

ME531: Advanced Mechanics of Solids

Motion, Strain and Stress

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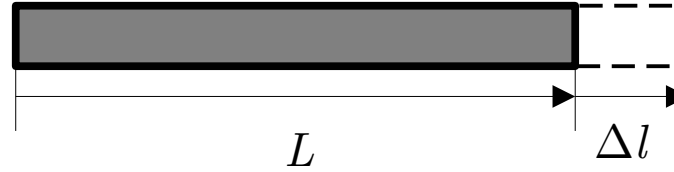
Room No. # 106

Measure of Deformation in solids

From our knowledge of Strength of materials (UG course), we know that in linear stress-strain analysis the deformation of a continuum body is measured in terms of small strains.

For instance, small strain in axial loading is defined as

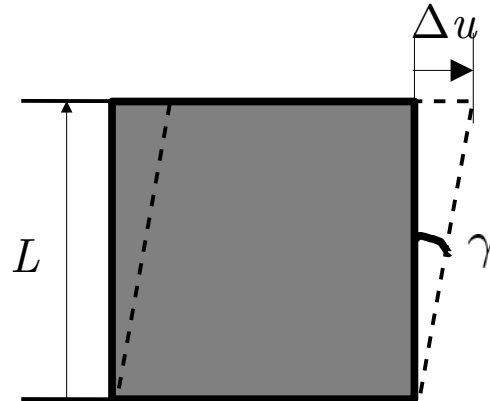
$$\epsilon = \frac{\Delta l}{L},$$



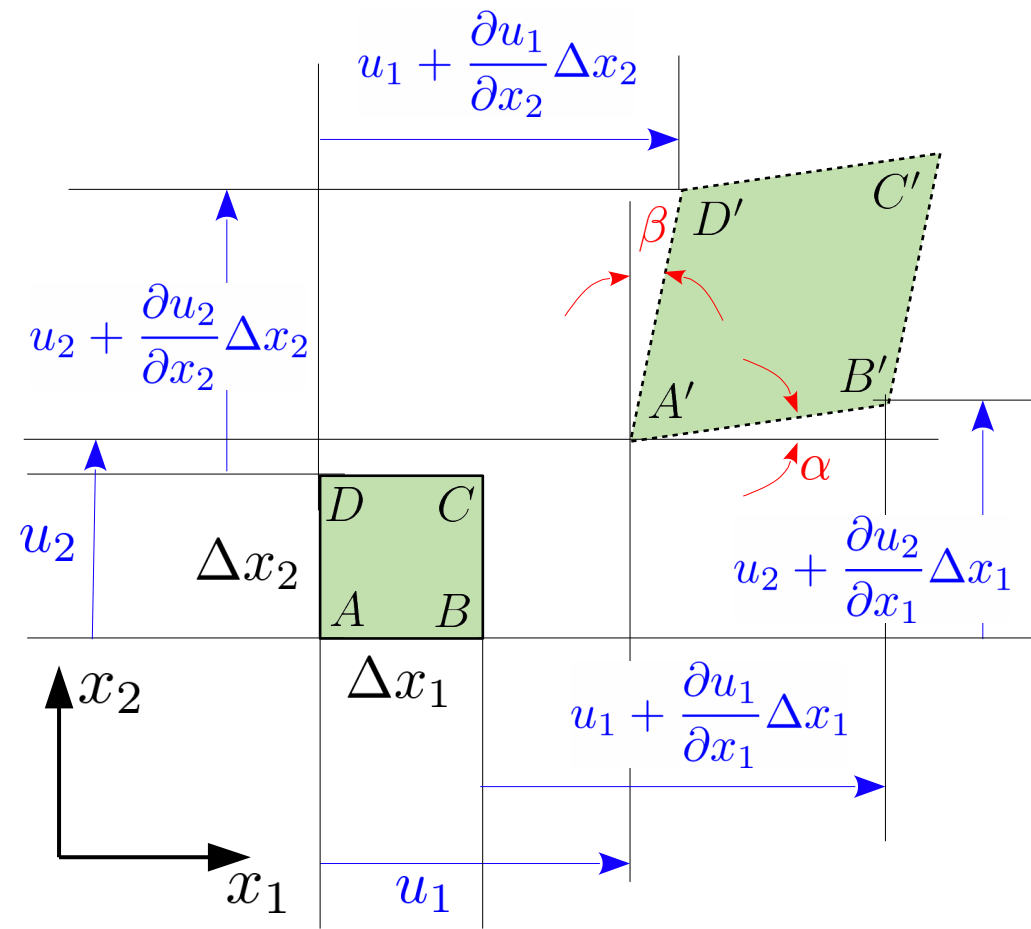
where L is the initial length and Δl is the elongation.

