ME632: Fracture Mechanics

Timings	
Monday	10:00 to 11:20
Thursday	08:30 to 09:50

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Room No. # 106

Approximate determination of crack-tip plastic zone

Strictly speaking, the plastic zone should be determined from an elastic-plastic analysis of the stress field around the crack tip. However, such analysis is quite complex and involved. Hence, we can obtain some useful results regarding the shape of the plastic zone from the approximate calculation.

A first estimate of the extent of the plastic zone attending the crack tip can be obtained by determining the locus of points where the elastic stress field satisfies the yield criterion. This calculation is very approximate, since yielding leads to stress redistribution and modifies the size and shape of the plastic zone. To apply the yield criterion let us first determine principle stresses. Stresses at the crack-tip are

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right),$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right),$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}} \sin\theta \sin\frac{3\theta}{2}\cos\frac{3\theta}{2},$$

$$\sigma_z = 0. \quad \text{for plane stress,}$$

$$\sigma_z = \nu(\sigma_x + \sigma_y) \quad \text{for plane strain.}$$

.................(1)

Principal stresses are

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \sigma_{xy}^2} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\right]$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \sigma_{xy}^2} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\right]$$

$$\sigma_3 = 0 \quad \text{(for plane stress)} \qquad \cdots\cdots\cdots\cdots(2)$$

$$\sigma_3 = 2\nu \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \quad \text{(for plane strain)}$$

By applying the Von-Mises criterion for yielding,

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \geq 2\sigma_Y^2 \qquad \cdots\cdots\cdots\cdots(3)$$

Substituting (2) in (3) and simplifying we get,

$$\frac{K_I^2}{2\pi r}\left(1+\cos\theta+\frac{3}{2}\sin^2\theta\right)\geq 2\sigma_Y^2 \qquad \text{(for plane stress)}$$

$$\frac{K_I^2}{2\pi r}\left[(1-2\nu)^2(1+\cos\theta)+\frac{3}{2}\sin^2\theta\right]\geq 2\sigma_Y^2 \qquad \text{(for plane strain)}$$

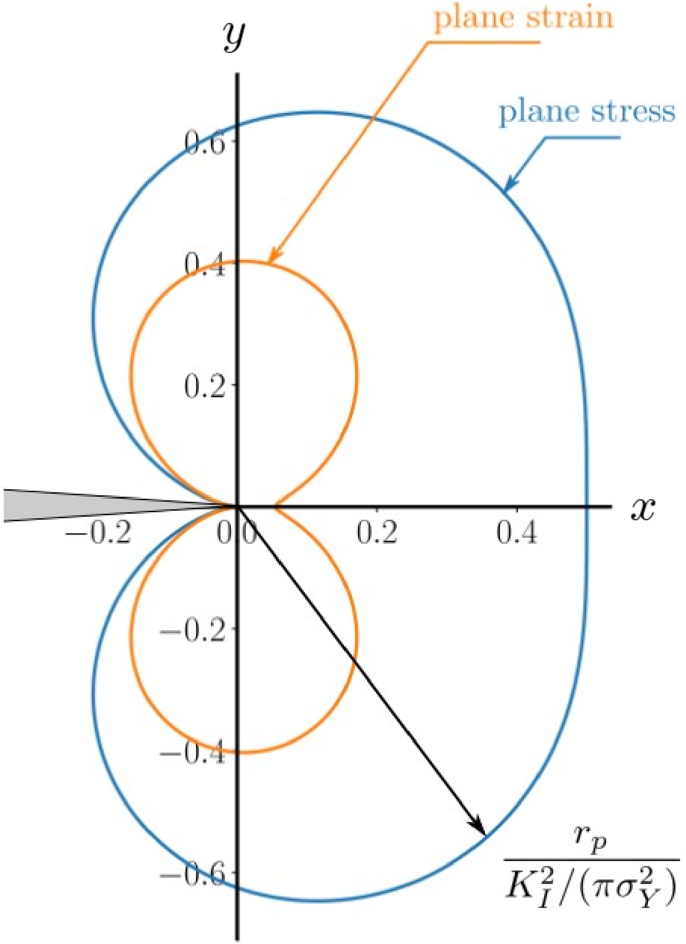
Thus the boundary of elastic plastic interface can be obtained as

