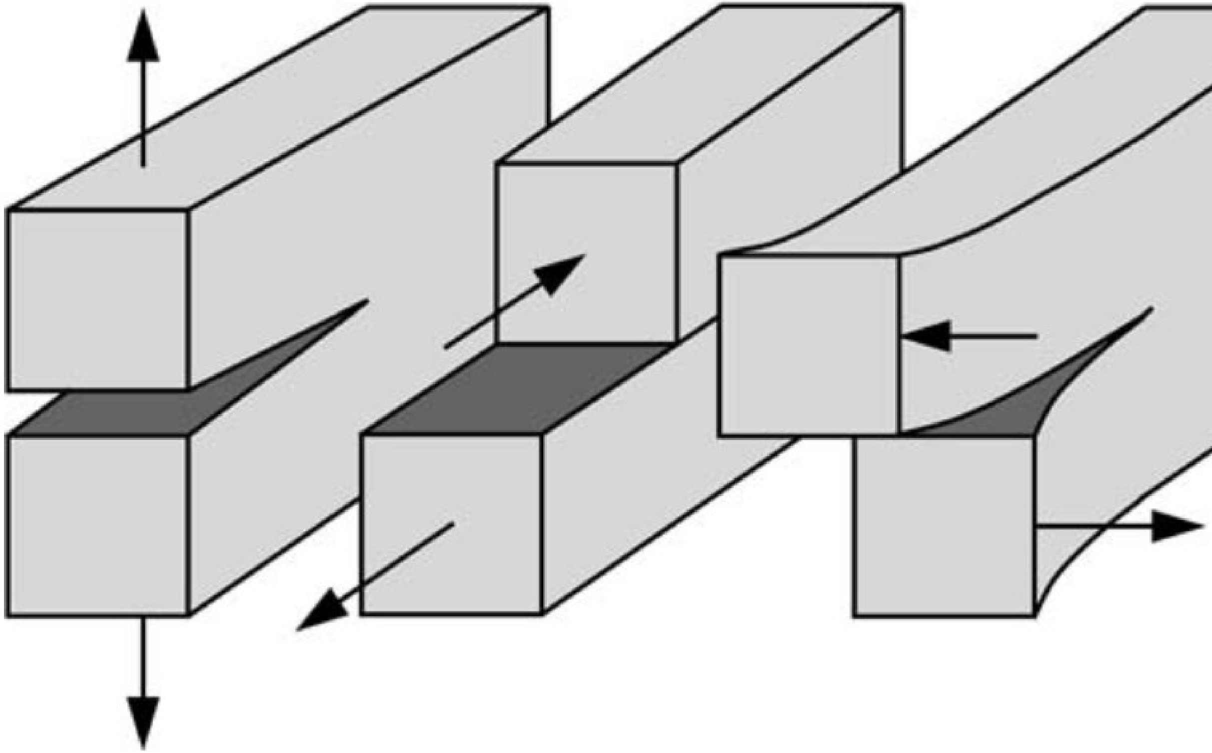
ME632: Fracture Mechanics

Timings	
Monday	10:00 to 11:20
Thursday	08:30 to 09:50

Anshul Faye
afaye@iitbhilai.ac.in
Room No. # 106

Modes of fracture



Mode I
(Opening Mode)

Mode II
(Sliding Mode)

Mode III
(Tearing Mode)

Stresses near the crack-tip

- In previous classes we studied effects of far-field loading on a crack. We considered energy or change in energy of overall system due to applied load and we could correlated them with crack growth and stability of crack growth.
- Now, we will concentrate on the stresses near the crack-tip. In the vicinity of a crack-tip stresses are very high. Knowledge of the stress or displacement field near a crack-tip may be very useful.
- This knowledge may help material scientists to develop new materials which can diffuse high stresses at the crack-tip.
- It may help designers to modify features such as notches, cutouts, keyways, etc., to minimize stresses.
- Experimentalists can devise methods of characterizing cracks by measuring stresses or strains near the crack tip.
- One of the biggest advantages is that stress analysis leads to define a parameter, stress intensity factor (SIF) to characterize a crack. In comparison to energy release rate, SIF is simpler for a designer and easier for laboratory measurements, so as to determine material properties.

Field equations of elasticity

Equilibrium equations:

For plane problems (plane stress or plane strain) in the absence of body forces equilibrium equations are,

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0,$$
$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0. \qquad \cdots\cdots\cdots\cdots(1)$$

Strain-displacement and Compatibility relations:

For plane problems there are three strain-displacement relations and only one compatibility equation.

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1},$$
$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2},$$
$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \qquad \cdots\cdots\cdots\cdots(2)$$

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0. \qquad \cdots\cdots\cdots\cdots(3)$$

Stress-strain relations:

For linear isotropic materials stress-strain relations are,

$$
\begin{aligned}
\varepsilon_{11} &= \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22} + \sigma_{33}\right)\right], \\
\varepsilon_{22} &= \frac{1}{E}\left[\sigma_{22} - \nu\left(\sigma_{11} + \sigma_{33}\right)\right], \\
\varepsilon_{33} &= \frac{1}{E}\left[\sigma_{33} - \nu\left(\sigma_{11} + \sigma_{22}\right)\right], \\
\varepsilon_{12} &= \frac{\sigma_{12}}{2\mu} = \frac{1+\nu}{E}\sigma_{12}.
\end{aligned}
\qquad \cdots\cdots\cdots\cdots(4)
$$

where E is the Young's Modulus, μ is the shear modulus and ν is the Poisson's Ratio.

