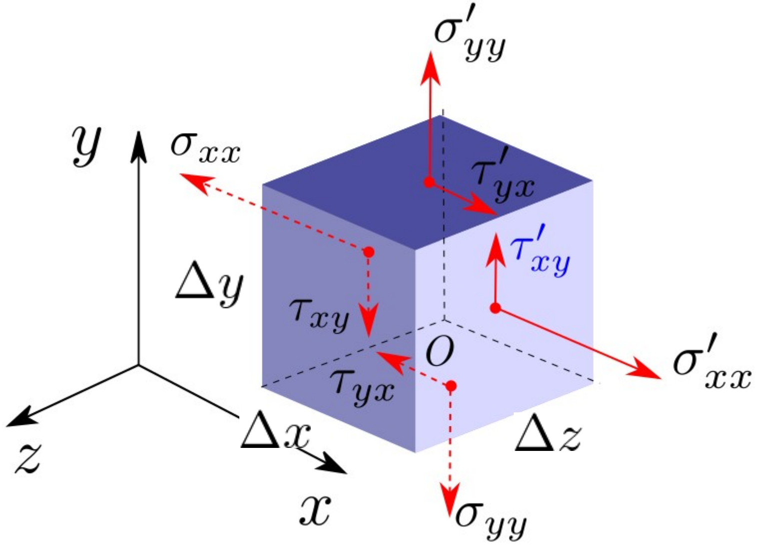
ME231: Solid Mechanics-I

Stress and Strain

Equilibrium of a differential element in plane stress



A small element from a body is shown. For a plane stress case stresses working on the element is also shown. If the body is in equilibrium then the element should also be in equilibrium.

If stress components at the negative x-face is σ_x and τ_{xy}, then the components at a parallel face (at a distance of Δx) can be approximated as,

$$\sigma'_{xx} = \sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\Delta x, \quad \tau'_{xy} = \tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\Delta x. \qquad \cdots\cdots\cdots\cdot(9)$$

Similarly, stress components at the positive y-face will be,

$$\sigma'_{yy} = \sigma_{yy} + \frac{\partial \sigma_{yy}}{\partial y}\Delta y, \quad \tau'_{yx} = \tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\Delta y. \qquad \cdots\cdots\cdots\cdot(10)$$

Applying equilibrium conditions for the element as,

$$\sum F_x = (\sigma'_{xx} - \sigma_{xx})\Delta z\Delta y + (\tau'_{yx} - \tau_{yz})\Delta x\Delta z = 0. \qquad \cdots\cdots\cdots(11)$$

Using (9) and (10) in (11),

$$\left[\sigma_{xx} + \frac{\partial\sigma_{xx}}{\partial x}\Delta x - \sigma_{xx}\right]\Delta z\Delta y + \left[\tau_{yx} + \frac{\partial\tau_{yx}}{\partial y}\Delta y - \tau_{yx}\right]\Delta x\Delta z = 0,$$

$$\Rightarrow \left[\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y}\right]\Delta x\Delta y\Delta z = 0,$$

which gives one of the condition of equilibrium of the element as,

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} = 0. \qquad \cdots\cdots\cdots(12)$$

Similarly, consider the equilibrium condition in y-direction, i.e., $\Sigma F_y=0$ will give another equilibrium condition as, `

$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} = 0. \qquad \cdots\cdots\cdots(13)$$

Now let us consider the moment equilibrium about the center of the element, i.e., ΣM=0, as

$$\left[\tau'_{xy}\frac{\Delta x}{2}+\tau_{xy}\frac{\Delta x}{2}\right]\Delta y\Delta z-\left[\tau'_{yx}\frac{\Delta y}{2}+\tau_{yx}\frac{\Delta y}{2}\right]\Delta x\Delta z=0$$

Using (9) and (10),

$$\frac{1}{2}\left[\tau_{xy}+\frac{\partial\tau_{xy}}{\partial x}\Delta x+\tau_{xy}\right]\Delta x\Delta y\Delta z-\frac{1}{2}\left[\tau_{yx}+\frac{\partial\tau_{yx}}{\partial y}\Delta y+\tau_{yx}\right]\Delta x\Delta y\Delta z=0,$$

$$\Rightarrow\tau_{xy}+\frac{\partial\tau_{xy}}{\partial x}\frac{\Delta x}{2}-\tau_{yx}-\frac{\partial\tau_{yx}}{\partial y}\frac{\Delta y}{2}=0, \qquad \cdots\cdots\cdots\cdots(14)$$

when Δx and Δy tend to zero, then (14) yields, $\tau_{xy}=\tau_{yx}$. ··········(15)

Equation (15) suggests that for a body in plane stress the shear-stress components on perpendicular faces must be equal in magnitude. Using (15) conditions for equilibrium i.e., (12) and (13), become,

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}=0.$$

$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}=0.$$

In the similar manner, considering the moment equilibrium of an element with all stress components, following conditions are obtained.

