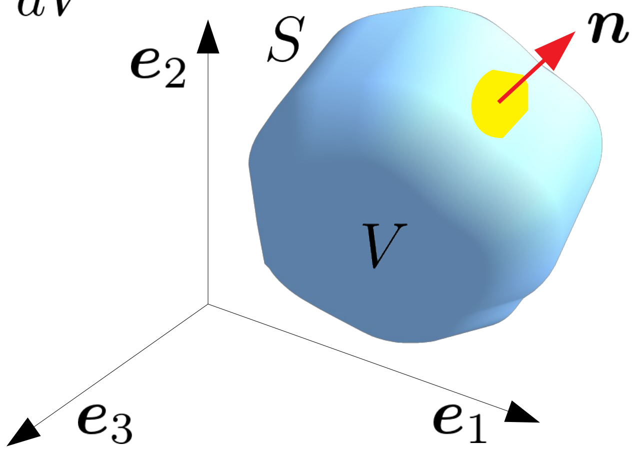
Introduction to Tensors

Integral theorems

We introduce two important integral theorems. First one is known as *Gauss' divergence theorem* which transforms a surface integral into volume integral and states that,

$$\int_S \boldsymbol{u} \cdot \boldsymbol{n}\, dS = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{u}\, dV \quad \text{or} \quad \int_S u_i n_i\, dS = \int_V \frac{\partial u_i}{\partial x_i}\, dV$$



where $\boldsymbol{u}(\boldsymbol{x})$ is a smooth vector field defined in space.

Similarly for a smooth tensor field $\boldsymbol{A}(\boldsymbol{x})$ in space,

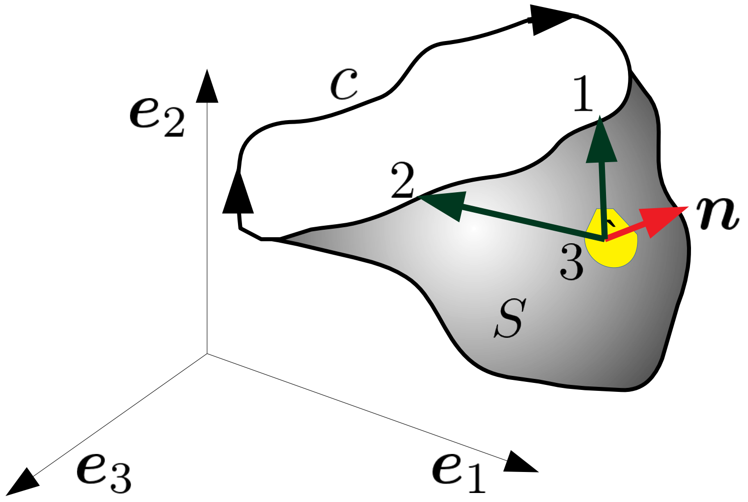
$$\int_S \boldsymbol{A} \cdot \boldsymbol{n}\, dS = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{A}\, dV \quad \text{or} \quad \int_S A_{ij} n_j\, dS = \int_V \frac{\partial A_{ij}}{\partial x_j}\, dV$$

Another theorem is known as *Stoke's theorem* which is related to *open surfaces*. It relates the *surface integral over the open surface* to the *line integral around the bounding closed curve* in space.

$$\oint_c \boldsymbol{u} \cdot d\boldsymbol{x} = \int_S (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot \boldsymbol{n} dS$$

or

$$\oint_c u_k dx_k = \int_S e_{ijk} \frac{\partial u_k}{\partial x_j} n_i dS$$



Note that the sence of curve $\boldsymbol{c}$ and the direction of normal $\boldsymbol{n}$ will be such that the vectors connecting points 1,2, and 3 form a right handed set of vectors.

Eigenvalues and Eigenvectors

For a tensor $\boldsymbol{A}$, if there exists scalars λ_i and corresponding normalized vectors $\hat{\boldsymbol{n}}_i$ such that

$$\boldsymbol{A}\hat{\boldsymbol{n}}_i = \lambda_i \hat{\boldsymbol{n}}_i, \quad (i = 1, 2, 3; \text{ no summation})$$

then λ_i are called *eigenvalues* (or principal values) and $\hat{\boldsymbol{n}}_i$ are called *eigenvectors* (or principal directions or principal axes) of tensor $\boldsymbol{A}$.

A set of homogeneous algebric equation to determine unknown eigenvalues λ_i (i=1,2,3) and unknown eigenvectors $\hat{\boldsymbol{n}}_i$ (i=1,2,3) are,

$$(\boldsymbol{A} - \lambda_i \boldsymbol{I})\hat{\boldsymbol{n}}_i = \boldsymbol{o}, \; (i = 1, 2, 3; \text{ no summation})$$

Eigenvalues characterize the physical nature of a tensor. They do not depend upon the coordinates. For a *positive definite tensor* all eigenvalues are *real* (and *positive).* Also, the set of eigenvectors of a symmetric tensor form a *mutually orthogonal (or orthonormal) basis* $\{\hat{\boldsymbol{n}}_i\}$.

The trivial solution of the system given by $(\boldsymbol{A}-\lambda_i\boldsymbol{I})\hat{\boldsymbol{n}}_i=\boldsymbol{o}$ is $\hat{\boldsymbol{n}}_i=\boldsymbol{o}$.

