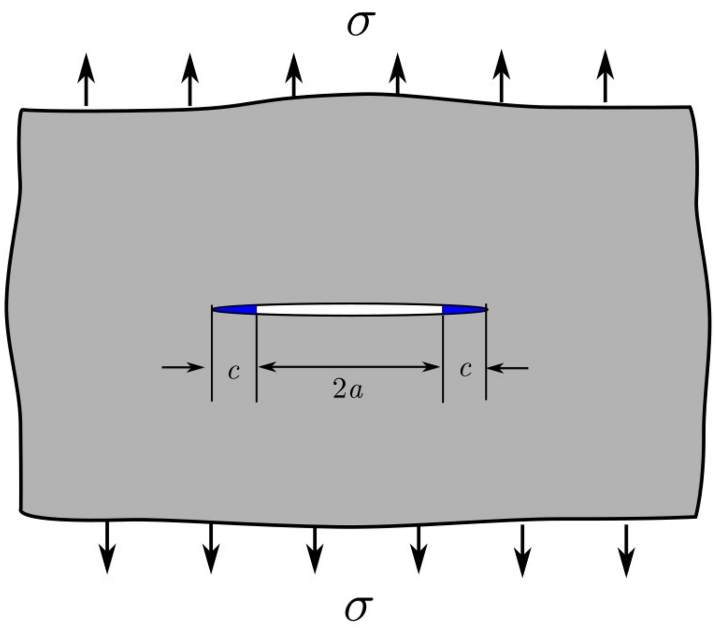
ME632: Fracture Mechanics

Timings	
Monday	10:00 to 11:20
Thursday	08:30 to 09:50

Anshul Faye
afaye@iitbhilai.ac.in
Room No. # 106

Dugdale's model



Dugdale (1960) and Barenblatt (1962) individually proposed a simplified method to determine the size of the plastic zone ahead of crack-tip. Following are the assumption.

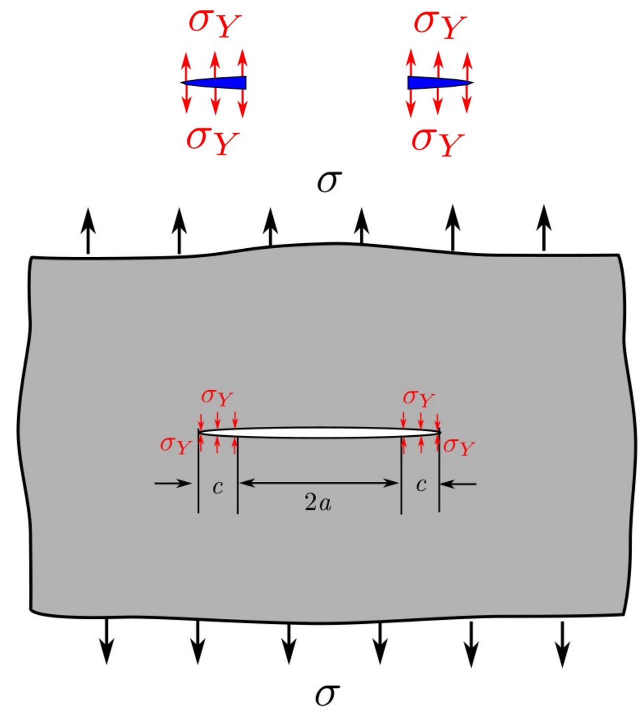
(i) Applicable to very thin plates (i.e., plane stress conditions).

(ii) Infinite plate having crack of length $2a$, and subjected to uniform tensile load σ.

(ii) Material is elastic-perfectly plastic and it obeys the Tresca yield criterion.

(iii) All plastic deformation concentrates in a line in front of the crack.

(iv) The crack has an effective length of $2(a+c)$, which exceeds that of the physical crack ($2a$) by the length of the plastic zone.



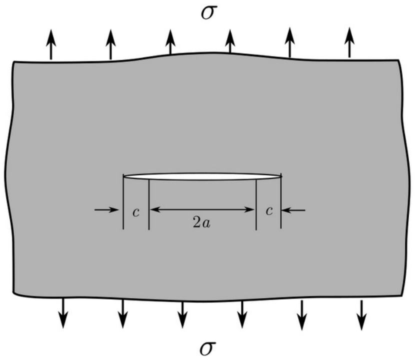
Isolate the strips of material which has gone under the plastic deformation froσm the plate. Stresses on these strips are the yield stress.

