ME231: Solid Mechanics-I

Stress and Strain

Strain-displacement relations(in 2D)

Engineering strain components,

$$\epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \cdots\cdots\cdots\cdots(34)$$

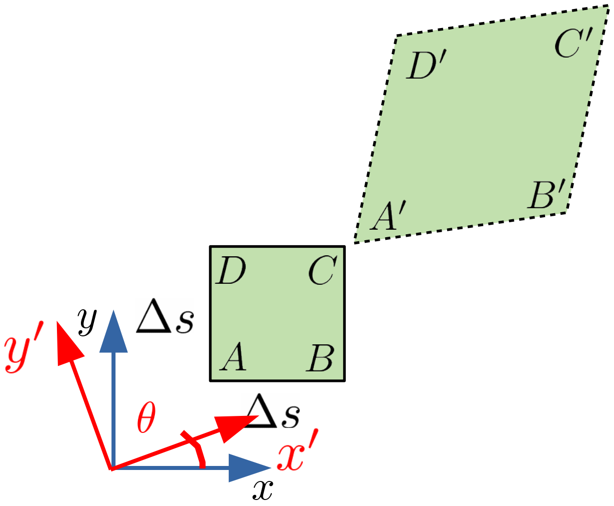
Similar to stress, strain is also a second order tensor. In plane strain case, strain matrix is defined as,

$$[\boldsymbol{\epsilon}] = \begin{bmatrix} \epsilon_{xx} & \gamma_{xy}/2 \\ \gamma_{yx}/2 & \epsilon_{yy} \end{bmatrix} \qquad \cdots\cdots\cdots\cdots(36)$$

Strain transformation

Being a second order tensor, strain tensor follows general rules of transformations.

$$[\boldsymbol{\epsilon}]' = [\boldsymbol{Q}]^T[\boldsymbol{\epsilon}][\boldsymbol{Q}],$$



Following this equation transformed strains will be given as

$$\epsilon'_{xx} = \frac{(\epsilon_{xx} + \epsilon_{yy})}{2} + \frac{(\epsilon_{xx} - \epsilon_{yy})}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta,$$
$$\epsilon'_{yy} = \frac{(\epsilon_{xx} + \epsilon_{yy})}{2} - \frac{(\epsilon_{xx} - \epsilon_{yy})}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta,$$
$$\gamma'_{xy} = \frac{\gamma_{xy}}{2}\cos 2\theta - \frac{(\epsilon_{xx} - \epsilon_{yy})}{2}\sin 2\theta.$$

..........(37)

Strain transformation equations (35) can also be derived from purely geometrical relations. Relations between the coordinate and displacements in xy-system and $x'y'$-system is given as

$$\begin{aligned} x &= x'\cos\theta - y'\sin\theta \qquad & u' &= u\cos\theta + v\sin\theta \\ y &= x'\sin\theta + y'\cos\theta. \qquad & v' &= -u\sin\theta + v\cos\theta. \end{aligned} \quad \cdots\cdots\cdots(38)$$

Now, using strain-displacement relations,

$$\begin{aligned} \epsilon'_x &= \frac{\partial u'}{\partial x'} = \frac{\partial u'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial x'} = \frac{\partial u'}{\partial x}\cos\theta + \frac{\partial u'}{\partial y}\sin\theta \\ &\Rightarrow \left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial v}{\partial x}\sin\theta\right)\cos\theta + \left(\frac{\partial u}{\partial y}\cos\theta + \frac{\partial v}{\partial y}\sin\theta\right)\sin\theta \\ &\Rightarrow \frac{\partial u}{\partial x}\cos^2\theta + \frac{\partial v}{\partial y}\sin^2\theta + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\sin\theta\cos\theta, \\ &\Rightarrow \epsilon_x\cos^2\theta + \epsilon_y\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta. \qquad \cdots\cdots\cdots(39) \end{aligned}$$