Plane problems:

For linear isotropic materials stress-strain relations are,

Plane stress

$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$$

$$\begin{aligned}
\varepsilon_{11} &= \frac{1}{E}\left[\sigma_{11} - \nu\sigma_{22}\right], \\
\varepsilon_{22} &= \frac{1}{E}\left[\sigma_{22} - \nu\sigma_{11}\right], \\
\varepsilon_{12} &= \frac{\sigma_{12}}{2\mu} = \frac{1+\nu}{E}\sigma_{12}, \\
\varepsilon_{33} &= -\frac{\nu}{E}\left(\sigma_{11} + \sigma_{22}\right).
\end{aligned} \qquad \cdots\cdots\cdots\cdots(5)$$

Plane strain

$$\varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0$$

$$\begin{aligned}
\varepsilon_{11} &= \frac{1-\nu^2}{E}\left[\sigma_{11} - \frac{\nu}{1-\nu}\sigma_{22}\right] = \frac{1}{E'}\left[\sigma_{11} - \nu'\sigma_{22}\right], \\
\varepsilon_{22} &= \frac{1-\nu^2}{E}\left[\sigma_{22} - \frac{\nu}{1-\nu}\sigma_{11}\right], = \frac{1}{E'}\left[\sigma_{22} - \nu'\sigma_{11}\right], \\
\varepsilon_{12} &= \frac{\sigma_{12}}{2\mu} = \frac{1+\nu}{E}\sigma_{12} = \frac{1+\nu'}{E'}\sigma_{12}, \\
\sigma_{33} &= \nu\left(\sigma_{11} + \sigma_{22}\right).
\end{aligned} \qquad \cdots\cdots\cdots\cdots(6)$$

Observe that plane stress relations can be converted to plane strain relations by replacing E with E' and ν with ν' where

$$E' = \frac{E}{1-\nu^2} \quad \text{and} \quad \nu' = \frac{\nu}{1-\nu}. \qquad \cdots\cdots\cdots\cdots(7)$$

To solve boundary value problems in elasticity all these equation need to be considered.

We can reduce the number of equations by combining some of them. We substitute (5) into (3) and obtain following equation,

$$\frac{\partial^2}{\partial x_2^2}(\sigma_{11} - \nu\sigma_{22}) + \frac{\partial^2}{\partial x_1^2}(\sigma_{22} - \nu\sigma_{11}) - 2(1+\nu)\frac{\partial^2 \sigma_{12}}{\partial x_1 \partial x_2} = 0. \qquad \text{............(8)}$$

Equation (61) has three dependent variables. To make the solution of (61) more convenient we represent all stress components in terms of a new function Φ, which is called Airy's stress function, as follows,

$$\sigma_{11} = \frac{\partial^2 \Phi}{\partial x_2^2}, \quad \sigma_{22} = \frac{\partial^2 \Phi}{\partial x_1^2}, \quad \sigma_{12} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2}. \qquad \text{............(9)}$$

The reason to choose the particular form (9) is that it satisfy equilibrium equations (1).

Now we substitute (9) into (8) and get the following differential equation,

$$\frac{\partial^4 \Phi}{\partial x_1^2} + 2\frac{\partial^4 \Phi}{\partial^2 x_1 \partial^2 x_2} + \frac{\partial^4 \Phi}{\partial x_2^2} = 0. \qquad \text{............(10)}$$

We have now reduced the number of unknown function to one, i.e., Φ.

Equation (63) can be written is a compact form as follows,

$$\nabla^2\left(\nabla^2\Phi\right)=0 \quad \text{or} \quad \nabla^4\Phi=0, \qquad \cdots\cdots\cdots\cdots(11)$$

where $\nabla^2=\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}$ is the Laplace operator. Equation (11) is called Biharmonic equation.

We derived (11) by using stress-strain relation (5) for plane stress case; however same equation can be derived for plane strain case using (6). Functions Φ satisfying (11) are called Biharmonic equation. We will use this equation to determine the stress field near crack-tip.

Note that in Polar coordinate system,

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}. \qquad \cdots\cdots\cdots\cdots(12)$$

and

$$\sigma_{rr}=\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}, \quad \sigma_{\theta\theta}=\frac{\partial^2\Phi}{\partial r^2}, \quad \sigma_{r\theta}=-\frac{\partial\Phi}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right). \qquad \cdots\cdots\cdots\cdots(13)$$