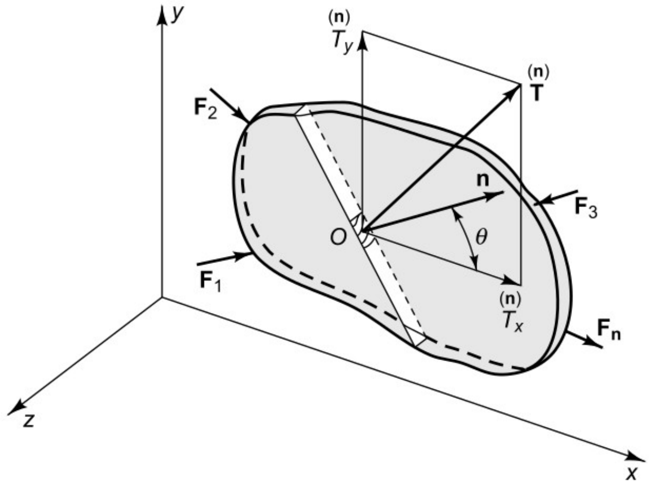
$$\tau_{xy} = \tau_{yx}, \qquad \tau_{yz} = \tau_{zy}, \qquad \text{and} \qquad \tau_{xz} = \tau_{zx}. \qquad \cdots\cdots\cdots\cdot(16)$$

Using force equilibrium conditions and with the use of (15), equilibrium equations in three-dimensions can be derived, which are as follows.

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} &= 0, \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} &= 0, \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} &= 0. \end{aligned} \qquad \cdots\cdots\cdots\cdot(17)$$

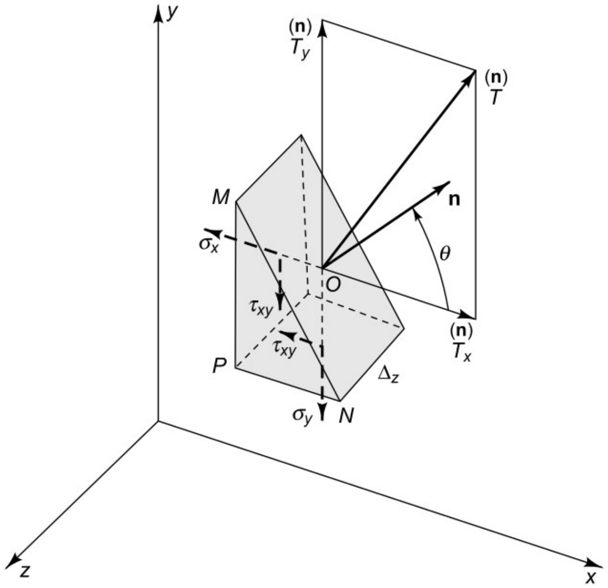
Stress components associated with arbitrarily oriented faces in plane stress

While deriving the stress components at a point corresponding to x, y and z faces, we discussed that stress components at arbitrary plane passing through O can be related to these components. Let us find the relations for plane stress case.



Let us assume that stress component at point O associated with x and y planes are σ_x, σ_y, and τ_{xy}. Consider an arbitrary plane having a normal $\boldsymbol{n}$ and which passes through O.