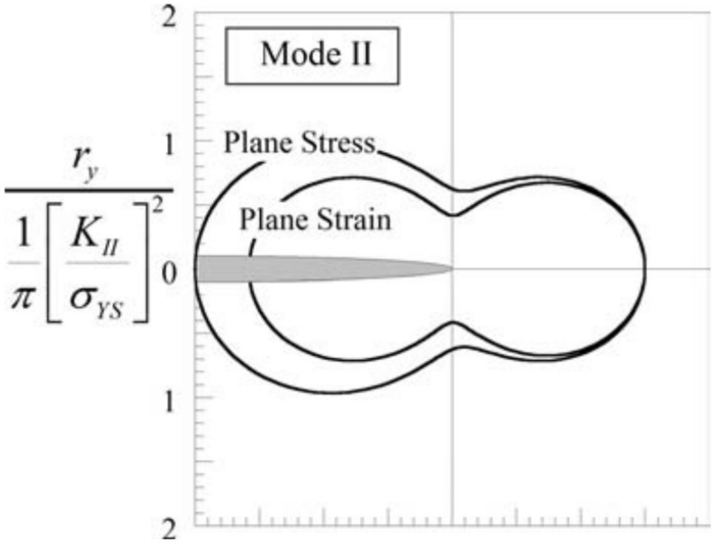
$$r_p(\theta)=\frac{K_I^2}{4\pi\sigma_Y^2}\left(1+\cos\theta+\frac{3}{2}\sin^2\theta\right) \qquad \text{(for plane stress)}$$

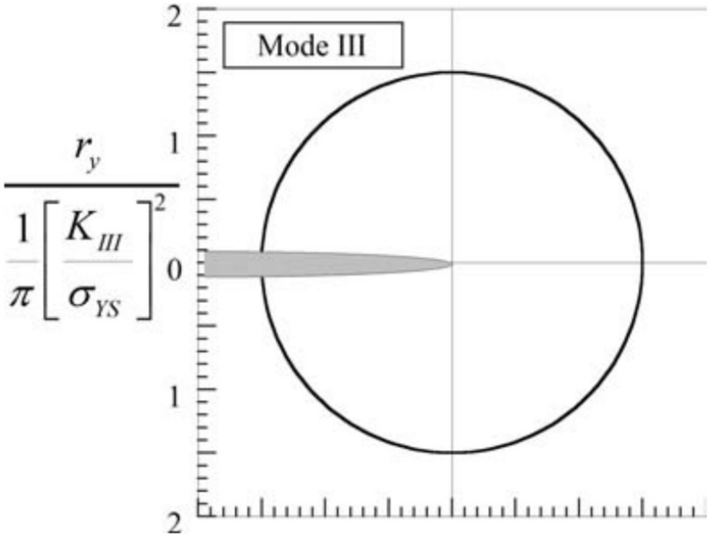
$$r_p(\theta)=\frac{K_I^2}{4\pi\sigma_Y^2}\left[(1-2\nu)^2(1+\cos\theta)+\frac{3}{2}\sin^2\theta\right] \qquad \text{(for plane strain)}$$

..................(4)

(for $\nu = 1/3$)

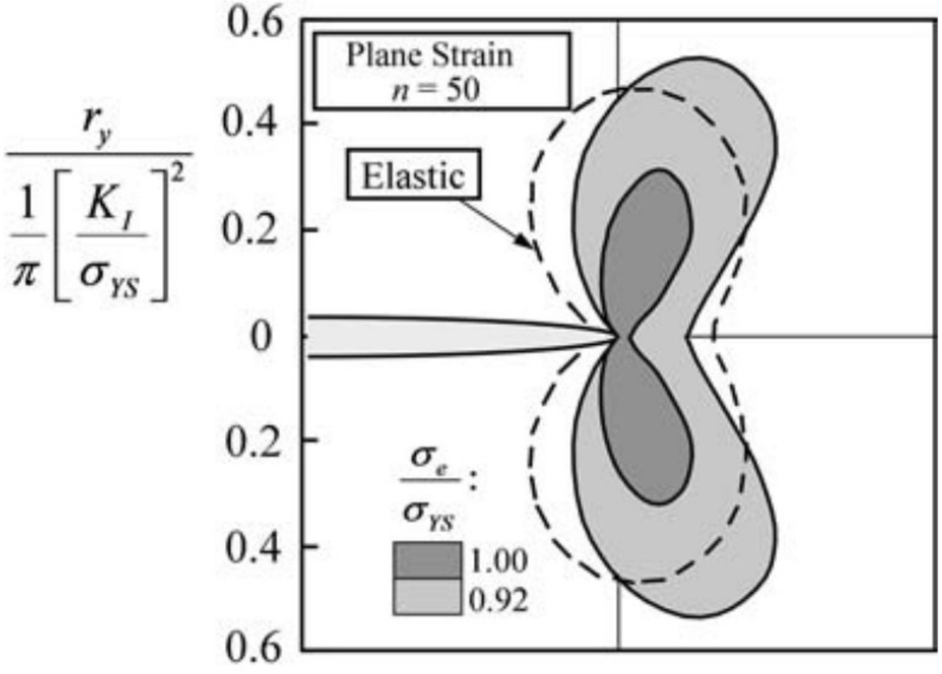






Plastic zone through finite element simulation, using the following stress-strain relation:

$$\frac{\varepsilon}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \alpha \left(\frac{\sigma}{\sigma_0}\right)^n$$