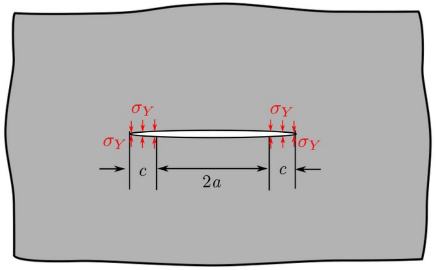
Equal and opposite pressure distribution will work on the plate from where the stripes have been isolated.

Thus we convert the elastic-plastic crack problem to a elastic problem having crack-length $2(a+c)$. The SIF for the problem will be the superpostion of the SIFs for the following problems.

(i) SIF for infinite plate having crack length $2(a+c)$ and subjected to tensile load σ (K_σ).

(ii) SIF for Infinite plate having crack length $2(a+c)$ and subjected with pressure σ_Y on the crack faces from length $\pm a$ to $\pm(a+c)$ ($K_{\sigma Y}$).





Thus, $$K = K_\sigma + K_{\sigma_Y}$$

$$K = \sigma\sqrt{\pi(a+c)} - \frac{2\sigma_Y\sqrt{\pi(a+c)}}{\pi}\cos^{-1}\frac{a}{a+c}$$

The effect of plasticity ahead of the crack-tip is to kill the singularity there. Hence, the K for the problem will be zero.

$$K = \sigma\sqrt{\pi(a+c)} - \frac{2\sigma_Y\sqrt{\pi(a+c)}}{\pi}\cos^{-1}\frac{a}{a+c} = 0$$

$$\Rightarrow \frac{a}{a+c} = \cos\frac{\pi\sigma}{2\sigma_Y}$$

$$\Rightarrow c = a\left(\sec\frac{\pi\sigma}{2\sigma_Y} - 1\right) \qquad \cdots\cdots\cdots\cdots\cdots\cdots(12)$$

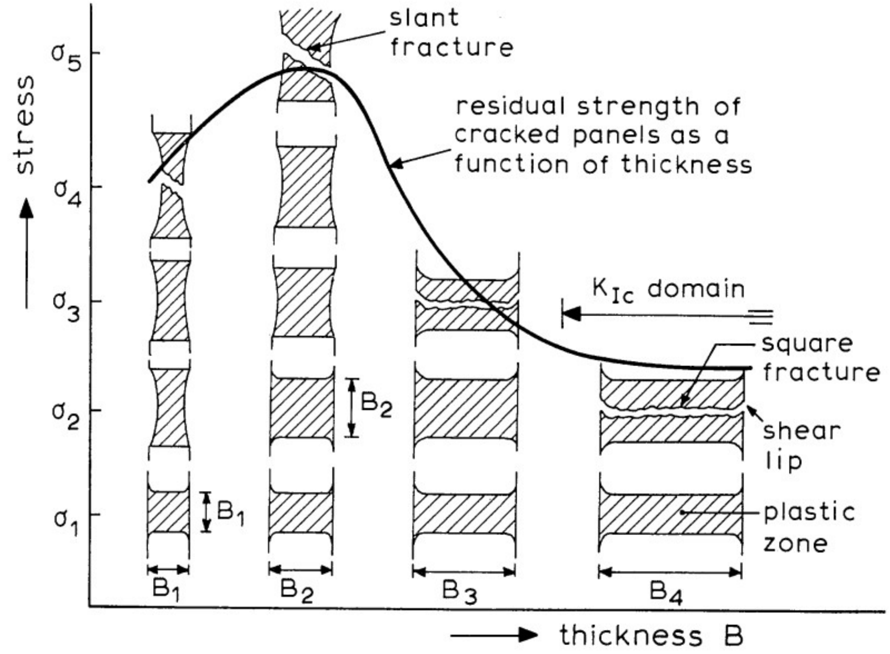
Equation (12) can be simplified to determine the size of plastic zone required to remove the singularity at the crack-tip. When $\sigma << \sigma_Y$ and $c << a$, then the above relation is approximated to

