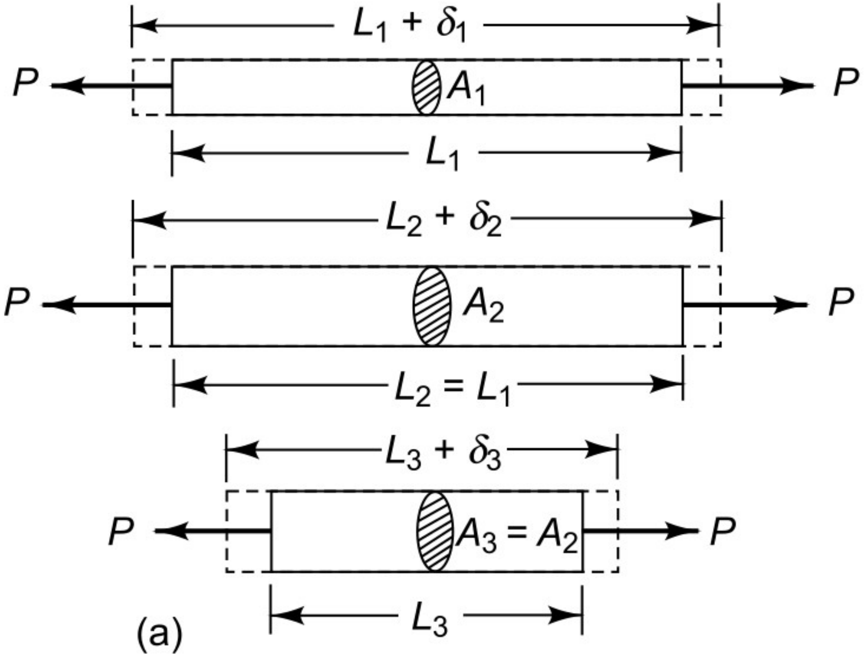
ME231: Solid Mechanics-I

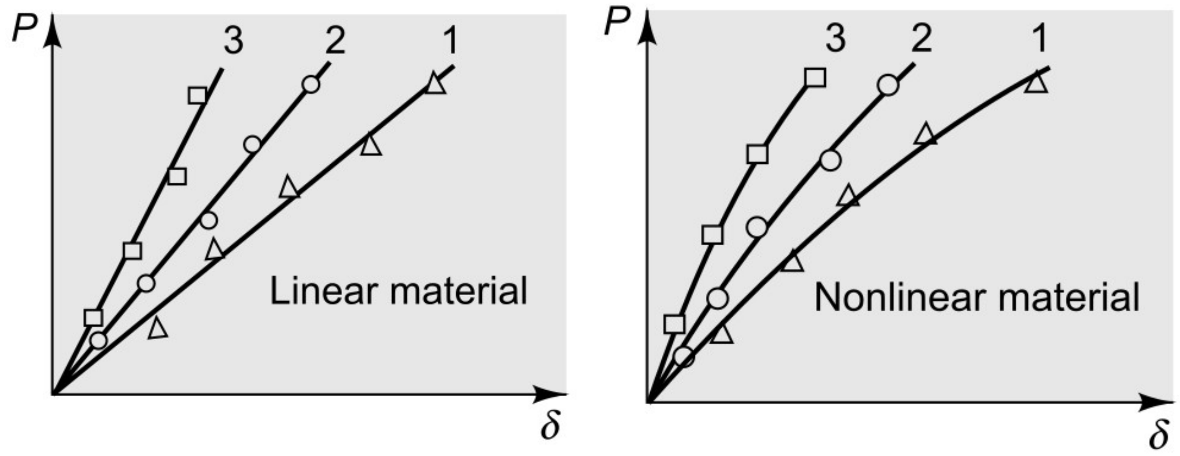
Introduction to Mechanics of Deformable Bodies

