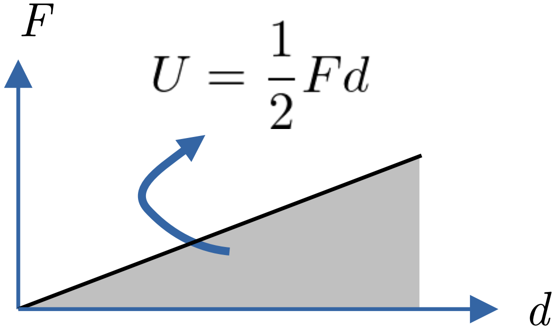
Strain energy in an elastic body

We will understand the strain energy in linear elastic bodies subjected to small deformation.

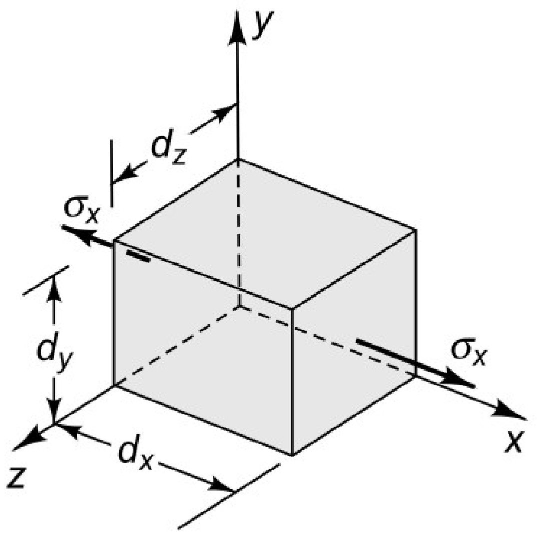
The elastic energy U stored in a linear spring is given in three forms:

- in terms of the deflection d,
- in terms of the force F, or
- in terms of the deflection d and the force F.

Because of the linearity, force and deflection grow in proportion during the loading process, and thus the total work done is just one-half the product of the final force and the final deflection.

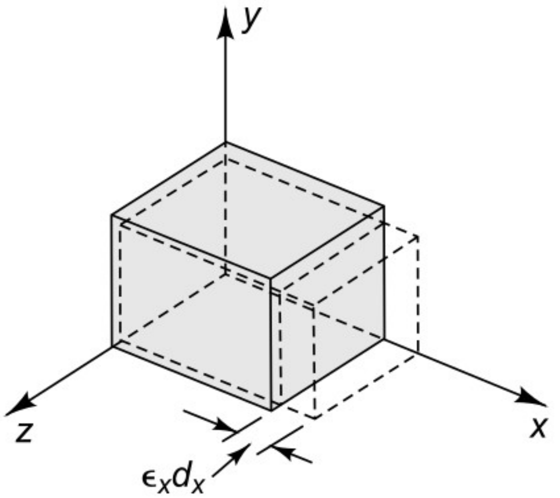


We apply this concept to an infinitesimal element of a linear elastic body. Figure shows a uniaxial stress component σ_x and corresponding deformation for a cuboidal element. The elastic energy stored in such an element is commonly called strain energy.
In this case the force $\sigma_x\ dy\ dz$ acting on the positive x face does work as the element undergoes the elongation $\varepsilon_x\ dx$.
As strain is proportional to the stress, the strain energy dU stored in the element, is



$$dU = \frac{1}{2}(\sigma_x dydz)(\varepsilon_x dx) = \frac{1}{2}\sigma_x \epsilon_x dV.$$

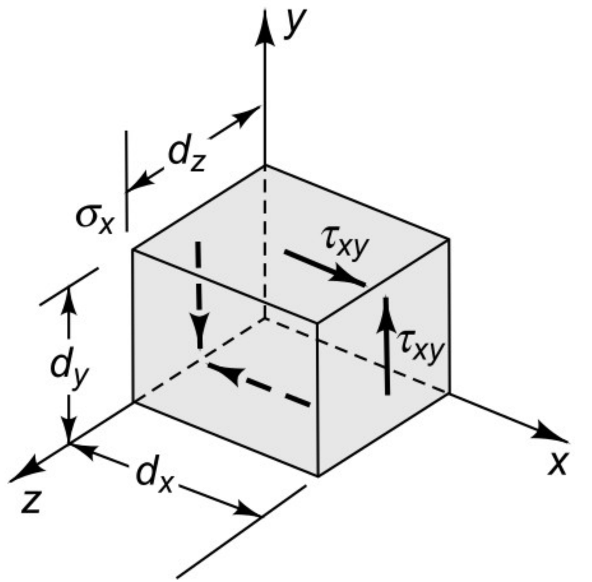
For a linear uniaxial member stress and strain can be taken to be uniform through the volume of the member.
Thus, $\sigma_x = \mathrm{P/A}$ and $\varepsilon_x = \delta/L$, then we have the total strain energy as,



$$U = \int dU = \int_V \frac{1}{2}\sigma_x \epsilon_x dV = \sigma_x \epsilon_x V = \frac{P}{A}\frac{\delta}{L}(AL) = \frac{1}{2}P\delta,$$

