Elasticity Tensor

Consider the nonlinear second Piola-Kirchhoff stress tensor $\boldsymbol{S}$ of a point at a certain time t. We look at $\boldsymbol{S}$ as a nonlinear tensor-valued tensor function of one variable. We assume this variable to be right Cauchy-Green tensor $\boldsymbol{C}$.

Then from the definition of differentiation of a tensor valued function, we can write

$$d\boldsymbol{S} = \frac{\partial \boldsymbol{S}}{\partial \boldsymbol{C}} : d\boldsymbol{C}, \quad \text{or} \quad d\boldsymbol{S} = 2\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{C}} : \frac{1}{2} d\boldsymbol{C}, \qquad \cdots\cdots\cdots\cdots(27)$$

where,

$$\mathbb{C} = 2\frac{\partial \boldsymbol{S}(\boldsymbol{C})}{\partial \boldsymbol{C}}, \qquad \cdots\cdots\cdots\cdots(28)$$

is characterizes the gradient of function S and relates the work conjugate pairs of stress and strain tensors. It is a fourth order tensor, which measures the change in stress which results from a change in strain and is referred to as **the elasticity tensor in the material description** or the **referential tensor of elasticities**. Recall that a fourth order tensor has 81 components.

Due to the symmetry of stress and strain tensor (i.e., $\boldsymbol{S}$ and $\boldsymbol{C}$), the elasticity tensor $\mathbb{C}$ is always symmetric in its first and second slots, i.e. *ij* and in its third and fourth slots, i.e. *kl*.

$$\text{i.e., } \mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{ijlk}. \qquad \cdots\cdots\cdots\cdots\cdots(29)$$

This is called the **minor symmetries** of $\mathbb{C}$, which reduced the number of independent components from 81 to 36 at each strain state. Note that, the minor symmetry condition is independent of the existence of a strain-energy function and holds for all elastic materials.

Now, If we consider hyperelasticity, then their exist of a strain energy function Ψ, and $\boldsymbol{S}$ may be derived from it. Hence,

$$\mathbb{C} = 2\frac{\partial \boldsymbol{S}(\boldsymbol{C})}{\partial \boldsymbol{C}} = 4\frac{\partial^2 \Psi(\boldsymbol{C})}{\partial \boldsymbol{C} \partial \boldsymbol{C}}, \quad \text{or} \quad \mathbb{C} = 4\frac{\partial^2 \Psi}{\partial C_{ij} \partial C_{kl}}. \qquad \cdots\cdots\cdots\cdots\cdots(30)$$

Now, we can observe that $\mathbb{C}_{ijkl} = \mathbb{C}_{klij}$, which is said to be **major symmetry** of $\mathbb{C}$. This symmetry reduced the number of independent constants from 36 to 21.

The elasticity tensor in the spatial description or the spatial tensor of elasticities, denoted by c, is defined as the push-forward operation of $\mathbb{C}$ times a factor of J^{-1} as

$$c_{pqrs} = J^{-1} F_{pi} F_{qj} F_{rk} F_{sl} \mathbb{C}_{ijkl}. \qquad \cdots\cdots\cdots\cdots\cdots(31)$$

$\boldsymbol{c}$ also possesses major and minor symmetries similar to $\mathbb{C}$.

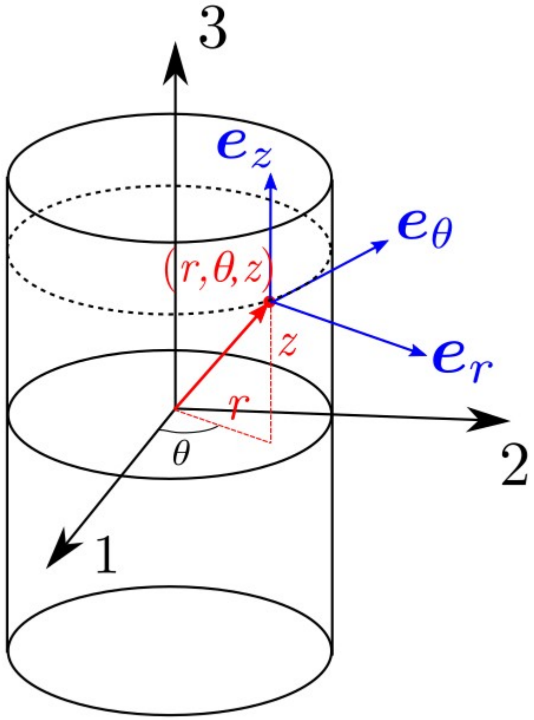
The number of independent components is further reduced if the material possesses any material symmetry. For isotropic material most general form of fourth order tensor is as follows,

$$\mathbb{C}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}. \qquad \text{....................(32)}$$

Cylindrical Coordinate System

Selection of appropriate coordinate system can facilitate easier solution to many problems. We will rewrite some important relation in Cylindrical coordinate system.