For system to have solutions $\hat{\boldsymbol{n}}_i\neq\boldsymbol{o}$ following condition should be satisfied,

$$\det(\boldsymbol{A}-\lambda_i\boldsymbol{I})=0$$

where $\det(\boldsymbol{A}-\lambda_i\boldsymbol{I})=-\lambda_i^3+I_1\lambda_i^2-I_2\lambda_i+I_3$.This requires to solve a cubic equation

$$\lambda^3-I_1\lambda^2+I_2\lambda-I_3=0,$$

which is known as characteristic equation of $\boldsymbol{A}$. The solution of the equations are the eigenvalues λ_i (i =1,2,3). *Scaler Invarients* (or *principal scalar invarients*) I_1, I_2 and, I_3 in terms of eigenvalues are,

.

$$\begin{aligned}
I_1 &= \mathrm{tr}\boldsymbol{A}=A_{ii}=\lambda_1+\lambda_2+\lambda_3\\
I_2 &= \frac{1}{2}\left[I_1^2-\mathrm{tr}(\boldsymbol{A}^2)\right]=\frac{1}{2}\left[A_{ii}A_{jj}-A_{ij}A_{ji}\right]=\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1\\
I_3 &= \det\boldsymbol{A}=\lambda_1\lambda_2\lambda_3
\end{aligned}$$

Application of $\boldsymbol{A}$ to the equation $\boldsymbol{A}\hat{\boldsymbol{n}}_i = \lambda_i \hat{\boldsymbol{n}}_i$ results,

$$\boldsymbol{A}^2 \hat{\boldsymbol{n}}_i = \lambda_i \boldsymbol{A}\hat{\boldsymbol{n}}_i = \lambda_i^2 \hat{\boldsymbol{n}}_i.$$

Operating the above equation with tensor $\boldsymbol{A}$ again gives, $\boldsymbol{A}^3 \hat{\boldsymbol{n}}_i = \lambda_i^3 \hat{\boldsymbol{n}}_i$.

Similarly, after repeated application of A, we can write a general relation as,

$$\boldsymbol{A}^\alpha \hat{\boldsymbol{n}}_i = \lambda_i^\alpha \hat{\boldsymbol{n}}_i, \text{ where } a \text{ is a positive integer.}$$

By multiplying the characteristic equation by $\hat{\boldsymbol{n}}_i$ and using the above relation, we obtain the *Caley-Hamilton equation*.

$$\boldsymbol{A}^3 - I_1 \boldsymbol{A}^2 + I_2 \boldsymbol{A} - I_3 = 0.$$

Spectral decomposition of a tensor

Any symmetric tensor $\boldsymbol{A}$ can be represented using its eigenvalue and eigenvectors as basis vectors $\{\hat{\boldsymbol{n}}_i\}$. A unit tensor in $\{\hat{\boldsymbol{n}}_i\}$ basis vectors is represented as,

$$\boldsymbol{I} = \hat{\boldsymbol{n}}_i \otimes \hat{\boldsymbol{n}}_i.$$

Tensor A can be written as,

$$\boldsymbol{A} = \boldsymbol{A}\boldsymbol{I} = (\boldsymbol{A}\hat{\boldsymbol{n}}_i) \otimes \hat{\boldsymbol{n}}_i = \sum_{i=1}^{3} \lambda_i \hat{\boldsymbol{n}}_i \otimes \hat{\boldsymbol{n}}_i$$

Now, let us find the $(ij)^{\text{th}}$ component of tensor $\boldsymbol{A}$ relative to the a basis of eigenvectors $\{\hat{\boldsymbol{n}}_i\}$,

$$A_{ij} = \hat{\boldsymbol{n}}_i \cdot (\boldsymbol{A}\hat{\boldsymbol{n}}_j) = \hat{\boldsymbol{n}}_i \cdot (\lambda_j \hat{\boldsymbol{n}}_j) = \lambda_j \delta_{ij}. \; (j = 1, 2, 3; \text{ no summation})$$

In matrix form,

$$[\boldsymbol{A}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

Example

For the given tensor $\boldsymbol{A}$, find the eigenvalues and corresponding eigenvectors.

$$\boldsymbol{A} = \alpha(\boldsymbol{I} - \boldsymbol{e}_1 \otimes \boldsymbol{e}_1) + \beta(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1)$$

First let us find the invarients of the tensor,

$$I_1 = 2a \ , \quad I_2 = a^2 - \beta^2 , \quad I_3 = -a\beta^2$$

Characteristic equation now become,

$$\lambda^3 - 2a\lambda^2 + (a^2 - \beta^2)\lambda - (-a\beta^2) = 0.$$

Roots of the equation are,

$$\lambda_1 = \alpha, \quad \lambda_2 = \frac{\sqrt{4\beta^2 + \alpha^2} + \alpha}{2}, \quad \lambda_3 = \frac{-\sqrt{4\beta^2 + \alpha^2} - \alpha}{2}.$$

To find the eigenvectors we use the following relation and substitute the value of λ in it,

$$\boldsymbol{A}\hat{\boldsymbol{n}}_i = \lambda_i \hat{\boldsymbol{n}}_i, \quad (i = 1, 2, 3; \text{ no summation})$$

Substituting $\lambda_1 = a$ in equation we get following set of equations,

$$-\alpha\hat{n}_1^1 + \beta\hat{n}_1^2 = 0$$
$$\beta\hat{n}_1^1 = 0$$

From above two equations we get, $\hat{n}_1^1 = \hat{n}_1^2 = 0.$

Two determine the third component, we have another equation as,

$$(\hat{n}_1^1)^2 + (\hat{n}_1^2)^2 + (\hat{n}_1^3)^2 = 1,$$

which gives us $n_1^3 = 1$. Thus, we have the eigenvector corresponding to eigenvalue $\lambda_1 = a$ is $\hat{\boldsymbol{n}}_1$={0,0,1}.

Similarly, eigenvectors $\hat{\boldsymbol{n}}_2$ and $\hat{\boldsymbol{n}}_3$ corresponding to λ_2 and λ_3 can also be determined.