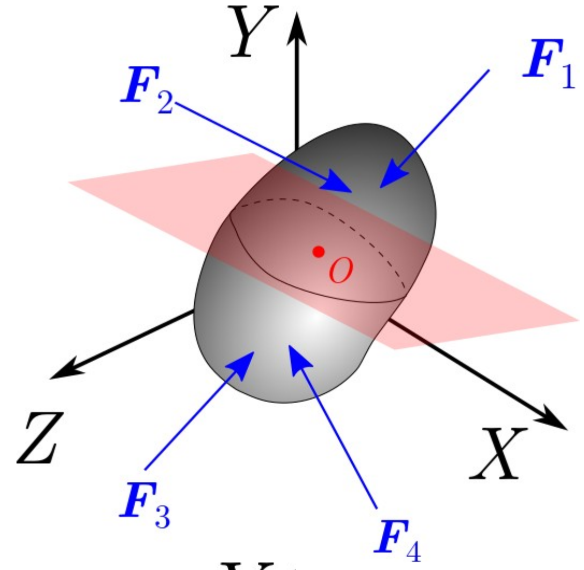
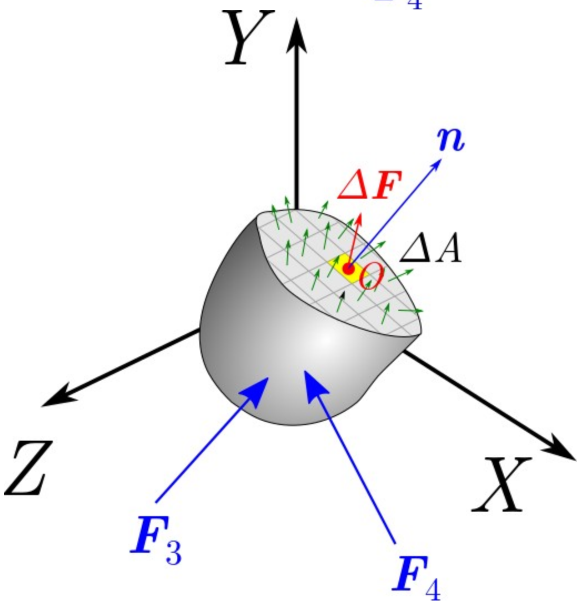
ME231: Solid Mechanics-I

Stress and Strain

Stress





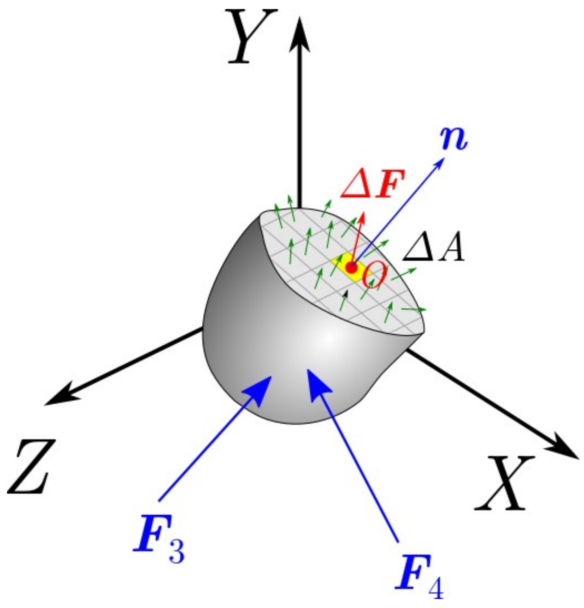
Consider a body acted upon by several external forces.

To determine the internal forces at some point O in the interior of the body, we cut the body by an imaginary plane passing though point O.

For the separate halves of the body to be in equilibrium there must be internal forces transmitted across the cutting plane.

We divide the plane into a number of small areas and measure the force acting on each of these. We will see that these forces in general vary from one area to the other.