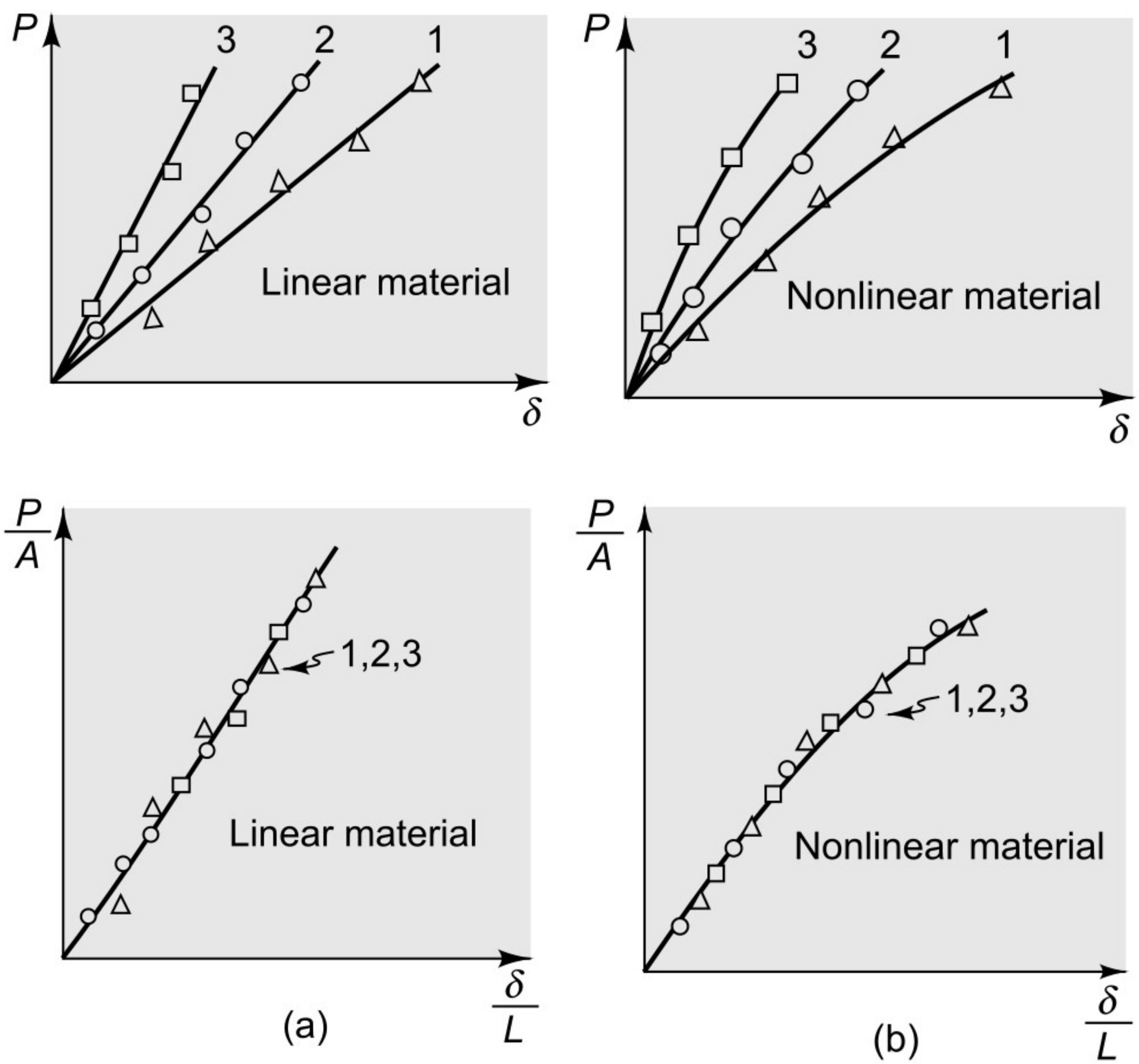
Uniaxial loading and deformation

One of the most basic loading, which can be considered on structures like bar or rod is the load applied along their axis. For such a loading load vs. deformation curve for a given specimen can be plotted.



Load – deformation behaviour





When the load (P) - deformation (δ) curves are re-plotted as (P/A) - (δ/L) curves, then it is observed that curves obtained for specimens of different dimensions overlap.

This suggests that the new curves obtained now are **independent of specimen dimensions.**

Thus, they represent **load-elongation characteristics of a particular material.**

For materials having a **linear** uni-axial load-deformation characteristics, we define **modulus of elasticity** as the slope of load-deformation curve, which is generally denoted as E.

Thus,

$$E = \frac{P/A}{\delta/L}. \qquad \cdots\cdots\cdots\cdots\cdots(1)$$

Dimensions of E is same as that of P/A, (denominator is unit-less), which is N/m^2 (Pa) in SI Units.

