Elasticity Problems

Total 15 unknown scalar variables (3 Displacements, 6 strain components and 6 stress components.

Available field Equations:
6 Strain Displacement relations

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathsf{T}}\right] , \quad \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

3 Equilibrium equations:

$$\nabla \cdot \boldsymbol{\sigma}^{\mathsf{T}} + \mathbf{f} = \mathbf{0} , \quad \sigma_{ji,j} + f_i = 0 ,$$

6 Constitutive equations:

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{I} , \quad \sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} .$$

Additionally 6 scalar compatibility equations

Lame-Navier Equation

Substituting strain-displacement relations into Linear-elastic constitutive relation, we get

$$\sigma_{ij} = \mu(u_{i,j} + u_{j,i}) + \lambda u_{k,k}\,\delta_{ij}\,.$$

Now, substituting this expression into equilibrium equation,

$$\begin{aligned} 0 &= \sigma_{ji,j} + f_i \\ &= \mu(u_{j,ij} + u_{i,jj}) + \lambda u_{k,kj}\,\delta_{ji} + f_i \\ &= \mu u_{k,ki} + \mu u_{i,jj} + \lambda u_{k,ki} + f_i\,, \end{aligned}$$

which can be rewritten as

$$\boxed{\begin{aligned} (\lambda+\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u} + \mathbf{f} = \mathbf{0}\,, \\ (\lambda+\mu)u_{k,ki} + \mu u_{i,jj} + f_i = 0\,. \end{aligned}}$$

These Lame-Navier equations, which represent the equilibrium equation expressed in terms of the displacement field, provide three scalar field equations for three unknown scalar displacement fields. For any displacement field satisfying, the corresponding strain field is given by the strain-displacement equations. The Lame-Navier equations are the field equations to use when the displacement field is the only unknown that one wishes to determine. In practice, however, boundary conditions will very often require one to determine the stress field as well, which somewhat limits the utility of the Lame-Navier equations.

Laplacian

If $\boldsymbol{T}$ is a tensor field of order n, then its Laplacian $\nabla^2\boldsymbol{T}$ is a tensor field of order n defined as the divergence of the gradient:

$$\nabla^2\mathbf{T} \equiv \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\mathbf{T})$$

Laplacian in Cartesian coordinates

If ϕ is a scalar, $\mathbf{v} = v_i\hat{\mathbf{e}}_i$ is a vector, and $\mathbf{S} = S_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j$ is a second-order tensor, then, in a Cartesian coordinate system,

$$\begin{aligned}
\nabla^2\phi &= \boldsymbol{\nabla}\cdot(\phi_{,i}\,\hat{\mathbf{e}}_i) = \phi_{,ii}\ , \\
\nabla^2\mathbf{v} &= \boldsymbol{\nabla}\cdot(v_{i,j}\,\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j) = v_{i,jj}\,\hat{\mathbf{e}}_i\ , \\
\nabla^2\mathbf{S} &= \boldsymbol{\nabla}\cdot(S_{ij,k}\,\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\hat{\mathbf{e}}_k) = S_{ij,kk}\,\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j
\end{aligned}$$

Laplacian in cylindrical coordinate system:

Laplacian of scalars and vectors

Recall that the Laplacian of a tensor field is defined to be the divergence of its gradient. Using the above results, it follows that for a scalar field f,

$$\boxed{\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}\,,} \tag{2.5.29}$$

and for a vector field $\mathbf{u}$,

$$\boxed{\begin{aligned}\nabla^2 \mathbf{u} = &\left(\nabla^2 u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{1}{r^2}u_r\right)\hat{\mathbf{e}}_r \\ &+ \left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{1}{r^2}u_\theta\right)\hat{\mathbf{e}}_\theta + (\nabla^2 u_z)\hat{\mathbf{e}}_z\,.\end{aligned}} \tag{2.5.30}$$

Laplacian in spherical coordinate system:

Laplacian of scalars and vectors

Recall that the Laplacian of a tensor field is defined to be the divergence of its gradient. Using the above results, it follows that for a scalar field f,

$$\boxed{\nabla^2 f = \frac{\partial^2 f}{\partial R^2} + \frac{2}{R}\frac{\partial f}{\partial R} + \frac{1}{R^2}\left(\frac{\partial^2 f}{\partial \theta^2} + \cot\theta \frac{\partial f}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}\right)\ ,} \tag{2.5.59}$$

and for a vector field $\mathbf{u}$,

$$\boxed{\begin{aligned}\nabla^2 \mathbf{u} = &\left[\nabla^2 u_R - \frac{2}{R^2}\left(u_R + u_\theta \cot\theta + \frac{\partial u_\theta}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right)\right]\hat{\mathbf{e}}_R \\ &+ \left[\nabla^2 u_\theta + \frac{1}{R^2}\left(2\frac{\partial u_R}{\partial \theta} - \frac{1}{\sin^2\theta}u_\theta - 2\frac{\cos\theta}{\sin^2\theta}\frac{\partial u_\phi}{\partial \phi}\right)\right]\hat{\mathbf{e}}_\theta \\ &+ \left[\nabla^2 u_\phi + \frac{1}{R^2\sin\theta}\left(2\frac{\partial u_R}{\partial \phi} + 2\cot\theta\frac{\partial u_\theta}{\partial \phi} - \frac{1}{\sin\theta}u_\phi\right)\right]\hat{\mathbf{e}}_\phi\ .\end{aligned}} \tag{2.5.60}$$