Now, consider an elemental area ΔA in the neighborhood of point O. The force on the area is $\Delta \boldsymbol{F}$, and normal to the area is $\boldsymbol{n}$.



We define a **traction (or stress) vector $\boldsymbol{t}^{\boldsymbol{n}}$** corresponding to normal $\boldsymbol{n}$ and point O is defined as,

$$\boldsymbol{t}^{\boldsymbol{n}} = \lim_{\Delta A \to 0} \frac{\Delta \boldsymbol{F}}{\Delta A}. \qquad \text{......................}(1)$$

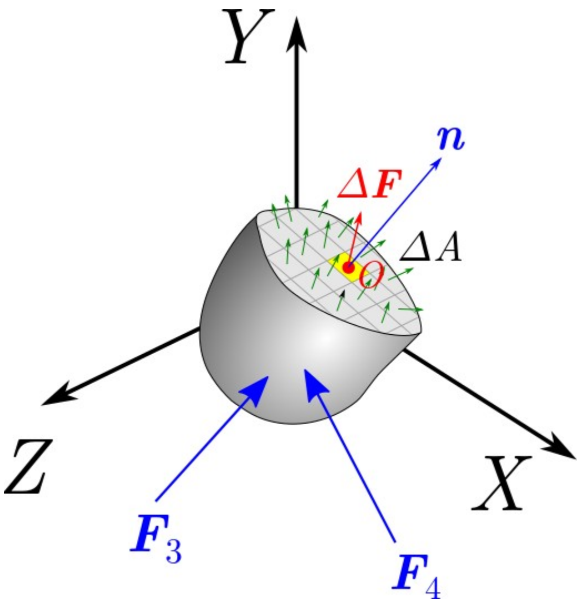
Thus, $\boldsymbol{t}^{\boldsymbol{n}}$ is the stress vector at point O acting on a plane having normal $\boldsymbol{n}$.

Following points must be noted about the stress:

1. The physical dimensions of the stress is force per unit area,

2. Stress is defined at a point upon an imaginary plane, dividing the material into two parts

3. Stress is a vector equivalent to the action of one part of the material upon another.

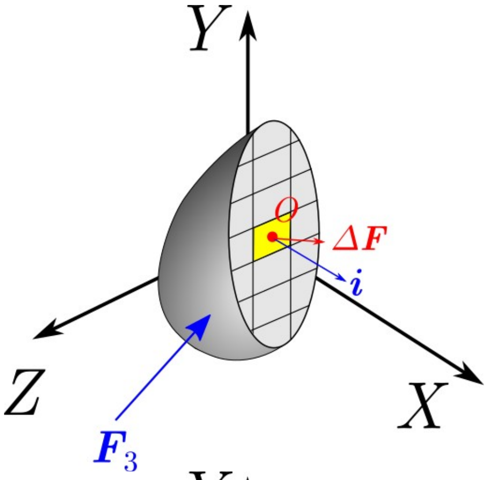


Stress vector in terms of its component along coordinate axes can be written as,

$$\boldsymbol{t^n} = t_x^n\,\boldsymbol{i} + t_y^n\,\boldsymbol{j} + t_z^n\,\boldsymbol{k}. \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots(2)$$

It is convenient to express the stress vector $\boldsymbol{t^n}$ in terms of stress vectors on planes perpendicular to coordinate axes.

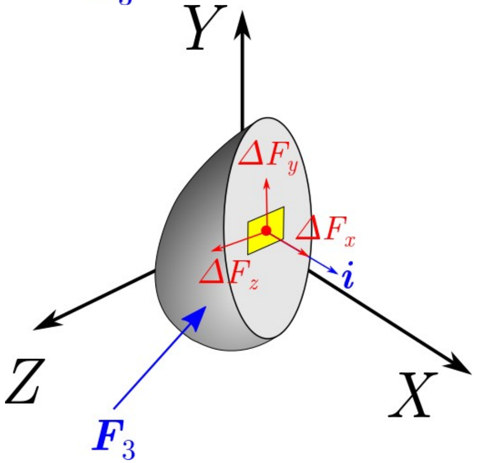
Once the components are obtained along these plane we can relate it to the components of $\boldsymbol{t^n}$ in (2).