Material	*E, psi*	*E, kN/m²*
Tungsten carbide	$60–100 \times 10^6$	$410–690 \times 10^6$
Tungsten	58×10^6	400×10^6
Molybdenum	40×10^6	275×10^6
Aluminum oxide	47×10^6	325×10^6
Steel and iron	$28–30 \times 10^6$	$194–205 \times 10^6$
Brass	15×10^6	103×10^6
Aluminum	10×10^6	69×10^6
Glass	10×10^6	69×10^6
Cast iron	$10–20 \times 10^6$	$69–138 \times 10^6$
Wood	$1–2 \times 10^6$	$6.9–13.8 \times 10^6$
Nylon, epoxy, etc.	$4–8 \times 10^4$	$27.5–55 \times 10^4$
Collagen	$2–15 \times 10^3$	$13.8–103 \times 10^3$
Soft rubber	$2–8 \times 10^2$	$13.8–55 \times 10^2$
Smooth muscle	2–150	13.8–1034
Elastin	50–100	345–690

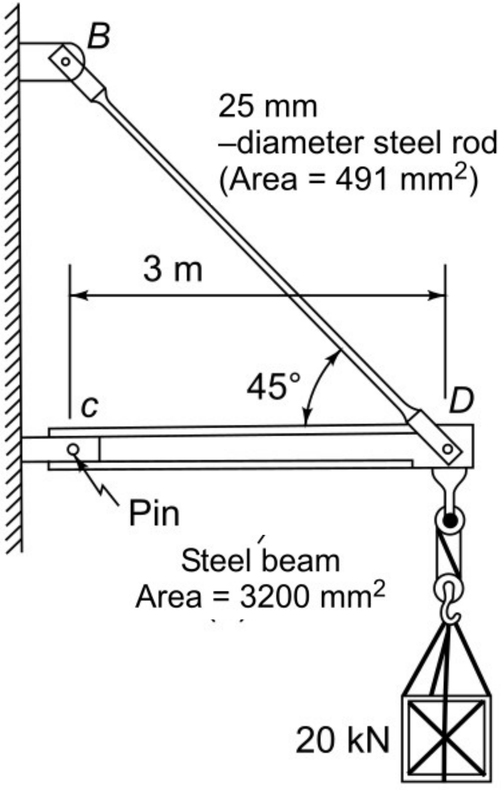
1 N/m^2 = pascal (Pa)

Materials having nonlinear uni-axial load-deformation characteristics can not be characterized by just one constant; in-fact actual curve is required to specify the load-deformation behaviour. Therefore for non-linear materials analytical work become complicated and materials showing small amount of non-linearity are **approximated as linear materials**.

For most materials, elongation under uni-axial tensile load is same as the shortening or compression under the compressive load. Hence, (1) is assumed to be applicable under compression also.

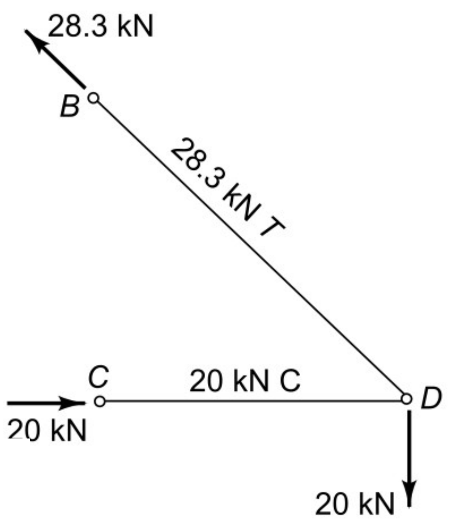
Example 3

A triangular frame supporting a load of 20 kN. Our aim is to estimate the displacement at the point D due to the 20-kN load carried by the chain hoist.