- Observe that the plane stress zone is much larger than the plane strain zone because of the higher constraint for plane strain.
- Plastic zone is proportional to the square of SIF and inversely proportional to the square of the yield stress.
- Whenever yield stress of a metal is increased, say by an appropriate heat treatment, its plastic zone size decreases considerably; this in turn makes the material more prone to crack growth.
- The increase in yield stress may please a conventional designer, because he usually designs structural components based on a yield criterion. But as far as the toughness of a material is concerned, the designer is left with an inferior material. The designer msut explore a satisfactory compromise between yield stress and toughness, while choosing a material and its heat treatment.
- Let us try to understand the yielding behaviour. Tresca criteria helps us in understanding it in a simple manner.

For plane strain (considering $\nu = 1/3$)

$$\sigma_{\max} = \sigma_1 = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\right]$$

$$\text{for } \theta \leq 38.9°,\ \sigma_{\min} = \sigma_3 = 2\nu\frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}$$

$$\text{for } \theta \geq 38.9°,\ \sigma_{\min} = \sigma_2 = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\right]$$

Thus,

$$\text{for } \theta \leq 38.9°,\quad \tau_{\max} = \frac{\sigma_{\max} - \sigma_{\min}}{2} = \frac{1}{2}\frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[1 - 2\nu + \sin\frac{\theta}{2}\right]$$

$$\text{for } \theta \geq 38.9°,\quad \tau_{\max} = \frac{\sigma_{\max} - \sigma_{\min}}{2} = \frac{1}{2}\frac{K_I}{\sqrt{2\pi r}}\sin\theta$$

Hence, from Tresca criteria, ...(5)

$$\text{for } \theta \leq 38.9°,\quad r_p = \frac{K_I^2}{2\pi\sigma_Y^2}\cos^2\frac{\theta}{2}\left[1 - 2\nu + \sin\frac{\theta}{2}\right]^2, \quad \text{and} \quad \text{for } \theta \leq 38.9°,\quad r_p = \frac{K_I^2}{2\pi\sigma_Y^2}\sin^2\theta.$$

For plane stress,

$$\sigma_{\max} = \sigma_1 = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\right]$$

$$\sigma_{\min} = \sigma_3 = 0$$

Now, as per Tresca criteria for yielding

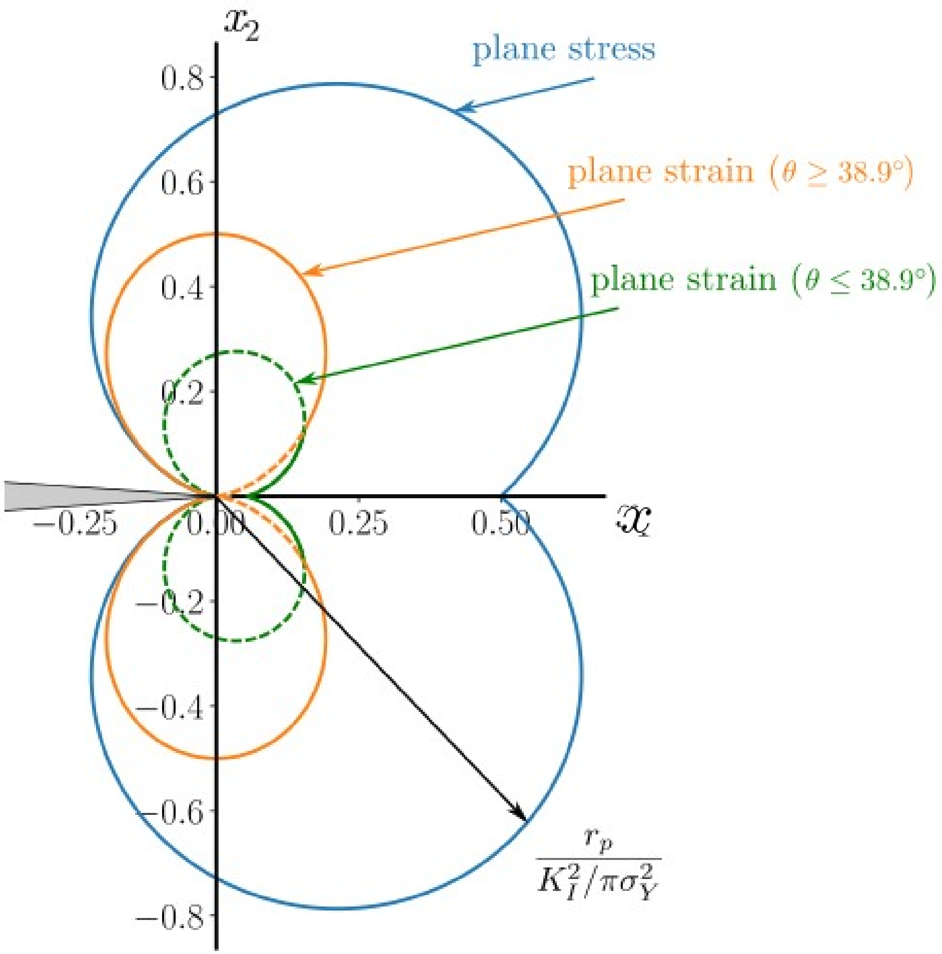
$$\tau_{\max} = \frac{\sigma_{\max} - \sigma_{\min}}{2} \geq \frac{\sigma_Y}{2},$$