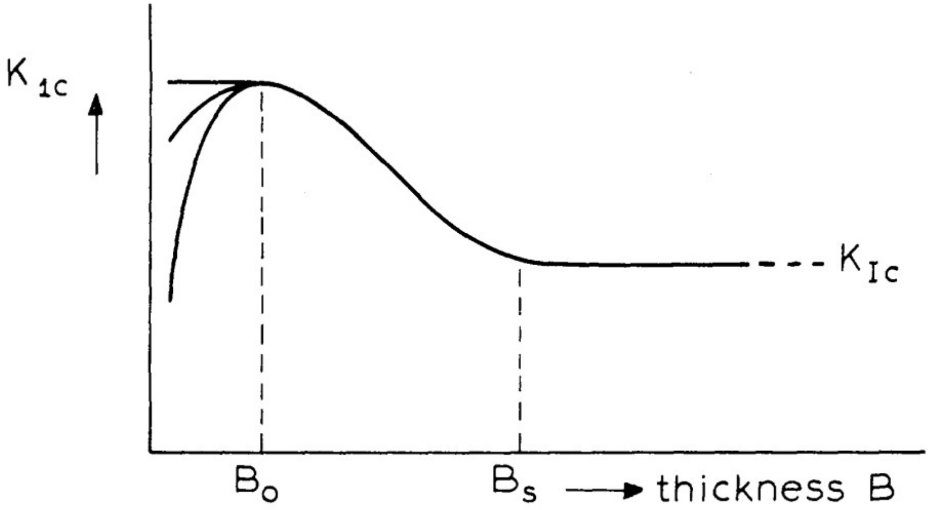
$$1 - \frac{c}{a} = 1 - \frac{\pi^2\sigma^2}{8\sigma_Y^2}$$

$$\Rightarrow c = \frac{\pi^2\sigma^2 a}{8\sigma_Y^2} = \frac{\pi K_I^2}{8\sigma_Y^2} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(13)$$

Equation (13) overestimates the plastic zone size by about 20%, compared to Irwin model for $\sigma << \sigma_Y$.

Effect of plate thickness

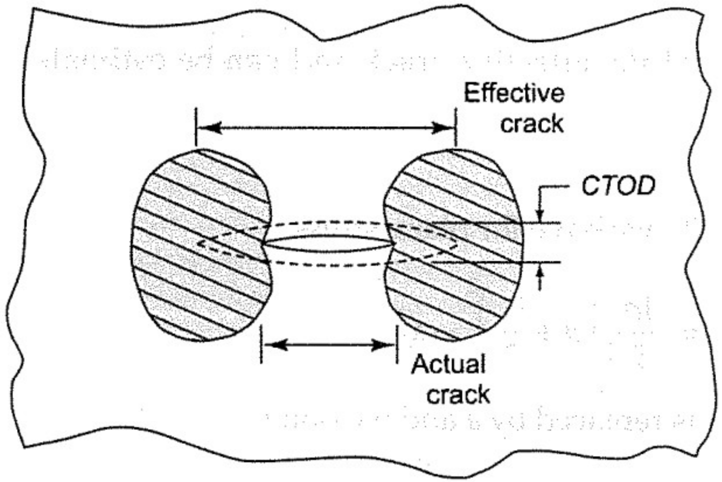


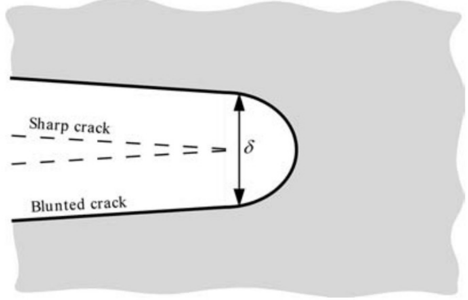


Source: Elementary Engineering Fracture Mechanics, David Broek

Crack tip opening displacement (CTOD)

Crack-tip opening displacement (CTOD) is another parameter suitable to characterize a crack. It can be used for both linear elastic fracture mechanics (LEFM) and elastic-plastic fracture mechanics (EPFM).





We have already seen that in the presence of plasticity the structure becomes more complaint and this effect can be simulated by considering an effective crack-length which is longer than the actual crack length.
With this assumption it can be observed that the tip of the actual crack in inside the effective crack and there is some opening at the actual crack-tip. The displacement of effective crack faces at the localtion of actual crack tip is called the crack-tip opening displacement (CTOD).
In reality, there is hardly any opening of the crack tip; only the tip may become more rounded as the plastic deformation increases, which is called the blunting of the crack-tip.

