ME232: Dynamics

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Room # 106

Case 1: ellipse ($e < 1$):

From (60) we deduce that r is a minimum when $\theta = 0$ and is a maximum when $\theta = \pi$. Thus,

$$2a = r_{\min} + r_{\max} = \frac{ed}{1+e} + \frac{ed}{1-e} \quad \text{or} \quad a = \frac{ed}{1-e^2}.$$

With the distance d expressed in terms of a, (60) and the maximum and minimum values of r may be written as

$$\frac{1}{r} = \frac{1+e\cos\theta}{a(1-e^2)}, \quad r_{\min} = a(1-e), \quad r_{\max} = a(1+e). \qquad \cdots\cdots\cdots(62)$$

Equation (62) is an expression of **Kepler's first law**, which says that **the planets move in elliptical orbits around the sun as a focus**. The period τ for the elliptical orbit is the total area A of the ellipse divided by the constant rate $\dot{A}$ at which the area is swept through.

Thus, from (56),

$$\tau = \frac{A}{\dot{A}} = \frac{\pi ab}{\frac{1}{2}r^2\dot{\theta}} \quad \text{or} \quad \tau = \frac{2\pi ab}{h}.$$

Eliminate $\dot{\theta}$ or h in the expression for τ by substituting (61), the identity $d = 1/C$, the geometric relationships $a = ed/(1-e^2)$ and $b = a(1-e^2)^{1/2}$ for the ellipse, and the equivalence $Gm_0 = gR^2$. The result after simplification is

$$\tau = 2\pi\frac{a^{3/2}}{R\sqrt{g}}. \qquad \cdots\cdots\cdots(63)$$

In this equation note that R is the mean radius of the central attracting body and g is the absolute value of the acceleration due to gravity at the surface of the attracting body. Equation (63) expresses **Kepler's third law** of planetary motion which states that **the square of the period of motion is proportional to the cube of the semi-major axis of the orbit**.

Case 2: parabola ($e = 1$):

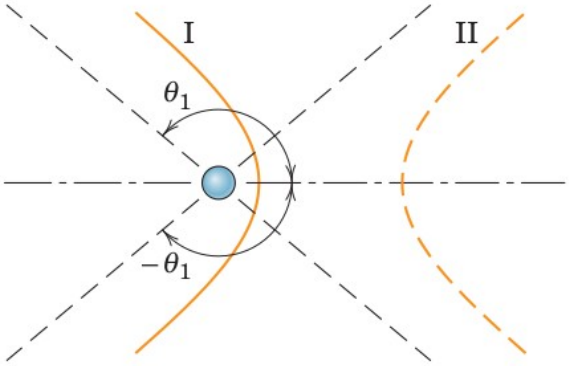
Equations (60) and (61) become,

$$\frac{1}{r} = \frac{1}{d}(1 + \cos\theta) \quad \text{and} \quad h^2 C = G m_0. \qquad \cdots\cdots\cdots(64)$$

The radius vector becomes infinite as θ approaches π, so the dimension a is infinite.

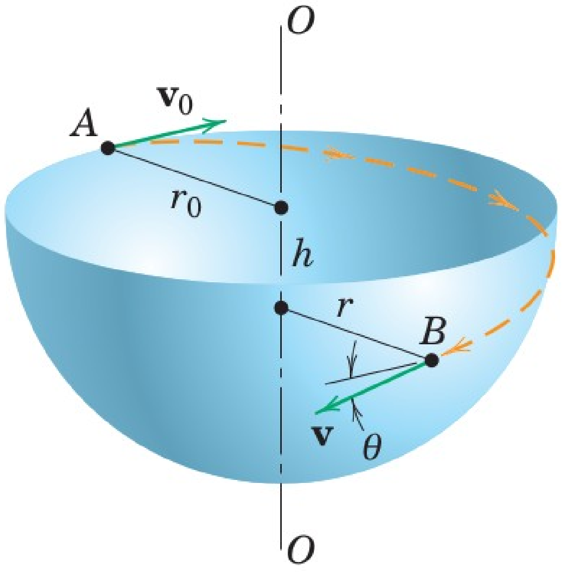
Case 3: hyperbola ($e > 1$):

From (60) we see that the radial distance r becomes infinite for the two values of the polar angle θ_1 and $-\theta_1$ defined by $\cos\theta_1 = -1/e$. Only branch I corresponding to $-\theta_1 < \theta < \theta_1$ represents a physically possible motion.



