ME232: Dynamics

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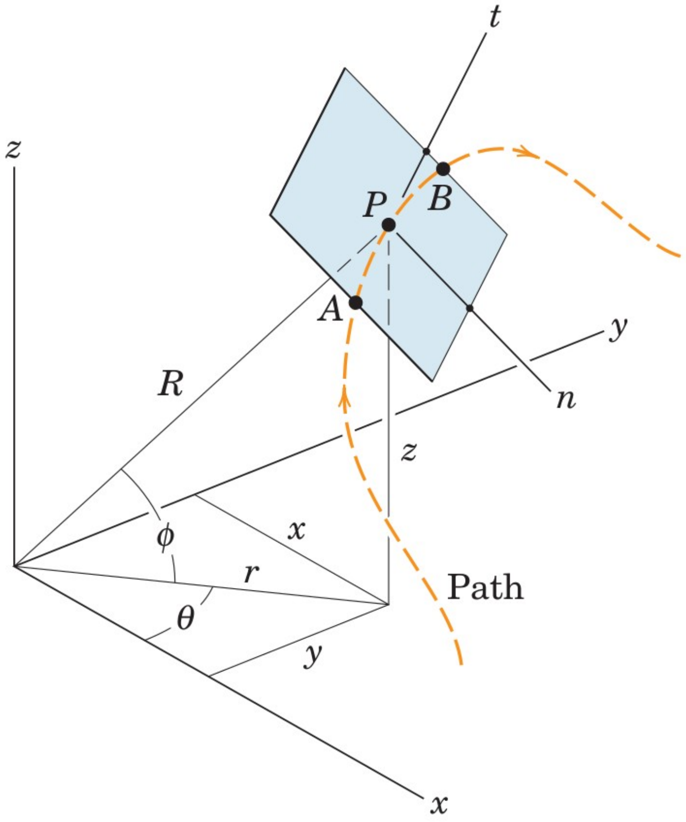
Room # 106

Mechanics

- *Statics:* Deals with the effects of forces on bodies at rest
- *Dynamics:* Deals with the motion of bodies under the action of forces
    - *Kinematics:* The study of motion without reference to the forces causing motion
    - *Kinetics:* Relates the action of forces on bodies to their resulting motions
- Depending upon the application a body under study can be modeled as
    - A particle with the whole mass concentrated at the center of mass
    - A system of particles with mass concentrated at more than one point
    - A body with its actual shape or volume with distributed mass

Kinematics of particles



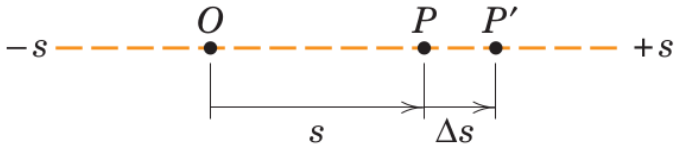
Types of motion:

- Constrained motion
- Unconstrained motion

Types of coordinate system:

- Rectangular coordinates (x–y–z)
- Polar coordinates (r–θ–z)
- Spherical coordinates (R–θ–ϕ)
- Normal-tangent coordinates (t–n)

Rectilinear motion



The average velocity of the particle during the interval $\Delta t = v_{\text{av}} = \Delta s/\Delta t$.

For $\Delta t \to 0$ the instantaneous velocity of the particle

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} = \dot{s}. \qquad \cdots\cdots\cdots\cdots(1)$$

The average acceleration of the particle during the interval $\Delta t = a_{\text{av}} = \Delta v/\Delta t$.

