ME531: Advanced Mechanics of Solids

Motion, Strain and Stress

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Principal stress

Normal and shear stresses vary in magnitude, direction and location. For design of structure knowledge of extremal values are required. We will determine the extremal (maximum and minimum) values of stresses.

Maximum and minium values of normal stresses:

Consider the Cauchy traction vector $\boldsymbol{t}(\boldsymbol{x},\boldsymbol{n})=\boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{n}$ at any arbitrary plane at a point $\boldsymbol{x}$. We first try to find the unit vector $\boldsymbol{n}$ at a point $\boldsymbol{x}$ indicating the maximum and minimum values of normal stresses. In order to obtain the maximum and minimum values of normal stress σ, we apply Lagrange-multiplier method and claim ther stationary position of the following functional,

$$\mathcal{L}(\boldsymbol{n},\lambda)=\boldsymbol{n}\cdot\boldsymbol{\sigma}\boldsymbol{n}-\lambda(|\boldsymbol{n}|^2-1),\ \text{or}$$

$$\mathcal{L}(\boldsymbol{n},\lambda)=n_i\sigma_{ij}n_j-\lambda(n_kn_k-1).$$

Here, $|\boldsymbol{n}|^2-1$ is a *constraint condition* and λ is the *Lagrange multiplier.*

For stationary position of the functional following derivatives must vanish, $\frac{\partial \mathcal{L}}{\partial n_i}$, and $\frac{\partial \mathcal{L}}{\partial \lambda}$.

i.e. $$\frac{\partial \mathcal{L}}{\partial n_k} = \sigma_{ij}\left(\delta_{jk} n_i + \delta_{ik} n_j\right) - \lambda(2 n_i \delta_{ik}) = 2(\sigma_{ki} n_i - \lambda n_k) = 0.$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = n_k n_k - 1 = 0.$$

To derive the above expression, we have used the fact that $\boldsymbol{\sigma}$ is symmetric. Above two equations in tensorial form are

$$(\boldsymbol{\sigma} - \lambda \boldsymbol{I})\boldsymbol{n} = 0 \text{ and } |\boldsymbol{n}|^2 - 1 = 0.$$

Above is an eigenvalue problem, where Lagrange parameter λ is identified as eigenvalues. Corresponding eigenvalues are σ_1, σ_2 and σ_3, which are called principal stresess, includes both maximum and minimum values of normal stresses. Corresponding eigenvectors $\boldsymbol{n}_i$, i=1,2,3 are called principal directions. Planes normal to the eigenvectors are called princiapal plane. At principal planes shear stresses vanish and stress tensor in the basis of eigenvectors is written as,

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \sum_{i=1}^{3} \sigma_i \boldsymbol{n}_i \otimes \boldsymbol{n}_i.$$

Maximum and minimum values of shear stresses:

Next we determine the direction n at point x that gives maximum or minimum values of shear stresses.

We start with the representation of stress tensor in eigenvector basis as,

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \sum_{i=1}^{3} \sigma_i \boldsymbol{n}_i \otimes \boldsymbol{n}_i.$$

Then traction vector along a vector $\boldsymbol{N} = N_1\boldsymbol{n}_1 + N_2\boldsymbol{n}_2 + N_3\boldsymbol{n}_3$, which is a unit normal vector to an arbitrary oriented plane, will be

$$\boldsymbol{t} = \boldsymbol{N} \cdot \boldsymbol{\sigma}(\boldsymbol{x}) = \sigma_1 N_1 \boldsymbol{n}_1 + \sigma_2 N_2 \boldsymbol{n}_2 + \sigma_3 N_3 \boldsymbol{n}_3,$$

Now the magnitude of normal and shear stress on the plane will be,

$$\sigma = \boldsymbol{N} \cdot \boldsymbol{t} = \sigma_1 N_1^2 + \sigma_2 N_2^2 + \sigma_3 N_3^2, \text{ and}$$

$$\tau^2 = |\boldsymbol{t}|^2 - \sigma^2 = \sigma_1^2 N_1^2 + \sigma_2^2 N_2^2 + \sigma_3^2 N_3^2 - (\sigma_1 N_1^2 + \sigma_2 N_2^2 + \sigma_3 N_3^2)^2.$$