Similarly, shear strain is defined as

$$\tan \gamma \approx \gamma = \frac{\Delta u}{L}.$$



Small strains in 2D



$$\epsilon_{11} = \lim_{\Delta x_1 \rightarrow 0} \frac{A'B' - AB}{AB} = \lim_{\Delta x_1 \rightarrow 0} \frac{\Delta x_1 + \frac{\partial u_1}{\partial x_1} \Delta x_1 - \Delta x_1}{\Delta x_1},$$

$$\Rightarrow \epsilon_{11} = \frac{\partial u_1}{\partial x_1},$$

$$\epsilon_{22} = \lim_{\Delta x_2 \rightarrow 0} \frac{A'D' - AD}{AD} = \lim_{\Delta x_2 \rightarrow 0} \frac{\Delta x_2 + \frac{\partial u_2}{\partial x_2} \Delta x_2 - \Delta x_2}{\Delta x_2},$$

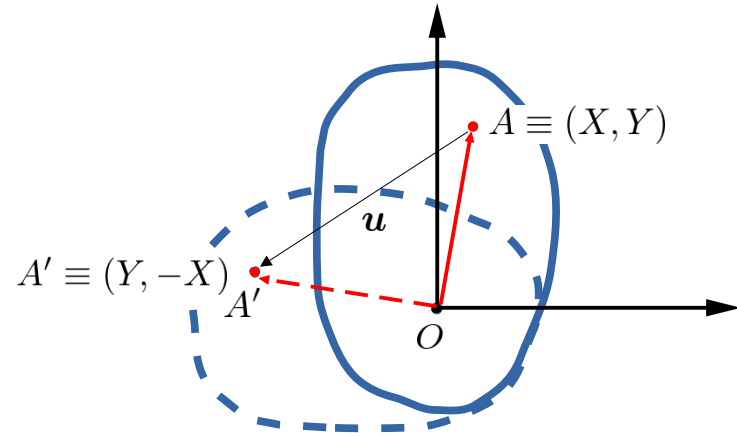
$$\Rightarrow \epsilon_{22} = \frac{\partial u_2}{\partial x_2},$$

$$\gamma_{12} = \lim_{\substack{\Delta x_1 \rightarrow 0 \\ \Delta x_2 \rightarrow 0}} \frac{\pi}{2} - \angle B'A'D' = \alpha + \beta$$

$$\Rightarrow \gamma_{12} = \lim_{\substack{\Delta x_1 \rightarrow 0 \\ \Delta x_2 \rightarrow 0}} \frac{\frac{\partial u_2}{\partial x_1} \Delta x_1}{\Delta x_1 + \frac{\partial u_1}{\partial x_1} \Delta x_1} + \frac{\frac{\partial u_1}{\partial x_2} \Delta x_2}{\Delta x_2 + \frac{\partial u_2}{\partial x_2} \Delta x_2}$$

$$\Rightarrow \gamma_{12} \approx \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}.$$

What if displacements are large?



Consider a body which goes under a rotation of 90° in the anti-clockwise direction. During the rotation, the point A which had its coordinates as (X, Y) before the rotation, take a new position A' with its coordinates as $(-Y, X)$ after the rotation.

Displacements of point A along X and Y directions are,

$$u_x = Y - X \quad \text{and} \quad u_y = -X - Y.$$

Let us determine the components of strain for this case as,

$$\varepsilon_x = \frac{\partial u_x}{\partial X} = -1, \quad \varepsilon_Y = \frac{\partial u_y}{\partial Y} = -1, \quad \text{and} \quad \gamma_{xy} = \frac{\partial u_x}{\partial Y} + \frac{\partial u_y}{\partial X} = 0.$$

Are these strains correct?

