ME531: Advanced Mechanics of Solids

Motion, Strain and Stress

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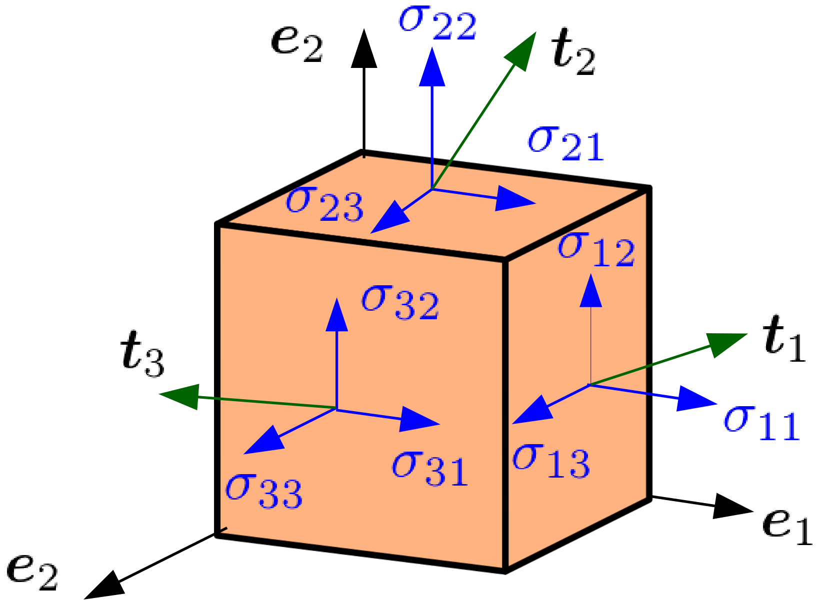
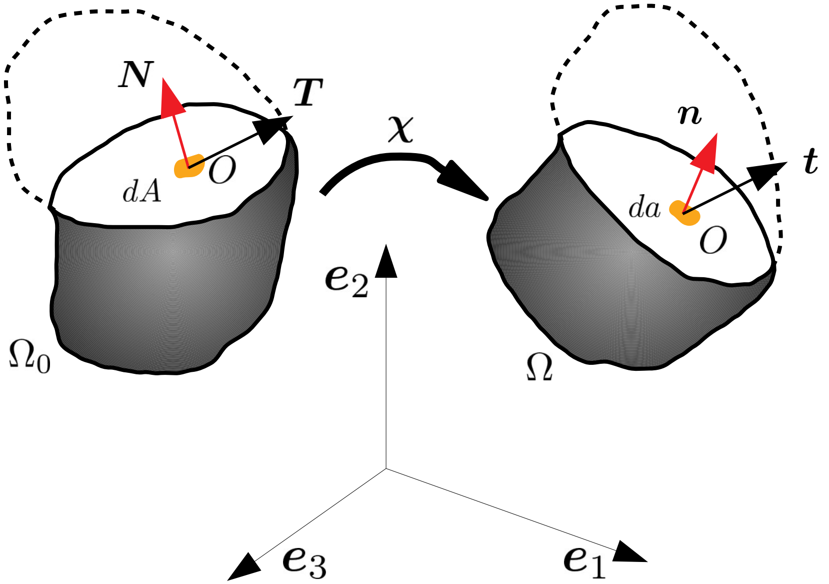
Now, we find the traction vectors at planes having basis vectors $\{\boldsymbol{e}_i\}$ as normal,

$$
\begin{aligned}
\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{e}_1) &= \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{e}_1 = \sigma_{11}\boldsymbol{e}_1 + \sigma_{12}\boldsymbol{e}_2 + \sigma_{13}\boldsymbol{e}_3, \\
\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{e}_2) &= \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{e}_2 = \sigma_{21}\boldsymbol{e}_1 + \sigma_{22}\boldsymbol{e}_2 + \sigma_{23}\boldsymbol{e}_3, \\
\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{e}_3) &= \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{e}_3 = \sigma_{31}\boldsymbol{e}_1 + \sigma_{32}\boldsymbol{e}_2 + \sigma_{33}\boldsymbol{e}_3.
\end{aligned}
$$

State of stress at a point can be represented by considering an infinitesimal cubic element surrounding it as,





Now, consider a body under stress in its reference and current configuration. Traction vector in the current configuration and the corresponding normal is $\boldsymbol{t}$ and $\boldsymbol{n}$, respectively. Corresponding normal and (pseudo) traction vector is the reference configurations are $\boldsymbol{T}$ and $\boldsymbol{N}$ respectively.