We start by having an imaginary cut on the body by a plane perpendicular to X-axis and passing through point O, as shown in figure.

Now, the stress vector at point O acting on a plane having normal $\boldsymbol{i}$ become,

$$\boldsymbol{t}^{\boldsymbol{i}} = \lim_{\Delta A_x \to 0} \frac{\Delta \boldsymbol{F}}{\Delta A_x}.$$



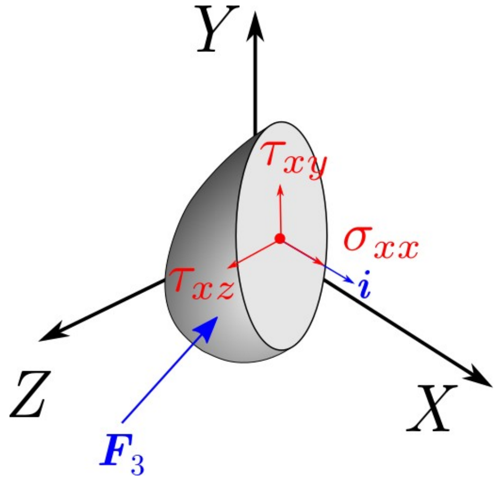
Force $\Delta \boldsymbol{F}$ can further be resolved along the coordinate axes, then,

$$\boldsymbol{t}^{\boldsymbol{i}} = \lim_{\Delta A_x \to 0} \frac{1}{\Delta A_x} \left(\Delta F_x \, \boldsymbol{i} + \Delta F_y \, \boldsymbol{j} + \Delta F_z \, \boldsymbol{k} \right), \quad \text{or}$$

$$\boldsymbol{t}^{\boldsymbol{i}} = \sigma_{xx} \, \boldsymbol{i} + \tau_{xy} \, \boldsymbol{j} + \tau_{xz} \, \boldsymbol{k}, \qquad \cdots\cdots\cdots\cdots\cdots(3)$$

where,

$$\sigma_{xx} = \lim_{\Delta A_x \to 0} \frac{\Delta F_x}{\Delta A_x}, \quad \tau_{xy} = \lim_{\Delta A_x \to 0} \frac{\Delta F_y}{\Delta A_x}, \text{ and } \tau_{xz} = \lim_{\Delta A_x \to 0} \frac{\Delta F_z}{\Delta A_x}. \qquad \cdots\cdots\cdots\cdots\cdots(4)$$



σ_{xx}, τ_{xy} and τ_{xz} are called stress components associated with the x-face at point O. These stress components are shown in figure.

In the similar manner, now we cut the body with a plane, which is perpendicular to y-axis, and passing through point O, to obtain the traction vector,

$$\boldsymbol{t}^{\boldsymbol{j}} = \tau_{yx}\, \boldsymbol{i} + \sigma_{yy}\, \boldsymbol{j} + \tau_{yz}\, \boldsymbol{k},$$

where, $$\tau_{yx} = \lim_{\Delta A_y \to 0} \frac{\Delta F_x}{\Delta A_y}, \quad \sigma_{yy} = \lim_{\Delta A_y \to 0} \frac{\Delta F_y}{\Delta A_y}, \text{ and } \tau_{yz} = \lim_{\Delta A_y \to 0} \frac{\Delta F_z}{\Delta A_y}. \qquad \cdots\cdots\cdots(5)$$

