Introduction to Tensors

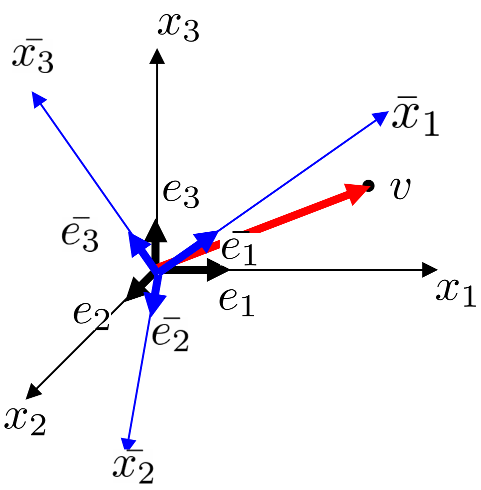
Transformation laws of tensors

- Through a tensor is invarient with respect to coordinate system, its components change if the coordinate system changes.

- Consider two coordinate system denoted by x_i and $\bar{x}_i$ with base vectors $\boldsymbol{e}_i$ and $\bar{e}_i$, respectively.

- As shown in the figure the components of a vector $\boldsymbol{v}$ are $v_i = \boldsymbol{e}_i \cdot \boldsymbol{v}$ in the first system, and $\bar{v}_i = \bar{e}_i \cdot \boldsymbol{v}$ in the second system.

- Similarly, the components of a tensor $\boldsymbol{A}$ is $A_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{A}\boldsymbol{e}_j$ in the first system and $\bar{A}$ is $\bar{A}_{ij} = \bar{e}_i \cdot \boldsymbol{A}\bar{e}_j$ in the second system. We will derive the transformation laws for a tensor, i.e. the relationship between the components A_{ij} and $\bar{A}_{ij}$.



Transformation laws of tensors

Let $\boldsymbol{Q}$ be an orthogonal tensor and the components Q_{ij} of tensor are defined as,

$$\boldsymbol{Q} = \cos(\boldsymbol{e}_i, \bar{\boldsymbol{e}}_j)$$

Tensor $\boldsymbol{Q}$ transforms the basis vectors $\boldsymbol{e}_i$ to $\bar{\boldsymbol{e}}_i$ i.e.,

$$\bar{\boldsymbol{e}}_i = \boldsymbol{Q}\boldsymbol{e}_i, \text{ and } \boldsymbol{e}_i = \boldsymbol{Q}^T\bar{\boldsymbol{e}}_i.$$

Consider a vector $\boldsymbol{u}$, which can be written as,

$$\boldsymbol{u} = u_i\boldsymbol{e}_i = \bar{u}_i\bar{\boldsymbol{e}}_i.$$

Using the relation between $\boldsymbol{e}_i$ and $\bar{\boldsymbol{e}}_i$, we can write,

$$\begin{aligned}
\bar{u}_i &= \boldsymbol{u} \cdot \bar{\boldsymbol{e}}_i = \boldsymbol{u} \cdot \boldsymbol{Q}\boldsymbol{e}_i \\
&\Rightarrow u_k\boldsymbol{e}_k \cdot (Q_{mn}\boldsymbol{e}_m\boldsymbol{e}_n)\,\boldsymbol{e}_i \\
&\Rightarrow u_k\boldsymbol{e}_k \cdot Q_{mn}\boldsymbol{e}_m\delta_{ni} = u_k\boldsymbol{e}_k \cdot Q_{mi}\boldsymbol{e}_m \\
&\Rightarrow Q_{mi}u_k\delta_{km} = Q_{ki}u_k = \bar{u}_i
\end{aligned}$$

This is the transformation laws for vectors, which in vector notation can be written as,

$$\{\bar{\boldsymbol{u}}\}^T = [\boldsymbol{Q}]^T\{\boldsymbol{u}\}$$

Transformation laws of tensors

The transformation laws for vector can be used to derive the transformation laws for tensors. For a second order tensor, we can write,

$$\boldsymbol{A} = A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \bar{A}_{ij}\bar{\boldsymbol{e}}_i \otimes \bar{\boldsymbol{e}}_j.$$

which follows that,

$$\begin{aligned}\bar{A}_{ij} &= \bar{\boldsymbol{e}}_i \cdot \boldsymbol{A}\bar{\boldsymbol{e}}_j = \boldsymbol{Q}\boldsymbol{e}_i \cdot \boldsymbol{A}\left(\boldsymbol{Q}\boldsymbol{e}_j\right) \\ &\Rightarrow Q_{mi}\boldsymbol{e}_m \cdot \boldsymbol{A}Q_{nj}\boldsymbol{e}_n = Q_{mi}Q_{nj}\boldsymbol{e}_m \cdot \boldsymbol{A}\boldsymbol{e}_n \\ &\Rightarrow Q_{mi}Q_{nj}A_{mn}\end{aligned}$$

Thus, $\bar{A}_{ij} = Q_{mi}Q_{nj}A_{mn}$ which is the transformation law for second order tensor. In the matrix form it can be written as

$$[\bar{\boldsymbol{A}}] = [\boldsymbol{Q}]^T[\boldsymbol{A}][\boldsymbol{Q}]$$

The transformation law can be generalized to a n^{th} order tensor $\boldsymbol{\mathcal{A}}$ as,

$$\bar{\mathcal{A}}_{i_1 i_2 i_3 \ldots i_n} = Q_{m_1 i_1}Q_{m_2 i_2}Q_{m_3 i_3}\cdots Q_{m_n i_n}\mathcal{A}_{m_1 m_2 m_3 \ldots m_n}$$

Invariants of a tensor

For a second order tensor $\boldsymbol{A}$ following can be defined,

$$I_1(\boldsymbol{A}) = \mathrm{tr}\boldsymbol{A} = A_{ii}$$

$$I_2(\boldsymbol{A}) = \frac{1}{2}\left[(\mathrm{tr}\boldsymbol{A})^2 - \mathrm{tr}\boldsymbol{A}^2\right] = \frac{1}{2}(A_{ii}A_{jj} - A_{ij}A_{ji})$$

$$I_3(\boldsymbol{A}) = \det \boldsymbol{A} = e_{ijk}A_{1i}A_{2j}A_{3k}$$

I_1, I_2, I_3 are three invarients of tensor $\boldsymbol{A}$, as they remain constant with transformation of tensor.

Here, we can introduce Cayley-Hamilton equation, which states every second order tensor $\boldsymbol{A}$ will statisfy the following equation (called characterstic equation of $\boldsymbol{A}$)

$$\boldsymbol{A}^3 - I_1\boldsymbol{A}^2 + I_2\boldsymbol{A} - I_3\boldsymbol{I} = \boldsymbol{O}.$$