Example 10

A small mass particle is given an initial velocity $\boldsymbol{v}_0$ tangent to the horizontal rim of a smooth hemispherical bowl at a radius r_0 from the vertical centerline, as shown at point A. As the particle slides past point B, a distance h below A and a distance r from the vertical centerline, its velocity v makes an angle θ with the horizontal tangent to the bowl through B. Determine θ.



Forces acting on the particles are the weight and the normal reaction force exerted by the surface of the bowl. Neither of the force exerts a moment about axis O-O, hence the angular momentum remain conserved about that axis. Thus from the conservation of angular momentum about O-O,

$$(H_O)_1 = (H_O)_2 \qquad m v_0 r_0 = m v r \cos\theta. \qquad \cdots\cdots\cdots\cdots\text{(a)}$$

Note that energy is also conversed, i.e., $E_1 = E_2$. Thus

$$\frac{1}{2} m v_0^2 + mgh = \frac{1}{2} m v^2 + 0 \qquad \cdots\cdots\cdots\cdots\text{(b)}$$

$$v = \sqrt{v_0^2 + 2gh} \qquad \cdots\cdots\cdots\cdots\text{(c)}$$

Also, $$r^2 = r_0^2 - h^2 \qquad \cdots\cdots\cdots\cdots\text{(d)}$$

θ can be obtained by solving equations (a)-(d).

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Kinetics of system of particles

- The principles of dynamics, which are applicable for a single particle can be extended to rigid bodies as well as nonrigid systems.
- A rigid body is a solid system of particles wherein the distances between particles remain essentially unchanged. The overall motions found with machines, land and air vehicles, rockets and spacecraft, and many moving structures provide examples of rigid-body problems.
- Many times it is required to study the time-dependent changes in the shape of a **nonrigid, but solid,** body due to elastic or inelastic deformations. Another example of a nonrigid body is a defined mass of liquid or gaseous particles flowing at a specified rate.
- Examples are the air and fuel flowing through the turbine of an aircraft engine, the burned gases issuing from the nozzle of a rocket motor, or the water passing through a rotary pump.

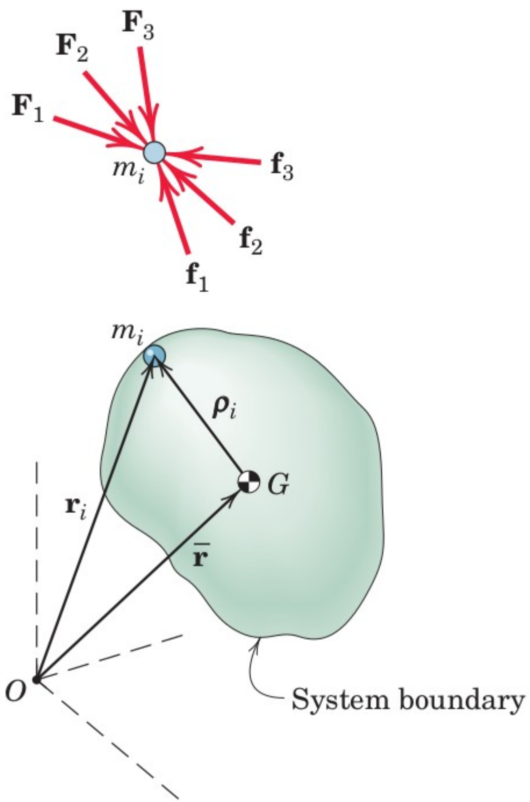
Generalized Newton's Second Law

We now extend Newton's second law of motion to cover a general mass system which is modeled by considering n mass particles bounded by a closed surface in space. This bounding envelope, for example,

- may be the exterior surface of a given rigid body,
- the bounding surface of an arbitrary portion of the body,
- the exterior surface of a rocket containing both rigid and flowing particles,
- a particular volume of fluid particles.