$$
\begin{aligned}
\epsilon'_y &= \frac{\partial v'}{\partial y'} = \frac{\partial v'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial y'} = -\frac{\partial v'}{\partial x}\sin\theta + \frac{\partial v'}{\partial y}\cos\theta \\
&\Rightarrow -\left(-\frac{\partial u}{\partial x}\sin\theta + \frac{\partial v}{\partial x}\cos\theta\right)\sin\theta + \left(-\frac{\partial u}{\partial y}\sin\theta + \frac{\partial v}{\partial y}\cos\theta\right)\cos\theta \\
&\Rightarrow \frac{\partial v}{\partial y}\cos^2\theta + \frac{\partial u}{\partial x}\sin^2\theta - \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\sin\theta\cos\theta \\
&\Rightarrow \epsilon_y\cos^2\theta + \epsilon_x\sin^2\theta - \gamma_{xy}\sin\theta\cos\theta \qquad \cdots\cdots\cdots\cdots(40)
\end{aligned}
$$

$$
\begin{aligned}
\gamma'_{xy} &= \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'} = \left(\frac{\partial v'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial x'}\right) + \left(\frac{\partial u'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial y'}\right) \\
&\Rightarrow \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)\sin\theta\cos\theta + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\left(\cos^2\theta - \sin^2\theta\right) \\
&\Rightarrow (\epsilon_y - \epsilon_x)\sin\theta\cos\theta + \gamma_{xy}\left(\cos^2\theta - \sin^2\theta\right) \qquad \cdots\cdots\cdots\cdots(41)
\end{aligned}
$$

Use of trigonometric identities will results in equations identical to the equations derived from tensor transformations.

Principal strains and maximum shear strain

- Planes at which **shear strain is zero** are called **principal planes** of strains and the normal strains at these planes are called **principal strains.** Inclination of principal planes from x-axis are given by

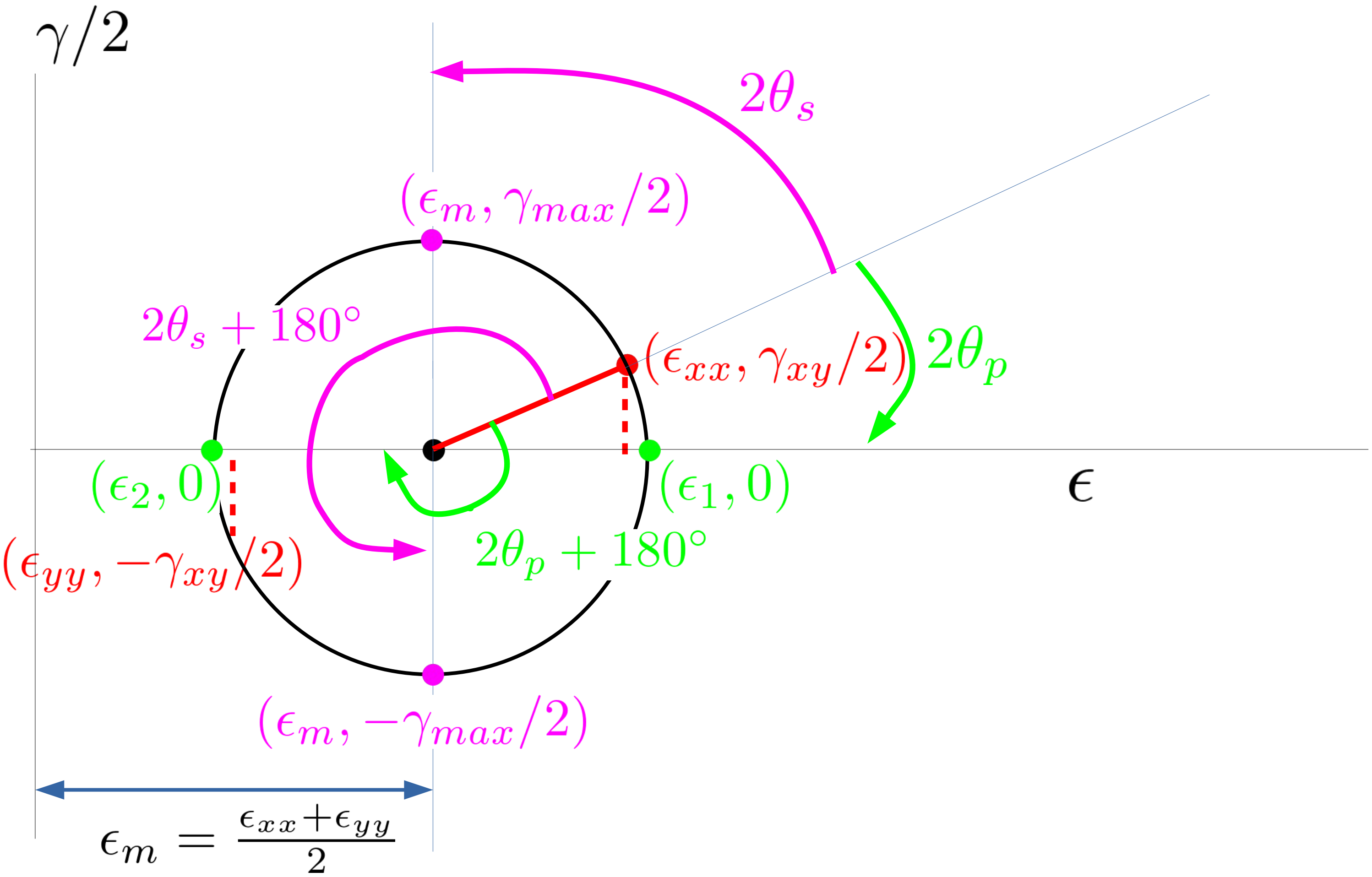
$$\tan 2\theta_p = \frac{\gamma_{xy}}{\epsilon_{xx} - \epsilon_{yy}} \qquad \cdots\cdots\cdots\cdots(42)$$

