ME632: Fracture Mechanics

Timings	
Monday	10:00 to 11:20
Thursday	08:30 to 09:50

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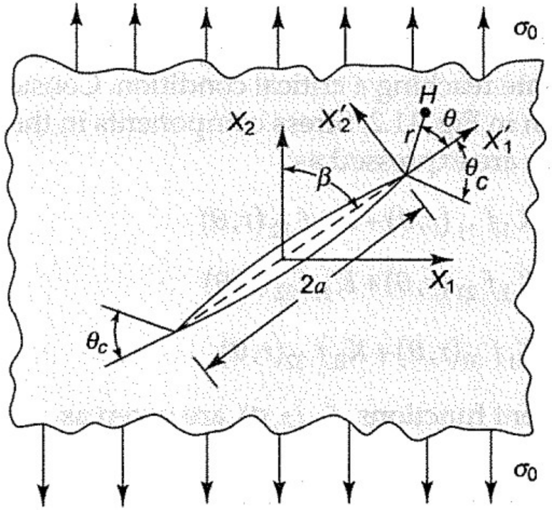
Mixed-mode crack growth

When direction of load is not aligned with the orientation of the crack, the crack-tip stress fields are governed by a combination of stress intensity factors K_{I}, K_{II} and K_{III}. The direction of crack initiation also depends on a failure criterion which is a function of K_{I}, K_{II} and K_{III}, resulting in a curved crack path. We will discuss the following criteria:

(a) Maximum Tangential Stress (MTS) criterion

(b) Strain Energy Density (SED) criterion

MTS criterion



This criterion was proposed by Erdogan and Sih in 1963.

$$\sigma_{rr} = K_I f_{11}(r,\theta) + K_{II} f_{12}(r,\theta)$$

$$\sigma_{\theta\theta} = K_I f_{21}(r,\theta) + K_{II} f_{22}(r,\theta)$$

$$\tau_{r\theta} = K_I f_{31}(r,\theta) + K_{II} f_{32}(r,\theta)$$

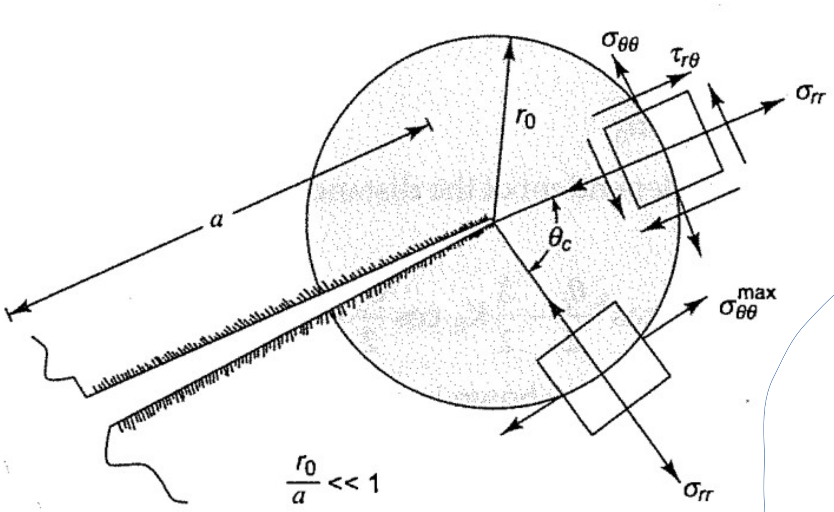
$$f_{11}(r,\theta) = \frac{1}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right) \qquad f_{22}(r,\theta) = \frac{1}{\sqrt{2\pi r}}\left(-\frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$f_{12}(r,\theta) = \frac{1}{\sqrt{2\pi r}}\left(-\frac{5}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right) \qquad f_{31}(r,\theta) = \frac{1}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right)$$

$$f_{21}(r,\theta) = \frac{1}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right) \qquad f_{32}(r,\theta) = \frac{1}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right)$$

According to MTS criterion, crack extension will occur in the direction where tangential stress component $\sigma_{\theta\theta}$ at an infinitesimal radial distance r_0 from the crack tip is maximum and the extension will take place when the maximum tangential stress reaches a critical value which is a material dependent parameter.

Crack extension direction:



$$\sigma_{\theta\theta} = \frac{K_I}{4\sqrt{2\pi r}}\left(3\cos\frac{\theta}{2} + \cos\frac{3\theta}{2}\right) - \frac{3K_{II}}{4\sqrt{2\pi r}}\left(\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right)$$

$$\frac{\partial \sigma_{\theta\theta}}{\partial \theta} = 0$$

$$\frac{\partial^2 \sigma_{\theta\theta}}{\partial \theta^2} < 0.$$

$$K_I \sin\theta_c + K_{II}\left(3\cos\theta_c - 1\right) = 0$$

Note that at $\theta = \theta_c$ shear stress $\tau_{r\theta}$ become zero; hence σ_{rr} and $\sigma_{\theta\theta}$ become principle stresses in that direction.

