ME231: Solid Mechanics-I

Deflections due to bending

The moment-curvature relationship

For a symmetrical, linearly elastic beam element subjected to pure bending, we have derived the following moment-curvature relationship.

$$\frac{1}{\rho} = \frac{d\phi}{ds} = \frac{M_b}{EI_{zz}} \qquad \text{..........................(1)}$$

The curvature of the neutral axis completely defines the deformation of an element in pure bending.

We can extend this to the case of general bending where the bending moment varies along the length of the beam.

We **assume that the shear forces do not contribute** significantly to the overall deformation.

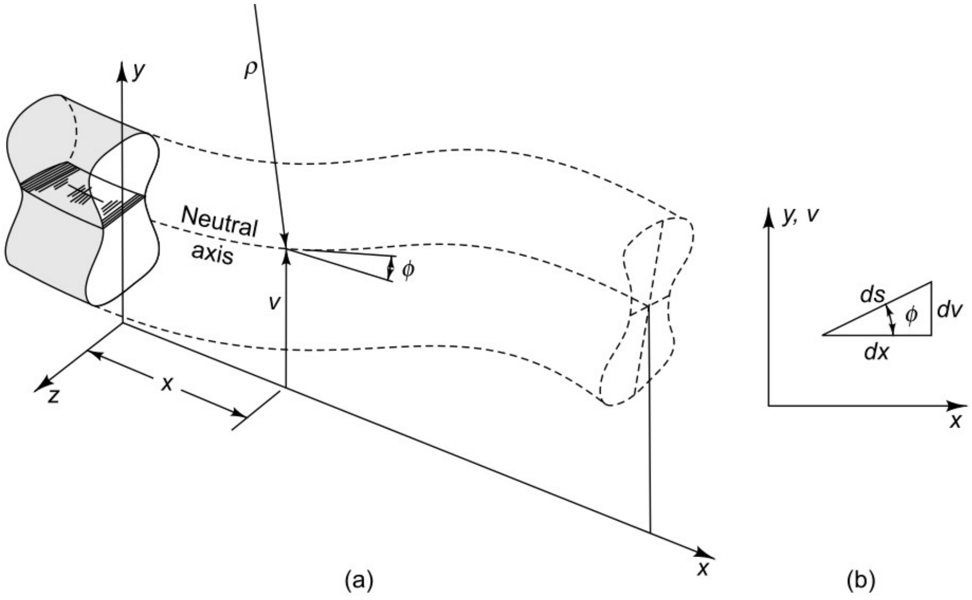
Thus we assume that the **deformation is still defined by the curvature** and that the curvature is still given by above relation.

Accordingly, if we know how the bending moment varies along the length of the beam, we will then know how the curvature varies.

To determine the bent shape of the beam, we deduce the deflection of the neutral axis from the knowledge of its curvature. To facilitate this, we first derive a differential equation relating the curvature $d\phi/ds$ to the deflection $v(x)$.

We start with the definition of the slope of the neutral axis in

$$\frac{dv}{dx} = \tan\phi. \qquad \cdots\cdots\cdots\cdots(2)$$



Differentiating (2) w.r.t. s,

$$\frac{d^2v}{dxds} = \frac{d^2v}{dx^2}\frac{dx}{ds} = \sec^2\phi\frac{d\phi}{ds},$$

or the curvature,

$$\frac{d\phi}{ds} = \cos^2\phi\frac{d^2v}{dx^2}\frac{dx}{ds} = \frac{d^2v/dx^2}{\left[1+(dv/dx)^2\right]^{3/2}}, \qquad \left(\because \cos\phi = \frac{dx}{ds} = \frac{1}{\left[1+(dv/dx)^2\right]^{1/2}}\right)$$

$$\cdots\cdots\cdots\cdots(3)$$

From (1) and (3) we get,

$$\frac{M_b}{EI_{zz}} = \frac{d^2v/dx^2}{[1+(dv/dx)^2]^{3/2}}.$$

M(x)

When the slope angle ϕ is small, then dv/dx is small compared to unity. If we neglect $(dv/dx)^2$ in the denominator, we obtain a simple approximation as

$$\frac{M_b}{EI_{zz}} \approx \frac{d^2v}{dx^2}. \quad \cdots\cdots\cdots\cdots(4)$$

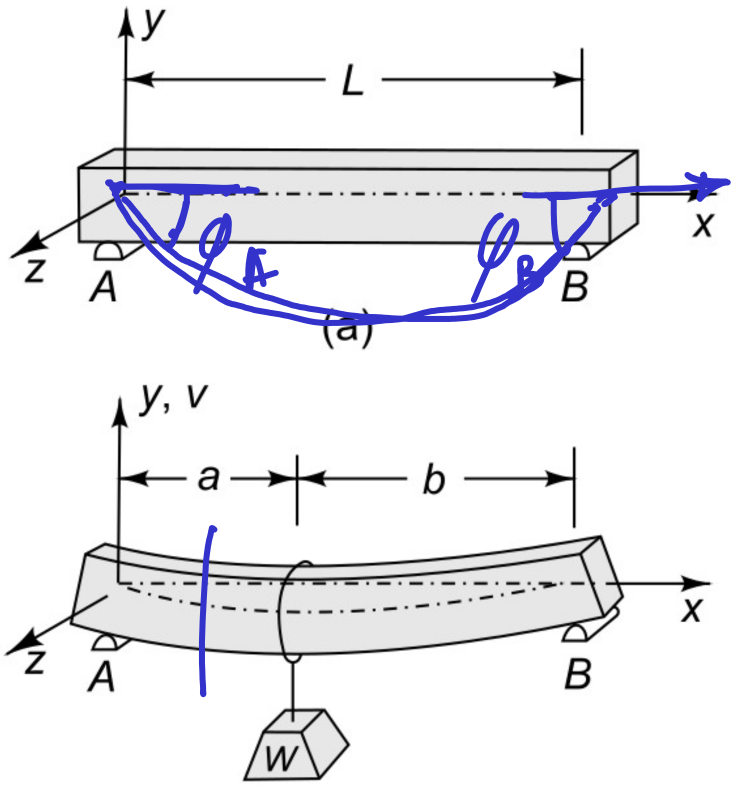
v(x)

Equation (4) relates the bending moment to the transverse displacement. Although (4) involves an approximation to the curvature which is valid only for small bending angles, we call it the **moment-curvature relation**. It is essentially a "force-deformation" or "stress-strain" relation in which the bending moment is the "force" or "stress" and the approximate curvature is the resulting "deformation" or "strain."