Position vector of a point, its coordinates, and basis vectors in Cartesian coordinate system and Cylindrical coordinate system are related in the following manner.



$$x_1 = r\cos\theta$$

$$x_2 = r\sin\theta$$

$$x_3 = z$$

$$r = \sqrt{x_1^2 + x_2^2}$$

$$\theta = \tan^{-1}(x_2/x_1)$$

$$\boldsymbol{e}_r = \boldsymbol{e}_1\cos\theta + \boldsymbol{e}_2\sin\theta$$

$$\boldsymbol{e}_\theta = -\boldsymbol{e}_1\sin\theta + \boldsymbol{e}_2\cos\theta$$

$$\boldsymbol{e}_z = \boldsymbol{e}_3$$

$$\boldsymbol{r} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 = r\boldsymbol{e}_r + z\boldsymbol{e}_z$$

It can be observed that the basis vector $\boldsymbol{e}_r$ lies along the radial direction, and $\boldsymbol{e}_\theta$ is always perpendicular to $\boldsymbol{e}_r$. Thus $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ change with change in θ. Thus, it can be shown that,

$$\frac{\partial \boldsymbol{e}_r}{\partial \theta} = \boldsymbol{e}_\theta, \quad \frac{\partial \boldsymbol{e}_\theta}{\partial \theta} = -\boldsymbol{e}_r.$$

Derivative with respect to r and θ can be changed to the derivative w.r.t. x_1 and x_2 using chain rule of differentiation as,

$$\begin{aligned}
\frac{\partial}{\partial r} &= \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial r} + \frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial r} = \frac{\partial}{\partial x_1}\cos\theta + \frac{\partial}{\partial x_2}\sin\theta, \\
\frac{\partial}{\partial \theta} &= \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial \theta} + \frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial \theta} = -r\sin\theta\frac{\partial}{\partial x_1} + r\cos\theta\frac{\partial}{\partial x_2},
\end{aligned}$$

or vice-versa as,

$$\begin{aligned}
\frac{\partial}{\partial x_1} &= \frac{\partial}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial}{\partial \theta}\sin\theta, \\
\frac{\partial}{\partial x_2} &= \frac{\partial}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial}{\partial \theta}\cos\theta.
\end{aligned}$$

Using the definition of Gradient operator in the Cylindrical coordinates can be written as,

$$\boldsymbol{\nabla} = \frac{\partial}{\partial x_1}\boldsymbol{e}_1 + \frac{\partial}{\partial x_2}\boldsymbol{e}_2 + \frac{\partial}{\partial x_3}\boldsymbol{e}_3,$$

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial}{\partial \theta}\sin\theta\right)\boldsymbol{e}_1 + \left(\frac{\partial}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial}{\partial \theta}\cos\theta\right)\boldsymbol{e}_2 + \frac{\partial}{\partial z}\boldsymbol{e}_z,$$

$$\boldsymbol{\nabla} = \frac{\partial}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\boldsymbol{e}_\theta + \frac{\partial}{\partial z}\boldsymbol{e}_z.$$

$$\boldsymbol{\nabla}(\bullet) = \frac{\partial(\bullet)}{\partial x_i}\boldsymbol{e}_i = \frac{\partial(\bullet)}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial(\bullet)}{\partial \theta}\boldsymbol{e}_\theta + \frac{\partial(\bullet)}{\partial z}\boldsymbol{e}_z.$$

