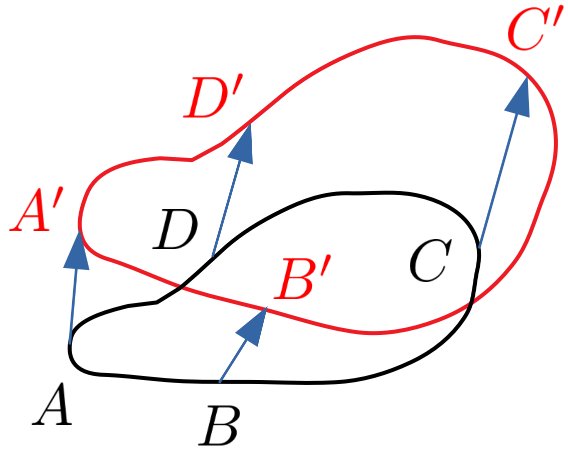
ME231: Solid Mechanics-I

Stress and Strain

Deformation



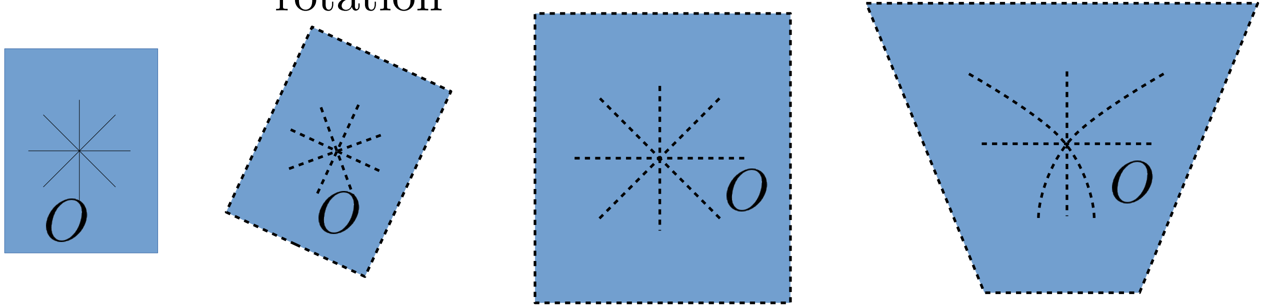
Displacement of a continuous body may consist of,

- Rigid body displacement
- Rigid body rotation
- Relative displacements between points (deformation)

Rigid body rotation

Uniform deformation

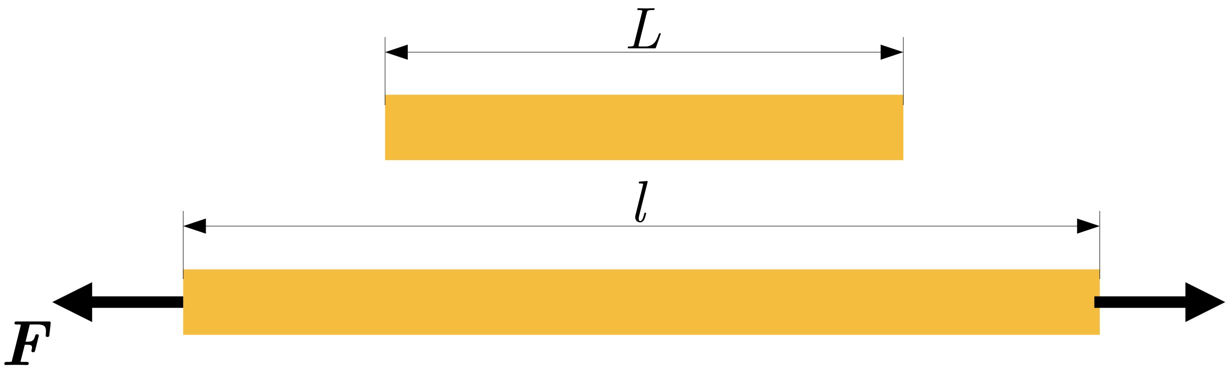
Non-uniform deformation



For negligible area surrounding point O, deformation can be assumed to be uniform even if overall deformation is non-uniform.