i.e.,

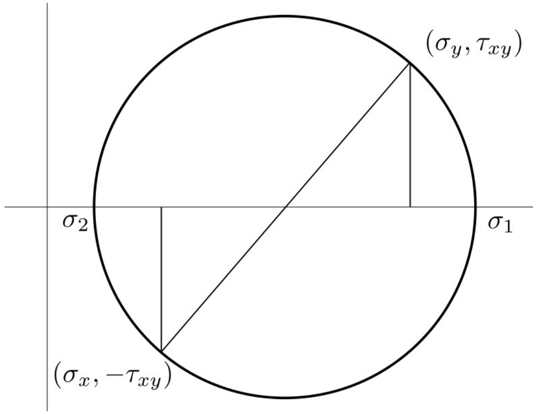
$$\frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\right] \geq \sigma_Y.$$

Thus,

$$r_p(\theta) = \frac{K_I^2}{2\pi\sigma_Y^2} \cos^2\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\right]^2. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(6)$$



Let us now use Mohr's circle to visualize the planes of yielding in plane stress and plane strain cases under Mode-I loading. During the plastic deformation there is no change in volume and hence, $\nu = 0.5$ will be considered here.

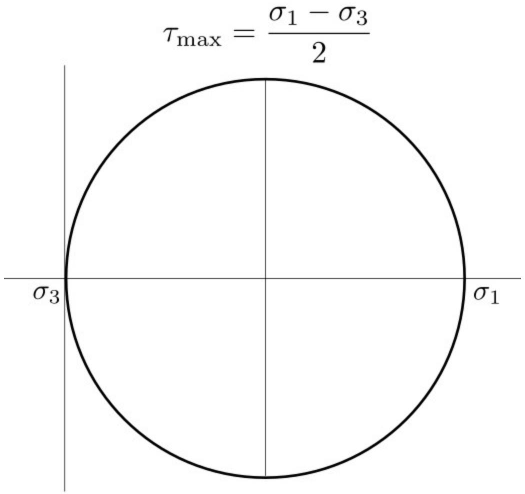


The state of stress for an element at $\theta = 0°$ plane is,

$$\sigma_x = \sigma_y = \frac{K_I}{\sqrt{2\pi r}} \quad \text{and} \quad \sigma_{xy} = 0.$$

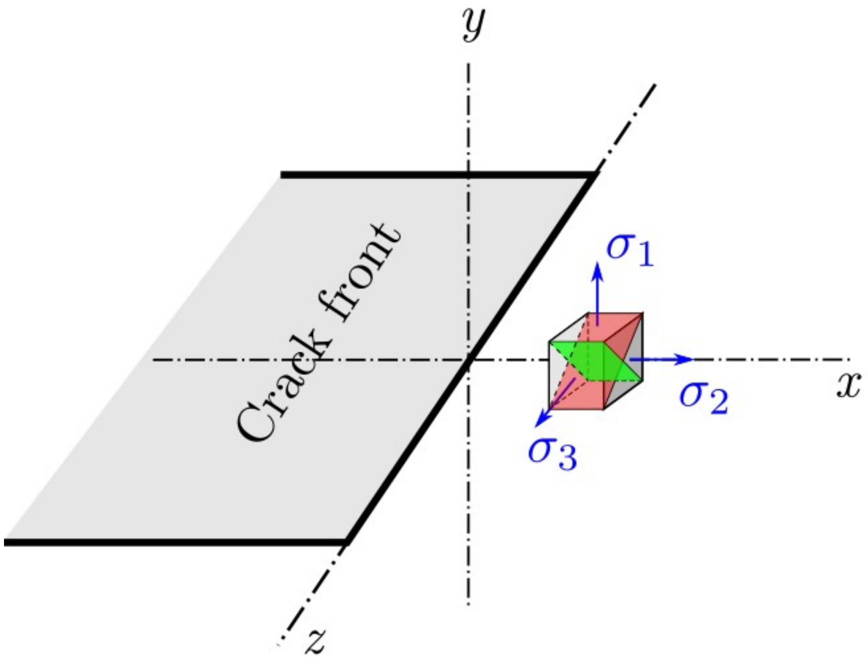
In this case, stress in x- and y- direction are equal. However, if we consider a small deviation from $\theta = 0°$ then σ_x and σ_y will be different and $\sigma_y > \sigma_x$. In this case the maximum principle stress σ_1 is in y- direction and the minimum principal stress is in the z- direction which is equal to zero.

