ME232: Dynamics

Vibrations

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Applications:

Vibration-measuring instruments such as **seismometers and accelerometers** are frequently encountered applications of harmonic excitation. The elements of this class of instruments are shown in figure.

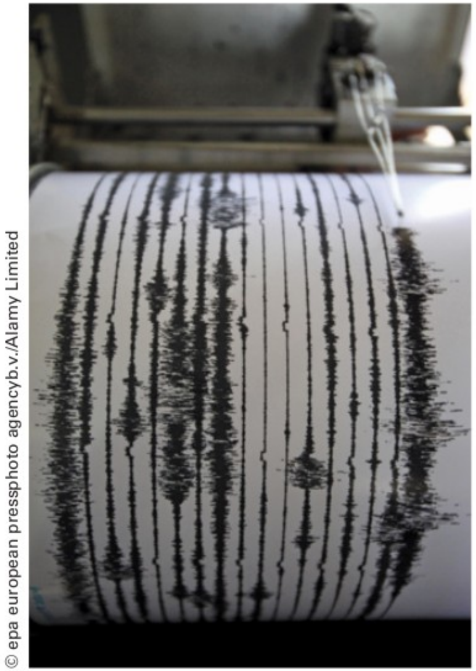
The entire system is subjected to the motion x_B of the frame. Letting x denote the position of the mass **relative to the frame**, apply Newton's second law to obtain

$$-kx - c\dot{x} = m(\ddot{x} + \ddot{x}_B) \quad \text{or} \quad m\ddot{x} + c\dot{x} + kx = -m\ddot{x}_B.$$

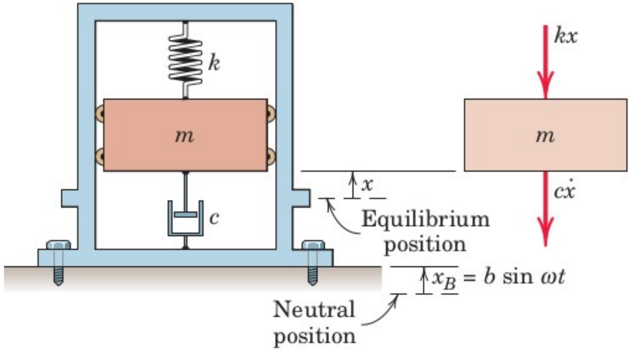
where $(x+x_B)$ is the inertial displacement of the mass. If $x_B = b \sin \omega t$, then our equation of motion with the usual notation is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = b\omega^2 \sin\omega t, \quad \cdots\cdots\cdots\cdots(30)$$

which is similar to (26).



The seismograph is a useful application of the principles of this article.



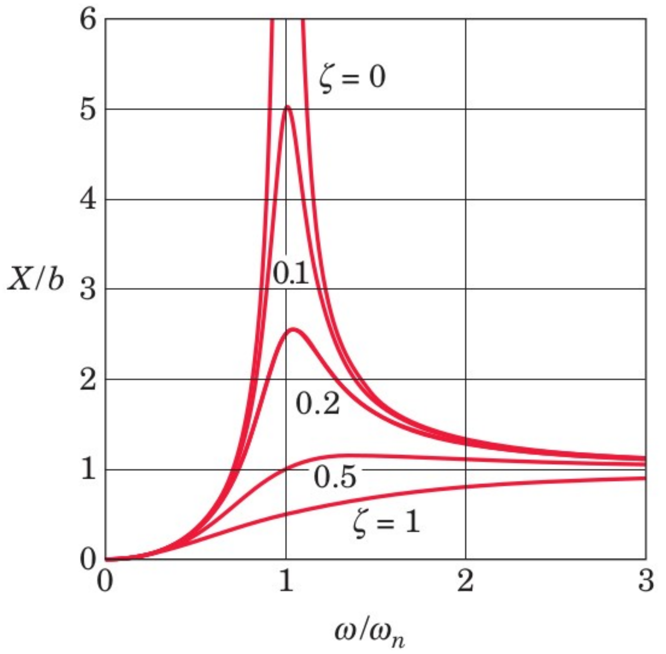
Again , we are interested in steady-state solution x_p. Thus from (27), we have

$$x_p = \frac{b(\omega/\omega_n)^2}{\sqrt{[1-(\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}} \sin(\omega t - \phi). \qquad \cdots\cdots\cdots\cdots(31)$$

If X represents the amplitude of the relative response x_p , then the non-dimensional ratio X/b is

$$X/b = (\omega/\omega_n)^2 M, \qquad \cdots\cdots\cdots\cdots(32)$$

where M is the magnification ratio given by (29). A plot of X/b as a function of the driving-frequency ratio ω/ω_n is shown.