- The displacements associated with rigid-body motion can be either large or small, while the displacements associated with deformation usually are small.
- The description and analysis of rigid-body motion is important in dynamics where the forces required to produce different time rates of rigid-body motion are of interest.
- The description and analysis of deformation is important in our present study of the mechanics of deformable bodies where the forces required to produce different distortions are of interest.
- To start with, we focus our attention on a body whose particles all lie in the same plane and which deforms only in this plane. This type of deformation is called **plane strain.**

Measurement of deformation: Strain



Strain is a measure of deformation. For uniaxial condition strain is defined as follows. **Engineering or nominal strain** is defined as **deformation per unit original length**.

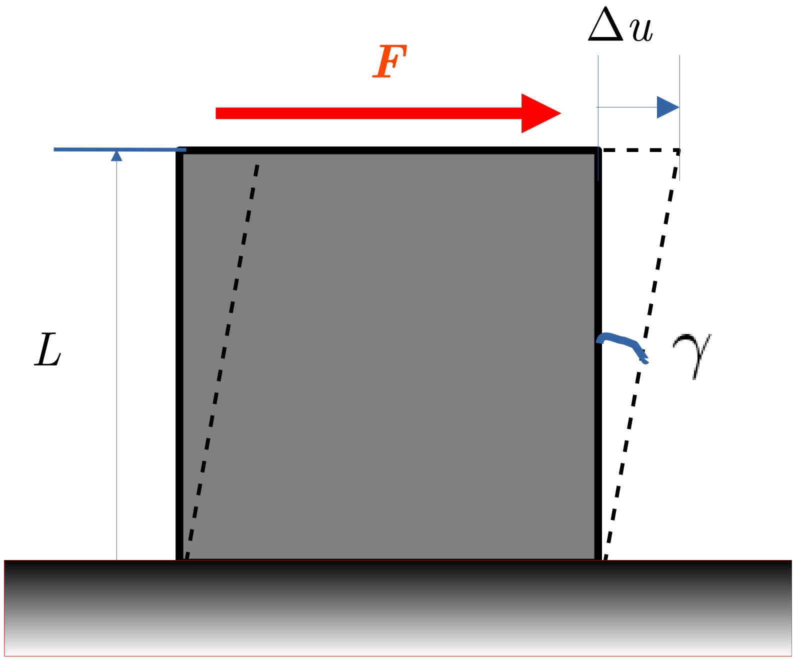
$$\varepsilon = \frac{l - L}{L} = \frac{\Delta L}{L}. \qquad \text{..........(32)}$$

Another definition of strain is called **true or logarithmic strain** defined as,

