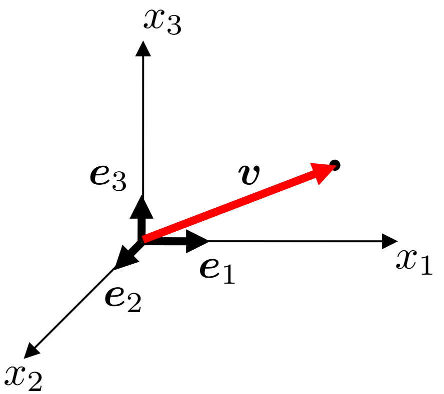
Introduction to Tensors

Indical Notations



A vector $\boldsymbol{v}$ can be represented as,

$$\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3$$

$(\boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3)$ are unit basis vectors of a right-handed, orthonormal system (Cartesian), which are given as,

$$\boldsymbol{e}_1 = \langle 1, 0, 0 \rangle\,, \quad \boldsymbol{e}_2 = \langle 0, 1, 0 \rangle\,, \quad \boldsymbol{e}_3 = \langle 0, 0, 1 \rangle\,,$$

and $(v_1,\ v_2,\ v_3)$ are components of vector $\boldsymbol{v}$ along $(\boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3)$, respectively. We can use indical notations to represent the vector $\boldsymbol{v}$ as,

$$\boldsymbol{v} = v_i\boldsymbol{e}_i.$$

where $\{v_i\}_{i\in\{1,2,3\}}$ and $\{\boldsymbol{e}_i\}_{i\in\{1,2,3\}}$denotes components of vector $\boldsymbol{v}$ and base vectors.

For three dimensional Cartesian coordinate system i takes value as [1,2,3]. In-fact, it can take values [1,2,3…N] for an N-dimensional coordinate system.
In that case,

$$\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3 \cdots + v_N\boldsymbol{e}_N.$$

Indical Notations

Now, consider a system of linear equations,

$$y_1 = a_{11}x_1 + a_{12}x_2$$
$$y_2 = a_{21}x_1 + a_{22}x_2.$$

We can denote these equations as,

$$y_i = a_{i1}x_1 + a_{i2}x_2.$$

In summation form,

$$y_i = \sum_{j=1}^{2} a_{ij}x_j,$$

Which can also be written as,

$$y_i = a_{ij}x_j.$$

For writing this form we have used *summation convention*, which says that whenever an index is *repeated* in the same term, it deontes a *summation* over the *range* of the index.

Here, index i, which is not summed up is known as *free* index. It appears on both side of the equation. The index j, which is summed up, is called a *dummy* index. The dummy index can be replaced by any other index; this does not alter the summation. For equation, the last equation can also be written as,

$$y_i = a_{im}x_m.$$

Indical Notations

In our course we will be working in three dimensional cartesian coordinate system. So all indices i, j, k, l, m, ... etc. will take values [1,2,3]

- Thus, for vector (or first order tensor) $\boldsymbol{v}$, indical notation is v_i, which denotes 3 terms,
- For a second order tensor $\boldsymbol{A}$, indicial notation is A_{ij} , which will denote 9 terms,
- Similar for a fourth order tensor $\boldsymbol{\mathcal{A}}$, indicial notation is $\mathcal{A}_{ijkl}$, which denots 81 (3^4) terms.

Kronecker delta

The *Kronecker delta* is an important tensor, which is defined as,

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

which implies,

$$\begin{aligned} &\delta_{11} = 1, \quad \delta_{12} = 0, \quad \delta_{13} = 0, \\ &\delta_{21} = 0, \quad \delta_{22} = 1, \quad \delta_{23} = 0, \\ &\delta_{31} = 0, \quad \delta_{32} = 0, \quad \delta_{33} = 1. \end{aligned}$$

It can be used to denote the dot product of two orthogoal basis vectors as,

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$$

Kronecker delta

Kronecker delta also works as a *replacement operator*. The index on u_i becomes j when the components u_i are multiplied with δ_{ij}, i.e.

$$\delta_{ij}u_i = u_j$$

An important application of Kronecker delta is in *factoring* and *contraction* of tensors.

