ME232: Dynamics

Vibrations

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Basic concepts

- **Space:** Geometric region occupied by bodies.
- **Primary inertial system or Astronomical frame of reference:**
  - The basic frame of reference for the laws of Newtonian mechanics.
  - An imaginary set of rectangular axes assumed to have **no translation or rotation in space.**
  - The laws of Newtonian mechanics are valid for this reference system as long as any velocities involved are negligible compared with the speed of light (i.e., 300,000 km/s).
  - Measurements made with respect to this reference are said to be **absolute**, and this reference system may be considered **fixed in space**.
- A reference frame attached to the surface of the earth has a somewhat complicated motion in the primary system, and **a correction to the basic equations of mechanics** must be applied for measurements made relative to the reference frame of the earth.

Basic concepts

- **Particle:** A body can be treated as particle if
  - it has negligible dimensions.
  - when the **dimensions of a body are irrelevant to the description of its motion** or the action of forces on it; e.g., an airplane may be treated as a particle for the description of its flight path.
- **Rigid body:** A body is treated as rigid body when **changes in shape are negligible** compared with the overall dimensions of the body or with the changes in position of the body as a whole.

Newton's Laws

- **Newton's Laws:**
  - ***Law I.*** A particle remains at rest or continues to move with uniform velocity (in a straight line with a constant speed) if there is no unbalanced force acting on it.
  - ***Law II.*** The acceleration of a particle is proportional to the resultant force acting on it and is in the direction of this force.
  - ***Law III.*** The forces of action and reaction between interacting bodies are equal in magnitude, opposite in direction, and collinear.
- The first two laws hold for measurements made in an absolute frame of reference, and they are subject to some correction when the motion is measured relative to a reference system having acceleration, such as one attached to the surface of the earth.
- The correction become **insignificant for most engineering problems** involving machines and structures which remain on the surface of the earth and in such cases these **laws may be applied directly with measurements made relative to the earth.**

Newton's Laws

- Newton's second law forms the basis for most of the analysis in dynamics. For a particle of mass m subjected to a resultant force F, the law may be stated as

$$\boldsymbol{F} = m\boldsymbol{a},$$

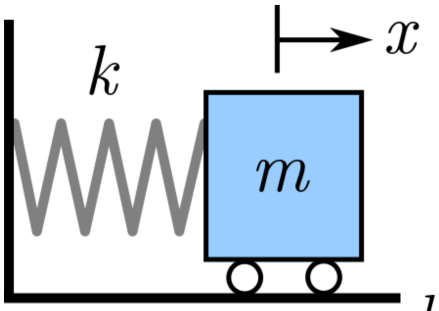
where $\boldsymbol{a}$ is the resulting acceleration measured in a non-accelerating frame of reference. Newton's first law is a consequence of the second law since there is no acceleration when the force is zero, and so the particle is either at rest or is moving with constant velocity.

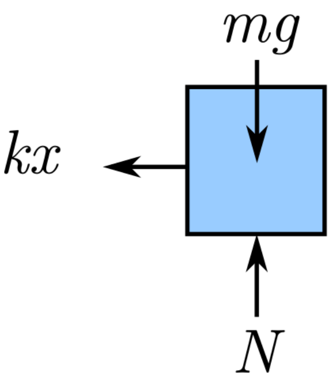
Vibration

- Vibration of mechanical systems
  - concerns the linear and angular oscillatory motions of bodies
  - motion in response to applied disturbances in the presence of restoring forces.
- For example:
  - response of an engineering structure to earthquakes,
  - the time response of the plucked string of a musical instrument,
  - the vibration of an unbalanced rotating machine,
  - the wind-induced vibration of power lines,
  - the flutter of aircraft wings.
- In many cases, presence of excessive vibration levels may lead to **human discomfort, non-smooth and noisy operation of machines, and fatigue**, which may ultimately lead to failure.

Free vibration of particles

- When a spring-mounted body is disturbed from its equilibrium position, its ensuing motion **in the absence of any imposed external forces** is termed **free vibration**.
- In practice, some **retarding or damping force** always exists which tends to diminish the motion. Common damping forces are those due to **mechanical and fluid friction**.
- We will first consider the ideal case where the **damping forces are small enough** to be neglected.