Let us considered the equilibrium of a wedge element which obtained when plane $\boldsymbol{n}$ slices the elemental cuboid around point O.



$$\sum F_x = t_x^{\boldsymbol{n}} \Delta z \overline{MN} - \sigma_{xx} \overline{MP} \Delta z - \tau_{xy} \Delta z \overline{PN} = 0,$$

$$\Rightarrow t_x^{\boldsymbol{n}} \Delta z \overline{MN} - \sigma_{xx} \overline{MN} \cos\theta - \tau_{xy} \Delta z \overline{MN} \sin\theta = 0,$$

$$\Rightarrow t_x^{\boldsymbol{n}} = \sigma_{xx} \cos\theta + \tau_{xy} \sin\theta. \qquad \cdots\cdots\cdots\cdots(18)$$

$$\sum F_y = t_y^{\boldsymbol{n}} \Delta z \overline{MN} - \sigma_{yy} \overline{PN} \Delta z - \tau_{xy} \Delta z \overline{MP} = 0,$$

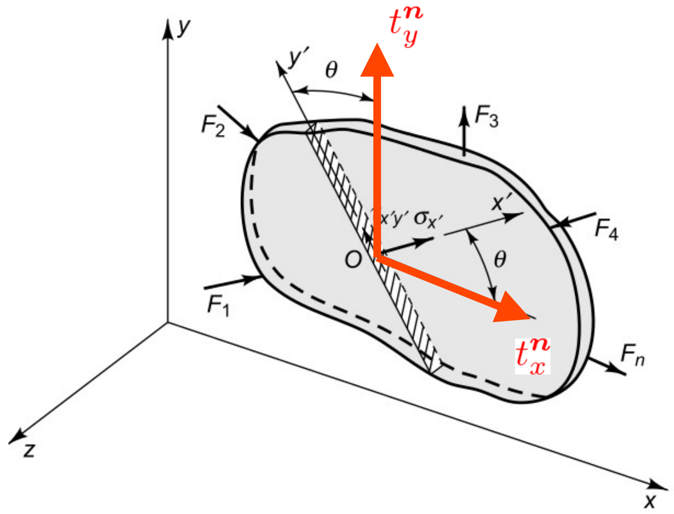
$$\Rightarrow t_y^{\boldsymbol{n}} \Delta z \overline{MN} - \sigma_{yy} \overline{MN} \sin\theta - \tau_{xy} \Delta z \overline{PN} \cos\theta = 0,$$

$$\Rightarrow t_y^{\boldsymbol{n}} = \sigma_{yy} \sin\theta + \tau_{xy} \cos\theta. \qquad \cdots\cdots\cdots\cdots(19)$$

Thus, (18) and (19) relates the stress components at $\boldsymbol{n}$–plane to stress components at x – and y – planes.

$$\begin{aligned} t_x^{\boldsymbol{n}} &= \sigma_{xx} \cos\theta + \tau_{xy} \sin\theta. \\ t_y^{\boldsymbol{n}} &= \sigma_{yy} \sin\theta + \tau_{xy} \cos\theta. \end{aligned} \qquad \cdots\cdots\cdots\cdots(20)$$

Components $t_x^{\boldsymbol{n}}$ and $t_y^{\boldsymbol{n}}$ directs along x- and y- coordinate axes. However, when talking about any plane, knowledge of stress components perpendicular and parallel to the plane (along x' and y') is important.



With the knowledge of $t_x^{\boldsymbol{n}}$ and $t_y^{\boldsymbol{n}}$, we can simply transform forces in x and y direction along x' and y' direction to find the stress components σ'_{xx} and τ'_{xy}.

Thus,

$$\sigma'_{xx}\Delta A = t_x^{\boldsymbol{n}}\Delta A\cos\theta + t_y^{\boldsymbol{n}}\Delta A\sin\theta,$$
$$\text{and } \tau'_{xy}\Delta A = -t_x^{\boldsymbol{n}}\Delta A\sin\theta + t_y^{\boldsymbol{n}}\Delta A\cos\theta. \qquad \text{..........(21)}$$

Substituting (20) in to (21) and simplifying, we get,

$$\sigma'_{xx} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta.$$
$$\tau'_{xy} = (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta). \qquad \text{..........(22)}$$

Note that another method to derive (22) is to consider the equilibrium of the infinitesimal wedge element near point O in x' and y' directions.

Another component σ'_{yy} can be determined by considering another plane having normal perpendicular to $\boldsymbol{n}$ and considering the equilibrium of wedge element near point O.