For $\Delta t \to 0$ the instantaneous acceleration of the particle

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \dot{v}, \quad \text{or} \quad a = \frac{d^2 s}{dt^2} = \ddot{s}. \qquad \cdots\cdots\cdots\cdots(2)$$

From (1) and (2) we can derive

$$v\,dv = a\,ds \quad \text{or} \quad \dot{s}\,d\dot{s} = \ddot{s}\,ds. \qquad \cdots\cdots\cdots\cdots(3)$$

Application of analytical expressions:

- The position coordinate (s) is known for all values of the time (t):
  - Successive differentiation w.r.t. t gives the velocity (v) and acceleration (a).
- Acceleration (a) is known for all values of time (t):
  - Successive integration in t gives the velocity (v) and position (s).
- Acceleration is determined by the forces acting on moving bodies
- The acceleration may be specified as a function of time, velocity, or position, or as a combined function of these quantities.

(a) Constant Acceleration:

When a is constant (2) and (3) can be integrated directly.

With initial position $s = s_0$, and initial velocity $v = v_0$ at time t=0, we get

$$\int_{v_0}^{v} dv = \int_{0}^{t} a\,dt, \quad \Rightarrow \quad v = v_0 + at \qquad \cdots\cdots\cdots\cdots\cdots\cdots(4)$$

$$\int_{v_0}^{v} v\,dv = \int_{s_0}^{s} a\,ds \quad \Rightarrow \quad v^2 = v_0^2 + 2a(s - s_0). \qquad \cdots\cdots\cdots\cdots\cdots\cdots(5)$$

Substituting (4) in (1), and integrating, we get

$$\int_{s_0}^{s} ds = \int_{0}^{t} (v_0 + at)\,dt, \quad \Rightarrow \quad s = s_0 + v_0 t + \frac{1}{2}at^2. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(6)$$

Caution: **Above relations are necessarily restricted to the special case where the acceleration is constant.**

(b) Acceleration given as a function of time, $a = f(t)$:

If initial conditions are $s = s_0$, $v = v_0$ at time t=0, then for a time interval t the integration of (2) gives,

$$\int_{v_0}^{v} dv = \int_{0}^{t} f(t)\, dt, \quad \Rightarrow \quad v = v_0 + \int_{0}^{t} f(t)\, dt, \qquad \text{.................}(7)$$

Using (7) as a function of t can be substituted in (1) to obtain the displacement as,

$$\int_{s_0}^{s} ds = \int_{0}^{t} v\, dt, \quad \Rightarrow \quad s = s_0 + \int_{0}^{t} v\, dt. \qquad \text{.................}(8)$$

(c) Acceleration given as a function of velocity, $a = f(v)$:

With initial position $s = s_0$, and initial velocity $v = v_0$ at time t=0, by substituting the function in (2), we get,

$$a = f(v) = \frac{dv}{dt}.$$

Separating the variable and integrating as,

$$\int_0^t dt = t = \int_v^{v_0} \frac{dv}{f(v)}. \qquad \text{.................(9)}$$

Above result gives t as a function of v, which can be used to obtain v as a function of t. Then equation (1) can be used to obtain s as a function of t as

$$\int_{s_0}^{s} ds = \int_0^t v\, dt, \quad \Rightarrow \quad s = s_0 + \int_0^t v\, dt. \qquad \text{.................(10)}$$

(c) Acceleration given as a function of displacement, $a = f(s)$:

With initial position $s = s_0$, and initial velocity $v = v_0$ at time t=0, by substituting the function in (3) and integrating,

$$\int_{v_0}^{v} v\,dv = \int_{s_0}^{s} f(s)\,ds, \quad \Rightarrow \quad v_0^2 = v^2 + 2\int_{s_0}^{s} f(s)\,ds. \qquad \cdots\cdots\cdots\cdots(11)$$