Cutting the body with a plane, which is perpendicular to z – axis, and passing through point O, we get the traction vector,

$$\boldsymbol{t}^{\boldsymbol{k}} = \tau_{zx}\, \boldsymbol{i} + \tau_{zy}\, \boldsymbol{j} + \sigma_{zz}\, \boldsymbol{k},$$

where,

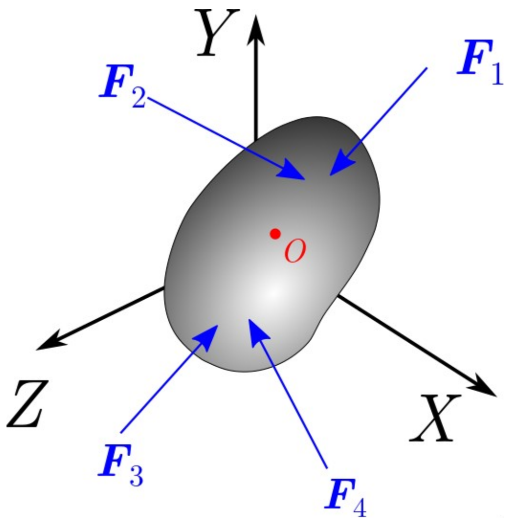
$$\tau_{zx} = \lim_{\Delta A_z \to 0} \frac{\Delta F_x}{\Delta A_z}, \quad \tau_{zy} = \lim_{\Delta A_z \to 0} \frac{\Delta F_y}{\Delta A_z}, \text{ and } \sigma_{zz} = \lim_{\Delta A_z \to 0} \frac{\Delta F_z}{\Delta A_z}. \qquad \cdots\cdots\cdots(6)$$

Now, at point O we see that the state of stress is dependent on nine stress components, which are

$$\begin{matrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz}. \end{matrix} \quad \cdots\cdots\cdots\cdots(6)$$

Later, we will see how to relate the components of traction vector $\boldsymbol{t}^n$ to these nine components.

Now, think about the overall continuous body again.



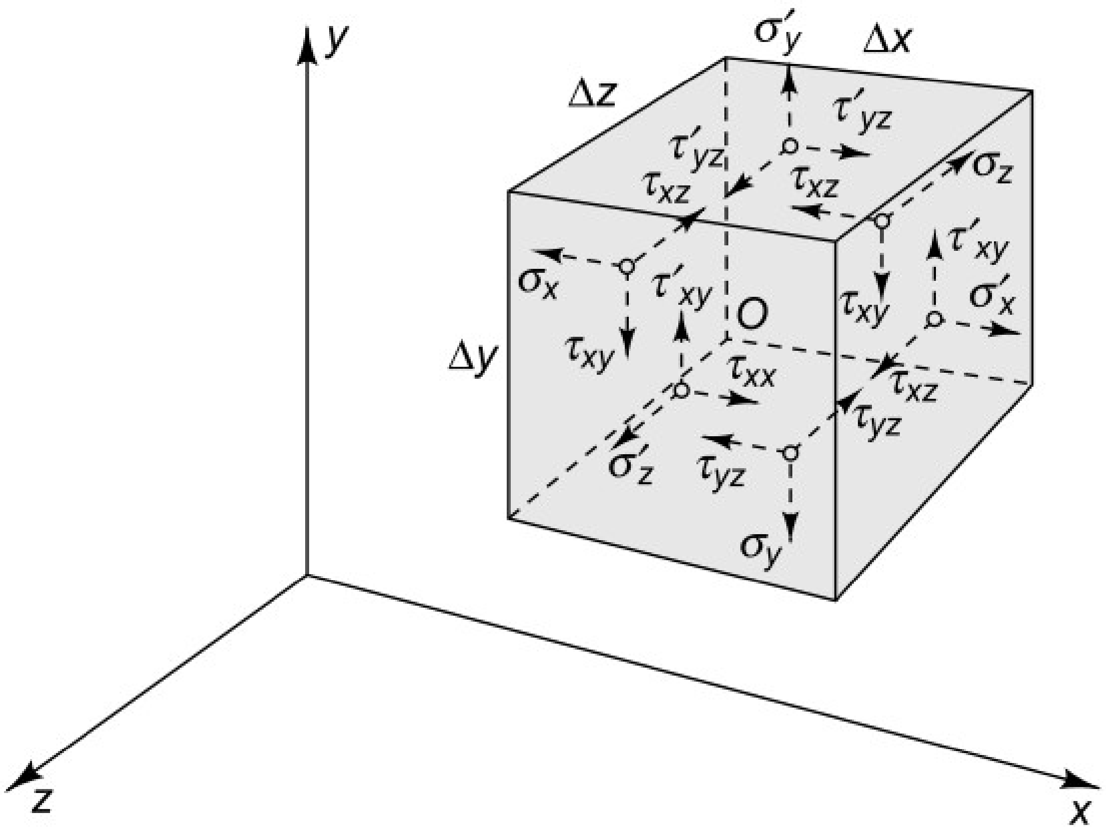
Along with the plane perpendicular to x-axis through O, we pass another plane perpendicular to x-axis at a distance Δx from O in the x-direction.

