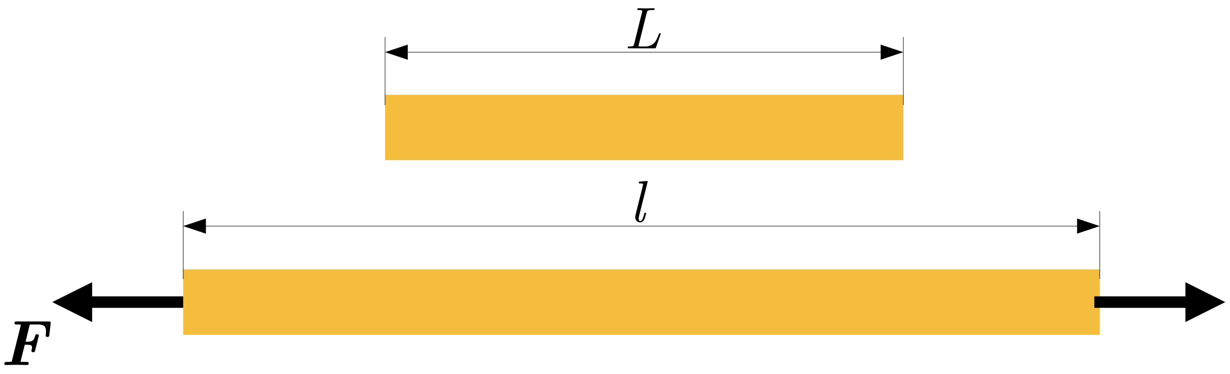
ME231: Solid Mechanics-I

Stress and Strain

Measurement of deformation: Strain



Strain is a measure of deformation. For uniaxial condition strain is defined as follows.

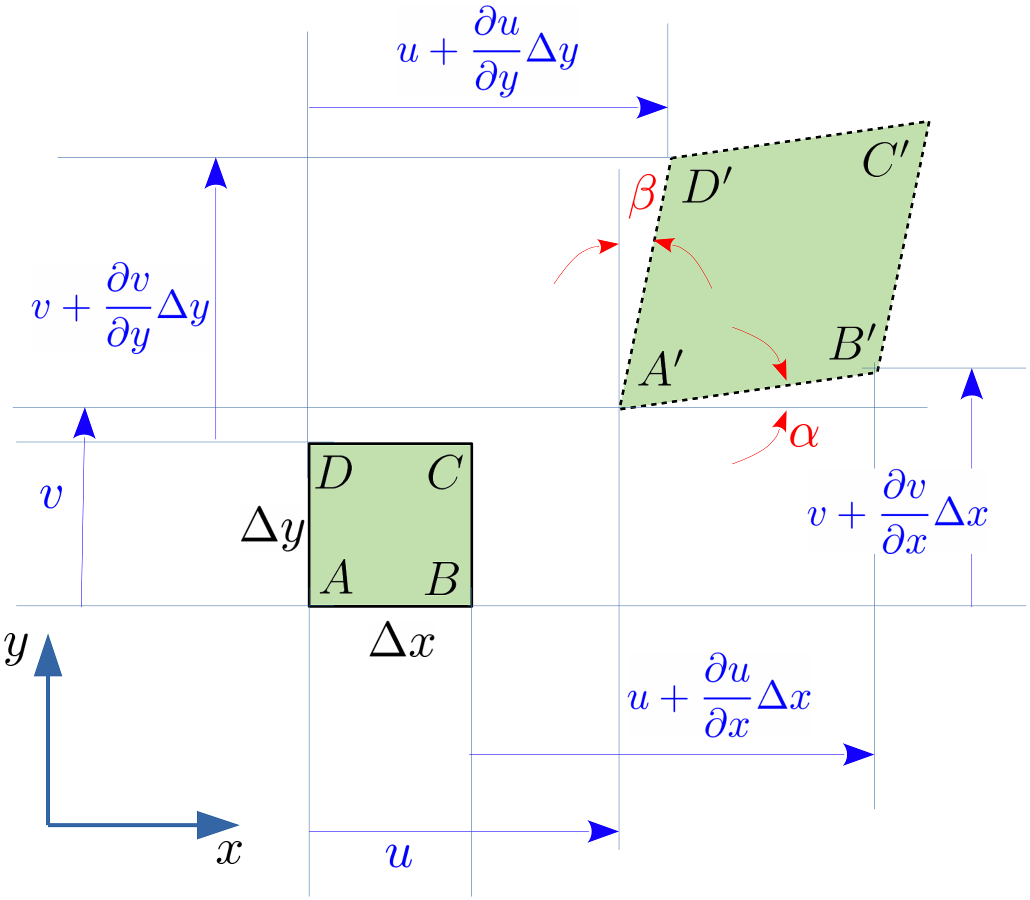
Engineering or nominal strain is defined as **deformation per unit original length**.