The relation is a linear one; the constant of proportionality EI is sometimes called the **flexural rigidity or the bending modulus.**

Example 1

The simply supported beam of uniform cross section shown in figure is subjected to a concentrated load W. It is desired to obtain the deflection curve of the deformed neutral axis.



From the bending moment analysis of the beam, it can be shown that the distribution of bending moment is as follows,

$\left.\frac{dv}{dx}\right)_{x=0}$, $\left.\frac{dv}{dx}\right|_{x=L}$

For length AC,

$$M_b = \frac{Wb}{L}x. \qquad \cdots\cdots\cdots\cdots(1.a)$$

For length CB

$$M_b = \frac{Wb}{L}x - W(x-a) \qquad \cdots\cdots\cdots\cdots(1.b)$$

For determining the displacement, we use the moment-curvature relationship (4).
First for part AC $(x < a)$as,

$$EI\frac{d^2v}{dx^2} = M_b = \frac{Wb}{L}x \qquad \cdots\cdots\cdots\cdots(1.c)$$

Since EI is constant throughout the beam, integration of (1.c) gives following,

$$EI\frac{dv}{dx} = \frac{Wb}{L}\frac{x^2}{2} + c_1 \qquad \cdots\cdots\cdots\cdots(1.c)$$

$$EIv = \frac{Wb}{L}\frac{x^3}{6} + c_1x + c_2 \qquad \cdots\cdots\cdots\cdots(1.d)$$

For part CD $(x>a)$,

$$EI\frac{d^2v}{dx^2} = M_b = \frac{Wb}{L}x - W(x-a)$$

Integrating twice we get,

$$EI\frac{dv}{dx} = \frac{Wb}{L}\frac{x^2}{2} - W\frac{(x-a)^2}{2} + c_3 \qquad \cdots\cdots\cdots\cdots(1.e)$$

$$EIv = \frac{Wb}{L}\frac{x^3}{6} - W\frac{(x-a)^3}{6} + c_3x + c_4 \qquad \cdots\cdots\cdots\cdots(1.f)$$

Now we apply boundary conditions to determine the constants of integration as follows:
(I) at x=0, v=0, which gives $c_2 = 0$

(II) at x=L, v=0, which gives $$\frac{WbL^2}{6} - W\frac{b^3}{6} + c_3x + c_4 = 0 \qquad \text{..........(1.g)}$$

(III) at x=a, $v_{AC} = v_{CB}$ $$\frac{Wb}{L}\frac{a^3}{6} + c_1a = \frac{Wb}{L}\frac{a^3}{6} + c_3x + c_4 \qquad \text{..........(1.h)}$$

(IV) at $x = a$, $$\left.\frac{dv}{dx}\right|_{AC} = \left.\frac{dv}{dx}\right|_{CB}$$

$$\frac{Wb}{L}\frac{a^2}{2} + c_1 = \frac{Wb}{L}\frac{a^2}{2} + c_3 \quad \Rightarrow \quad c_1 = c_3. \qquad \text{..........(1.i)}$$

Finally, we get

$$v = -\frac{W}{6EI}\left[\frac{bx}{L}\left(L^2 - b^2 - x^2\right)\right] \quad (0 \le x \le a) \qquad \text{..........(1.j)}$$

$$v = -\frac{W}{6EI}\left[\frac{bx}{L}\left(L^2 - b^2 - x^2\right) + (x-a)^3\right] \quad (a \le x \le L) \qquad \text{..........(1.k)}$$

Check deflection and slope for the given data, which correspond to a very common case in small-house construction. Verify that the assumptions considered for the derivation are satisfied.

$$L = 3.70 \text{ m},\ a = b = 1.85 \text{ m},\ W = 1.8 \text{ kN},\ E = 11 \text{ GPa},\ I = 3.33 \times 10^7 \text{ mm}^4$$