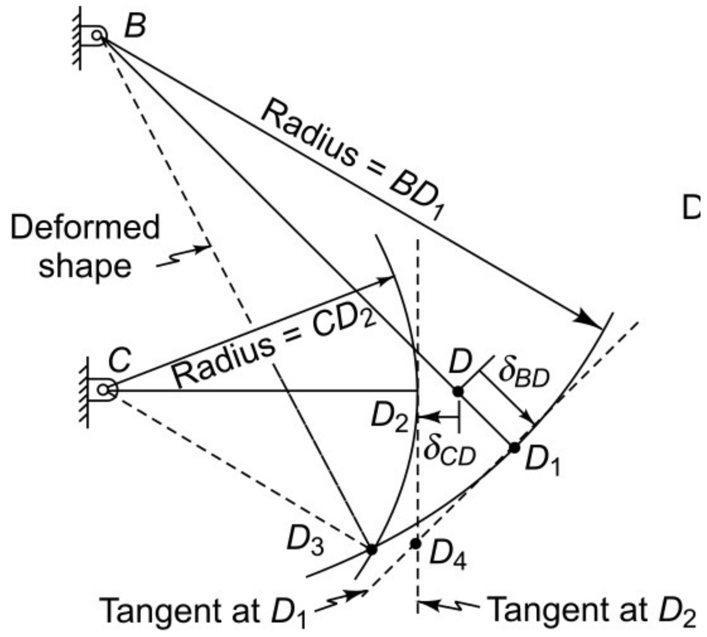


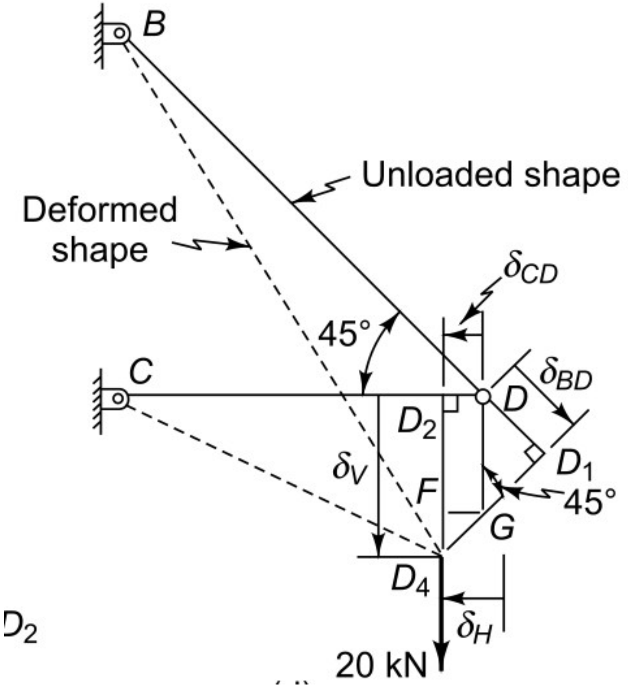
Application of equilibrium conditions:

With the application of equilibrium conditions for the frame as well as for individual links of the frame reaction forces and load in link BD and CD can be calculated (already solved).



Geometric compatibility:





Application of force-displacement relation:

Considering axial load in link BD and CD, axial deflections of links can be calculated as,

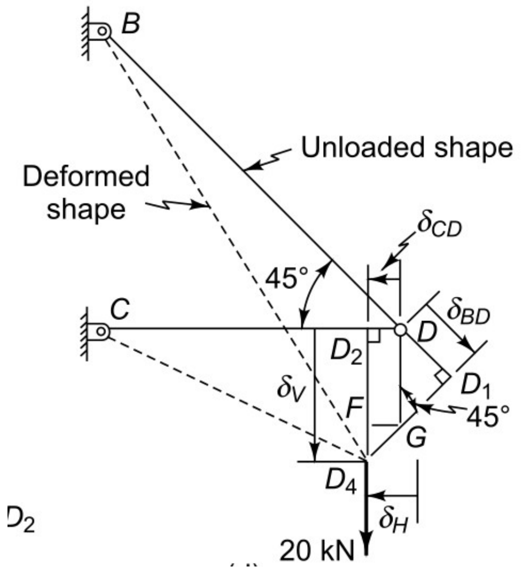
$$\delta_{\mathrm{BD}} = \left(\frac{FL}{AE}\right)_{\mathrm{BD}}, \quad \delta_{\mathrm{CD}} = \left(\frac{FL}{AE}\right)_{\mathrm{CD}}. \qquad \cdots\cdots\cdots\cdots\cdots(\mathrm{f})$$

Substituting corresponding given values in (f) deflection can be determined.