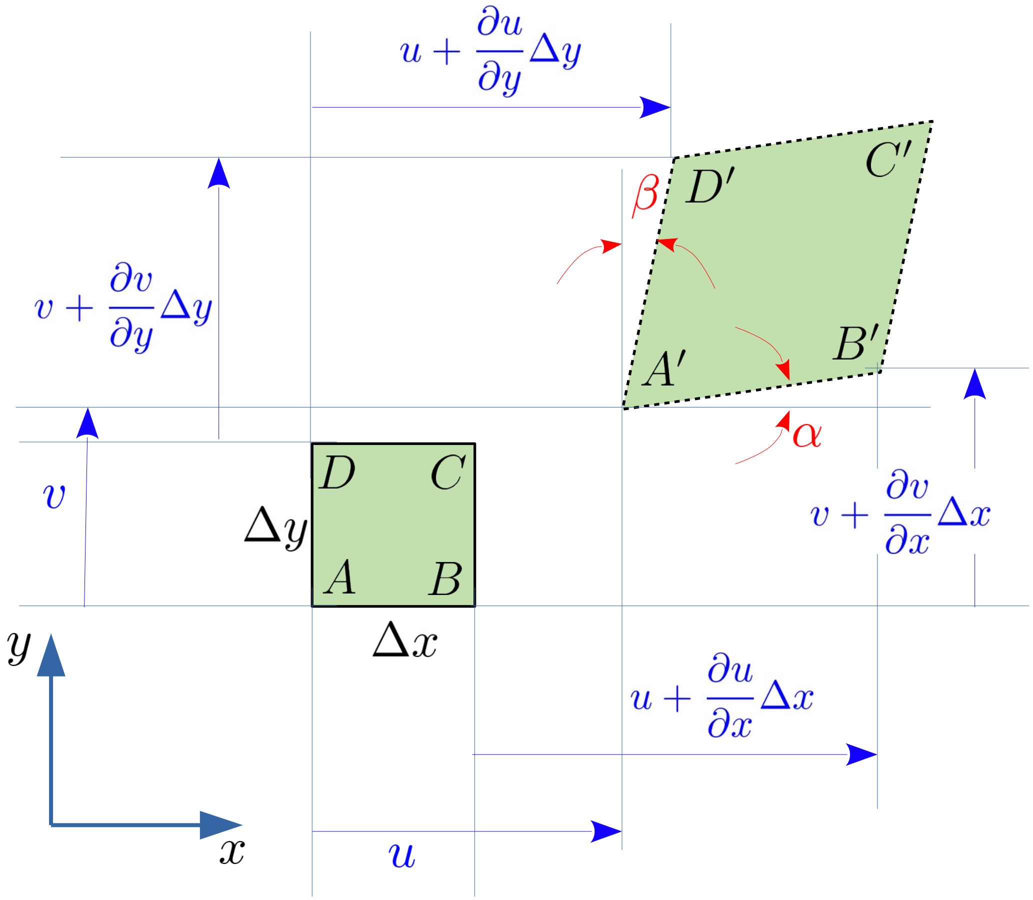
$$e = \int_L^l \frac{dl}{l} = \ln \frac{l}{L}. \qquad \text{..........(33)}$$

In case of tangential force, **shear strain** is defined as

$$\tan\gamma \approx \gamma = \frac{\Delta u}{L}.$$



Plane strain in case of small deformations



$$\epsilon_x = \lim_{\Delta x \to 0} \frac{A'B' - AB}{AB} = \lim_{\Delta x \to 0} \frac{\Delta x + \frac{\partial u}{\partial x}\Delta x - \Delta x}{\Delta x},$$

$$\Rightarrow \epsilon_x = \frac{\partial u}{\partial x},$$

$$\epsilon_y = \lim_{\Delta y \to 0} \frac{A'D' - AD}{AD} = \lim_{\Delta y \to 0} \frac{\Delta y + \frac{\partial v}{\partial y}\Delta y - \Delta y}{\Delta y},$$

$$\Rightarrow \epsilon_y = \frac{\partial v}{\partial y},$$

$$\gamma_{xy} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\pi}{2} - \angle B'A'D' = \alpha + \beta$$

$$\Rightarrow \gamma_{xy} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\frac{\partial v}{\partial x}\Delta x}{\Delta x + \frac{\partial u}{\partial x}\Delta x} + \frac{\frac{\partial u}{\partial y}\Delta y}{\Delta y + \frac{\partial v}{\partial y}\Delta y}$$

$$\Rightarrow \gamma_{xy} \approx \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.$$

Strain-displacement relations(in 2D)

Engineering strain components,

$$\epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \text{..........(34)}$$

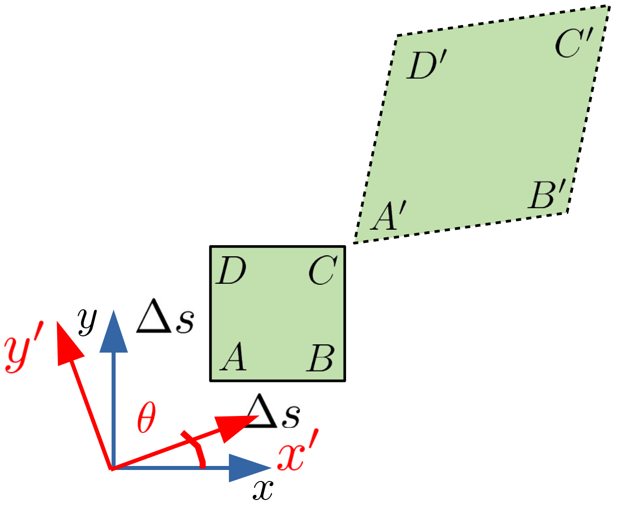
Similar to stress, strain is also a second order tensor. In plane strain case, strain matrix is defined as,

$$[\boldsymbol{\epsilon}] = \begin{bmatrix} \epsilon_{xx} & \gamma_{xy}/2 \\ \gamma_{yx}/2 & \epsilon_{yy} \end{bmatrix} \qquad \text{..........(35)}$$