Above result give $v = g(s)$. Using the relation,

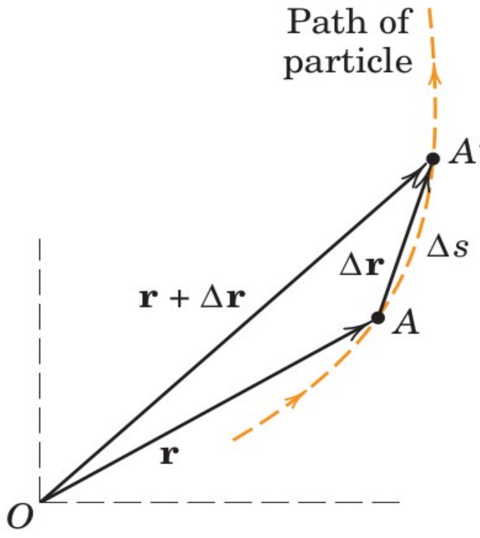
$$v = g(s) = \frac{ds}{dt},$$

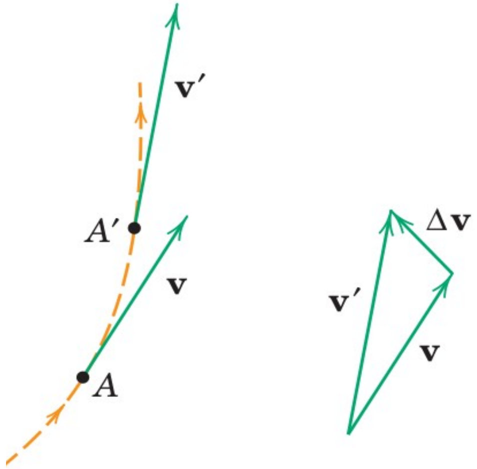
and integrating in the following form,

$$\int_{0}^{t} dt = t = \int_{s}^{s_0} \frac{ds}{g(s)}. \qquad \cdots\cdots\cdots\cdots(12)$$

which gives t as a function of s. Terms can be rearranged to get s as a function of t.

Plane curvilinear motion





$\Delta \boldsymbol{r}$ is the vector displacement

Δs is the scalar distance

Between A and A'

The **average velocity** $= \boldsymbol{v}_{\text{av}} = \Delta \boldsymbol{r}/\Delta t$

The **average speed** $= \Delta s/\Delta t$

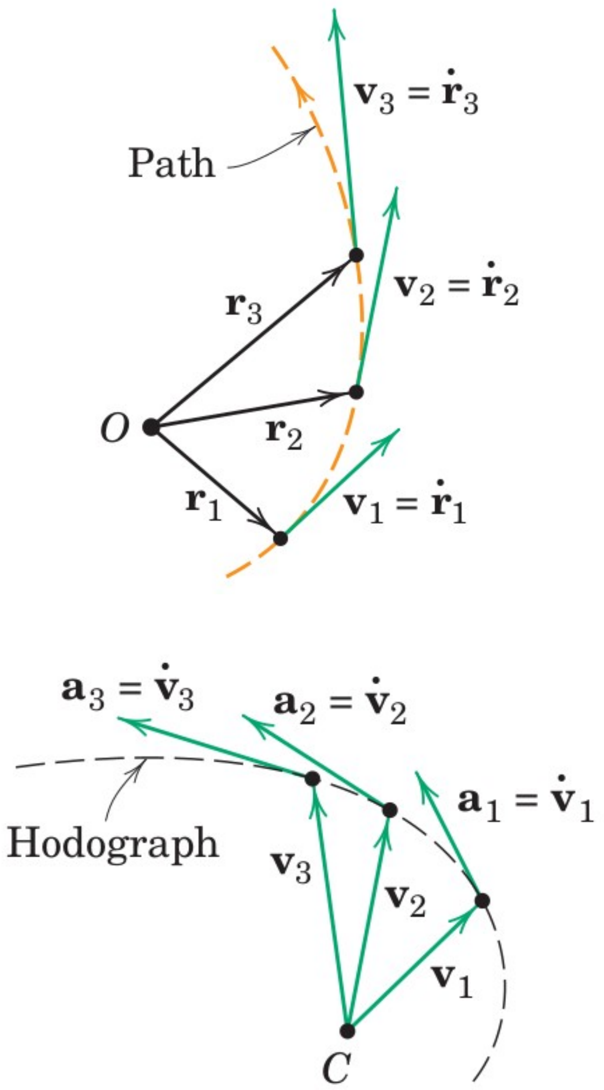
For $\Delta t \to 0$, the instantaneous velocity

$$\boldsymbol{v} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t} = \frac{d\boldsymbol{r}}{dt} = \dot{\boldsymbol{r}} \qquad \cdots\cdots\cdots\cdots(13)$$