- Maximum value of shear strain is given as,

$$\gamma_{\max} = \sqrt{(\epsilon_{xx} - \epsilon_{yy})^2 + \gamma_{xy}^2}, \text{ at angle } \theta \text{ satisfying } \tan 2\theta_s = -\frac{\epsilon_{xx} - \epsilon_{yy}}{\gamma_{xy}}.$$

- Similar to stresses, Mohr's circle can be drawn for strains also. ....(43)

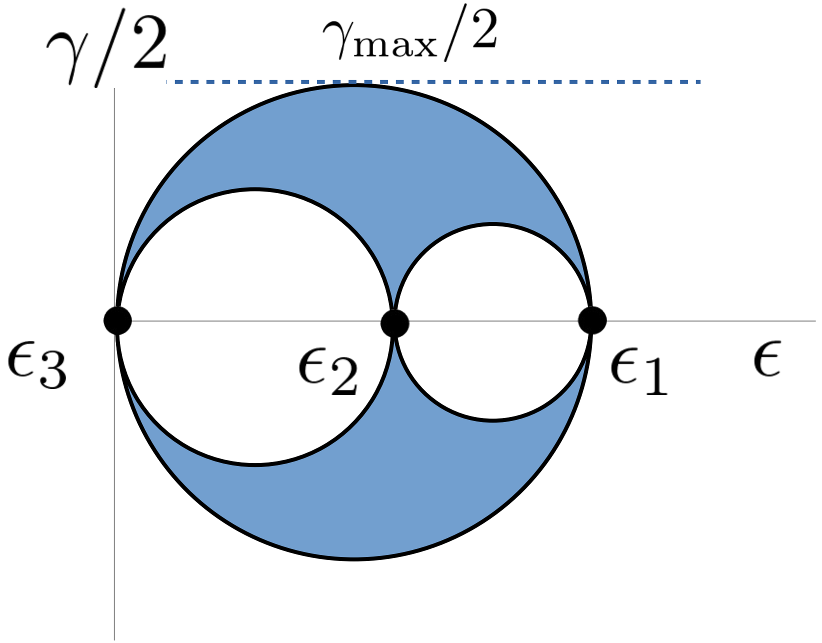
Mohr's circle for strains



Principal strains in plane strain

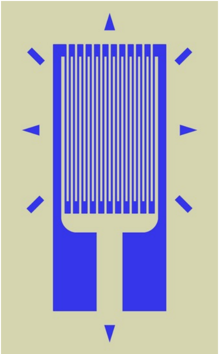
For plane stress, $\varepsilon_{zz} = 0$. In case ε_1 and, ε_2 >0, then maximum stresses will be calculated as,