Considering the horizontal vibration of the simple friction-less spring-mass system. The variable x denotes the displacement of the mass from the **equilibrium position**, which is also the **position of zero spring deflection.**

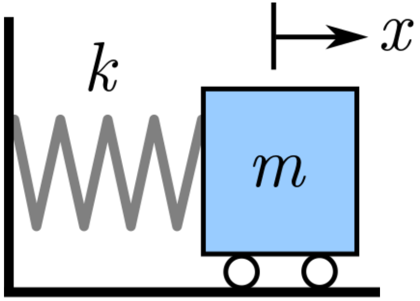
Figure shows a plot of the force F_s necessary to deflect the spring vs. the corresponding spring deflection for three types of springs.

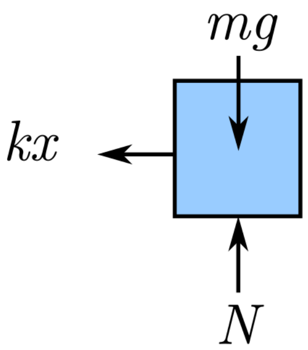
If we consider a linear spring, then the restoring force exerted by the spring on the mass is $-kx$, i.e., when the mass is displaced to the right, the spring force is to the left, and vice versa.

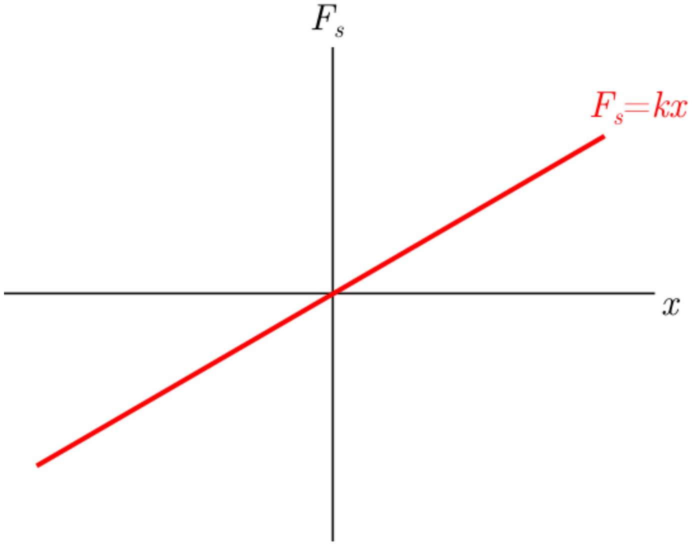
The constant of proportionality k is called the **spring constant, modulus, or stiffness** and has the units N/m.

The equation of motion for the body of is obtained by applying Newton's second law,

$$\sum F_x = m\ddot{x} \quad \Rightarrow \quad -kx = m\ddot{x} \quad \text{or} \quad m\ddot{x} + kx = 0. \qquad \text{.................(1)}$$







The oscillation of a mass subjected to a linear restoring force described by (1) is called **simple harmonic motion** and is characterized by acceleration which is proportional to the displacement but of opposite sign.

Normally, we write (1) as following:

$$\ddot{x} + \omega_n^2 x = 0, \quad \text{where} \quad \omega_n = \sqrt{k/m}, \qquad \cdots\cdots(2)$$

Because we anticipate an oscillatory motion, we look for a solution which gives x as a periodic function of time. Thus, a logical choice is

$$x = A\cos\omega_n t + B\sin\omega_n t, \quad \text{or} \qquad \cdots\cdots(3)$$

$$\text{alternatively} \quad x = C\sin(\omega_n t + \psi). \qquad \cdots\cdots(4)$$

The constants A and B, or C and ψ, can be determined from knowledge of the initial displacement x_0 and initial velocity $\dot{x}_0$ of the mass. For example, if we work with the solution form of (3) and evaluate x and $\dot{x}$ at time $t = 0$, we obtain,

$$x_0 = A, \qquad \text{and} \qquad \dot{x}_0 = \omega_n B$$

Substitution of these values of A and B into (4) yields

$$x = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(5)$$

The constants C and ψ of (4) can be determined in terms of given initial conditions in a similar manner. Evaluation of (4) and its first time derivative at $t = 0$ gives