In each case, the system to be considered is the mass within the envelope, and that mass must be clearly defined and isolated.

- A representative particle of mass m_i of the system is isolated. Forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$, ... acting on m_i from sources external to the envelope, and forces $\boldsymbol{f}_1$, $\boldsymbol{f}_2$, $\boldsymbol{f}_3$, ... acting on m_i from sources internal to the system boundary. The external forces are due to contact with external bodies or to external gravitational, electric, or magnetic effects.
- The particle of mass m_i is located by its position vector $\boldsymbol{r}_i$ measured from the non-accelerating origin O of a Newtonian set of reference axes.
- The center of mass G of the isolated system of particles is located by the position vector $\bar{\boldsymbol{r}}$ which is given by



$$m\bar{\boldsymbol{r}} = \sum m_i \boldsymbol{r}_i, \qquad \cdots\cdots\cdots\cdots\cdots(1)$$

where the total system mass is $m = \Sigma m_i$. The summation is over all n particles.

Newton's second law, when applied to m_i gives

$$\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \cdots + \boldsymbol{f}_1 + \boldsymbol{f}_2 + \boldsymbol{f}_3 + \cdots = m_i \ddot{\boldsymbol{r}}_i \qquad \text{................(2)}$$

where $\ddot{\boldsymbol{r}}_i$ is the acceleration of m_i . A similar equation may be written for each of the particles of the system. If these equations written for all particles of the system are added together, the result is

$$\Sigma \boldsymbol{F} + \Sigma \boldsymbol{f} = \Sigma m_i \ddot{\boldsymbol{r}}_i \qquad \text{................(3)}$$

Note that vector sum of all forces on all particles produced by the internal actions and reactions between particles (i.e. $\Sigma \boldsymbol{f}$) is identically zero since all internal forces occur in pairs of equal and opposite actions and reactions.

By differentiation (1) twice w.r.t. to time, we get, $m\ddot{\bar{\boldsymbol{r}}} = \sum m_i \ddot{\boldsymbol{r}}_i$. Using this relation (3) becomes,

$$\sum \boldsymbol{F} = m\ddot{\bar{\boldsymbol{r}}}, \quad \text{or} \quad \sum \boldsymbol{F} = m\bar{\boldsymbol{a}}, \qquad \text{................(4)}$$

where, $\bar{\boldsymbol{a}}$ is the acceleration of the center of mass of the system.

Equation (4) is is the generalized Newton's second law of motion for a mass system and is called **the equation of motion of *m***. This law expresses the **principle of motion of the mass center**.

Equation (4) may be expressed in component form using x-y-z coordinates as,

$$\sum F_x = m\bar{a}_x, \quad \sum F_y = m\bar{a}_y, \quad \sum F_z = m\bar{a}_z. \qquad \text{................(4a)}$$

Although (4) as a vector equation, requires that the acceleration vector $\boldsymbol{a}$ have the same direction as the resultant external force $\Sigma\boldsymbol{F}$, it does not follow that $\Sigma\boldsymbol{F}$ necessarily passes through G. In general, in fact, $\Sigma\boldsymbol{F}$ does not pass through G.

Work-Energy Relation

The work-energy relation was developed for a single particle, and we saw that it applies to a system of two joined particles too.

Now consider the general system of particles, where the work-energy relation for the representative particle of mass m_i is $(U_{1\text{-}2})_i = \Delta T_i$. Here $(U_{1\text{-}2})_i$ is the work done on m_i during an interval of motion by all forces $\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \cdots$ applied from sources external to the system and by all forces $\boldsymbol{f}_1 + \boldsymbol{f}_2 + \boldsymbol{f}_3 + \cdots$ applied from sources internal to the system.

