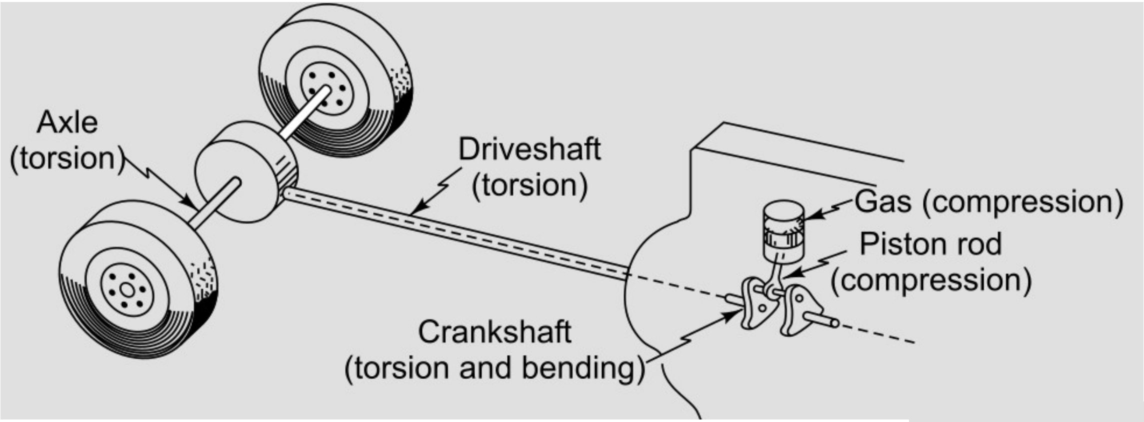
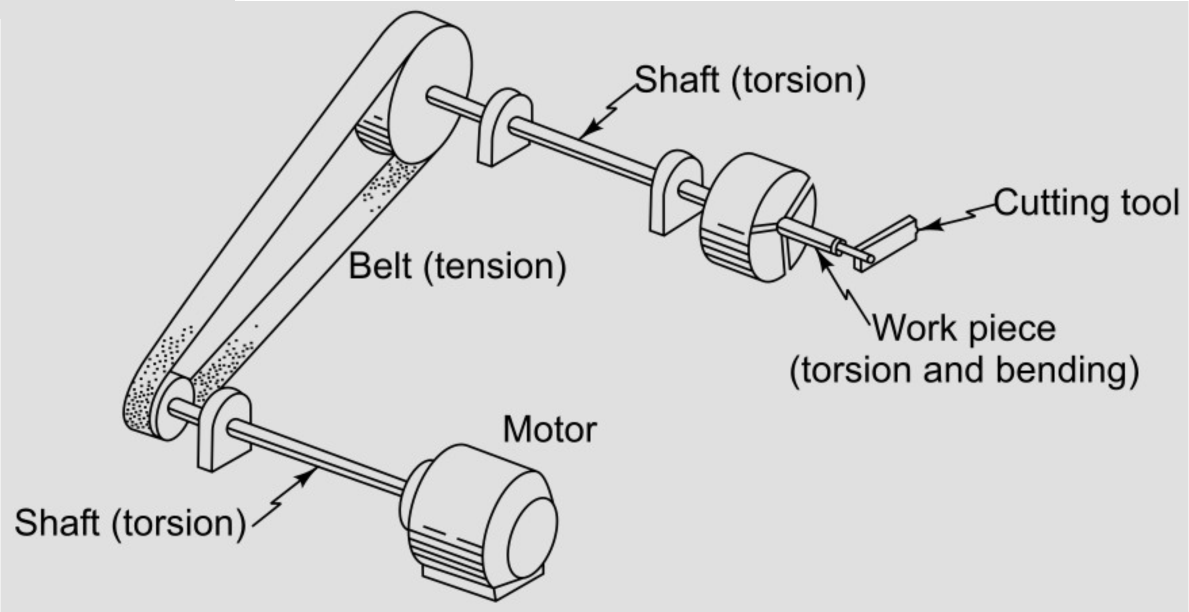
ME231: Solid Mechanics-I