Williams' asymptotic method

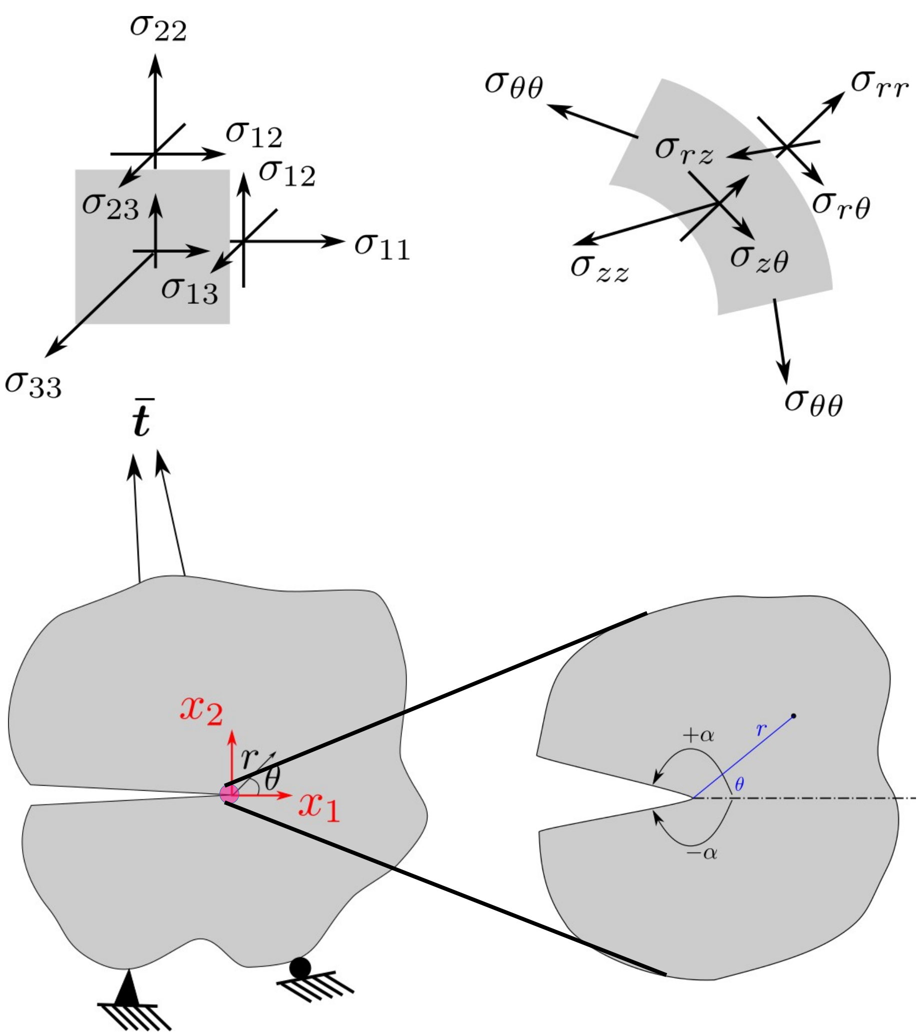


Figure shows a body with a notch. It is loaded by tractions on the remote boundaries. Williams developed a method of exploring the nature of the stress field near the crack-tip by defining a set of polar coordinates centered at the tip and expanding the stress field as an asymptotic series in powers of r.

We are concerned only with the stress components in the notch at very small values of r and hence we imagine looking at the corner through a strong microscope, so that we see the wedge. We magnify upto an extent so that the other surfaces of the body, including the loaded boundaries, appear far enough away for us and we can treat the wedge as infinite which is 'loading at infinity'.

Williams (1952) proposed the following form of Φ for crack-field solution,

$$\Phi = r^{\lambda+1}F(\theta). \qquad \cdots\cdots\cdots\cdots(14)$$

Φ in (14) must satisfy (11). So substituting (14) in (11), we get

$$\nabla^2\left(\nabla^2\Phi\right) = \nabla^2\left(\nabla^2 r^{\lambda+1}F(\theta)\right) = 0. \qquad \cdots\cdots\cdots\cdots(15)$$

Now using (12)

$$\begin{aligned}
&\nabla^2\left(r^{\lambda+1}F(\theta)\right)\\
\Rightarrow&\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(r^{\lambda+1}F(\theta)\right)\\
\Rightarrow&\lambda(\lambda+1)r^{\lambda-1}F(\theta) + (\lambda+1)r^{\lambda-1}F(\theta) + r^{\lambda-1}F''(\theta)\\
\Rightarrow&\left[\frac{\partial^2}{\partial\theta^2} + (\lambda+1)^2\right]F(\theta)r^{\lambda-1}. \qquad \cdots\cdots\cdots\cdots(17)
\end{aligned}$$

Similarly using (17) it can be shown that

$$\nabla^4\Phi = \left[\frac{\partial^2}{\partial\theta^2} + (\lambda+1)^2\right]\left[\frac{\partial^2}{\partial\theta^2} + (\lambda-1)^2\right]F(\theta)r^{\lambda-3} = 0 \qquad \cdots\cdots\cdots\cdots(18)$$

Equation (18) can be satisfied if,

$$\left[\frac{\partial^2}{\partial\theta^2}+(\lambda-1)^2\right]F(\theta)r^{\lambda-3}=0, \Rightarrow F(\theta)=c_1\cos\left[(\lambda-1)\theta\right]+c_2\sin\left[(\lambda-1)\theta\right]. \qquad \cdots\cdots\cdots\cdots(19)$$

or,

$$\left[\frac{\partial^2}{\partial\theta^2}+(\lambda+1)^2\right]F(\theta)r^{\lambda-3}=0, \Rightarrow F(\theta)=c_3\cos\left[(\lambda+1)\theta\right]+c_4\sin\left[(\lambda+1)\theta\right]. \qquad \cdots\cdots\cdots\cdots(20)$$

Thus, the general solution can now be written as,

$$F(\theta)=c_1\cos\left[(\lambda-1)\theta\right]+c_2\sin\left[(\lambda-1)\theta\right]+c_3\cos\left[(\lambda+1)\theta\right]+c_4\sin\left[(\lambda+1)\theta\right]. \qquad \cdots\cdots\cdots\cdots(21)$$

Here c_1, c_2, c_3 and c_4 are constant which can be determined by applying boundary conditions. Stress components are

Also,

$$\begin{aligned}F'(\theta)=&-c_1(\lambda-1)\sin\left[(\lambda-1)\theta\right]+c_2(\lambda-1)\cos\left[(\lambda-1)\theta\right]\\&-c_3(\lambda+1)\sin\left[(\lambda+1)\theta\right]+c_4(\lambda+1)\cos\left[(\lambda+1)\theta\right].\end{aligned} \qquad \cdots\cdots\cdots\cdots(22)$$

$$\begin{aligned}F''(\theta)=&-c_1(\lambda-1)^2\cos\left[(\lambda-1)\theta\right]-c_2(\lambda-1)^2\sin\left[(\lambda-1)\theta\right]\\&-c_3(\lambda+1)^2\cos\left[(\lambda+1)\theta\right]-c_4(\lambda+1)^2\sin\left[(\lambda+1)\theta\right].\end{aligned} \qquad \cdots\cdots\cdots\cdots(23)$$