Using the geometry, horizontal as well as vertical deflection of point D is,

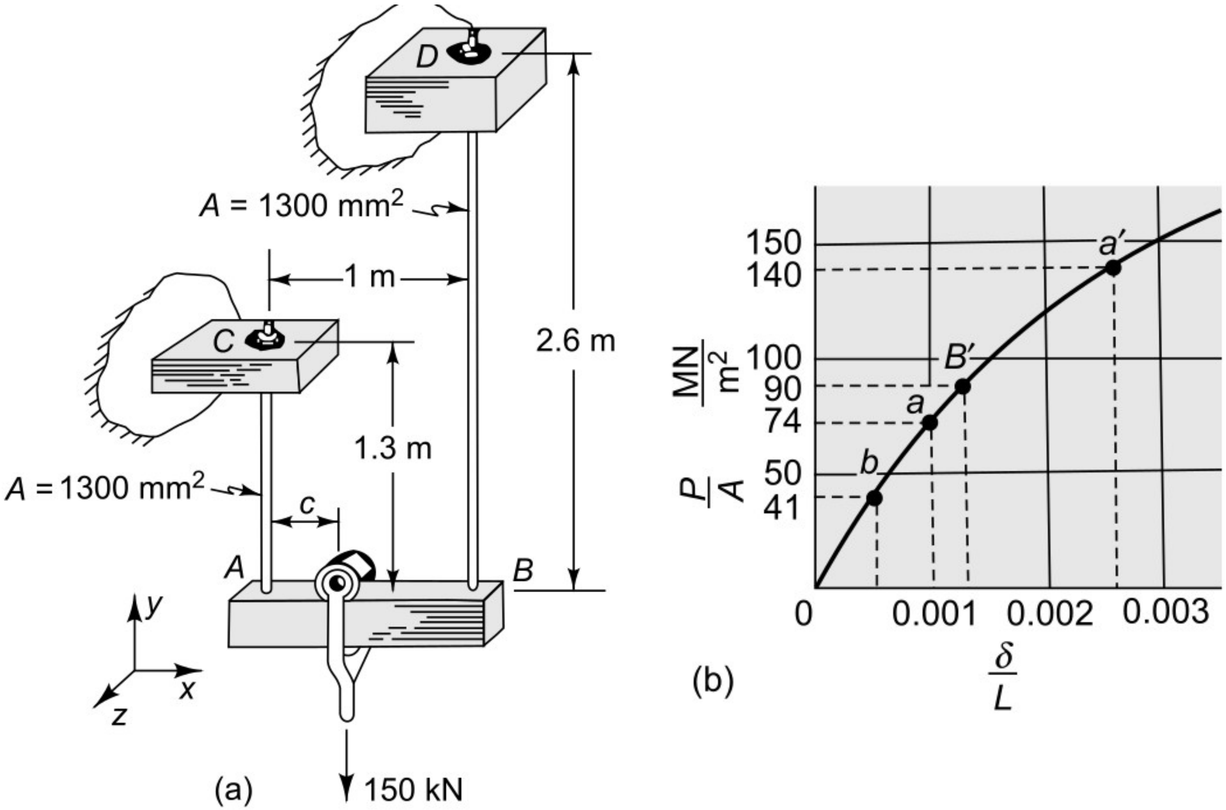
$$\delta_H = \delta_{\mathrm{CD}}$$

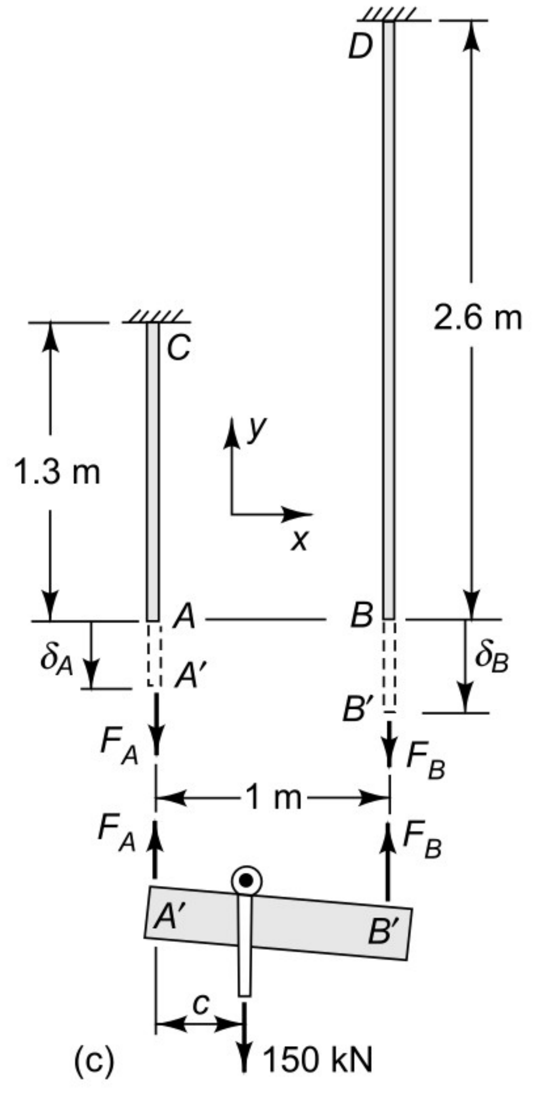
$$\delta_V = \mathrm{D_2F} + \mathrm{FD_4} = \sqrt{2}\delta_{\mathrm{BD}} + \delta_{\mathrm{CD}}$$



Example 4

The stiff horizontal beam AB in is supported by two soft copper rods AC and BD of the same cross-sectional area but of different lengths. The load-deformation diagram for the copper is also shown. A vertical load of 150 kN is to be suspended from a roller which rides on the horizontal beam. We do not want the roller to move after the load is put on, so we wish to find out where to locate the roller so that the beam will still be horizontal in the deflected position.





As a very first step, isolate the beam AB and draw its FBD.

