ME632: Fracture Mechanics

Timings	
Monday	10:00 to 11:20
Thursday	08:30 to 09:50

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Antiplane strain

In addition to plane stress and plane strain case there is another class of plane problems which is called Antiplane strain. In this case there are only out-of-plane deformations exists. Hence the assumed displacement fields are,

$$u_1 = u_2 = 0 \text{ and } u_3 = u_3(x_1, x_2). \tag{53}$$

Thus, strain components are $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{12} = \varepsilon_{33} = 0$, and $\varepsilon_{13} = \dfrac{1}{2}\dfrac{\partial u_3}{\partial x_1}, \varepsilon_{23} = \dfrac{1}{2}\dfrac{\partial u_3}{\partial x_2}$. .....(54)

Accordingly stress components are $\sigma_{11} = \sigma_{22} = \sigma_{12} = \sigma_{33} = 0$, and $\sigma_{13} = 2\mu\varepsilon_{13}, \sigma_{23} = 2\mu\varepsilon_{23}$. .....(55)

It must be observed that in Antiplane strain case the only equilibrium equation (in the absence of body force) which need to be satisfied is,

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} = 0. \tag{56}$$

Using (54) and (55), equation (56) can be written in terms of u_3 as,

$$\nabla^2 u_3 = 0, \tag{57}$$

which is nothing but the Navier's equation for Antiplane strain case.

Mode-III crack

Mode-III crack problem is solved as an Antiplane strain problem. We assume the following form of the solution for out-of-plane displacement,

$$u_z = r^\lambda F(\theta). \qquad \cdots\cdots\cdots\cdots\cdots(58)$$

Substituting (58) in (57) (remember to use the polar coordinate form of ∇^2) we get,

$$\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} = 0. \qquad \cdots\cdots\cdots\cdots\cdots(59)$$

Boundary conditions are

$$\text{at } \theta = \pm\pi, \;\; \sigma_{\theta z} = \mu \frac{\partial u_z}{\partial \theta} = 0. \qquad \cdots\cdots\cdots\cdots\cdots(60)$$

Substituting (58) in (60), we obtain the following characteristic equation for nontrivial solution

$$\lambda^2 F(\theta) + F''(\theta) = 0. \qquad \cdots\cdots\cdots\cdots\cdots(61)$$

General solution of (61) is

$$F(\theta) = A\cos(\lambda\theta) + B\sin(\lambda\theta). \qquad \cdots\cdots\cdots\cdots\cdots(62)$$

Hence, the displacement field is

$$u_z = r^\lambda \left[A\cos(\lambda\theta) + B\sin(\lambda\theta)\right]. \qquad \cdots\cdots\cdots\cdots\cdots(63)$$

Now, note that λ=0 gives u_z=A+$B\theta$ (for small θ).

The constant terms correspond to rigid body motion, which we do not consider as it does not lead to any stress/strain. Next term $B\theta$ leads to strain components

$$\varepsilon_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} = \frac{B}{r}.$$

This strain leads to an infinite strain energy as $r\to 0$, which is non-physical, hence λ=0 is not allowed.

Negative values of λ are not allowed, since they lead to non-phyiscal infinite displacement at the crack-tip.

Hence, to determine other values of λ we use boundary conditions (60) to obtain the following equations,

$$\begin{aligned} r^{-\lambda}\left[-\lambda A\sin\left(\lambda\pi\right)+\lambda B\cos\left(\lambda\pi\right)\right]&=0,\\ r^{-\lambda}\left[-\lambda A\sin\left(-\lambda\pi\right)+\lambda B\cos\left(-\lambda\pi\right)\right]&=0. \end{aligned} \quad \text{...............}(64)$$

For non-trivial solutions of

$$\begin{vmatrix} -\lambda\sin(\lambda\pi) & \lambda\cos(\lambda\pi) \\ -\lambda\sin(-\lambda\pi) & \lambda\cos(-\lambda\pi) \end{vmatrix} =0$$

$$\Rightarrow \sin\left(2\lambda\pi\right)=0 \Rightarrow \lambda = n/2, \quad \text{where } n=1,2,3,\ldots$$

From (64),

$$-A\sin\frac{n\pi}{2}+B\cos\frac{n\pi}{2}=0, \qquad \cdots\cdots\cdots\cdots\cdots(65)$$
$$-A\sin\frac{-n\pi}{2}+B\cos\frac{-n\pi}{2}=0.$$

which leads to,

$$A=0 \text{ for } n=1,3,5,... \qquad \text{and} \qquad B=0 \text{ for } n=2,4,6,.... \qquad \cdots\cdots\cdots\cdots\cdots(66)$$

Thus, the displacement field is

$$u_z(r,\theta)=\sum_{n=1,3,5,\ldots} r^{n/2}B_n\sin\frac{n\theta}{2}+\sum_{n=2,4,6,\ldots} r^{n/2}A_n\cos\frac{n\theta}{2}. \qquad \cdots\cdots\cdots\cdots\cdots(67)$$