$$\sigma_{\max} = \sigma_1 = \sigma_y, \quad \sigma_{\min} = \sigma_3 = \sigma_z = 0,$$

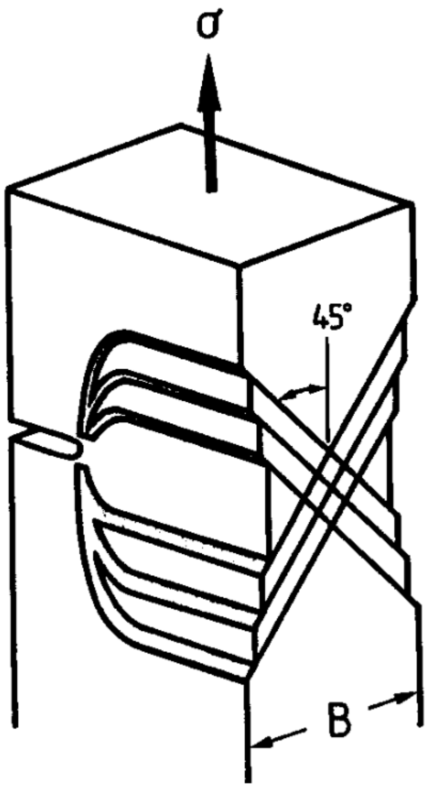


Thus, the plane of maximum shear stress at planes parallel to x-axis and inclined at an angle of $\pm 45°$ from the direction of σ_1 (i.e., y-axis)

Planes having maximum shear stress under mode-I loading for plane stress case

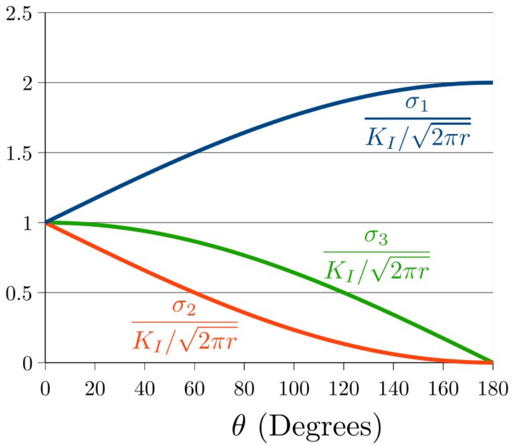


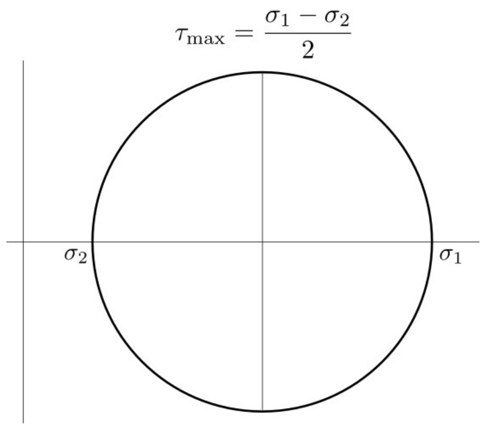
Slip-planes around a mode-I crack for plane stress case



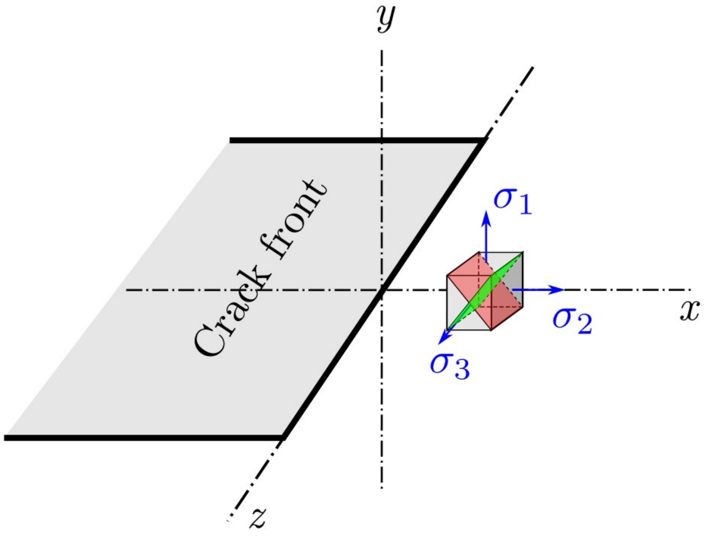
Source: Fracture Mechanics, E. E. Gdoutos