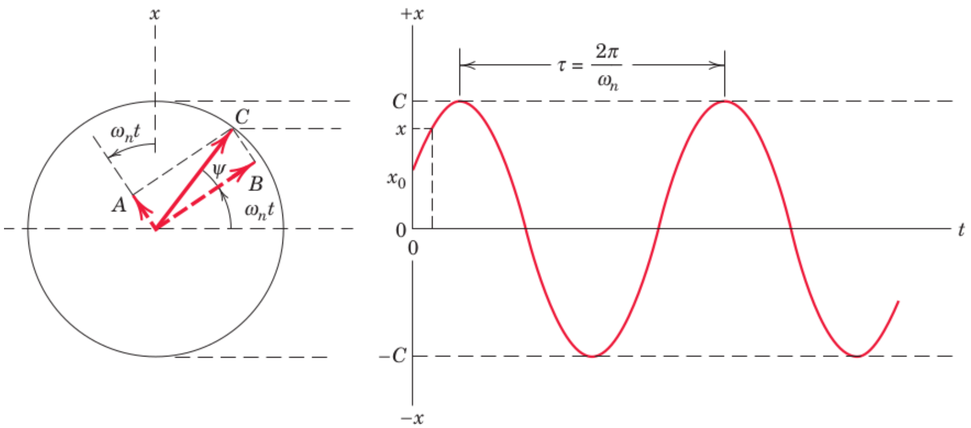
$$x_0 = C \sin \psi, \qquad \text{and} \qquad \dot{x}_0 = C\omega_n \cos \psi. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(6)$$

Solving for C and ψ yields,

$$C = \sqrt{x_0 + (\dot{x}_0/\omega_n)^2}, \qquad \psi = \tan^{-1}(x_0 \omega_n / \dot{x}_0). \qquad \cdots\cdots\cdots\cdots\cdots\cdots(7)$$

Equations (3) and (4) represent two different mathematical expressions for the same time-dependent motion.

Graphical representation of motion



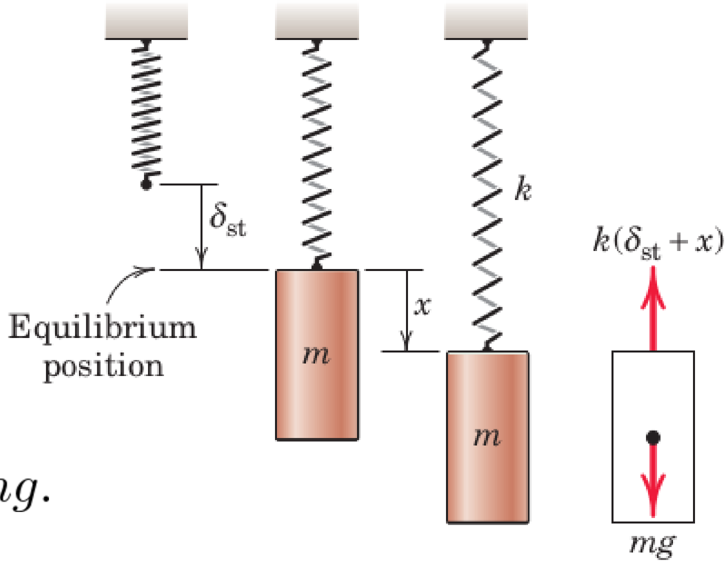
The motion may be represented graphically, where x is seen to be the projection onto a vertical axis of the rotating vector of length C. The vector rotates at the constant angular velocity $\omega_n = \sqrt{(k/m)}$, which is called the **natural circular frequency** and has the units radians per second.

The **number of complete cycles per unit time** is the **natural frequency** $\boldsymbol{f_n = \omega_n / 2\pi}$ and is expressed in hertz (1 hertz (Hz)=1 cycle per second).

The **time required for one complete motion cycle** (one rotation of the reference vector) is **the period of the motion** and is given by $\tau = 1/f_n = 2\pi/\omega_n$.

Also note that x is the sum of the projections onto the vertical axis of two perpendicular vectors whose magnitudes are A and B and whose vector sum C is the amplitude. Vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ rotate together with the constant angular velocity ω_n. Thus, as we have already seen, $C = (A^2 + B^2)^{\frac{1}{2}}$ and $\psi = \tan^{-1}(A/B)$.

Now if the motion of mass is vertical rather than horizontal, the equation of motion (and therefore all system properties) is **unchanged** if we continue to define x as the **displacement from the equilibrium position**. The equilibrium position now involves a nonzero spring deflection δ_{st}. From the free-body diagram, Newton's second law gives



$$-k(x + \delta_{st}) + mg = m\ddot{x} \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(8)$$