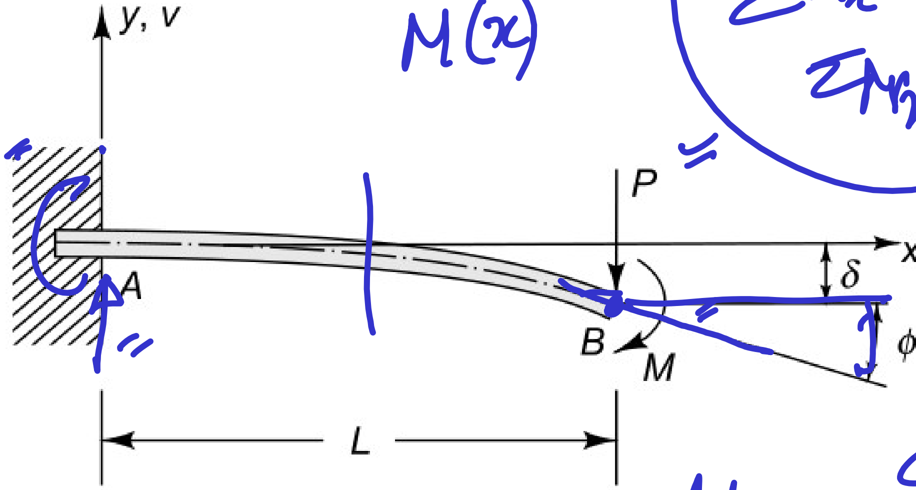
Example 2

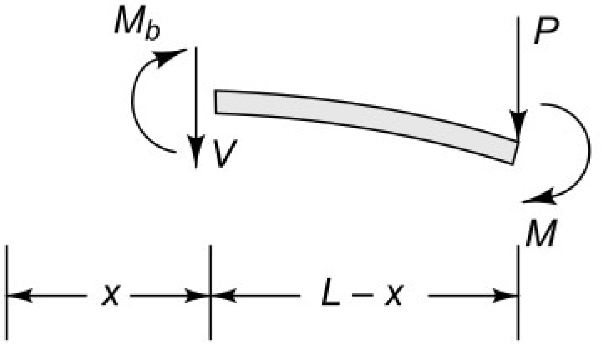
A uniform cantilever beam has bending modulus EI and length L. It is built in at A and subjected to a concentrated force P and moment M applied at B, as shown in figure. We shall find the deflection δ and the slope angle ϕ at B due to these loads.

$M(x)$

$\Sigma F_x = 0$

$\Sigma M_x = 0$





$$\boxed{M_b = -P(L-x) - M}$$

C_1

C_2

$\frac{dv}{dx} = C_1$

$v = C_1 x + C_2$

B.C.,

$$\boxed{\begin{aligned} &v = 0 \text{ at } x = 0 \text{ and} \\ &\phi = dv/dx = 0 \text{ at } x = 0. \end{aligned}}$$

Use moment-curvature relations, integrate and apply boundary conditions to find the expression of the deflection as follows.

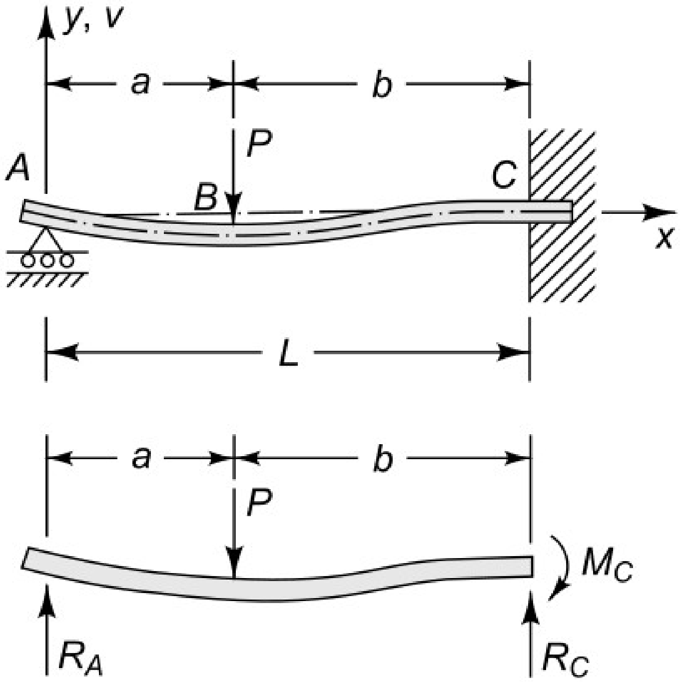
$$v = -\frac{1}{EI}\left[\frac{Px^2}{6}(3L - x) + \frac{Mx^2}{2}\right]$$

$$\delta = -v|_{x=L} = \frac{PL^3}{3EI} + \frac{ML^2}{2EI}$$

$$\phi = -\left.\frac{dv}{dx}\right|_{x=L} = \frac{PL^2}{2EI} + \frac{ML}{EI}$$

Example 3

Figure shows a beam whose neutral axis coincided with the x axis before the load P was applied. The beam has a simple support at A and a clamped or built-in support at C. The bending modulus EI is constant along the length of the beam.



From equilibrium conditions:

$$R_C = P - R_A \qquad \text{(3.a)}$$

$$M_C = Pb - R_A L \qquad \text{(3.b)}$$

$$M_b = R_A x \quad (x < a) \qquad \text{(3.c)}$$

$$M_b = R_A x - P(x - a) \quad (x \geq a) \qquad \text{(3.d)}$$

Boundary conditions

$$v = 0 \text{ at } x = 0 \qquad \text{(3.e)}$$

$$v = 0 \text{ at } x = L \qquad \text{(3.f)}$$

$$dv/dx = 0 \text{ at } x = L \qquad \text{(3.g)}$$

For part AB,

$$EI\frac{d^2v}{dx^2} = M_b = R_A x \quad \cdots\cdots\cdots\cdots (3.i)$$

$$EI\frac{dv}{dx} = R_A\frac{x^2}{2} + c_1 \quad \cdots\cdots\cdots\cdots (3.j)$$

$$EIv = R_A\frac{x^3}{6} + c_1 x + c_2 \cdots\cdots\cdots\cdots (3.k)$$

For part BC,

$$EI\frac{d^2v}{dx^2} = M_b = R_A x - P(x-a) \quad \cdots\cdots\cdots\cdots (3.l)$$

$$EI\frac{dv}{dx} = R_A\frac{x^2}{2} - P\frac{(x-a)^2}{2} + c_3 \cdots\cdots\cdots\cdots (3.m)$$

$$EIv = R_A\frac{x^3}{6} - P\frac{(x-a)^3}{6} + c_3 x + c_4 \cdots\cdots\cdots\cdots (3.n)$$