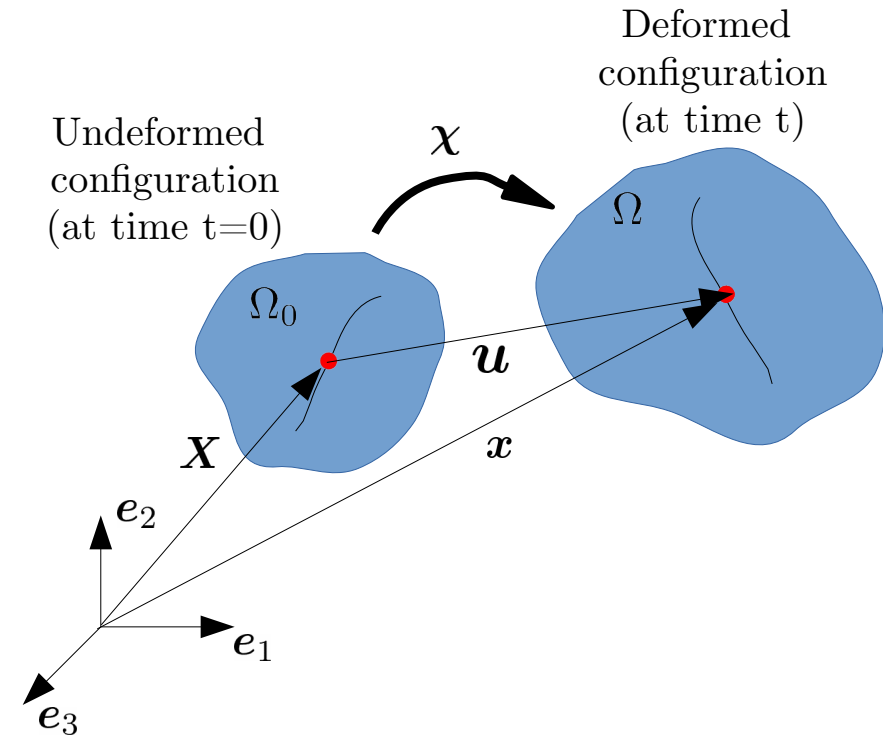
Motion and Deformation

Consider a body in its *undeformed* (or *initial*) configuration with a volume Ω_0 . After time t , the body moves to a new position occupying a volume Ω in the space. The configuration at time t is called *deformed* (or *current*) configuration.

A function χ , which maps a point \mathbf{X} in Ω_0 to another point \mathbf{x} in Ω is called *motion* and the motion is assumed to be uniquely *invertible*, i.e.

$$\mathbf{x} = \chi(\mathbf{X}, t) \text{ and } \mathbf{X} = \chi^{-1}(\mathbf{x}, t).$$

It can be observed that, $\mathbf{x}(\mathbf{X}, t) = \mathbf{X} + \mathbf{u}(\mathbf{X}, t)$, where \mathbf{u} is the *displacement vector*.



Lagrangian and Eulerian Description

Characterization of motion (or any other quantity) with respect to material (or referential) coordinates X_i is called *material* or *Lagrangian* description. *Spatial* or *Eulerian* description is the characterized with respect to spatial (or current) coordinates x_i .

E.g. Displacement field in the Lagrangian form is $\mathbf{U}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X}$, and Eulerian form of the displacement is $\mathbf{u}(\mathbf{x}, t) = \mathbf{x} - \mathbf{X}(\mathbf{x}, t)$.

For a very simple motion given as,

$$\begin{aligned}x_1(X_1, X_2, X_3, t) &= X_1 + \gamma X_2, \\x_2(X_1, X_2, X_3, t) &= X_2, \\x_3(X_1, X_2, X_3, t) &= X_3,\end{aligned}$$