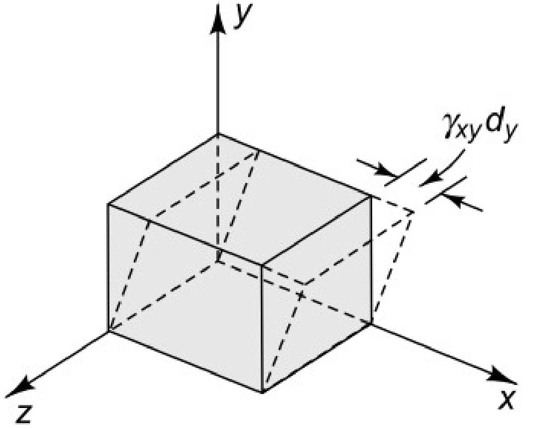
which is same as the relation for the stored energy in a linear elastic spring.

Now, consider the shear-stress component τ_{xy} acting on the infinitesimal element. The corresponding deformation due to the shear-strain component γ_{xy} . In this case the force τ_{xy} dx dz acting on the positive y face does work as that face translates through the distance γ_{xy} dy. As τ_{xy} and γ_{xy} are proportional to each other, the strain energy stored in the element is,



$$dU = \frac{1}{2}(\tau_{xy}dxdz)(\gamma_{xy}dy) = \frac{1}{2}\tau_{xy}\gamma_{xy}dV.$$

Similar expressions will be valid for other components of stresses and strains.



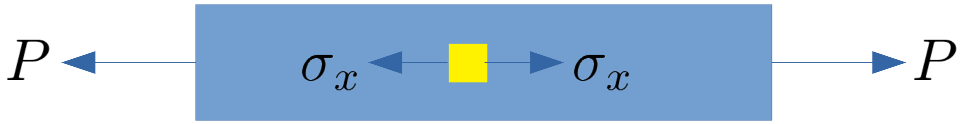
Finally, for an element with all components of stress and stress present working on it, we can write the strain energy stored in the element as,

$$dU = \frac{1}{2}\left[\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right]dV.$$

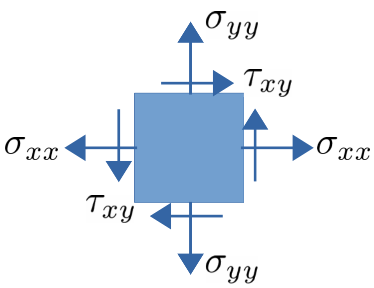
The total energy is then $U = \int_V dU.$

Theories of failure

Structural elements and machine components, when subjected to uni-axial loading will yield/break at the yield/ultimate stress of the material.



When structural elements and machine components are subjected to multi-axial loading, state of stress is different from the uni-axial condition. Yielding/failure in such cases can not be predicted from uni-axial tests. We need to develop some criteria to predict actual failure considering multiple stress components.



Yield criteria for ductile materials

Maximum shear stress criterion

This criterion is based on the observations that the yield in ductile materials is caused by slippage of material along oblique planes, which is primarily due to shear stresses.

As per this criterion a component is safe when

maximum shear stress in the component	$\leq$	Maximum shear stress in tensile specimen during yield