Stress components are

$$\sigma_{rr} = r^{\lambda-1}\left[(\lambda+1)F(\theta) + F''(\theta)\right],$$

$$\sigma_{\theta\theta} = \lambda(\lambda+1)r^{\lambda-1}F(\theta), \qquad \cdots\cdots\cdots\cdots(24)$$

$$\sigma_{r\theta} = -\lambda r^{\lambda-1}F'(\theta).$$

We apply boundary conditions that crack faces are traction free, i.e.,

$$\sigma_{\theta\theta} = \sigma r\theta = 0 \text{ at } \theta = \pm\alpha. \qquad \cdots\cdots\cdots\cdots(25)$$

Using (22)-(24) we get,

$$\sigma_{\theta\theta}|_{\theta=\alpha} = \lambda(\lambda+1)r^{\lambda-1}\left[c_1\cos(\lambda-1)\alpha + c_2\sin(\lambda-1)\alpha + c_3\cos(\lambda+1)\alpha + c_4\sin(\lambda+1)\alpha\right] = 0$$

$$\sigma_{\theta\theta}|_{\theta=-\alpha} = \lambda(\lambda+1)r^{\lambda-1}\left[c_1\cos(\lambda-1)\alpha - c_2\sin(\lambda-1)\alpha + c_3\cos(\lambda+1)\alpha - c_4\sin(\lambda+1)\alpha\right] = 0$$

$$\begin{aligned}\sigma_{r\theta}|_{\theta=\alpha} = -\lambda r^{\lambda-1}[&-c_1(\lambda-1)\sin(\lambda-1)\alpha + c_2(\lambda-1)\cos(\lambda-1)\alpha\\ &-c_3(\lambda+1)\sin(\lambda+1)\alpha + c_4(\lambda+1)\cos(\lambda+1)\alpha] = 0\end{aligned}$$

$$\begin{aligned}\sigma_{r\theta}|_{\theta=-\alpha} = -\lambda r^{\lambda-1}[&c_1(\lambda-1)\sin(\lambda-1)\alpha + c_2(\lambda-1)\cos(\lambda-1)\alpha\\ &+c_3(\lambda+1)\sin(\lambda+1)\alpha + c_4(\lambda+1)\cos(\lambda+1)\alpha] = 0\end{aligned} \qquad \cdots\cdots\cdots\cdots(26)$$

(26) is a set of 4 homogeneous equations, which will have non-trivial solution only for few values of λ. λ=0 is one solution as it is common in all four equation. To determine other values of λ we first simplify these equations by adding and subtraction first two equations and last two equations and get

$$
\begin{aligned}
(\lambda+1)\left[c_1\cos(\lambda-1)\alpha + c_3\cos(\lambda+1)\alpha\right] &= 0\\
(\lambda+1)\left[c_2\sin(\lambda-1)\alpha + c_4\sin(\lambda+1)\alpha\right] &= 0\\
c_2(\lambda-1)\cos(\lambda-1)\alpha + c_4(\lambda+1)\cos(\lambda+1)\alpha &= 0\\
c_1(\lambda-1)\sin(\lambda-1)\alpha + c_3(\lambda+1)\sin(\lambda+1)\alpha &= 0
\end{aligned}
\qquad \cdots\cdots\cdots\cdots(27)
$$

This procedure gave us two independent matrix equations as follows,

$$
\begin{bmatrix}(\lambda+1)\cos(\lambda-1)\alpha & (\lambda+1)\cos(\lambda+1)\alpha\\ (\lambda-1)\sin(\lambda-1)\alpha & (\lambda+1)\sin(\lambda+1)\alpha\end{bmatrix}\begin{Bmatrix}c_1\\ c_3\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix} \qquad \cdots\cdots\cdots\cdots(28)
$$

$$
\begin{bmatrix}(\lambda+1)\sin(\lambda-1)\alpha & (\lambda+1)\sin(\lambda+1)\alpha\\ (\lambda-1)\cos(\lambda-1)\alpha & (\lambda+1)\cos(\lambda+1)\alpha\end{bmatrix}\begin{Bmatrix}c_2\\ c_4\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix} \qquad \cdots\cdots\cdots\cdots(29)
$$

For non-trivial solutions of c_1 and c_3, determinant of coefficient matrix of (28) must be zero, which gives characteristic equation as,

$$\lambda \sin 2\alpha + \sin 2\lambda\alpha = 0. \qquad \cdots\cdots\cdots\cdots(30)$$

Similar for non-trivial solutions of c_2 and c_4, determinant of coefficient matrix of (29) must be zero and the characteristic equation is,

$$\lambda \sin 2\alpha - \sin 2\lambda\alpha = 0. \qquad \cdots\cdots\cdots\cdots(31)$$

From (30) and (31) we conclude that,

$$\sin 2\lambda\alpha = 0, \quad \text{and} \quad \lambda \sin 2\alpha = 0. \qquad \cdots\cdots\cdots\cdots(32)$$

Now we consider the case of sharp crack, which means $\alpha = \pi$. In this case second equation of (32) is trivially satisfied and the first equation gives,

$$\sin 2\lambda\pi = 0 = \sin n\pi$$

$$\Rightarrow \lambda = n/2, \quad \text{where} \quad n = 0, \pm 1, \pm 2, \pm 3 \ldots \qquad \cdots\cdots\cdots\cdots(33)$$

One should realize that the displacements near the crack-tip will be proportional to r^{λ}. Then all negative values of λ lead to infinite displacements at the crack-tip, which is not physical. Hence negative values of λ are not allowed.