Consider force equilibrium of beam AB as,

$$\sum F = F_A + F_B - 150 = 0, \qquad \text{..............(g)}$$

$$\sum M_A = F_B \cdot 1 - 150 \cdot c = 0, \qquad \text{..............(h)}$$

Now, geometric requirement for the beam to be in horizontal position is

$$\delta_A = \delta_B. \qquad \text{..............(i)}$$

(h) can also be written as,

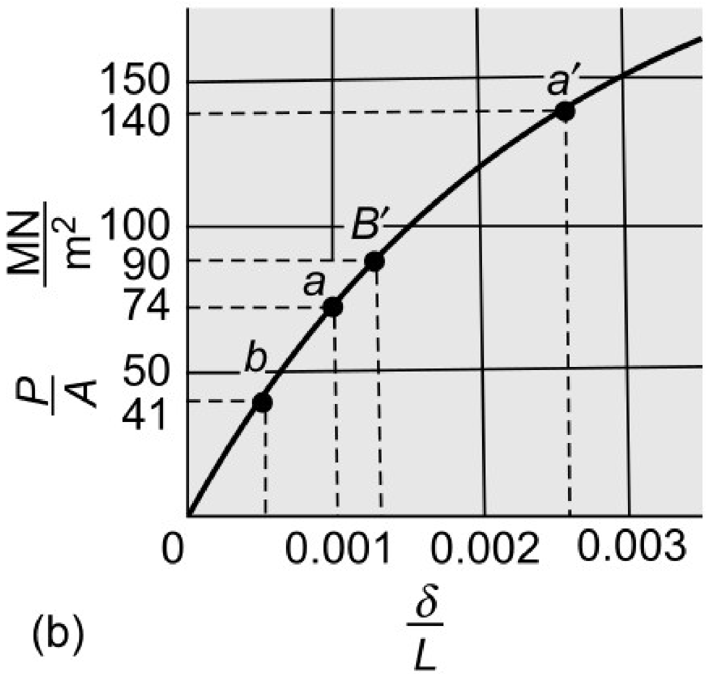
$$\frac{\delta_A}{L_A} = \frac{\delta_B}{L_A} = \frac{\delta_B}{L_B}\frac{L_B}{L_A} = 2\frac{\delta_B}{L_B}. \qquad \text{..............(j)}$$

We can also write (g) as,

$$\frac{F_A}{A_A} + \frac{F_B}{A_A} = \frac{150}{A_A} \quad \text{or} \quad \frac{F_A}{A_A} + \frac{F_B}{A_B} = 115\ \mathrm{MN/m^2}$$

..............(k)

As we do not have an analytical expression for relation between load and deflection, solution for this problem will not be easy. An iterative approach will be required for the solution.



First assume a value of δ_A/L_A, and find the value of δ_B/L_B using (j).

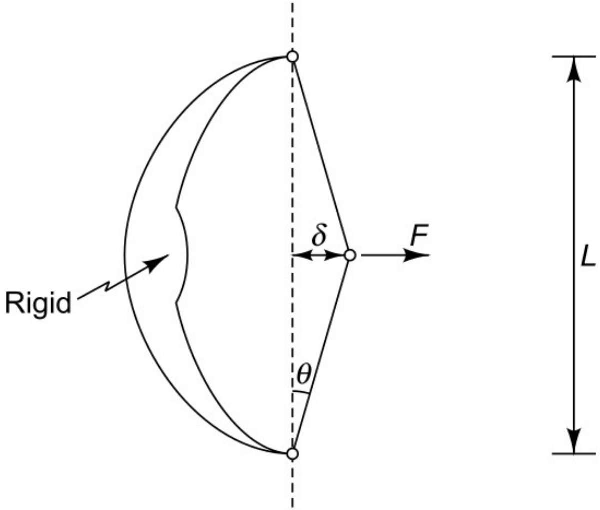
Find the corresponding values of F_A/A_A and F_B/A_B from the curve.

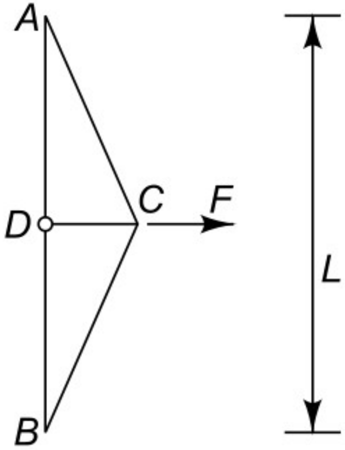
Substitute these values in (k) and check if these values satisfy the equation. If yes, F_A and F_B are known.

If not, then assume a different value of δ_A/L_A, and repeat the previous steps till (k) is satisfied.