Strain transformation

Being a second order tensor, strain tensor follows general rules of transformations.

$$[\boldsymbol{\epsilon}]' = [\boldsymbol{Q}]^T[\boldsymbol{\epsilon}][\boldsymbol{Q}],$$



Following this equation transformed strains will be given as

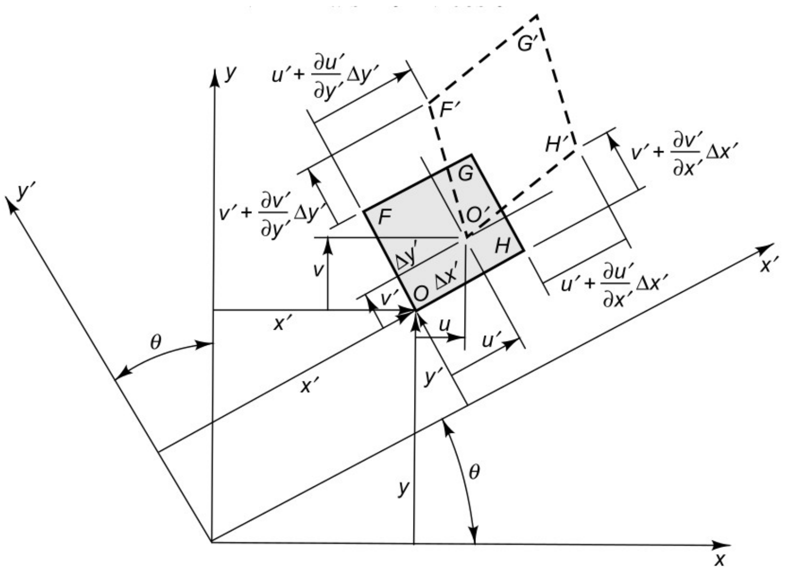
$$\epsilon'_{xx} = \frac{(\epsilon_{xx} + \epsilon_{yy})}{2} + \frac{(\epsilon_{xx} - \epsilon_{yy})}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta,$$
$$\epsilon'_{yy} = \frac{(\epsilon_{xx} + \epsilon_{yy})}{2} - \frac{(\epsilon_{xx} - \epsilon_{yy})}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta,$$
$$\gamma'_{xy} = \frac{\gamma_{xy}}{2}\cos 2\theta - \frac{(\epsilon_{xx} - \epsilon_{yy})}{2}\sin 2\theta.$$

..........(36)

Strain transformation equations (36) can also be derived from purely geometrical relations. Relations between the coordinate and displacements in xy-system and $x'y'$-system is given as

$$
\begin{aligned}
x &= x'\cos\theta - y'\sin\theta & \qquad u' &= u\cos\theta + v\sin\theta \\
y &= x'\sin\theta + y'\cos\theta. & \qquad v' &= -u\sin\theta + v\cos\theta.
\end{aligned}
\qquad \cdots\cdots\cdots(38)
$$

Now, using strain-displacement relations,



$$
\begin{aligned}
\epsilon'_x &= \frac{\partial u'}{\partial x'} = \frac{\partial u'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial x'} = \frac{\partial u'}{\partial x}\cos\theta + \frac{\partial u'}{\partial y}\sin\theta \\
&\Rightarrow \left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial v}{\partial x}\sin\theta\right)\cos\theta + \left(\frac{\partial u}{\partial y}\cos\theta + \frac{\partial v}{\partial y}\sin\theta\right)\sin\theta \\
&\Rightarrow \frac{\partial u}{\partial x}\cos^2\theta + \frac{\partial v}{\partial y}\sin^2\theta + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\sin\theta\cos\theta, \\
&\Rightarrow \epsilon_x\cos^2\theta + \epsilon_y\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta.
\end{aligned}
$$