A first order tensor $\boldsymbol{u}$ and a second order tensor $\boldsymbol{\sigma}$ in Cylindrical coordinate system is given as,

$$\boldsymbol{u} = u_r\boldsymbol{e}_r + u_\theta\boldsymbol{e}_\theta + u_z\boldsymbol{e}_z$$

$$\begin{aligned}\boldsymbol{\sigma} =& \sigma_{rr}\boldsymbol{e}_r\boldsymbol{e}_r + \sigma_{r\theta}\boldsymbol{e}_r\boldsymbol{e}_\theta + \sigma_{rz}\boldsymbol{e}_r\boldsymbol{e}_z + \sigma_{\theta r}\boldsymbol{e}_\theta\boldsymbol{e}_r + \sigma_{\theta\theta}\boldsymbol{e}_\theta\boldsymbol{e}_\theta + \sigma_{\theta z}\boldsymbol{e}_\theta\boldsymbol{e}_z + \\ & \sigma_{zr}\boldsymbol{e}_z\boldsymbol{e}_r + \sigma_{z\theta}\boldsymbol{e}_z\boldsymbol{e}_\theta + \sigma_{zz}\boldsymbol{e}_z\boldsymbol{e}_z.\end{aligned}$$

Gradient of vector $\boldsymbol{u}$ is now defined as,

$$\boldsymbol{\nabla}\boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial x_i}\boldsymbol{e}_i = \frac{\partial \boldsymbol{u}}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial \boldsymbol{u}}{\partial \theta}\boldsymbol{e}_\theta + \frac{\partial \boldsymbol{u}}{\partial z}\boldsymbol{e}_z.$$

$$\begin{aligned}\boldsymbol{\nabla}\boldsymbol{u} =& \frac{\partial u_r}{\partial r}\boldsymbol{e}_r\boldsymbol{e}_r + \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right)\boldsymbol{e}_r\boldsymbol{e}_\theta + \frac{\partial u_r}{\partial z}\boldsymbol{e}_r\boldsymbol{e}_z + \\ & \frac{\partial u_\theta}{\partial r}\boldsymbol{e}_\theta\boldsymbol{e}_r + \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right)\boldsymbol{e}_\theta\boldsymbol{e}_\theta + \frac{\partial u_\theta}{\partial z}\boldsymbol{e}_\theta\boldsymbol{e}_z + \\ & \frac{\partial u_z}{\partial r}\boldsymbol{e}_z\boldsymbol{e}_r + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\boldsymbol{e}_z\boldsymbol{e}_\theta + \frac{\partial u_z}{\partial z}\boldsymbol{e}_z\boldsymbol{e}_z\end{aligned}$$

Using the above expression, small strain components in Cylindrical coordinates are given as,

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right), \quad \varepsilon_z = \frac{\partial u_z}{\partial z}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right), \quad \varepsilon_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right), \quad \varepsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right).$$

Divergence of second order tensor $\boldsymbol{\sigma}$ is

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \frac{\partial \boldsymbol{\sigma}}{\partial r} \cdot \boldsymbol{e}_r + \frac{1}{r}\frac{\partial \boldsymbol{\sigma}}{\partial \theta} \cdot \boldsymbol{e}_\theta + \frac{\partial \boldsymbol{\sigma}}{\partial z} \cdot \boldsymbol{e}_z.$$

Let us first derive the second term (i.e. $\partial\boldsymbol{\sigma}/\partial\theta$)

