ME232: Dynamics

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Room # 106

Conservation of Linear Momentum

If the resultant force on a particle is zero during an interval of time then its linear momentum $\boldsymbol{G}$ remain constant and we say that the linear momentum of the particle is **conserved**.

Linear momentum may be conserved in one coordinate direction, such as x, but not necessarily in the y- or z-direction. A careful examination of the impulse-momentum diagram of the particle will disclose whether the total linear impulse on the particle in a particular direction is zero. If it is, the corresponding linear momentum is unchanged (conserved) in that direction.

Consider, the motion of two particles a and b which interact during an interval of time. If the interactive forces $\boldsymbol{F}$ and $-\boldsymbol{F}$ between them are the only unbalanced forces acting on the particles during the interval, it follows that the linear impulse on particle a is the negative of the linear impulse on particle b. Therefore, from (37), the change in linear momentum of a is negative of the change in linear momentum of particle b. So we have,

$$\Delta \boldsymbol{G}_a = -\Delta \boldsymbol{G}_b \text{ or } \Delta(\boldsymbol{G}_a + \boldsymbol{G}_b) = 0.$$

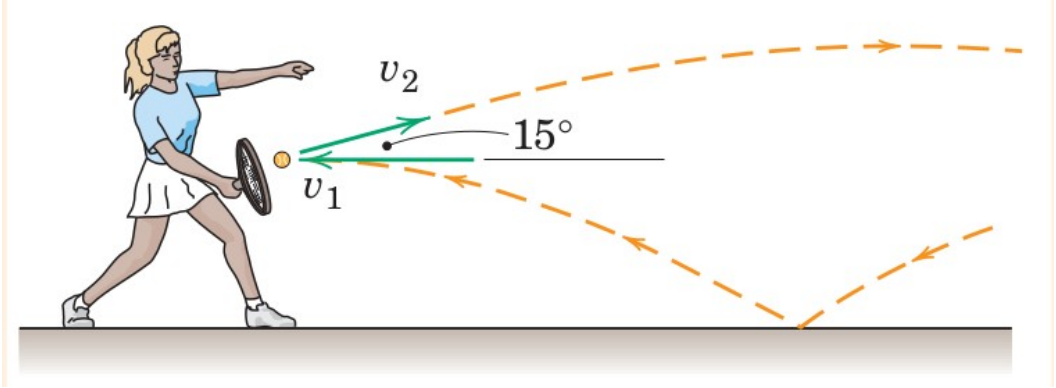
Thus, the total linear momentum $\boldsymbol{G} = \boldsymbol{G}_a + \boldsymbol{G}_b$ for the system of the two particles remains constant during the interval, and we write

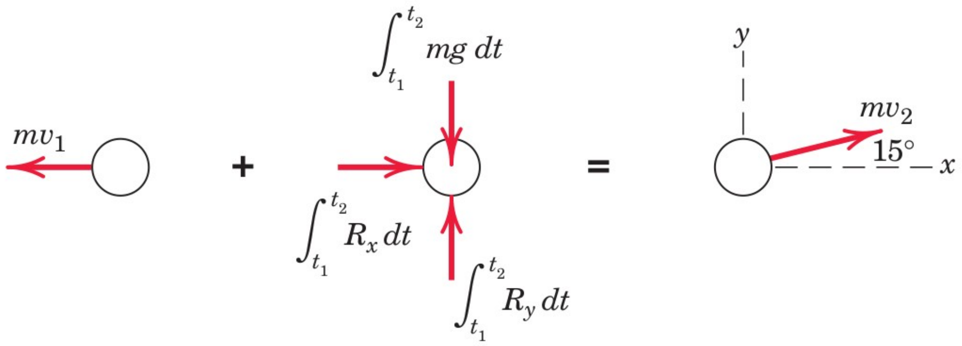
$$\Delta \boldsymbol{G} = \boldsymbol{0} \qquad \text{or} \qquad \boldsymbol{G}_1 = \boldsymbol{G}_2 . \qquad \cdots\cdots\cdots(38)$$

Equation (38) expresses the principle of **conservation of linear momentum.**

Example 9

A tennis player strikes the tennis ball with her racket when the ball is at the uppermost point of its trajectory as shown. The horizontal velocity of the ball just before impact with the racket is $v_1 = 15$ m/sec, and just after impact its velocity is $v_2 = 21$ m/sec directed at the 15° angle as shown. If the 60 g ball is in contact with the racket for 0.02 sec, determine the magnitude of the average force $\boldsymbol{R}$ exerted by the racket on the ball. Also determine the angle β made by $\boldsymbol{R}$ with the horizontal.