The average acceleration $= \boldsymbol{a}_{\text{av}} = \Delta \boldsymbol{v} / \Delta t$

For $\Delta t \to 0$, the **instantaneous acceleration**

$$\boldsymbol{a} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}}{\Delta t} = \frac{d\boldsymbol{v}}{dt} = \dot{\boldsymbol{v}} \qquad \cdots\cdots\cdots\cdots(14)$$



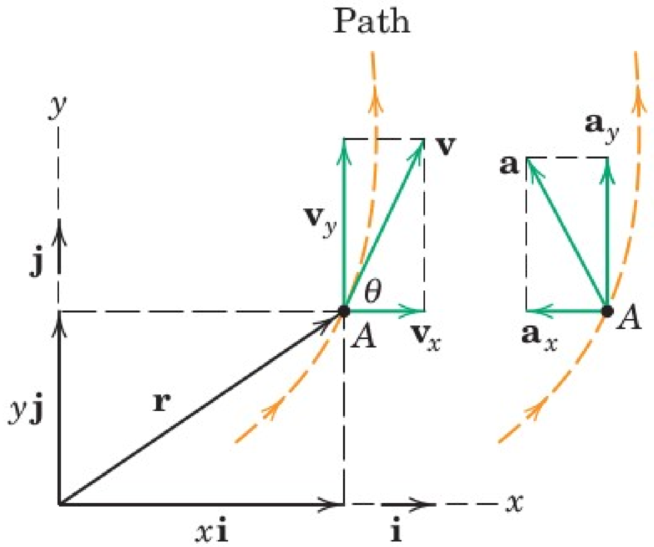
A graphical approach to the visualization of acceleration is shown.

Velocity vectors are tangent to the path corresponding to each position vector, and the relation is $\boldsymbol{v} = \dot{\boldsymbol{r}}$.

Similarly, if these velocity vectors are now plotted from some arbitrary point C, a curve, called the **hodograph**, is formed. The derivatives of these velocity vectors are the acceleration vectors $\boldsymbol{a} = \dot{\boldsymbol{v}}$, which are **tangent to the hodograph**.

Rectangular coordinates (x-y)

This system of coordinates is particularly useful for describing motions where the x- and y-components of acceleration are independently generated or determined.



The vectors $\boldsymbol{r}$, $\boldsymbol{v}$, and $\boldsymbol{a}$ in terms of their x- and y-components are written as,

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$$

$$\dot{\boldsymbol{r}} = \boldsymbol{v} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} \qquad v_x = \dot{x}, \quad v_y = \dot{y}$$

$$\ddot{\boldsymbol{r}} = \dot{\boldsymbol{v}} = \boldsymbol{a} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} \qquad a_x = \ddot{x}, \quad a_y = \ddot{y},$$

Also, ...(15)

$$v = \sqrt{v_x^2 + v_y^2}, \quad a = \sqrt{a_x^2 + a_y^2}, \text{ and } \tan\theta = v_y/v_x.$$