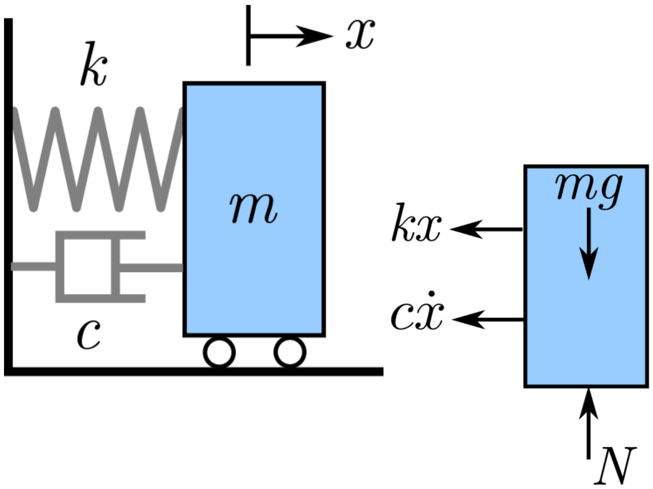
Note that at the equilibrium position (i.e., at $x = 0$), $k\delta_{st} = mg$. Thus (8) becomes,

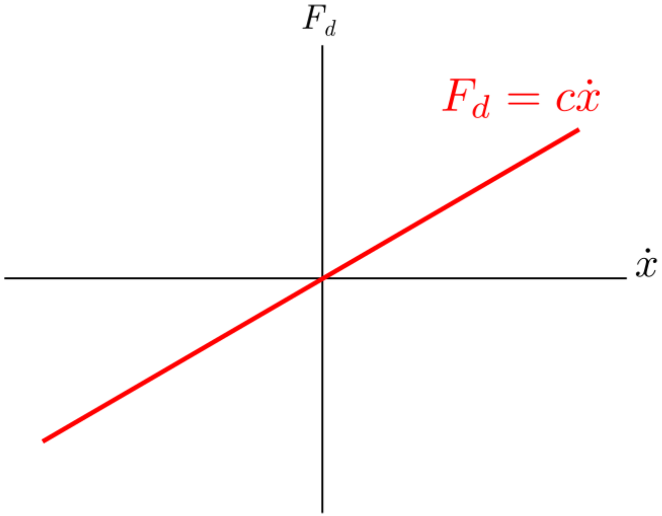
$$-kx = m\ddot{x} \quad \text{or} \quad m\ddot{x} + kx = 0. \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(9)$$

which is same as (1).

The lesson here is that by defining **the displacement variable to be zero at equilibrium** rather than at the position of zero spring deflection, **we may ignore the equal and opposite forces associated with equilibrium.**

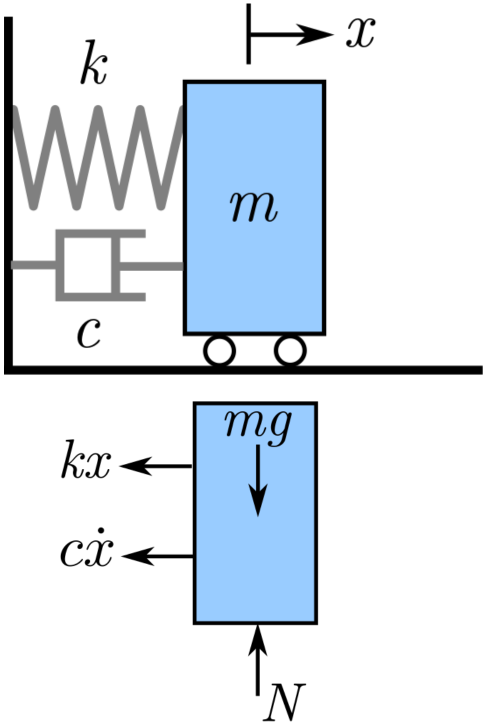
Damped free vibration





Every mechanical system possesses some forces which dissipates mechanical energy. A dashpot or viscous damper is a device added to systems for limiting or retarding vibration. It consists of a cylinder filled with a viscous fluid and a piston with holes or other passages by which the fluid can flow from one side of the piston to the other.

A simple linear dashpot is shown, which exert a force F_d whose magnitude is proportional to the velocity of the mass. The constant of proportionality c is called the **viscous damping coefficient** and has units of N·s/m. The direction of the **damping force applied to the mass is opposite that of the velocity $\dot{x}$.** Thus, the force on the mass is $-c\dot{x}$.