Critical condition:

$$\sigma_{\theta\theta}^{\max} = (\sigma_{\theta\theta})_{\theta=\theta_c}$$

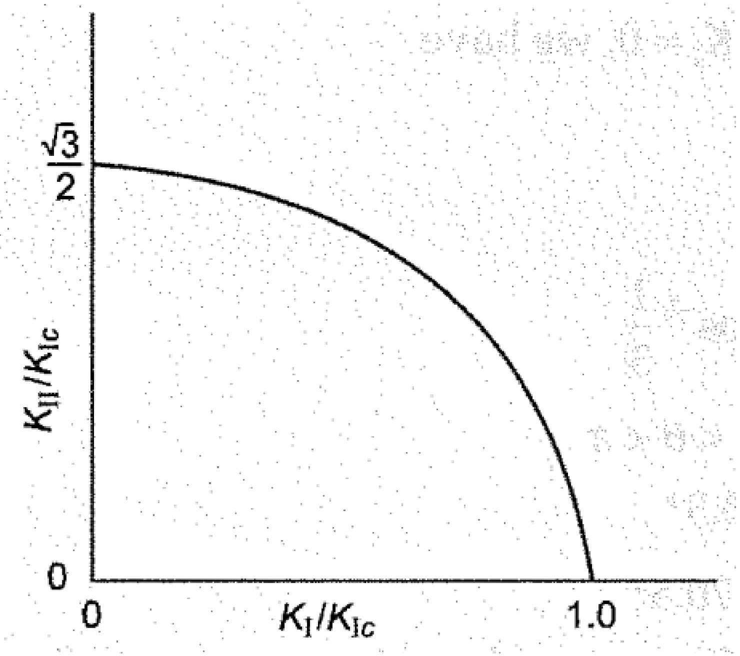
$$\sigma_{\theta\theta}^{\max} = \frac{K_I}{\sqrt{2\pi r_0}}\cos^3\frac{\theta_c}{2} - \frac{3}{2}\frac{K_{II}}{\sqrt{2\pi r_0}}\cos\frac{\theta_c}{2}\sin\theta_c$$

Crack extension occurs when $\sigma_{\theta\theta}^{\max}$ reaches a critical value σ_c which is a material property; σ_c is usually obtained from pure Mode I loading where $\theta_c = 0$ and $K_I = K_{Ic}$, that is,

$$\sigma_c = \frac{K_{Ic}}{\sqrt{2\pi r_0}}$$

Failure will occur when, $\sigma_{\theta\theta}^{\max} = \sigma_c$, which gives the equation of failure surface as,

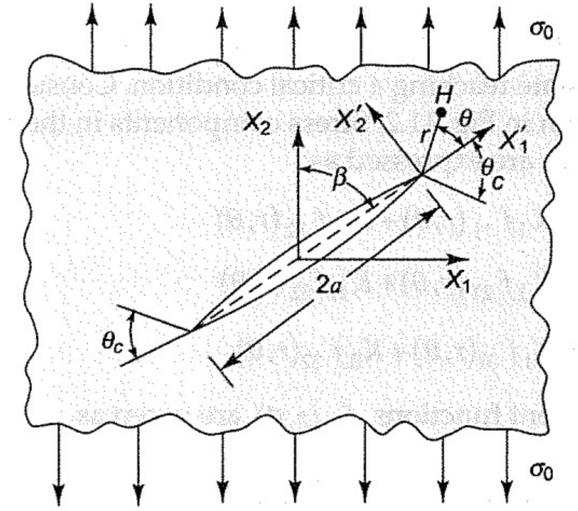
$$K_I\cos^3\frac{\theta_c}{2} - \frac{3}{2}K_{II}\cos\frac{\theta_c}{2}\sin\theta_c = K_{Ic}$$



SED criterion

Based on energy principles, Sih in 1973 proposed Strain Energy Density (SED) criterion. Consider a crack subjected to Modes I and II loading. Strain energy density is defined as the strain energy per unit volume at a given point in solid, which can be obtained from stress and strain field as:

$$W = \frac{dU}{dV} = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij}$$



For plane linear elasticity problems W can be written in the following form in terms of stress components

$$W = \frac{(1+\nu)}{2E}\left[\frac{\kappa+1}{4}(\sigma_{11}+\sigma_{22})^2 - 2(\sigma_{11}\sigma_{22} - \tau_{12}^2)\right]$$

where

$$\kappa = 3-4\nu \qquad \text{(for plane strain)}$$
$$= (3-\nu)/(1+\nu) \qquad \text{(for plane stress)}$$