The kinetic energy of m_i is $T_i = \frac{1}{2} m_i v_i^2$, where v_i is the magnitude of the particle velocity $\boldsymbol{v}_i = \dot{\boldsymbol{r}}_i$. For the entire system, the sum of the work-energy equations written for all particles is $\Sigma(U_{1\text{-}2})_{\boldsymbol{i}} = \Sigma\Delta T_i$, which may be represented as,

$$U_{1\text{-}2} = \Delta T \quad \text{or} \quad T_1 + U_{1\text{-}2} = T_2 . \qquad \cdots\cdots\cdots\cdots\cdots(5)$$

where $U_{1\text{-}2} = \Sigma(U_{1\text{-}2})_i$, the work done by all forces, external and internal, on all particles, and ΔT is the change in the total kinetic energy $T = \Sigma T_i$ of the system.[7]

For a nonrigid mechanical system which includes elastic members, a part of the work done by the external forces goes into changing the internal elastic potential energy V_e. Also, if the work done by the gravity forces is excluded from the work term and is accounted for instead by the changes in gravitational potential energy V_g, then we may equate the work $U'_{1\text{-}2}$ done on the system during an interval of motion to the change ΔE in the total mechanical energy of the system. Thus,

$$U'_{1\text{-}2} = \Delta E \qquad \text{or} \qquad U'_{1\text{-}2} = \Delta T + \Delta V, \qquad \cdots\cdots\cdots\cdots\cdots\cdots(6)$$

or

$$T_1 + V_1 + U'_{1\text{-}2} = T_2 + V_2, \qquad \cdots\cdots\cdots\cdots\cdots\cdots(7)$$

where $V = V_e + V_g$ is the total potential potential energy.

Kinetic Energy

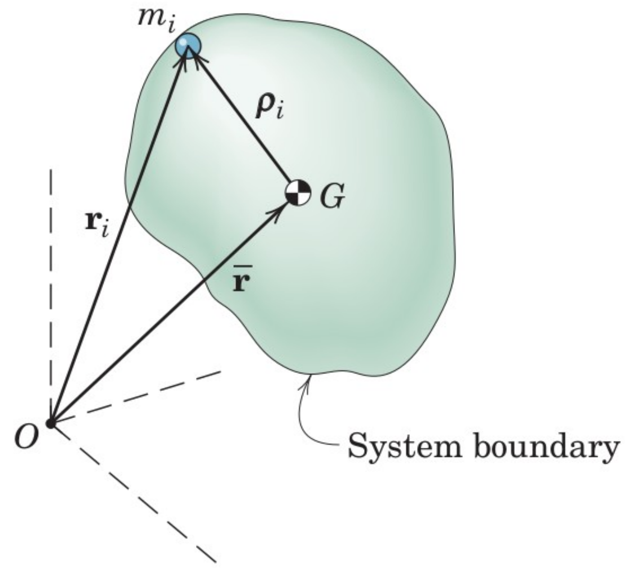
The expression for kinetic energy of the system can be written as,

$$T = \sum \frac{1}{2} m_i \left(\boldsymbol{v}_i \cdot \boldsymbol{v}_i \right) = \sum \frac{1}{2} m_i v_i^2 . \qquad \cdots\cdots\cdots\cdots(8)$$

Using the concept of relative velocity we can write,

$$\boldsymbol{v}_i = \bar{\boldsymbol{v}} + \dot{\boldsymbol{\rho}}_i , \qquad \cdots\cdots\cdots\cdots(9)$$

where $\bar{\boldsymbol{v}}$ is the velocity of the mass center G and $\dot{\boldsymbol{\rho}}_i$ is the velocity of m_i with respect to a translating reference frame moving with the mass center G.



Substituting (9) in (8)

$$T = \sum \frac{1}{2} m_i \left[\left(\bar{\boldsymbol{v}} + \boldsymbol{\rho}_i \right) \cdot \left(\bar{\boldsymbol{v}} + \boldsymbol{\rho}_i \right) \right] = \sum \frac{1}{2} m_i \bar{v}^2 + \sum \frac{1}{2} m_i \left| \dot{\boldsymbol{\rho}}_i \right|^2 + \sum m_i \bar{\boldsymbol{v}} \cdot \dot{\boldsymbol{\rho}}_i$$