Cartesian Coordinates

$$\mu\nabla^2 u + (\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+F_x=0$$
$$\mu\nabla^2 v + (\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+F_y=0$$
$$\mu\nabla^2 w + (\lambda+\mu)\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+F_z=0$$

Cylindrical Coordinates

$$\mu\left(\nabla^2 u_r - \frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right)+(\lambda+\mu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial z}\right)+F_r=0$$
$$\mu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right)+(\lambda+\mu)\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial z}\right)+F_\theta=0$$
$$\mu\nabla^2 u_z+(\lambda+\mu)\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial z}\right)+F_z=0$$

Spherical Coordinates

$$\mu\left(\nabla^2 u_R-\frac{2u_R}{R^2}-\frac{2}{R^2}\frac{\partial u_\phi}{\partial\phi}-\frac{2u_\phi\cot\phi}{R^2}-\frac{2}{R^2\sin\phi}\frac{\partial u_\theta}{\partial\theta}\right)$$
$$+(\lambda+\mu)\frac{\partial}{\partial R}\left(\frac{1}{R^2}\frac{\partial}{\partial R}(R^2u_R)+\frac{1}{R\sin\phi}\frac{\partial}{\partial\phi}(u_\phi\sin\phi)+\frac{1}{R\sin\phi}\frac{\partial u_\theta}{\partial\theta}\right)+F_R=0$$

General Field Equation System

(15 Equations, 15 Unknowns: u_i, e_{ij}, σ_{ij})

$$\Im\{u_i, e_{ij}, \sigma_{ij}; \lambda, \mu, F_i\} = 0$$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$\sigma_{ij,j} + F_i = 0$$

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$$

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0$$

Stress Formulation

(6 Equations, 6 Unknowns: σ_{ij})

$$\Im^{(t)}\{\sigma_{ij}; \lambda, \mu, F_i\}$$

$$\sigma_{ij,j} + F_i = 0$$

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = -\frac{\nu}{1-\nu}\delta_{ij}F_{k,k} - F_{i,j} - F_{j,i}$$

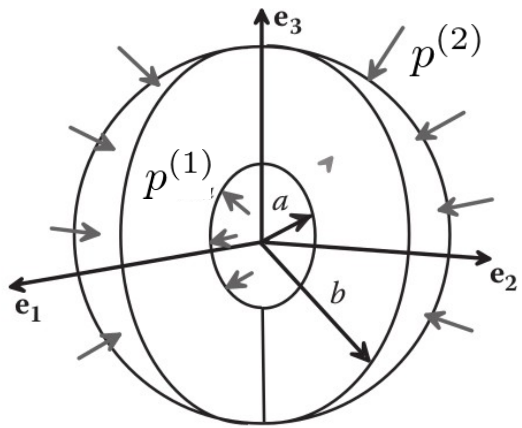
Displacement Formulation

(3 Equations, 3 Unknowns: u_i)

$$\Im^{(u)}\{u_i; \lambda, \mu, F_i\}$$

$$\mu u_{i,kk} + (\lambda + \mu)u_{k,ki} + F_i = 0$$

Pressurized hollow sphere



- No body forces act on the sphere
- The inner surface $R{=}a$ is subjected to pressure $p^{(1)}$, which implies

$$\sigma_{R\theta} = \sigma_{R\phi} = 0 \,, \quad \sigma_{RR} = -p^{(1)} \quad \text{on } R = a$$

- The outer surface $R{=}b$ is subjected to pressure $p^{(2)}$, which implies,

$$\sigma_{R\theta} = \sigma_{R\phi} = 0 \,, \quad \sigma_{RR} = -p^{(2)} \quad \text{on } R = b \,.$$

Let us solve for displacements first.

As the geometry and loading is symmetric, we also assume that the solution displacement field exhibits the spherical symmetry. The displacement components are then,

$$u_R = U(R) \,, \quad u_\theta = u_\phi = 0$$

Symmetry also implies that the problem is independent of and ϕ direction.

Then, the Navier's equation become,

$$(\lambda + 2\mu)\frac{d}{dR}\left[\frac{1}{R^2}\frac{d}{dR}(R^2U)\right] = 0$$