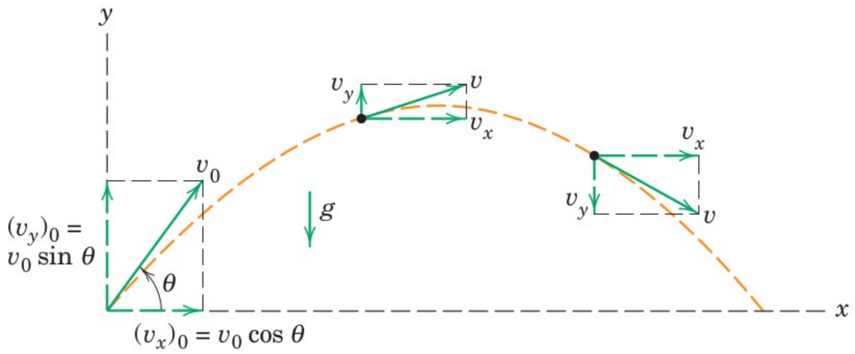
Observed that the rectangular coordinate representation of curvilinear motion is the superposition of the components of two simultaneous rectilinear motions in the x- and y-directions. Therefore, everything applicable for rectilinear motion can be applied separately to the x-motion and to the y-motion.

Projectile motion

Assumptions:

Aerodynamic drag and the curvature and rotation of the earth is neglected.

The altitude change is small enough so that the acceleration due to gravity can be considered constant.



The acceleration components are
a_x=0, and a_y=-g.

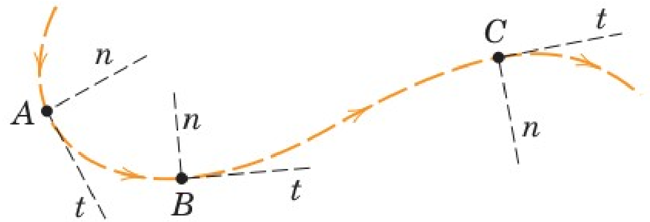
Since, this is a case of constant acceleration, using (4), (5) and (6) velocity and displacement components can be obtained as,

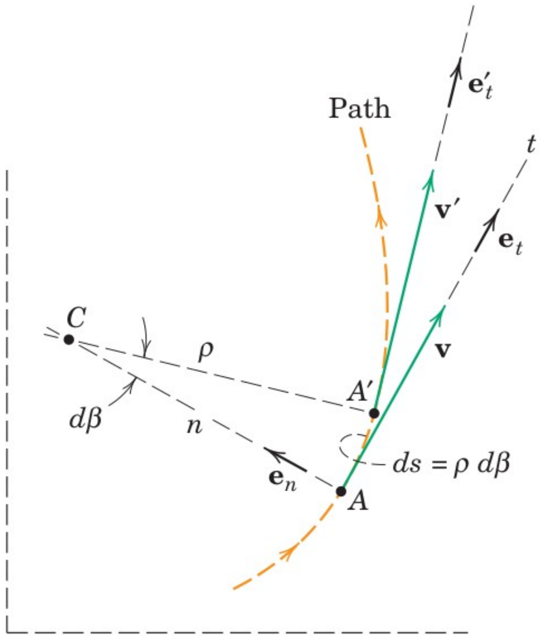
$$v_x = (v_x)_0, \quad v_y = (v_y)_0 - gt,$$

$$v_y^2 = (v_y)_0^2 - 2g(y - y_0),$$

$$x = x_0 + (v_x)_0 t, \quad y = y_0 + (v_y)_0 t - \frac{1}{2}gt^2.$$

Normal and tangential coordinates (n-t)





We now use the coordinates $\boldsymbol{n}$ and $\boldsymbol{t}$ to describe the velocity $\boldsymbol{v}$ and acceleration $\boldsymbol{a}$.

Unit vectors $\boldsymbol{e}_n$ and $\boldsymbol{e}_t$ are defined in the n-and t-direction, respectively at point A.