Similarly. along with the plane perpendicular to y-axis through O, we pass another plane perpendicular to y-axis at a distance Δy from O in the y-direction.

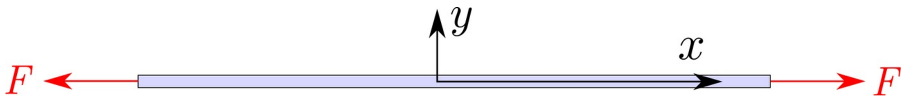
And along with the plane perpendicular to z-axis through O, we pass another plane perpendicular to z-axis at a distance Δz from O in the z-direction.

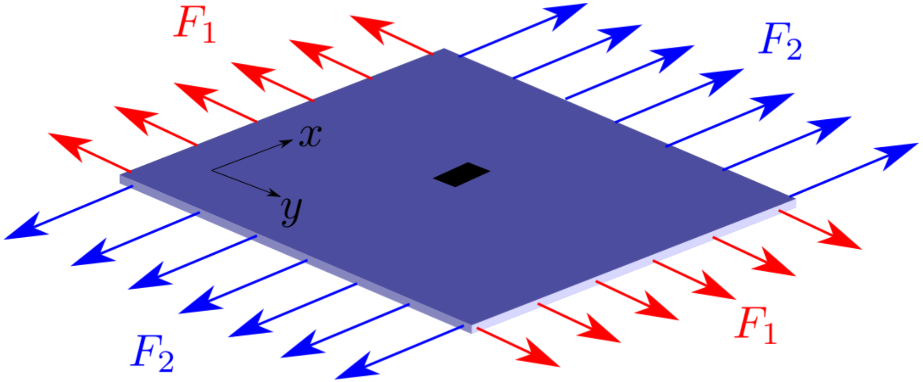
By doing this we get a cuboid having length Δx, Δy, and Δz in x-, y- and z- directions respectively. Corresponding to every face of the cuboid stress components are shown.

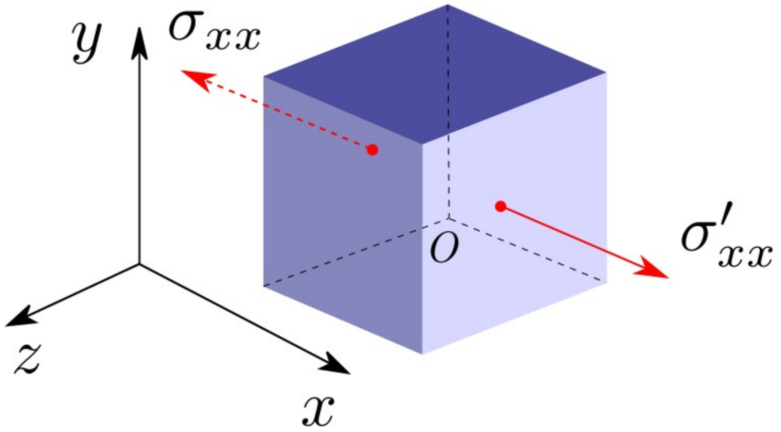
Note the direction of stress. Sign convention for faces and stress components follows same rules as that of the force components discussed earlier.

