ME531: Advanced Mechanics of Solids

Motion, Strain and Stress

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Cauchy-Green deformation tensors

Let us consider the squared length of a line elements in the current configuration,

$$dx^2 = d\boldsymbol{x} \cdot d\boldsymbol{x} = (\boldsymbol{F} d\boldsymbol{X}) \cdot (\boldsymbol{F} d\boldsymbol{X}) = d\boldsymbol{X}(\boldsymbol{F}^T \boldsymbol{F}) d\boldsymbol{X} = d\boldsymbol{X} \cdot (\boldsymbol{C} d\boldsymbol{X}),$$

where $\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F}$ or $C_{IJ} = F_{kI} F_{kJ}$, is the Right Cauchy-Green deformation tensor. It should be noted that $\boldsymbol{C}$ is a *symmetric and positive definite tensor*. We observe that $\det \boldsymbol{C} = (\det\ \boldsymbol{F})^2 = J^2 > 0$.

Similarly, we can also find,

$$dX^2 = d\boldsymbol{X} \cdot d\boldsymbol{X} = (\boldsymbol{F}^{-1} d\boldsymbol{x}) \cdot (\boldsymbol{F}^{-1} d\boldsymbol{x}) = d\boldsymbol{x} \cdot (\boldsymbol{F}^{-T} \boldsymbol{F}^{-1}) d\boldsymbol{x} = d\boldsymbol{x} \cdot (\boldsymbol{b}^{-1} d\boldsymbol{x}),$$

where $\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^T$ or $b_{ij} = F_{iK} F_{jK}$, is the Left Cauchy-Green deformation tensor. Similar to $\boldsymbol{C}$, $\boldsymbol{b}$ is also a *symmetric and positive definite tensor* and $\det \boldsymbol{b} = (\det\ \boldsymbol{F})^2 = J^2 > 0$.

Green-Lagrange and Euler-Almansi strain tensors can be written in terms of Cauchy-Green strain tensors as,

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}), \text{ and } \boldsymbol{e} = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{b}^{-1}\right)$$

Cauchy-Green deformation tensors

Left and right Cauchy-Green deformation tensors can be physically understood in the following manner. We define line element $d\boldsymbol{X}$ in terms of its magnitude dX, and a unit vector in the direction of $d\boldsymbol{X}$ as, $d\boldsymbol{X} = dX\ \boldsymbol{N}$ and similar $d\boldsymbol{x} = dx\ \boldsymbol{n}$, where $\boldsymbol{N}$ and $\boldsymbol{n}$ are unit vectors in the directions of $\mathrm{d}\boldsymbol{X}$ and $d\boldsymbol{x}$, respectively.

Now we can write,

$$dx^2 = dX\boldsymbol{N} \cdot (\boldsymbol{C} dX\boldsymbol{N}) = dX^2\boldsymbol{N} \cdot (\boldsymbol{C}\boldsymbol{N}) \text{ or } dx^2/dX^2 = \boldsymbol{N} \cdot (\boldsymbol{C}\boldsymbol{N}).$$

Similarly,

$$dX^2 = dx\boldsymbol{n} \cdot (\boldsymbol{b}^{-1} dx\boldsymbol{n}) = dx^2\boldsymbol{n} \cdot (\boldsymbol{b}^{-1}\boldsymbol{n}) \text{ or } dX^2/dx^2 = \boldsymbol{n} \cdot (\boldsymbol{b}^{-1}\boldsymbol{n}).$$

Thus components of $\boldsymbol{C}$ (and $\boldsymbol{b}$) denote square of ratio of element length in current configuration to its length in reference configuration (and vice versa) in the specific directions. Ratio of element length in current configuration to its length in reference configurationn is known as *stretch*, i.e. $dx/dX = \lambda$ or $dx = \lambda dX$.

Rotation, stretch tensor

Deformation gradient tensor $\boldsymbol{F}$ can be multiplicatively decomposed into a *pure stretch tensor* and a *pure rotation tensor* as,

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{v}\boldsymbol{R}, \text{ or } F_{iJ} = R_{iK}U_{KJ} = v_{ik}R_{kJ}.$$

where $\boldsymbol{R}$ is a pure rotation tensor ($\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}$), and $\boldsymbol{U}$ and $\boldsymbol{v}$ are pure stretch tensors. This decomposition is called the *polar decomposition*.