Two more boundary conditions comes from the requirement of compatibility, which are

$$(v|_{x=a})_{AB} = (v|_{x=a})_{BC} \cdots\cdots\cdots\cdots (3.o)$$

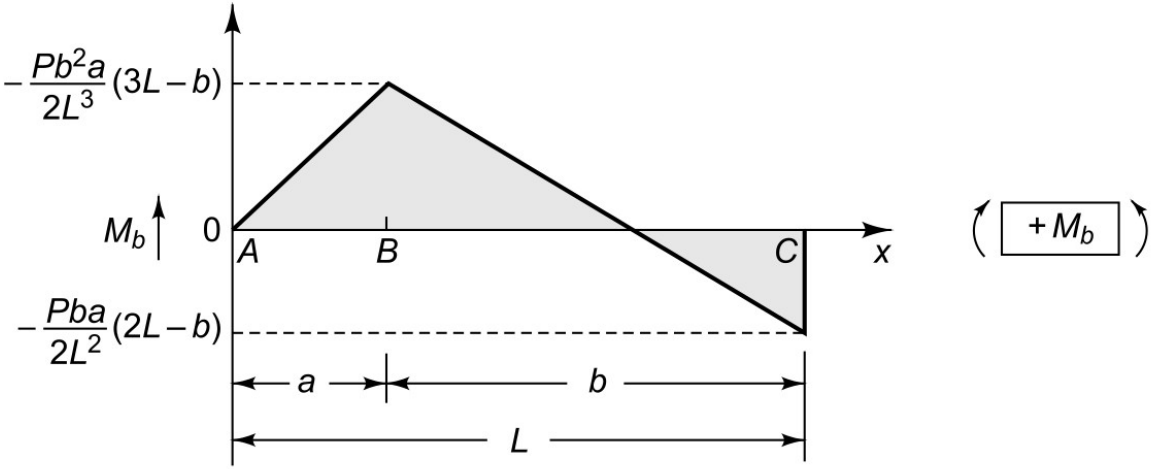
$$\left(\left.\frac{dv}{dx}\right|_{x=a}\right)_{AB} = \left(\left.\frac{dv}{dx}\right|_{x=a}\right)_{BC} \cdots\cdots\cdots\cdots (3.p)$$

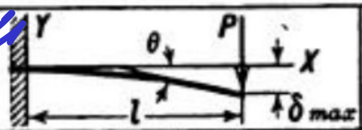
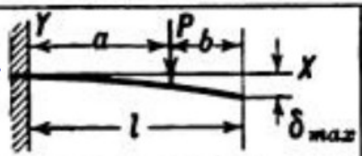
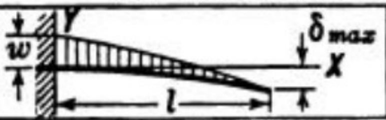
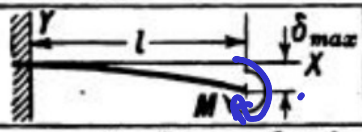
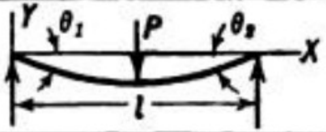
5

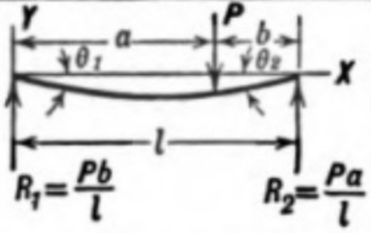
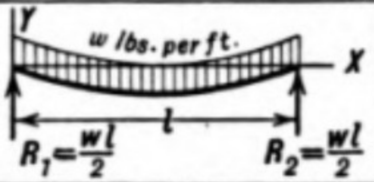
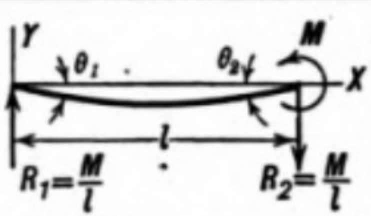
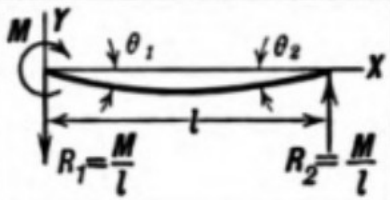
Note that reaction force R_A is also an unknown, as it can not be determined from the equilibrium equations. Thus, boundary conditions (3.e)-(3.g), (3.o) and (3.p) will provide five necessary equations to determine five unknowns, i.e., four constants of integration c_1 to c_4 and reaction force R_A. R_A is as follows,

$$R_A = \frac{Pb^2}{2L^3}(3L - b) \quad \text{..........(3.q)}$$

Now bending moment diagram can be drawn for this beam.



ART. 49	Slope at free end.	Deflection at any section in terms of x: δ is positive downward.	Maximum deflection.
1. Cantilever Beam — Concentrated load P at the free end.			
− 	$\theta = \frac{Pl^2}{2EI}$	$\delta = \frac{Px^2}{6EI}(3l - x)$	$\delta_{max} = \frac{Pl^3}{3EI}$
2. Cantilever Beam — Concentrated load P at any point.			
	$\theta = \frac{Pa^2}{2EI}$	$\delta = \frac{Px^2}{6EI}(3a - x)$ for $0 < x < a$ $\delta = \frac{Pa^2}{6EI}(3x - a)$ for $a < x < l$	$\delta_{max} = \frac{Pa^2}{6EI}(3l - a)$
3. Cantilever Beam — Uniformly distributed load of w lbs. per ft. over entire length.			
	$\theta = \frac{wl^3}{6EI}$	$\delta = \frac{wx^2}{24EI}(x^2 + 6l^2 - 4lx)$	$\delta_{max} = \frac{wl^4}{8EI}$
4. Cantilever Beam — Uniformly varying load; maximum intensity w lbs. per ft.			
	$\theta = \frac{wl^3}{24EI}$	$\delta = \frac{wx^2}{120lEI}(10l^3 - 10l^2x + 5lx^2 - x^3)$	$\delta_{max} = \frac{wl^4}{30EI}$
5. Cantilever Beam — Couple M applied at the free end.			
− 	$\theta = \frac{Ml}{EI}$	$\delta = \frac{Mx^2}{2EI}$	$\delta_{max} = \frac{Ml^2}{2EI}$
6. Beam freely supported at ends — Concentrated load P at the center.			
	$\theta_1 = \theta_2 = \frac{Pl^2}{16EI}$	$\delta = \frac{Px}{12EI}\left(\frac{3l^2}{4} - x^2\right)$ for $0 < x < \frac{l}{2}$	$\delta_{max} = \frac{Pl^3}{48EI}$

