# ME531: Advanced Mechanics of Solids

# Motion, Strain and Stress

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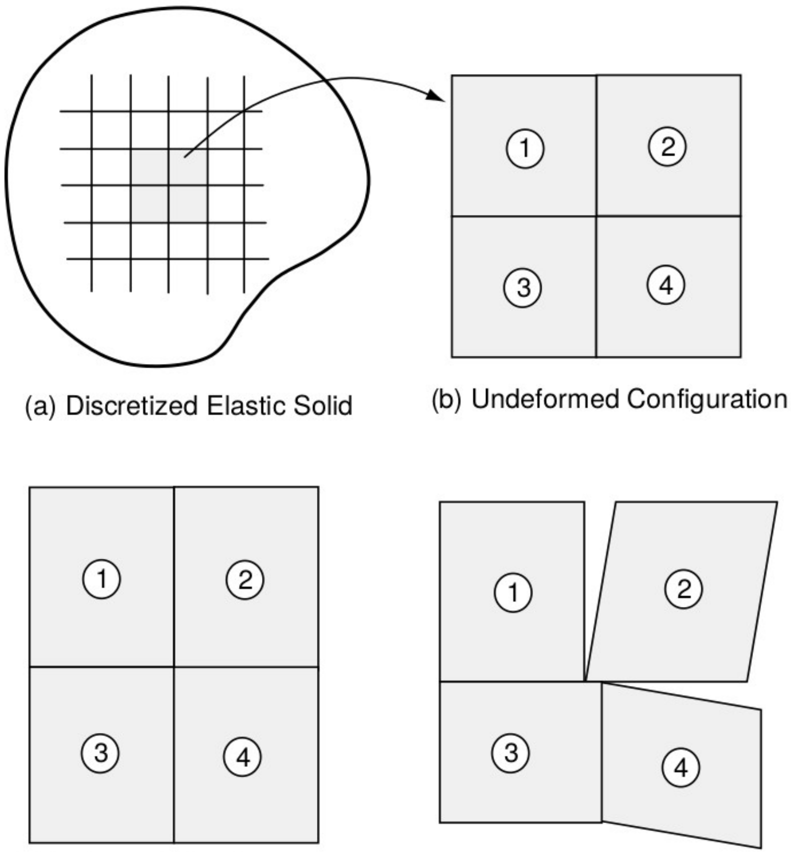
# Strain compatibility for small strains

- Strain-displacement relations are *six* equations in terms of three displacements.
- If we specify 3 single valued displacement components $u$, $v$, and $w$ then through differentiation resulting strain field will be well behaved.
- However converse is not true always; i.e. if strain fields are given then integration of these equations getting does not necessarily produce a single-valued continuous displacements fields. It is because we are trying to solve six equations for only three unknown displacement components.
- Thus, to ensure continuous, single-valued displacements, the strains must satisfy additional relations called compatibility equations.
- Also remember that, we can never completely recover the displacement field that gives rise to a particular strain field. Any rigid motion produces no strain, so the displacements can only be completely determined if there is some additional information (besides the strain) that will tell us how much the solid has rotated and translated. However, integrating the strain field can tell us the displacement field to within an arbitrary rigid motion.

# Physical interpretation of strain compatibility




(a) Discretized Elastic Solid

(b) Undeformed Configuration

(c) Deformed Configuration
Continuous Displacements

(d) Deformed Configuration
Discontinuous Displacements

To develop the additional conditions so that integration of strain yields continuous single-valued displacement fields, we start with the definition of small strains,

$$\varepsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right).$$

The idea is to eliminate displacements from these equations. For that we differentiate the equation twice w.r.t. $x$, as

$$\varepsilon_{ij,kl} = \frac{1}{2}\left(u_{i,jkl} + u_{j,ikl}\right).$$

Simple exchange of subscript in the previous equation results in following additional relations,

$$\varepsilon_{kl,ij} = \frac{1}{2}\left(u_{k,lij} + u_{l,kij}\right), \ \varepsilon_{jl,ik} = \frac{1}{2}\left(u_{j,lik} + u_{l,jik}\right), \ \ \varepsilon_{ik,jl} = \frac{1}{2}\left(u_{i,kjl} + u_{k,ijl}\right).$$

With the assumption of continuous displacement field $\boldsymbol{u}$, order of differentiation on $\boldsymbol{u}$ can be interchanged and displacements can be eliminated from the previous equations, which results in the following equations, called *Saint Vinent Compatibility* equations.