Here, $\boldsymbol{U}$ and $\boldsymbol{v}$ are *unique*, *symmetric*, *positive definite* tensors, which are called *right (or material) stretch* and *left (or spatial) stretch tensors, respectively* They represent local elongation (or contraction) along their mutually orthogonal Eigen vectors. It should be noted that,

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} = (\boldsymbol{R}\boldsymbol{U})^T(\boldsymbol{R}\boldsymbol{U}) = \boldsymbol{U}^T\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{U} = \boldsymbol{U}\boldsymbol{I}\boldsymbol{U} = \boldsymbol{U}\boldsymbol{U} = \boldsymbol{U}^2, \text{ and}$$
$$\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^T = (\boldsymbol{v}\boldsymbol{R})(\boldsymbol{v}\boldsymbol{R})^T = \boldsymbol{v}\boldsymbol{R}\boldsymbol{R}^T\boldsymbol{v}^T = \boldsymbol{v}\boldsymbol{I}\boldsymbol{v} = \boldsymbol{v}\boldsymbol{v} = \boldsymbol{v}^2.$$

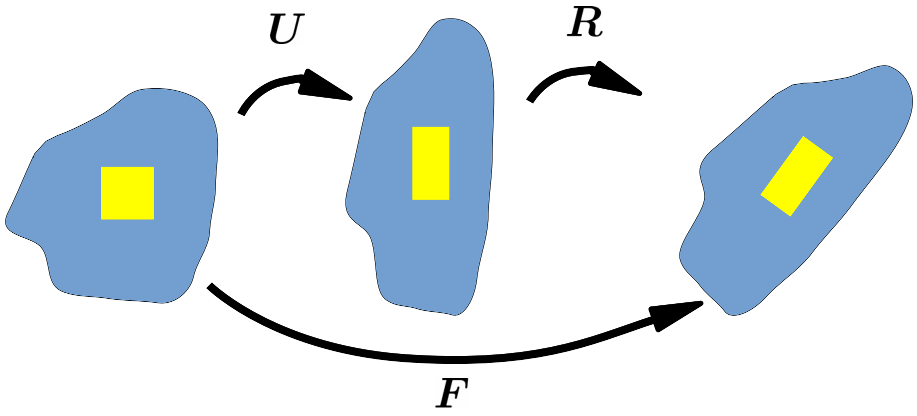
Also, $\det\boldsymbol{U} = (\det\boldsymbol{C})^{1/2} = J > 0$, $\det\boldsymbol{v} = (\det\boldsymbol{b})^{1/2} = J > 0$, and $\det\boldsymbol{R} = 1$.

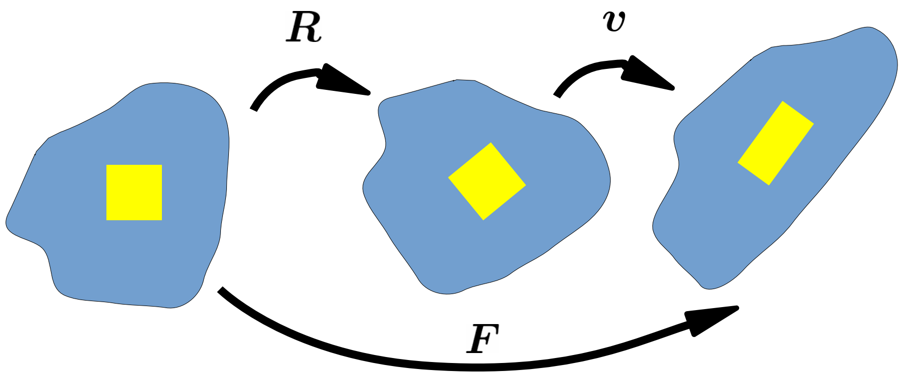
Polar decomposition can be visualized in the following manner.

- A general motion can be represented as a combination of elongation (or contraction) is some directions and then rotation of the body resulting from the deformation in the first step.
- First the initial configuration can be rotated and then the rotated body can be stretched (or contracted) in particular directions.

E.g. for a vector $\boldsymbol{X}$ in the reference configuration, a deformation is given as

$$\boldsymbol{x} = \boldsymbol{F}\boldsymbol{X} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{X} = \boldsymbol{v}\boldsymbol{R}\boldsymbol{x}$$





Eigenvalues and Eigenvectors of strain tensors

Consider the right or material stretch tensor $\boldsymbol{U}$. Eigenvalues and mutually orthogonal and normalized set of Eigenvectors of $\boldsymbol{U}$ are λ_i and $\hat{\boldsymbol{N}}_i$ respectively. Thus,

$$\boldsymbol{U}\hat{\boldsymbol{N}}_i = \lambda_i \hat{\boldsymbol{N}}_i. \quad (i = 1, 2, 3; \text{ no summation})$$

From the properties of eigenvalues and eigenvectors, we can write

$$\boldsymbol{U}^2\hat{\boldsymbol{N}}_i = \boldsymbol{C}\hat{\boldsymbol{N}}_i = \lambda_i^2 \hat{\boldsymbol{N}}_i. \quad (i = 1, 2, 3; \text{ no summation})$$

Thus $\boldsymbol{U}$ and $\boldsymbol{C}$ have the same orthogonal eigenvectors $\hat{\boldsymbol{N}}_i$ which are called the *principal referential directions* (or *principal referential axes*). Whereas the eigenvalues of $\boldsymbol{U}$ are λ_i which are called the *principal stretches* and the eigenvalues of $\boldsymbol{C}$ are square of principal stretches, i.e. λ_i^2.