Torsion

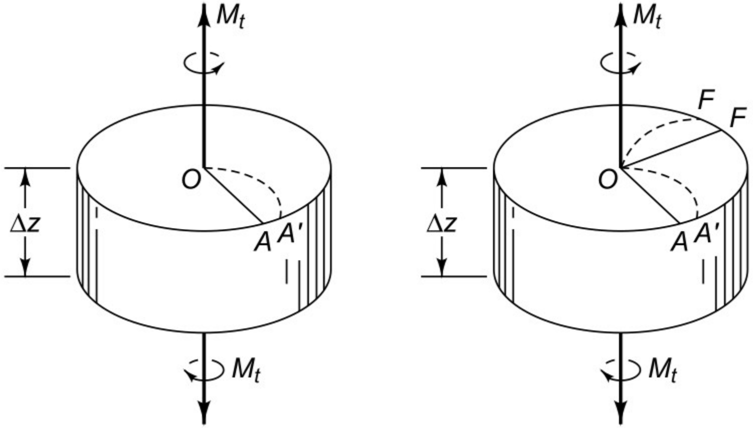
Introduction



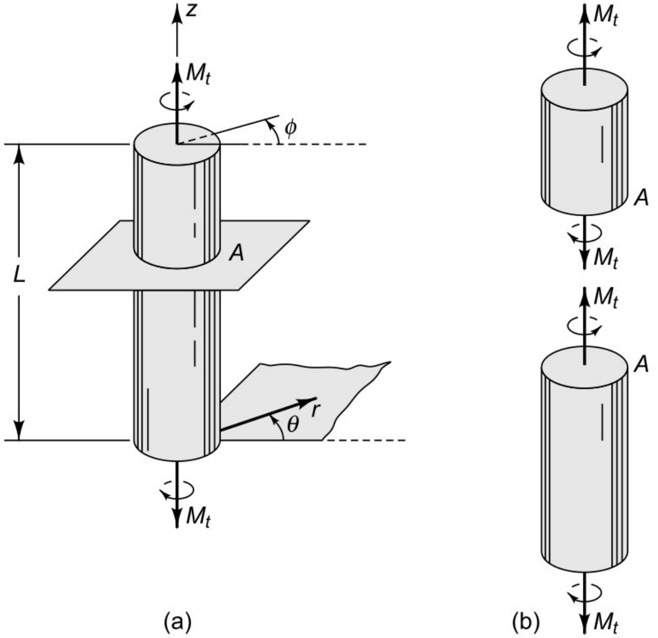


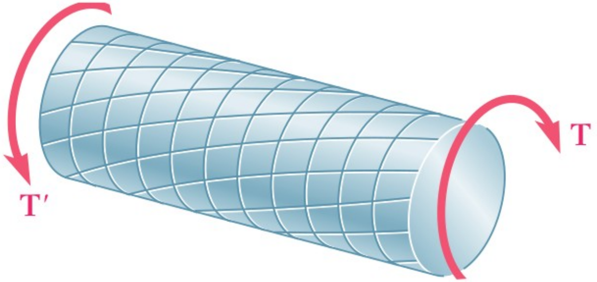
Geometry of deformation of a twisted circular shaft

Consider a uniform circular shaft loaded only by twisting couples M_t, at its ends. We will use the the argument of symmetry to deduce a plausible mode of deformation of any section of the shaft.