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0.$$

Saint Vinent compatibility equations derived in the previous slide are total of 81 equations, out of which only 6 are meaningful (others are simple identities or repetition). These equations are following.

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 x_2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = 0, \qquad \frac{\partial^2 \varepsilon_{11}}{\partial x_2 x_3} - \frac{\partial}{\partial x_1}\left(-\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3}\right) = 0,$$

$$\frac{\partial^2 \varepsilon_{22}}{\partial x_3^2} - 2\frac{\partial^2 \varepsilon_{23}}{\partial x_1 x_3} + \frac{\partial^2 \varepsilon_{33}}{\partial x_2^2} = 0, \qquad \frac{\partial^2 \varepsilon_{22}}{\partial x_3 x_1} - \frac{\partial}{\partial x_2}\left(-\frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3} + \frac{\partial \varepsilon_{23}}{\partial x_1}\right) = 0,$$

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_3^2} - 2\frac{\partial^2 \varepsilon_{13}}{\partial x_1 x_3} + \frac{\partial^2 \varepsilon_{33}}{\partial x_1^2} = 0, \qquad \frac{\partial^2 \varepsilon_{33}}{\partial x_1 x_2} - \frac{\partial}{\partial x_3}\left(-\frac{\partial \varepsilon_{12}}{\partial x_3} + \frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2}\right) = 0.$$

It can be shown that these six equations are equivalent to three independent fourth-order relations.

# Cauchy-Green deformation tensors

Let us consider the squared length of a line elements in the current configuration,

$$dx^2 = d\boldsymbol{x} \cdot d\boldsymbol{x} = (\boldsymbol{F} d\boldsymbol{X}) \cdot (\boldsymbol{F} d\boldsymbol{X}) = d\boldsymbol{X}(\boldsymbol{F}^T \boldsymbol{F}) d\boldsymbol{X} = d\boldsymbol{X} \cdot (\boldsymbol{C} d\boldsymbol{X}),$$

where $\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F}$ or $C_{IJ} = F_{kI} F_{kJ}$, is the Right Cauchy-Green deformation tensor. It should be noted that $\boldsymbol{C}$ is a *symmetric and positive definite tensor*. We observe that $\det \boldsymbol{C} = (\det \boldsymbol{F})^2 = J^2 > 0$.

Similarly, we can also find,

$$dX^2 = d\boldsymbol{X} \cdot d\boldsymbol{X} = (\boldsymbol{F}^{-1} d\boldsymbol{x}) \cdot (\boldsymbol{F}^{-1} d\boldsymbol{x}) = d\boldsymbol{x} \cdot (\boldsymbol{F}^{-T} \boldsymbol{F}^{-1}) d\boldsymbol{x} = d\boldsymbol{x} \cdot (\boldsymbol{b}^{-1} d\boldsymbol{x}),$$

where $\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^T$ or $b_{ij} = F_{iK} F_{jK}$, is the Left Cauchy-Green deformation tensor. Similar to $\boldsymbol{C}$, $\boldsymbol{b}$ is also a *symmetric and positive definite tensor* and $\det \boldsymbol{b} = (\det \boldsymbol{F})^2 = J^2 > 0$.

Green-Lagrange and Euler-Almansi strain tensors can be written in terms of Cauchy-Green strain tensors as,

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}), \text{ and } \boldsymbol{e} = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{b}^{-1}\right)$$

# Cauchy-Green deformation tensors

Left and right Cauchy-Green deformation tensors can be physically understood in the following manner. We define line element $d\boldsymbol{X}$ in terms of its magnitude $dX$, and a unit vector in the direction of $d\boldsymbol{X}$ as, $d\boldsymbol{X} = dX\ \boldsymbol{N}$ and similar $d\boldsymbol{x} = dx\ \boldsymbol{n}$, where $\boldsymbol{N}$ and $\boldsymbol{n}$ are unit vectors in the directions of $\mathrm{d}\boldsymbol{X}$ and $d\boldsymbol{x}$, respectively.