Once F_B is known, (h) can be used to determine the value of c.

Example 5





For the bow shown in figure, plot Force vs displacement relation. The string is under initial tension. The string follows a linear load-deflection relation.

The force in the string AC can be calculated as,

$$F_{AC} = (F/2)/\sin\theta \qquad \cdots\cdots\cdots\cdots\cdots(\mathrm{l})$$

Elongation of the string AC,

$$\delta_{\mathrm{AC}} = AC - AB/2 = (L/2)/\cos\theta - L/2 \cdots\cdots\cdots\cdots\cdots(\mathrm{m})$$

Using load-deflection relationship for string as,

$$F_{\mathrm{AC}} = k\delta_{\mathrm{AC}} \qquad \cdots\cdots\cdots\cdots\cdots(\mathrm{n})$$

Using (l)-(n), we get

$$\frac{F}{2\sin\theta} = \frac{kL}{2}\frac{1-\cos\theta}{\cos\theta} \Rightarrow F = 2k\frac{L\tan\theta}{2}(1-\cos\theta)$$

$$\Rightarrow F = 2k\delta_{\mathrm{CD}}(1-\cos\theta) = 4k\delta_{\mathrm{CD}}\sin^2(\theta/2) = k\delta_{\mathrm{CD}}\theta^2 = k/L^2\delta_{\mathrm{CD}}^3.$$

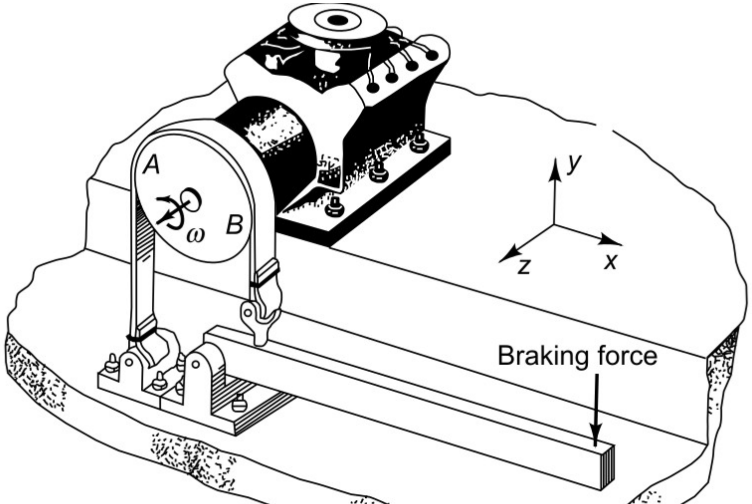
$$(\text{for small } \theta,\ \tan\theta \approx \theta = \delta_{\mathrm{CD}}/L)$$

$$\Rightarrow F = \frac{k\delta_{\mathrm{CD}}^3}{L^2}.$$

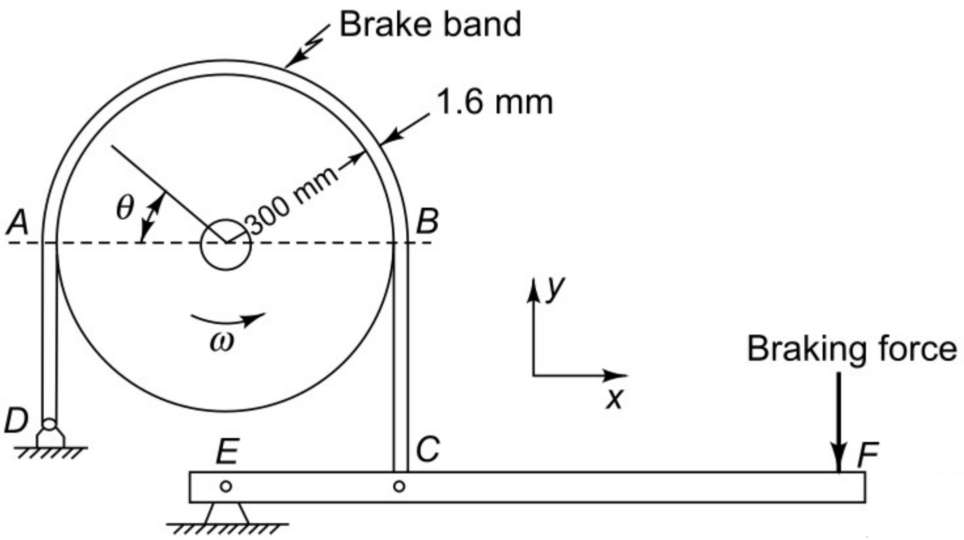
We thus get a non-linear response. Notice importantly that it is possible to obtain a non-linear response from linear springs as shown.