ART. 49	Slope at ends.	Deflection at any section in terms of x: δ is positive downward.	Maximum and center deflections.
7. Beam freely supported at the ends — Concentrated load at any point.			
	Left End. $\theta_1=\frac{Pb(l^2-b^2)}{6lEI}$ Right End. $\theta_2=\frac{Pab(2l-b)}{6lEI}$	To the left of load P: $\delta=\frac{Pbx}{6lEI}(l^2-x^2-b^2)$ To the right of load P: $\delta=\frac{Pb}{6lEI}\left[\frac{l}{b}(x-a)^3+(l^2-b^2)x-x^3\right]$	$\delta_{max}=\frac{Pb(l^2-b^2)^{3/2}}{9\sqrt{3}\,l\,EI}$ at $x=\sqrt{\frac{l^2-b^2}{3}}$ At center, if $a>b$ $\delta=\frac{Pb}{48EI}(3l^2-4b^2)$
8. Beam freely supported at the ends—Uniformly distributed load of w lbs. per ft.			
	$\theta_1=\theta_2=\frac{wl^3}{24EI}$	$\delta=\frac{wx}{24EI}(l^3-2lx^2+x^3)$	$\delta_{max}=\frac{5wl^4}{384EI}$
9. Beam freely supported at the ends — Couple M at the right end.			
	$\theta_1=\frac{Ml}{6EI}$ $\theta_2=\frac{Ml}{3EI}$	$\delta=\frac{Mlx}{6EI}\left(1-\frac{x^2}{l^2}\right)$	$\delta_{max}=\frac{Ml^2}{9\sqrt{3}EI}$ at $x=\frac{l}{\sqrt{3}}$ At center $\delta=\frac{Ml^2}{16EI}$
10. Beam freely supported at the ends—Couple M at the left end.			
	$\theta_1=\frac{Ml}{3EI}$ $\theta_2=\frac{Ml}{6EI}$	$\delta=\frac{Mx}{6lEI}(l-x)(2l-x)$	$\delta_{max}=\frac{Ml^2}{9\sqrt{3}EI}$ at $x=\left(1-\frac{1}{\sqrt{3}}\right)l$ At center $\delta=\frac{Ml^2}{16EI}$

Method of superposition

- It can be observed in the foregoing examples that the relationship between load and the deflection is linear.
- This linearity depends upon the linearity of moment-curvature relationship and the fact that we are considering a linear elastic material.
- It also depends upon the fact that we assumed that deflections are small, which lead to linearity between deflection and curvature.
- This linear relationship between deflection and moment allows superposition of deflections; i.e., total deflection due to a number of loads is equal to the sum of the deflections due to each load acting separately. It can also be shown analytically as follows.

Let the total bending moment is the sum of a number of individual bending moments as $M_b = M_{b1} + M_{b2} + ...$, and v_1, v_2 ... be the deflection to due individual moments, then we can write,

$$EI\frac{d^2v_1}{dx^2} = M_{b1}$$

$$EI\frac{d^2v_2}{dx^2} = M_{b2}$$

$$\cdots\cdots\cdots\cdots$$

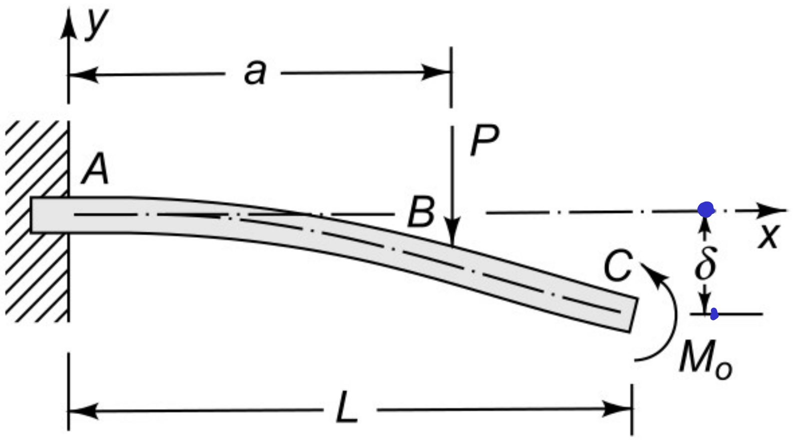
Total deflection, i.e., the sum of v_1, v_2 ... will also satisfy the deflection equation. Hence,

$$EI\frac{d^2v}{dx^2} = EI\frac{d^2}{dx^2}\left(v_1 + v_2 + \cdots\right)$$

$$EI\frac{d^2v}{dx^2} = EI\frac{d^2v_1}{dx^2} + EI\frac{d^2v_2}{dx^2} + \cdots = M_{b1} + M_{b2} + \cdots = M_b \quad \cdots\cdots\cdots(5)$$

Example 4

A cantilever beam carries a concentrated load P and an end moment M_O. It is desired to predict the deflection δ at the free end C in terms of the constant bending modulus EI and the dimensions shown. Use method of superpostion.