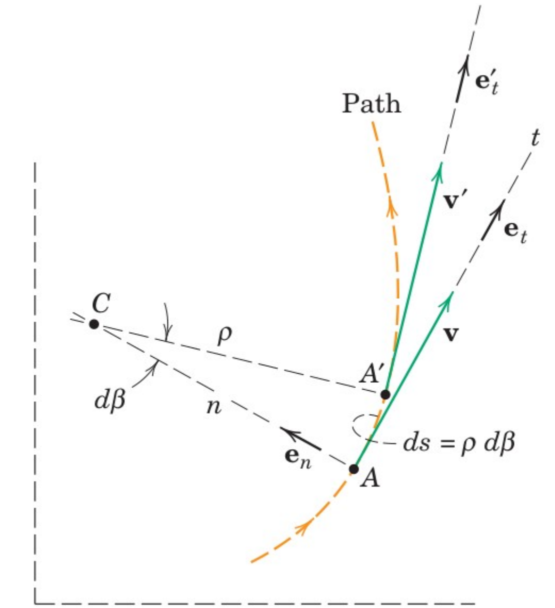
During a differential increment of time dt, the particle moves a differential distance ds along the curve from A to A'.

$\boldsymbol{\rho}$ being the radius of curvature of the path at A, $ds = \boldsymbol{\rho}\, d\beta$, where β is in radians. (change in $\boldsymbol{\rho}$ is neglected)

Thus, the magnitude of the velocity can be written

$v = ds\,/dt = \boldsymbol{\rho}\; d\beta/dt$, and the velocity vector is

$$\boldsymbol{v} = v\boldsymbol{e}_t = \rho\dot{\beta}\boldsymbol{e}_t. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(16)$$



The acceleration $\boldsymbol{a}$ of the particle $\boldsymbol{a} = d\boldsymbol{v}\,/dt$, which is obtained as,

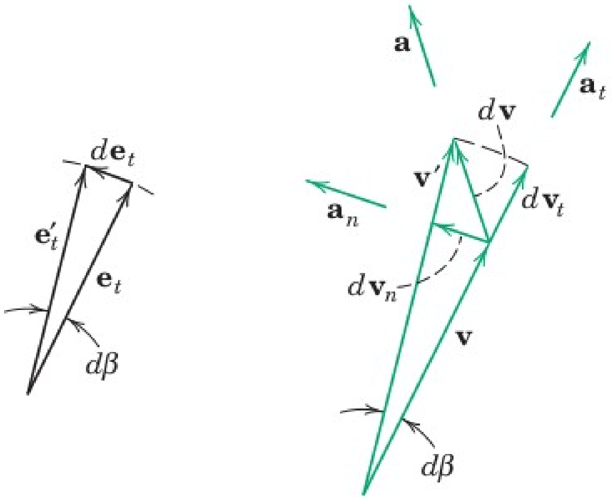
$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{d(v\boldsymbol{e}_t)}{dt} = v\dot{\boldsymbol{e}}_t + \dot{v}\boldsymbol{e}_t, \qquad \cdots\cdots\cdots\cdots(17)$$

Note that $\dot{\boldsymbol{e}}_t \neq 0$ because it changes the direction.

Change is et during small time increment dt is $d\boldsymbol{e}_t$ in the direction of $\boldsymbol{e}_n$. Thus, we can write

$$|d\boldsymbol{e}_t| = |\boldsymbol{e}_t| d\beta = d\beta.$$

$$d\boldsymbol{e}_t = |d\boldsymbol{e}_t|\boldsymbol{e}_n = d\beta\boldsymbol{e}_n \Rightarrow \dot{\boldsymbol{e}}_t = \dot{\beta}\boldsymbol{e}_n. \qquad \cdots\cdots\cdots\cdots(18)$$