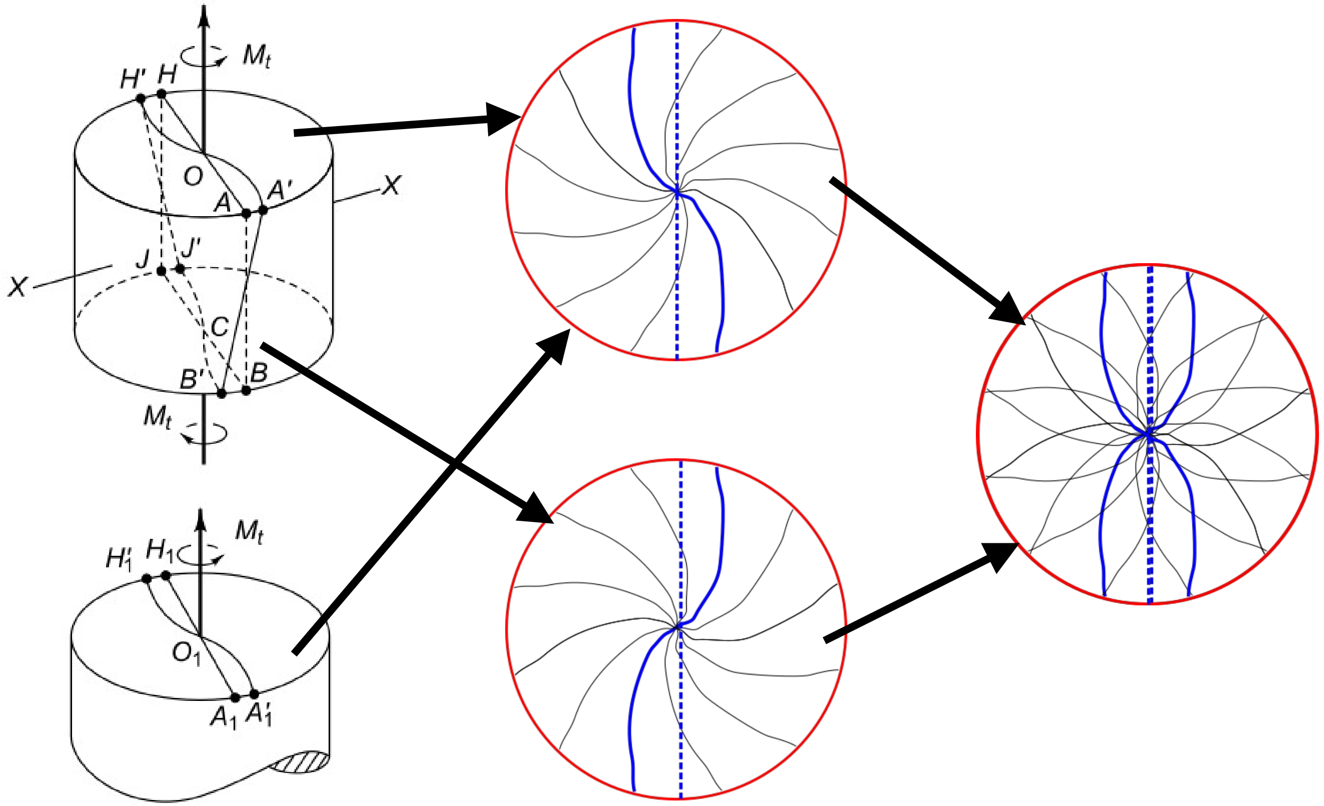


- All radii are deformed into identical curved lines.
- Cross-section must remain plane.



