ME232: Dynamics

Plane Kinetics of Rigid Bodies

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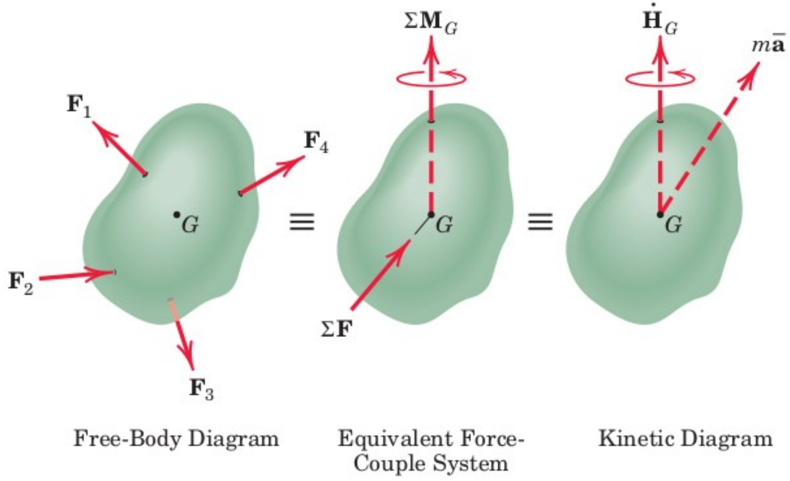
General equations of motion

- For a general rigid body in three dimensions, the force equation following equations can be written,

$$\sum \boldsymbol{F} = m\bar{\boldsymbol{a}} \qquad \text{...........(1)}$$
$$\sum \boldsymbol{M}_G = \dot{\boldsymbol{H}}_G$$



- For plane motion,

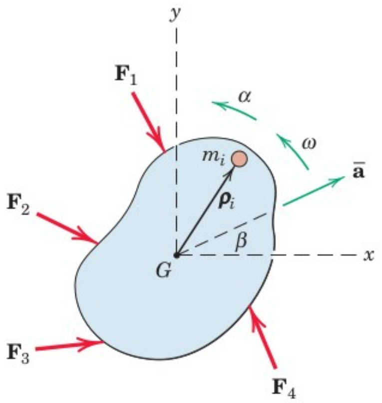
$$\boldsymbol{H}_G = \sum \boldsymbol{\rho}_i \times m_i\dot{\boldsymbol{\rho}}_i = \sum \boldsymbol{\rho}_i \times m_i(\boldsymbol{\omega} \times \boldsymbol{\rho}_i)$$

where, magnitude of $H_G = \sum m_i\rho_i^2\omega$.

Here, $\sum m_i\rho_i^2 = \int \rho_i^2 dm = \bar{I}$ is the mass moment of inertia of the body about the z-axis through G. Thus, $H_G = \bar{I}\omega$.

Hence, for a rigid body in plane motion general equations of motion become,

$$\sum \boldsymbol{F} = m\bar{\boldsymbol{a}} \qquad \text{...........(1a)}$$
$$\sum \boldsymbol{M}_G = \bar{I}\alpha$$



Alternate moment equations

Moment about any arbitrary point P can be written as (already derived),

$$\sum \boldsymbol{M}_P = \dot{\boldsymbol{H}}_G + \overline{\boldsymbol{\rho}} \times m\overline{\boldsymbol{a}} \qquad \cdots\cdots\cdots\cdots(2)$$

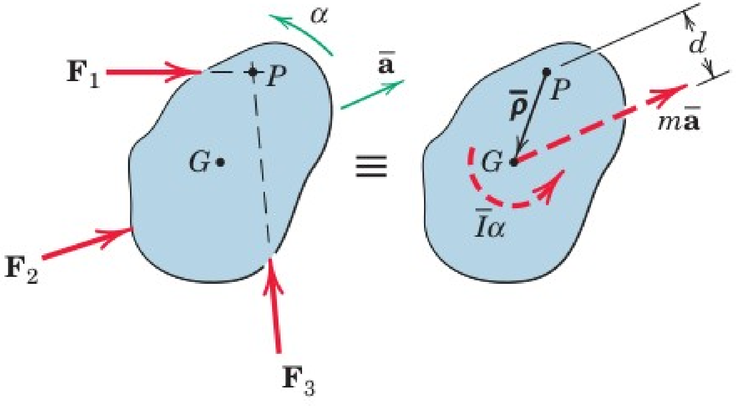
For a two-dimensional body,

$$\sum M_P = \overline{I}\alpha + m\overline{a}d$$

Alternatively, we can also write (already derived),

$$\sum \boldsymbol{M}_P = (\dot{\boldsymbol{H}}_P)_{\mathrm{rel}} + \overline{\boldsymbol{\rho}} \times m\overline{\boldsymbol{a}}_P \qquad \cdots\cdots\cdots\cdots(2a)$$

where, $(\dot{\boldsymbol{H}}_P)_{\mathrm{rel}} = I_P\boldsymbol{\alpha}$.