Material description of displacement is

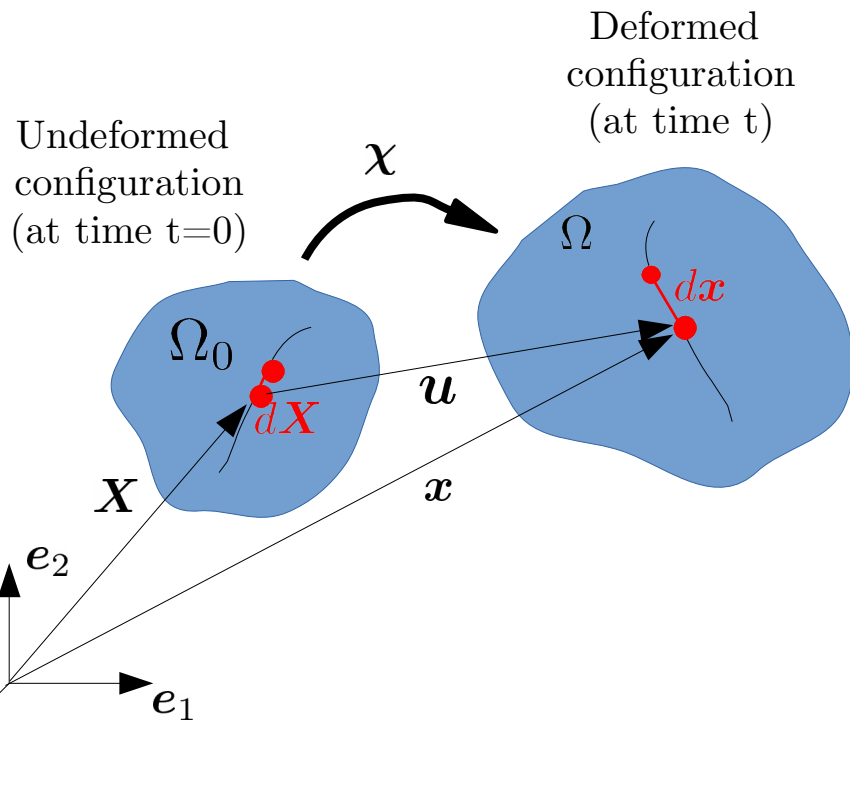
$$\begin{aligned}U_1(X_1, X_2, X_3, t) &= x_1 - X_1 = \gamma X_2, \\U_2(X_1, X_2, X_3, t) &= 0, \\U_3(X_1, X_2, X_3, t) &= 0,\end{aligned}$$

and spatial description of is

$$\begin{aligned}u_1(x_1, x_2, x_3, t) &= x_1 - X_1 = \gamma X_2 = \gamma x_2, \\u_2(x_1, x_2, x_3, t) &= 0, \\u_3(x_1, x_2, x_3, t) &= 0.\end{aligned}$$

It should be noticed that $\mathbf{u}(\mathbf{x}, t) = \mathbf{U}(\mathbf{X}, t)$.

Deformation gradient



Consider a line element $d\mathbf{X}$ in Ω_0 which deforms to a line element $d\mathbf{x}$. *Deformation gradient* tensor maps the line element $d\mathbf{X}$ to $d\mathbf{x}$ as,

$$d\mathbf{x} = \mathbf{F} d\mathbf{X} \text{ or } dx_i = F_{iJ} dX_J,$$

where, $\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$ or $F_{iJ} = \frac{\partial x_i}{\partial X_J}$,
or $\mathbf{F} = \text{Grad } \mathbf{x} = \nabla_{\mathbf{X}} \mathbf{x}$

is an invertible tensor. \mathbf{F}^{-1} is the *inverse deformation gradient* tensor, defined as

$$\mathbf{F}^{-1} = \frac{\partial \mathbf{X}}{\partial \mathbf{x}} \text{ or } F_{Ij}^{-1} = \frac{\partial X_I}{\partial x_j}.$$

or $\mathbf{F}^{-1} = \text{grad } \mathbf{X} = \nabla_{\mathbf{x}} \mathbf{X}$

It relates line elements as,

$$d\mathbf{X} = \mathbf{F}^{-1} d\mathbf{x} \text{ or } dX_I = F_{Ij}^{-1} dx_j.$$

Example

Consider a two dimensional motion given by two equations as

$$x_1 = 4 - 2X_1 - X_2$$

$$x_2 = 2 + 1.5X_1 - 0.5X_2$$

Deformation gradient for the given motion is calculated as,

$$[\mathbf{F}] = \begin{bmatrix} -2 & -1 \\ 1.5 & -0.5 \end{bmatrix}, \text{ and } [\mathbf{F}]^{-1} = \frac{1}{5} \begin{bmatrix} -1 & 2 \\ -3 & -4 \end{bmatrix}$$

Now consider following vectors $\mathbf{a}_0 = [1,0]$, $\mathbf{b}_0 = [0,1]$ and $\mathbf{c}_0 = [0.707, 0.707]$ in the reference configuration. Current or deformed vectors can be obtained as $\mathbf{a} = \mathbf{F}\mathbf{a}_0$, $\mathbf{b} = \mathbf{F}\mathbf{b}_0$ and $\mathbf{c} = \mathbf{F}\mathbf{c}_0$.

Consider another vector $\mathbf{d} = [1,0]$ in the current configuration. Reference configuration \mathbf{d}_0 for vector \mathbf{d} can be obtained as $\mathbf{d}_0 = \mathbf{F}^{-1}\mathbf{d}$.