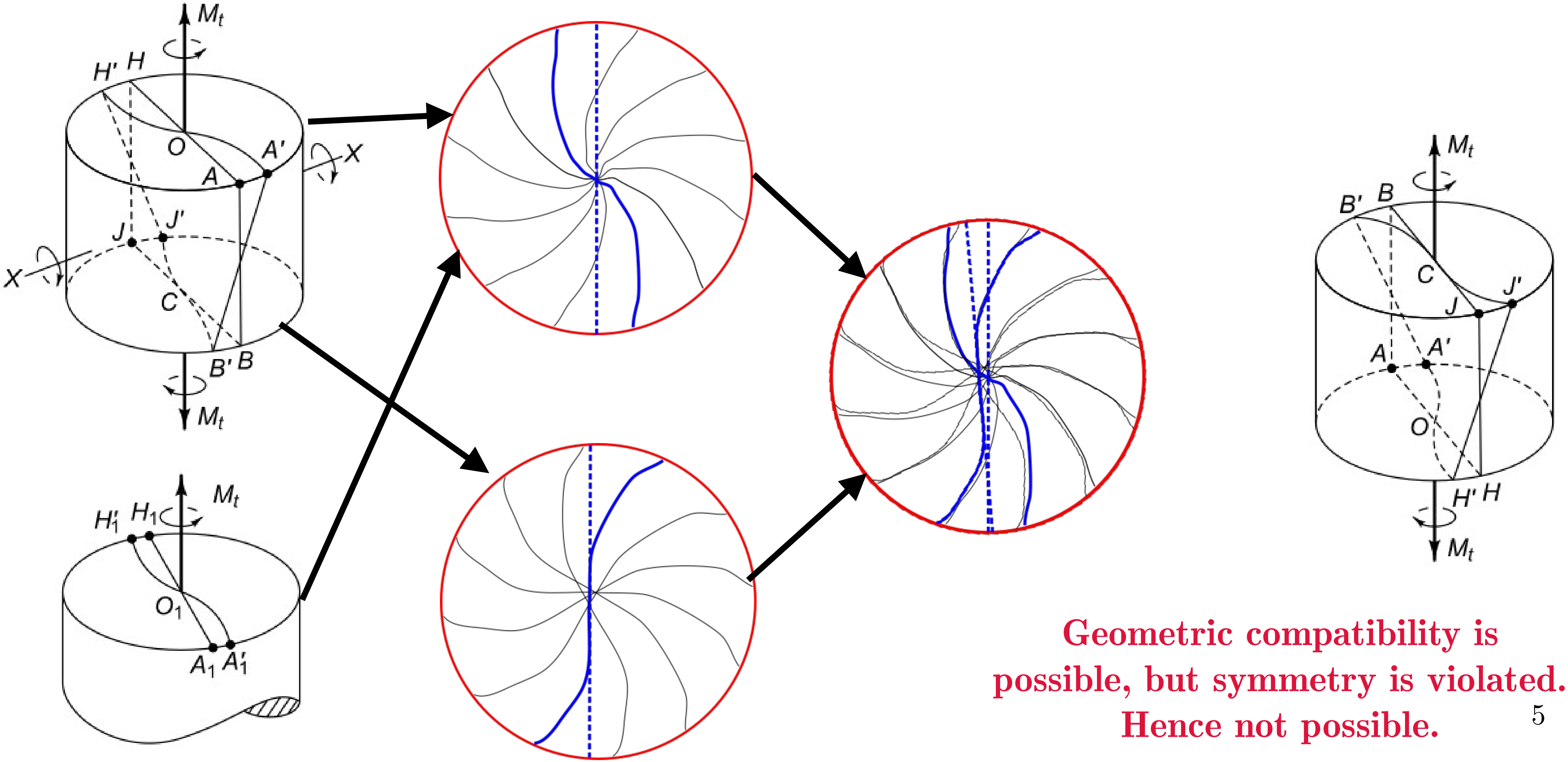


In-plane deformation – Hypothesis 1



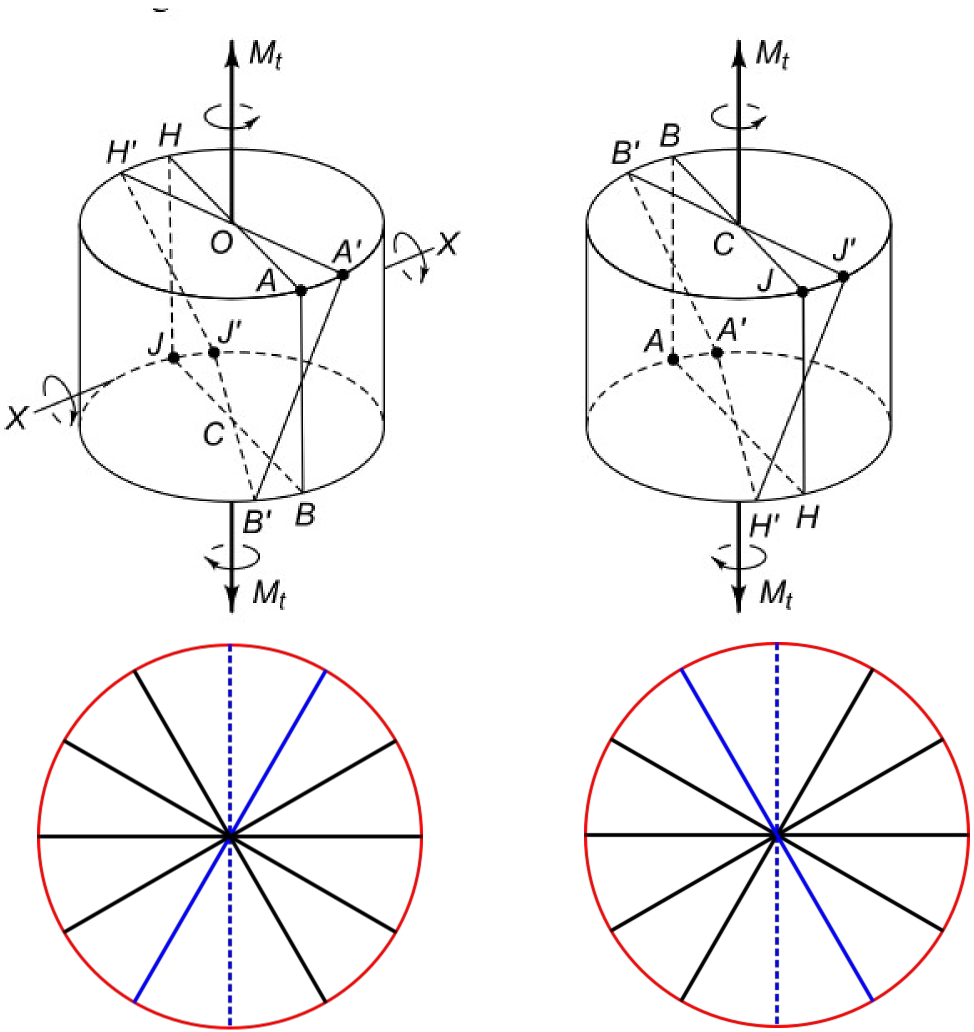
Geometric compatibility is violated. Hence not possible.

In-plane deformation – Hypothesis 2



Geometric compatibility is possible, but symmetry is violated. Hence not possible.

In-plane deformation – Hypothesis 3



Geometric compatibility as well as symmetry is satisfied.

Hence it is concluded that the deformation must have this nature; i.e., straight diameters are carried into straight diameters by the twisting deformation.

To summarize, circular shaft must deform such that each plane cross section originally normal to the axis remains plane and normal.

We have not ruled out a symmetrical expansion or contraction of the circular cross section or a lengthening or shortening of the cylinder. However, we shall make the **tentative assumption** that **the extensional strains all vanish. i.e.,**

$$\varepsilon_r = \varepsilon_\theta = \varepsilon_z = 0.$$

We will see that this assumption leads to a consistent theory meeting all the requirements of the theory of elasticity, **provided the amount of twist is small;** which is true for most structural materials.