The equation of motion for the body with damping is given by Newton's second as

$$-kx - c\dot{x} = m\ddot{x} \quad \text{or} \quad m\ddot{x} + c\dot{x} + kx = 0. \qquad \text{......................(10)}$$

In addition to the substitution $\omega_n = \sqrt{k/m}$, it is convenient to introduce the combination of constants $\zeta = c/(2m\omega_n)$. The quantity ζ (zeta) is called the **viscous damping factor or damping ratio** and is a measure of the severity of the damping. It should be noted that ζ is non-dimensional. (10) may now be written as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0. \qquad \text{......................(11)}$$

In order to solve the equation of motion (11) we assume solutions of the form

$$x = Ae^{\lambda t}. \qquad \text{......................(12)}$$

Substituting (12) in (11) yields, $\quad \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0. \qquad$ ....(13)

which is called **the characteristic equation**. Its roots are

$$\lambda_1 = \omega_n(-\zeta + \sqrt{\zeta^2 - 1}), \quad \lambda_2 = \omega_n(-\zeta - \sqrt{\zeta^2 - 1}). \qquad \text{......................(14)}$$

Linear systems have the **property of superposition**, which means that the general solution is the **sum of the individual solutions** each of which corresponds to one root of the characteristic equation. Thus, the general solution is

$$x = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} = A_1 e^{(-\zeta+\sqrt{\zeta^2-1})\omega_n t} + A_2 e^{(-\zeta-\sqrt{\zeta^2-1})\omega_n t}. \quad \cdots\cdots\cdots\cdots(14)$$

Categories of damped motion:

Because $0 \leq \zeta \leq \infty$, the radicand $(\zeta^2 - 1)$ may be positive, negative, or even zero, giving rise to the following three categories of damped motion:

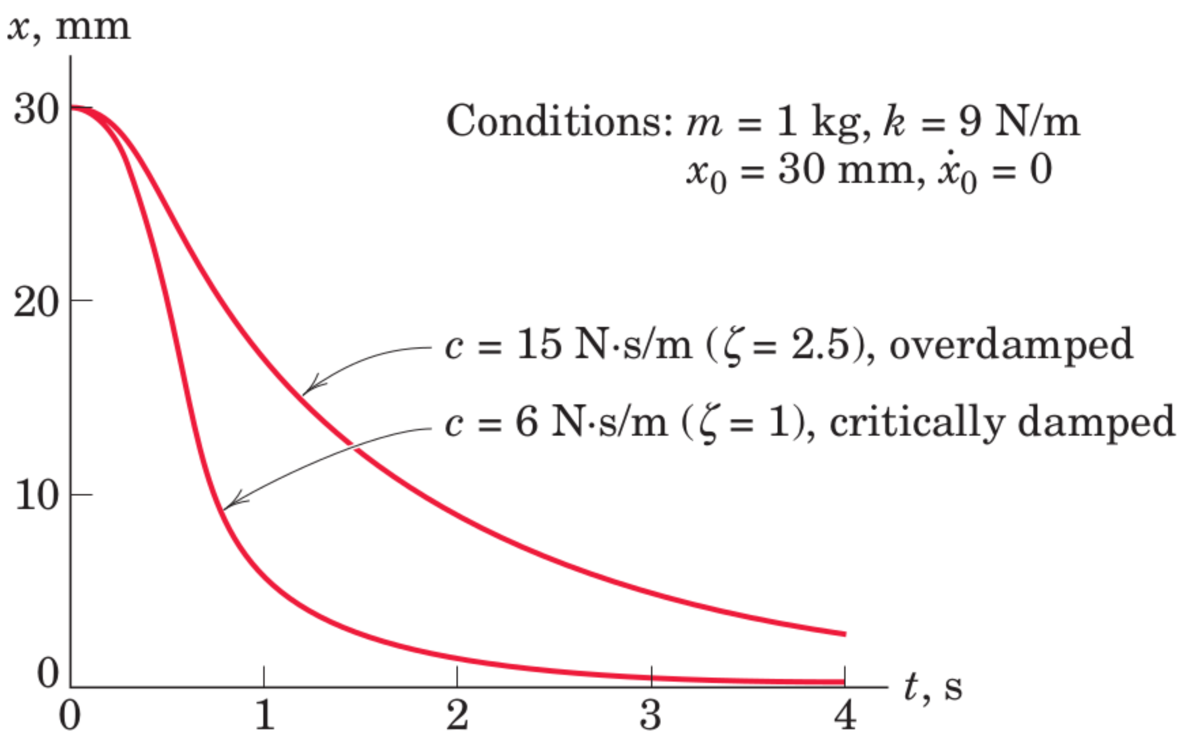
I. $\zeta > 1$ (overdamped): The roots λ_1 and λ_2 are **distinct, real, and negative numbers**. The motion as given by (14) decays so that x **approaches zero for large values of time** t**.** There is **no oscillation** and therefore no period associated with the motion.