Non-zero stress components are

$$\sigma_{rz}=\sum_{n=1,3,5,\ldots}\mu\frac{n}{2}r^{n/2-1}B_n\sin\frac{n\theta}{2}+\sum_{n=2,4,6,\ldots}\mu\frac{n}{2}r^{n/2-1}A_n\cos\frac{n\theta}{2}, \qquad \cdots\cdots\cdots\cdots\cdots(68)$$

$$\sigma_{\theta z}=\sum_{n=1,3,5,\ldots}\mu\frac{n}{2}r^{n/2-1}B_n\cos\frac{n\theta}{2}+\sum_{n=2,4,6,\ldots}-\mu\frac{n}{2}r^{n/2-1}A_n\sin\frac{n\theta}{2}. \qquad \cdots\cdots\cdots\cdots\cdots(69)$$

Similar to mode-I and mode-II cases the dominant term correspond to n=1, which are

$$\sigma_{rz} = \frac{\mu}{2\sqrt{r}} B_1 \sin\frac{\theta}{2} = \frac{K_{III}}{\sqrt{2\pi r}} \sin\frac{\theta}{2},$$

$$\sigma_{\theta z} = \frac{\mu}{2\sqrt{r}} B_1 \cos\frac{\theta}{2} = \frac{K_{III}}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \qquad \text{................(70)}$$

where $K_{III} = B_1 \mu \sqrt{\frac{\pi}{2}}$, is mode-III stress intensity factor,

and,

$$u_z = \sqrt{\frac{2}{\pi}} \frac{K_{III}}{\mu} \sqrt{r} \sin\frac{\theta}{2}. \qquad \text{................(71)}$$

Westergaard's approach

There is another method to solve the stress and displacement field at the crack-tip. Westergaard (1939) gave a solution using complex variable approach. Let us first look into some of the basics of complex variable theory. An advantage complex variable offer is that it reduces the number of variable from two to one.

A complex variable is given as

$$z = x_1 + i x_2. \qquad \cdots\cdots\cdots(72)$$

In polar coordinate complex variable is $z = re^{i\theta}$.

The complex conjugate $\bar{z}$ of the variable z is $\bar{z} = x_1 - i x_2 = re^{i\theta}$. $\cdots\cdots\cdots(73)$

Using the definition of complex variable following differential operators can be defined.

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial z}\frac{\partial z}{\partial x_1} + \frac{\partial}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x_1} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \qquad \frac{\partial}{\partial z} = \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial z} + \frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right),$$

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial z}\frac{\partial z}{\partial x_2} + \frac{\partial}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x_2} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right). \qquad \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial \bar{z}} + \frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right).$$

$\cdots\cdots\cdots(74)$

A function of complex function variables may be written as

$$f(z) = f(x_1 + ix_2) = u(x_1, x_2) + iv(x_1, x_2), \qquad \cdots\cdots\cdots\cdots\cdots(75)$$

where $u(x_1, x_2) = \mathrm{Re}(f)$ and $v(x_1, x_2) = \mathrm{Im}(f)$.

A complex conjugate function is defined as,

$$\overline{f(z)} = \bar{f}(\bar{z}) = u(x_1, x_2) - iv(x_1, x_2), \qquad \cdots\cdots\cdots\cdots\cdots(76)$$

For e.g.,

$$f(z) = az + bz^2$$
$$\Rightarrow a(x_1 + ix_2) + b(x_1 + ix_2)^2$$
$$\Rightarrow (ax_1 + bx_1^2 - bx_2^2) + i(ax_2 + 2bx_1x_2).$$

Here,

$$u(x_1, x_2) = (ax_1 + bx_1^2 - bx_2^2),$$
$$v(x_1, x_2) = (ax_2 + 2bx_1x_2).$$

Thus, $f(z) = u(x_1, x_2) + iv(x_1, x_2)$.

$$\overline{f(z)} = \bar{f}(\bar{z}) = a\bar{z} + b\bar{z}^2$$
$$\Rightarrow a(x_1 - ix_2) + b(x_1 - ix_2)^2$$
$$\Rightarrow (ax_1 + bx_1^2 - bx_2^2) - i(ax_2 + 2bx_1x_2).$$

Thus,

$$\overline{f(z)} = u(x_1, x_2) - iv(x_1, x_2).$$

Differentiation of complex function are

$$f'(z) = \frac{\partial}{\partial z}(u+iv) = \frac{1}{2}\left(\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial x_2}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x_1} - \frac{\partial u}{\partial x_2}\right). \qquad \cdots\cdots\cdots\cdots\cdots(77)$$

Following the basic definition of differentiation, it can be shown that Cauchy-Riemann equations for analyticity of function *f* is

$$\frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_2}, \quad \frac{\partial u}{\partial x_2} = -\frac{\partial v}{\partial x_1}. \qquad \cdots\cdots\cdots\cdots\cdots(78)$$

By simple differentiation of (78) we can show that,

$$\nabla^2 u = 0, \nabla^2 v = 0, \qquad \cdots\cdots\cdots\cdots\cdots(79)$$

Thus real and imaginary part of an analytic complex function must be a solution of Laplace's equation and hence they are harmonic functions.

We have already seen that solution of boundary value problems in elasticity can be obtained in the form of Airy stress function Φ and it must satisfy the biharmonic equation.

One way to solve the biharmonic equation is to represent Φ in terms of another complex functions. Westergaard suggested function Φ for mode-I and mode-II problems.