$$mv_{1x} + \int_{t_1}^{t_2} \sum F_x dt = mv_{2x},$$

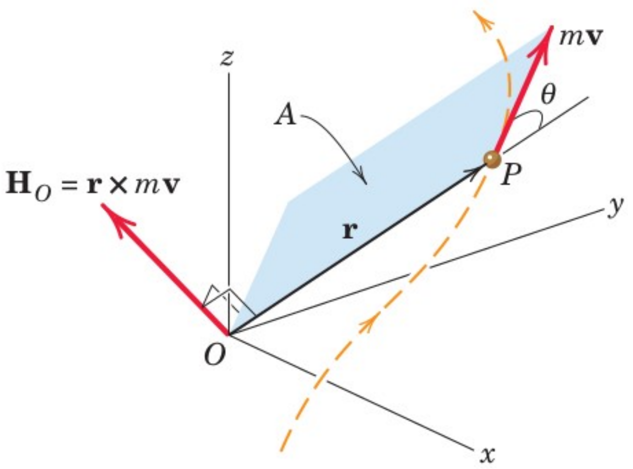
$$mv_{1y} + \int_{t_1}^{t_2} \sum F_y dt = mv_{2y},$$

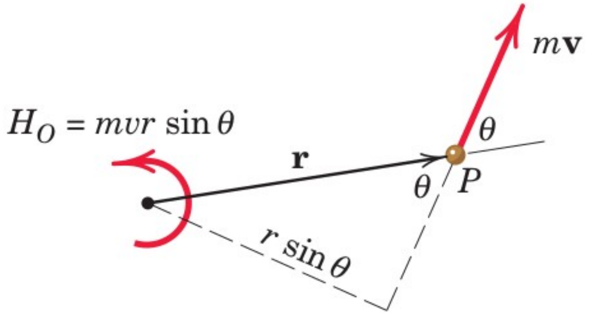
$R_x = 105.9$ N

$R_y = 16.89$ N

$R = 107.2$ N, angle $9.07°$

Angular momentum





A particle P of mass m moves along a curve in space. The particle is located by its position vector $\boldsymbol{r}$ as shown. The velocity of the particle is $\boldsymbol{v} = \dot{\boldsymbol{r}}$, and its linear momentum is $\boldsymbol{G} = m\boldsymbol{v}$.

The **moment of the linear momentum vector** $m\boldsymbol{v}$ about the origin O is defined as the **angular momentum** $\boldsymbol{H}_O$ of P about O and is given by the following cross-product

$$\boldsymbol{H}_0 = \boldsymbol{r} \times m\boldsymbol{v}. \qquad \cdots\cdots\cdots(39)$$

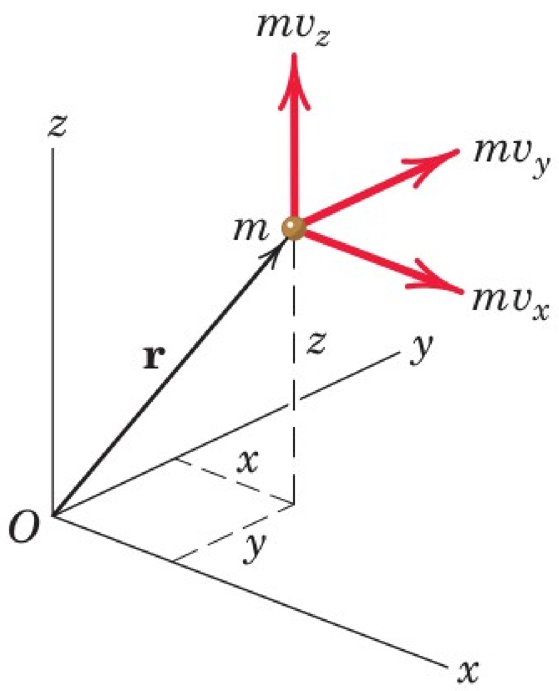
The angular momentum is a vector perpendicular to the plane A defined by $\boldsymbol{r}$ and $\boldsymbol{v}$. The sense of $\boldsymbol{H}_O$ is defined by the right-hand rule for cross products.