With our assumption, the only possibility of deformation is that the cross sections of the shaft remain undeformed but rotate relative to each other.

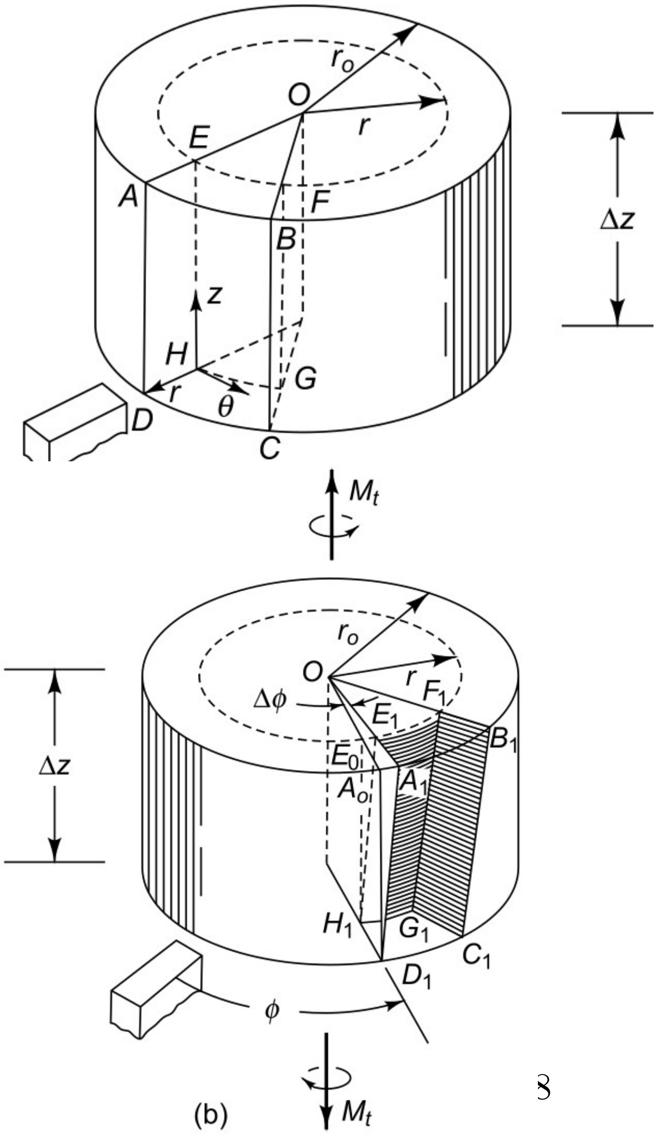
A slice of length Δz is shown before and after twisting. After twisting, the bottom section has rotated through the angle ϕ and the top section has rotated through the angle $\phi + \Delta\phi$. The relative rotation causes rectangular elements $EFGH$ to shear into parallelograms $E_1F_1G_1H_1$.

The originally right angle EHG is sheared into the acute angle $E_1H_1G_1$. This shear deformation is denoted by

$$\gamma_{\theta z} = \lim_{\Delta z \to 0} \frac{E_1E_0}{H_1E_0} = \lim_{\Delta z \to 0} \frac{r\Delta\phi}{\Delta z} = r\frac{d\phi}{dz}. \qquad \cdots\cdots(1)$$

Note that this states that the **shear strain varies in direct proportion to the radius**, from no shear at the center to a greatest shear at the out side.

We have already noted that each slice deforms in the same way as any other, so that we can conclude that $d\phi/dz$ is a constant along a uniform section of shaft subjected to twisting moments at the ends. **$d\boldsymbol{\phi}/dz$ is called the twist per unit length, or the rate of twist.**



The assumption regarding no distortion within the plane of the cross section implies that the right angle DHG goes over into a right angle $D_1H_1G_1$; also the right angle EHD goes over into the right angle $E_1H_1D_1$ (at least for small deformation). This implies that,

