ME531: Advanced Mechanics of Solids

Motion, Strain and Stress

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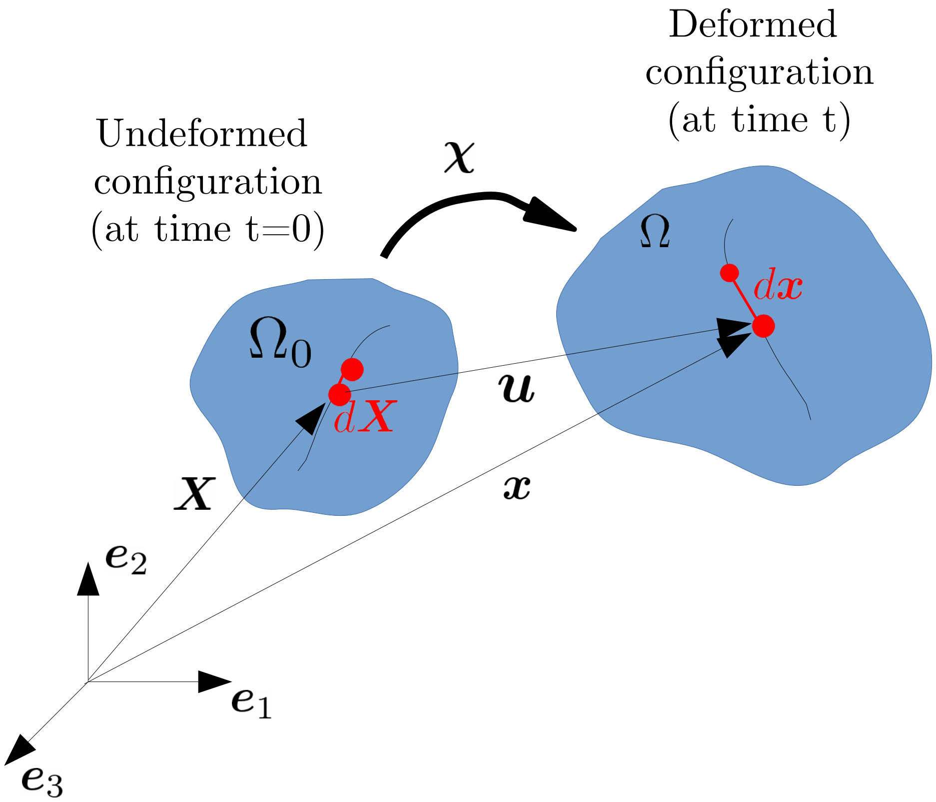
Deformation gradient



Consider a line element $d\boldsymbol{X}$ in Ω_0 which deforms to a line element $d\boldsymbol{x}$. *Deformation gradient* tensor maps the line element $d\boldsymbol{X}$ to $d\boldsymbol{x}$ as,

$$d\boldsymbol{x} = \boldsymbol{F} d\boldsymbol{X} \text{ or } dx_i = F_{iJ} dX_J,$$

$$\text{where, } \boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} \text{ or } F_{iJ} = \frac{\partial x_i}{\partial X_J},$$

$$\text{or } \boldsymbol{F} = \text{Grad } \boldsymbol{x} = \boldsymbol{\nabla}_X \boldsymbol{x}$$

is an invertible tensor. $\boldsymbol{F}^{-1}$ is the *inverse deformation gradient* tensor, defined as

$$\boldsymbol{F}^{-1} = \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}} \text{ or } F^{-1}_{Ij} = \frac{\partial X_I}{\partial x_j}.$$

$$\text{or } \boldsymbol{F}^{-1} = \text{grad } \boldsymbol{X} = \boldsymbol{\nabla} \boldsymbol{X}$$

It relates line elements as,

$$d\boldsymbol{X} = \boldsymbol{F}^{-1} d\boldsymbol{x} \text{ or } dX_I = F^{-1}_{Ij} dx_j.$$