The scalar components of angular momentum may be obtained from the expansion

$$\boldsymbol{H}_0 = \boldsymbol{r} \times m\boldsymbol{v} = m(v_z y - v_y z)\boldsymbol{i} + m(v_x z - v_z x)\boldsymbol{j} + m(v_y x - v_x y)\boldsymbol{k}, \qquad \cdots\cdots\cdots(40)$$

so that,

$$H_x = m(v_z y - v_y z), \quad H_y = m(v_x z - v_z x), \quad H_z = m(v_y x - v_x y). \qquad \cdots\cdots\cdots(41)$$

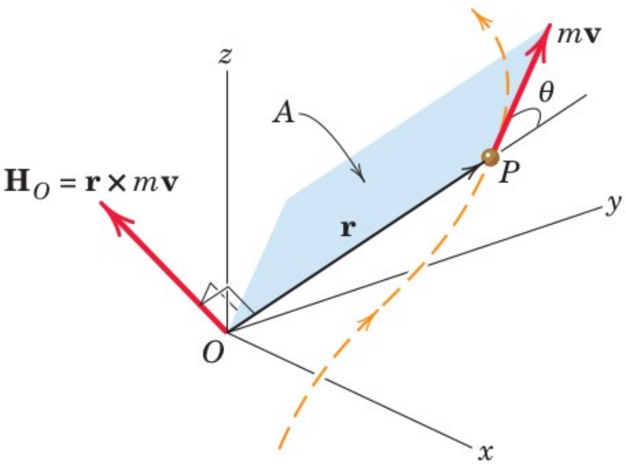


Expressions (41) for angular momentum may be visualized from the figure as the moments of three linear-momentum components components about the respective axes.

In SI units, angular momentum has the units

kg $\cdot$(m/s)$\cdot$m = kg$\cdot$m^2/s = N$\cdot$m$\cdot$s.

Rate of angular momentum



The moment of the forces acting on the particle P can be related to its angular momentum. If $\Sigma\boldsymbol{F}$ represents the resultant of all forces acting on the particle P, the moment $\boldsymbol{M}_O$ about the origin O is the vector cross product

$$\sum \boldsymbol{M}_0 = \boldsymbol{r} \times \sum \boldsymbol{F} = \boldsymbol{r} \times m\dot{\boldsymbol{v}}, \qquad \cdots\cdots\cdots(42)$$

Now differentiate (39) with time, using the rule for the differentiation of a cross product,

$$\dot{\boldsymbol{H}}_O = \dot{\boldsymbol{r}} \times m\boldsymbol{v} + \boldsymbol{r} \times m\dot{\boldsymbol{v}} = \cancelto{0}{\boldsymbol{v} \times m\boldsymbol{v}} + \boldsymbol{r} \times m\dot{\boldsymbol{v}},$$

Thus,

$$\sum \boldsymbol{M}_0 = \dot{\boldsymbol{H}}_0. \qquad \cdots\cdots\cdots(43)$$

Equation (43) states that **the moment about the fixed point O of all forces acting on m equals the time rate of change of angular momentum of m about O**. Scalar components of (43) are,

$$\sum M_{0_x} = \dot{H}_{0_x}, \quad \sum M_{0_y} = \dot{H}_{0_y}, \quad \sum M_{0_z} = \dot{H}_{0_z}. \qquad \cdots\cdots\cdots(44)$$

The Angular Impulse-Momentum Principle

Equation (43) gives the instantaneous relation between the moment and the time rate of change of angular momentum. To obtain the effect of the moment $\Sigma \boldsymbol{M}_O$ on the angular momentum of the particle over a finite period of time, we integrate (43) from time t_1 to time t_2. Multiplying (43) by dt gives $\Sigma \boldsymbol{M}_O\ dt = d\boldsymbol{H}_O$ and integrating,

$$\int_{t_1}^{t_2} \sum \boldsymbol{M}_0 dt = (\boldsymbol{H}_0)_2 - (\boldsymbol{H}_0)_1 = \Delta \boldsymbol{H}_0. \qquad \cdots\cdots\cdots(45)$$

where $(\boldsymbol{H}_\text{O})_2 = \boldsymbol{r}_2 \times \text{m}\boldsymbol{v}_2$ and $(\boldsymbol{H}_O)_1 = \boldsymbol{r}_1 \times m\boldsymbol{v}_1$. The product of moment and time is defined as **angular impulse** and (45) states that **the total angular impulse on *m* about the fixed point *O* equals the corresponding change in angular momentum of *m* about *O*.** (45) can also be written as,

$$(\boldsymbol{H}_0)_1 + \int_{t_1}^{t_2} \sum \boldsymbol{M}_0 dt = (\boldsymbol{H}_0)_2. \qquad \cdots\cdots\cdots(45a)$$

The SI units of angular impulse are same as that of angular momentum, which are N·m·s or kg·m²/s.