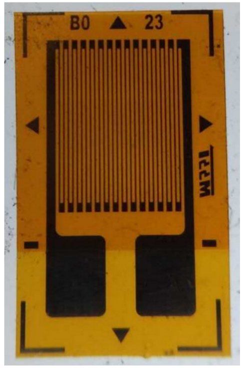
$$\gamma_{\max} = |\epsilon_{\max} - \epsilon_{\min}| = |\epsilon_3 - \epsilon_1|$$

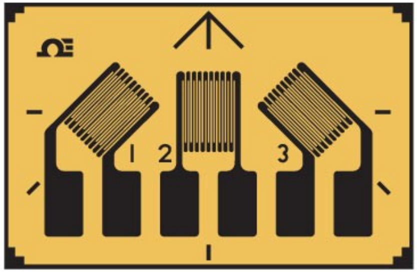


Measurement of strains

- **Electrical strain gages** are used to measure strains.
- Strain gages work on the principal that certain metals exhibit a **change in electrical resistance with change in mechanical strain**.
- Strain gages are bonded on any surface and it measure **strain the axial direction** of the gage.
- **Strain rosette** are combination of strain gages, used to measure strains in different directions and calculate principal strains.







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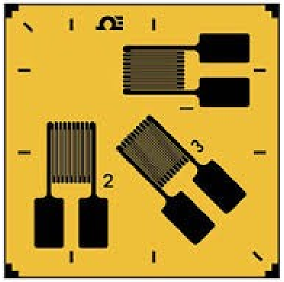

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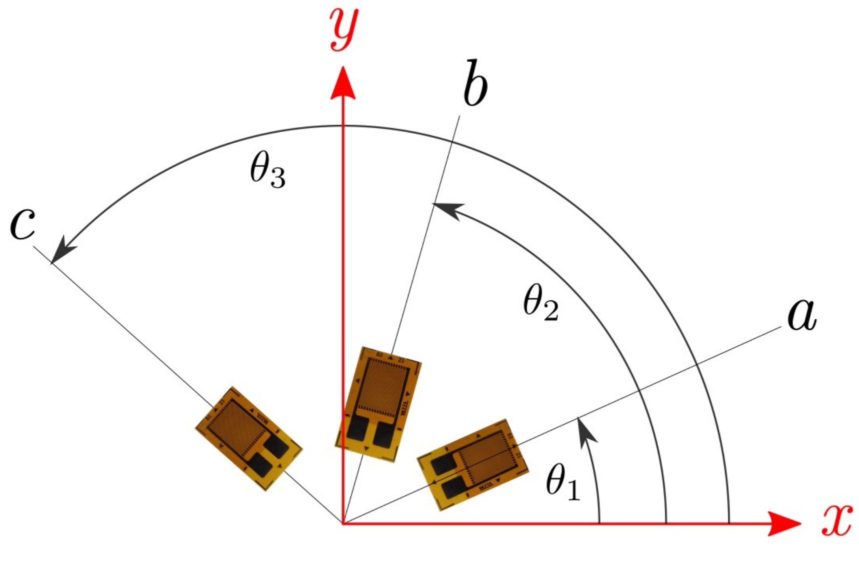
Strain rosette



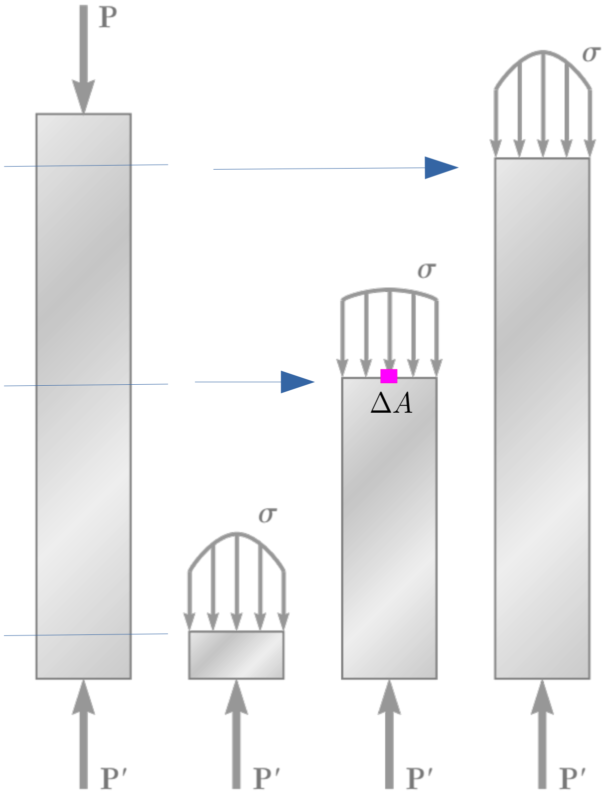
If, in-plane strain components are ε_x, ε_y, γ_{xy}, then strains in the direction of a, b, and c can be expression using in-plane strains and the inclination from x-axis, as