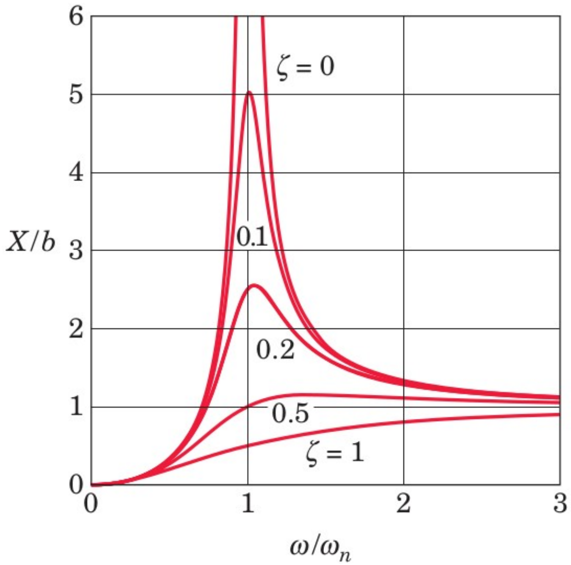


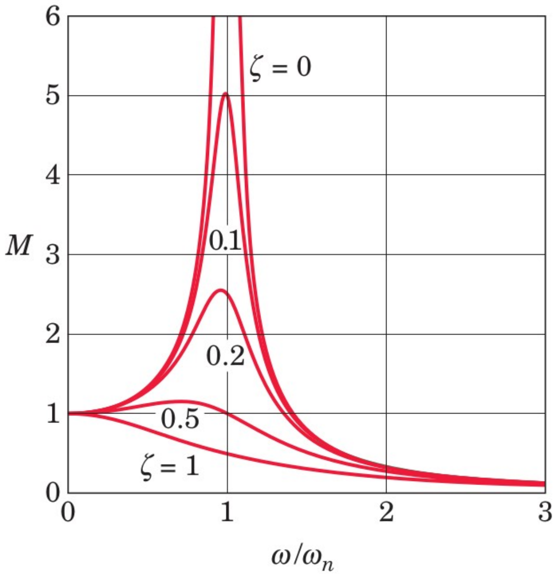
Note the different between the amplification factor for force based and displacement excited vibration.

If the frequency ratio ω/ω_n is large, then $X/b \cong 1$ for all values of the damping ratio ζ. Under these conditions, the displacement of the mass relative to the frame is approximately the same as the absolute displacement of the frame, and the **instrument acts as a displacement meter.**

To obtain a high value of ω/ω_n, we need a **small value of $\omega_n = \sqrt{(k/m)}$**, which **means a soft spring and a large mass.** With such a combination, the mass will tend to stay inertially fixed. Displacement meters generally have **very light damping.**

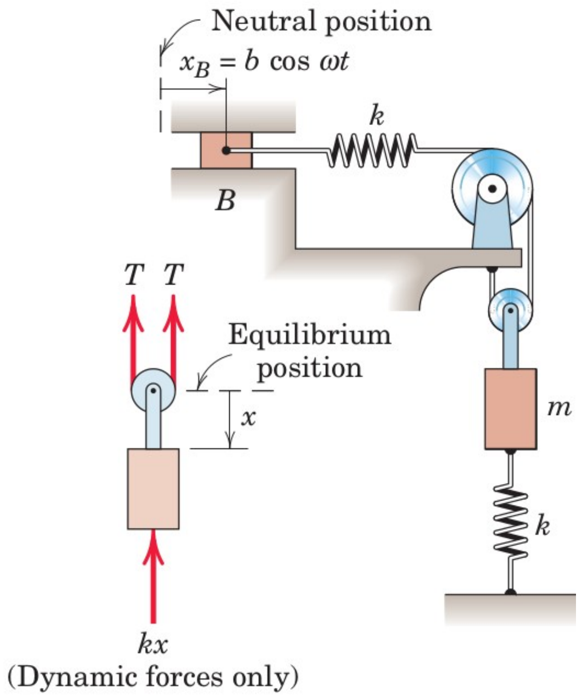
On the other hand, if the **frequency ratio ω/ω_n is small,** then M approaches unity and $X/b \cong (\omega/\omega_n)^2$ or $X \cong b(\omega/\omega_n)^2$. But **$b\omega^2$ is the maximum acceleration of the frame.** Thus, X is proportional to the maximum acceleration of the frame, and the instrument may be used as **an accelerometer.** The damping ratio is generally selected so that **M approximates unity over the widest possible range of ω/ω_n.** We see that a damping factor somewhere between ζ=0.5 and ζ=1 would meet this criterion.