$$\frac{|\sigma_{\max} - \sigma_{\min}|}{2} \leq \frac{\sigma_y}{2}$$

If principal stresses are $\sigma_1 > \sigma_2 > \sigma_3$, then

$$|\sigma_1 - \sigma_3| \leq \sigma_y$$

Maximum shear stress criterion

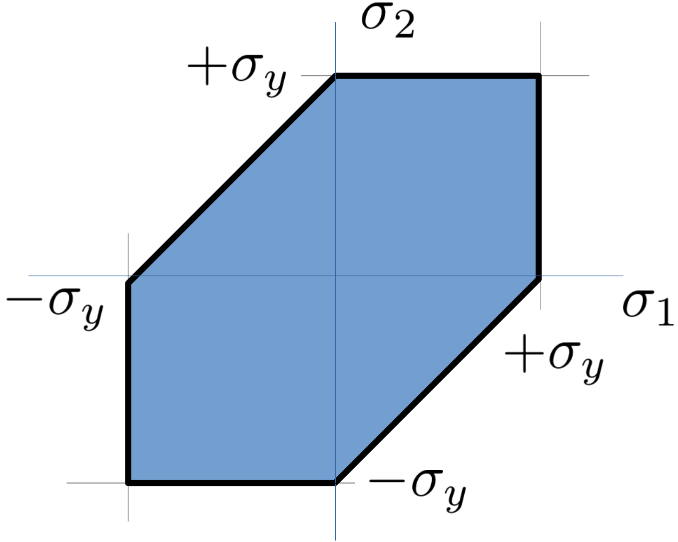
For plane stress when $\sigma_3 = 0$,

if $\sigma_1 > 0$ and $\sigma_2 < 0$ then $|\sigma_1 - \sigma_2| \leq \sigma_y$

if $\sigma_1 > \sigma_2 > 0$, then $|\sigma_1 - 0| \leq \sigma_y \Rightarrow |\sigma_1| \leq \sigma_y$

if $\sigma_2 < \sigma_1 < 0$, then $|0 - (-\sigma_2)| \leq \sigma_y \Rightarrow |\sigma_2| \leq \sigma_y$

These relations when represented graphically



All state of stresses for which (σ_1, σ_2) point fall inside the shaded area structure remain safe; whereas when the point fall outside it fails.

The hexagon associated with the initiation of yield is known as *Tresca's hexagon* after the French engineer Henri Edouard Tresca and this criteria is aka *Tresca's criterion.*

von Mises yield criterion

This criterion (after the German-American applied mathematician Richard von Mises) suggest that for a three dimensional state of stress, when principal stresses are σ_1, σ_2, σ_3, then yielding occurs when von Mises stress in three dimensional condition is less than or equal to the corresponding von Mises stress in uniaxial stress during yield.

In terms of principal stresses von Mises stress is given as

$$\sigma_e = \sqrt{\frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]}$$

In terms of stresses in *xy*-coordinates, von Mises stress is

$$\sigma_e = \sqrt{\frac{1}{3}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 3\tau_{xy}^2 + 3\tau_{yz}^2 + 3\tau_{xz}^2\right]}$$

von Mises yield criterion

According to the criterion structure remain safe when,

$$\sqrt{\frac{1}{3}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]} \leq \sqrt{\frac{2}{3}\sigma_Y^2}$$

or

$$\sqrt{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]} \leq \sigma_Y$$

or

$$\sqrt{\frac{1}{3}\left[(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+3\tau_{xy}^2+3\tau_{yz}^2+3\tau_{xz}^2\right]} \leq \sqrt{\frac{2}{3}}\sigma_Y$$

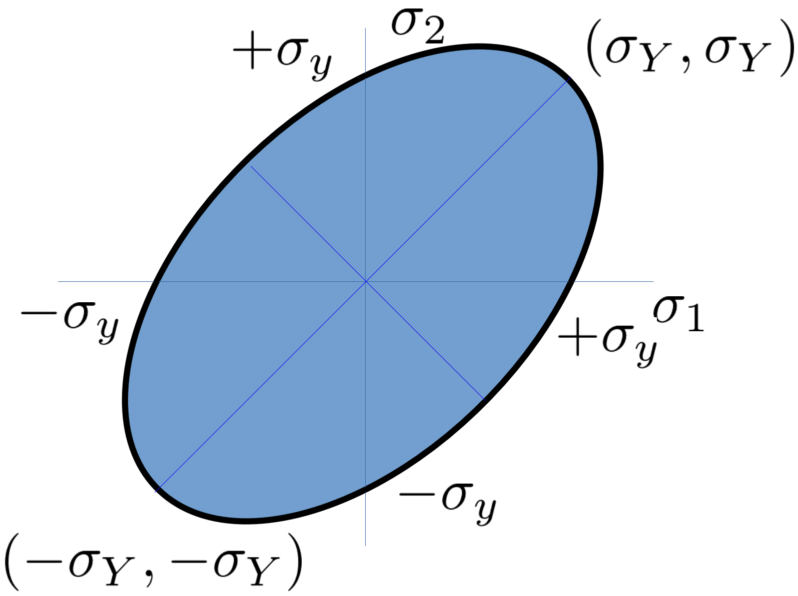
or

$$\sqrt{\frac{1}{2}\left[(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+3\tau_{xy}^2+3\tau_{yz}^2+3\tau_{xz}^2\right]} \leq \sigma_Y$$

von Mises yield criteria

For plane stress when $\sigma_3 = 0$, criterion becomes

$$\sqrt{\frac{1}{2}\left(\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2\right)} \leq \sigma_Y \quad \text{or} \quad \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 \leq 2\sigma_Y^2$$



Again, all state of stresses for which (σ_1, σ_2) point fall inside the shaded area structure remain safe; for the point falling outside it fails.

This criterion is also known as *Distortion energy criterion* or *Octahedral shear stress criterion.*

Comparison of both yield criteria