From incompressibility we have

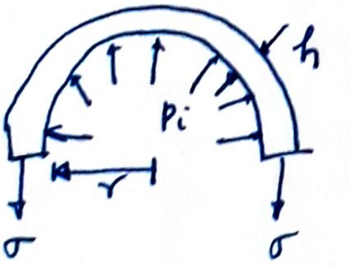
$$4\pi R^2 H = 4\pi r^2 h \quad \text{—— (e)}$$

$$\Rightarrow \frac{h}{H} = \frac{R^2}{r^2} \Rightarrow \lambda_3 = \frac{R^2}{r^2} \qquad \left\{\lambda_3 = \frac{h}{H}\right\}$$

now, $\frac{r}{R} = \lambda$ (circumferential stretch) → How? Think

$$\therefore \lambda_3 = \frac{1}{\lambda^2} \quad \text{—— (f)}$$

Now, we want to relate pressure to the circumferential stress. ~~...~~
For that we take upper half of the ballon and apply equilibrium, ~~equation~~



From equilibrium

$$\sigma \times h \times 2\pi r = p_i \times \pi r^2$$

$$\therefore p_i = \sigma \times \frac{2h}{r} \quad \text{—— (g)}$$

$$\Rightarrow p_i = \sigma \times \frac{2}{r} \times H \times \frac{R^3}{r^3} \qquad \{\text{from (e)}\}$$

$$\Rightarrow p_i = \frac{2\sigma H}{R} \times \frac{R^3}{r^3} = \frac{2\sigma H}{R\lambda^3} \quad \text{—— (h)}$$

from (h) & (d) we get

$$\cancel{\sigma_1 = \sigma_2 = \sigma = \frac{p_i \, R\lambda^3}{2H}}$$

$$\boxed{p_i = \frac{2H}{R\lambda^3} \sum_{i=1}^{N} \left(\lambda^{\alpha_i} - \lambda^{-2\alpha_i}\right) = \frac{2H}{R} \sum_{i=1}^{N} \left(\lambda^{\alpha_i - 3} - \lambda^{-2\alpha_i - 3}\right)}$$

—— (i)

Elasticity Tensor

We have derived that $S(\underset{\sim}{C})$, then we can write

$$d\underset{\sim}{S} = \frac{\partial S(\underset{\sim}{C})}{\partial \underset{\sim}{C}} : d\underset{\sim}{C} \quad \text{or} \quad 2\frac{\partial S(\underset{\sim}{C})}{\partial \underset{\sim}{C}} : \frac{d\underset{\sim}{C}}{2}$$

we write the above relation as

$$d\underset{\sim}{S} = \mathbb{C} : \frac{1}{2} d\underset{\sim}{C}$$

where $$\boxed{\underset{\sim}{\mathbb{C}} = \mathcal{C}_{IJKL} = 2\frac{\partial S(\underset{\sim}{C})}{\partial \underset{\sim}{C}} = 2\frac{\partial S_{IJ}}{\partial C_{KL}}} \quad \text{—— (1)}$$

Tensor $\mathbb{C}$ is called elasticity tensor, which measured the change in stress resulting from change in strain.

from chain rule, we can derive

$$\boxed{\underset{\sim}{\mathbb{C}} = \frac{\partial \underset{\sim}{S}}{\partial \underset{\sim}{E}}} \quad \text{—— (2)} \qquad \text{where } \left\{ \underset{\sim}{E} = \frac{1}{2}(\underset{\sim}{C} - \underset{\sim}{I}) \right\}$$

As $\underset{\sim}{S}$ and $\underset{\sim}{E}$ are symmetric tensor

$$dS_{ij} = \mathbb{C}_{ijkl}\, dE_{kl}$$

$$dS_{ij} = dS_{ji} = \mathbb{C}_{jikl}\, dE_{kl} \quad \& \quad dS_{ij} = \mathbb{C}_{ijkl}\, dE_{kl} = \mathbb{C}_{ijlk}\, dE_{lk}$$

$$\therefore \boxed{\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{ijlk}} \quad \text{——— } (3)$$

Symmetric in first and second slot. (Called Minor Symmetry)

Hence in general total 36 independent Components

→ Note that till now, nowhere we brought the strain-energy function into picture

→ Now, for hyperelasticity we know that

$$\underset{\sim}{S} = 2\frac{\partial \Psi(\underset{\sim}{C})}{\partial \underset{\sim}{C}} \quad \text{——— } (4)$$

∴ From (3), we get

$$\mathbb{C} = 4\frac{\partial^2 \Psi(\underset{\sim}{C})}{\partial \underset{\sim}{C}\,\partial \underset{\sim}{C}}$$

or

$$\mathbb{C}_{ijkl} = 4\frac{\partial^2 \Psi}{\partial C_{ij}\,\partial C_{kl}}$$

→ which is same as

$$\boxed{\mathbb{C}_{klij} = \mathbb{C}_{ijkl}} \rightarrow \text{Major Symmetry}$$

This symmetry brings down the number of elastic constants from 36 to 21.

The spacial form of elasticity tensor can be obtained by Piola transformation as

$$\boxed{c_{ijkl} = J^{-1} F_{iP} F_{jQ} F_{kR} F_{lS} C_{PQRS}}$$

Spacial form also possesses major and minor symmetry. and has 21 independent material constant in general.

→ These numbers of constant further decreases based on the symmetry of material structures.

→ For an isotropic material C_{ijkl} takes a form

$$C_{ijkl} = \alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk}$$

To maintain the minor symmetry i.e $C_{ijkl} = C_{jikl}$ or $C_{ijkl} = C_{ijlk}$

it is required that $\beta = \gamma$.

→ Hence, we conclude that there are only two independent constant for an isotropic elastic material.

$$\therefore \; C_{ijkl} = \overset{\lambda}{\cancel{\alpha}} \delta_{ij} \delta_{kl} + \overset{\mu}{\cancel{\beta}} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

These two constants are denoted as λ & μ, and are called Lame's constant