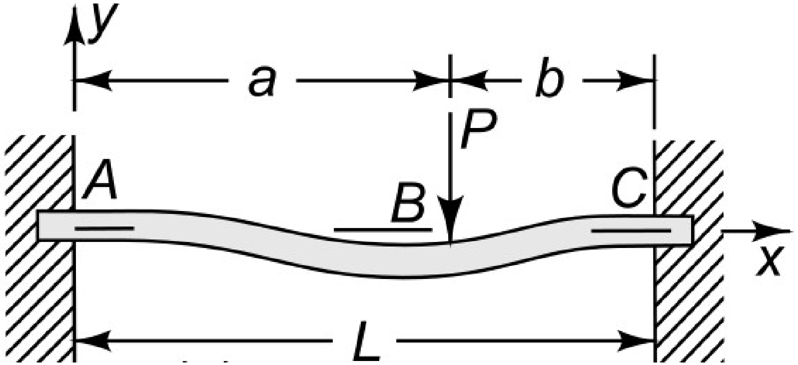
$$\delta = \delta_P + \delta_{M_O}$$

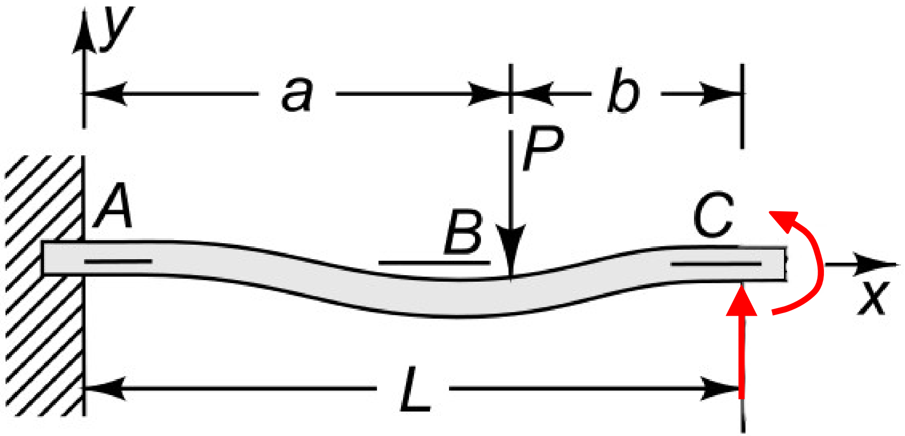
$$\delta_P = \frac{Pa^2}{6EI}(3L - a)$$

$$\delta_{M_O} = \frac{-M_O L^2}{2EI}$$

Example 5

A uniform beam which is built-in at the ends carries a concentrated load P. It is desired to obtain the bending-moment diagram.





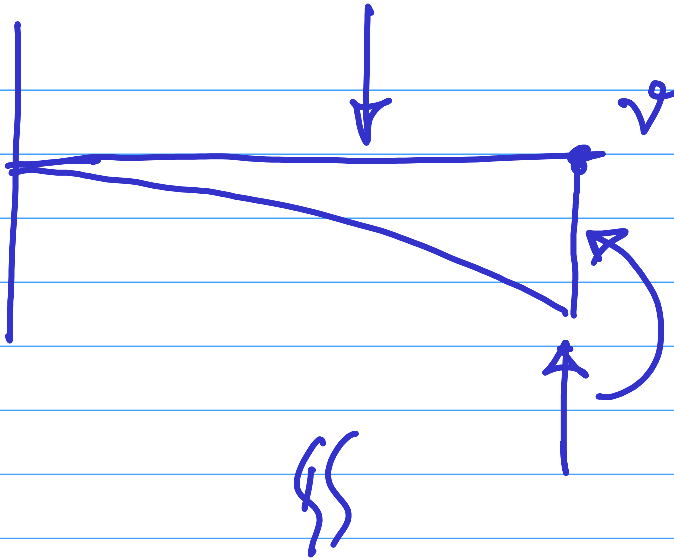
$$\delta|_C = \delta_P + \delta_{R_C} + \delta_{M_C} = 0 \qquad \cdots\cdots\cdots\cdots(5.a)$$

$$\phi|_C = \phi_P + \phi_{R_C} + \phi_{M_C} = 0 \qquad \cdots\cdots\cdots\cdots(5.b)$$

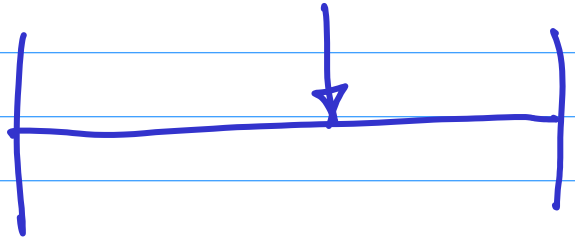
$$\delta_P = \frac{Pa^2(3L-a)}{6EI}, \quad \phi_P = \frac{Pa^2}{2EI}, \qquad \cdots\cdots\cdots\cdots(5.c)$$

$$\delta_{R_C} = -\frac{R_C L^3}{3EI}, \quad \phi_{R_C} = -\frac{R_C L^2}{2EI}, \qquad \cdots\cdots\cdots\cdots(5.d)$$