The equation of angular impulse and angular momentum is a vector equation where changes in direction as well as magnite may occur during the interval of integration. Component form of (45) in x-direction is,

$$(H_{0x})_1 + \int_{t_1}^{t_2} \sum M_{0x} dt = (H_{0x})_2, \quad \text{or,}$$
$$m(v_z y - v_y z)_1 + \int_{t_1}^{t_2} \sum M_{0x} dt = m(v_z y - v_y z)_2. \qquad \cdots\cdots\cdots(46)$$

where the subscripts 1 and 2 refer to the values of the respective quantities at times t_1 and t_2. Similar expressions can be written for the y- and z-components of the angular impulse-momentum equation.

We consider the application of angular impulse-momentum principal for plane-motion problems where moments are taken about a single axis normal to the plane of motion. In this case, the angular momentum may change magnitude and sense, but the direction of the vector remains unaltered.

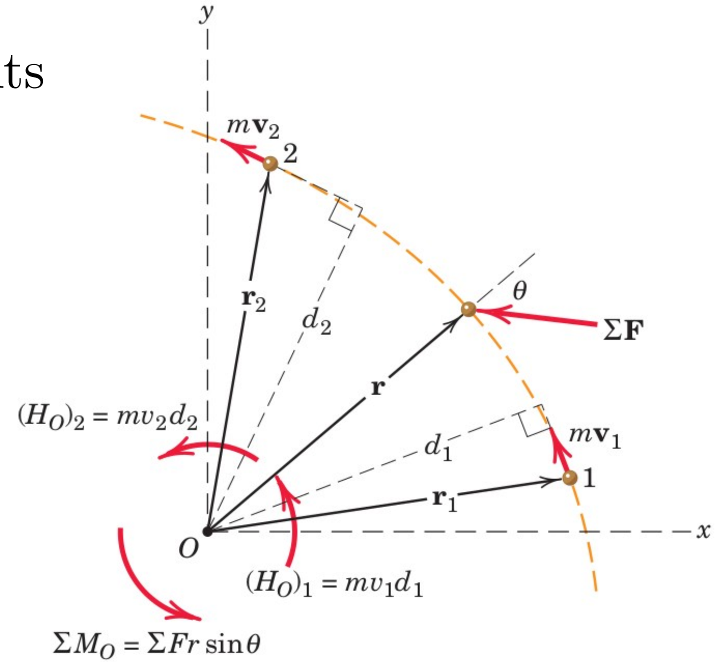
Thus, for a particle of mass m moving along a curved path in the x-y plane, the angular momenta about O at points 1 and 2 have the magnitudes $(H_O)_1 = |\boldsymbol{r}_1 \times m\boldsymbol{v}_1| = mv_1 d_1$ and $(H_O)_2 = |\boldsymbol{r}_2 \times m\boldsymbol{v}_2| = mv_2 d_2$, respectively.

The scalar form (46) applied to the motion between points 1 and 2 during the time interval t_1 to t_2 becomes,

$$(H_0)_1 + \int_{t_1}^{t_2} \sum M_0 dt = (H_0)_2,$$

$$\Rightarrow m_1 v_1 d_1 + \int_{t_1}^{t_2} \sum Fr\sin\theta dt = m_2 v_2 d_2,$$

.........(47)



Conservation of Angular Momentum

If the resultant moment about a fixed point O of all forces acting on a particle is zero during an interval of time, then (45) requires that its angular momentum $\boldsymbol{H}_O$ about that point remain constant. In this case, the angular momentum of the particle is said to be conserved. Angular momentum may be conserved about one axis but not about another axis.