System of interconnected bodies

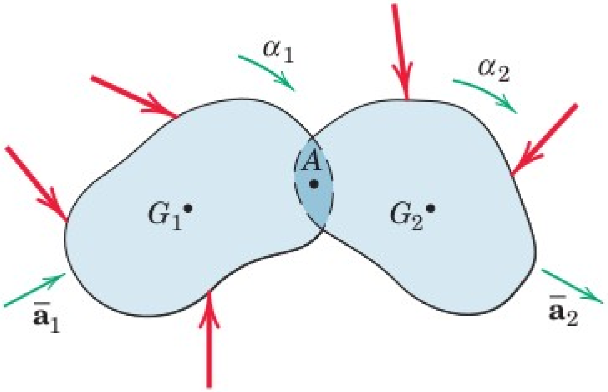
Many times it is required to deal with two or more connected rigid bodies whose motions are related kinematically. It is convenient to analyze the bodies as an entire system.

Two rigid bodies, which are hinged at A, and are subjected to external forces are shown.

For the two bodies we can write,

$$\sum \boldsymbol{F} = \sum m\bar{\boldsymbol{a}} \qquad \cdots\cdots\cdots\cdots(3)$$

$$\sum M_P = \sum \bar{I}\alpha + \sum m\bar{a}d$$



Free-Body Diagram of System

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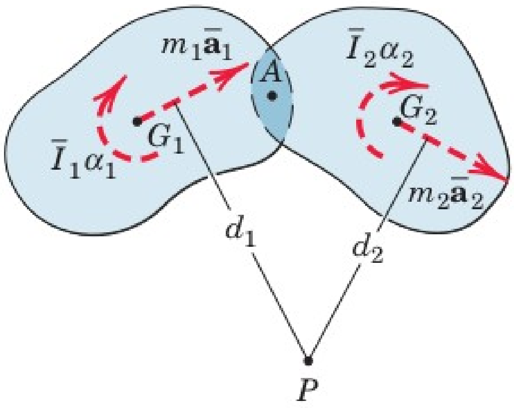


Kinetic Diagram of System

Translation

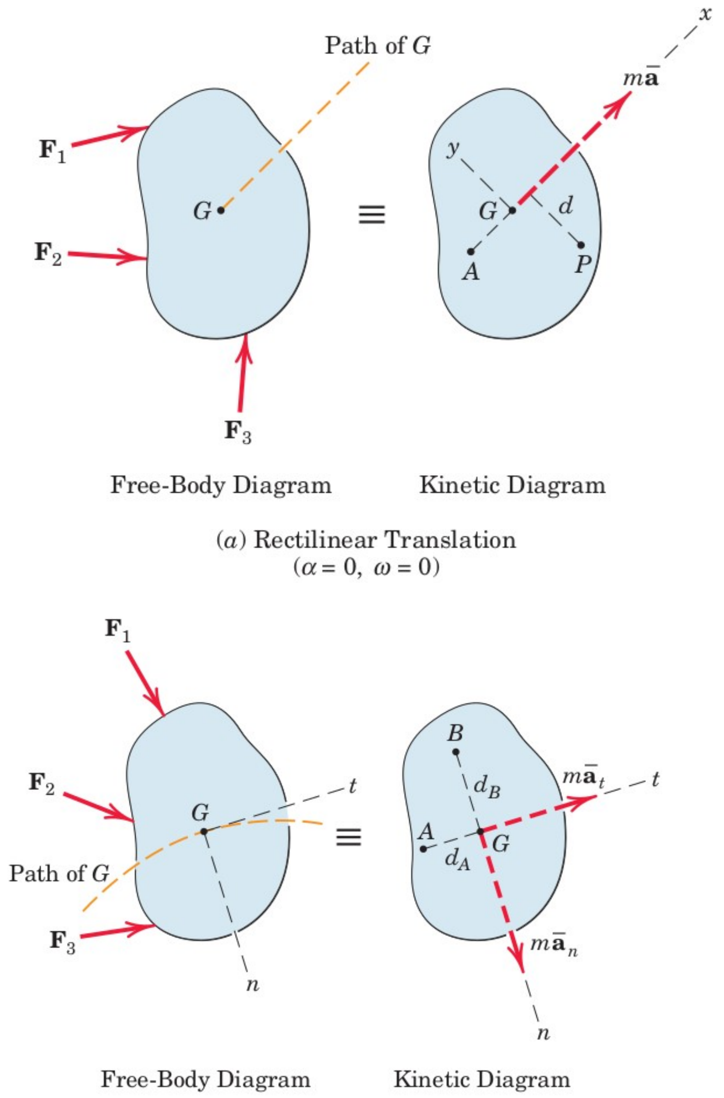
We already studied kinematics of rigid body translation in plane motion.

During translation (rectilinear or curvilinear) all points of the body moves either in straight lines or on congruent curved paths. In either case there is no angular motion of the body, so that both ω and α are zero.

Hence, from (1a), general equations for rigid bodies in plane motions are

$$\sum \boldsymbol{F} = m\bar{\boldsymbol{a}} \qquad \cdots\cdots\cdots\cdots(4)$$

$$\sum \boldsymbol{M}_G = \bar{I}\alpha = 0$$



(*a*) Rectilinear Translation
($\alpha = 0,\ \omega = 0$)

(*b*) Curvilinear Translation
($\alpha = 0,\ \omega = 0$)