Substituting (18) in (17), the acceleration becomes,

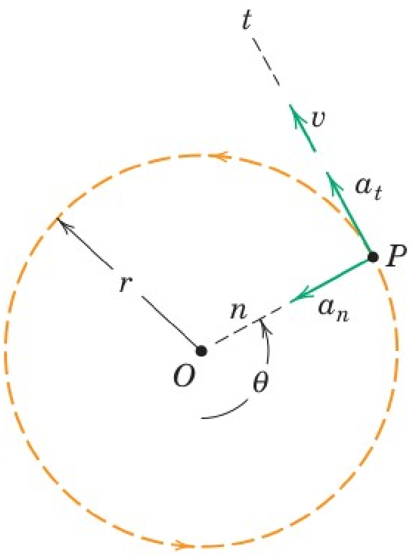
$$\boldsymbol{a} = \frac{v^2}{\rho}\boldsymbol{e}_n + \dot{v}\boldsymbol{e}_t = a_n\boldsymbol{e}_n + a_t\boldsymbol{e}_t. \qquad \cdots\cdots\cdots\cdots(19)$$

where,

$$a_n = v^2/\rho = \rho\dot{\beta}^2 = v\dot{\beta}, \quad a_t = \dot{v} = \ddot{s}. \qquad \cdots\cdots\cdots\cdots(20)$$

Circular motion

Circular motion is an important special case of plane curvilinear motion where the radius of curvature ρ becomes the constant radius r of the circle and the angle β is replaced by the angle measured from any convenient radial reference to OP. The velocity and the acceleration components for the circular motion of the particle P become

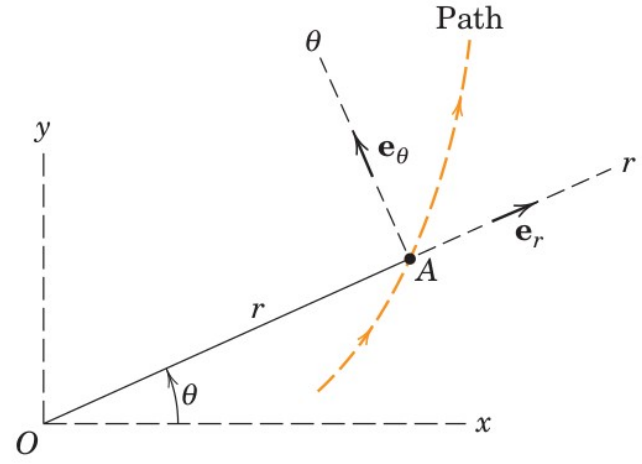


$$v = r\dot{\theta}$$

$$a_n = v^2/r = r\dot{\theta}^2 = v\dot{\theta} \qquad \cdots\cdots\cdots\cdots(21)$$

$$a_t = r\ddot{\theta}$$

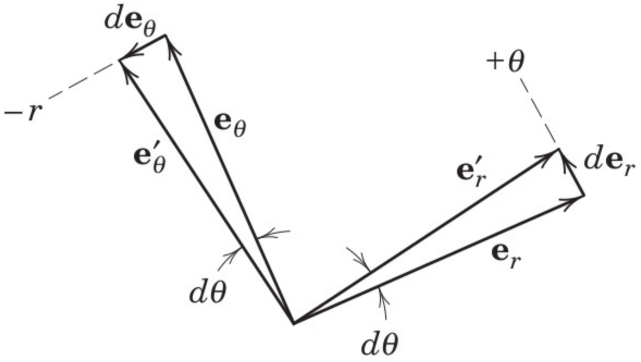
Polar coordinates (r-θ)



Polar coordinates are particularly useful when a motion is constrained through the control of a radial distance and an angular position or when an unconstrained motion is observed by measurements of a radial distance and an angular position.

$\boldsymbol{r}$ is the position vector to the particle at A, which is expressed as

$$\boldsymbol{r} = r\boldsymbol{e}_r. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(22)$$



During time dt the coordinate directions as well as unit vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}$ rotate through an angle $d\theta$to $\boldsymbol{e}'_r$ and $\boldsymbol{e}'_\theta$. We can write

$$\frac{d\boldsymbol{e}_r}{d\theta} = \boldsymbol{e}_\theta, \quad \text{and} \quad \frac{d\boldsymbol{e}_\theta}{d\theta} = -\boldsymbol{e}_r, \text{ or,} \quad \cdots\cdots\cdots\cdots\cdots\cdots(23)$$

$$\dot{\boldsymbol{e}}_r = \dot{\theta}\boldsymbol{e}_\theta, \quad \text{and} \quad \dot{\boldsymbol{e}}_\theta = -\dot{\theta}\boldsymbol{e}_r. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(24)$$