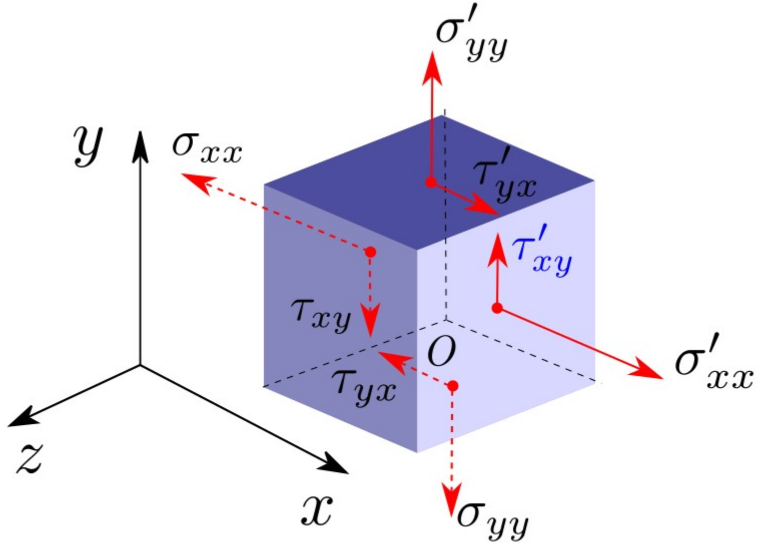


Some simpler stress situations









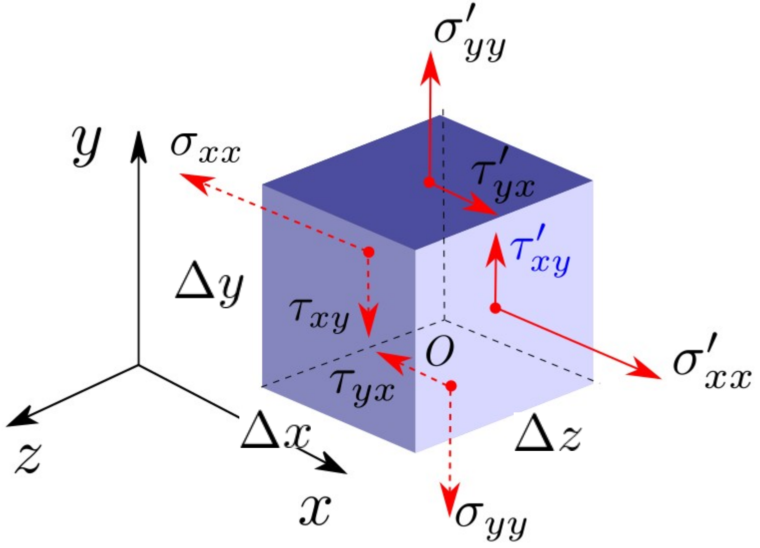
For a thin sheet there is no force in the z-direction. Hence stress components at the z-faces are zero. As the thickness is small the z-direction stress components are almost constant throughout the thickness and hence it can be assumed these stress components are zero throughout the thickness. For such case state of stress at a point is given as

$$\begin{matrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{matrix} \qquad \cdots\cdots\cdots\cdots(8)$$

where stress components are only functions of **only $\boldsymbol{x}$ and $\boldsymbol{y}$**. This combination of stress components is called **plane stress** in the xy plane.

Plane stress condition is of practical importance in several applications, which will be discussed later.

Equilibrium of a differential element in plane stress



A small element from a body is shown. For a plane stress case stresses working on the element is also shown. If the body is in equilibrium then the element should also be in equilibrium.