$$\cdots\cdots\cdots(39)$$

$$\epsilon'_y = \frac{\partial v'}{\partial y'} = \frac{\partial v'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial y'} = -\frac{\partial v'}{\partial x}\sin\theta + \frac{\partial v'}{\partial y}\cos\theta$$

$$\Rightarrow -\left(-\frac{\partial u}{\partial x}\sin\theta + \frac{\partial v}{\partial x}\cos\theta\right)\sin\theta + \left(-\frac{\partial u}{\partial y}\sin\theta + \frac{\partial v}{\partial y}\cos\theta\right)\cos\theta$$

$$\Rightarrow \frac{\partial v}{\partial y}\cos^2\theta + \frac{\partial u}{\partial x}\sin^2\theta - \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\sin\theta\cos\theta$$

$$\Rightarrow \epsilon_y\cos^2\theta + \epsilon_x\sin^2\theta - \gamma_{xy}\sin\theta\cos\theta \qquad \cdots\cdots\cdots\cdots(40)$$

$$\gamma'_{xy} = \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'} = \left(\frac{\partial v'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial x'}\right) + \left(\frac{\partial u'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial y'}\right)$$

$$\Rightarrow \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)\sin\theta\cos\theta + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)(\cos^2\theta - \sin^2\theta)$$

$$\Rightarrow (\epsilon_y - \epsilon_x)\sin\theta\cos\theta + \gamma_{xy}\left(\cos^2\theta - \sin^2\theta\right) \qquad \cdots\cdots\cdots\cdots(41)$$

Use of trigonometric identities will results in equations identical to the equations derived from tensor transformations.

Principal strains and maximum shear strain

- Planes at which **shear strain is zero** are called **principal planes** of strains and the normal strains at these planes are called **principal strains.** Inclination of principal planes from x-axis are given by

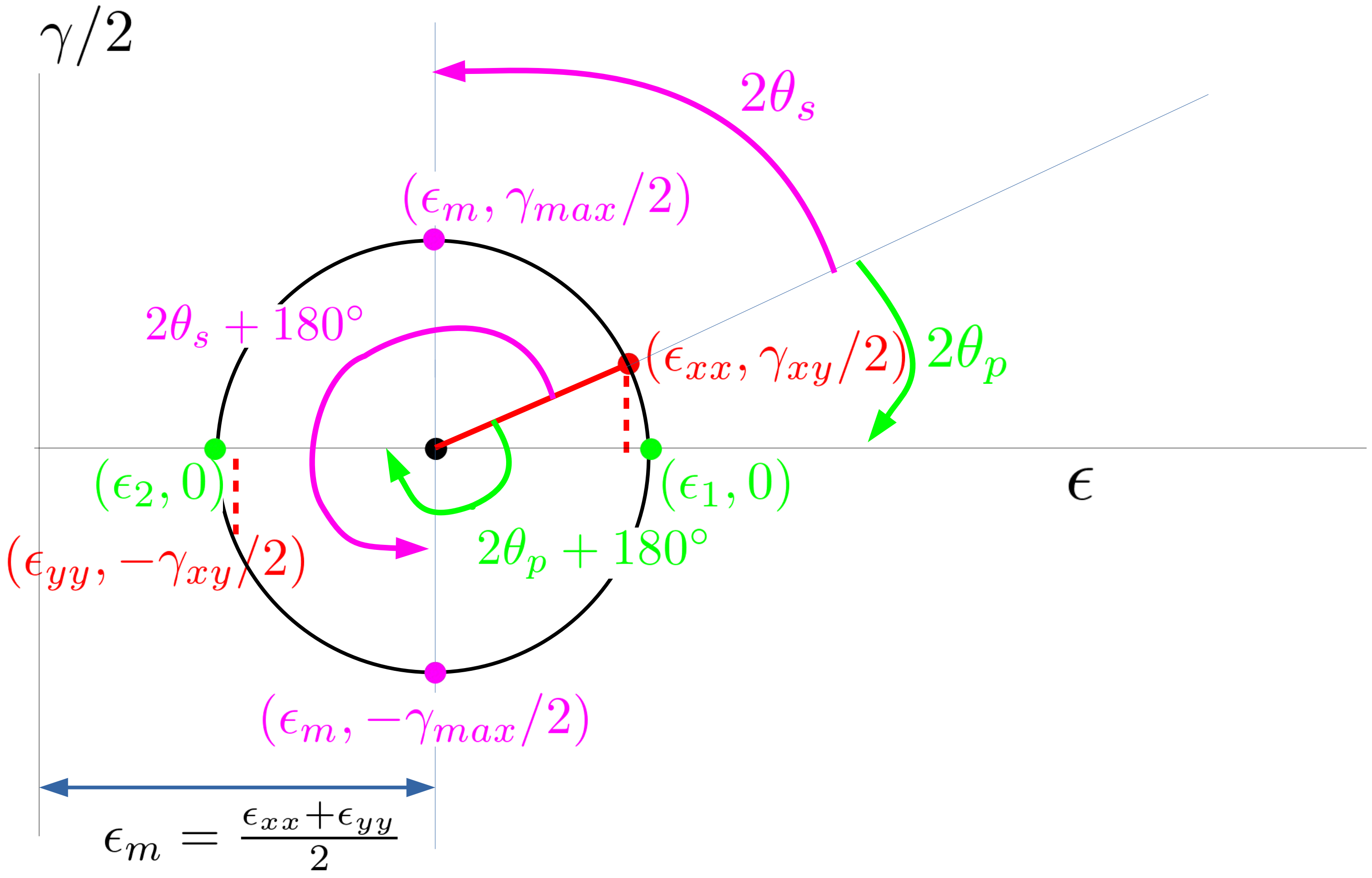
$$\tan 2\theta_p = \frac{\gamma_{xy}}{\epsilon_{xx} - \epsilon_{yy}} \qquad \cdots\cdots\cdots(42)$$

