ME232: Dynamics

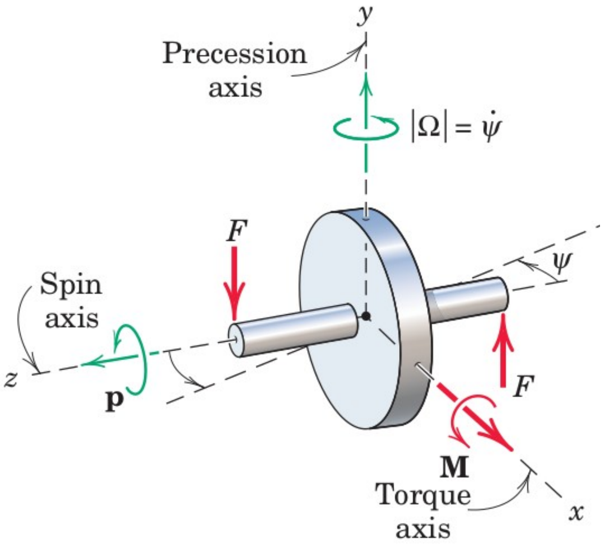
3D dynamics of rigid bodies

Anshul Faye
afaye@iitbhilai.ac.in
Room # 106

Gyroscopic motion: Steady precession

- Gyroscopic motion is one of the most interesting of all problems in dynamics.
- This motion occurs whenever the axis about which a body is spinning is itself rotating about another axis.
- Although the complete description of this motion involves considerable complexity, we will focus on a most common and useful examples of gyroscopic motion occur when the axis of a rotor spinning at constant speed turns (precesses) about another axis at a steady rate.
- Let us first understand the dynamics of this motion. Then we will look at the engineering applications of gyroscopic motion (or gyroscope)

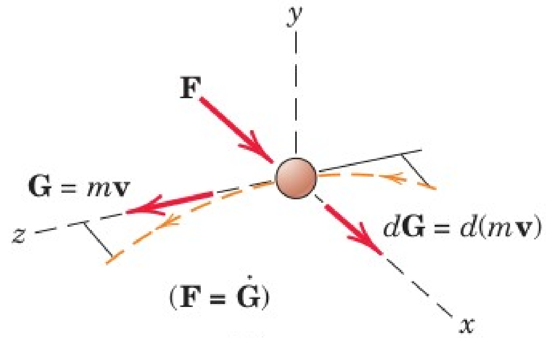
Figure shows a symmetrical rotor spinning about the z-axis with a large angular velocity $\boldsymbol{p}$, known as the **spin velocity**. If we apply two forces $\boldsymbol{F}$ to the rotor axle to form a couple $\boldsymbol{M}$ whose vector is directed along the x-axis, we will find that the rotor shaft rotates in the x-z plane about the y-axis in the sense indicated, with a relatively slow angular velocity Ω known as the **precession velocity**. Thus, we identify the spin axis ($\boldsymbol{p}$), the torque axis ($\boldsymbol{M}$), and the precession axis ($\boldsymbol{\Omega}$), where the usual right-hand rule identifies the sense of the rotation vectors.



Note that, the rotor shaft does not turn about the x-axis in the sense of $\boldsymbol{M}$, as it would if the rotor were not spinning.

For understanding this phenomenon, a direct analogy may be made between the rotation vectors and the familiar vectors describing the curvilinear motion of a particle

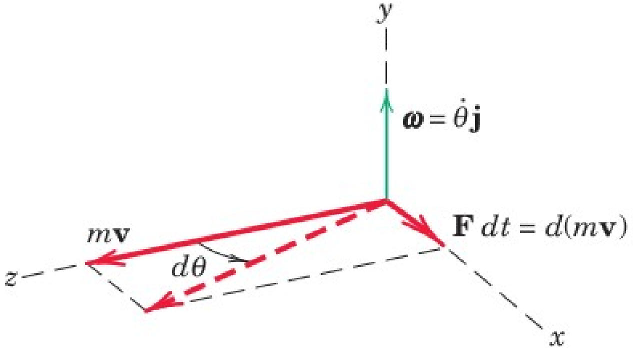
Figure shows a particle of mass m moving in the x-z plane with constant speed v. The application of a force $\boldsymbol{F}$ normal to its linear momentum $\boldsymbol{G} = m\boldsymbol{v}$ causes a change $d\boldsymbol{G} = d(m\boldsymbol{v})$ in its momentum. We see that $d\boldsymbol{G}$, and thus d$\boldsymbol{v}$, is a vector in the direction of the normal force $\boldsymbol{F}$ according to Newton's second law $\boldsymbol{F} = \dot{\boldsymbol{G}}$, which may be written as $\boldsymbol{F}dt = d\boldsymbol{G}$.