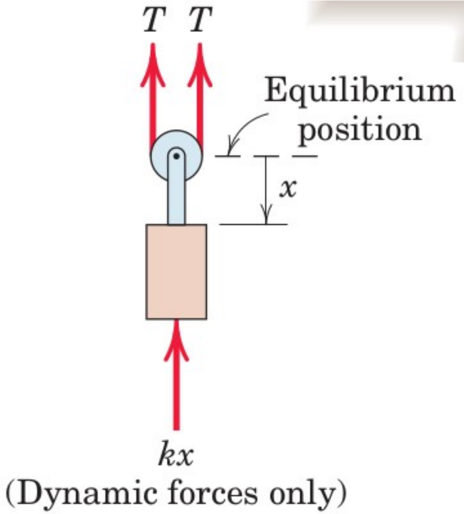


Example 3

The spring attachment point B is given a horizontal motion $x_B = b\cos\omega t$. Determine the critical driving frequency ω_c for which the oscillations of the mass m tend to become excessively large. Neglect the friction and mass associated with the pulleys. The two springs have the same stiffness k.



The free-body diagram is drawn for arbitrary positive displacements x and x_B . The motion variable x is measured downward from the position of static equilibrium defined as that which exists when $x_B = 0$. The additional stretch in the upper spring, beyond that which exists at static equilibrium, is $2x - x_B$. Therefore, the dynamic spring force in the upper spring, and hence the dynamic tension T in the cable, is $k(2x - x_B)$. Summing forces in the x-direction gives



$$\sum F_x = m\ddot{x} \qquad -2k(2x - x_B) - kx = m\ddot{x}$$

which gives

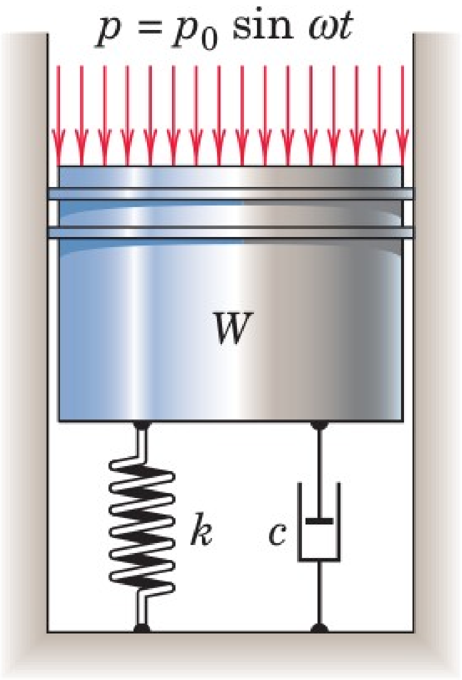
$$\ddot{x} + \frac{5k}{m}x = \frac{2kb}{m}\cos\omega t.$$

The natural frequency of system is then, $\omega_n = \sqrt{5k/m}$.

Thus, $\omega_c = \omega_n = \sqrt{5k/m}.$

Example 4

The 45 kg piston is supported by a spring of modulus $k = 35$ kN/m. A dashpot of damping coefficient $c = 1250$ N·s/m acts in parallel with the spring. A fluctuating pressure $p = 4000 \sin 30t$ in Pa acts on the piston, whose top surface area is 50×10^{-3} m^2. Determine the steady-state displacement as a function of time and the maximum force transmitted to the base.