Consider the last term in the previous equation,

$$\sum m_i \bar{\boldsymbol{v}} \cdot \dot{\boldsymbol{\rho}}_i = \bar{\boldsymbol{v}} \cdot \sum m_i \dot{\boldsymbol{\rho}}_i = \bar{\boldsymbol{v}} \cdot \frac{d}{dt}\left(\sum m_i \boldsymbol{\rho}_i\right).$$

As $\boldsymbol{\rho}_i$ is measured from the mass center G, $\sum m_i \boldsymbol{\rho}_i = 0$, hence, the total kinetic energy becomes,

$$T = \sum \frac{1}{2} m_i \bar{v}^2 + \sum \frac{1}{2} m_i \left|\dot{\boldsymbol{\rho}}_i\right|^2 . \qquad \text{..........(10)}$$

This equation states that the total kinetic energy of a mass system equals the **kinetic energy of mass-center translation of the system as a whole plus the kinetic energy due to motion of all particles relative to the mass center.**

Impulse-momentum

The linear momentum of the representative particle of the system is $\boldsymbol{G}_i = m_i\boldsymbol{v}_\mathrm{i}$ where the velocity of m_i is $\boldsymbol{v}_i = \dot{\boldsymbol{r}}_i$.

The linear momentum of the system is defined as the vector sum of the linear momenta of all of its particles, or $\boldsymbol{G} = \Sigma m_i\boldsymbol{v}_\mathrm{i}$. By substituting the relative-velocity relation (9) and noting again that $\sum m_i\boldsymbol{\rho}_i = 0$, we obtain

$$\boldsymbol{G} = \sum m_i\left(\bar{\boldsymbol{v}} + \dot{\boldsymbol{\rho}}_i\right) = \sum m_i\bar{\boldsymbol{v}} + \frac{d}{dt}\sum m_i\boldsymbol{\rho}_i = \bar{\boldsymbol{v}}\sum m_i + \frac{d}{dt}(0),$$

$$\boldsymbol{G} = m\bar{\boldsymbol{v}} \qquad \cdots\cdots\cdots(11)$$

Thus, **the linear momentum of any system of constant mass is the product of the mass and the velocity of its center of mass.**

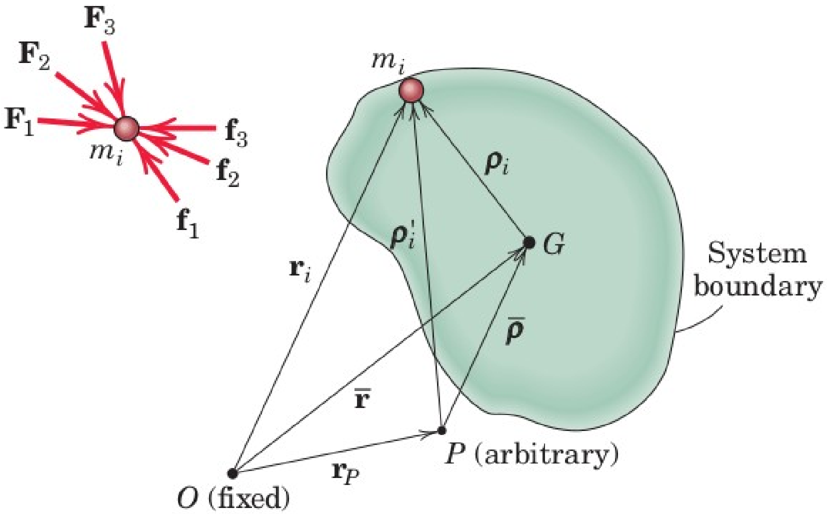
The time derivative of $\boldsymbol{G}$ is $m\dot{\bar{\boldsymbol{v}}} = m\bar{\boldsymbol{a}}$. Thus by (4), $\sum \boldsymbol{F} = \dot{\boldsymbol{G}}$. $\qquad\cdots\cdots\cdots(12)$

Equation (12) has the same form as that for a single particle, which states that **the resultant of the external forces on any mass system equals the time rate of change of the linear momentum of the system.**

Angular momentum

Earlier we defined angular momentum of a particle about fixed point O.

