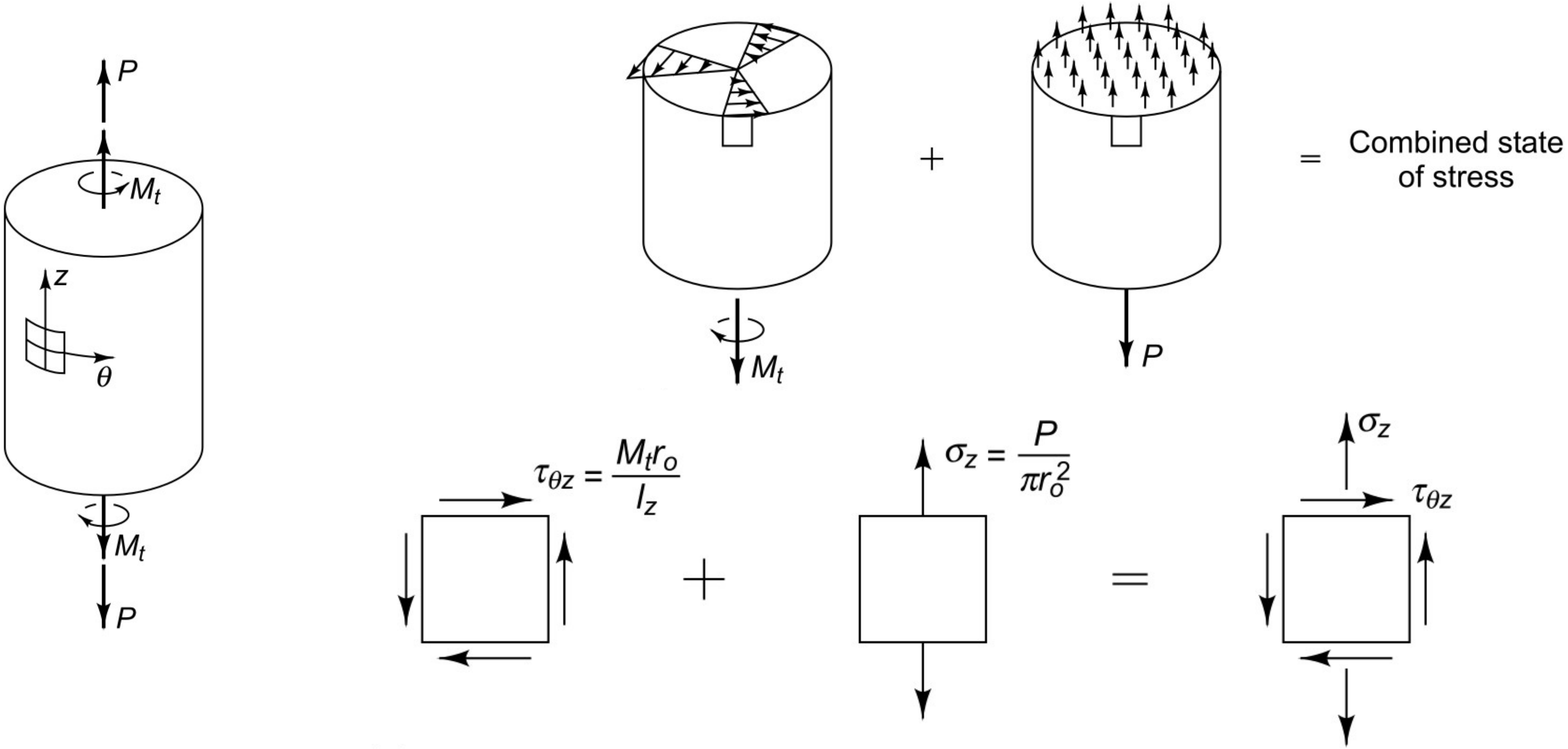
# ME231: Solid Mechanics-I

# Torsion

# Example 3: Combined stresses

A uniform, homogeneous, circular shaft is subjected simultaneously to an axial tensile force $P$ and a twisting moment $M_t$. Combined effect of both the loading can be determined by using the principle of superposition.