First compute the system natural frequency and damping ratio,

$$\omega_n = \sqrt{k/m} = 27.9 \text{ rad/s},$$

$$\zeta = c/2m\omega_n = 0.498 \text{ (underdamped)}$$

The steady-state amplitude and phase angle can be calculated by (27) as

$$X = \frac{F_0/k}{\sqrt{[1-(\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}} = 5.28 \text{ mm}, \quad \text{and}$$

$$\phi = \tan^{-1}\left(\frac{2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2}\right) = 1.716 \text{ rad}.$$

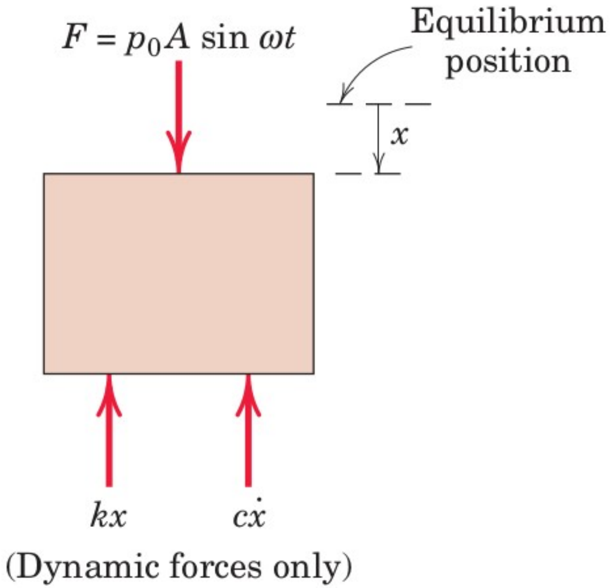
The steady-state motion is then given by, $x_p = X \sin(\omega t - \phi)$.

The force transmitted to the base is then

$$F_{\text{tr}} = kx_p + c\dot{x}_p = kX\sin(\omega t - \phi) + cX\omega\cos(\omega t - \phi).$$

The maximum value of F_{tr} is then,

$$(F_{\text{tr}})_{\max} = \sqrt{(kX)^2 + (c\omega X)^2} = 271 \text{ N}.$$



Vibration of rigid bodies

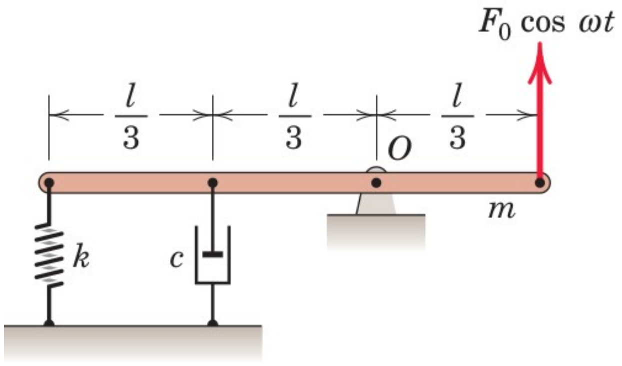
The subject of planar rigid-body vibrations is entirely analogous to that of particle vibrations.

In particle vibrations, the variable of interest is one of **translation** (x), while in rigid-body vibrations, the variable of primary concern may be one of **rotation** (θ).

Thus, the principles of rotational dynamics play a central role in the development of the equation of motion. We will see that the equation of motion for rotational vibration of rigid bodies has a mathematical form identical to that developed for translational vibration of particles.

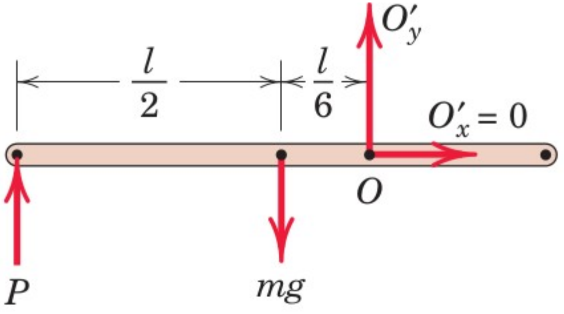
Rotational vibration of a bar:

Consider the rotational vibration of the uniform slender bar. Figure depicts the free-body diagram associated with the horizontal position of static equilibrium.