Now, we determine the angular momentum of our general mass system about the fixed point O, about the mass center G, and about an arbitrary point P, which may have an acceleration $\boldsymbol{a}_P = \ddot{\boldsymbol{r}}_P$.



About a Fixed Point O :

The angular momentum of the mass system about the point O, fixed in the Newtonian reference system, is defined as the vector sum of the moments of the linear momenta about O of all particles of the system and is

$$\boldsymbol{H}_O = \Sigma(\boldsymbol{r}_i \times m_i \boldsymbol{v}_i)$$

The time derivative of the vector product is

$$\dot{\boldsymbol{H}}_O = \Sigma(\dot{\boldsymbol{r}}_i \times m_i \boldsymbol{v}_i) + \Sigma(\boldsymbol{r}_i \times m_i \dot{\boldsymbol{v}}_i).$$

The first summation vanishes since the cross product of two parallel vectors $\dot{\boldsymbol{r}}_i$ and $m_i v_i$ is zero.

The second summation is $\Sigma(\boldsymbol{r}_i \times m_i \boldsymbol{a}_i) = \Sigma(\boldsymbol{r}_i \times \boldsymbol{F}_i)$, which is the vector sum of the moments about O of all forces acting on all particles of the system.

This moment sum $\Sigma \boldsymbol{M}_O$ represents only the moments of forces external to the system, since the internal forces cancel one another and their moments add up to zero. Thus, the moment sum is

$$\Sigma \boldsymbol{M}_O = \dot{\boldsymbol{H}}_O, \qquad \cdots\cdots\cdots(13)$$

which has the same form as for a single particle.

It states that **the resultant vector moment about any fixed point of all external forces on any system of mass equals the time rate of change of angular momentum of the system about the fixed point.**

About center of mass G :

The angular momentum of the mass system about the mass center G is the sum of the moments of the linear momenta about G of all particles and is

$$\boldsymbol{H}_G = \sum (\boldsymbol{\rho}_i \times m_i \boldsymbol{v}_i). \qquad \cdots\cdots\cdots(14)$$

Absolute velocity $\boldsymbol{v}_i = \bar{\boldsymbol{v}} + \dot{\boldsymbol{\rho}}_i$, so $\boldsymbol{H}_G$ becomes

$$\boldsymbol{H}_G = \sum (\boldsymbol{\rho}_i \times m_i \bar{\boldsymbol{v}}) + \sum (\boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i). \qquad \cdots\cdots\cdots(14a)$$

The first term can be rewritten as, $\sum m_i \boldsymbol{\rho}_i \times \bar{\boldsymbol{v}}$ which is zero because $\sum m_i \boldsymbol{\rho}_i = 0$.

Thus,

$$\boldsymbol{H}_G = \sum (\boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i). \qquad \cdots\cdots\cdots(14b)$$

The expression (14) is called the **absolute** angular momentum because the absolute velocity $\boldsymbol{v}_i$ is used. The expression (14a) is called the **relative** angular momentum because the relative velocity $\dot{\boldsymbol{\rho}}_i$ is used.

With the mass center G as a reference, the absolute and relative angular momenta are seen to be identical.

Differentiating (14) w.r.t. time once,

$$\dot{\boldsymbol{H}}_G = \sum (\dot{\boldsymbol{\rho}}_i \times m_i \boldsymbol{v}_i) + \sum (\boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{v}}_i).$$

Now, using the expression of velocity as $\boldsymbol{v}_i = \bar{\boldsymbol{v}} + \dot{\boldsymbol{\rho}}_i$, we get

$$\dot{\boldsymbol{H}}_G = \sum (\dot{\boldsymbol{\rho}}_i \times m_i (\bar{\boldsymbol{v}} + \dot{\boldsymbol{\rho}}_i)) + \sum (\boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{v}}_i)$$

$$\Rightarrow \dot{\boldsymbol{H}}_G = \sum (\dot{\boldsymbol{\rho}}_i \times m_i \bar{\boldsymbol{v}}) + \sum (\boldsymbol{\rho}_i \times m_i \boldsymbol{a}_i) = \sum m_i \dot{\boldsymbol{\rho}}_i \times \bar{\boldsymbol{v}} + \sum (\boldsymbol{\rho}_i \times m_i \boldsymbol{a}_i)$$