Now we use the constraint $N_1^2 + N_2^2 + N_3^2 = 1$ to replace N_3 with N_1 and N_2 in the previous equation, so that τ^2 become a function of N_1 and N_2 only. To identify the extremal values of τ^2 the expression is partially differentiated w.r.t N_1 and N_2 and equated to zero as,

$$\frac{\partial \tau^2}{\partial N_1} = 2N_1(\sigma_1 - \sigma_3)\left[\sigma_1 - \sigma_3 - 2\left\{(\sigma_1 - \sigma_3)N_1^2 + (\sigma_2 - \sigma_3)N_2^2\right\}\right] = 0,$$

$$\frac{\partial \tau^2}{\partial N_2} = 2N_2(\sigma_2 - \sigma_3)\left[\sigma_2 - \sigma_3 - 2\left\{(\sigma_1 - \sigma_3)N_1^2 + (\sigma_2 - \sigma_3)N_2^2\right\}\right] = 0.$$

To solve for the direction of extremal shear stress the very first choice can be N_1=N_2=0, with N_3=±1. Similar if we choose N_1=N_3=0, corresponding component is N_2=±1 and for N_2=N_3=0, corresponding component is N_1=±1. Thus we obtain three unit vector which are same as eigenvectors. Basically these vector corresponds to the minimum solution as in the principal direction magnitude of shear stress is zero.

Another solution is obtained by setting N_1=0, which results N_2=$\pm 1/\sqrt{2}$ from the last set of equations and $N_3 = \pm 1/\sqrt{2}$ by the condition that $|\boldsymbol{N}| = 1$.

Similarly by setting N_2=0, we obtain $N_1 = N_3 = \pm 1/\sqrt{2}$ and, by setting N_3=0, $N_1 = N_2 = \pm 1/\sqrt{2}$. These vector putting back in the equation for τ^2 results in the following extremal values.

$$\boldsymbol{N} = \pm \frac{1}{\sqrt{2}}\boldsymbol{n}_2 \pm \frac{1}{\sqrt{2}}\boldsymbol{n}_3, \quad \tau^2 = \frac{1}{4}\left(\sigma_2 - \sigma_3\right)^2,$$

$$\boldsymbol{N} = \pm \frac{1}{\sqrt{2}}\boldsymbol{n}_1 \pm \frac{1}{\sqrt{2}}\boldsymbol{n}_2, \quad \tau^2 = \frac{1}{4}\left(\sigma_1 - \sigma_2\right)^2,$$

$$\boldsymbol{N} = \pm \frac{1}{\sqrt{2}}\boldsymbol{n}_1 \pm \frac{1}{\sqrt{2}}\boldsymbol{n}_3, \quad \tau^2 = \frac{1}{4}\left(\sigma_1 - \sigma_3\right)^2.$$

Consequently maximum value of shear stress is given by maximum of above three, i.e., $\tau_{\max} = \frac{1}{2}|\sigma_{\max} - \sigma_{\min}|$, where $\sigma_{\max}$ and $\sigma_{\min}$ indicate the maximum and minimum values of principal stresses.

It is important to note that the maximum shear stress acts on a plane that is shifted about an angle of $\pm 45°$ to the principal plane in which the maximum and minimum principal stresses act.

By substituting the it can be shown that the normal stress associated with $\tau_{\max}$ has the value

$$\sigma = \frac{1}{2}|\sigma_{\max} + \sigma_{\min}|.$$

Stress invariants

Following are the invariant of stress tensor.

$$I_1 = \mathrm{tr}\boldsymbol{\sigma} = \sigma_{ii}$$

$$I_2 = \frac{1}{2}\left[(\mathrm{tr}\boldsymbol{\sigma})^2 - \mathrm{tr}\boldsymbol{\sigma}^2\right] = \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji})$$

$$I_3 = \det \boldsymbol{\sigma}$$

Hydrostatic and Deviatoric stress

Stress being a second order tensor can be splitted in to a spherical and deviatoric part as,

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + \boldsymbol{\sigma}' \text{ or } \sigma_{ij} = -p\delta_{ij} + \sigma'_{ij},$$

where $p = -\mathrm{tr}\boldsymbol{\sigma},$ is the hydrostatic stress component and $\boldsymbol{\sigma}'$ is called the deviatoric stress. Hydrostatic stress is the volume changing and shape conserving part, whereas deviatoric stress is volume conserving and shape changing part of the stress.