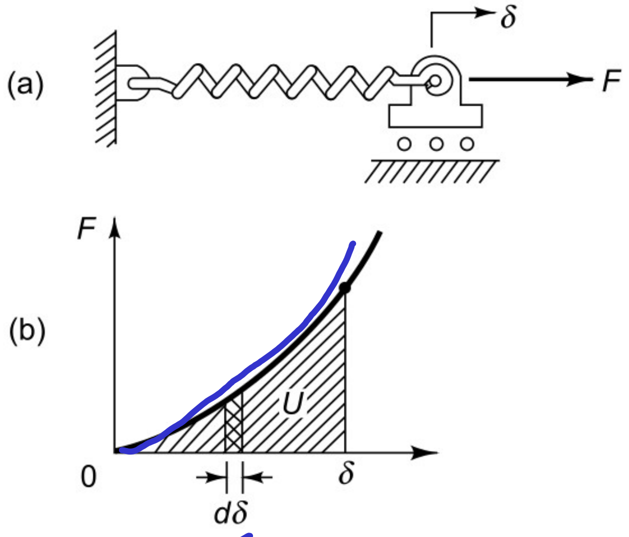
$$\delta_{M_C} = -\frac{M_C L^2}{2EI}, \quad \phi_{M_C} = -\frac{M_C L}{EI}, \qquad \cdots\cdots\cdots\cdots(5.e)$$



$v = 0, \frac{dw}{dx} = 0$



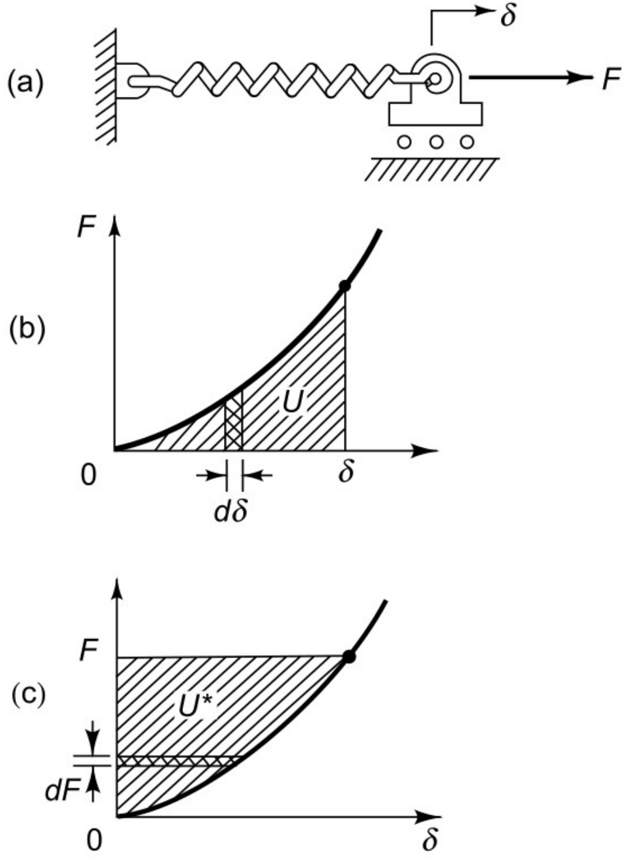
Energy methods



Consider an elastic spring (not necessarily linear). The spring undergo a gradual elongation under the action of an external force F, which remain in equilibrium with the internal force in the spring during the elongation. The work done by the external force F is stored in the elastic body as potential energy U which is given as,

$$U = \int \boldsymbol{F} \cdot \boldsymbol{ds} = \int_0^{\delta} F d\delta. \qquad \cdots\cdots\cdots\cdots(6)$$

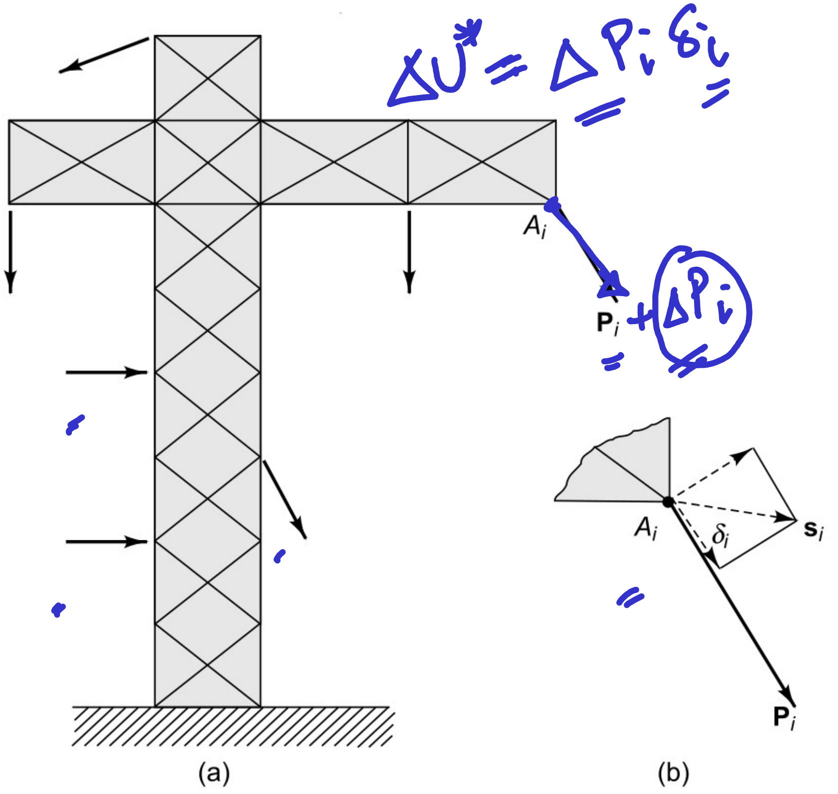
Potential energy U appears as the area under the force-deflection curve.



During the loading process of the spring the complementary energy is given as,

$$U^* = \int \boldsymbol{s} \cdot \boldsymbol{dF} = \int_0^F \delta dF. \qquad \cdots\cdots\cdots\cdots(7)$$

The shaded area above the force-deflection curve given the complementary energy.