$$\epsilon_a = \epsilon_x \cos^2\theta_1 + \epsilon_y \sin^2\theta_1 + \gamma_{xy} \sin\theta_1 \cos\theta_1,$$
$$\epsilon_b = \epsilon_x \cos^2\theta_2 + \epsilon_y \sin^2\theta_2 + \gamma_{xy} \sin\theta_2 \cos\theta_2,$$
$$\epsilon_c = \epsilon_x \cos^2\theta_3 + \epsilon_y \sin^2\theta_3 + \gamma_{xy} \sin\theta_3 \cos\theta_3.$$

..........(44)

These three equations can be solved to find three unknowns i.e. ε_x, ε_y, and, γ_{xy}. With the knowledge of all in-plane strain components, principal strains and their directions can also be calculated.

Simple state of stress – Axial stress



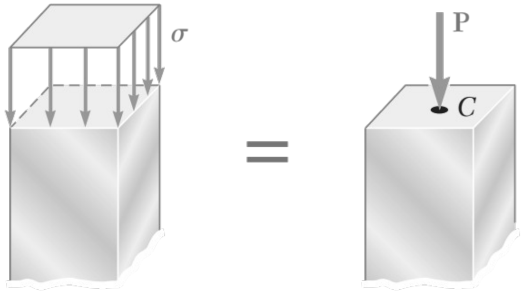
$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$