$$\gamma_{r\theta} = \gamma_{rz} = 0.$$

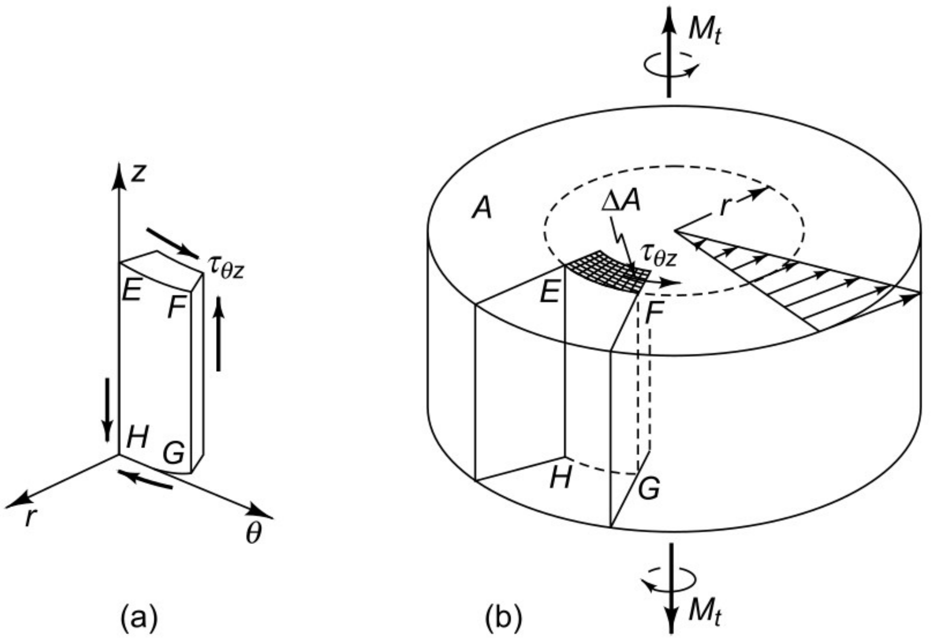
Thus, from symmetry and our assumptions following is the state of strain at any point in a circular shaft subjected to twisting moment at the end.

$$\varepsilon_r = \varepsilon_\theta = \varepsilon_z = \gamma_{r\theta} = \gamma_{rz} = 0, \quad \text{and} \quad \gamma_{\theta z} = r\frac{d\phi}{dz}. \qquad \cdots\cdots(2)$$

Stresses

With the knowledge of state of strain (i.e., equation (2)) we can now derive the stress components from the generalized Hooke's law. Stress components are as follows,

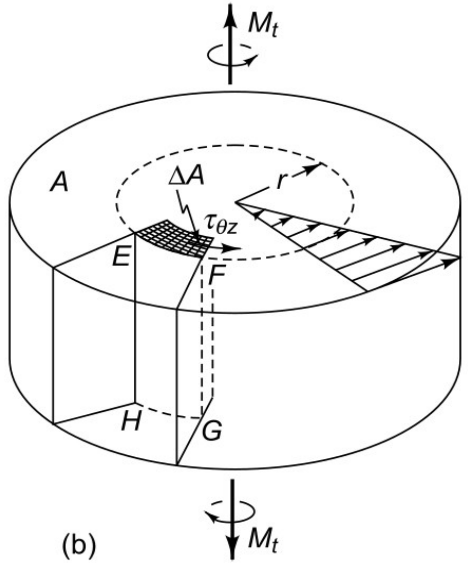
$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{r\theta} = \tau_{rz} = 0, \quad \text{and} \quad \tau_{\theta z} = Gr\frac{d\phi}{dz}. \qquad \cdots\cdots(3)$$



Equilibrium requirements

So we have seen that the shear strain $\gamma_{\theta z}$ and the shear stress $\tau_{\theta z}$ are proportional to the rate of twist $d\phi/dz$ (which is still unknown). Now we require that our stresses meet the conditions of equilibrium.

First of all, note that the stress distribution given by (2) leaves the external cylindrical surface of the shaft free of stress (i.e., $\sigma_r = \tau_{rz} = \tau_{r\theta} = 0$), as it should.



Inside the shaft, each element has only stress component $\tau_{\theta z}$. Note that $\tau_{\theta z}$ does not vary in θ-direction (because of symmetry) as well as z-direction (because of the uniformity of the deformation and stress pattern along the length of the shaft) and hence all equilibrium equations are satisfied **(Check it)**.

On each cross section of the shaft the resultant of the stress distribution must be equal to the applied twisting moment M_t. Hence,

$$M_t = \int_A r(\tau_{\theta z} dA). \qquad \cdots\cdots(4)$$

Stress and deformation in a twisted elastic circular shaft

Let us find some relations for twisted circular shafts by correlating the relations obtained for stress, strain, deformation and equilibrium.

Let us substitute the expression for stress (3) in to (4),

$$M_t = \int_A r\left(Gr\frac{d\phi}{dz}dA\right) = G\frac{d\phi}{dz}\int_A r^2 dA = G\frac{d\phi}{dz}I_z, \qquad \cdots\cdots\cdot(5)$$

where I_z is the polar moment of inertia of the cross-sectional area about the axis of the shaft.