λ=0 gives displacement field which is independent of distance from the crack-tip, hence not allowed (because this is also not physical).

So now for other possible values of λ, we determine the values of constants c_1, c_2, c_3 and c_4 using (27).

$$c_{1n}\cos\left(\frac{n}{2}-1\right)\pi + c_{3n}\cos\left(\frac{n}{2}+1\right)\pi = 0$$

$$c_{2n}\sin\left(\frac{n}{2}-1\right)\pi + c_{4n}\sin\left(\frac{n}{2}+1\right)\pi = 0$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdots\cdots\cdots\cdots(34)$$

$$c_{2n}\left(\frac{n}{2}-1\right)\cos\left(\frac{n}{2}-1\right)\pi + c_{4n}\left(\frac{n}{2}+1\right)\cos\left(\frac{n}{2}+1\right)\pi = 0$$

$$c_{1n}\left(\frac{n}{2}-1\right)\sin\left(\frac{n}{2}-1\right)\pi + c_{3n}\left(\frac{n}{2}+1\right)\sin\left(\frac{n}{2}+1\right)\pi = 0$$

For $n = 1, 3, 5,\ldots$ second and fourth equation of (34) gives,

$$c_{4n} = -c_{2n},$$

and

$$c_{3n} = -c_{1n}\frac{n-2}{n+2}. \qquad \cdots\cdots\cdots\cdots(35)$$

For $n = 2, 4, 6,...$ first and third equation of (34) gives,

$$c_{3n} = -c_{1n},$$

and ....(36)

$$c_{4n} = -c_{2n}\frac{n-2}{n+2}.$$

So finally stress components are,

$$\sigma_{rr} = \sum_n r^{n/2-1}\left[\left\{\left(\frac{n}{2}+1\right)-\left(\frac{n}{2}-1\right)^2\right\}\left\{c_{1n}\cos\left(\frac{n}{2}-1\right)\theta + c_{2n}\sin\left(\frac{n}{2}-1\right)\theta\right\}+\right.$$
$$\left.\left\{\left(\frac{n}{2}+1\right)-\left(\frac{n}{2}+1\right)^2\right\}\left\{c_{3n}\cos\left(\frac{n}{2}+1\right)\theta + c_{4n}\sin\left(\frac{n}{2}+1\right)\theta\right\}\right]$$

............(37)

$$
\begin{aligned}
\sigma_{r\theta} = & \sum_{n=1,3,5,\ldots} \frac{n}{2}\left(\frac{n}{2}-1\right) r^{n/2-1} \left[c_{1n}\left(\frac{n}{2}-1\right)\left\{ \sin\left(\frac{n}{2}-1\right)\theta - \frac{n-2}{n+2}\left(\frac{n}{2}+1\right)\sin\left(\frac{n}{2}+1\right)\theta \right\} + \right. \\
& \left. c_{2n}\left\{ -\left(\frac{n}{2}-1\right)\cos\left(\frac{n}{2}-1\right)\theta - \left(\frac{n}{2}+1\right)\cos\left(\frac{n}{2}+1\right)\theta \right\} \right] + \\
& \sum_{n=2,4,6,\ldots} \frac{n}{2} r^{n/2-1} \left[c_{1n}\left(\frac{n}{2}-1\right)\left\{ \sin\left(\frac{n}{2}-1\right)\theta - \left(\frac{n}{2}+1\right)\sin\left(\frac{n}{2}+1\right)\theta \right\} + \right. \\
& \left. c_{2n}\left\{ \left(\frac{n}{2}-1\right)\cos\left(\frac{n}{2}-1\right)\theta + \frac{n-2}{n+2}\left(\frac{n}{2}+1\right)\sin\left(\frac{n}{2}+1\right)\theta \right\} \right] \qquad \cdots\cdots\cdots\cdots(38)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{\theta\theta} = & \sum_{n=1,3,5,\ldots} \frac{n}{2}\left(\frac{n}{2}+1\right) r^{n/2-1} \left[c_{1n}\left\{ \cos\left(\frac{n}{2}-1\right)\theta - \frac{n-2}{n+2}\cos\left(\frac{n}{2}+1\right)\theta \right\} + \right. \\
& \left. c_{2n}\left\{ \sin\left(\frac{n}{2}-1\right)\theta - \sin\left(\frac{n}{2}+1\right)\theta \right\} \right] + \\
& \sum_{n=2,4,6,\ldots} \frac{n}{2}\left(\frac{n}{2}-1\right) r^{n/2-1} \left[c_{1n}\left\{ \cos\left(\frac{n}{2}-1\right)\theta - \cos\left(\frac{n}{2}+1\right)\theta \right\} + \right. \\
& \left. c_{2n}\left\{ \sin\left(\frac{n}{2}-1\right)\theta - \frac{n-2}{n+2}\sin\left(\frac{n}{2}+1\right)\theta \right\} \right] \qquad \cdots\cdots\cdots\cdots(39)
\end{aligned}
$$

Note that in all the stress components dominant term corresponds to n=1. Thus,

$$\sigma_{rr} = \frac{1}{\sqrt{r}}\left[c_{11}\cos\frac{\theta}{2}\left(2-\cos^2\frac{\theta}{2}\right)+c_{21}\sin\frac{\theta}{2}\left(2-3\sin^2\frac{\theta}{2}\right)\right] \qquad \cdots\cdots\cdots\cdots(40)$$

$$\sigma_{\theta\theta} = \frac{3}{4\sqrt{r}}\left[c_{11}\left(\cos\frac{\theta}{2}+\frac{1}{3}\cos\frac{3\theta}{2}\right)-c_{21}\left(\sin\frac{\theta}{2}+\sin\frac{3\theta}{2}\right)\right] \qquad \cdots\cdots\cdots\cdots(41)$$

$$\sigma_{r\theta} = \frac{1}{4\sqrt{r}}\left[c_{11}\left(\sin\frac{\theta}{2}+\sin\frac{3\theta}{2}\right)+c_{21}\left(\cos\frac{\theta}{2}+3\cos\frac{3\theta}{2}\right)\right] \qquad \cdots\cdots\cdots\cdots(42)$$