Now consider a general elastic structure, which is loaded by an arbitrary number of loads. At a typical point A_i the load is $\boldsymbol{P}_i$, and the equilibrium displacement due to all loads is $\boldsymbol{s}_i$. If during the loading process $\boldsymbol{s}_i$ are permitted to grow slowly through a sequence of equilibrium configurations, the total potential energy U stored by all the internal elastic members will be

$$U = \sum_i \int_0^{\boldsymbol{s}_i} \boldsymbol{P}_i \cdot \boldsymbol{d}s_i \qquad \cdots\cdots\cdots\cdots(8)$$

If the loads are gradually increased from zero so that the system passed through a succession of equilibrium states, the total complimentary energy stored by all the internal elastic members will be

$$U^* = \sum_i \int_0^{\boldsymbol{P}_i} \boldsymbol{s}_i \cdot \boldsymbol{d}P_i = \sum_i \int_0^{\boldsymbol{P}_i} \delta_i dP_i \qquad \cdots\cdots\cdots\cdots(9)$$

Now if the structure is under equilibrium position with the complementary energy (9), we consider a small increment ΔP_i of the load P_i while all other loads remain fixed. The internal forces will change slightly to maintain force equilibrium, and the increment in complementary work will equal the increment in complementary energy ΔU^*. For small ΔP_i we have, approximately

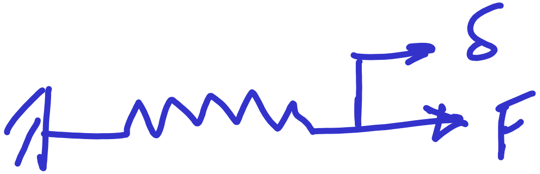
$$\Delta U^* = \delta_i \Delta P_i \quad \text{or} \quad \delta_i = \frac{\Delta U^*}{\Delta P_i}. \qquad \cdots\cdots\cdots\cdots(10)$$

In the limit $\Delta P_i \rightarrow 0$, we have

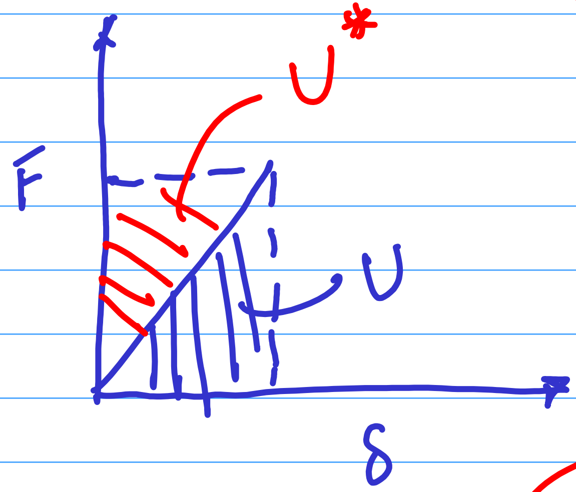
$$\delta_i = \frac{\partial U^*}{\partial P_i}. \qquad \cdots\cdots\cdots\cdots(11)$$

Equation (11) is a form of Castigliano's theorem, which states that the total complementary energy U* of a loaded elastic system is expressed in terms of the loads, the in-line deflection at any particular loading point is obtained by differentiating U* with respect to the load at that point. It can be extended to include moment load M_i, in which case the in-line displacement is the angle of rotation ϕ_i and

$$\phi_i = \frac{\partial U^*}{\partial M_i}. \qquad \cdots\cdots\cdots\cdots(12)$$



$$F = K\delta$$



$$U = U^*$$

$$\delta_i = \frac{\partial U}{\partial P_i}$$

$$\phi_i = \frac{\partial U}{\partial M_i}$$

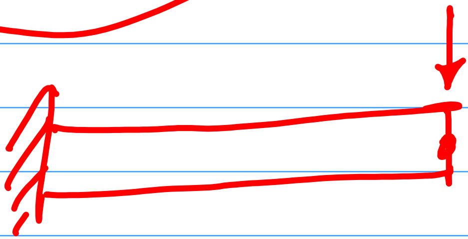
U

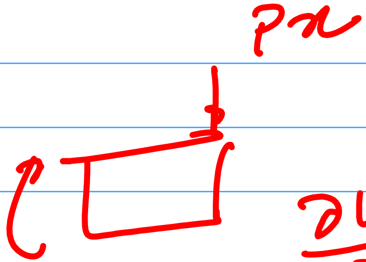
$$\delta = \frac{PL^3}{3EI}$$

$$U = \int_0^L \frac{M^2}{2EI}\,dx$$

$$= \int_0^L \frac{P^2x^2}{2EI}\,dx$$

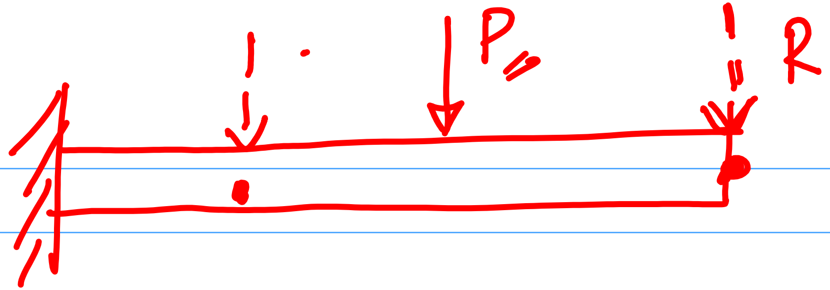
$$= \frac{P^2L^3}{6EI}$$





$$U = \frac{P^2L^3}{6EI}$$

$$\frac{\partial U}{\partial P} = \frac{2PL^3}{6EI}$$



$$\delta_i = \frac{\partial U}{\partial P_i}$$

$$U = U(P, R)$$

$$\boxed{\delta_R = \frac{\partial U(P,R)}{\partial R} \Bigg|_{R=0}}$$