$$\int dF = \int_A \sigma dA$$

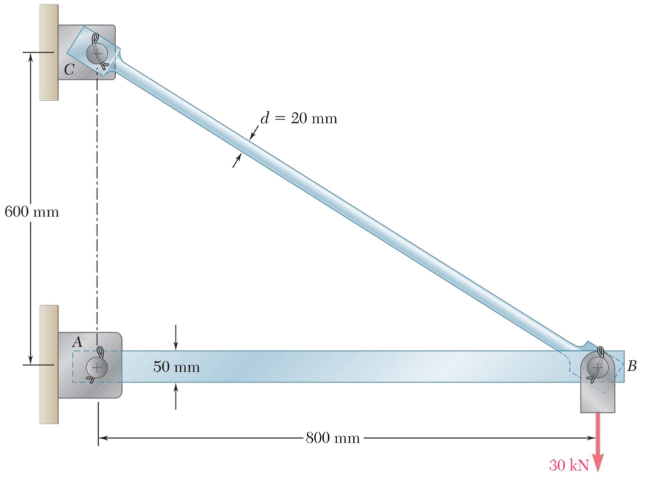
From equilibrium

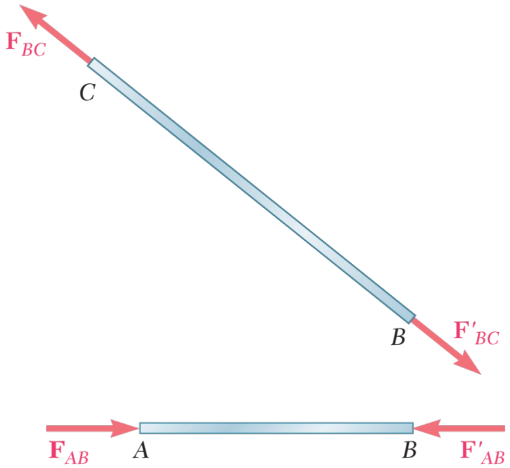
$$P = \int dF = \int_A \sigma dA$$

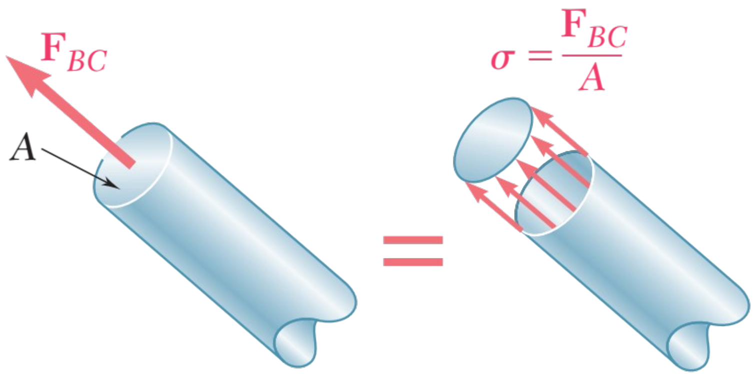
Idealization



Axial stress

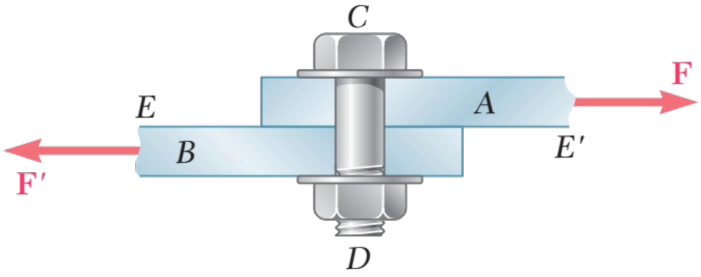


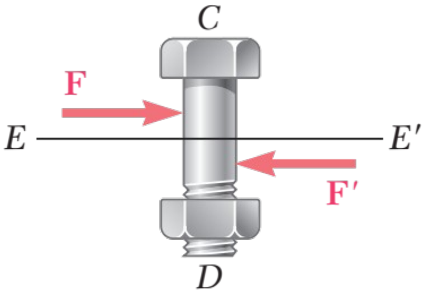


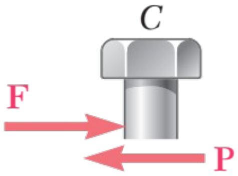


Only normal stress in axial direction is non-zero. All other stress components are zero