$\boldsymbol{T}$ is known as *first Piola-Kirchhoff traction vector* and works in the same directions as $\boldsymbol{t}$. It should be noted that $\boldsymbol{T}$ does not describe the actual intensity as it works on the reference volume Ω_0 and is a function of reference position $\boldsymbol{X}$. Cauchy's stress theorem is also applicable for traction vector $\boldsymbol{T}(\boldsymbol{X},\boldsymbol{N})$ and normal vector $\boldsymbol{N}$, which is as follows,

$$\boldsymbol{T}(\boldsymbol{X},\boldsymbol{N}) = \boldsymbol{N} \cdot \boldsymbol{P}(\boldsymbol{X}), \quad \text{or} \quad T_i = P_{ji} N_j,$$

Here, $\boldsymbol{P}$ is a second order tensor and known as *first-Piola Kirchhoff stress tensor*.

It should be noted that the force on the element area can be expressed as,

$$\mathrm{d}\boldsymbol{f} = \boldsymbol{T}\mathrm{d}A = \boldsymbol{t}\,\mathrm{d}a.$$

Above relation can be used to establish a relation between the Cauchy stress tensor, and the first Piola-Kirchhoff stress tensor. We use the Cauchy's stress theorem and write the previous equation as,

$$\boldsymbol{N}\cdot\boldsymbol{P}(\boldsymbol{X})\mathrm{d}A = \boldsymbol{n}\cdot\boldsymbol{\sigma}(\boldsymbol{x})\mathrm{d}a.$$

Using Nanson's relations, we write,

$$\boldsymbol{P}(\boldsymbol{X}) = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}(\boldsymbol{x}), \qquad P_{ij} = JF_{ik}^{-1}\sigma_{kj}.$$

We can also write,

$$\boldsymbol{\sigma} = J^{-1}\boldsymbol{F}\boldsymbol{P}, \qquad \sigma_{ij} = J^{-1}F_{ik}P_{kj}.$$

Later, we will prove that the Cauchy stress tensor is symmetric under the assumption of zero resultant couples. i.e.

$$\boldsymbol{\sigma} = J^{-1}\boldsymbol{F}\boldsymbol{P} = \boldsymbol{\sigma}^T, \qquad \sigma_{ij} = J^{-1}P_{ij} = F_{ik}\sigma_{kj},$$

which implies that $\boldsymbol{F}\boldsymbol{P} = \boldsymbol{P}^T\boldsymbol{F}^T$, which also suggests that the tensor $\boldsymbol{P}$ in general is not symmetric.

Example

Deformation of a body is described by,

$$x_1 = \text{-}6X_2,\ x_2 = 0.5X_1, \qquad x_3 = 1/3\ X_3.$$

The Cauchy stress tensor for certain part of the body is given by the matrix representation as,

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathrm{kg/cm}^2$$

Determine the Cauchy traction vector $\boldsymbol{t}$ and the first Piola-Kirchhoff traction vector $\boldsymbol{T}$ acting on a plane, which is characterized by the outward unit normal $\boldsymbol{n} = \boldsymbol{e}_2$ in the current configuration.

For the given deformation,

$$[\boldsymbol{F}] = \begin{bmatrix} 0 & -6 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}, \quad [\boldsymbol{F}]^{-1} = \begin{bmatrix} 0 & 2 & 0 \\ -1/6 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{and } \det \boldsymbol{F} = 1.$$

The components for first Piola-Kirchoff stress tensor will be

$$[\boldsymbol{P}] = J[\boldsymbol{F}]^{-1}[\boldsymbol{\sigma}] = \begin{bmatrix} 0 & 100 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{kN/cm}^2.$$

Outward unit normal $\boldsymbol{N}$ in the reference configuration will be related to $\boldsymbol{n}$ by Nanson's formula as,

Thus, for $\boldsymbol{n}=\boldsymbol{e}_2$, $\boldsymbol{N}dS = J^{-1}\boldsymbol{F}^T\boldsymbol{n}ds \Rightarrow \boldsymbol{N}dS = \dfrac{\boldsymbol{e}_1}{2}ds.$

Hence, $\boldsymbol{N} = \boldsymbol{e}_1$ and $dS = ds/2$.

Finally, using Cauchy's stress theorem,

$$\{\boldsymbol{t}\} = [\boldsymbol{\sigma}]^T[\boldsymbol{n}] = \begin{bmatrix} 0 \\ 50 \\ 0 \end{bmatrix} \text{kN/cm}^2, \ \{\boldsymbol{T}\} = [\boldsymbol{P}]^T[\boldsymbol{N}] = \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix} \text{kN/cm}^2.$$

i.e. $\boldsymbol{t} = 50\boldsymbol{e}_2$ and $\boldsymbol{T} = 100\boldsymbol{e}_2$.

It can be observed that both $\boldsymbol{t}$ and $\boldsymbol{T}$ have same direction, but the magnitudes of $\boldsymbol{T}$ is twice that of $\boldsymbol{t}$, as deformed area is half the undeformed area.