From (5), rate of twist can be related to the applied moment, geometry and material property of the shaft as,

$$\frac{d\phi}{dz} = \frac{M_t}{GI_z}. \qquad \left(I_z = \frac{\pi r_0^4}{2} = \frac{\pi d^4}{32}\right) \qquad \cdots\cdots\cdot(6)$$

Thus for a shaft of length L, with twisting moment applied at the ends, twist angle between two ends can be determined as,

$$\phi = \int\limits_0^L \frac{M_t}{GI_z}dz = \frac{M_t L}{GI_z}. \qquad \cdots\cdots\cdot(7)$$

From (6) and (3), we can write the expression for stress as,

$$\tau_{\theta z} = \frac{M_t r}{I_z}. \qquad \cdots\cdots(8)$$

Equation (8) gives us stress distribution in shaft when twisting moment M_t is applied.

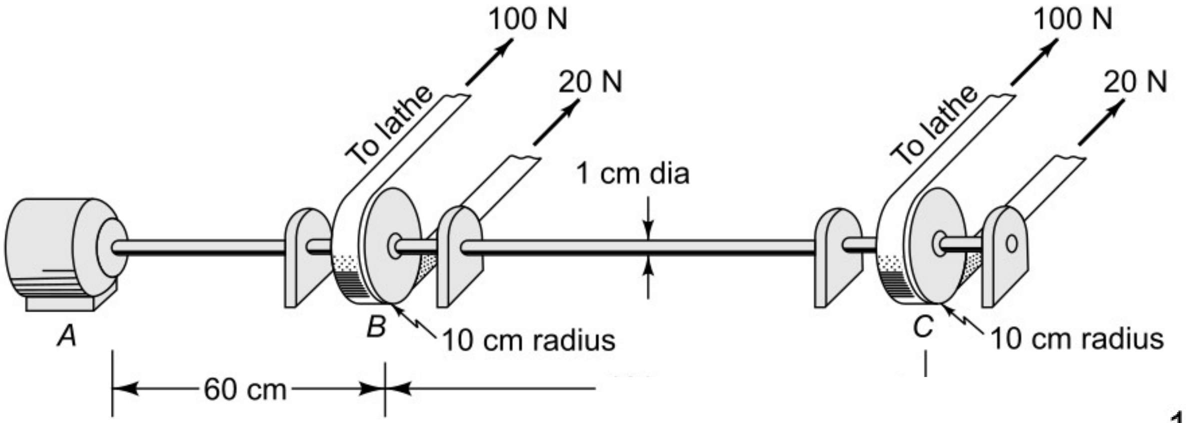
From (7), we find the ration of twisting moment to the twist, i.e., twisting moment per radian of twist.

$$\frac{M_t}{\phi} = \frac{G I_z}{L}. \qquad \cdots\cdots(9)$$

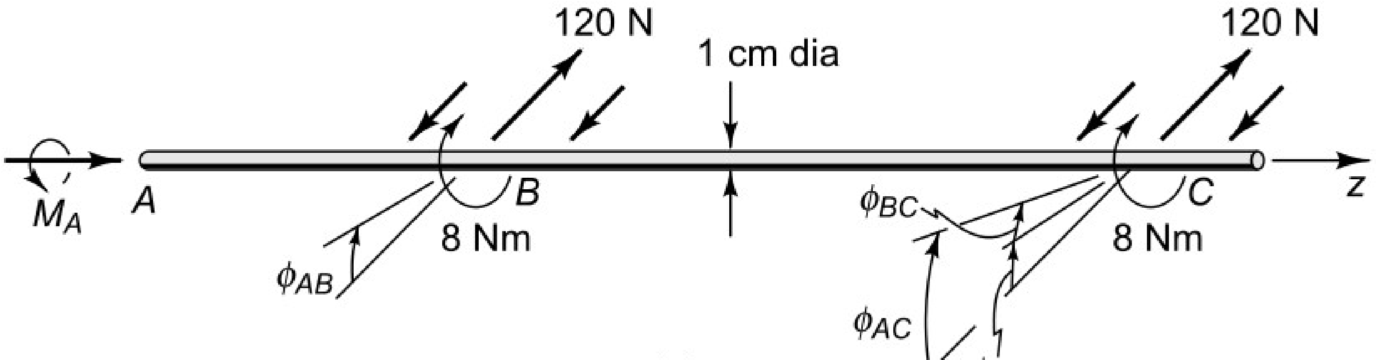
This ratio is analogous to a spring constant which gives tensile force per unit length of stretch. The ratio in (9) is called the **torsional stiffness of the shaft** and is often denoted by the symbol k or c.

Example 1

Two small lathes are driven by the same motor through a 1 cm diameter steel shaft, as shown in Figure. We wish to know the maximum shear stress in the shaft due to twisting and the angle of twist between the two ends of the shaft.