Fixed-axis rotation

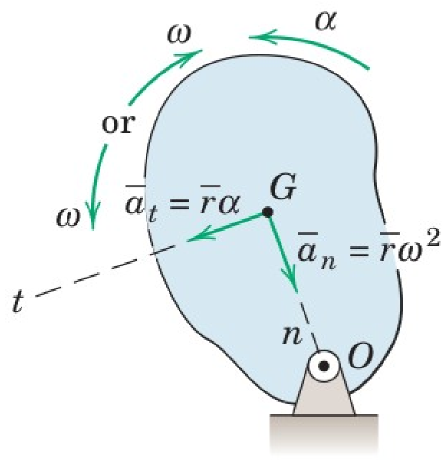
For rigid body rotation about fixed axis we have already seen that all points of the body rotates in a circle about the rotation axis, and all lines of the body have same angular velocity ω and acceleration α. Thus, the general equation for plane motions are directly applicable.

$$\sum \boldsymbol{F} = m\bar{\boldsymbol{a}}$$

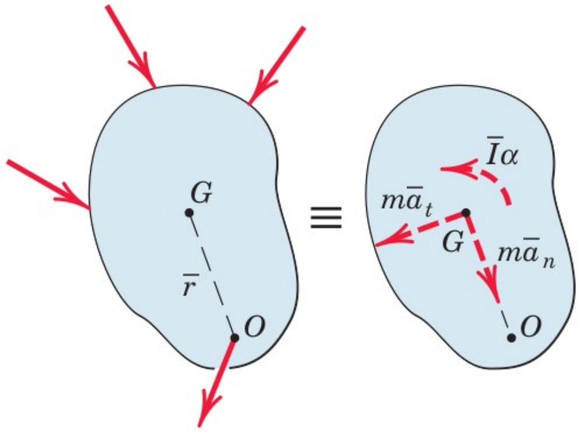
$$\sum \boldsymbol{M}_G = \bar{I}\alpha$$

For fixed-axis roation, it is generally useful to apply a moment equation direction about the rotation axis O, which yield the following equation,

$$\sum M_O = I_O\alpha.$$



Fixed-Axis Rotation
(a)

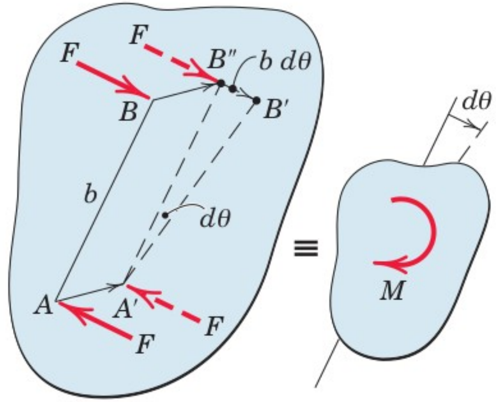


Free-Body Diagram
(b)
Kinetic Diagram
(c)

Work and Energy

- We have already developed the principles of work and energy and their application to the motion of a particle or system of particles.
- For finite displacements, the work-energy method eliminates the necessity for determining the acceleration and integrating it over the interval to obtain the velocity change.
- These same advantages are realized after extending the work-energy principles to describe rigid-body motion.

Work of forces and couples



The work done by a force $\boldsymbol{F}$ for a displacement of d$\boldsymbol{r}$ at the point of application of load $\boldsymbol{F}$ is given by

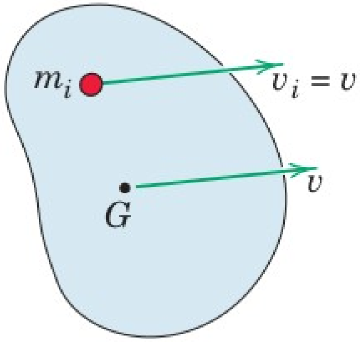
$$U = \int \boldsymbol{F} \cdot d\boldsymbol{r}.$$

Figure shows a couple $\boldsymbol{M} = \boldsymbol{Fb}$ acting on a rigid body which moves in the plane of the couple. Work done by $\boldsymbol{M}$ is determined as follows.

During time dt the body rotates through an angle $d\theta$, and line AB moves to $A'B'$. We may consider this motion in two parts, first a translation to $A'B''$ and then a rotation d about A'. It can be seen that **during the translation the work done by one of the forces cancels that done by the other force**, so that the net work done is $dU = F(b\ d\theta\ ) = M\ d\theta$ due to the rotational part of the motion.

If the couple acts in the sense opposite to the rotation, the work done is negative. During a finite rotation, the work done by a couple M whose plane is parallel to the plane of motion is, therefore, $U = \int M \cdot d\theta$.

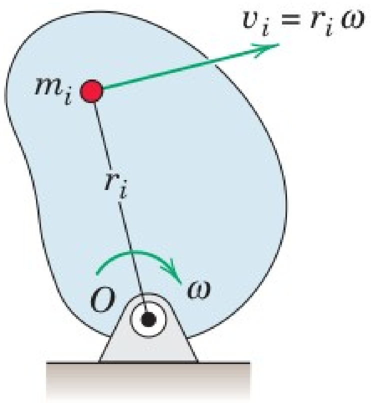
Kinetic energy



(a) Translation

The kinetic energy of a rigid body under pure translation is

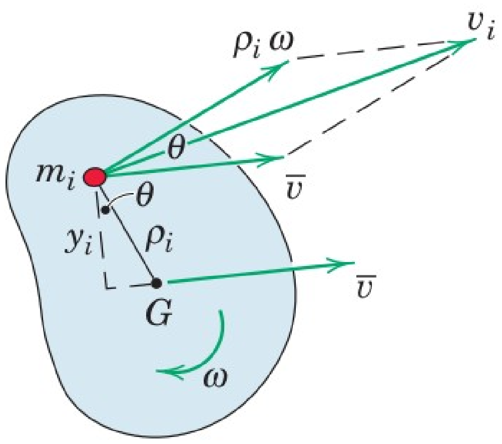
$$T = \sum \tfrac{1}{2} m_i v^2 = \tfrac{1}{2}\left(\sum m_i\right) v^2 = \tfrac{1}{2} M v^2.$$