We have already derived the expression for COD for a center crack of length $2a$ under Mode-I loading as,

$$\mathrm{COD} = \frac{4\sigma}{E}\sqrt{a^2 - x_1^2}, \qquad \text{(for plane stress)}$$

For effective crack length a_{eff}, COD at a distance of a from the center from the crack-tip is

$$\begin{aligned}
\mathrm{CTOD} &= \frac{4\sigma}{E}\sqrt{a_{\text{eff}}^2 - a^2} = \frac{4\sigma}{E}\sqrt{\left(a + \frac{r_p}{2}\right)^2 - a^2}, \\
&= \frac{4\sigma}{E}\sqrt{\frac{r_p^2}{4} + ar_p} \\
&\approx \frac{4\sigma}{E}\sqrt{ar_p} \quad \text{(for small } r_p\text{)} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(14)
\end{aligned}$$

From (9), length of plastic zone, $r_p = \dfrac{K_I^2}{\pi\sigma_Y^2} = \dfrac{\sigma^2\pi a}{\pi\sigma_Y^2} = \dfrac{\sigma^2 a}{\sigma_Y^2}.$ (for plane stress)

Thus,

$$\mathrm{CTOD} = \frac{4\sigma^2 a}{\sigma_Y E} = \frac{4K_I^2}{\pi\sigma_Y E} \qquad \text{(for plane stress)} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(15)$$

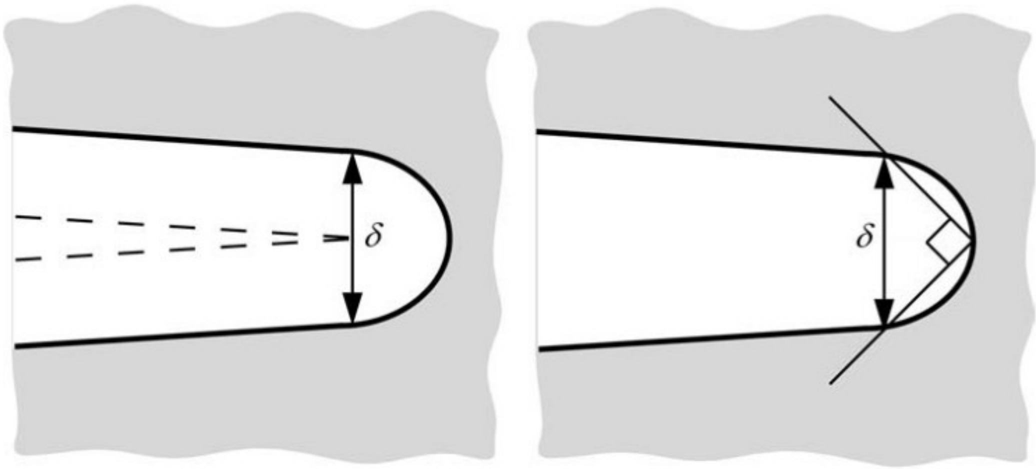
If Dugdale model is used then, $\mathrm{CTOD} = \dfrac{4\sigma^2 a}{\sigma_Y E} = \dfrac{K_I^2}{\sigma_Y E} = \dfrac{G_I}{\sigma_Y}.$ $\cdots\cdots\cdots\cdots\cdots\cdots(16)$

Thus in general CTOD can be written as,

$$\mathrm{CTOD} = \frac{K_I^2}{\lambda \sigma_Y E}, \qquad \text{.................}(17)$$

where value of λ depends upon the type of model used for finding the plastic zone size. From the direct experimental method it has been found that λ is close to unity.

The two most common method of measuring CTOD are shown, are the displacement at the original crack tip and the 90° intercept. Note that these two definitions are equivalent if the crack blunts in a semicircle.



Path independent integrals

Consider a elastic (linear or nonlinear), homogeneous, anisotropic solid body which is in a state of static equilibrium under the action of surface tractions. The body occupies a region Ω in the space which is bounded by surface $\partial\Omega$. For simplification we assume that the deformations remain small.

For elastic body the stress tensor can be obtained from the elastic strain energy ω as,

$$\sigma_{ij} = \frac{\partial w}{\partial \varepsilon_{ij}}, \quad w(0) = 0, \qquad \text{.................(17)}$$

where ε_{ij} are the components of small strain tensor. The strain energy density of the solid is

$$w = \int_0^{\varepsilon_{kl}} \sigma_{ij} d\varepsilon_{ij}. \qquad \text{.................(18)}$$

Note that this integral is path independent since the material is elastic.

In the absence of any body forces equilibrium equation and traction vector on surface are

$$\sigma_{ij,i} = \frac{\partial \sigma_{ij}}{\partial x_i} = 0, \quad t_j = \sigma_{ij} n_i, \qquad \text{.................(19)}$$

where n_i is the normal to the surface.