Now we can write,

$$dx^2 = dX\boldsymbol{N} \cdot (\boldsymbol{C} dX\boldsymbol{N}) = dX^2\boldsymbol{N} \cdot (\boldsymbol{C}\boldsymbol{N}) \text{ or } dx^2/dX^2 = \boldsymbol{N} \cdot (\boldsymbol{C}\boldsymbol{N}).$$

Similarly,

$$dX^2 = dx\boldsymbol{n} \cdot (\boldsymbol{b}^{-1} dx\boldsymbol{n}) = dx^2\boldsymbol{n} \cdot (\boldsymbol{b}^{-1}\boldsymbol{n}) \text{ or } dX^2/dx^2 = \boldsymbol{n} \cdot (\boldsymbol{b}^{-1}\boldsymbol{n}).$$

Thus components of $\boldsymbol{C}$ (and $\boldsymbol{b}$) denote square of ratio of element length in current configuration to its length in reference configuration (and vice versa) in the specific directions. Ratio of element length in current configuration to its length in reference configurationn is known as *stretch*, i.e. $dx/dX = \lambda$ or $dx = \lambda dX$.

# Rotation, stretch tensor

Deformation gradient tensor $\boldsymbol{F}$ can be multiplicatively decomposed into a *pure stretch tensor* and a *pure rotation tensor* as,

$$\boldsymbol{F} = \boldsymbol{RU} = \boldsymbol{vR}, \text{ or } F_{iJ} = R_{iK}U_{KJ} = v_{ik}R_{kJ}.$$

where $\boldsymbol{R}$ is a pure rotation tensor ($\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}$), and $\boldsymbol{U}$ and $\boldsymbol{v}$ are pure stretch tensors. This decomposition is called the *polar decomposition*.

Here, $\boldsymbol{U}$ and $\boldsymbol{v}$ are *unique*, *symmetric*, *positive definite* tensors, which are called *right (or material) stretch* and *left (or spatial) stretch tensors, respectively* They represent local elongation (or contraction) along their mutually orthogonal Eigen vectors. It should be noted that,

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} = (\boldsymbol{RU})^T(\boldsymbol{RU}) = \boldsymbol{U}^T\boldsymbol{R}^T\boldsymbol{RU} = \boldsymbol{UIU} = \boldsymbol{UU} = \boldsymbol{U}^2, \text{ and}$$
$$\boldsymbol{b} = \boldsymbol{FF}^T = (\boldsymbol{vR})(\boldsymbol{vR})^T = \boldsymbol{vRR}^T\boldsymbol{v}^T = \boldsymbol{vIv} = \boldsymbol{vv} = \boldsymbol{v}^2.$$

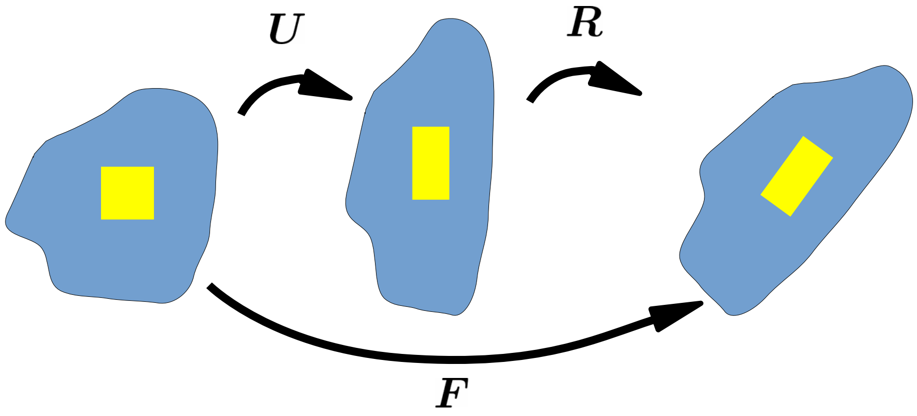
Also, $\det \boldsymbol{U} = (\det \boldsymbol{C})^{1/2} = J > 0$, $\det \boldsymbol{v} = (\det \boldsymbol{b})^{1/2} = J > 0$, and $\det \boldsymbol{R} = 1$.