II. $\zeta = 1$ (critically damped): The roots λ_1 and λ_2 are **equal, real, and negative numbers** ($\lambda_1 = \lambda_2 = -\omega_n$). The solution to the differential equation for the special case of equal roots is given by

$$x = (A_1 + A_2 t)e^{-\zeta\omega_n t}. \quad \cdots\cdots\cdots\cdots(15)$$

Again, the motion **decays with x approaching zero for large** t, and the motion is nonperiodic.

A critically damped system, when excited with an initial velocity or displacement (or both), will approach equilibrium faster than will an overdamped system.



III. $\zeta < 1$ (underdamped): So the radicand $(\zeta^2 - 1)$ is negative and we may rewrite (14) as

$$x = e^{-\zeta\omega_n t}[A_1 e^{i\sqrt{1-\zeta^2}\omega_n t} + A_2 e^{-i\sqrt{1-\zeta^2}\omega_n t}].$$

It is convenient to let a new variable ω_d represent the combination $\omega_n\sqrt{1-\zeta^2}$. Thus,

$$x = e^{-\zeta\omega_n t}[A_1 e^{i\omega_d t} + A_2 e^{-i\omega_d t}]. \qquad \text{................(16a)}$$

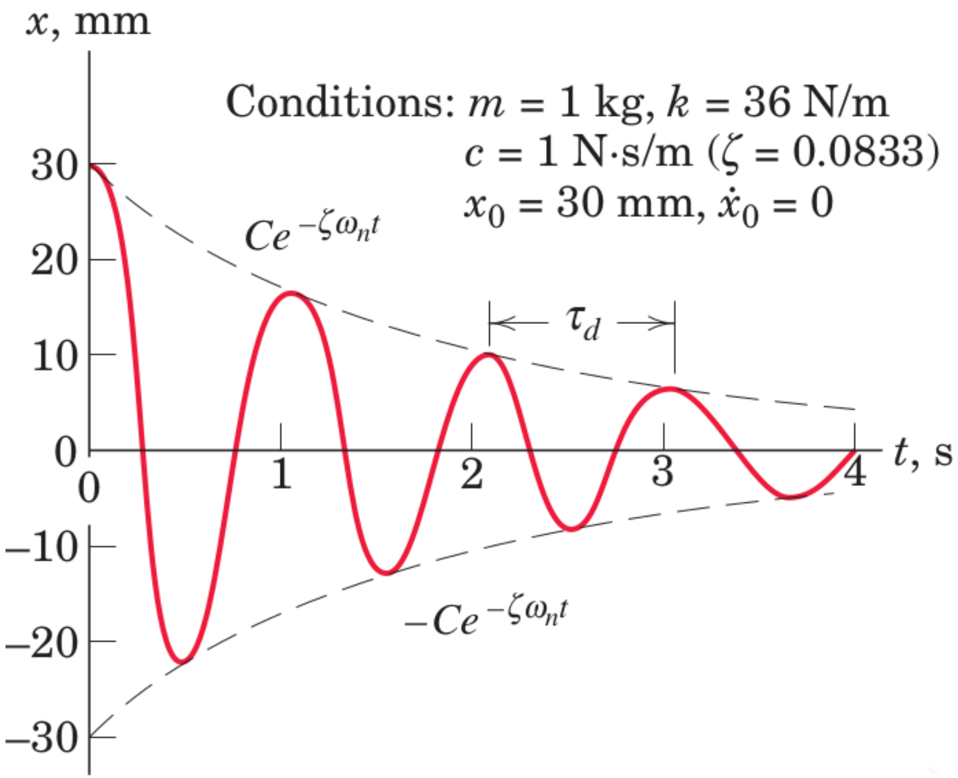
Use of the Euler formula (16) can be rewritten as

$$x = e^{-\zeta\omega_n t}[A_3 \cos\omega_d t + A_4 \sin\omega_d t], \qquad \text{................(16b)}$$

where $A_3 = (A_1 + A_2)$ and $A_4 = i\,(A_1 - A_2)$. Alternatively we can also write,

$$x = e^{-\zeta\omega_n t}[C\sin(\omega_d t + \psi)],$$

$$\text{or} \qquad x = Ce^{-\zeta\omega_n t}\sin(\omega_d t + \psi). \qquad \text{.............(16c)}$$



(16) represents an exponentially decreasing harmonic function, as shown in figure for specific numerical values. The frequency