We see that, in the limit, $\tan d\theta = d\theta = Fdt/mv$ or $F = mv\dot{\theta}$.

In vector notation with $\boldsymbol{\omega} = \dot{\theta}\boldsymbol{j}$, the force becomes

$$\boldsymbol{F} = m\boldsymbol{\omega} \times \boldsymbol{v} \qquad \cdots\cdots\cdots\cdots\cdots(15)$$

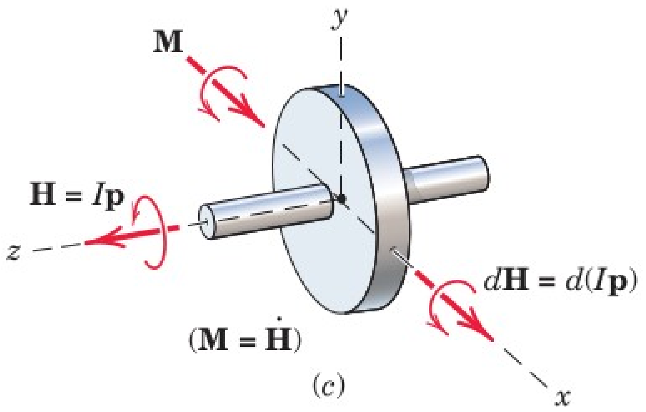


which is the vector equivalent of scalar relation $F_n = ma_n$ for the normal force on the particle.

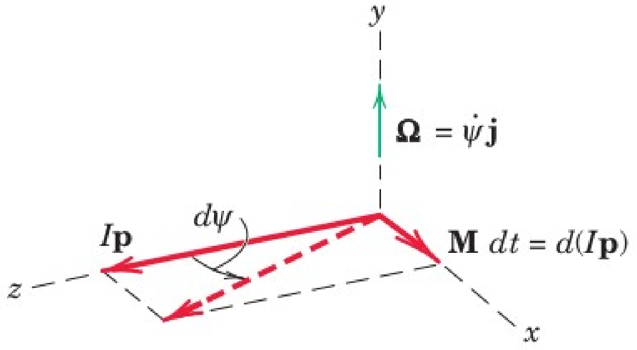
With these relations in mind, now we discuss the problem of rotation. Recall now the analogous equation $\boldsymbol{M} = \dot{\boldsymbol{H}}$ which we developed for any prescribed mass system, rigid or nonrigid, referred to its mass center or to a fixed point O.

Now apply this relation to symmetrical rotor. For a high rate of spin $\boldsymbol{p}$ and a low precession rate $\boldsymbol{\Omega}$ about the y-axis, the angular momentum is represented by the vector $\boldsymbol{H} = I\boldsymbol{p}$, where $I = I_{zz}$ is the moment of inertia of the rotor about the spin axis.

Initially, we neglect the small component of angular momentum about the y-axis which accompanies the slow precession. The application of the couple $\boldsymbol{M}$ normal to $\boldsymbol{H}$ causes a change $d\boldsymbol{H} = d(I\boldsymbol{p})$ in the angular momentum. We see that $d\boldsymbol{H}$, and thus $d\boldsymbol{p}$, is a vector in the direction of the couple $\boldsymbol{M}$ since $\boldsymbol{M} = \dot{\boldsymbol{H}}$, which may also be written $\boldsymbol{M}dt = d\boldsymbol{H}$. Just as the change in the linear-momentum vector of the particle is in the direction of the applied force, so is the change in the angular-momentum vector of the gyro in the direction of the couple.