Now consider the surface integrals

$$Q_j = \int_{\partial\Omega} (w n_j - t_k u_{k,j})\, dS, \quad j = 1, 2, 3. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(20)$$

where $\partial\Omega$ is a closed surface bounding a region Ω which is assumed to be free of singularities. Substituting (19) in (20),

$$Q_j = \int_{\partial\Omega} (w n_j - \sigma_{lk} n_l u_{k,j})\, dS = \int_{\partial\Omega} (w \delta_{jl} - \sigma_{lk} u_{k,j})\, n_l dS. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(21)$$

Applying Gauss divergence theorem,

$$Q_j = \int_{\Omega} \frac{\partial}{\partial x_l} (w \delta_{jl} - \sigma_{lk} u_{k,j})\, dV = \int_{\Omega} (w \delta_{jl} - \sigma_{lk} u_{k,j})_{,l}\, dV. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(22)$$

We look at the integrand,

$$\begin{aligned}
(w \delta_{jl} - \sigma_{lk} u_{k,j})_{,l} &= w_{,l} \delta_{jl} - \cancelto{0}{\sigma_{lk,l} u_{k,j}} - \sigma_{lk} u_{k,jl} = w_{,j} - \sigma_{lk} u_{k,jl} \\
&= \frac{\partial w}{\partial \varepsilon_{lk}} \varepsilon_{lk,j} - \sigma_{lk} u_{k,jl} = \sigma_{lk} \varepsilon_{lk,j} - \sigma_{lk} u_{k,jl} \\
&= \sigma_{lk} (\varepsilon_{lk,j} - u_{k,jl}) = \sigma_{lk} (\varepsilon_{lk} - u_{k,l})_{,j}
\end{aligned}$$

$$
\begin{aligned}
\left(w\delta_{jl} - \sigma_{lk}u_{k,j}\right)_{,l} &= \sigma_{lk}\left[\frac{1}{2}\left(u_{l,k} + u_{k,l}\right) - u_{k,l}\right]_{,j} \\
&= \sigma_{lk}\frac{1}{2}\left(u_{l,k} - u_{k,l}\right)_{,j} = \sigma_{lk}\omega_{lk,j} = 0
\end{aligned}
$$

(Since, $\boldsymbol{\omega}$ is an antisymmetric tensor and contraction of a symmetric and anti-symmetric tensor is zero).

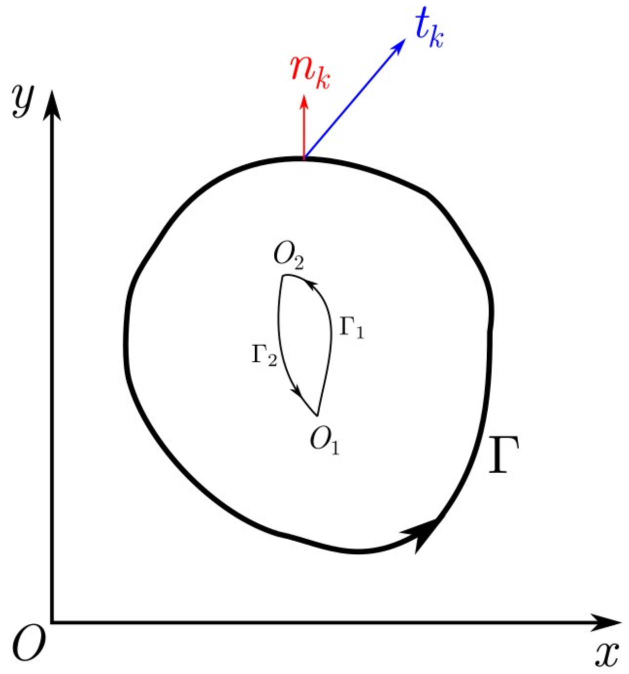
Thus,

$$Q_j = 0. \qquad \text{.................(23)}$$

i.e., the integral (20) is equal to zero for any surface $\partial\Omega$.

J-integral

For the particular case of the two-dimensional plane elastic problem, consider the integral



$$J = Q_1 = \int_\Gamma \left(w n_1 - t_k \frac{\partial u_k}{\partial x} \right) dS, \quad \text{where} \quad n_1 = dy/dS,$$

Thus,

$$J = \int_\Gamma w dy - t_k \frac{\partial u_k}{\partial x} dS, \quad (k = 1, 2) \quad \cdots\cdots\cdots\cdots\cdots\cdots (24)$$

Equation (24) defines the J-integral along a closed contour in the two-dimensional space. From (23) it follows that $J = 0$.

J=0 for any other closed integral which does not include singularity within. Hence,

$$J = J_{\Gamma_1} + J_{\Gamma_2} = 0 \Rightarrow J_{\Gamma_1} = -J_{\Gamma_2}.$$