The Cartesian stress components in the vicinity of crack tip in terms of polar coordinate system are given as:

$$\sigma_{11} = K_I f_{11}(r,\theta) + K_{II} f_{12}(r,\theta)$$
$$\sigma_{22} = K_I f_{21}(r,\theta) + K_{II} f_{22}(r,\theta)$$
$$\tau_{12} = -K_I f_{31}(r,\theta) + K_{II} f_{32}(r,\theta)$$

$$f_{11}(r,\theta) = \frac{1}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \qquad f_{22}(r,\theta) = \frac{1}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$f_{12}(r,\theta) = -\frac{1}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\left(2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right) \qquad f_{31}(r,\theta) = \frac{1}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$f_{21}(r,\theta) = \frac{1}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \qquad f_{32}(r,\theta) = \frac{1}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right).$$

Strain energy density function can be written in terms of stress field as,

$$W = \frac{1}{\pi r}\left[g_{11}K_I^2 + 2g_{12}K_I K_{II} + g_{22}K_{II}^2\right]$$

$$g_{11} = \frac{1}{16\mu}(1+\cos\theta)(\kappa-\cos\theta)$$

$$g_{12} = \frac{1}{16\mu}\sin\theta\left[2\cos\theta-(\kappa-1)\right]$$

$$g_{22} = \frac{1}{16\mu}\left[(\kappa+1)(1-\cos\theta)+(1+\cos\theta)(3\cos\theta-1)\right]$$

$$\mu = \frac{E}{2(1+\nu)}$$

Strain energy density function poses singularity of order one at the crack tip. Sih proposed a strain energy density factor S in a quadratic form which is independent of the coordinate r and it is defined as:

$$S(\theta) = \left(g_{11} K_{\mathrm{I}}^2 + 2 g_{12} K_{\mathrm{I}} K_{\mathrm{II}} + g_{22} K_{\mathrm{II}}^2\right)/\pi$$

According to SED criterion, crack extension will occur in the direction of minimum strain energy density $S(\theta)$ and the extension will occur when the $S(\theta)$ reaches a critical value S, which is a material property.

Crack extension direction:

$$\frac{\partial W}{\partial \theta} = 0 \qquad \text{and} \qquad \frac{\partial^2 W}{\partial \theta^2} > 0$$

or

$$\frac{\partial S}{\partial \theta} = 0 \qquad \text{and} \qquad \frac{\partial^2 S}{\partial \theta^2} > 0$$

These conditions leads to the following conditions which yields the direction of crack initiation,

$$\left[2\cos\theta-(\kappa-1)\right]\sin\theta\, K_{\mathrm{I}}^2+2\left[2\cos 2\theta-(\kappa-1)\cos\theta\right]K_{\mathrm{I}}\,K_{\mathrm{II}}\,+$$

$$\left[(\kappa-1-6\cos\theta)\sin\theta\right]K_{\mathrm{II}}^2=0$$

$$\left[2\cos 2\theta-(\kappa-1)\cos\theta\right]K_{\mathrm{I}}^2+2\left[(\kappa-1)\sin\theta-4\sin 2\theta\right]K_{\mathrm{I}}\,K_{\mathrm{II}}\,+$$

$$\left[(\kappa-1)\cos\theta-6\cos 2\theta\right]K_{\mathrm{II}}^2>0$$

Critical condition:

Crack extension will occur when minimum value of strain energy density function ($S_{\min}$) reaches a critical value of strain energy density factor S_c Thus, the condition is expressed as:

$$\left(S_{\min}\right)\geq S_c$$

S_c is usually obtained from pure Mode I loading where $\theta_c=0$ and $K_{\mathrm{I}}=K_{\mathrm{Ic}}$, that is,

$$S_c=\frac{(1+\nu)(\kappa-1)}{4\pi E}K_{\mathrm{Ic}}^2$$

Exercise:

1. Find out the stress intensity factor at crack initiation and direction of crack initiation for pure mode-I and pure mode-II loading

(a) Using MTS criterion

(b) Using SED criterion

2. Consider an infinite plate with a crack of length $2a = 80$ mm, inclined at angle β with the applied tensile stress σ_0. K_{Ic} of the material is known to be 40 MPa.√m, its elastic constants are $E = 200$ GPa and $\nu = 0.3$, and the plate is subjected to plane strain.

(i) Determine initial crack extension direction using MTS and SED fracture criteria for $\beta = 60°$, (ii) find the applied stress σ_0 corresponding to the crack initiation using MTS and SED fracture criteria for $\beta = 60°$ and,

(iii) determine relations θ_c vs. β and critical σ_0 vs. β for both fracture criteria for β varying between 10° and 90°.