For plane strain case (considering $\nu = 0.5$) the maximum principal stress is σ_1 and minimum principal stress is always σ_2 hence the maximum shear stress will be occurs in planes which are parallel to z-axis and inclined at an angle of $\pm 45°$ from the direction of σ_1 (i.e., y-axis)

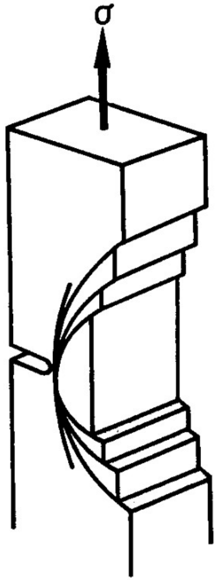




Planes having maximum shear stress under mode-I loading for plane stain case

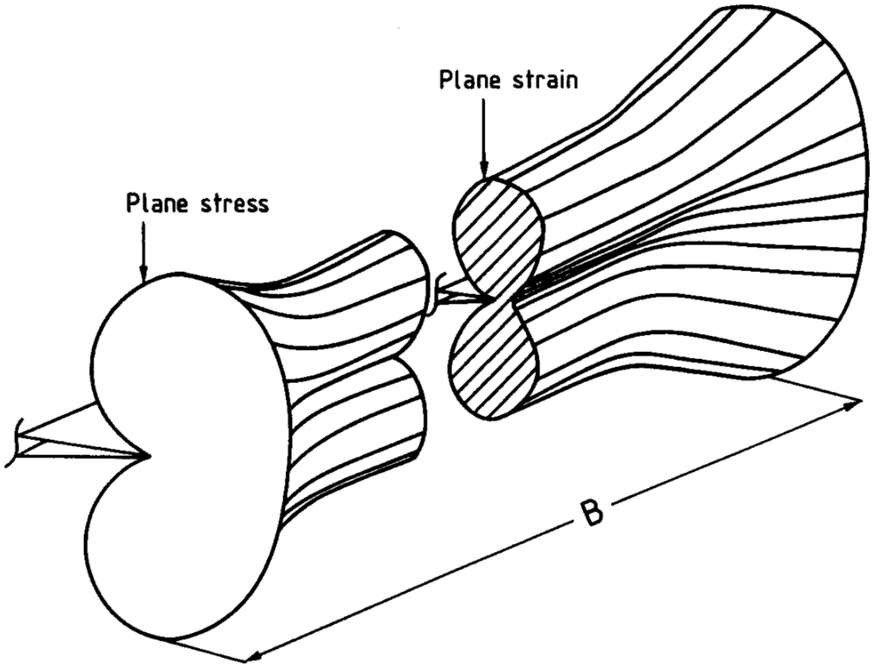


Slip-planes around a mode-I crack for plane stress case

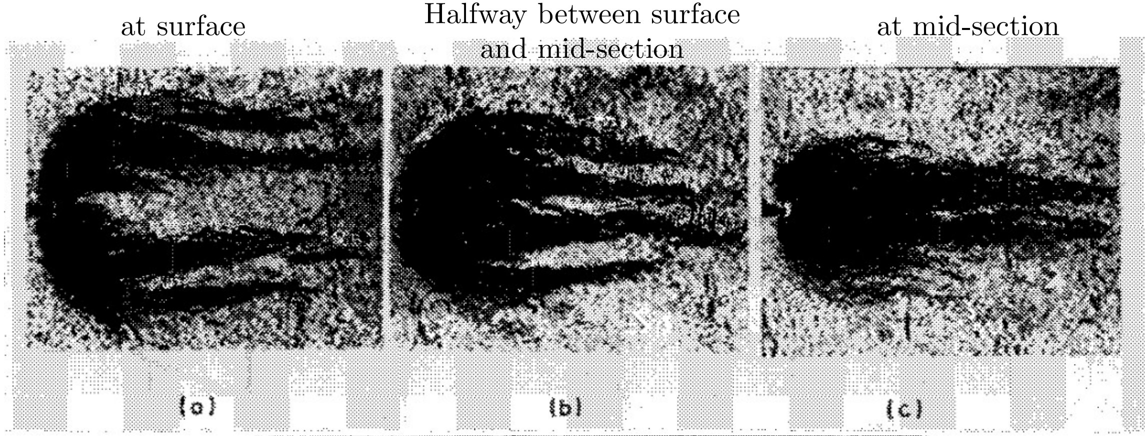


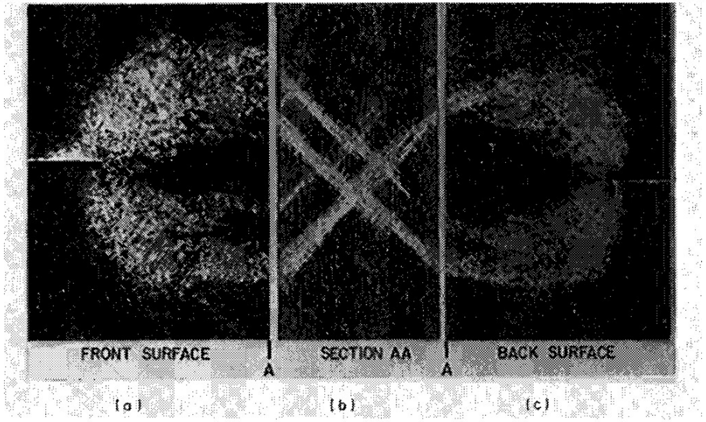
Source: Fracture Mechanics, E. E. Gdoutos