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

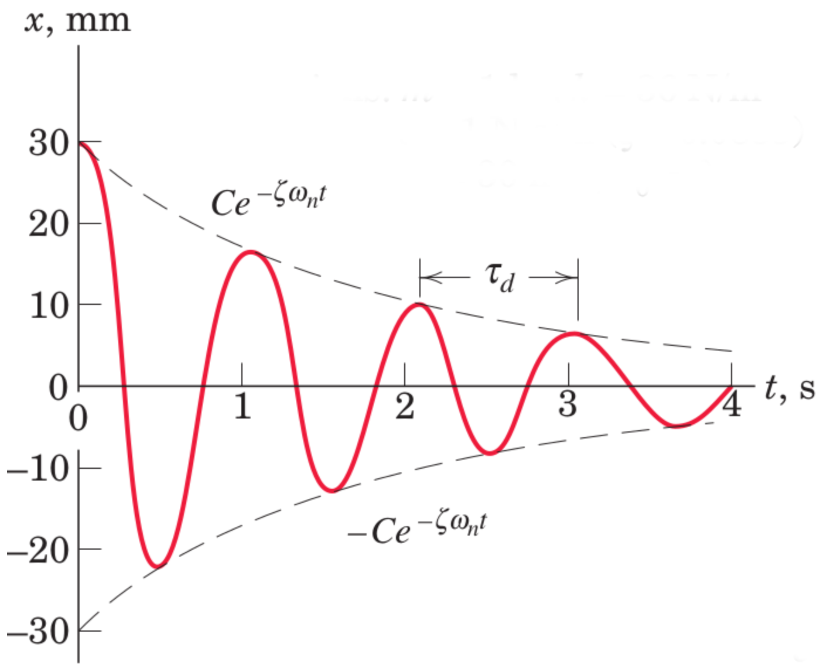
is called the **damped natural frequency**. The damped period is given by $\tau_d = 2\pi/\omega_d$.

To find C and ψ if damping is present we use (16) and apply initial conditions, i.e., at $t = 0$, initial displacement is x_0 and initial velocity is $\dot{x}_0$, respectively.

Determination of damping by experiment:

The value of the damping ratio ζ need to be **determined experimentally** for an underdamped system because the value of the viscous damping coefficient c is not well known.

To determine the damping, we may **excite the system by initial conditions** and obtain a plot of the displacement x versus time t. We then measure two amplitudes x_1 and x_{N+1}, N full cycle apart and compute their ratio



$$\frac{x_1}{x_{N+1}} = \frac{Ce^{-\zeta\omega_n t_1}}{Ce^{-\zeta\omega_n (t_1+N\tau_d)}} = e^{\zeta\omega_n N\tau_d}. \qquad \text{.............(17)}$$

The logarithmic decrement δ is defined as, $\delta = \frac{1}{N}\ln\left(\frac{x_1}{x_{N+1}}\right) = \zeta\omega_n\tau_d = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$. ......(18)

From this equation, we may solve for ζ and obtain $\zeta = \frac{\delta}{\sqrt{(2\pi)^2+\delta^2}}$. ....(19)

Example 1

A body weighing 10 kg is suspended from a spring of constant $k = 2.5$ kN/m. At time $t = 0$, it has a downward velocity of 0.5 m/sec as it passes through the position of static equilibrium. Determine

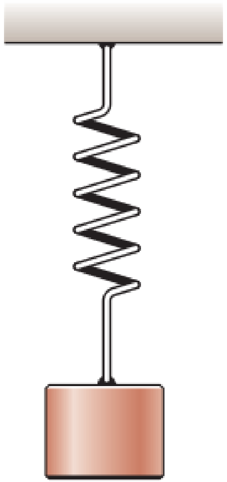
(a) the static spring deflection δ_{st}

(b) the natural frequency of the system in both rad/sec (ω_n) and cycles/sec (f_n)

(c) the system period τ

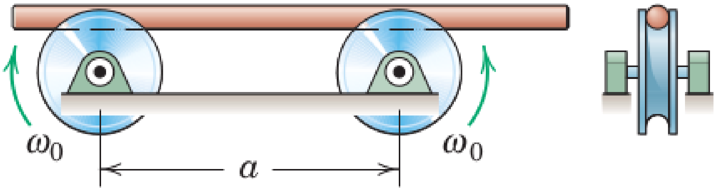
(d) the displacement x as a function of time, where x is measured from the position of static equilibrium