Consider the motion of two particles a and b which interact during an interval of time. If the interactive forces $\boldsymbol{F}$ and -$\boldsymbol{F}$ between them are the only unbalanced forces acting on the particles during the interval, then the moments of the equal and opposite forces about any fixed point O not on their line of action are equal and opposite. If we apply (45) to particle a and then to particle b and add the two equations, we obtain $\Delta H_a + \Delta H_b = 0$ (where all angular momenta are referred to point O). Thus, the total angular momentum for the system of the two particles remains constant during the interval, and we write

$$\Delta \boldsymbol{H}_O = \boldsymbol{0} \quad \text{or} \quad (\boldsymbol{H}_O)_1 = (\boldsymbol{H}_O)_2. \qquad \cdots\cdots\cdots(48)$$

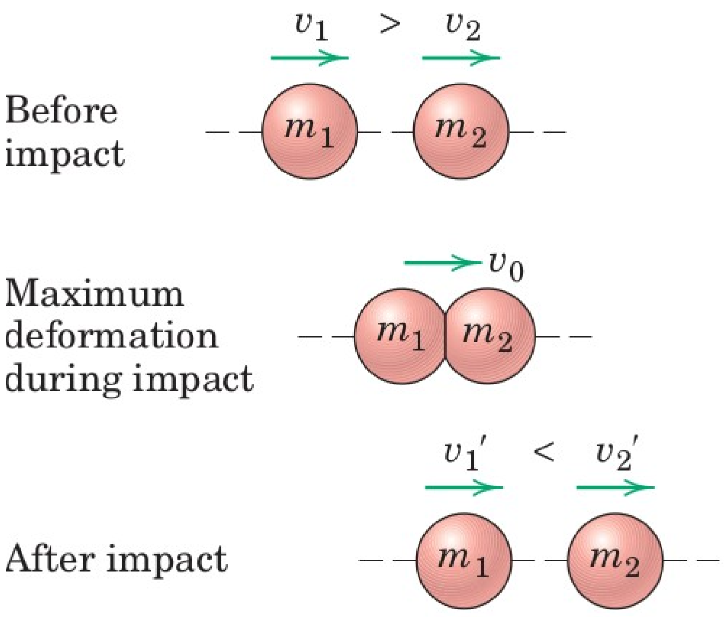
(48) expresses the principle of **conservation of angular momentum**.

Application: Impact

- Impact refers to the **collision between two bodies** and is characterized by the generation of relatively large contact forces which act over a very short interval of time.
- The principles of impulse and momentum have important use in describing the impact behaviour.
- It is important to realize that an impact is a very complex event involving material deformation and recovery and the generation of heat and sound. Small changes in the impact conditions may cause large changes in the impact process and thus in the conditions immediately following the impact.

Direct Central Impact:

We first consider the collinear motion of two spheres of masses m_1 and m_2, traveling with velocities v_1 and v_2. If v_1 is greater than v_2, collision occurs with the contact forces directed along the line of centers. This condition is called **direct central impact**.



Following initial contact, a short period of increasing deformation takes place until the contact area between the spheres ceases to increase. At this instant, both spheres, are moving with the same velocity v_0.

During the remainder of contact, a period of restoration occurs during which the contact area decreases to zero. In the final condition the spheres now have new velocities v'_1 and v'_2, where v'_1 must be less than v'_2. All velocities are arbitrarily assumed positive to the right, so that with this scalar notation a velocity to the left would carry a negative sign.

If the impact is not overly severe and if the spheres are highly elastic, they will regain their original shape following the restoration.

With a more severe impact and with less elastic bodies, a permanent deformation may result.

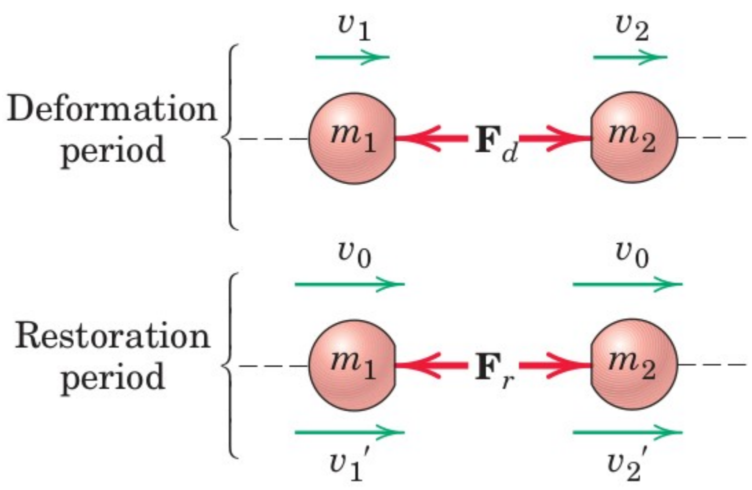
Because the contact forces are equal and opposite during impact, the linear momentum of the system remains unchanged. Thus, we apply the law of conservation of linear momentum and write

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2. \qquad \cdots\cdots\cdots(49)$$

While writing (49) we assume that any forces acting on the spheres during impact, other than the large internal forces of contact, are relatively small and produce negligible impulses compared with the impulse associated with each internal impact force.