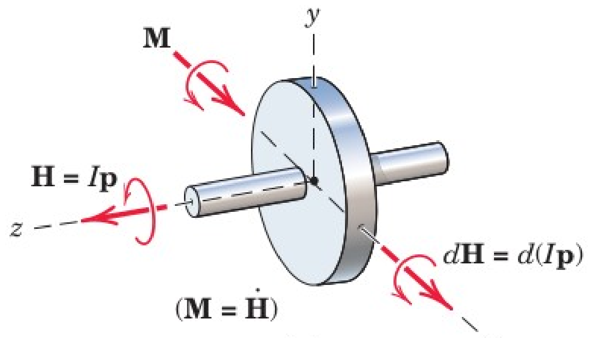


(c)



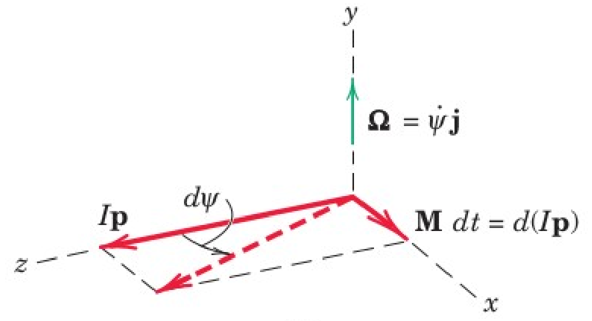
(d)

Thus, we see that the vectors $\boldsymbol{M}$, $\boldsymbol{H}$, and $d\boldsymbol{H}$ are analogous to the vectors $\boldsymbol{F}$, $\boldsymbol{G}$, and $d\boldsymbol{G}$. With this insight, it is no longer strange to see the rotation vector undergo a change which is in the direction of $\boldsymbol{M}$, thereby causing the axis of the rotor to precess about the y-axis.



(c)

It can be seen that during time dt the angular-momentum vector $\boldsymbol{Ip}$ has swung through the angle $d\psi$, so that in the limit with $\tan d\psi = d\psi$, we have



(d)

$$d\psi = \frac{M dt}{IP} \quad \text{or} \quad M = I\frac{d\psi}{dt} = I\Omega p. \qquad \text{……………}(16)$$

Note that $\boldsymbol{M}$, $\boldsymbol{\Omega}$, and $\boldsymbol{p}$ are mutually perpendicular vectors, and it can be written as

$$\boldsymbol{M} = \boldsymbol{I\Omega} \times \boldsymbol{p}, \qquad \text{……………}(16a)$$

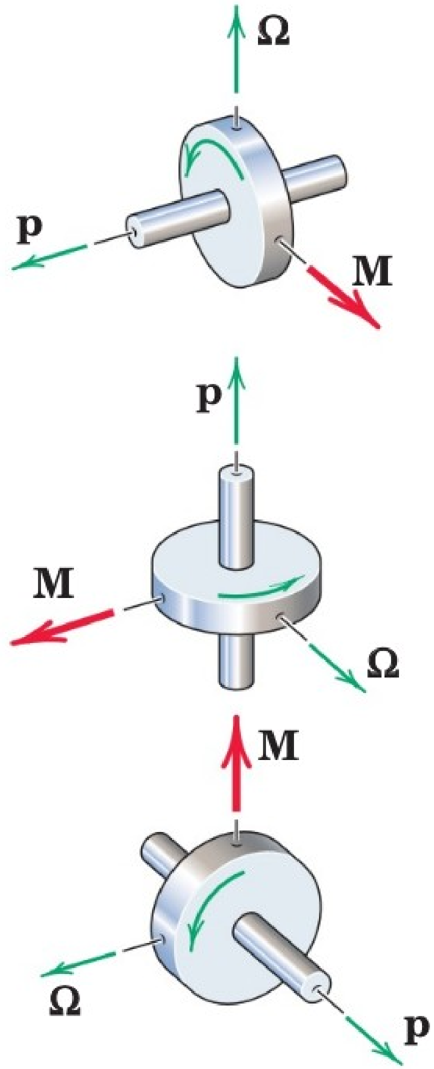
which is analogous to the relation (15) applicable for curvilinear motion of particle.

The correct spatial relationship among the three vectors may be remembered from the fact that $d\boldsymbol{H}$, and thus $\mathrm{d}\boldsymbol{p}$, is in the direction of $\boldsymbol{M}$, which establishes the correct sense for the precession $\boldsymbol{\Omega}$. Therefore, the spin vector $\boldsymbol{p}$ always tends to rotate toward the torque vector $\boldsymbol{M}$.

Figure shows three orientations of the three vectors which are consistent with their correct order.