Equating to zero the moment sum about O yields

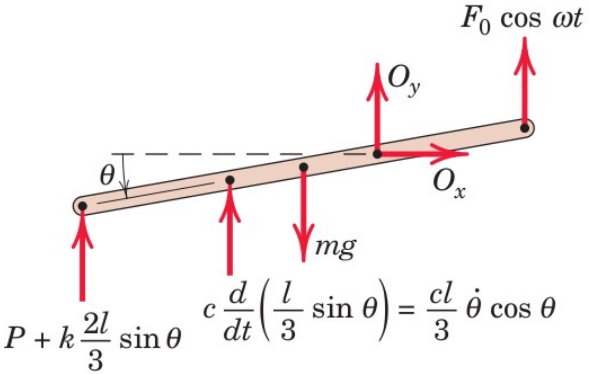
$$-P\left(\frac{l}{2}+\frac{l}{6}\right)+mg\frac{l}{6}=0 \Rightarrow P=\frac{mg}{4},$$



where P is the magnitude of the static spring force. Next figure depicts the free-body diagram associated with an arbitrary positive angular displacement . Using the equation of rotational motion $\sum M_O = I_O\ddot{\theta}$, we write

$$mg\cdot\frac{l}{6}\cos\theta-\frac{cl}{3}\dot{\theta}\cos\theta\cdot\frac{l}{3}\cos\theta$$

$$-P+k\frac{2l}{3}\sin\theta\cdot\frac{2l}{3}\cos\theta+F_0\cos\theta\left(\frac{l}{3}\cos\theta\right)=\frac{1}{9}ml^2\ddot{\theta},$$



…………(33)

For small angular deflection sin≈ and cos≈1 may be used. Thus, upon consideration of static equilibrium condition (33) can be rewritten as,

$$\ddot{\theta} + \frac{c}{m} + 4\frac{k}{m} = \frac{(F_0 l/3)\cos\omega t}{ml^2/9}. \qquad \text{............}(34)$$

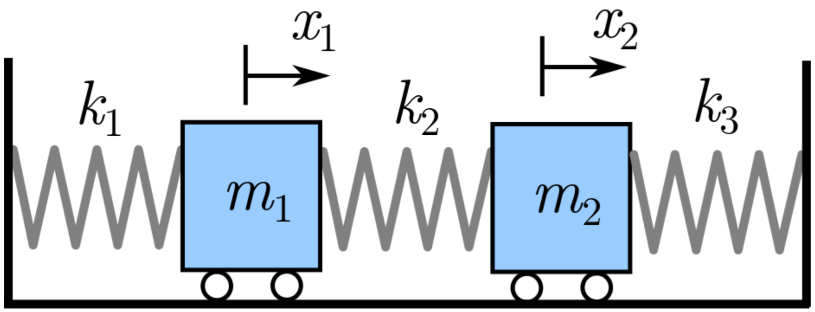
Note that the two equal and opposite **moments associated with static equilibrium forces canceled** on the left side of the equation of motion. Thus, it is **not necessary to include the static-equilibrium forces and moments** in the analysis.

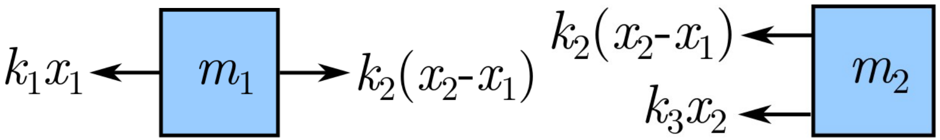
Rotational counterparts of Translational Vibration:

It should be observed that (34) is identical in form to (26) for the translational case. Thus, we may use all of the relations developed for translational case merely by replacing the translational quantities with their rotational counterparts. The following table shows the results of this procedure as applied to the current problem of rotating bar.