$$\begin{aligned}
\frac{\partial \boldsymbol{\sigma}}{\partial \theta} =& \frac{\partial \sigma_{rr}}{\partial \theta}\boldsymbol{e}_r\boldsymbol{e}_r + \sigma_{rr}\frac{\partial \boldsymbol{e}_r}{\partial \theta}\boldsymbol{e}_r + \sigma_{rr}\boldsymbol{e}_r\frac{\partial \boldsymbol{e}_r}{\partial \theta} + \frac{\partial \sigma_{r\theta}}{\partial \theta}\boldsymbol{e}_r\boldsymbol{e}_\theta + \sigma_{r\theta}\frac{\partial \boldsymbol{e}_r}{\partial \theta}\boldsymbol{e}_\theta + \sigma_{r\theta}\boldsymbol{e}_r\frac{\partial \boldsymbol{e}_\theta}{\partial \theta} + \\
& \frac{\partial \sigma_{rz}}{\partial \theta}\boldsymbol{e}_r\boldsymbol{e}_z + \sigma_{rz}\frac{\partial \boldsymbol{e}_r}{\partial \theta}\boldsymbol{e}_z + \sigma_{rz}\boldsymbol{e}_r\frac{\partial \boldsymbol{e}_z}{\partial \theta} + \frac{\partial \sigma_{\theta r}}{\partial \theta}\boldsymbol{e}_\theta\boldsymbol{e}_r + \sigma_{\theta r}\frac{\partial \boldsymbol{e}_\theta}{\partial \theta}\boldsymbol{e}_r + \sigma_{\theta r}\boldsymbol{e}_\theta\frac{\partial \boldsymbol{e}_r}{\partial \theta} + \\
& \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\boldsymbol{e}_\theta\boldsymbol{e}_\theta + \sigma_{\theta\theta}\frac{\partial \boldsymbol{e}_\theta}{\partial \theta}\boldsymbol{e}_\theta + \sigma_{\theta\theta}\boldsymbol{e}_\theta\frac{\partial \boldsymbol{e}_\theta}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial \theta}\boldsymbol{e}_\theta\boldsymbol{e}_z + \sigma_{\theta z}\frac{\partial \boldsymbol{e}_\theta}{\partial \theta}\boldsymbol{e}_z + \sigma_{\theta z}\boldsymbol{e}_\theta\frac{\partial \boldsymbol{e}_z}{\partial \theta} + \\
& \frac{\partial \sigma_{zr}}{\partial \theta}\boldsymbol{e}_z\boldsymbol{e}_r + \sigma_{zr}\frac{\partial \boldsymbol{e}_z}{\partial \theta}\boldsymbol{e}_r + \sigma_{zr}\boldsymbol{e}_z\frac{\partial \boldsymbol{e}_r}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial \theta}\boldsymbol{e}_z\boldsymbol{e}_\theta + \sigma_{z\theta}\frac{\partial \boldsymbol{e}_z}{\partial \theta}\boldsymbol{e}_\theta + \sigma_{z\theta}\boldsymbol{e}_z\frac{\partial \boldsymbol{e}_\theta}{\partial \theta} + \\
& \frac{\partial \sigma_{zz}}{\partial \theta}\boldsymbol{e}_z\boldsymbol{e}_z + \sigma_{zz}\frac{\partial \boldsymbol{e}_z}{\partial \theta}\boldsymbol{e}_z + \sigma_{zz}\boldsymbol{e}_z\frac{\partial \boldsymbol{e}_z}{\partial \theta}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \boldsymbol{\sigma}}{\partial \theta} =& \frac{\partial \sigma_{rr}}{\partial \theta}\boldsymbol{e}_r\boldsymbol{e}_r + \sigma_{rr}\boldsymbol{e}_\theta\boldsymbol{e}_r + \sigma_{rr}\boldsymbol{e}_r\boldsymbol{e}_\theta + \frac{\partial \sigma_{r\theta}}{\partial \theta}\boldsymbol{e}_r\boldsymbol{e}_\theta + \sigma_{r\theta}\boldsymbol{e}_\theta\boldsymbol{e}_\theta - \sigma_{r\theta}\boldsymbol{e}_r\boldsymbol{e}_r + \\