# Strain energy due to torsion

We observed that for torsion of an isotropic elastic shaft of circular cross section, the only nonvanishing stress and strain components are $\tau_{\theta z}$ and $\gamma_{\theta z}$. Hence, the total strain energy reduces to

$$U = \frac{1}{2}\int_V \tau_{\theta z}\gamma_{\theta z}dV = \frac{1}{2}\int_V \frac{\gamma_{\theta z}^2}{G}. \qquad \cdots\cdots(11)$$

Using (3) and (6), we can write,

$$U = \frac{1}{2}\int_V \frac{1}{G}\left(\frac{M_t r}{I_z}\right)^2 dV = \frac{1}{2}\int_L \frac{1}{G}\left(\frac{M_t}{I_z}\right)^2 dz \int_A r^2 dA = \frac{1}{2}\int_L \frac{M_t^2}{GI_z}dz.$$

Thus,

$$U = \frac{1}{2}\int_L \frac{M_t^2}{GI_z}dz. \qquad \cdots\cdots(12)$$

# Torsion of rectangular shaft

When cross section of slender members subjected to torsion are not circular, symmetry arguments breaks down.

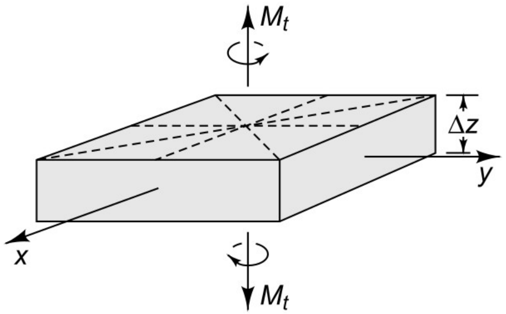
If a shaft having a square cross section is twisted, symmetry arguments can be used to show that only the four dotted lines shown in the figure remain straight. However, all other lines in the cross section can deform when the shaft is twisted without violating the requirements of symmetry.

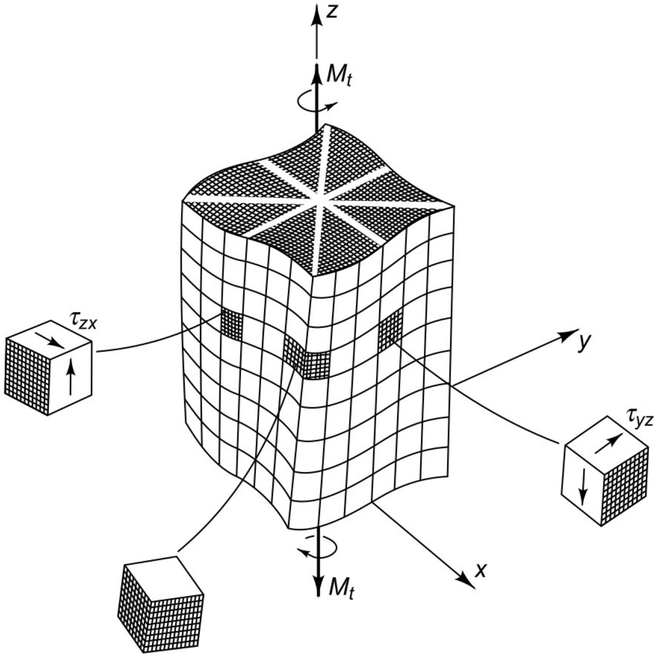
Consider a corner element for the grid shown in the figure. It can not have any stress and hence remain undeformed in the twisted shaft.

The originally plane cross sections have deformed or warped out of their own planes.

Exact solutions for the torsion of rectangular shafts using the equations of elasticity are available; however they are out of scope of this course.

We simply reports some important results there.



For a long, rectangular shaft with cross-section dimensions $a$ and $b$ with $b \leq a$, the maximum shearing stress neglecting end effects occurs in the middle of the side $a$ and has the magnitude