(b) Fixed-Axis Rotation

The kinetic energy of a rigid body under rotation about a fixed-axis through O is

$$T = \sum \tfrac{1}{2} m_i r_i^2 \omega^2 = \tfrac{1}{2}\left(\sum m_i r_i^2\right) \omega^2 = \tfrac{1}{2} I_O \omega^2.$$



(c) General Plane Motion

For general plane motion, kinetic energy of the body is

$$T = \sum \tfrac{1}{2} m_i v_i^2,$$

$$T = \sum \tfrac{1}{2} m_i (\overline{v}^2 + \rho_i^2 \omega^2 + 2\overline{v}\rho_i \omega \cos\theta),$$

It can be shown that the third term is zero. Thus,

$$T = \tfrac{1}{2} m\overline{v}^2 + \tfrac{1}{2}\overline{I}\omega^2.$$

The kinetic energy of plane motion may also be expressed in terms of the rotational velocity about the instantaneous center C of zero velocity. Thus,

$$T = \tfrac{1}{2} I_C \omega^2.$$

Potential energy and the work-energy equation

The work-energy relation studied earlier is also applicable to rigid bodies in motion, as

$$T_1 + U_{1\text{-}2} = T_2, \text{ or}$$

$$T_1 + V_1 + U'_{1\text{-}2} = T_2 + V_2,$$

where, T is the kinetic energy of the system, $V = V_e + V_g$ is the total potential potential energy, and $U'_{1\text{-}2}$ is the work of all forces external to the system.

Power:

Power is the time rate at which work is performed. If the force $\boldsymbol{F}$ and the couple M is acting on a body and v is the velocity of the point of application of force $\boldsymbol{F}$ and ω is the angular velocity of the body, then, the total instantaneous power is

$$P = \boldsymbol{F} \cdot \boldsymbol{V} + M\omega.$$

From work-energy relation, it can be derived that,

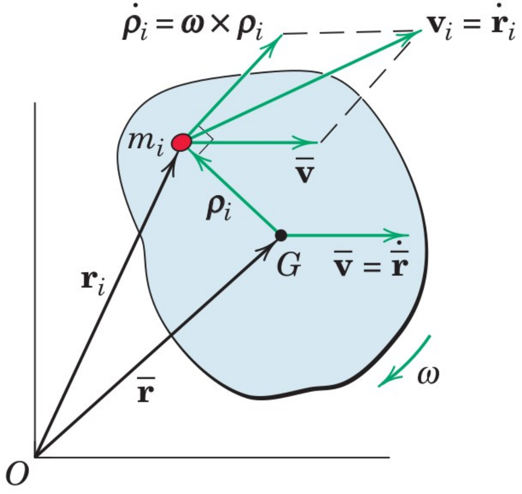
$$P = \frac{dU}{dt} = \dot{T}.$$

Thus, the power developed by the active forces and couples equals the rate of change of kinetic energy of the body or system of bodies.

$$\begin{aligned}\dot{T} &= \frac{d}{dt}\left(\frac{1}{2}m\overline{\boldsymbol{v}}\cdot\overline{\boldsymbol{v}} + \frac{1}{2}\overline{I}\omega^2\right)\\ &= \frac{1}{2}m(\overline{\boldsymbol{a}}\cdot\overline{\boldsymbol{v}} + \overline{\boldsymbol{v}}\cdot\overline{\boldsymbol{a}}) + \overline{I}\omega\dot{\omega}\\ &= m\overline{\boldsymbol{a}}\cdot\overline{\boldsymbol{v}} + \overline{I}\alpha\omega = \boldsymbol{R}\cdot\overline{\boldsymbol{v}} + \overline{M}\omega\end{aligned}$$

where $\boldsymbol{R}$ is the resultant of all forces acting on the body and $\overline{M}$ is the resultant moment about the mass center G of all forces.

Impulse and momentum



Expressions of impulse-momentum principle can be extended to motion of bodies in two-dimensions.

Linear momentum:

Linear momentum of a body under general plane motion can be defined as,

$$\boldsymbol{G} = m\overline{\boldsymbol{v}}$$

where m is the overall mass of the body, and $\overline{\boldsymbol{v}}$ is the velocity of mass center.

Impulse-Linear Momentum principle:

From Newton's generalized second law we get the impulse-linear momentum principle as

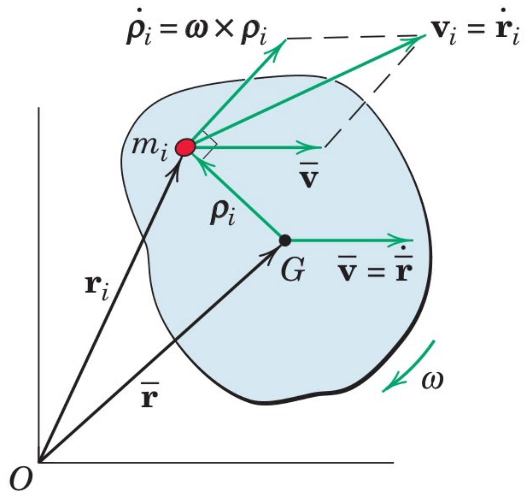
$$\boldsymbol{G}_1 + \int_{t_1}^{t_2} \sum \boldsymbol{F} dt = \boldsymbol{G}_2$$

or in component form

$$(G_x)_1 + \int_{t_1}^{t_2} \sum F_x dt = (G_x)_2$$

$$(G_y)_1 + \int_{t_1}^{t_2} \sum F_y dt = (G_y)_2$$

Angular Momentum:



Angular momentum is defined as the moment of linear momentum.