Factoring: Consider the following equation,

$$A_{ij}n_j = \lambda n_i \Rightarrow A_{ij}n_j - \lambda n_i = 0,$$

Following the property of Kronecker delta, we can write n_i as,

$$n_i = \delta_{ij}n_j,$$

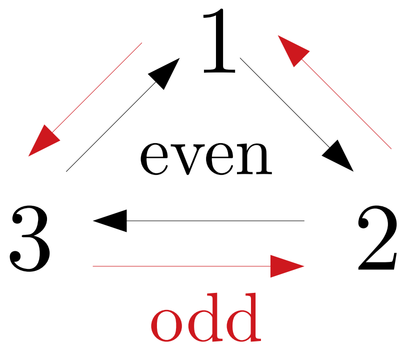
which follows, $A_{ij}n_j - \lambda\delta_{ij}n_j = 0 \Rightarrow (A_{ij} - \lambda\delta_{ij})\, n_j = 0.$

Contraction: $A_{mn}\delta_{mn} = A_{mm} = A_{11} + A_{22} + A_{33}$

Permutation symbol

The *Permutation symbol* is another important tensor, which is defined as,

$$e_{ijk} = \begin{cases} 1, & \text{for } \textbf{even} \text{ permutations of } (i,j,k) \text{ i.e. } 123,\ 231,\ 312, \\ -1, & \text{for } \textbf{odd} \text{ permutations of } (i,j,k) \text{ i.e. } 132,\ 213,\ 321, \\ 0, & \text{there is a repeated index.} \end{cases}$$



It can be used to denote the cross product of two orthogoal basis vectors as follows,

$$\boldsymbol{e}_i \times \boldsymbol{e}_j = e_{ijk}\boldsymbol{e}_k,$$

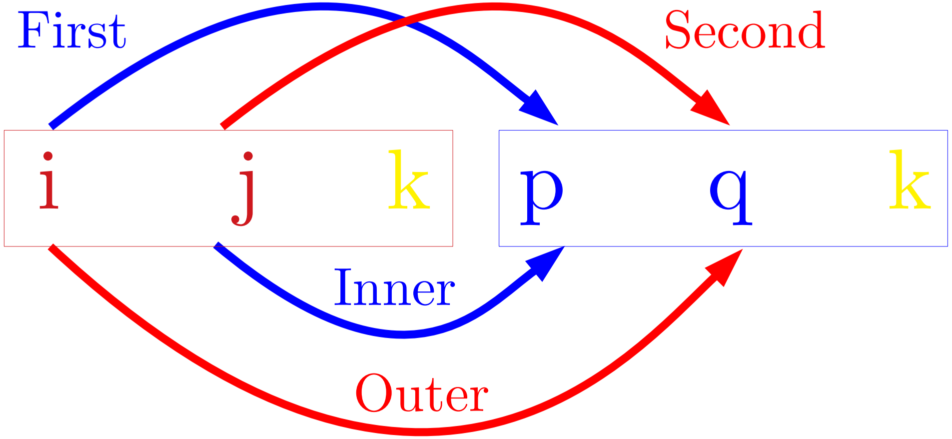
$$\boldsymbol{e}_i \times \boldsymbol{e}_j = \begin{cases} \boldsymbol{e}_k, & \text{for } \textbf{even} \text{ permutations of } (i,j,k), \\ -\boldsymbol{e}_k, & \text{for } \textbf{odd} \text{ permutations of } (i,j,k), \\ 0, & \text{otherwise.} \end{cases}$$

e-δ indentity

The e-δ indentity relates the permutation symbol with Kronecker delta. It can be shown that,

$$e_{ijk}e_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

An easy way to remember it,



$$e_{ijk}e_{pqk} = (\text{First})(\text{Second}) - (\text{Outer})(\text{Inner})$$

Examples

Write following expression in indical form.