- Maximum value of shear strain is given as,

$$\gamma_{\max} = \sqrt{(\epsilon_{xx} - \epsilon_{yy})^2 + \gamma_{xy}^2}, \text{ at angle } \theta \text{ satisfying } \tan 2\theta_s = -\frac{\epsilon_{xx} - \epsilon_{yy}}{\gamma_{xy}}.$$

- Similar to stresses, Mohr's circle can be drawn for strains also. $\cdots\cdots\cdots(43)$

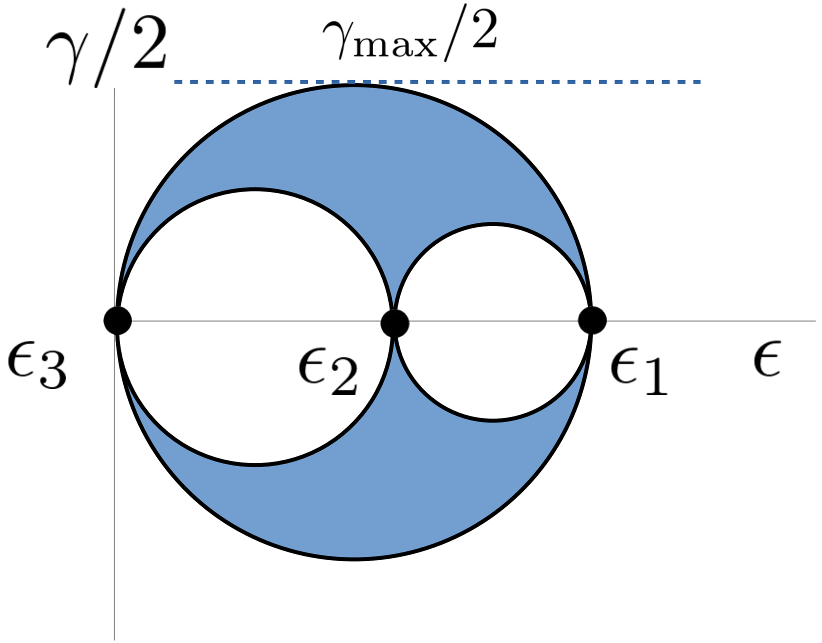
Mohr's circle for strains



Principal strains in plane strain

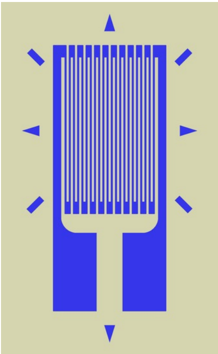
For plane strain, $\varepsilon_{zz} = 0$. In case ε_1 and, ε_2 >0, then maximum stresses will be calculated as,