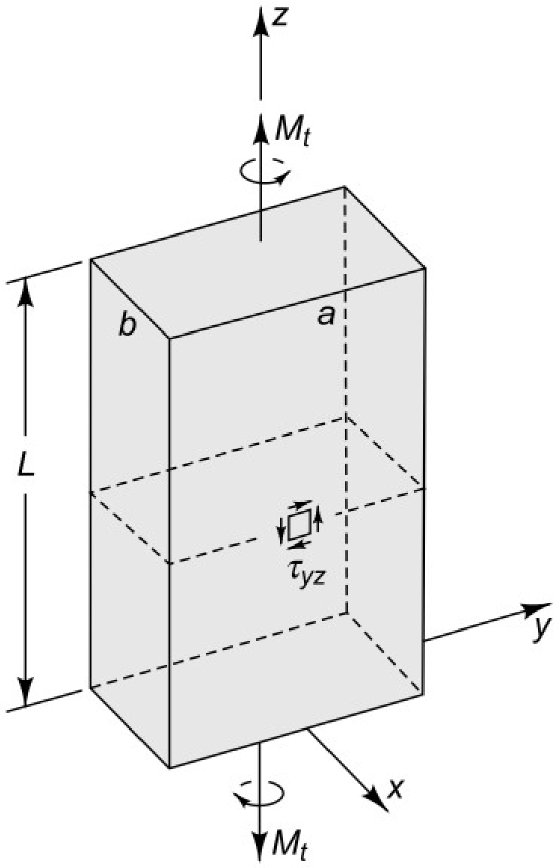
$$(\tau_{yz})_{\max} = c_1 \frac{M_t}{ab^2}. \qquad \cdots\cdots\cdot(13)$$

The torsional stiffness of a long rectangular shaft is

$$\frac{M_t}{\phi} = c_2 \frac{Gab^3}{L}. \qquad \cdots\cdots\cdot(14)$$

Constants $c_1$ and $c_2$ are functions of the ratio $a/b$ and varies as follows.

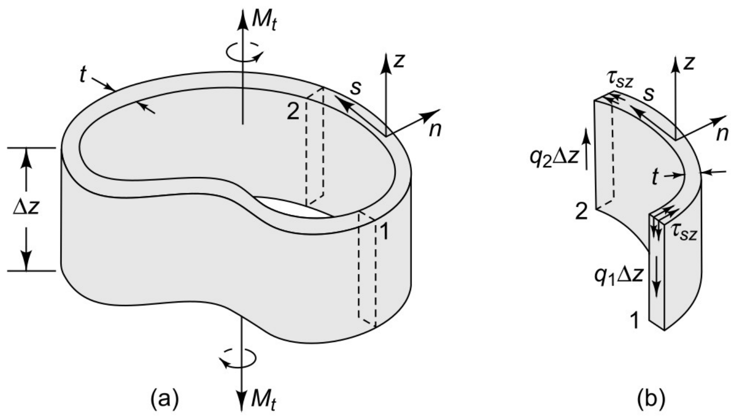
| $a/b$ | $c_1$ | $c_2$ |
|---|---|---|
| 1 | 4.81 | 0.141 |
| 1.5 | 4.33 | 0.196 |
| 2 | 4.06 | 0.229 |
| 3 | 3.74 | 0.263 |
| 5 | 3.44 | 0.291 |
| 10 | 3.20 | 0.312 |



# Torsion of hollow, thin walled shafts

Consider a long, hollow, cylindrical shaft of noncircular section and with a wall thickness $t$ which need not be constant around the circumference but which is small compared with the overall dimensions of the cross section.

A small element of length $\Delta z$, subject to twisting moment is shown in the figure. Let us try to make plausible assumption regarding the state of stress in this shaft.

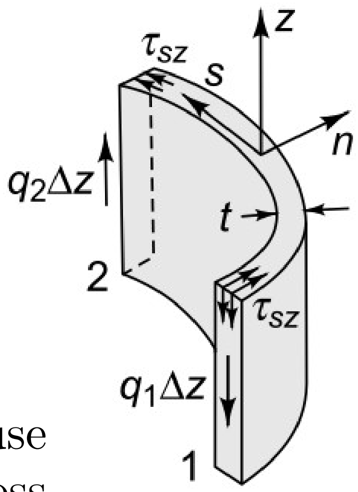



With the analysis of circular shaft, we have already shown the existence of shear stress component $\tau_{sz}$.

As inner and outer surfaces of shafts are traction free, stress components $\sigma_{nn}, \tau_{ns}$ and $\tau_{nz}$ on the outer and inner surfaces are zero; and because thickness of the shaft is very small these stress components can be assumed to be negligible.

Other components are $\sigma_{ss}$ and $\sigma_{zz}$. Similar to the assumption for circular shaft, we can assume these stress components equal to zero for small angle of twists.