Other stress measures

There are other definitions of stress which of practical applications in analysis. Most of them do not have a direct physical interpretation.

Kirchhoff stress tensor: It is defined the as Cauchy stress tensor time the volume ratio J.

$$\boldsymbol{\tau} = J\boldsymbol{\sigma} \text{ or } \tau_{ij} = J\sigma_{ij}.$$

Second Piola-Kirchhoff stress tensor: It is denoted as $\boldsymbol{S}$ and defined as,

$$\boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{\tau}\boldsymbol{F}^{-T} \text{ or } S_{ij} = F_{im}^{-1}F_{jn}^{-1}\tau_{mn}.$$

With its inverse,

$$\boldsymbol{\sigma} = J^{-1}\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^{T} \text{ or } \sigma_{ij} = J^{-1}F_{im}F_{jn}^{-1}S_{mn}.$$

Second Piola-Kirchhoff stress tensor in Lagrangian description and is a symmetric tensor. It is related with the First Piola-Kirchhoff stress tensor as

$$\boldsymbol{P} = \boldsymbol{S}\boldsymbol{F}^{T} \text{ or } P_{ij} = S_{ik}F_{jk}.$$

2^{nd} PK stress derivation

Considering that, the force is a vector quantity, we use following transformation rule for mapping of a force in the deformed configuration to the undeformed configuration,

$$d\boldsymbol{F} = \boldsymbol{F}^{-1} d\boldsymbol{f} \quad \text{or} \quad dF_i = F_{ij}^{-1} df_j .$$

Force in the deformation configuration is $df_j = t_j da = \sigma_{kj} n_k da$. Thus,

$$dF_i = F_{ij}^{-1} \sigma_{kj} n_k da,$$

$$dF_i = F_{ij}^{-1} \sigma_{kj} J F_{pk}^{-1} N_p dA.$$

(Using Nanson's relationship, i.e., $n_k da = J F_{pk}^{-1} N_p dA$)

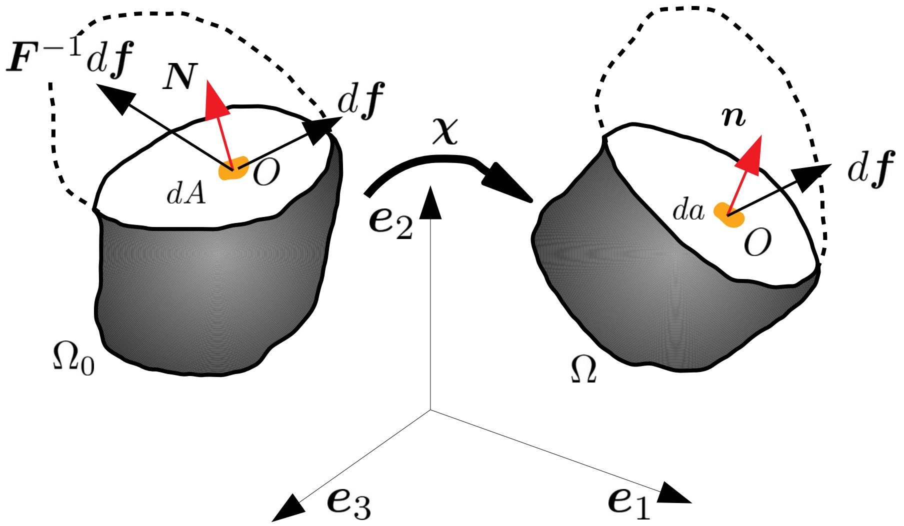
$dF_i = T_i dA = F_{ij}^{-1} \sigma_{kj} J F_{pk}^{-1} N_p dA$, or

$T_i = (J F_{ij}^{-1} \sigma_{kj} F_{pk}^{-1}) N_p, \quad \text{or} \quad T_i = S_{pi} N_p.$

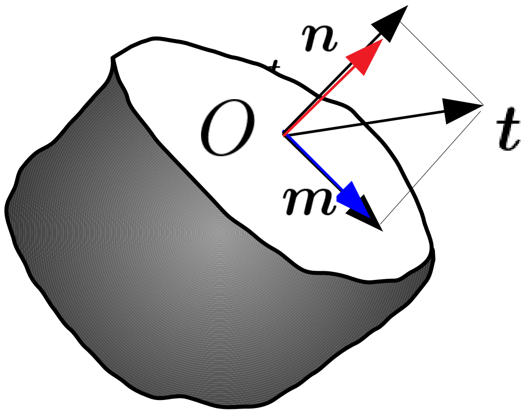
where, $S_{pi} = J F_{ij}^{-1} \sigma_{kj} F_{pk}^{-1}$ or $\boldsymbol{S} = J \boldsymbol{F}^{-1} \boldsymbol{\sigma} \boldsymbol{F}^{-T}$, is Second P-K Stress Tensor