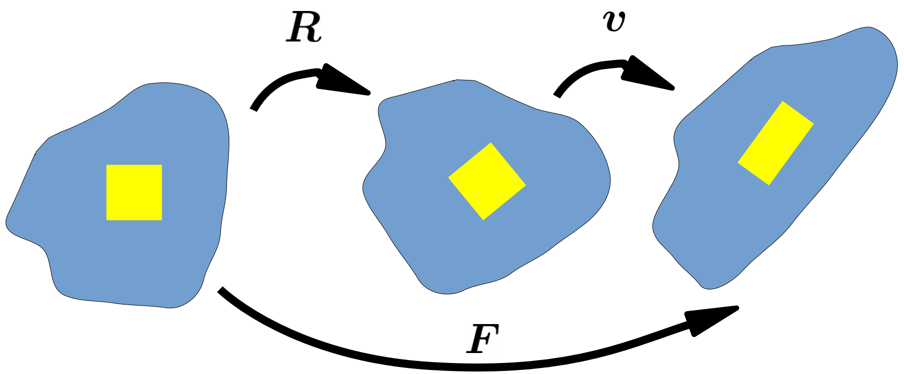
Polar decomposition can be visualized in the following manner.

- A general motion can be represented as a combination of elongation (or contraction) is some directions and then rotation of the body resulting from the deformation in the first step.
- First the initial configuration can be rotated and then the rotated body can be stretched (or contracted) in particular directions.

E.g. for a vector $\boldsymbol{X}$ in the reference configuration, a deformation is given as

$$\boldsymbol{x} = \boldsymbol{F}\boldsymbol{X} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{X} = \boldsymbol{v}\boldsymbol{R}\boldsymbol{x}$$



# Eigenvalues and Eigenvectors of strain tensors

Consider the right or material stretch tensor $\boldsymbol{U}$. Eigenvalues and mutually orthogonal and normalized set of Eigenvectors of $\boldsymbol{U}$ are $\lambda_i$ and $\hat{\boldsymbol{N}}_i$ respectively. Thus,

$$\boldsymbol{U}\hat{\boldsymbol{N}}_i = \lambda_i \hat{\boldsymbol{N}}_i. \quad (i = 1, 2, 3; \text{ no summation})$$

From the properties of eigenvalues and eigenvectors, we can write

$$\boldsymbol{U}^2\hat{\boldsymbol{N}}_i = \boldsymbol{C}\hat{\boldsymbol{N}}_i = \lambda_i^2 \hat{\boldsymbol{N}}_i. \quad (i = 1, 2, 3; \text{ no summation})$$

Thus $\boldsymbol{U}$ and $\boldsymbol{C}$ have the same orthogonal eigenvectors $\hat{\boldsymbol{N}}_i$ which are called the *principal referential directions* (or *principal referential axes*). Whereas the eigenvalues of $\boldsymbol{U}$ are $\lambda_i$ which are called the *principal stretches* and the eigenvalues of $\boldsymbol{C}$ are square of principal stretches, i.e. $\lambda_i^2$ .

To determine the Eigenvalue of left stretch tensor $\boldsymbol{v}$, we use the following relation,

$$\boldsymbol{v} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^T.$$

We write the eigenvalue problem for $\boldsymbol{v}$ as,

$$\boldsymbol{v}(\boldsymbol{R}\boldsymbol{N}_i) = \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^T(\boldsymbol{R}\boldsymbol{N}_i) = \boldsymbol{R}\boldsymbol{U}\boldsymbol{N}_i = \lambda_i \boldsymbol{R}\boldsymbol{N}_i.$$

Using above relation, we can write the eigenvalue for for $\boldsymbol{b}$ as,

$$\boldsymbol{v}^2\boldsymbol{R}\hat{\boldsymbol{N}}_i = \boldsymbol{b}\boldsymbol{R}\hat{\boldsymbol{N}}_i = \lambda_i^2 \boldsymbol{R}\hat{\boldsymbol{N}}_i. \quad (i = 1, 2, 3; \text{ no summation})$$

It can be observed that $\boldsymbol{v}$ and $\boldsymbol{b}$ have same eigenvectors $\boldsymbol{n}_i = \boldsymbol{R}\hat{\boldsymbol{N}}_i$, which are called *principal spatial directions*, whereas eigenvalues are $\lambda_i$ and $\lambda_i^2$, respectively.

To summarize, we can summarize the spectral decomposition of the tensors $\boldsymbol{U}$, $\boldsymbol{v}$, $\boldsymbol{C}$ and $\boldsymbol{b}$ in the following way when, $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$

$$\boldsymbol{U}^2 = \boldsymbol{C} = \sum_{i=1}^{3} \lambda_i^2 \boldsymbol{N}_i \boldsymbol{N}_i, \quad \boldsymbol{v}^2 = \boldsymbol{b} = \sum_{i=1}^{3} \lambda_i^2 \boldsymbol{n}_i \boldsymbol{n}_i.$$