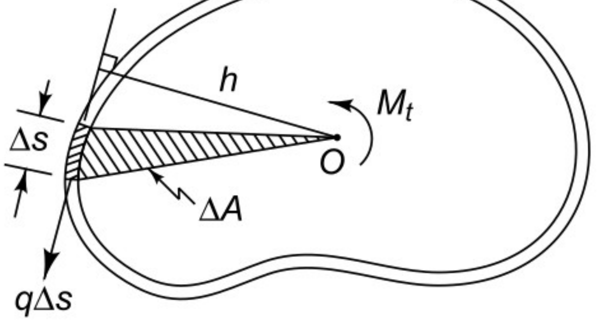
Direction of the shear stress component $\tau_{sz}$ are shown along the top and bottom and on the vertical faces 1 and 2. We introduce a concept of the shear flow $q$, which is defined as follows:




$$q = \int_{-t/2}^{t/2} \tau_{sz} dn. \qquad \cdots\cdots\cdot(15)$$

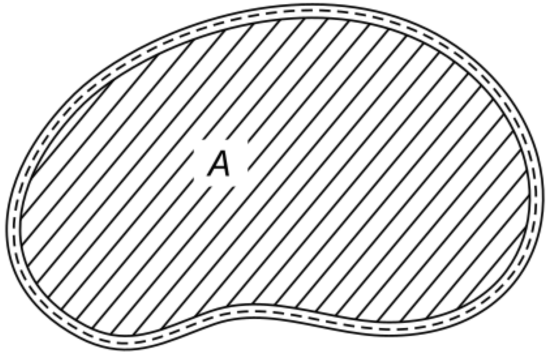
Shear flow is the stress force per unit length of the shaft. It is convenient to use the shear flow in discussions where the precise distribution of shear stress across the thickness is unknown or unimportant. From equilibrium of element it can be shown that $q_1 = q_2$. This means that the shear flow is constant around the cross section of the shaft.

Shear flow in the shaft can be related to the applied twisting moment as follows. Consider the moment of shear force working on small length $\Delta s$, as




$$\Delta M_t = (q\Delta s)h \approx 2q\Delta A$$

Hence the total twisting moment,




$$M_t = 2qA, \qquad \cdots\cdots(16)$$

where $A$ is the total area enclosed by the shaft. Optimum accuracy is obtained by extending $A$ to the mid-thickness of the wall.

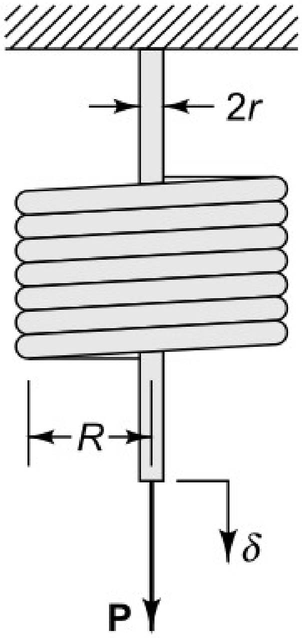
Now, if we consider the uniform distribution of $\tau_{sz}$ across the thickness. Then $q = \tau_{sz} t$.

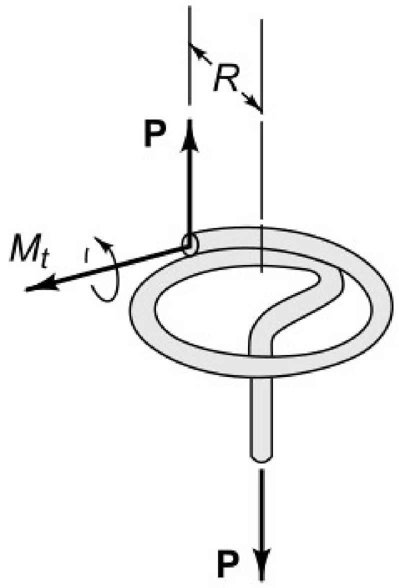
Hence, from (16) we get

$$\tau_{sz} = \frac{M_t}{2At}. \qquad \cdots\cdots(17)$$

# Example 4: Strain energy of a coiled spring

Consider a closely wound coil spring of radius $R$ loaded by a force $P$. The spring consists of $n$ turns of wire with wire radius $r$. We wish to find the strain energy stored in the spring due to twisting moment.




$$U = \frac{1}{2}\int_L \frac{M_t^2}{GI_z}dz.$$

$$U = \frac{1}{2}\int_L \frac{P^2R^2}{GI_z}dz = \frac{1}{2}\int_0^{2\pi n} \frac{P^2R^2}{GI_z}Rd\theta$$

$$U = \frac{P^2R^3}{2GI_z}2\pi n.$$