TRANSLATIONAL	ANGULAR (for current problem)
$\ddot{x} + \dfrac{c}{m}\dot{x} + \dfrac{k}{m}x = \dfrac{F_0 \cos \omega t}{m}$	$\ddot{\theta} + \dfrac{c}{m}\dot{\theta} + \dfrac{4k}{m}\theta = \dfrac{M_0 \cos \omega t}{I_O}$
$\omega_n = \sqrt{k/m}$	$\omega_n = \sqrt{4k/m} = 2\sqrt{k/m}$
$\zeta = \dfrac{c}{2m\omega_n} = \dfrac{c}{2\sqrt{km}}$	$\zeta = \dfrac{c}{2m\omega_n} = \dfrac{c}{4\sqrt{km}}$
$\omega_d = \omega_n\sqrt{1-\zeta^2} = \dfrac{1}{2m}\sqrt{4km - c^2}$	$\omega_d = \omega_n\sqrt{1-\zeta^2} = \dfrac{1}{2m}\sqrt{16km - c^2}$
$x_c = Ce^{-\zeta\omega_n t} \sin(\omega_d t + \psi)$	$\theta_c = Ce^{-\zeta\omega_n t} \sin(\omega_d t + \psi)$
$x_p = X \cos(\omega t - \phi)$	$\theta_p = \Theta \cos(\omega t - \phi)$
$X = M\left(\dfrac{F_0}{k}\right)$	$\Theta = M\left(\dfrac{M_0}{k_\theta}\right) = M\dfrac{F_0(l/3)}{\frac{4}{9}kl^2} = M\dfrac{3F_0}{4kl}$

Two degree of freedom system





Now, consider the system with two masses (m_1 and m_2) connected with springs as shown. Displacement of both the masses from the equilibrium position is measured as x_1 and x_2. Similar to previous cases, let us first write the equation of motion for both masses as,

$$\begin{aligned} k_2(x_2 - x_1) - k_1x_1 &= m_1\ddot{x}_1 \\ -k_3x_2 - k_2(x_2 - x_1) &= m_2\ddot{x}_2 \end{aligned} \quad \text{or} \quad \begin{aligned} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 &= 0 \\ m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 &= 0 \end{aligned} \quad \cdots\cdots\cdots\cdots(35)$$

Equations (24) can be written in matrix form as,

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{0}, \quad \text{where,} \quad \boldsymbol{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \text{ and } \boldsymbol{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22}. \end{bmatrix}$$

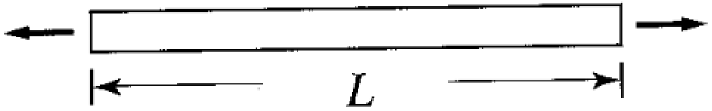
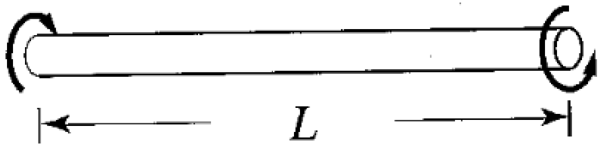
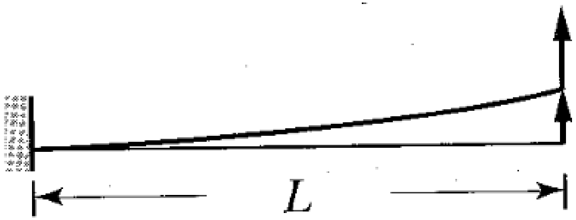
$$\cdots\cdots\cdots\cdots(36)$$

Here $\boldsymbol{M}$ is known as *mass matrix* and $\boldsymbol{K}$ is known as *stiffness matrix.*

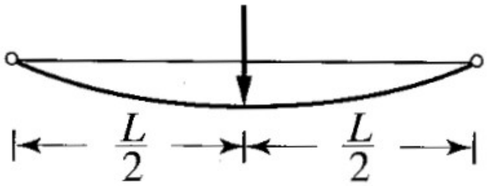
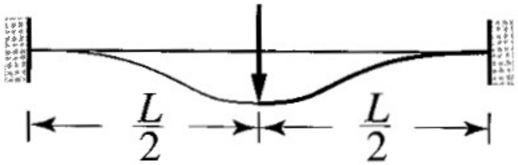
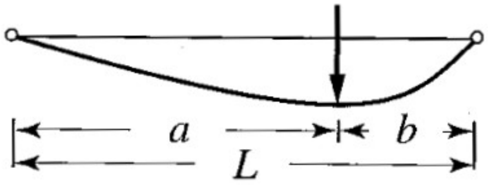
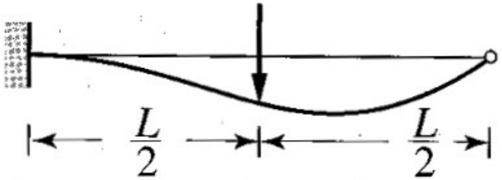
- These system of equations can be solved for assumed harmonic response and it can be shown that the system has two natural frequencies.
- Corresponding to each natural frequency there exists a particular type of oscillatory motion which is called mode shape of vibration. So there are two modes of vibration corresponding to two natural frequencies of the system.
- In case of forced vibration resonance occurs when the excitation frequency matches with any of the natural frequency of the system.
- With an appropriate combination of spring stiffness and masses vibration absorbers can be designed.