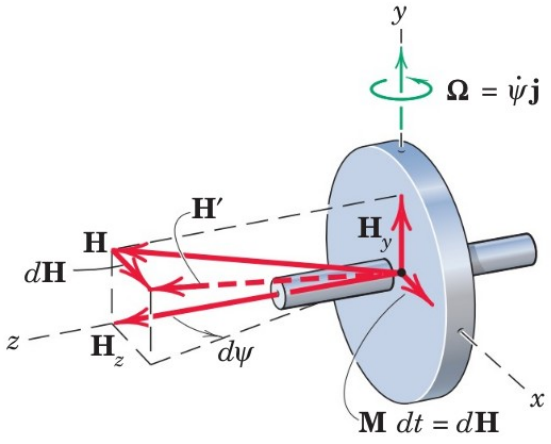
Remember that, $\boldsymbol{M} = I\boldsymbol{\alpha}$, is an equation of motion, where $\boldsymbol{M}$ represents the couple due to all forces acting on the rotor, as disclosed by a correct free-body diagram of the rotor.

When a rotor is forced to precess, as occurs with the turbine in a ship which is executing a turn, the **motion will generate a gyroscopic couple $\boldsymbol{M}$** which obeys (16a) in both magnitude and sense.



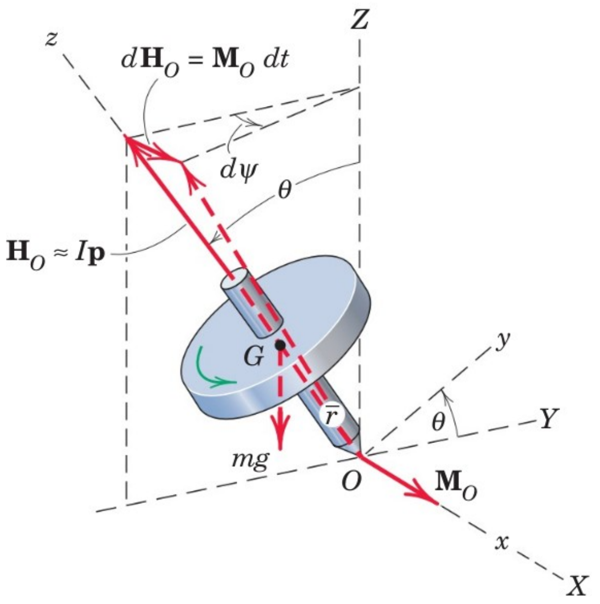
In the foregoing discussion of gyroscopic motion, it was assumed that **the spin was large and the precession was small.** Although we can see from (16) that for given values of I and $\boldsymbol{M}$, the precession $\boldsymbol{\Omega}$ must be small if $\boldsymbol{p}$ is large, let us now examine the influence of $\boldsymbol{\Omega}$ on the momentum relations.

We restrict our attention to steady precession, where $\boldsymbol{\Omega}$ has a constant magnitude. Figure shows the same rotor again. Because it has a moment of inertia about the y-axis and an angular velocity of precession about this axis, there will be an additional component of angular momentum about the y-axis. Thus, we have the two components $H_z = Ip$ and $H_y = I_O\Omega$, where $I_O = I_{yy}$ and $I = I_{zz}$. The total angular momentum is $\boldsymbol{H}$ as shown.



The change in $\boldsymbol{H}$ remains $d\boldsymbol{H} = \boldsymbol{M}\ dt$ as previously, and the precession during time dt is the angle $d\psi = Mdt/H_z = M\ dt/(Ip)$ as before. Thus, (16) is still valid and for steady precession is an exact description of the motion as long as the spin axis is perpendicular to the axis around which precession occurs.

Consider now the steady precession of a symmetrical top spinning about its axis with a high angular velocity $\boldsymbol{p}$ and supported at its point O. Here the spin axis makes an angle θ with the vertical Z-axis around which precession occurs. Again, we will neglect the small angular-momentum component due to the precession and consider $\boldsymbol{H} = I\boldsymbol{p}$, the angular momentum about the axis of the top associated with the spin only. The moment about O is due to the weight and is $mgr \sin\theta$, where r is the distance from O to the mass center G. From the diagram, we see that the angular-momentum vector $\boldsymbol{H}_O$ has a change $d\boldsymbol{H}_O = \boldsymbol{M}_O\, dt$ in the direction of M_O during time dt and that θ is unchanged. The increment in precessional angle around the Z-axis is