In cracked plates, conditions of plane stress dominate at the traction-free surfaces, while plane strain prevails in the interior. This results in a variation of the plastic zone size through the plate thickness, which decreases from the surface to the interior of the plate.



Source: Fracture Mechanics, E. E. Gdoutos

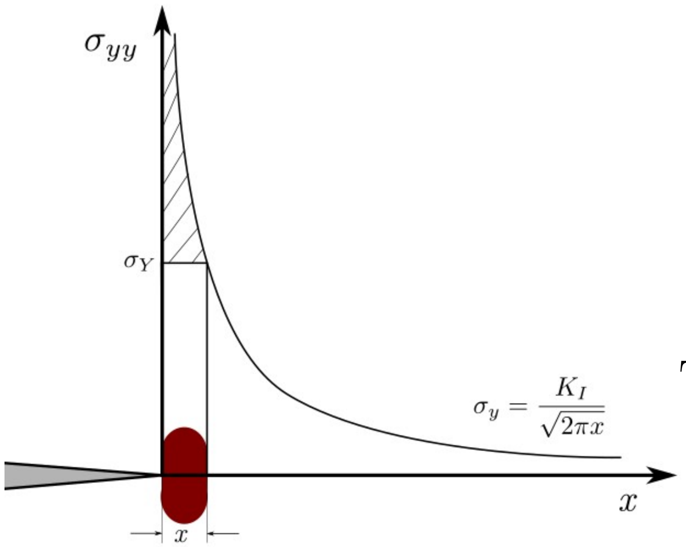




Source: Fracture Mechanics, E. E. Gdoutos

Approximate method for plastic zone calculation

In a very approximate manner the size of plastic zone can be determined by identifying the distance ahead of the crack-tip at which vertical stress reaches yield stress. We first use the Von-Mises yield criteria to determine the the amount of stress in y-direction during yield. Principal stresses ahead of the crack-tip at $\theta = 0°$is



Thus the distance at which yield stress reaches,

For plane stress

$$\sigma_1 = \sigma_{xx} (= \sigma_{yy})$$
$$\sigma_2 = \sigma_{yy}$$
$$\sigma_3 = 0.$$

Thus from Von-mises criteria,

$$2\sigma_{yy}^2 = 2\sigma_Y^2$$
$$\Rightarrow \sigma_{yy} = \sigma_Y.$$
$$x = \frac{K_I^2}{2\pi\sigma_Y^2}$$

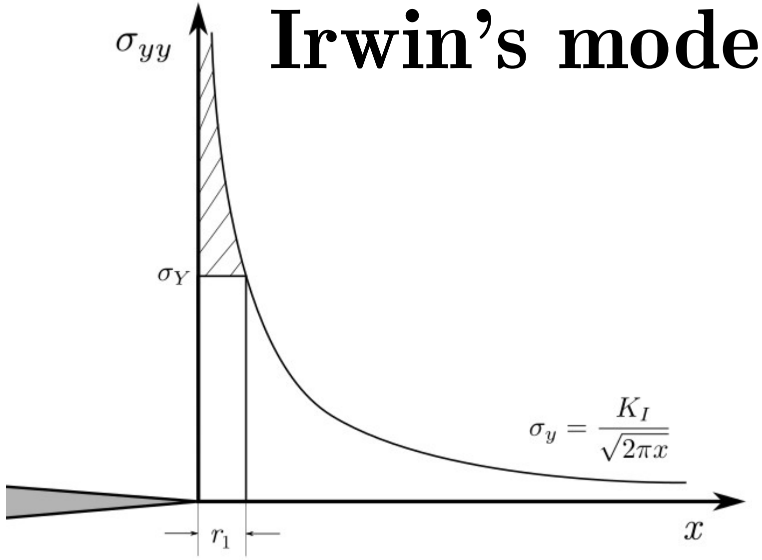
For plane strain (for $\nu = 1/3$)

$$\sigma_1 = \sigma_{xx} (= \sigma_{yy})$$
$$\sigma_2 = \sigma_{yy}$$
$$\sigma_3 = 2\nu(\sigma_1 + \sigma_2) = \frac{4}{3}\sigma_{yy}$$

Thus from Von-mises criteria,

$$2\left(\frac{4}{3}\sigma_{yy} - \sigma_{yy}\right)^2 = 2\sigma_Y^2$$
$$\Rightarrow \sigma_{yy} = 3\sigma_Y.$$
$$x = \frac{K_I^2}{18\pi\sigma_Y^2}$$

Irwin's model for plastic zone calculation



Irwin presented a simplified model for the determination of the plastic zone at the crack tip under small-scale yielding.