$$\varepsilon = \frac{l-L}{L} = \frac{\Delta L}{L}. \qquad \cdots\cdots\cdots\cdots(32)$$

Another definition of strain is called **true or logarithmic strain** defined as,

$$e = \int_L^l \frac{dl}{l} = \ln \frac{l}{L}. \qquad \cdots\cdots\cdots\cdots(33)$$

Plane strain in case of small deformations



$$\epsilon_x = \lim_{\Delta x \to 0} \frac{A'B' - AB}{AB} = \lim_{\Delta x \to 0} \frac{\Delta x + \frac{\partial u}{\partial x}\Delta x - \Delta x}{\Delta x},$$

$$\Rightarrow \epsilon_x = \frac{\partial u}{\partial x},$$

$$\epsilon_y = \lim_{\Delta y \to 0} \frac{A'D' - AD}{AD} = \lim_{\Delta y \to 0} \frac{\Delta y + \frac{\partial v}{\partial y}\Delta y - \Delta y}{\Delta y},$$

$$\Rightarrow \epsilon_y = \frac{\partial v}{\partial y},$$

$$\gamma_{xy} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\pi}{2} - \angle B'A'D' = \alpha + \beta$$

$$\Rightarrow \gamma_{xy} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\frac{\partial v}{\partial x}\Delta x}{\Delta x + \frac{\partial u}{\partial x}\Delta x} + \frac{\frac{\partial u}{\partial y}\Delta y}{\Delta y + \frac{\partial v}{\partial y}\Delta y}$$

$$\Rightarrow \gamma_{xy} \approx \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.$$

Strain-displacement relations(in 2D)

Engineering strain components,

$$\epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \cdots\cdots\cdots\cdots(34)$$

Tensorial strain components,

$$e_x = \frac{\partial u}{\partial x}, \quad e_y = \frac{\partial v}{\partial y}, \quad e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \cdots\cdots\cdots\cdots(35)$$

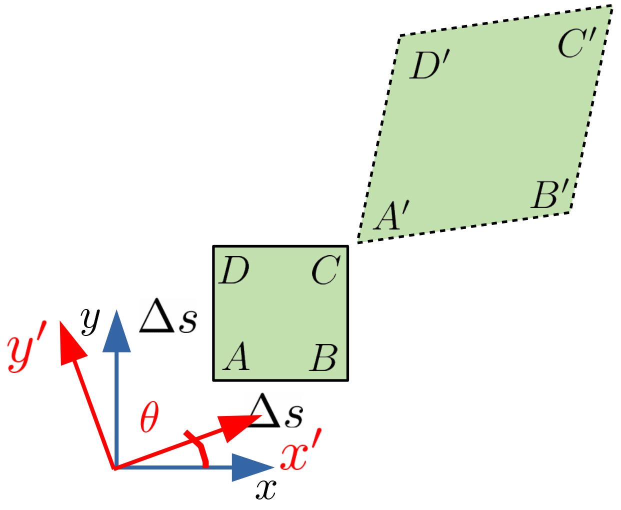
Similar to stress, strain is also a second order tensor. In plane strain case,

$$[\boldsymbol{e}] = \begin{bmatrix} e_{xx} & e_{xy} \\ e_{yx} & e_{yy} \end{bmatrix} \quad \text{or} \quad [\boldsymbol{\epsilon}] = \begin{bmatrix} \epsilon_{xx} & \gamma_{xy}/2 \\ \gamma_{yx}/2 & \epsilon_{yy} \end{bmatrix} \qquad \cdots\cdots\cdots\cdots(36)$$