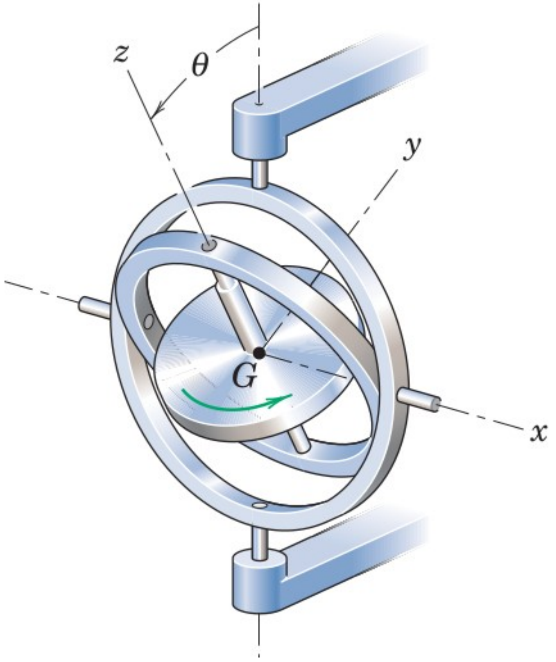
$$d\psi = \frac{M_O dt}{I p \sin\theta}.$$

Substituting, $M_O = mg\bar{r}\sin\theta$ and $\Omega = d\psi/dt$, we get, $mg\bar{r}\sin\theta = I\Omega p\sin\theta$ or $mg\bar{r} = I\Omega p$, which is independent of θ. Introducing the radius of gyration so that $I = mk^2$ and solving for the precessional velocity give

$$\Omega = \frac{g\bar{r}}{k^2 p}. \qquad \text{...............(17)}$$

(17) is an approximation based on the assumption that the angular momentum associated with $\mathbf{\Omega}$ is negligible compared with that associated with $\boldsymbol{p}$. On the basis of our analysis, the top will have a steady precession at the constant angle only if it is set in motion with a value of $\mathbf{\Omega}$ which satisfies (17). When these conditions are not met, the precession becomes unsteady, and may oscillate with an amplitude which increases as the spin velocity decreases. The corresponding rise and fall of the rotation axis is called **nutation**.

Applications of Gyroscope



The gyroscope has important engineering applications.

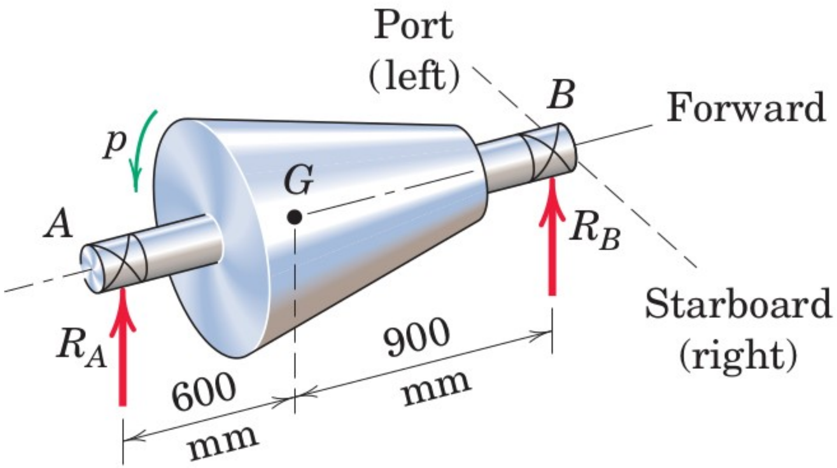
With a mounting in gimbal rings, the gyro is free from external moments, and its axis will retain a fixed direction in space regardless of the rotation of the structure to which it is attached. In this way, the gyro is used for inertial guidance systems and other directional control devices.

The gyroscope has also found important use as a stabilizing device. The controlled precession of a large gyro mounted in a ship is used to produce a gyroscopic moment to counteract the rolling of a ship at sea.

The gyroscopic effect is also an extremely important consideration in the design of bearings for the shafts of rotors which are subjected to forced precessions.

Example 5

The turbine rotor in a ship's power plant has a mass of 1000 kg, with center of mass at G and a radius of gyration of 200 mm. The rotor shaft is mounted in bearings A and B with its axis in the horizontal fore-and-aft direction and turns counterclockwise at a speed of 5000 rev/min when viewed from the stern. Determine the vertical components of the bearing reactions at A and B if the ship is making a turn to port (left) of 400-m radius at a speed of 25 knots (1 knot $=$ 0.514 m/s). Does the bow of the ship tend to rise or fall because of the gyroscopic action?