The angular momentum about the mass center of any prescribed system of mass as

$$\boldsymbol{H}_G = \sum \boldsymbol{\rho}_i \times m_i \boldsymbol{v}_i = \sum \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i$$

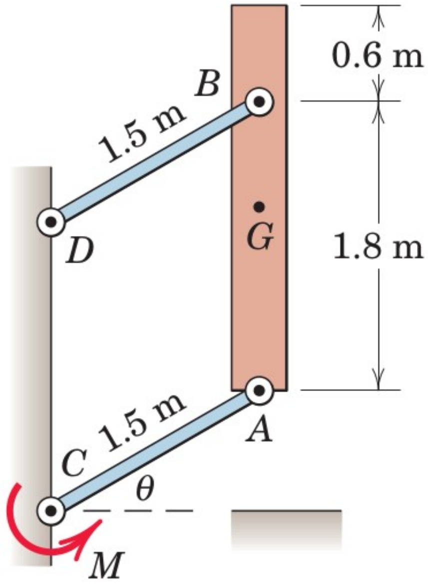
$$\boldsymbol{H}_G = \sum m_i \rho_i^2 \omega \boldsymbol{k} = \overline{I} \omega \boldsymbol{k}$$

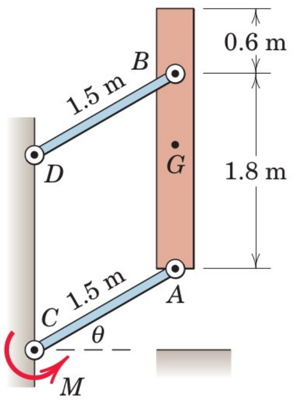
where $\overline{I} = \sum m_i \rho_i^2$ is the mass moment of inertia of the body about its mass center. Again from Newton's generalized second law, we get

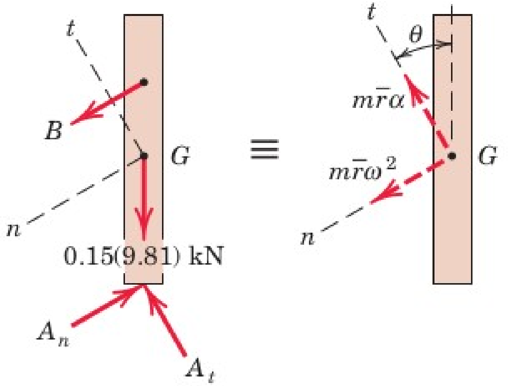
$$\sum M_G = \dot{H}_G \qquad \text{or} \qquad (H_G)_1 + \int_{t_1}^{t_2} \sum M_G dt = (H_G)_2$$

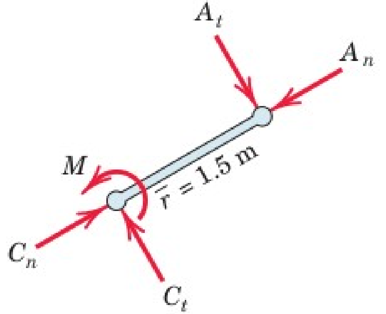
Example 1

The vertical bar AB has a mass of 150 kg with center of mass G midway between the ends. The bar is elevated from rest at $\theta = 0°$ by means of the parallel links of negligible mass, with a constant couple $M = 5$ kN·m applied to the lower link at C. Determine the angular acceleration of the links as a function of θ and find the force B in the link DB at the instant when $\theta = 30°$.









The bar translates on curvilinear path. With the circular motion of the mass center G, let us choose n- and t-coordinates for description of motion.

Considering negligible mass of the links, component A_t can be obtained from the analysis of link AC ($\Sigma M_C = 0$).

Now applying kinetic equations to link AB.

$\sum F_t = m\bar{a}_t$ gives relation between α and θ.

With α as a function of θ, angular velocity ω as a function of can be obtained as follows,