Strain transformation

Being a second order tensor, strain tensor follows general rules of transformations.

$$[\boldsymbol{\epsilon}]' = [\boldsymbol{Q}]^T[\boldsymbol{\epsilon}][\boldsymbol{Q}],$$



Following this equation transformed strains will be given as

$$\epsilon'_{xx} = \frac{(\epsilon_{xx} + \epsilon_{yy})}{2} + \frac{(\epsilon_{xx} - \epsilon_{yy})}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta,$$

$$\epsilon'_{yy} = \frac{(\epsilon_{xx} + \epsilon_{yy})}{2} - \frac{(\epsilon_{xx} - \epsilon_{yy})}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta,$$

$$\gamma'_{xy} = \frac{\gamma_{xy}}{2}\cos 2\theta - \frac{(\epsilon_{xx} - \epsilon_{yy})}{2}\sin 2\theta.$$

..........(37)

Strain transformation equations (35) can also be derived from purely geometrical relations. Relations between the coordinate and displacements in xy-system and $x'y'$-system is given as

$$
\begin{aligned}
x &= x'\cos\theta - y'\sin\theta \qquad & u' &= u\cos\theta + v\sin\theta \\
y &= x'\sin\theta + y'\cos\theta. \qquad & v' &= -u\sin\theta + v\cos\theta.
\end{aligned}
\qquad \cdots\cdots\cdots(38)
$$

Now, using strain-displacement relations,

$$
\begin{aligned}
\epsilon'_x &= \frac{\partial u'}{\partial x'} = \frac{\partial u'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial x'} = \frac{\partial u'}{\partial x}\cos\theta + \frac{\partial u'}{\partial y}\sin\theta \\
&\Rightarrow \left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial v}{\partial x}\sin\theta\right)\cos\theta + \left(\frac{\partial u}{\partial y}\cos\theta + \frac{\partial v}{\partial y}\sin\theta\right)\sin\theta \\
&\Rightarrow \frac{\partial u}{\partial x}\cos^2\theta + \frac{\partial v}{\partial y}\sin^2\theta + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\sin\theta\cos\theta, \\
&\Rightarrow \epsilon_x\cos^2\theta + \epsilon_y\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta. \qquad \cdots\cdots\cdots(39)
\end{aligned}
$$

$$
\begin{aligned}
\epsilon'_y &= \frac{\partial v'}{\partial y'} = \frac{\partial v'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial y'} = -\frac{\partial v'}{\partial x}\sin\theta + \frac{\partial v'}{\partial y}\cos\theta \\
&\Rightarrow -\left(-\frac{\partial u}{\partial x}\sin\theta + \frac{\partial v}{\partial x}\cos\theta\right)\sin\theta + \left(-\frac{\partial u}{\partial y}\sin\theta + \frac{\partial v}{\partial y}\cos\theta\right)\cos\theta \\
&\Rightarrow \frac{\partial v}{\partial y}\cos^2\theta + \frac{\partial u}{\partial x}\sin^2\theta - \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\sin\theta\cos\theta \\
&\Rightarrow \epsilon_y\cos^2\theta + \epsilon_x\sin^2\theta - \gamma_{xy}\sin\theta\cos\theta \qquad \cdots\cdots\cdots\cdots(40)
\end{aligned}
$$

$$
\begin{aligned}
\gamma'_{xy} &= \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'} = \left(\frac{\partial v'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial x'}\right) + \left(\frac{\partial u'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial y'}\right) \\
&\Rightarrow \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)\sin\theta\cos\theta + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\left(\cos^2\theta - \sin^2\theta\right) \\
&\Rightarrow (\epsilon_y - \epsilon_x)\sin\theta\cos\theta + \gamma_{xy}\left(\cos^2\theta - \sin^2\theta\right) \qquad \cdots\cdots\cdots\cdots(41)
\end{aligned}
$$

Use of trigonometric identities will results in equations identical to the equations derived from tensor transformations.