& \frac{\partial \sigma_{rz}}{\partial \theta}\boldsymbol{e}_r\boldsymbol{e}_z + \sigma_{rz}\boldsymbol{e}_\theta\boldsymbol{e}_z + \sigma_{rz}\boldsymbol{e}_r \cdot 0 + \frac{\partial \sigma_{\theta r}}{\partial \theta}\boldsymbol{e}_\theta\boldsymbol{e}_r - \sigma_{\theta r}\boldsymbol{e}_r\boldsymbol{e}_r + \sigma_{\theta r}\boldsymbol{e}_\theta\boldsymbol{e}_\theta + \\
& \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\boldsymbol{e}_\theta\boldsymbol{e}_\theta - \sigma_{\theta\theta}\boldsymbol{e}_r\boldsymbol{e}_\theta - \sigma_{\theta\theta}\boldsymbol{e}_\theta\boldsymbol{e}_r + \frac{\partial \sigma_{\theta z}}{\partial \theta}\boldsymbol{e}_\theta\boldsymbol{e}_z - \sigma_{\theta z}\boldsymbol{e}_r\boldsymbol{e}_z + \sigma_{\theta z}\boldsymbol{e}_\theta \cdot 0 + \\
& \frac{\partial \sigma_{zr}}{\partial \theta}\boldsymbol{e}_z\boldsymbol{e}_r + \sigma_{zr} \cdot 0\boldsymbol{e}_r + \sigma_{zr}\boldsymbol{e}_z\boldsymbol{e}_\theta + \frac{\partial \sigma_{z\theta}}{\partial \theta}\boldsymbol{e}_z\boldsymbol{e}_\theta + \sigma_{z\theta} \cdot 0\boldsymbol{e}_\theta - \sigma_{z\theta}\boldsymbol{e}_z\boldsymbol{e}_r + \\
& \frac{\partial \sigma_{zz}}{\partial \theta}\boldsymbol{e}_z\boldsymbol{e}_z + \sigma_{zz} \cdot 0\boldsymbol{e}_z + \sigma_{zz}\boldsymbol{e}_z \cdot 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \boldsymbol{\sigma}}{\partial \theta} \cdot \boldsymbol{e}_\theta = \Bigg(& \frac{\partial \sigma_{rr}}{\partial \theta}\boldsymbol{e}_r\boldsymbol{e}_r + \sigma_{rr}\boldsymbol{e}_\theta\boldsymbol{e}_r + \sigma_{rr}\boldsymbol{e}_r\boldsymbol{e}_\theta + \frac{\partial \sigma_{r\theta}}{\partial \theta}\boldsymbol{e}_r\boldsymbol{e}_\theta + \sigma_{r\theta}\boldsymbol{e}_\theta\boldsymbol{e}_\theta - \sigma_{r\theta}\boldsymbol{e}_r\boldsymbol{e}_r + \\
& \frac{\partial \sigma_{rz}}{\partial \theta}\boldsymbol{e}_r\boldsymbol{e}_z + \sigma_{rz}\boldsymbol{e}_\theta\boldsymbol{e}_z + \frac{\partial \sigma_{\theta r}}{\partial \theta}\boldsymbol{e}_\theta\boldsymbol{e}_r - \sigma_{\theta r}\boldsymbol{e}_r\boldsymbol{e}_r + \sigma_{\theta r}\boldsymbol{e}_\theta\boldsymbol{e}_\theta + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\boldsymbol{e}_\theta\boldsymbol{e}_\theta - \\
& \sigma_{\theta\theta}\boldsymbol{e}_r\boldsymbol{e}_\theta - \sigma_{\theta\theta}\boldsymbol{e}_\theta\boldsymbol{e}_r + \frac{\partial \sigma_{\theta z}}{\partial \theta}\boldsymbol{e}_\theta\boldsymbol{e}_z - \sigma_{\theta z}\boldsymbol{e}_r\boldsymbol{e}_z + \frac{\partial \sigma_{zr}}{\partial \theta}\boldsymbol{e}_z\boldsymbol{e}_r + \sigma_{zr}\boldsymbol{e}_z\boldsymbol{e}_\theta + \\
& \frac{\partial \sigma_{z\theta}}{\partial \theta}\boldsymbol{e}_z\boldsymbol{e}_\theta\boldsymbol{e}_\theta - \sigma_{z\theta}\boldsymbol{e}_z\boldsymbol{e}_r + \frac{\partial \sigma_{zz}}{\partial \theta}\boldsymbol{e}_z\boldsymbol{e}_z \Bigg) \cdot \boldsymbol{e}_\theta
\end{aligned}
$$