Example 6

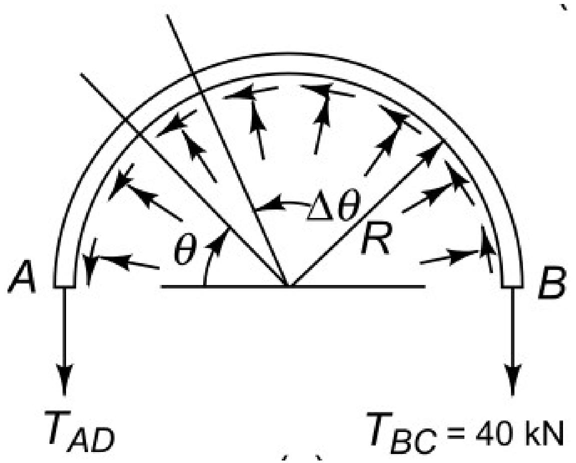
In a test on an engine, a braking force is supplied through a lever arm EF to a steel brake band $CBAD$ which is in contact with half the circumference of a 600-mm-diameter flywheel. The brake band is 1.6 mm thick and 50 mm wide and is lined with a relatively soft material which has a kinetic coefficient of friction $f = 0.4$ with respect to the rotating flywheel. The operator wishes to predict how much elongation there will be in the section AB of the brake band when the braking force is such that there is a tension of 40 kN in the section BC of the band.



(a)



As a very first step let us draw the FBD of section AB of the strap.

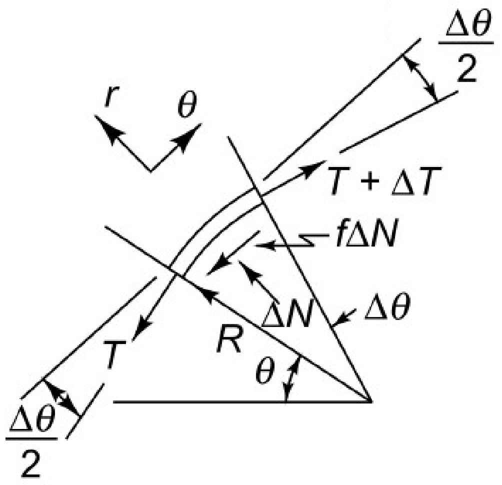


Distributed Normal force and the the friction force between the drum and the strap is shown.

Now consider a small portion of the strap. FBD of the section is shown. Writing equilibrium equation for the section,

$$\sum F_r = \Delta N - T\sin(\Delta\theta/2) - (T+\Delta T)\sin(\Delta\theta/2) = 0,$$

$$\Rightarrow \Delta N - 2T\sin(\Delta\theta/2) - \Delta T\sin(\Delta\theta/2) = 0, \qquad \cdots\cdots(\text{o})$$



$$\sum F_\theta = (T+\Delta T)\cos(\Delta\theta/2) - T\cos(\Delta\theta/2) - f\Delta N = 0,$$

$$\Rightarrow \Delta T\cos(\Delta\theta/2) - f\Delta N = 0, \qquad \cdots\cdots(\text{p})$$

For small angle $\Delta\theta$, we can write (o) as,

$$\Delta N = (2T + \Delta T)\Delta\theta/2,$$

Neglecting ΔT compared to T, $\Delta N = T\Delta\theta$.

Also (p) become, $\Delta T = f\Delta N = fT\Delta\theta$, or $\dfrac{\Delta T}{\Delta\theta} = fT$.

For $\Delta\theta \to 0$,

$$\lim_{\Delta\theta\to 0}\frac{\Delta T}{\Delta\theta} = \frac{dT}{d\theta} = fT.$$

Integrating above relation and applying the boundary condition $T = T_{AD}$ at $\theta = 0$,we get,

$$T = T_{AD}e^{f\theta},$$

T_{AD} can be obtained by considering that $T = T_{BC} = 40$ kN at $\theta = \pi$.

Finally,
$$T = 11.38e^{f\theta}. \qquad \text{............(p)}$$

Note that tension in the strap varies exponentially with angle θ. For fully wrapped strap (i.e., θ=2π) tension at one end will be even higher.

To determine the elongation in section AB, we first calculate the elongation in small element as,

$$\Delta\delta = \frac{TR\Delta\theta}{AE}. \qquad \text{............(q)}$$

Total elongation in section AB can be obtained by integration of (q) after substituting (p) as,

$$\delta_{AB} = \int_A^B d\delta = \int_{\theta=0}^{\pi}\frac{TR}{AE}d\theta.$$