$$\int_0^{\omega} \omega d\omega = \int_0^{\theta} \alpha(\theta) d\theta.$$

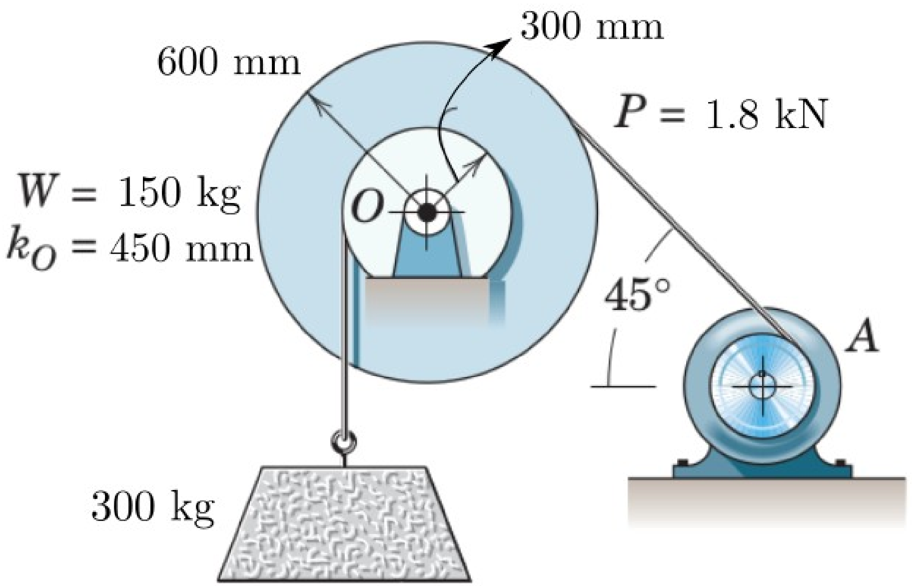
Now, force B at θ=30° can be obtained by taking moment about point A, as

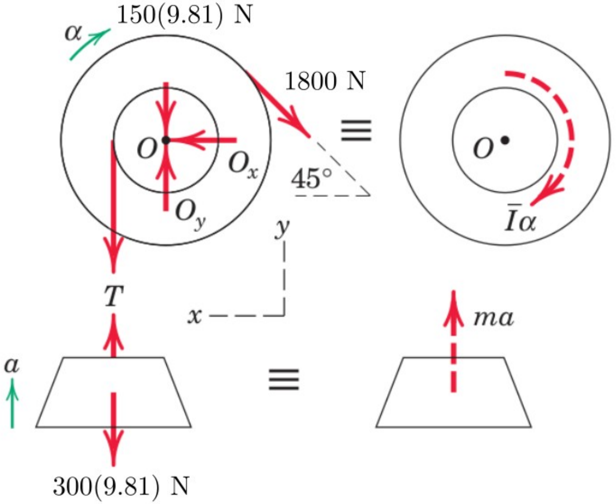
$$\sum M_A = m\bar{a}d$$

$$\Rightarrow 1.8 \cos 30° B = m\bar{r}\omega^2 (1.2 \cos 30°) + m\bar{r}\alpha (1.2 \sin 30°)$$

Example 2

The concrete block weighing 300 kg is elevated by the hoisting mechanism shown, where the cables are securely wrapped around the respective drums. The drums, which are fastened together and turn as a single unit about their mass center at O, have a combined weight of 150 kg and a radius of gyration about O of 450 mm. If a constant tension $P = 1.8$ kN is maintained by the power unit at A, determine the vertical acceleration of the block and the resultant force on the bearing at O.





The free-body and kinetic diagrams components showing all forces which act. The resultant of the force system on the drums for centroidal rotation is the couple

$\bar{I}\alpha = I_O\alpha$, where $\bar{I} = I_O = (0.45)^2 150 = 30.4$ kg·m^2

Now apply kinetic equations for the pulley and the block,

$$\sum M_G = \bar{I}\alpha \Rightarrow 1800(0.6) - T(0.3) = 30.4\alpha$$

$$\sum F_y = ma_y \Rightarrow T - 300(9.81) = 300a$$

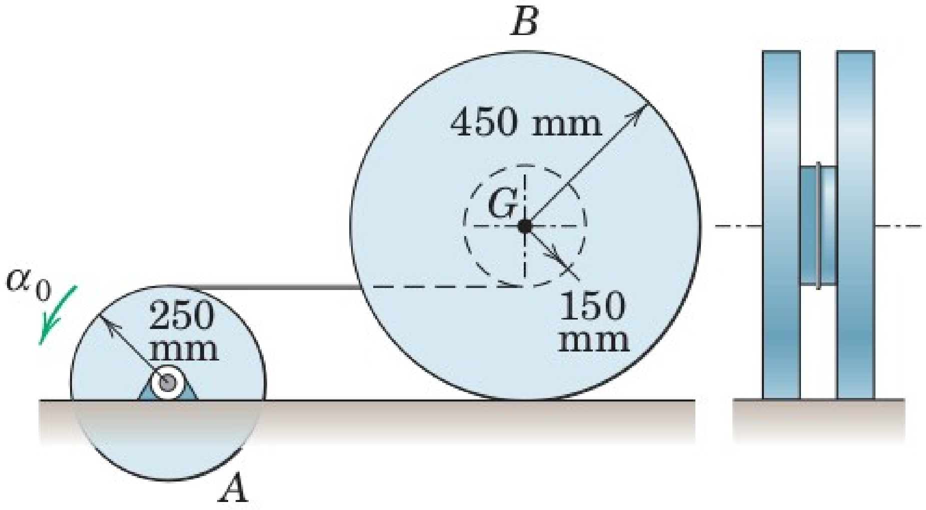
Note that, $a = r\alpha = 0.3\alpha$.