Idealization

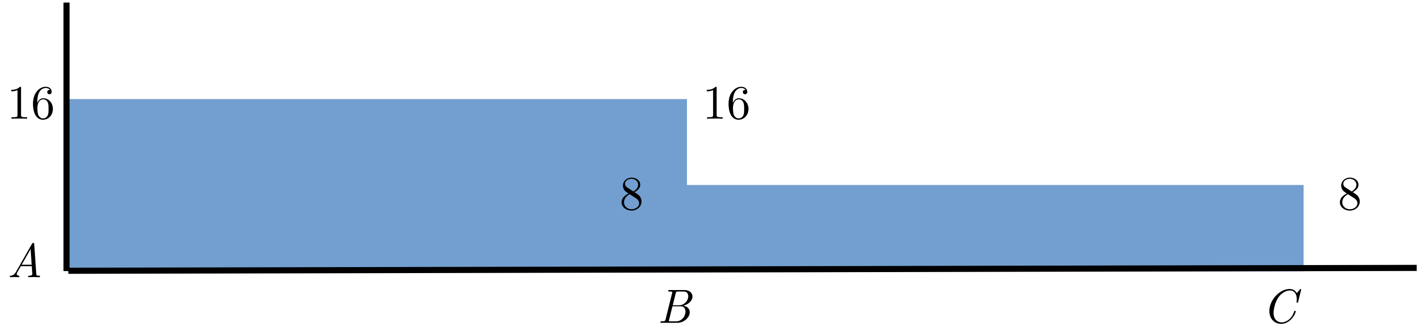


(b)

Equilibrium:

$\Sigma M_A = 0, \qquad M_A - M_B - M_C = 0, \qquad M_A = 16$ N·m

Thus, twisting moment in section AB is $M_{AB} = 16$ N·m and section BC is $M_{BC} = 8$ N·m



Geometric compatibility:

$$\phi_{AC} = \phi_{AB} + \phi_{BC}$$

Load – deformation relationship:

$$M_{AB} = \phi_{AB}\frac{GI_z}{L_{AB}} \quad \text{and} \quad M_{BC} = \phi_{BC}\frac{GI_z}{L_{BC}}.$$

Maximum shear stress:

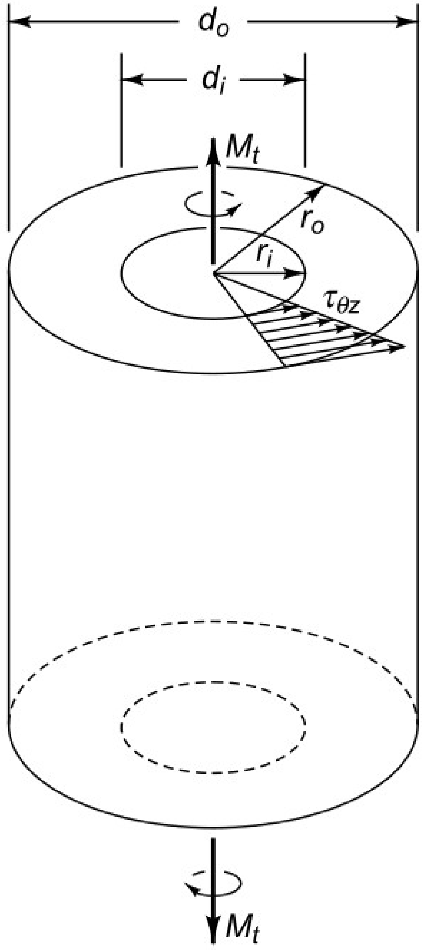
$$(\tau_{\theta z})_{\max} = \frac{M_{AB} r_0}{I_z}.$$

Torsion of elastic hollow circular shaft

All the equations derived for a solid shaft are equally applicable for hollow shafts too. For hollow shaft the polar moment of inertia of become,

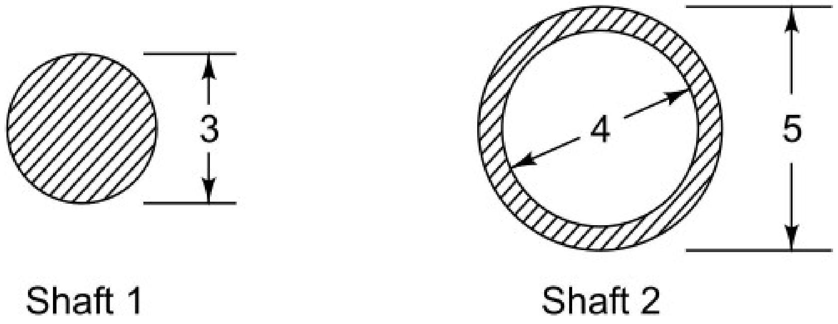
i.e.,

$$I_z = \frac{\pi}{2}(r_o^4 - r_i^4) = \frac{\pi}{2} r_o^4 \left(1 - \frac{r_i^4}{r_o^4}\right) \qquad \cdots\cdots(10)$$



Example 2

Consider the following two shafts which have the same cross-sectional area.



Show that for the same twisting moment, hollow shaft has lower maximum stress and higher stiffness than the solid shaft.