https://www.youtube.com/embed/87SQH1rGIG8

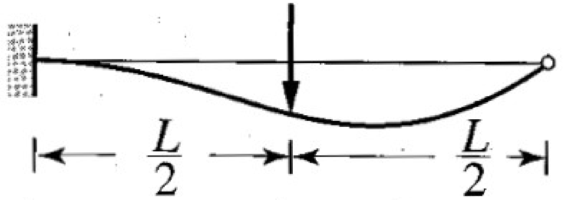
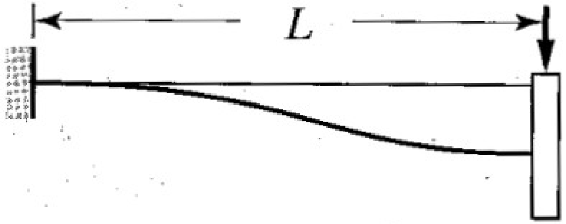
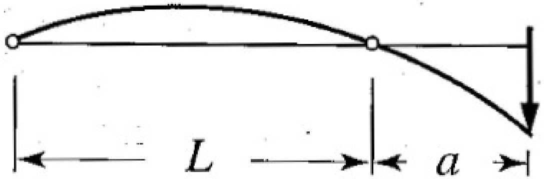
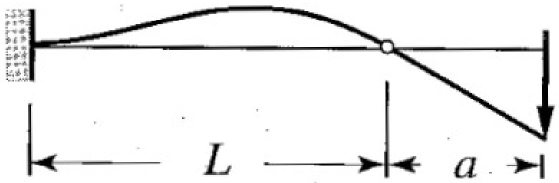
Equivalent spring constants

	Rod in axial deformation	$\frac{EA}{L}$
	Shaft in torsion	$\frac{GJ}{L}$
	Cantilever beam with a moment at the tip	$\frac{EI}{L}$
	Cantilever beam with a force at the tip	$\frac{3EI}{L^3}$