Now we define,

$$(\sigma_{rr})_{\text{symm}} = \frac{K_I}{\sqrt{2\pi r}}\left[\cos\frac{\theta}{2}\left(2-\cos^2\frac{\theta}{2}\right)\right], \qquad (\sigma_{rr})_{\text{asymm}} = \frac{K_{II}}{\sqrt{2\pi r}}\left[\sin\frac{\theta}{2}\left(2-3\sin^2\frac{\theta}{2}\right)\right], \quad \cdots\cdot(43)$$

$$(\sigma_{\theta\theta})_{\text{symm}} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{3}{4}\left(\cos\frac{\theta}{2}+\frac{1}{3}\cos\frac{3\theta}{2}\right)\right], (\sigma_{\theta\theta})_{\text{asymm}} = -\frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{3}{4}\left(\sin\frac{\theta}{2}+\sin\frac{3\theta}{2}\right)\right], \quad \cdots\cdot(44)$$

$$(\sigma_{r\theta})_{\text{symm}} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{1}{4}\left(\sin\frac{\theta}{2}+\sin\frac{3\theta}{2}\right)\right], \quad (\sigma_{r\theta})_{asymm} = \frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{1}{4}\left(\cos\frac{\theta}{2}+3\cos\frac{3\theta}{2}\right)\right], \quad \cdots\cdot(45)$$

where, $K_I = c_{11}\sqrt{2\pi}$ and $K_{II} = c_{21}\sqrt{2\pi}$ are mode-I and mode-II stress intensity factors.

Expressions for strains near the crack-tip can be obtained using (5) (for plane σ) and using (6) (for plane ε) as follows.

(for plane σ)

$$\varepsilon_{rr} = \frac{1}{E}\left[\sigma_{rr} - \nu\sigma_{\theta\theta}\right],$$
$$\varepsilon_{\theta\theta} = \frac{1}{E}\left[\sigma_{\theta\theta} - \nu\sigma_{rr}\right],$$
$$\varepsilon_{r\theta} = \frac{\sigma_{r\theta}}{2\mu} = \frac{1+\nu}{E}\sigma_{r\theta},$$
$$\varepsilon_{zz} = -\frac{\nu}{E}\left(\sigma_{rr} + \sigma_{\theta\theta}\right).$$

(for plane ε)

$$\varepsilon_{rr} = \frac{1}{E'}\left[\sigma_{rr} - \nu'\sigma_{\theta\theta}\right],$$
$$\varepsilon_{\theta\theta} = \frac{1-\nu^2}{E}\left[\sigma_{\theta\theta} - \frac{\nu}{1-\nu}\sigma_{rr}\right], = \frac{1}{E'}\left[\sigma_{\theta\theta} - \nu'\sigma_{rr}\right],$$
$$\varepsilon_{r\theta} = \frac{\sigma_{r\theta}}{2\mu} = \frac{1+\nu}{E}\sigma_{r\theta} = \frac{1+\nu'}{E'}\sigma_{r\theta},$$
$$\varepsilon_{zz} = 0.$$

Crack-tip displacements (u_r, u_θ) follows from strain using strain-displacement relations (2).

$$u_r = \frac{K_I\sqrt{r}}{4\mu\sqrt{2\pi}}\left[(2\kappa-1)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right], \quad u_r = \frac{K_{II}\sqrt{r}}{4\mu\sqrt{2\pi}}\left[-(2\kappa-1)\sin\frac{\theta}{2} + 3\sin\frac{3\theta}{2}\right], \qquad \cdots\cdots(47)$$

$$u_\theta = \frac{K_I\sqrt{r}}{4E\sqrt{2\pi}}\left[(2\kappa+1)\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right]. \quad u_\theta = \frac{K_{II}\sqrt{r}}{4E\sqrt{2\pi}}\left[-(2\kappa+1)\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}\right]. \qquad \cdots\cdots(48)$$

Here, $\kappa = 3 - 4\nu$ for plane ε and $\kappa = (3-\nu)/(1+\nu)$ for plane σ.

Solution in Cartesian Coordinate system

	(for Mode I)	(for Mode II)
$\sigma_{ij} = \frac{K_{I/II}}{\sqrt{2\pi r}}\hat{\sigma}_{ij}(\theta)$	$\hat{\sigma}_{xx} = \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$ $\hat{\sigma}_{yy} = \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$ $\hat{\sigma}_{xy} = \frac{1}{2}\sin\theta\cos\frac{3\theta}{2}$ ....(49)	$\hat{\sigma}_{xx} = \sin\frac{\theta}{2}\left(-2 - \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$ $\hat{\sigma}_{yy} = \frac{1}{2}\sin\theta\cos\frac{3\theta}{2}$ $\hat{\sigma}_{xy} = \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$ ....(50)
$u_i = \frac{K_{I/II}\sqrt{r}}{2\mu\sqrt{2\pi}}\hat{u}_i(\theta)$	$\hat{u}_x = \left[(\kappa - 1)\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2}\right]$ $\hat{u}_y = \left[(\kappa + 1)\sin\frac{\theta}{2} - \sin\theta\cos\frac{\theta}{2}\right]$ ....(51)	$\hat{u}_x = \sin\frac{\theta}{2}\left[\kappa + 1 + 2\cos^2\frac{\theta}{2}\right]$ $\hat{u}_y = \cos\frac{\theta}{2}\left[\kappa - 1 + 2\sin^2\frac{\theta}{2}\right]$ ....(52)