Simple state of stress – Shear stress







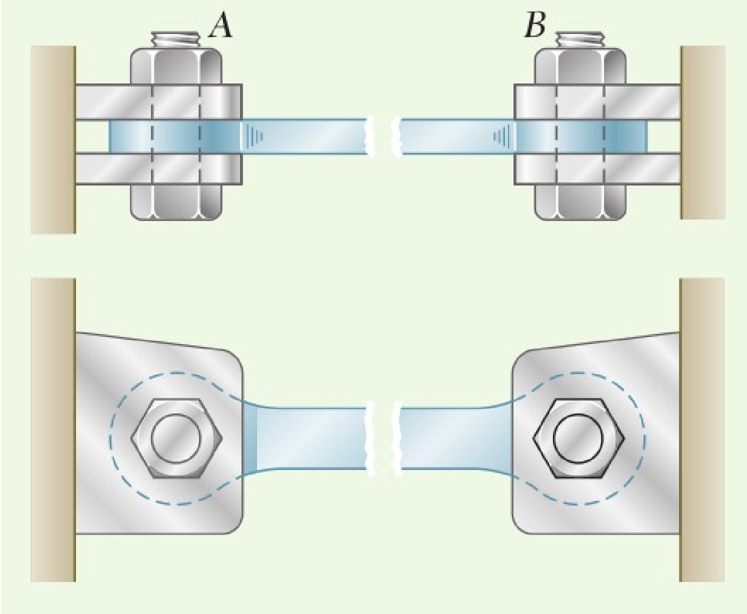
Average shear stress

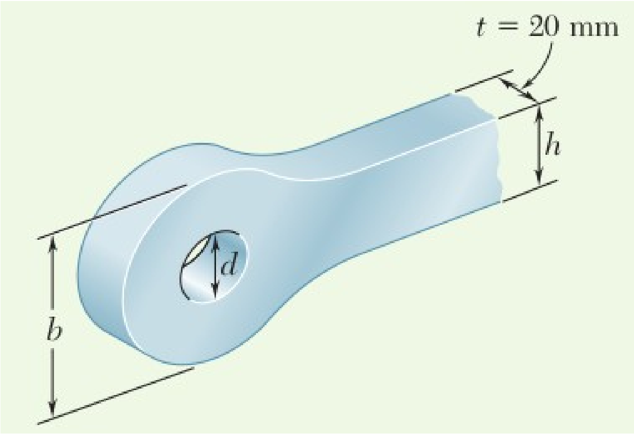
$$\tau_{\text{avg}} = \frac{P}{A} = \frac{F}{A}$$

Summarize

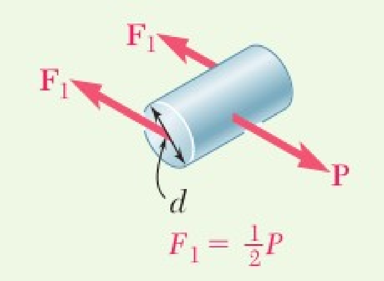
Example 1

The steel tie bar shown is to be designed to carry a tension force of magnitude $P = 120$ kN when bolted between double brackets at A and B. The bar will be fabricated from 20-mm-thick plate stock. The maximum allowable stresses are $\sigma = 175$ MPa, $\tau = 100$ MPa. Design the tie bar by determining the required values of (a) the diameter d of the bolt, (b) the dimension b at each end of the bar, and (c) the dimension h of the bar.





Diameter of the bolt:



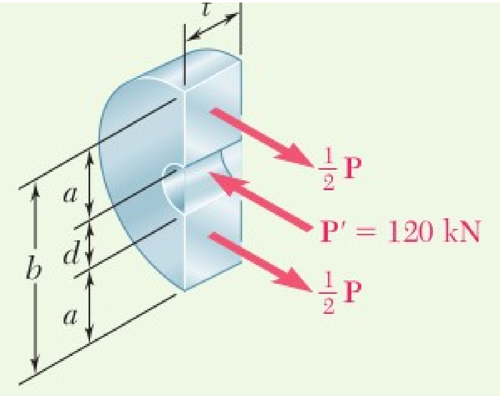
Shear stress at any of the cross section of the bolt is

$$\tau = F_1/A = P/(2A)$$

$$\tau = P/(2A) = 4P/(2\pi d^2) \leq \tau_{\text{allowable}}$$

Minimum required value of d can be obtained.

Dimension b of the bar:



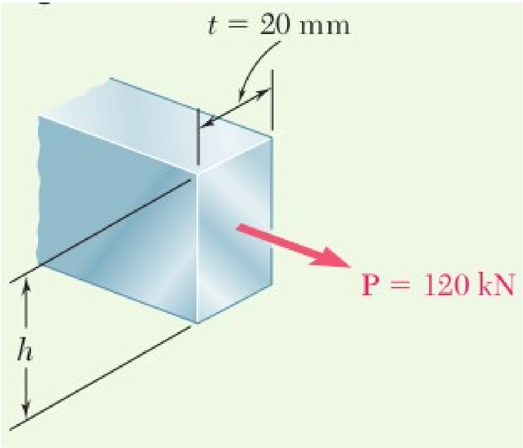
Normal stress at the cross section is

$$\sigma = \frac{P/2}{A} = \frac{P/2}{at}$$

$$\sigma = \frac{P}{2at} \leq \sigma_{\text{allowable}}$$

Thus for a given value of t, minimum required value of a can be obtained. Then $b = 2a + d$.

Dimension h of the bar:



Normal stress at the cross section is

$$\sigma = \frac{P}{A} = \frac{P}{th} \leq \sigma_{\text{allowable}},$$

which gives us the minimum value of h required.

Example 2

Two wooden planks, each 12 in. thick and 9 in. wide, are joined by the dry mortise joint shown. Knowing that the wood used shears off along its grain when the average shearing stress reaches 8 MPa, determine the magnitude P of the axial load that will cause the joint to fail.