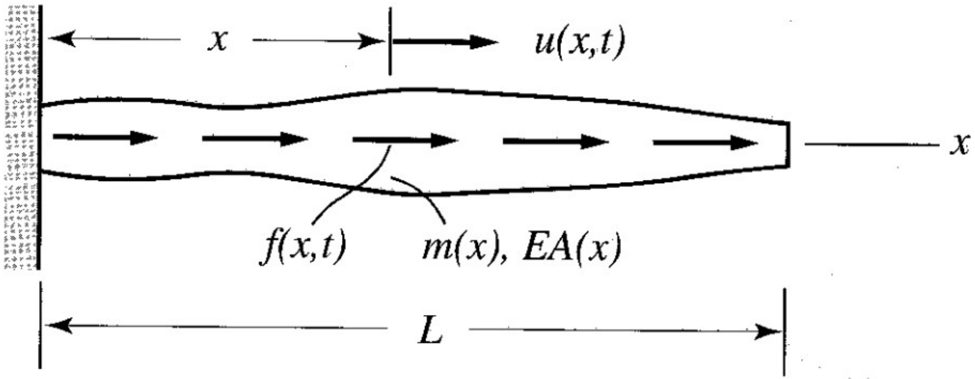
Equivalent spring constants

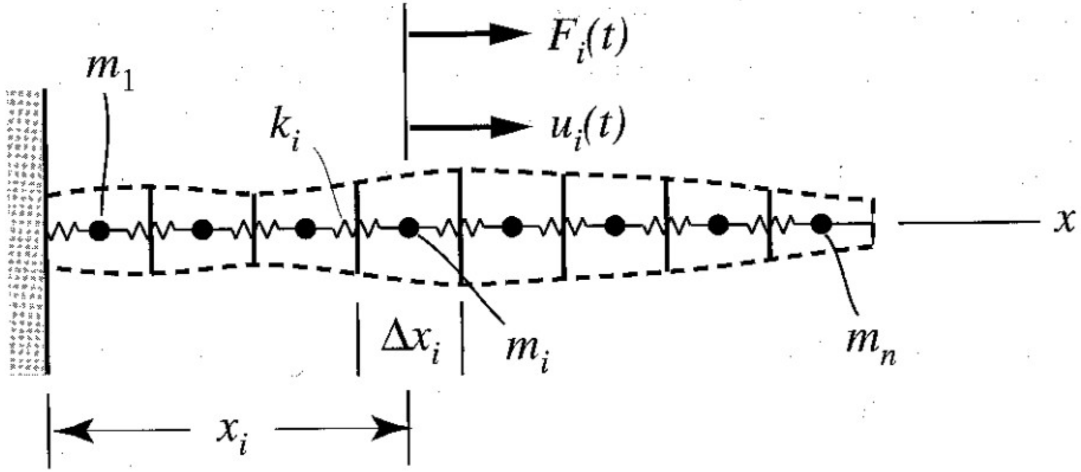
Component Sketch and Description		k_{eq}
	Pinned-pinned beam with a force at midspan	$\frac{48EI}{L^3}$
	Clamped-clamped beam with a force at midspan	$\frac{192EI}{L^3}$
	Pinned-pinned beam with an off-center force	$\frac{3EIL}{a^2b^2}$
	Clamped-pinned beam with a force at midspan	$\frac{768EI}{7L^3}$

Equivalent spring constants

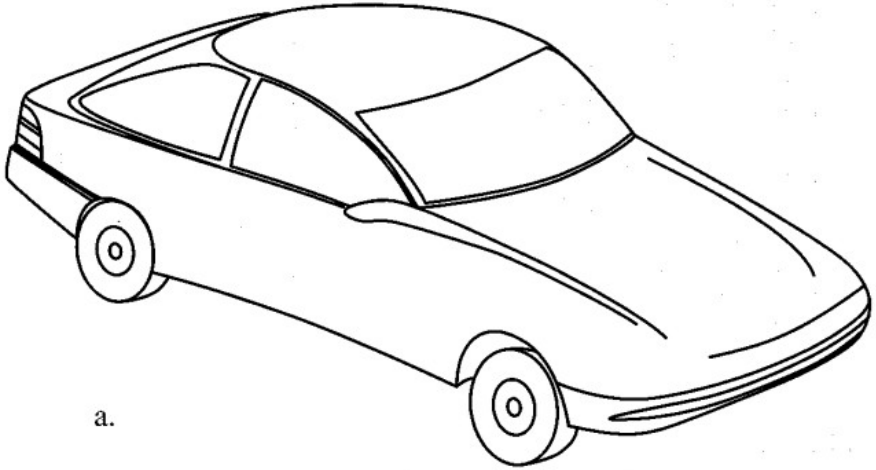
	Clamped-pinned beam with a force at midspan	$\frac{768EI}{7L^3}$
	Clamped-clamped beam with one end sagging under a force	$\frac{12EI}{L^3}$
	Pinned-pinned beam with an overhang and a force at the tip	$\frac{3EI}{a^2(L+a)}$
	Clamped-pinned beam with an overhang and a force at the tip	$\frac{12EI}{a^2(3L+4a)}$

Modeling of continuous systems as discrete systems

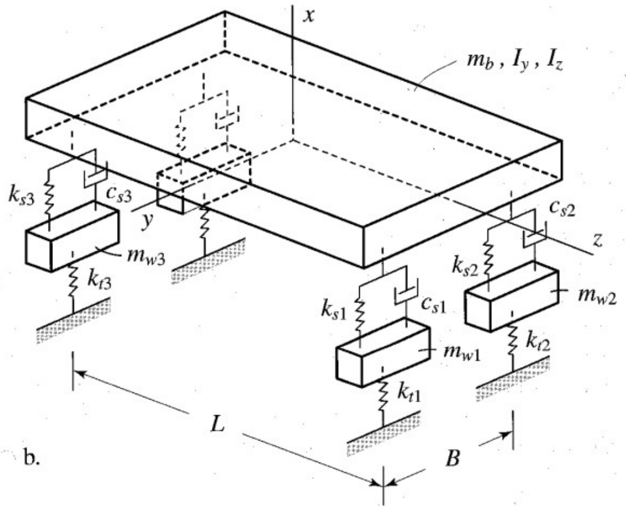




Different models of automobile for vibration analysis



a.



b.