The same can be obtained by replacing θ with (θ+90°) in (22). By doing that we get,

$$\sigma'_{yy} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta. \qquad \cdots\cdots\cdots(23)$$

To summarize, if stress components at a point in x-y coordinate system is known, then it is possible to know the stress components for all possible orientations of faces through the point using following equations. We say that we know the **state of stress** at the point.

$$\begin{aligned}
\sigma'_{xx} &= \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta.\\
\sigma'_{yy} &= \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta. \qquad \cdots\cdots\cdots(24)\\
\tau'_{xy} &= (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta).
\end{aligned}$$

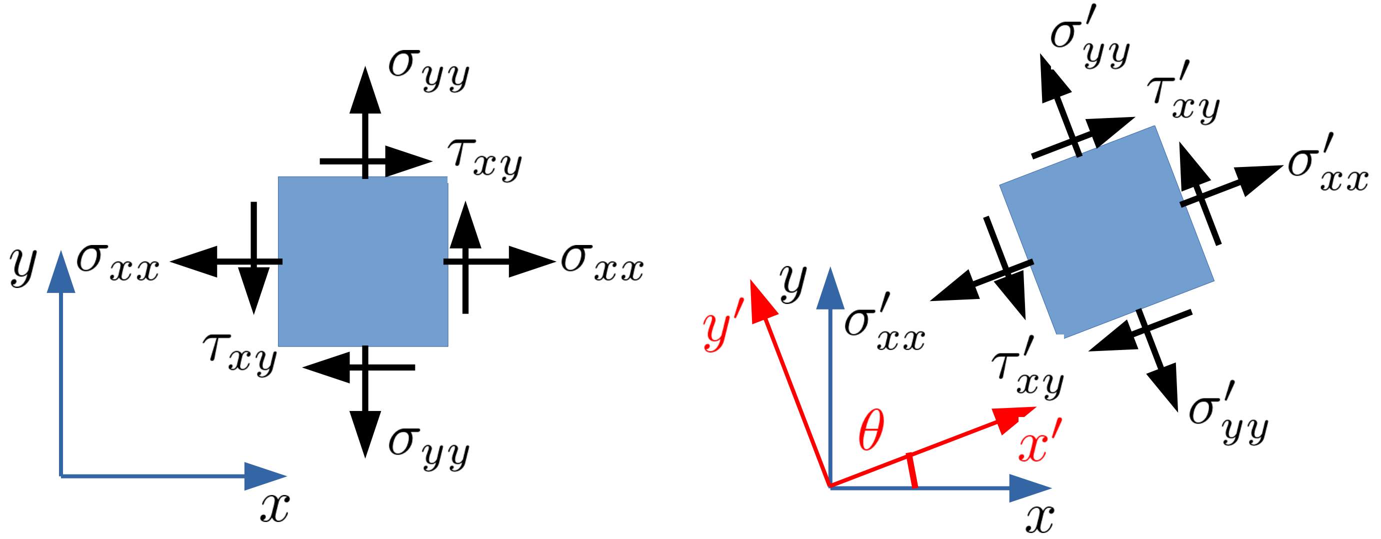
Be careful not to confuse a single stress component with the state of stress at the point.

Determining stress components in x'–y' coordinates system using known stress components in x–y direction is called **stress transformation.**

Using trigonometric identities (24) can be re-written as,

$$
\begin{aligned}
\sigma'_{xx} &= \frac{(\sigma_{xx} + \sigma_{yy})}{2} + \frac{(\sigma_{xx} - \sigma_{yy})}{2} \cos 2\theta + \tau_{xy} \sin 2\theta, \\
\sigma'_{yy} &= \frac{(\sigma_{xx} + \sigma_{yy})}{2} - \frac{(\sigma_{xx} - \sigma_{yy})}{2} \cos 2\theta - \tau_{xy} \sin 2\theta, \\
\tau'_{xy} &= \tau_{xy} \cos 2\theta - \frac{(\sigma_{xx} - \sigma_{yy})}{2} \sin 2\theta.
\end{aligned}
\qquad \cdots\cdots\cdots\cdot(25)
$$

Stress transformation can also be define using the transformation matrix as follow.



The transformation matrix is defined as,

$$[\boldsymbol{Q}] = \begin{bmatrix} \cos(x, x') & \cos(x, y') \\ \cos(y, x') & \cos(y, y') \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \text{..........(25)}$$

Now, the stress components in x'-y' coordinates is given as,

$$\begin{bmatrix} \sigma'_{xx} & \tau'_{xy} \\ \tau'_{yx} & \sigma'_{yy} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \qquad \text{..........(26)}$$