To determine the Eigenvalue of left stretch tensor $\boldsymbol{v}$, we use the following relation,

$$\boldsymbol{v} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^T.$$

We write the eigenvalue problem for $\boldsymbol{v}$ as,

$$\boldsymbol{v}(\boldsymbol{R}\boldsymbol{N}_i) = \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^T(\boldsymbol{R}\boldsymbol{N}_i) = \boldsymbol{R}\boldsymbol{U}\boldsymbol{N}_i = \lambda_i \boldsymbol{R}\boldsymbol{N}_i.$$

Using above relation, we can write the eigenvalue for for $\boldsymbol{b}$ as,

$$\boldsymbol{v}^2\boldsymbol{R}\hat{\boldsymbol{N}}_i = \boldsymbol{b}\boldsymbol{R}\hat{\boldsymbol{N}}_i = \lambda_i^2\boldsymbol{R}\hat{\boldsymbol{N}}_i. \quad (i = 1, 2, 3; \text{ no summation})$$

It can be observed that $\boldsymbol{v}$ and $\boldsymbol{b}$ have same eigenvectors $\boldsymbol{n}_i = \boldsymbol{R}\hat{\boldsymbol{N}}_i$, which are called *principal spatial directions*, whereas eigenvalues are λ_i and λ_i^2, respectively.

To summarize, we can summarize the spectral decomposition of the tensors $\boldsymbol{U}$, $\boldsymbol{v}$, $\boldsymbol{C}$ and $\boldsymbol{b}$ in the following way when, $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$

$$\boldsymbol{U}^2 = \boldsymbol{C} = \sum_{i=1}^{3} \lambda_i^2 \boldsymbol{N}_i\boldsymbol{N}_i, \quad \boldsymbol{v}^2 = \boldsymbol{b} = \sum_{i=1}^{3} \lambda_i^2 \boldsymbol{n}_i\boldsymbol{n}_i.$$

Procedure for polar decomposition of $\boldsymbol{F}$

1. First calculate $\boldsymbol{C}$ as, $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$.
2. Find the eigenvalues and eigenvectors of $\boldsymbol{C}$ as λ_i and $\boldsymbol{N}_i$.
3. Eigenvalues of $\boldsymbol{U}$ is then $\sqrt{\lambda_i}$ and eigenvector remain same as $\boldsymbol{N}_i$.
4. Tensor $\boldsymbol{U}$ can be obtained as,

$$\boldsymbol{U} = \sum_{i=1}^{3} \sqrt{\lambda_i}\boldsymbol{N}_i\boldsymbol{N}_i, \quad \text{where } \boldsymbol{N}_i = N_i^1\boldsymbol{e}_1 + N_i^2\boldsymbol{e}_2 + N_i^3\boldsymbol{e}_3.$$

5. Now, $\boldsymbol{R}$ can be obtained as, $\boldsymbol{R} = \boldsymbol{F}\boldsymbol{U}^{-1}$. Here $\boldsymbol{U}$ is in $\{\boldsymbol{e}_i\}$ basis vector, which in the matrix form can be obtained as $[\boldsymbol{V}][\boldsymbol{U}_d][\boldsymbol{V}]^T$. $[\boldsymbol{U}_d]$ is the diagonal form of stretch tensor with eigenvectors as basis and $[\boldsymbol{V}]$ is a matrix with i^{th} column corresponding to the i^{th} eigenvector.
6. Now, $\boldsymbol{v}$ can be determined as, $\boldsymbol{v} = \boldsymbol{F}\boldsymbol{R}^T$.

Exercise

For the given motion, find $\boldsymbol{F}$, $\boldsymbol{R}$, $\boldsymbol{U}$, and $\boldsymbol{v}$ tensors.

$$x_1 = 4 - 2X_1 - X_2$$
$$x_2 = 2 + 1.5x_1 - 0.5X_2$$

Also plot the configuration of a unit square having two opposite corners at (0,0) and (1,1) after the application of (i) $\boldsymbol{U}$ and then $\boldsymbol{R}$, (ii) $\boldsymbol{R}$ and then $\boldsymbol{v}$.