Summary

- The *Cauchy stress tensor* ($\boldsymbol{\sigma}$) is defined to be the *current force* ($d\boldsymbol{f}$) per unit *deformed area* (da).
- The *first Piola—Kirchhoff stress tensor* ($\boldsymbol{P}$) gives the *current force* ($d\boldsymbol{f}$) per unit *undeformed area* (dA).
- The *second Piola—Kirchhoff stress tensor* ($\boldsymbol{S}$) gives the *transformed current force* ($d\boldsymbol{F} = \boldsymbol{F}^{-1} d\boldsymbol{f}$) per unit *undeformed area* (dA).



Normal and shear stresses



Consider a traction vector $\boldsymbol{t}$ on at a surface having unit normal vector $\boldsymbol{n}$ and a tangential vector $\mathbf{m}$ which is perpendicular to $\boldsymbol{n}$.

Traction vector $\boldsymbol{t}$ can be decomposed into two vector along $\boldsymbol{n}$ and $\boldsymbol{m}$ directions namely *normal* and *shear traction*.

Normal traction is $\boldsymbol{t}_n = (\boldsymbol{n}\cdot\boldsymbol{t})\boldsymbol{n} = \sigma\boldsymbol{n}$, and

Shear traction is $\boldsymbol{t}_m = (\boldsymbol{m}\cdot\boldsymbol{t})\boldsymbol{m} = \tau\boldsymbol{m} = \boldsymbol{t} - \boldsymbol{t}_n$,

where σ and τ are magnitudes of normal and shear traction called normal and shear stress. If $\sigma > 0$ normal stresses are said to be tensile, while negative normal stresses (i.e. $\sigma < 0$) are call compressive stresses.

Now, $\boldsymbol{t} = \boldsymbol{t}_n + \boldsymbol{t}_m = \sigma\boldsymbol{n} + \tau\boldsymbol{m}$ and $|\boldsymbol{t}|^2 = \sigma^2 + \tau^2$.

Principal stress

Normal and shear stresses vary in magnitude, direction and location. For design of structure knowledge of extremal values are required. We will determine the extremal (maximum and minimum) values of stresses.

Maximum and minium values of normal stresses:

Consider the Cauchy traction vector $\boldsymbol{t}(\boldsymbol{x},\boldsymbol{n})=\boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{n}$ at any arbitrary plane at a point $\boldsymbol{x}$. We first try to find the unit vector $\boldsymbol{n}$ at a point $\boldsymbol{x}$ indicating the maximum and minimum values of normal stresses. In order to obtain the maximum and minimum values of normal stress σ, we apply Lagrange-multiplier method and claim ther stationary position of the following functional,

$$\mathcal{L}(\boldsymbol{n},\lambda) = \boldsymbol{n}\cdot\boldsymbol{\sigma}\boldsymbol{n} - \lambda(|\boldsymbol{n}|^2-1), \text{ or}$$

$$\mathcal{L}(\boldsymbol{n},\lambda) = n_i\sigma_{ij}n_j - \lambda(n_kn_k-1).$$

Here, $|\boldsymbol{n}|^2-1$ is a *constraint condition* and λ is the *Lagrange multiplier*.

For stationary position of the functional following derivatives must vanish, $\frac{\partial \mathcal{L}}{\partial n_i}$, and $\frac{\partial \mathcal{L}}{\partial \lambda}$.

i.e. $$\frac{\partial \mathcal{L}}{\partial n_k} = \sigma_{ij}\left(\delta_{jk} n_i + \delta_{ik} n_j\right) - \lambda(2 n_i \delta_{ik}) = 2(\sigma_{ki} n_i - \lambda n_k) = 0.$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = n_k n_k - 1 = 0.$$

To derive the above expression, we have used the fact that $\boldsymbol{\sigma}$ is symmetric. Above two equations in tensorial form are

$$(\boldsymbol{\sigma} - \lambda \boldsymbol{I})\boldsymbol{n} = 0 \text{ and } |\boldsymbol{n}|^2 - 1 = 0.$$

Above is an eigenvalue problem, where Lagrange parameter λ is identified as eigenvalues. Corresponding eigenvalues are σ_1, σ_2 and σ_3, which are called principal stresess, includes both maximum and minimum values of normal stresses. Corresponding eigenvectors $\boldsymbol{n}_i$, i=1,2,3 are called principal directions. Planes normal to the eigenvectors are called princiapal plane. At principal planes shear stresses vanish and stress tensor in the basis of eigenvectors is written as,

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \sum_{i=1}^{3} \sigma_i \boldsymbol{n}_i \otimes \boldsymbol{n}_i.$$