$$\Rightarrow \dot{\boldsymbol{H}}_G = \cancelto{0}{\frac{d}{dt} \sum (m_i \boldsymbol{\rho}_i)} \times \bar{\boldsymbol{v}} + \sum (\boldsymbol{\rho}_i \times m_i \boldsymbol{a}_i) = \sum \boldsymbol{\rho}_i \times (\boldsymbol{f}_i + \boldsymbol{F}_i)$$

$$\Rightarrow \dot{\boldsymbol{H}}_G = \sum \boldsymbol{\rho}_i \times \boldsymbol{F}_i = \sum \boldsymbol{M}_G$$

Thus,

$$\sum \boldsymbol{M}_G = \dot{\boldsymbol{H}}_G, \qquad \cdots\cdots\cdots\cdots(15)$$

where we may use either the absolute or the relative angular momentum.

Equations (13) and (15) are among the most powerful of the governing equations in dynamics and apply to any defined system of constant mass-rigid or nonrigid.

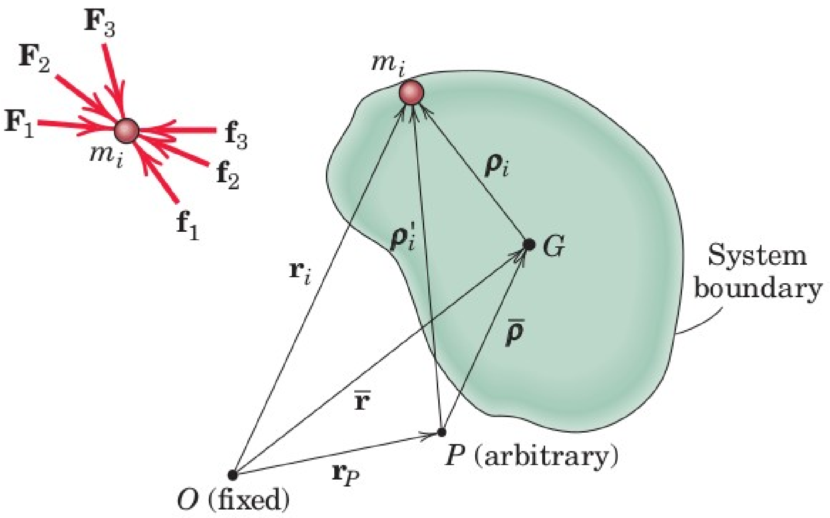
About an arbitrary point P :

The angular momentum about an arbitrary point P (which may have an acceleration $\ddot{\boldsymbol{r}}_P$) can be expressed as,

$$\boldsymbol{H}_P = \sum \boldsymbol{\rho}_i' \times m_i \boldsymbol{v}_i = \sum (\bar{\boldsymbol{\rho}} + \boldsymbol{\rho}_i) \times m_i \boldsymbol{v}_i,$$

$$\boldsymbol{H}_P = \bar{\boldsymbol{\rho}} \times \sum m_i \boldsymbol{v}_i + \sum \boldsymbol{\rho}_i \times m_i \boldsymbol{v}_i,$$

$$\boldsymbol{H}_P = \bar{\boldsymbol{\rho}} \times m\bar{\boldsymbol{v}} + \boldsymbol{H}_G. \qquad \text{..........(16)}$$



Equation (16) states that **the absolute angular momentum about any point P equals the angular momentum about G plus the moment about P of the linear momentum $m\bar{\boldsymbol{v}}$ of the system considered concentrated at G.**

Recall, from the knowledge of statics, a force system can be represented by a resultant force through any point, such as G, and a corresponding couple.