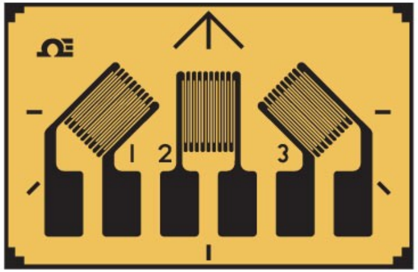
$$\gamma_{\max} = |\epsilon_{\max} - \epsilon_{\min}| = |\epsilon_3 - \epsilon_1|$$



Measurement of strains

- **Electrical strain gages** are used to measure strains.
- Strain gages work on the principal that certain metals exhibit a **change in electrical resistance with change in mechanical strain**.
- Strain gages are bonded on any surface and it measure **strain the axial direction** of the gage.
- **Strain rosette** are combination of strain gages, used to measure strains in different directions and calculate principal strains.





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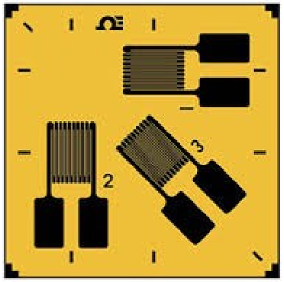

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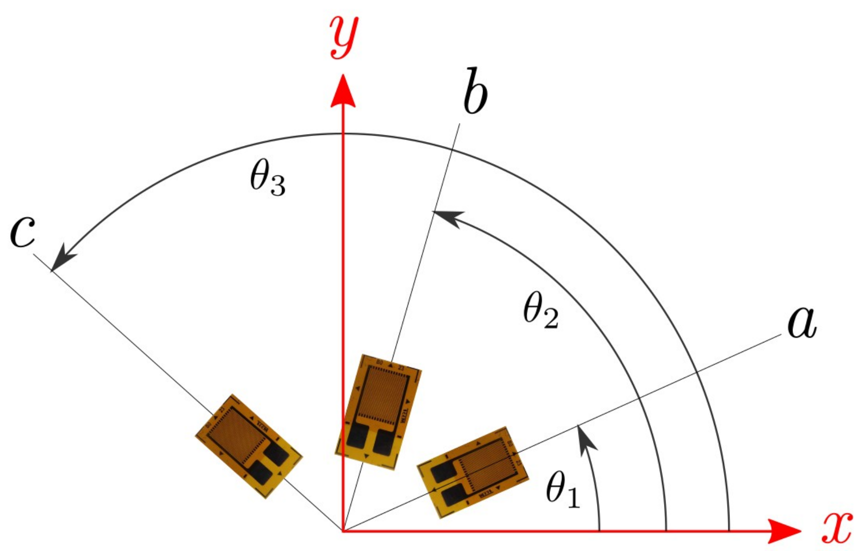
Strain rosette



If, in-plane strain components are ε_x, ε_y, γ_{xy}, then strains in the direction of a, b, and c can be expression using in-plane strains and the inclination from x-axis, as

$$\begin{aligned}
\epsilon_a &= \epsilon_x \cos^2\theta_1 + \epsilon_y \sin^2\theta_1 + \gamma_{xy}\sin\theta_1\cos\theta_1, \\
\epsilon_b &= \epsilon_x \cos^2\theta_2 + \epsilon_y \sin^2\theta_2 + \gamma_{xy}\sin\theta_2\cos\theta_2, \\
\epsilon_c &= \epsilon_x \cos^2\theta_3 + \epsilon_y \sin^2\theta_3 + \gamma_{xy}\sin\theta_3\cos\theta_3.
\end{aligned} \qquad \cdots\cdots\cdots\cdots(44)$$

These three equations can be solved to find three unknowns i.e. ε_x, ε_y, and, γ_{xy}. With the knowledge of all in-plane strain components, principal strains and their directions can also be calculated.