His focus was only on the extent along the crack axis and not on the shape of the plastic zone, for an elastic-perfectly plastic material.

Consider the elastic distribution of the σ_y along the crack axis for plane stress case. The extent of the plastic zone in front of the crack can be determined using (4a) as,

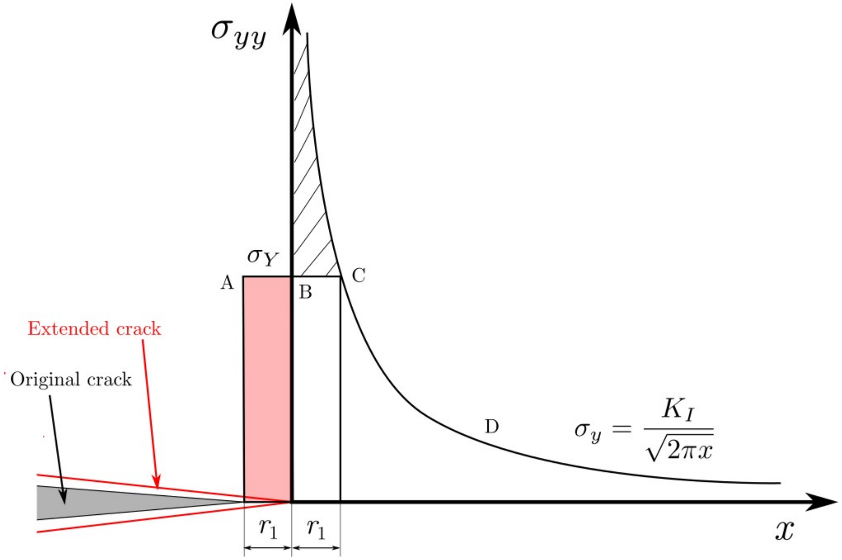
$$r_1 = \frac{K_I^2}{2\pi\sigma_Y^2}. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(7)$$

Let us calculate the load equivalent to of the elastic stress distribution ahead of the crack-tip till distance r_1, which is

$$P = \int_0^{r_1} B\sigma_{yy}dx = \int_0^{r_1} B\frac{K_I}{\sqrt{2\pi x}}dx = B\frac{K_I^2}{\pi\sigma_Y} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(8)$$

Now, If the stress upto a distance of r_1 is limited to constant yield stress of σ_Y then the equilibrium condition along the y-direction is violated and to maintain this equilibrium the load (8) must be maintained. We can rewrite (8) as,

$$P = B\frac{K_I^2}{\pi\sigma_Y} = B\left(\frac{K_I^2}{2\pi\sigma_Y^2}\right)(2\sigma_Y) = B(2\sigma_Y r_1). \qquad \text{(from 7)}$$



The result suggests that to satisfy equilibrium along the y-direction the original crack should be extended by a length r_1, so that, the σ_{yy} distribution is now represented by the curve ABCD, and the area under this curve is equal to the area underneath the σ_{yy} curve for earlier case. Thus, the length of the plastic zone c in front of the crack is

$$c = 2r_1 = \frac{K_I^2}{\pi\sigma_Y^2}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

These observations led Irwin to propose that the effect of plasticity makes the plate behave as if it had a crack longer than the actual crack size and the fictitious crack length is $(a+r_1)$.

Irwin's correction to the plane strain case is useful to determine the plastic zone size. Due to the plastic deformation the crack tip becomes rounded. Since the rounded tip acts as a free surface, σ_{xx} is released to zero, which also affect σ_{xx} stress upto some distance on the x-axis beyond the crack tip. Irwin found that σ_{yy} stress is no longer 3 times the yield stress (σ_Y) it is closer to $2\sqrt{2} \approx \sqrt{3}$ times, thus the plastic zone size

$$c = \frac{K_I^2}{3\pi\sigma_Y^2}. \qquad \text{.................}(10)$$

For an experimental determination of K_{IC} of material, plane strain conditions are assured by taking plate thickness to be much thicker than the plastic zone size.

According to the ASTM Standard E399 the stress condition is characterized as plane stress when $c = B$ and as plane strain when $c < B/25$, where B is the thickness of the plate. Therefore from (10) we obtain for plane strain case,

$$B > \frac{25K_I^2}{3\pi\sigma_Y^2} \approx \frac{2.5K_I^2}{\sigma_Y^2}. \qquad \text{.................}(11)$$