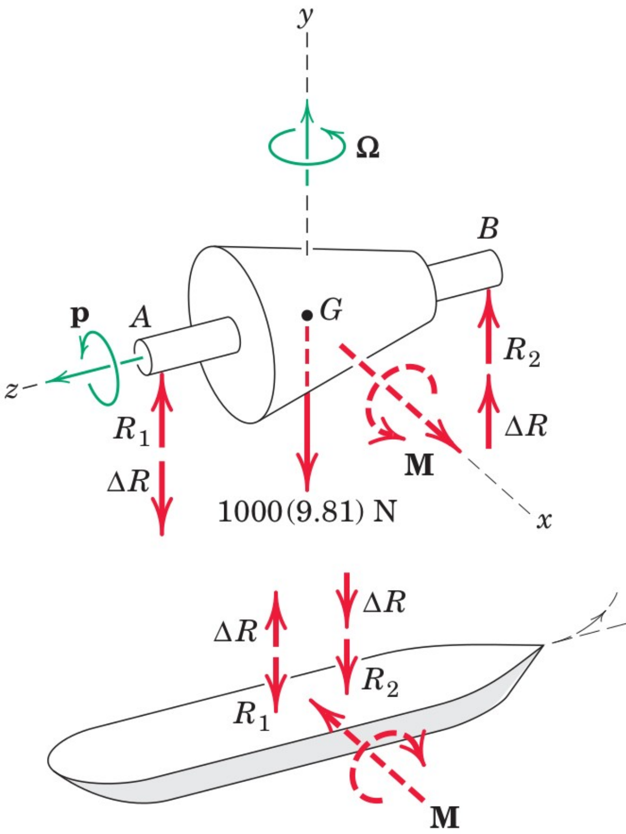
The vertical component of the bearing reactions will equal the static reactions R_1 and R_2 due to the weight of the rotor, plus or minus the increment ΔR due to the gyroscopic effect.

R_1 and R_2 can be easily calculated from the principles of statics (moment principle) as $R_1 = 5890$ N and $R_2 = 3920$ N.

The given directions of the spin velocity $\boldsymbol{p}$ and the precession velocity $\boldsymbol{\Omega}$ are shown with the free-body diagram of the rotor. Because the spin axis always tends to rotate toward the torque axis, the torque axis $\boldsymbol{M}$ points in the starboard direction.

The sense of the ΔR's is, therefore, up at B and down at A to produce the couple M. Thus, the bearing reactions at A and B are

$$R_A = R_1 - \Delta R \qquad \text{and} \qquad R_B = R_2 + \Delta R.$$



The precession velocity Ω is the speed of the ship divided by the radius of its turn, i.e.,

$$\Omega = v/\rho = 25(0.514)/400 = 0.321 \text{ rad/s}$$

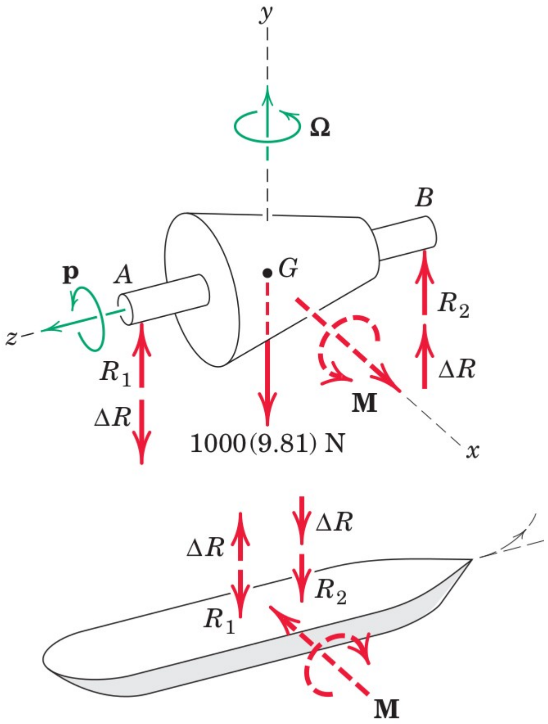
From the relation between M, Ω, and p, which is (about G)

$$M = I\,\Omega\,p$$

$1.5(\Delta R) = 1000(0.2)^2(0.0321)(2\pi\cdot 5000/60)$,

which gives

$\Delta R = 449$ N.



Now observe that the forces we just computed are exerted on the rotor shaft by the structure of the ship.

Consequently, from the principle of action and reaction, the equal and opposite forces are applied to the ship by the rotor shaft. Therefore, the effect of the gyroscopic couple is to generate the increments ΔR shown, and the bow will tend to fall and the stern to rise (but only slightly).