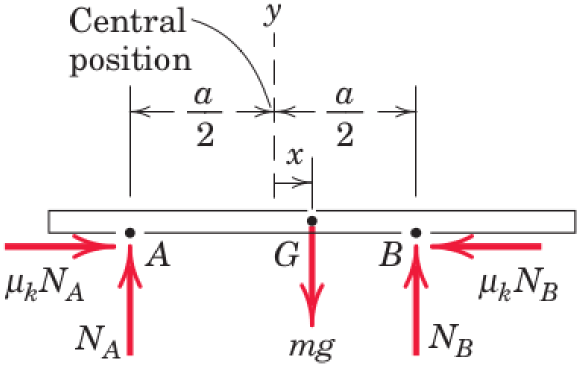
(e) the maximum velocity v_{max} attained by the mass.



Example 2

The two fixed counter rotating pulleys are driven at the same angular speed ω_0 . A round bar is placed off center on the pulleys as shown. Determine the natural frequency of the resulting bar motion. The coefficient of kinetic friction between the bar and pulleys is μ_k .





The governing equations are

$$\sum F_x = m\ddot{x} \qquad \mu_k N_A - \mu_k N_B = m\ddot{x},$$

$$\sum F_y = 0 \qquad N_A + N_B - mg = 0,$$

$$\sum M_A = 0 \qquad aN_B - (a/2 + x)mg = 0,$$

Eliminating N_A and N_B from the first equation yields

$$\ddot{x} + \frac{2\mu_k g}{a} x = 0.$$

Thus, natural frequency is,

$$\omega_n = \sqrt{\frac{2\mu_k g}{a}}.$$

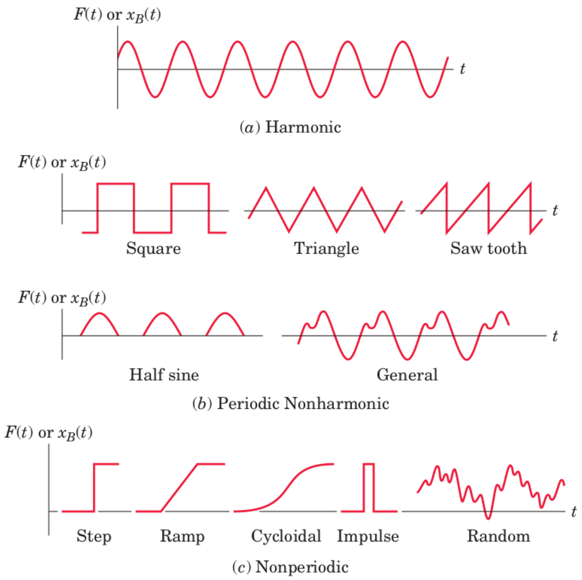
Forced vibration

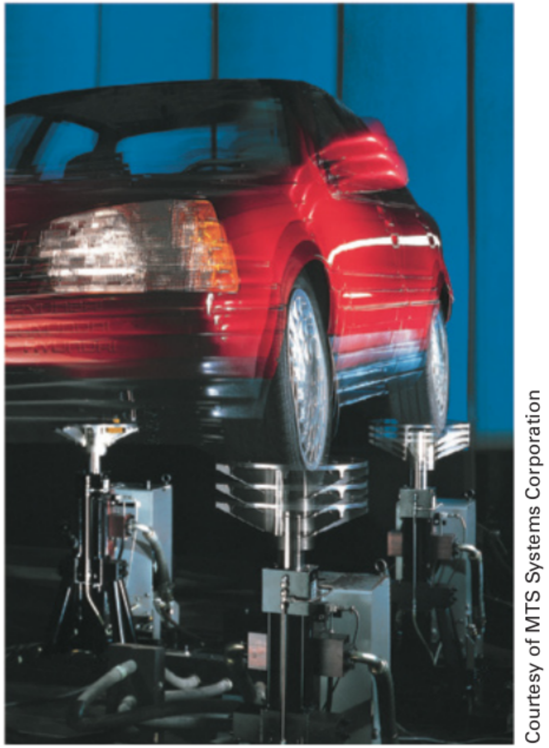
Although there are many significant applications of free vibrations, the most important class of vibration problems is that where the **motion is continuously excited by a disturbing force.**

The force may be **externally applied or may be generated within the system** by some means.

Forced vibrations may also be excited by the motion of the system foundation.

Various forms of excitation forcing functions $F = F(t)$ and foundation displacements $x_B = x_B(t)$ are shown in figure.





An automobile undergoing vibration testing of its suspension system.