$$\frac{\partial \boldsymbol{\sigma}}{\partial \theta} \cdot \boldsymbol{e}_\theta = \left(\sigma_{rr} - \sigma_{\theta\theta} + \frac{\partial \sigma_{r\theta}}{\partial \theta}\right)\boldsymbol{e}_r + \left(\sigma_{r\theta} + \sigma_{\theta r} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\right)\boldsymbol{e}_\theta + \left(\sigma_{zr} + \frac{\partial \sigma_{z\theta}}{\partial \theta}\right)\boldsymbol{e}_z$$

Other two terms are given as,

$$\frac{\partial \boldsymbol{\sigma}}{\partial r} \cdot \boldsymbol{e}_r = \frac{\partial \sigma_{rr}}{\partial r}\boldsymbol{e}_r + \frac{\partial \sigma_{\theta r}}{\partial r}\boldsymbol{e}_\theta + \frac{\partial \sigma_{zr}}{\partial r}\boldsymbol{e}_z$$

$$\frac{\partial \boldsymbol{\sigma}}{\partial z} \cdot \boldsymbol{e}_z = \frac{\partial \sigma_{rz}}{\partial z}\boldsymbol{e}_r + \frac{\partial \sigma_{\theta z}}{\partial z}\boldsymbol{e}_\theta + \frac{\partial \sigma_{zz}}{\partial z}\boldsymbol{e}_z$$

Substitute back into the original expression,

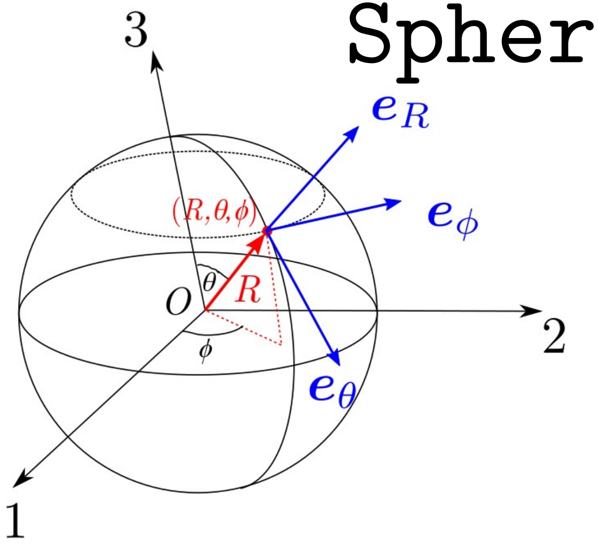
$$\begin{aligned}\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = &\left(\frac{\partial \sigma_{rr}}{\partial r}\boldsymbol{e}_r + \frac{\partial \sigma_{\theta r}}{\partial r}\boldsymbol{e}_\theta + \frac{\partial \sigma_{zr}}{\partial r}\boldsymbol{e}_z\right) + \\ &\frac{1}{r}\left(\sigma_{rr} - \sigma_{\theta\theta} + \frac{\partial \sigma_{r\theta}}{\partial \theta}\right)\boldsymbol{e}_r + \frac{1}{r}\left(\sigma_{r\theta} + \sigma_{\theta r} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\right)\boldsymbol{e}_\theta + \frac{1}{r}\left(\sigma_{zr} + \frac{\partial \sigma_{z\theta}}{\partial \theta}\right)\boldsymbol{e}_z + \\ &\left(\frac{\partial \sigma_{rz}}{\partial z}\boldsymbol{e}_r + \frac{\partial \sigma_{\theta z}}{\partial z}\boldsymbol{e}_\theta + \frac{\partial \sigma_{zz}}{\partial z}\boldsymbol{e}_z\right)\end{aligned}$$

Rearranging the terms will give the following expression for divergence of $\boldsymbol{\sigma}$,

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = & \left[\frac{\partial\sigma_{rr}}{\partial r}+\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{\partial\sigma_{rz}}{\partial z}+\frac{1}{r}\left(\sigma_{rr}-\sigma_{\theta\theta}\right)\right]\boldsymbol{e}_r+\\ & \left[\frac{\partial\sigma_{\theta r}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{\partial\sigma_{\theta z}}{\partial z}+\frac{1}{r}\left(\sigma_{r\theta}+\sigma_{\theta r}\right)\right]\boldsymbol{e}_\theta+\\ & \left[\frac{\partial\sigma_{zr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{z\theta}}{\partial\theta}+\frac{\partial\sigma_{zz}}{\partial z}+\frac{1}{r}\sigma_{zr}\right]\boldsymbol{e}_z.\end{aligned}$$

Above relation will be useful in writing the Cauchy's equations of motion in Cylindrical coordinate system.