Antiplane strain

In addition to plane stress and plane strain case there is another class of plane problems which is called Antiplane strain. In this case there are only out-of-plane deformations exists. Hence the assumed displacement fields are,

$$u_1 = u_2 = 0 \text{ and } u_3 = u_3(x_1, x_2). \qquad \cdots\cdots\cdots\cdots\cdots(53)$$

Thus, strain components are $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{12} = \varepsilon_{33} = 0$, and $\varepsilon_{13} = \dfrac{1}{2}\dfrac{\partial u_3}{\partial x_1}, \varepsilon_{23} = \dfrac{1}{2}\dfrac{\partial u_3}{\partial x_2}$. $\cdots\cdots(54)$

Accordingly stress components are $\sigma_{11} = \sigma_{22} = \sigma_{12} = \sigma_{33} = 0$, and $\sigma_{13} = 2\mu\varepsilon_{13}, \sigma_{23} = 2\mu\varepsilon_{23}$.

$\cdots\cdots(55)$

It must be observed that in Antiplane strain case the only equilibrium equation (in the absence of body force) which need to be satisfied is,

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} = 0. \qquad \cdots\cdots(56)$$

Using (54) and (55), equation (56) can be written in terms of u_3 as,

$$\nabla^2 u_3 = 0, \qquad \cdots\cdots\cdots\cdots\cdots(57)$$

which is nothing but the Navier's equation for Antiplane strain case.

Mode-III crack

Mode-III crack problem is solved as an Antiplane strain problem. We assume the following form of the solution for out-of-plane displacement,

$$u_z = r^\lambda F(\theta). \qquad \cdots\cdots\cdots\cdots\cdots(58)$$

Substituting (58) in (57) (remember to use the polar coordinate form of ∇^2) we get,

$$\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} = 0. \qquad \cdots\cdots\cdots\cdots\cdots(59)$$

Boundary conditions are

$$\text{at } \theta = \pm\pi,\ \sigma_{\theta z} = \mu\frac{\partial u_z}{\partial \theta} = 0. \qquad \cdots\cdots\cdots\cdots\cdots(60)$$

Substituting (58) in (60), we obtain the following characteristic equation for nontrivial solution

$$\lambda^2 F(\theta) + F''(\theta) = 0. \qquad \cdots\cdots\cdots\cdots\cdots(61)$$

General solution of (61) is

$$F(\theta) = A\cos(\lambda\theta) + B\sin(\lambda\theta). \qquad \cdots\cdots\cdots\cdots\cdots(62)$$

Hence, the displacement field is

$$u_z = r^\lambda\left[A\cos(\lambda\theta) + B\sin(\lambda\theta)\right]. \qquad \cdots\cdots\cdots\cdots\cdots(63)$$

Now, note that λ=0 gives u_z=A+B (for small).

The constant terms correspond to rigid body motion, which we do not consider as it does not lead to any stress/strain. Next term B leads to strain components

$$\varepsilon_{\theta z} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} = \frac{B}{r}.$$

This strain leads to an infinite strain energy as $r \to 0$, which is non-physical, hence λ=0 is not allowed.

Negative values of λ are not allowed, since they lead to non-phyiscal infinite displacement at the crack-tip.

Hence, to determine other values of λ we use boundary conditions (60) to obtain the following equations,

$$\begin{aligned} r^{-\lambda}\left[-\lambda A \sin(\lambda\pi) + \lambda B \cos(\lambda\pi)\right] &= 0, \\ r^{-\lambda}\left[-\lambda A \sin(-\lambda\pi) + \lambda B \cos(-\lambda\pi)\right] &= 0. \end{aligned} \qquad \text{...............(64)}$$

For non-trivial solutions of

$$\begin{vmatrix} -\lambda \sin(\lambda\pi) & \lambda \cos(\lambda\pi) \\ -\lambda \sin(-\lambda\pi) & \lambda \cos(-\lambda\pi) \end{vmatrix} = 0$$

$$\Rightarrow \sin(2\lambda\pi) = 0 \Rightarrow \lambda = n/2, \quad \text{where } n = 1, 2, 3, \ldots$$

From (64),

$$-A\sin\frac{n\pi}{2}+B\cos\frac{n\pi}{2}=0,$$
$$-A\sin\frac{-n\pi}{2}+B\cos\frac{-n\pi}{2}=0. \qquad \cdots\cdots\cdots\cdots\cdots(65)$$

which leads to,

$$A{=}0 \text{ for } n{=}1,3,5,\ldots \qquad \text{and} \qquad B{=}0 \text{ for } n{=}2,4,6,\ldots. \qquad \cdots\cdots\cdots\cdots\cdots(66)$$

Thus, the displacement field is

$$u_z(r,\theta)=\sum_{n=1,3,5,\ldots} r^{n/2}B_n\sin\frac{n\theta}{2}+\sum_{n=2,4,6,\ldots} r^{n/2}A_n\cos\frac{n\theta}{2}. \qquad \cdots\cdots\cdots\cdots\cdots(67)$$

Non-zero stress components are

$$\sigma_{rz}=\sum_{n=1,3,5,\ldots}\mu\frac{n}{2}r^{n/2-1}B_n\sin\frac{n\theta}{2}+\sum_{n=2,4,6,\ldots}\mu\frac{n}{2}r^{n/2-1}A_n\cos\frac{n\theta}{2}, \qquad \cdots\cdots\cdots\cdots\cdots(68)$$

$$\sigma_{\theta z}=\sum_{n=1,3,5,\ldots}\mu\frac{n}{2}r^{n/2-1}B_n\cos\frac{n\theta}{2}+\sum_{n=2,4,6,\ldots}-\mu\frac{n}{2}r^{n/2-1}A_n\sin\frac{n\theta}{2}. \qquad \cdots\cdots\cdots\cdots\cdots(69)$$