Now, the velocity

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{r}\boldsymbol{e}_r + r\dot{\boldsymbol{e}}_r = \dot{\boldsymbol{r}} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta, \qquad \text{[From (24)]} \qquad \cdots\cdots\cdots\cdots\cdots(25)$$

where, $v_r = \dot{r}, v_\theta = r\dot{\theta}$ and $v = \sqrt{v_r^2 + v_\theta^2}$.

The r-component of $\boldsymbol{v}$ is the rate at which the vector $\boldsymbol{r}$ stretches. The θ-component of $\boldsymbol{v}$ is due to the rotation of $\boldsymbol{r}$.

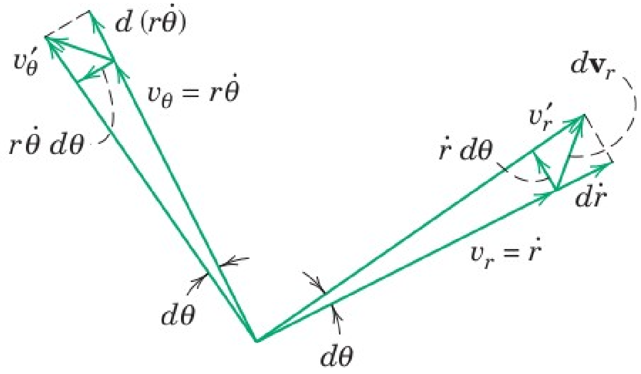
To obtain acceleration, we differentiate (25), so,

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = (\ddot{r}\boldsymbol{e}_r + \dot{r}\dot{\boldsymbol{e}}_r) + (\dot{r}\dot{\theta}\boldsymbol{e}_\theta + r\ddot{\theta}\boldsymbol{e}_\theta + r\dot{\theta}\dot{\boldsymbol{e}}_\theta).$$

Substituting derivatives of unit vectors from (24),

$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e}_\theta, \qquad \cdots\cdots\cdots\cdots\cdots(26)$$

where $a_r = (\ddot{r} - r\dot{\theta}^2), a_\theta = (r\ddot{\theta} + 2\dot{r}\dot{\theta})$, and $a = \sqrt{a_r^2 + a_\theta^2}$.



Following changes are shown in the figure:

(a) Magnitude change of $\boldsymbol{v_r}$: This change is the increase in length of v_r or $dv_r = d\dot{r}$, and the corresponding acceleration term is $d\dot{r}/dt = \ddot{r}$ in the positive r-direction.

(b) Direction change of $\boldsymbol{v_r}$: The magnitude of this change is $v_r d\theta = \dot{r} d\theta,$ and its contribution to the acceleration becomes $\dot{r} d\theta/dt = \dot{r}\dot{\theta},$ which is in the positive θ-direction.

(c) Magnitude change of $\boldsymbol{v_\theta}$: This term is the change in length of v_θ or $d(r\dot{\theta})$, and its contribution to the acceleration is $d(r\dot{\theta})/dt = r\ddot{\theta} + \dot{r}\dot{\theta}$, and is in the positive θ-direction.

(d) Direction Change of $\boldsymbol{v_\theta}$: The magnitude of this change is $v_\theta d\theta = r\dot{\theta} d\theta,$ and the corresponding acceleration term is observed to be $r\dot{\theta}(d\theta/dt) = r\dot{\theta}^2$, in the negative r-direction.

Circular motion

For motion in a circular path with r constant, the components of velocity and acceleration become,

$$v_r = 0, \quad v_\theta = r\dot{\theta},$$
$$a_r = -r\dot{\theta}^2, \quad a_\theta = r\ddot{\theta}.$$