Example

Consider a two dimensional motion given by two equations as

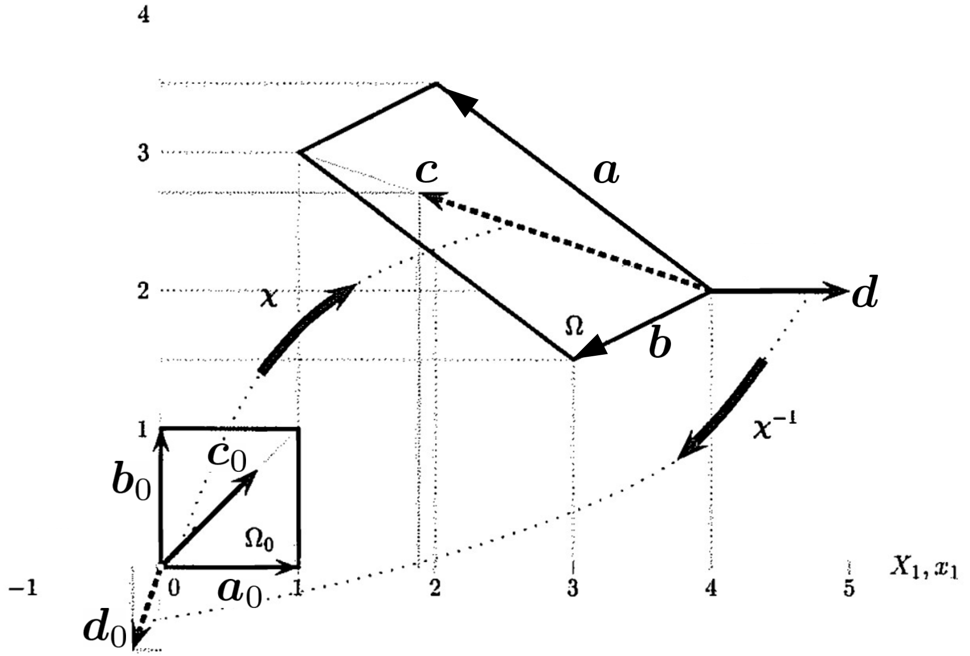
$$x_1 = 4 - 2X_1 - X_2$$
$$x_2 = 2 + 1.5X_1 - 0.5X_2$$

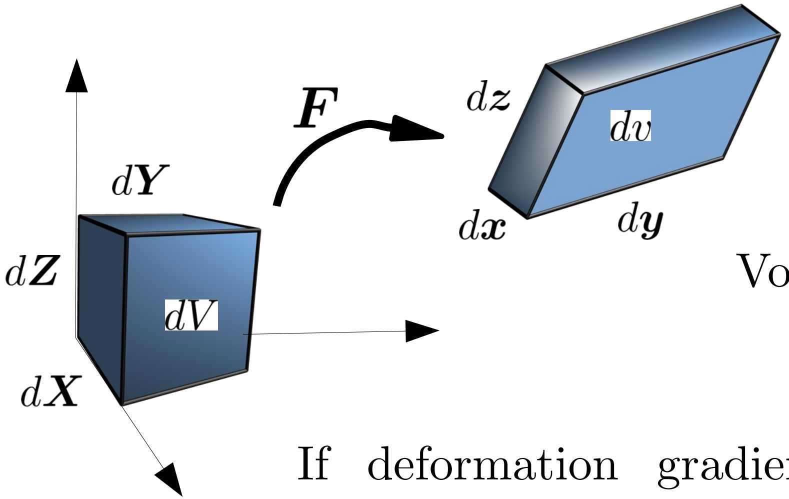
Deformation gradient for the given motion is calculated as,

$$[\boldsymbol{F}] = \begin{bmatrix} -2 & -1 \\ 1.5 & -0.5 \end{bmatrix}, \text{ and } [\boldsymbol{F}]^{-1} = \frac{1}{5}\begin{bmatrix} -1 & 2 \\ -3 & -4 \end{bmatrix}$$

Now consider following vectors $\boldsymbol{a}_0 = [1,0]$, $\boldsymbol{b}_0 = [0,1]$ and $\boldsymbol{c}_0 = [0.707, 0.707]$ in the reference configuration. Current or deformed vectors can be obtained as $\boldsymbol{a} = \boldsymbol{F}\boldsymbol{a}_0$, $\boldsymbol{b} = \boldsymbol{F}\boldsymbol{b}_0$ and $\boldsymbol{c} = \boldsymbol{F}\boldsymbol{c}_0$.

Consider another vector $\boldsymbol{d} = [1,0]$ in the current configuration. Reference configuration $\boldsymbol{d}_0$ for vector d can be obtained as $\boldsymbol{d}_0 = \boldsymbol{F}^{1}\boldsymbol{d}$.