Spherical coordinate system



Spherical coordinate variables (R, θ, ϕ) are related to Cartesian coordinate variables (x_1, x_2, x_3) as,

$$R = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad \theta = \tan^{-1} \frac{\sqrt{x_1^2 + x_2^2}}{x_3}, \quad \phi = \tan^{-1} \frac{x_2}{x_1},$$

and

$$x_1 = R \sin\theta \cos\phi, \quad x_2 = R \sin\theta \sin\phi, \quad x_3 = R\cos\theta.$$

An orthonormal basis can be defined in a spherical coordinate system with base vectors at a point in the space that are tangent to the curvilinear coordinate axes (i.e., lines along which two of the three coordinate variables are constant). Thus, base vector $\boldsymbol{e}_R$ is a unit vector that points in the direction of increasing R when θ and ϕ are held constant; similarly for $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\phi$. The base vectors are related to the base vectors in Cartesian coordinate system

$$\boldsymbol{e}_R = \sin\theta\cos\phi\boldsymbol{e}_1 + \sin\theta\sin\phi\boldsymbol{e}_2 + \cos\theta\boldsymbol{e}_3,$$
$$\boldsymbol{e}_\theta = \cos\theta\cos\phi\boldsymbol{e}_1 + \cos\theta\sin\phi\boldsymbol{e}_2 - \sin\theta\boldsymbol{e}_3,$$
$$\boldsymbol{e}_\phi = -\sin\phi\boldsymbol{e}_1 + \cos\phi\boldsymbol{e}_2$$

It can be observed that the basis vector changes with the change in angle θ and ϕ. Thus, similar to Cylindrical coordinate system, differentiation of basis vector w.r.t θ and ϕ will be considered. It can be shown that,

$$\frac{\partial \boldsymbol{e}_R}{\partial \theta} = \boldsymbol{e}_\theta, \quad \frac{\partial \boldsymbol{e}_\theta}{\partial \theta} = -\boldsymbol{e}_R,$$
$$\frac{\partial \boldsymbol{e}_R}{\partial \phi} = \sin\theta \boldsymbol{e}_\phi, \quad \frac{\partial \boldsymbol{e}_\theta}{\partial \phi} = -\cos\theta \boldsymbol{e}_\phi, \quad \frac{\partial \boldsymbol{e}_\phi}{\partial \phi} = -\sin\theta \boldsymbol{e}_R - \cos\theta \boldsymbol{e}_\theta.$$

It can be shown that small strain components in Spherical coordinates are,

$$\varepsilon_{RR} = \frac{\partial u_R}{\partial R}, \quad \varepsilon_{\theta\theta} = \frac{1}{R}\left(u_R + \frac{\partial u_\theta}{\partial \theta}\right), \quad \varepsilon_{\phi\phi} = \frac{1}{R\sin\theta}\left(\frac{\partial u_\phi}{\partial \phi} + u_R \sin\theta + u_\theta \cos\theta\right)$$
$$\varepsilon_{R\theta} = \frac{1}{2}\left(\frac{1}{R}\frac{\partial u_R}{\partial \theta} + \frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R}\right), \quad \varepsilon_{\theta\phi} = \frac{1}{2R}\left(\frac{1}{\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{\partial u_\phi}{\partial \theta} - u_\phi \cot\theta\right),$$
$$\varepsilon_{\phi r} = \frac{1}{2}\left(\frac{1}{R\sin\theta}\frac{\partial u_R}{\partial \phi} + \frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R}\right).$$

One can also derive the following Cauchy's equation of motion in Spherical coordinate system.

$$\frac{\partial \sigma_{RR}}{\partial R} + \frac{1}{R}\frac{\partial \sigma_{R\theta}}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial \sigma_{R\phi}}{\partial \phi} +$$
$$\frac{1}{R}\left(2\sigma_{RR} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{R\theta}\cot\theta\right) + b_r = \rho a_r$$
$$\frac{\partial \sigma_{\theta R}}{\partial R} + \frac{1}{R}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial \sigma_{\theta\phi}}{\partial \phi} +$$
$$\frac{1}{R}\left[(\sigma_{RR} - \sigma_{\phi\phi})\cot\theta + 2\sigma_{\theta R} + \sigma_{R\theta}\right] + b_\theta = \rho a_\theta$$
$$\frac{\partial \sigma_{\phi R}}{\partial R} + \frac{1}{R}\frac{\partial \sigma_{\phi\theta}}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} +$$
$$\frac{1}{R}\left[(\sigma_{\theta\phi} + \sigma_{\phi\theta})\cot\theta + 2\sigma_{\phi R} + \sigma_{R\phi}\right] + b_\phi = \rho a_\phi$$