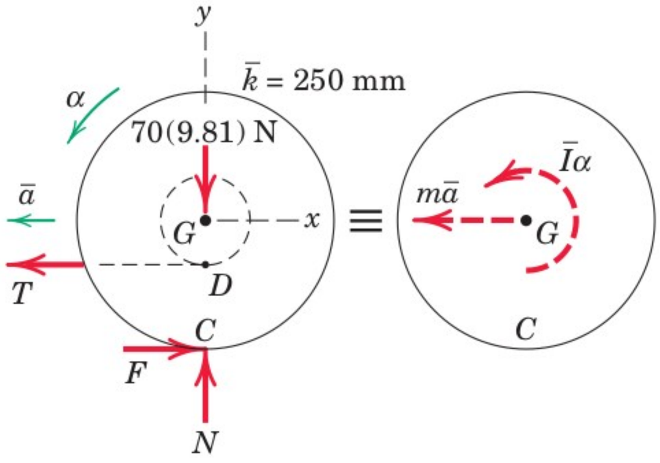
Thus from above equations, T, a and α can be determined. Bearing reaction can be computed by applying equations.

$\Sigma F_x = 0$ and $\Sigma F_y = 0$.

Example 3

The drum A is given a constant angular acceleration α_0 of 3 rad/s and causes the 70 kg spool B to roll on the horizontal surface by means of the connecting cable, which wraps around the inner hub of the spool. The radius of gyration k of the spool about its mass center G is 250 mm, and the coefficient of static friction between the spool and the horizontal surface is 0.25. Determine the tension T in the cable and the friction force F exerted by the horizontal surface on the spool.





The free-body diagram and the kinetic diagram of the spool are shown. The correct direction of the friction force may be assigned in this problem by observing from both diagrams that with counterclockwise angular acceleration, a moment sum about point G must be counterclockwise.

A point on the connecting cable has an acceleration $a_t = r\alpha = 0.25(3) = 0.75\ \mathrm{m/s^2}$, which is also the horizontal component of the acceleration of point D on the spool.

It will be assumed initially that the spool rolls without slipping, in which case it has a counterclockwise angular acceleration $\alpha = (a_D)_x / DC = 0.75/0.30 = 2.5\ \mathrm{rad/s^2}$.

The acceleration of the mass center G is, therefore, $a = r\alpha = 0.45(2.5) = 1.125\ \mathrm{m/s^2}$.

Apply equations of motion as

$$\sum F_x = m\bar{a}_x, \quad \sum F_y = m\bar{a}_y, \quad \sum M_G = \bar{I}\alpha.$$

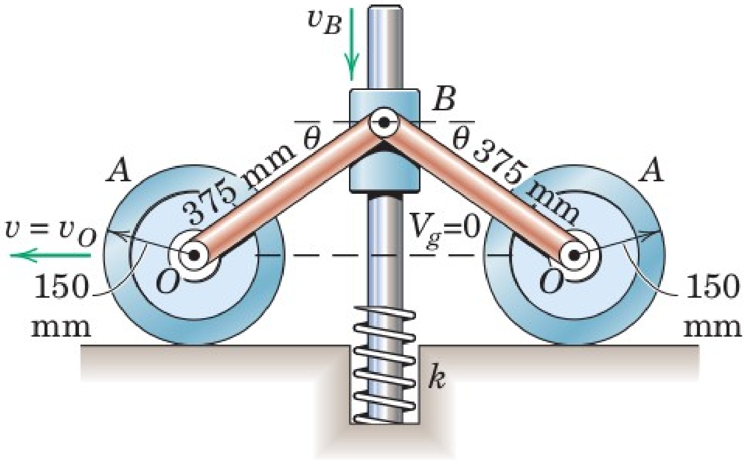
By solving above equations we obtain, $F = 75.8$ N, $T = 154.6$ N, and $N = 687$ N.

To establish the validity of our assumption of no slipping, we see that the surfaces are capable of supporting a maximum friction force $F_{max} = \mu_s N = 0.25(687) = 171.7$ N. Since only 75.8 N of friction force is required, we conclude that our assumption of rolling without slipping is valid.

If the coefficient of static friction had been 0.10, for example, then the friction force would have been limited to $0.10(687) = 68.7$ N, which is less than 75.8 N, and the spool would slip. In this event, the kinematic relation $a = r\alpha$ would no longer hold. With $(a_D)_x$ known, the angular acceleration would be $\alpha = [\overline{a} - (a_D)_x]/GD$. Using this relation along with $F = \mu_k N = 68.7$ N, we would then re-solve the three equations of motion for the unknowns T, a, and α.

Example 4

In the mechanism shown, each of the two wheels has a mass of 30 kg and a centroidal radius of gyration of 100 mm. Each link OB has a mass of 10 kg and may be treated as a slender bar. The 7-kg collar at B slides on the fixed vertical shaft with negligible friction. The spring has a stiffness $k = 30$ kN/m and is contacted by the bottom of the collar when the links reach the horizontal position. If the collar is released from rest at the position θ= 45° and if friction is sufficient to prevent the wheels from slipping, determine (a) the velocity v_B of the collar as it first strikes the spring and (b) the maximum deformation x of the spring.



The mechanism executes plane motion and is conservative with the neglect of kinetic friction losses. Three states are defined as, 1, 2 and 3 at θ= 45°, 0°, and maximum spring deflection, respectively. The datum for zero gravitational potential energy V_g is conveniently taken through O as shown.

(a) Note that for the interval from θ= 45° to θ= 0°, the initial and final kinetic energies of the wheels are zero since each wheel starts from rest and momentarily comes to rest at θ= 0°. Also, at position 2, each link is only rotating about its point O so that

$$T_2 = \left[2 \times \frac{1}{2} I_O \omega^2\right]_{\text{links}} + \left[\frac{1}{2} m v^2\right]_{\text{collar}} \qquad (I_O = \tfrac{1}{3} m L^2)$$

$\omega = 0.375\ v_B$, hence, $T_2 = 6.83 v_B^2$.