Consider an infinitesimally small volume element with reference volume dV going under deformation and the current volume is dv.

Volume of the element in current configuration can be given as,

$$dv = d\boldsymbol{z} \cdot (d\boldsymbol{x} \times d\boldsymbol{y}) \text{ or } dv = e_{ijk} dz_i dx_j dy_k$$

If deformation gradient for the motion is $\boldsymbol{F}$, then edges in the reference configuration can be mapped to the edges in undeformed configuration as,

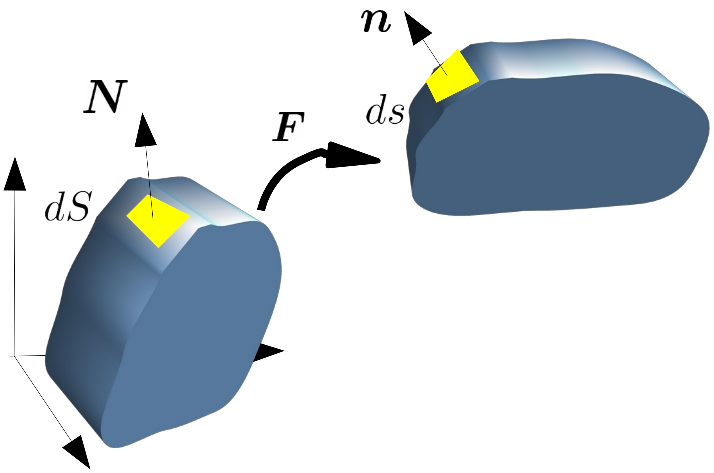
$$dx_i = F_{ij} dX_J, dy_i = F_{ij} dY_J, \text{ and } dz_i = F_{ij} dZ_J$$

Substituting above in the expression for volume, we get

$$dv = e_{ijk}(F_{ip} dZ_p)(F_{jq} dX_q)(F_{kr} dY_r)$$

It can be shown that, $e_{ijk} A_{ip} A_{jq} A_{kr} = e_{pqr} \det \boldsymbol{A}$.

Thus, $dv = e_{pqr} dZ_p dX_q dY_r \det \boldsymbol{F} \Rightarrow \boxed{dv = J dV,}$ where $J = \det \boldsymbol{F}$ is called the *Jacobian* of deformation gradient tensor. The Jacobian is the ratio of elemental volume deformed configuration to its volume in undeformed configuration.



Deformation gradient $\boldsymbol{F}$ is used to map a vector in undeformed configuration to the deformed configuration. However, the same does not apply to the unit normal vector $\boldsymbol{N}$ to an infinitesimal surface element dS. To find the relation between the normal vector $\boldsymbol{N}$ and $\boldsymbol{n}$ (deformed configuration of $\boldsymbol{N}$) we use the following relation.

$$dv = JdV$$

We now present the volume dV as $d\boldsymbol{S}\cdot d\boldsymbol{X}$, where $d\boldsymbol{X}$ is a line element in undeformed configuration, which deform to the line element $d\boldsymbol{x}$, and hence $dv{=}d\boldsymbol{s}\cdot d\boldsymbol{x}$.

Now, we can write,

$$d\boldsymbol{s}\cdot d\boldsymbol{x} = Jd\boldsymbol{S}\cdot d\boldsymbol{X}$$

Line element $d\boldsymbol{x}$ and $d\boldsymbol{X}$ are related as, $d\boldsymbol{x} = \boldsymbol{F}d\boldsymbol{X}$. Using this relation we can write,

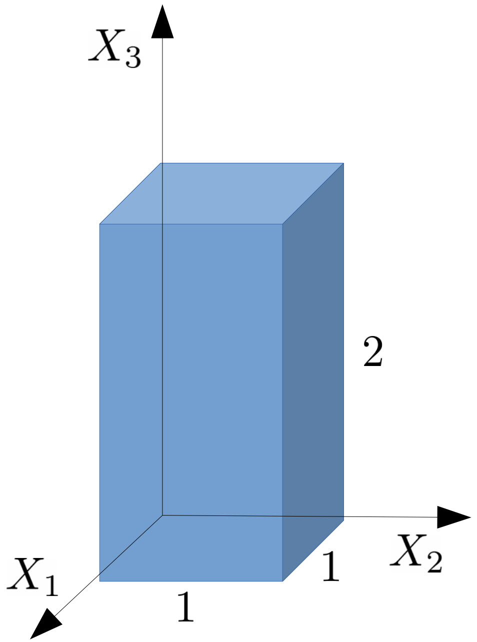
$$d\boldsymbol{s}\cdot \boldsymbol{F}d\boldsymbol{X} = Jd\boldsymbol{S}\cdot d\boldsymbol{X} \Rightarrow (\boldsymbol{F}^T d\boldsymbol{s} - Jd\boldsymbol{S})\cdot d\boldsymbol{X} = 0$$