Example 1:

$$\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v}$$

$$\Rightarrow u_i \boldsymbol{e}_i \times v_j \boldsymbol{e}_j$$

$$\Rightarrow u_i v_j \boldsymbol{e}_i \times \boldsymbol{e}_j$$

$$\Rightarrow u_i v_j e_{ijk} \boldsymbol{e}_k$$

$$\Rightarrow w_k \boldsymbol{e}_k = u_i v_j e_{ijk} \boldsymbol{e}_k.$$

Thus the component of $\boldsymbol{w}$ is

$$w_k = e_{ijk} u_i v_j$$

Example 2:

$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$$

$$\Rightarrow e_{ijk} u_i v_j \boldsymbol{e}_k \cdot w_m \boldsymbol{e}_m \quad \text{(from Example 1)}$$

$$\Rightarrow e_{ijk} u_i v_j w_m \ \boldsymbol{e}_k \cdot \boldsymbol{e}_m$$

$$\Rightarrow e_{ijk} u_i v_j w_m \delta_{km}$$

$$\Rightarrow e_{ijk} u_i v_j w_k.$$

Examples

Example 3: Prove the following vector identity using indical notations.

$$(\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w} = (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{w})\boldsymbol{u}$$

Let's start from LHS

$$(\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w}$$

$$\Rightarrow e_{ijm} u_i v_j \boldsymbol{e}_m \times w_k \boldsymbol{e}_k \quad \text{(from Example 1)}$$

$$\Rightarrow e_{ijm} u_i v_j w_k \ \boldsymbol{e}_m \times \boldsymbol{e}_k$$

$$\Rightarrow e_{ijm} e_{mkn} u_i v_j w_k \ \boldsymbol{e}_n$$

We will now make use of e-δ indentity,

$$e_{ijm} e_{mkn} = e_{ijm} e_{knm} \quad (\because e_{mkn} = e_{knm})$$

$$\Rightarrow e_{ijm} e_{knm} = \delta_{ik}\delta_{jn} - \delta_{in}\delta_{jk}$$

Now we can write

$$\begin{aligned}
&(\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w} = u_i v_j w_k \left(\delta_{ik}\delta_{jn} - \delta_{in}\delta_{jk}\right) \ \boldsymbol{e}_n \\
&\Rightarrow (u_i\delta_{ik})w_k(v_j\delta_{jn})\boldsymbol{e}_n - (u_i\delta_{in})(v_j\delta_{jk})w_k\boldsymbol{e}_n \\
&\Rightarrow u_k w_k v_n \boldsymbol{e}_n - v_k w_k u_n \boldsymbol{e}_n \\
&\Rightarrow (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{w})\boldsymbol{u}
\end{aligned}$$

In the last step we have used the following relation for dot product of two vectors (say $\boldsymbol{a}$ and $\boldsymbol{b}$)

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_i \boldsymbol{e}_i \cdot b_j \boldsymbol{e}_j = a_i b_j \boldsymbol{e}_i \cdot \boldsymbol{e}_j = a_i b_j \delta_{ij} = a_i b_i$$

Examples

Example 4: Consider the following,

$$y_i = a_{ij} x_j ,$$
$$x_i = b_{ij} z_j .$$

Express y_i in terms of z_i .

Note that in expression $x_i = b_{ij} z_j$, j is a dummy index, hence replacing it with any other index will not change the summation. So we write,

$$x_j = b_{jm} z_m ,$$

Now, we can express y_i as,

$$y_i = a_{ij} b_{jm} z_m .$$

Algebra of second order tensors

A second order tensor $\boldsymbol{A}$ is a linear transformation mapping of a vector to another vector, i.e.

$$\boldsymbol{u} = \boldsymbol{A}\boldsymbol{v}.$$

As $\boldsymbol{A}$ is a linear transformation, it implies

$$\boldsymbol{A}(\alpha \boldsymbol{u} + \beta \boldsymbol{v}) = \alpha \boldsymbol{A}\boldsymbol{u} + \beta \boldsymbol{A}\boldsymbol{v}.$$

The *tensor product* or *the dyad* of two vectors is a second order tensor defined as,

$$(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w} = \boldsymbol{u}(\boldsymbol{v} \cdot \boldsymbol{w}) = (\boldsymbol{v} \cdot \boldsymbol{w})\boldsymbol{u}$$

Note that the dot product is between the two immidiate adjacent vectors which are not connected by $\otimes$ symbol.

Sometimes dyad is simply written as $\boldsymbol{uv}$.

It also follows, $(\alpha \boldsymbol{u} + \beta \boldsymbol{v}) \otimes \boldsymbol{w} = \alpha \boldsymbol{u}\boldsymbol{w} + \beta \boldsymbol{v}\boldsymbol{w}$