If stress components at the negative x-face is σ_x and τ_{xy}, then the components at a parallel face (at a distance of Δx) can be approximated as,

$$\sigma'_{xx} = \sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\Delta x, \quad \tau'_{xy} = \tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\Delta x. \qquad \cdots\cdots\cdots\cdots(9)$$

Similarly, stress components at the positive y-face will be,

$$\sigma'_{yy} = \sigma_{yy} + \frac{\partial \sigma_{yy}}{\partial y}\Delta y, \quad \tau'_{yx} = \tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\Delta y. \qquad \cdots\cdots\cdots\cdots(10)$$

Applying equilibrium conditions for the element as,

$$\sum F_x = (\sigma'_{xx} - \sigma_{xx})\Delta z\Delta y + (\tau'_{yx} - \tau_{yz})\Delta x\Delta z = 0. \qquad \cdots\cdots\cdots\cdot(11)$$

Using (9) and (10) in (11),

$$\left[\sigma_{xx} + \frac{\partial\sigma_{xx}}{\partial x}\Delta x - \sigma_{xx}\right]\Delta z\Delta y + \left[\tau_{yx} + \frac{\partial\tau_{yx}}{\partial y}\Delta y - \tau_{yx}\right]\Delta x\Delta z = 0,$$

$$\Rightarrow \left[\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y}\right]\Delta x\Delta y\Delta z = 0,$$

which gives one of the condition of equilibrium of the element as,

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} = 0. \qquad \cdots\cdots\cdots\cdot(12)$$

Similarly, consider the equilibrium condition in y-direction, i.e., $\Sigma F_y=0$ will give another equilibrium condition as, `

$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} = 0. \qquad \cdots\cdots\cdots\cdot(13)$$

Now let us consider the moment equilibrium about the center of the element, i.e., ΣM=0, as

$$\left[\tau'_{xy}\frac{\Delta x}{2}+\tau_{xy}\frac{\Delta x}{2}\right]\Delta y\Delta z-\left[\tau'_{yx}\frac{\Delta y}{2}+\tau_{yx}\frac{\Delta y}{2}\right]\Delta x\Delta z=0$$

Using (9) and (10),

$$\frac{1}{2}\left[\tau_{xy}+\frac{\partial\tau_{xy}}{\partial x}\Delta x+\tau_{xy}\right]\Delta x\Delta y\Delta z-\frac{1}{2}\left[\tau_{yx}+\frac{\partial\tau_{yx}}{\partial y}\Delta y+\tau_{yx}\right]\Delta x\Delta y\Delta z=0,$$

$$\Rightarrow\tau_{xy}+\frac{\partial\tau_{xy}}{\partial x}\frac{\Delta x}{2}-\tau_{yx}-\frac{\partial\tau_{yx}}{\partial y}\frac{\Delta y}{2}=0, \qquad \cdots\cdots\cdots\cdot(14)$$

when Δx and Δy tend to zero, then (14) yields, $\tau_{xy}=\tau_{yx}$. ··········(15)

Equation (15) suggests that for a body in plane stress the shear-stress components on perpendicular faces must be equal in magnitude. Using (15) conditions for equilibrium i.e., (12) and (13), become,

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}=0.$$

$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}=0.$$

In the similar manner, considering the moment equilibrium of an element with all stress components, following conditions are obtained.

$$\tau_{xy} = \tau_{yx}, \qquad \tau_{yz} = \tau_{zy}, \qquad \text{and} \qquad \tau_{xz} = \tau_{zx}. \qquad \cdots\cdots\cdots\cdot(16)$$

Using force equilibrium conditions and with the use of (15), equilibrium equations in three-dimensions can be derived, which are as follows.

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} &= 0, \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} &= 0, \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} &= 0. \end{aligned} \qquad \cdots\cdots\cdots\cdot(17)$$