This relation holds for any arbitrary $d\boldsymbol{X}$, which implies $\boldsymbol{F}^T d\boldsymbol{s} - Jd\boldsymbol{S} = 0$.

Above gives us the ***Nanson's formula*** relating an area element in undeformed configuration to the area element in deformed configuration as $\boxed{d\boldsymbol{s} = J\boldsymbol{F}^{-T}d\boldsymbol{S}.}$

Exercise

Plot the deformed shape of the cube which goes under the following motion.



$$x_1 = X_1 \cos(\pi X_3/2) - X_2 \sin(\pi X_3/2)$$
$$x_2 = X_1 \sin(\pi X_3/2) + X_2 \cos(\pi X_3/2)$$
$$x_3 = X_3$$

Displacement gradient

Using displacements we can write, $\boldsymbol{x}(\boldsymbol{X},t) = \boldsymbol{X} + \boldsymbol{U}(\boldsymbol{X},t)$ or $x_i = X_i + U_i$.

From the definition of deformation gradient,

$$F_{iJ} = \frac{\partial x_i}{\partial X_J} = \frac{\partial}{\partial X_J}(X_i + U_i) = \frac{\partial X_i}{\partial X_J} + \frac{\partial U_i}{\partial X_J}$$

$$F_{iJ} = \delta_{ij} + \frac{\partial U_i}{\partial X_J}, \text{ or } \quad \boldsymbol{F} = \boldsymbol{I} + \boldsymbol{\nabla}_X \boldsymbol{U}$$

where, $\boldsymbol{\nabla}_X \boldsymbol{U} = \frac{\partial U_i}{\partial X_J}$ is called *displacement gradient tensor.*

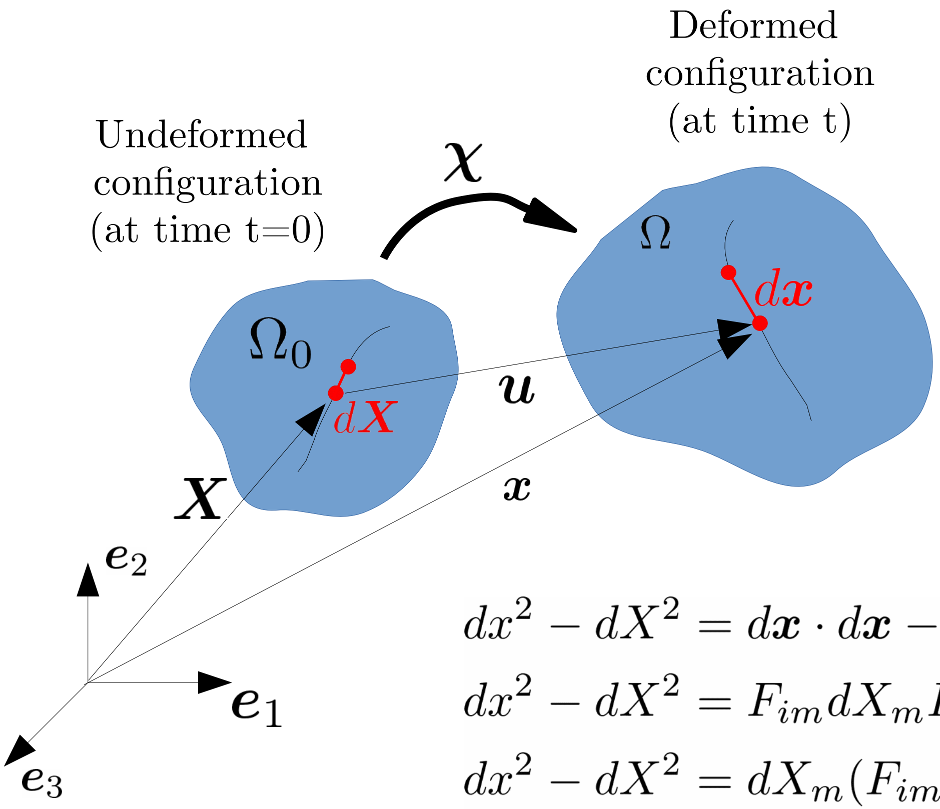
In material description, displacement gradient tensor is