Similar to mode-I and mode-II cases the dominant term correspond to n=1, which are

$$\sigma_{rz} = \frac{\mu}{2\sqrt{r}} B_1 \sin\frac{\theta}{2} = \frac{K_{III}}{\sqrt{2\pi r}} \sin\frac{\theta}{2},$$

$$\sigma_{\theta z} = \frac{\mu}{2\sqrt{r}} B_1 \cos\frac{\theta}{2} = \frac{K_{III}}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \qquad \text{...............(70)}$$

where $K_{III} = B_1 \mu \sqrt{\frac{\pi}{2}}$, is mode-III stress intensity factor,

and,

$$u_z = \sqrt{\frac{2}{\pi}} \frac{K_{III}}{\mu} \sqrt{r} \sin\frac{\theta}{2}. \qquad \text{...............(71)}$$

Westergaard's approach

There is another method to solve the stress and displacement field at the crack-tip. Westergaard (1939) gave a solution using complex variable approach. Let us first look into some of the basics of complex variable theory. An advantage complex variable offer is that it reduces the number of variable from two to one.

A complex variable is given as

$$z = x_1 + ix_2. \qquad \cdots\cdots\cdots\cdots\cdots(72)$$

In polar coordinate complex variable is $z = re^{i\theta}$.

The complex conjugate $\overline{z}$ of the variable z is $\overline{z} = x_1 - ix_2 = re^{i\theta}$. $\cdots\cdots\cdots\cdots\cdots(73)$

Using the definition of complex variable following differential operators can be defined.

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial z}\frac{\partial z}{\partial x_1} + \frac{\partial}{\partial \overline{z}}\frac{\partial \overline{z}}{\partial x_1} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}, \qquad \frac{\partial}{\partial z} = \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial z} + \frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right),$$

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial z}\frac{\partial z}{\partial x_2} + \frac{\partial}{\partial \overline{z}}\frac{\partial \overline{z}}{\partial x_2} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\right). \qquad \frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial \overline{z}} + \frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right).$$

$\cdots\cdots\cdots\cdots\cdots(74)$

A function of complex function variables may be written as

$$f(z) = f(x_1 + ix_2) = u(x_1, x_2) + iv(x_1, x_2), \qquad \cdots\cdots\cdots\cdots\cdots(75)$$

where $u(x_1, x_2) = \mathrm{Re}(f)$ and $v(x_1, x_2) = \mathrm{Im}(f)$.

A complex conjugate function is defined as,

$$\overline{f(z)} = \bar{f}(\bar{z}) = u(x_1, x_2) - iv(x_1, x_2), \qquad \cdots\cdots\cdots\cdots\cdots(76)$$

For e.g.,

$$\begin{aligned} &f(z) = az + bz^2 \\ \Rightarrow &a(x_1 + ix_2) + b(x_1 + ix_2)^2 \\ \Rightarrow &(ax_1 + bx_1^2 - bx_2^2) + i(ax_2 + 2bx_1x_2). \end{aligned}$$

Here,

$$\begin{aligned} u(x_1, x_2) &= (ax_1 + bx_1^2 - bx_2^2), \\ v(x_1, x_2) &= (ax_2 + 2bx_1x_2). \end{aligned}$$

Thus, $f(z) = u(x_1, x_2) + iv(x_1, x_2)$.

$$\begin{aligned} &\overline{f(z)} = \bar{f}(\bar{z}) = a\bar{z} + b\bar{z}^2 \\ \Rightarrow &a(x_1 - ix_2) + b(x_1 - ix_2)^2 \\ \Rightarrow &(ax_1 + bx_1^2 - bx_2^2) - i(ax_2 + 2bx_1x_2). \end{aligned}$$

Thus,

$$\overline{f(z)} = u(x_1, x_2) - iv(x_1, x_2).$$

Differentiation of complex function are

$$f'(z) = \frac{\partial}{\partial z}(u + iv) = \frac{1}{2}\left(\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial x_2}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x_1} - \frac{\partial u}{\partial x_2}\right). \qquad \cdots\cdots\cdots\cdots\cdots(77)$$

Following the basic definition of differentiation, it can be shown that Cauchy-Riemann equations for analyticity of function f is

$$\frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_2}, \quad \frac{\partial u}{\partial x_2} = -\frac{\partial v}{\partial x_1}, \quad \frac{\partial}{\partial x_1}\mathrm{Re}\, f = \frac{\partial}{\partial x_2}\,\mathrm{Im}\, f, \quad \frac{\partial}{\partial x_2}\mathrm{Re}\, f = -\frac{\partial}{\partial x_1}\,\mathrm{Im}\, f, \qquad \cdots\cdots\cdots\cdots\cdots(78)$$

By simple differentiation of (78) we can show that,

$$\nabla^2 u = 0, \nabla^2 v = 0, \qquad \cdots\cdots\cdots\cdots\cdots(79)$$

Thus real and imaginary part of an analytic complex function must be a solution of Laplace's equation and hence they are harmonic functions.

We have already seen that solution of boundary value problems in elasticity can be obtained in the form of Airy stress function Φ and it must satisfy the biharmonic equation.

One way to solve the biharmonic equation is to represent Φ in terms of another complex functions. Westergaard suggested function Φ for mode-I and mode-II problems.