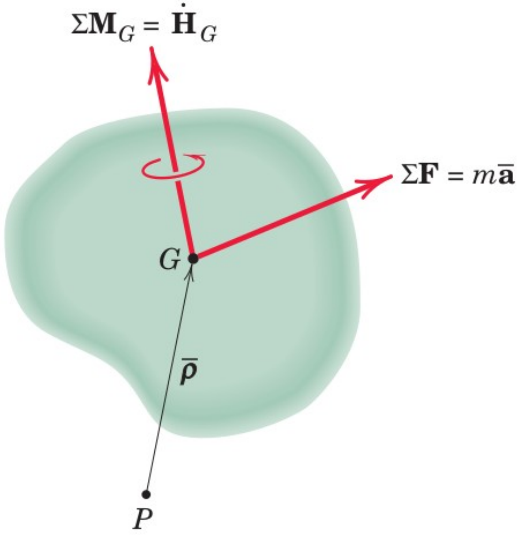


Figure represents the resultants of the external forces acting on the system expressed in terms of the resultant force $\Sigma\boldsymbol{F}$ through G and the corresponding couple $\Sigma\boldsymbol{M}_G$. We see that the sum of the moments about P of all forces external to the system must equal the moment of their resultants. Therefore, we may write

$$\sum \boldsymbol{M}_P = \sum \boldsymbol{M}_G + \bar{\boldsymbol{\rho}} \times \sum \boldsymbol{F} = \sum \boldsymbol{M}_G + \bar{\boldsymbol{\rho}} \times m\bar{\boldsymbol{a}}. \qquad \text{..........(17)}$$

Similar momentum relationships can be developed by using the momentum relative to P. Thus, from

$$(\boldsymbol{H}_P)_{\text{rel}} = \sum \boldsymbol{\rho}_i' \times m_i \dot{\boldsymbol{\rho}}_i', \qquad \text{..........(18)}$$

where $\dot{\boldsymbol{\rho}}_i'$ is the velocity of m_i relative to P. With the substitution $\boldsymbol{\rho}_i' = \bar{\boldsymbol{\rho}} + \boldsymbol{\rho}_i$ and $\dot{\boldsymbol{\rho}}_i' = \dot{\bar{\boldsymbol{\rho}}} + \dot{\boldsymbol{\rho}}_i$, (18) may be written as,

$$(\boldsymbol{H}_P)_{\text{rel}} = \sum \bar{\boldsymbol{\rho}} \times m_i \dot{\bar{\boldsymbol{\rho}}} + \sum \bar{\boldsymbol{\rho}} \times m_i \dot{\boldsymbol{\rho}}_i + \sum \boldsymbol{\rho}_i \times m_i \dot{\bar{\boldsymbol{\rho}}} + \sum \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i,$$

$$(\boldsymbol{H}_P)_{\text{rel}} = \sum \bar{\boldsymbol{\rho}} \times m_i \bar{\boldsymbol{v}}_{\text{rel}} + \cancelto{0}{\sum \bar{\boldsymbol{\rho}} \times m_i \dot{\boldsymbol{\rho}}_i} + \cancelto{0}{\sum m_i \boldsymbol{\rho}_i \times \dot{\bar{\boldsymbol{\rho}}}} + (\boldsymbol{H}_G),$$

$$(\boldsymbol{H}_P)_{\text{rel}} = \boldsymbol{H}_G + \sum \bar{\boldsymbol{\rho}} \times m_i \bar{\boldsymbol{v}}_{\text{rel}}, \qquad \text{..........(19)}$$

The moment equation about P may now be expressed in terms of the angular momentum relative to P by differentiating (18) as,

$$\sum \boldsymbol{M}_P = (\dot{\boldsymbol{H}}_P)_{\text{rel}} + \bar{\boldsymbol{\rho}} \times m \boldsymbol{a}_P, \qquad \text{..........(20)}$$ (Derive)

The form of (20) is convenient when a point P whose acceleration is known is used as a moment center. The equation reduces to the simpler form

$$\sum \boldsymbol{M}_P = (\dot{\boldsymbol{H}}_P)_{\text{rel}} \text{ if } \begin{cases} 1.\ \boldsymbol{a}_P = 0 \text{ [same as (13)]} \\ 2.\ \bar{\boldsymbol{\rho}} = 0 \text{ [same as (15)]} \\ 3.\ \bar{\boldsymbol{\rho}} \text{ and } \boldsymbol{a}_P \text{ are parallel} \end{cases} \qquad \text{..........(21)}$$