$$\boldsymbol{\nabla}_X \boldsymbol{U}(\boldsymbol{X},t) = \boldsymbol{F}(\boldsymbol{X},t) - \boldsymbol{I} \text{ or } \quad \frac{\partial U_i}{\partial X_J} = F_{iJ} - \delta_{ij}.$$

In spatial description, displacement gradient tensor is

$$\boldsymbol{\nabla} \boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{I} - \boldsymbol{F}^{-1}(\boldsymbol{x},t) \text{ or } \quad \frac{\partial u_i}{\partial x_J} = \delta_{ij} - F^{-1}_{iJ}.$$

Strain



Squared length of the line element $d\boldsymbol{X}$ is $dX^2 = d\boldsymbol{X} \cdot d\boldsymbol{X}$.

Similarly, squared length of the line element $d\boldsymbol{x}$ is $dx^2 = d\boldsymbol{x} \cdot d\boldsymbol{x}$.

Let's examine the difference between the squared length of $d\boldsymbol{x}$ and $d\boldsymbol{X}$ as,

$$dx^2 - dX^2 = d\boldsymbol{x} \cdot d\boldsymbol{x} - d\boldsymbol{X} \cdot d\boldsymbol{X} = dx_i dx_i - dX_j dX_j,$$

$$dx^2 - dX^2 = F_{im} dX_m F_{in} dX_n - dX_m \delta_{mn} dX_n,$$

$$dx^2 - dX^2 = dX_m (F_{im} F_{in} - \delta_{mn}) dX_n = d\boldsymbol{X} \cdot (\boldsymbol{F}^T \boldsymbol{F} - \boldsymbol{I}) d\boldsymbol{X},$$

$$dx^2 - dX^2 = dX_m (F_{im} F_{in} - \delta_{mn}) dX_n = d\boldsymbol{X} \cdot 2\boldsymbol{E} d\boldsymbol{X}.$$

where, $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T \boldsymbol{F} - \boldsymbol{I})$ or $E_{ij} = \frac{1}{2}(F_{ki} F_{kj} - \delta_{ij})$ is called *Green-Lagrange* strain tensor.

Using the relaton, $\boldsymbol{F} = \boldsymbol{I} + \boldsymbol{\nabla}_X \boldsymbol{U}$ or $F_{ij} = \delta_{ij} + \dfrac{\partial U_i}{\partial X_j}$, we write,

$$E_{ij} = \frac{1}{2}\left[\left(\delta_{ki} + \frac{\partial U_k}{\partial X_i}\right)\left(\delta_{kj} + \frac{\partial U_k}{\partial X_j}\right) - \delta_{ij}\right]$$

$$E_{ij} = \frac{1}{2}\left[\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} + \frac{\partial U_k}{\partial X_i}\frac{\partial U_k}{\partial X_j}\right] \quad \text{or} \quad \boldsymbol{E} = \frac{1}{2}\left[\boldsymbol{\nabla}_X \boldsymbol{U} + (\boldsymbol{\nabla}_X \boldsymbol{U})^T + (\boldsymbol{\nabla}_X \boldsymbol{U})^T \boldsymbol{\nabla}_X \boldsymbol{U}\right]$$

Eulerian description of the Green-Lagrange strain tensor is called *Euler-Almansi strain tensor* and defined as,

$$\boldsymbol{e} = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{F}^{-T}\boldsymbol{F}^{-1}\right) \quad \text{or} \quad e_{ij} = \frac{1}{2}\left(\delta_{ij} - F^{-1}_{ki} F_{kj}\right)$$

$$\boldsymbol{e} = \frac{1}{2}\left[\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T - (\boldsymbol{\nabla}\boldsymbol{u})^T \boldsymbol{\nabla}\boldsymbol{u}\right] \quad \text{or} \quad e_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right]$$