Reg. Potential energy, the collar at B travels a distance $0.375/\sqrt{2} = 0.265$ m so that

$V_1 = (2\times 10\text{g}\times 0.265/2) + (0.265\times 7\text{g}) = 44.2$ J, $\quad V_2 = 0$.

Also $U'_{1\text{-}2} = 0$.

Hence, $V_1 + T_1 + U'_{1\text{-}2} = V_2 + T_2$, and we get v_B as 2.54 m/s.

(b) At the condition of maximum deformation x of the spring, all parts are momentarily at rest, which makes $T_3 = 0$. Thus,

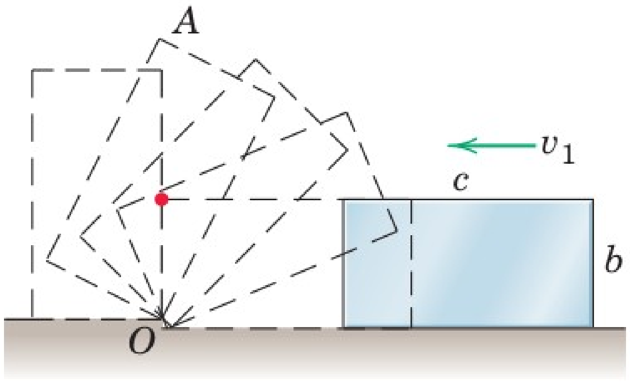
$$V_1 + T_1 + U'_{1\text{-}3} = V_3 + T_3,$$

$$0 + (2\times 10\text{g}\times 0.265/2) + (0.265\times 7\text{g}) = 0 - (2\times 10\text{g}\times x/2) - 7\text{g}\times x + 1/2\times(30\text{e}3)\times x^2$$

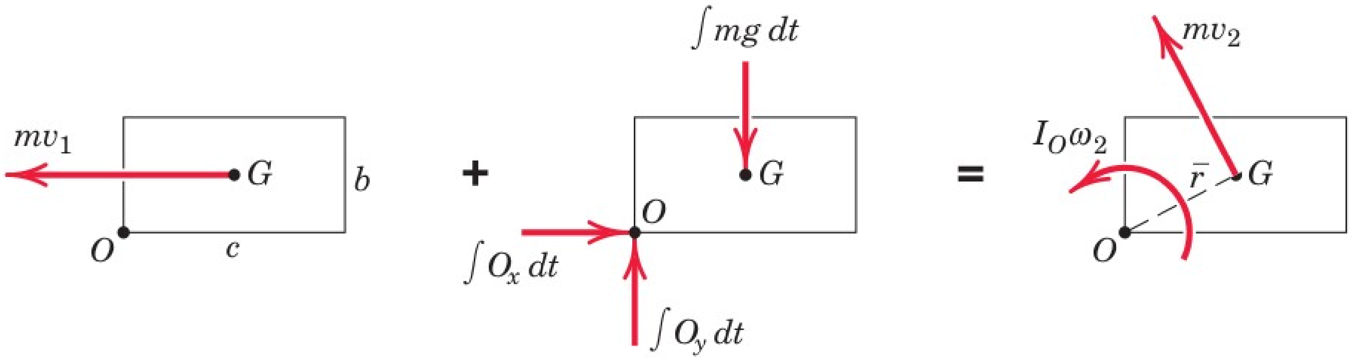
Solution for the positive value of x gives $x = 60.1$ mm.

Example 5

The uniform rectangular block of dimensions shown is sliding to the left on the horizontal surface with a velocity v_1 when it strikes the small step at O. Assume negligible rebound at the step and compute the minimum value of v_1 which will permit the block to pivot freely about O and just reach the standing position A with no velocity. Compute the percentage energy loss n for $b = c$.



We break the overall process into two subevents:
(i) the collision, (ii) and the subsequent rotation.



I. Collision. With the assumption that the weight mg is nonimpulsive, angular momentum about O is conserved. The initial angular momentum of the block about O just before impact is the moment about O of its linear momentum and is $(H_O)_1 = mv_1(b/2)$. The angular momentum about O just after impact when the block is starting its rotation about O is

$$(H_O)_2 = I_O\omega_2 = \frac{m}{3}(b^2 + c^2)\omega_2$$

From conservation of angular momentum i.e., $(H_O)_1 = (H_O)_2$ we get, $\omega_2 = \dfrac{3v_1 b}{2(b^2 + c^2)}$.

II. Rotation about *O*: With the assumptions that the rotation is like that about a fixed frictionless pivot and that the location of the effective pivot O is at ground level, mechanical energy is conserved during the rotation according to

$$T_2 + V_2 = T_3 + V_3$$

$$\frac{1}{2} I_O \omega_2^2 + 0 = 0 + mg\left[\sqrt{(b/2)^2 - (c/2)^2} - b/2\right]$$

Substituting ω_2 in terms of v_1 from part I, v_1 can now be calculated.

The percentage loss of energy during the impact is

$$n = \frac{|\Delta E|}{E} = \frac{\frac{1}{2}mv_1^2 - \frac{1}{2}I_O\omega_2^2}{\frac{1}{2}mv_1^2}$$