In addition, we assume that no appreciable change in the positions of the mass centers occurs during the short duration of the impact.

For given masses and initial conditions, the momentum equation contains two unknowns, v'_1 and v'_2. Thus, need an additional relationship to find the final velocities. This relationship must reflect the capacity of the contacting bodies to recover from the impact and can be expressed by the ratio e of the magnitude of the restoration impulse to the magnitude of the deformation impulse. This ratio is called **the coefficient of restitution**.



Let $\boldsymbol{F}_r$ and $\boldsymbol{F}_d$ represent the magnitudes of the contact forces during the restoration and deformation periods, respectively. For particle 1 the definition of e together with the impulse-momentum equation give

$$e = \frac{\int_{t_0}^{t} F_r\, dt}{\int_{0}^{t_0} F_d\, dt} = \frac{-m_1(v_1' - v_0)}{-m_1(v_0 - v_1)} = \frac{(v_0 - v_1')}{(v_1 - v_0)}, \qquad \cdots\cdots\cdots(50)$$

Similarly for particle 2,

$$e = \frac{\int_{t_0}^{t} F_r \, dt}{\int_0^{t_0} F_d \, dt} = \frac{m_2(v_2' - v_0)}{m_2(v_0 - v_2)} = \frac{(v_2' - v_0)}{(v_0 - v_2)}, \qquad \cdots\cdots\cdots(51)$$

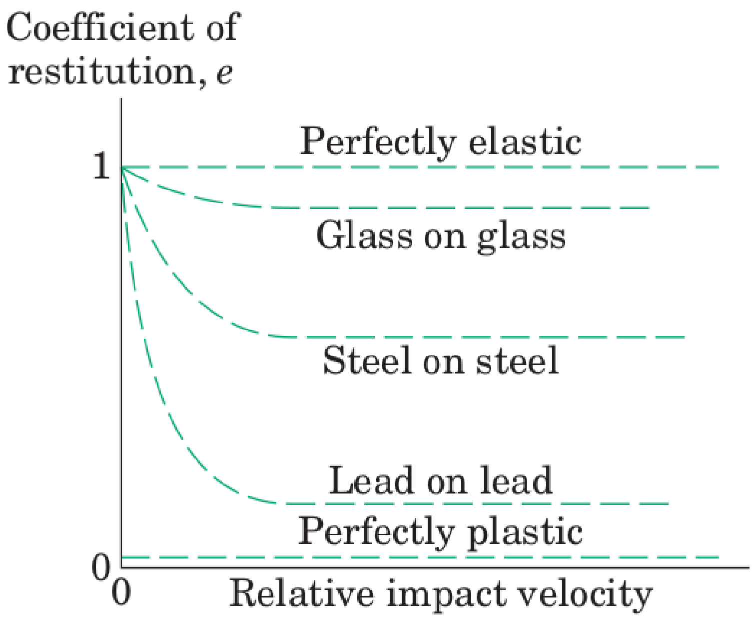
In these equations the change of momentum (and therefore Δv) is in the same direction as the impulse (and thus the force).
The time for the deformation is taken as t_0 and the total time of contact is t. Eliminating v_0 between the two expressions for e gives us

$$e = \frac{(v_2' - v_1')}{(v_1 - v_2)} = \frac{|\text{relative velocity of seperation}|}{|\text{relative velocity of approach}|}, \qquad \cdots\cdots\cdots(53)$$

If the two initial velocities v_1 and v_2 and the coefficient of restitution e are known, then (49) and (53) give us two equations in the two unknown final velocities v'_1 and v'_2.

Energy loss during impact:

Energy loss during the impact may be calculated by subtracting the kinetic energy of the system just after impact from that just before impact. Energy is lost through the generation of heat during the localized inelastic deformation of the material, through the generation and dissipation of elastic stress waves within the bodies, and through the generation of sound energy.



According to this classical theory of impact,

- the value $e = 1$ means that the capacity of the two particles to recover equals their tendency to deform. This condition is one of elastic impact with no energy loss.
- the value $e = 0$, on the other hand, describes inelastic or plastic impact where the particles cling together after collision and the loss of energy is a maximum.
- all impact conditions lie somewhere between these two extremes.