Balance Principles

In this chapter we discuss some important fundamental balance principles such as,

- conservation of mass,
- the momentum balance principle (linear and angular momentum), and
- balance of energy.

These principles are valid in all branches of continuum mechanics. They must be satisfied for all times.

We will also discuss some fundamentals laws which are inequalities such as second law of thermodynamics.

To obtain the expression for material description of momentum balance, we define reference body forces $\boldsymbol{B}(\boldsymbol{X},t)$ and its related with the body force $\boldsymbol{b}(\boldsymbol{x},t)$ in the following manner,

$$\int_{\Omega} \boldsymbol{b}(\boldsymbol{x},t)dv = \int_{\Omega_0} \boldsymbol{b}(\boldsymbol{\chi}(\boldsymbol{X},t),t)J(\boldsymbol{X},t)dV = \int_{\Omega_0} \boldsymbol{B}(\boldsymbol{X},t)dV,$$

or in the local form,

$$\boldsymbol{B}(\boldsymbol{X},t) = \boldsymbol{b}(\boldsymbol{x},t)J(\boldsymbol{X},t), \quad \text{or} \quad B_i = Jb_i.$$

Now, the linear and angular momentum balance principle in material co-ordinates can be written as,

$$\boxed{\begin{gathered}\frac{D}{Dt}\int_{\Omega_0} \rho_0 \boldsymbol{V}dV = \int_{\Omega_0} \rho_0 \dot{\boldsymbol{V}}dV = \int_{\partial\Omega_0} \boldsymbol{T}dS + \int_{\Omega_0} \boldsymbol{B}dV, \\ \frac{D}{Dt}\int_{\Omega_0} \boldsymbol{r}\times\rho_0 \boldsymbol{V}dV = \int_{\Omega_0} \boldsymbol{r}\times\rho_0 \dot{\boldsymbol{V}}dV = \int_{\partial\Omega_0} \boldsymbol{r}\times\boldsymbol{T}dS + \int_{\Omega_0} \boldsymbol{r}\times\boldsymbol{B}dV.\end{gathered}}$$

where, $\boldsymbol{T}(\boldsymbol{X},\boldsymbol{N},t)$ is the Piola-Kirchoff traction vector.

Equation of motion

Consider the spatial form of linear momentum balance equation,

$$\int_{\Omega} \rho \dot{\boldsymbol{v}} dv = \int_{\partial\Omega} \boldsymbol{t} ds + \int_{\Omega} \boldsymbol{b} dv.$$

By using Cauchy's stress theorem, we can write,

$$\int_{\Omega} \rho \dot{\boldsymbol{v}} dv = \int_{\partial\Omega} \boldsymbol{n} \cdot \boldsymbol{\sigma} ds + \int_{\Omega} \boldsymbol{b} dv, \quad \text{or} \quad \int_{\Omega} \rho \dot{v}_i dv = \int_{\partial\Omega} \sigma_{ji} n_j ds + \int_{\Omega} b_i dv.$$

Applying Gauss-divergence theorem,

$$\int_{\Omega} \rho \dot{\boldsymbol{v}} dv = \int_{\Omega} \left(\mathrm{div}\boldsymbol{\sigma} + \boldsymbol{b}\right) dv, \quad \text{or} \quad \int_{\Omega} \rho \dot{v}_i dv = \int_{\Omega} \left(\sigma_{ji,j} + b_i\right) dv.$$

$$\Rightarrow \boxed{\int_{\Omega} \left(\mathrm{div}\boldsymbol{\sigma} + \boldsymbol{b} - \rho \dot{\boldsymbol{v}}\right) dv = 0, \quad \text{or} \quad \int_{\Omega} \left(\sigma_{ji,j} + b_i - \rho \dot{v}_i\right) dv = 0.}$$

Above equation is known as **Cauchy's equation (or first equation) of motion** in the global form.

By applying the localization theorem as v is an arbitrary volume in region Ω local form is

$$\boxed{\mathrm{div}\boldsymbol{\sigma} + \boldsymbol{b} = \rho\dot{\boldsymbol{v}}, \quad \text{or} \quad \sigma_{ji,j} + b_i = \rho\dot{v}_i,}$$

for each point $\boldsymbol{x}$ of v at all time t.

If acceleration if assumed to be zero (i.e. constant velocity) then,

$$\boxed{\mathrm{div}\boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{O}, \quad \text{or} \quad \sigma_{ji,j} + b_i = 0,}$$

which is known as **Cauchy's equation of equilibrium.**

In the absence of body forces equilibrium equation become,

$$\mathrm{div}\boldsymbol{\sigma} = 0, \quad \text{or} \quad \sigma_{ji,j} = 0.$$

A spatial stress field satisfying the above equation is said to be **self-equilibrated.**

For solid bodies it is more convenient to work in material coordinates. Hence, material description of Cauchy's equation of motion is,

$$\boxed{\int_{\Omega_0} \left(\text{div}\boldsymbol{P} + \boldsymbol{B} - \rho_0 \dot{\boldsymbol{V}} \right) dV = 0, \quad \text{or} \quad \int_{\Omega_0} \left(P_{ji,j} + B_i - \rho_0 \dot{V}_i \right) dV = 0.}$$

Local form of the above equation is,

$$\boxed{\text{div}\boldsymbol{P} + \boldsymbol{B} = \rho_0 \dot{\boldsymbol{V}}, \quad \text{or} \quad P_{ji,j} + B_i = \rho_0 \dot{V}_i.}$$

For motions having zero acceleration,

$$\text{div}\boldsymbol{P} + \boldsymbol{B} = \boldsymbol{O}, \quad \text{or} \quad P_{ji,j} + B_i = 0.$$

In the absence of body forces, $\text{div}\boldsymbol{P} = \boldsymbol{O}$, or $P_{ji,j} = 0$, which is known as **Piola Identity.**

Symmetry of Cauchy's stress tensor

Start with the spatial form of angular momentum balance equation,

$$\int_\Omega \boldsymbol{r}\times\rho\dot{\boldsymbol{v}}dv = \int_{\partial\Omega}\boldsymbol{r}\times\boldsymbol{t}ds + \int_\Omega \boldsymbol{r}\times\boldsymbol{b}dv, \text{ or } \int_\Omega e_{ijk}\rho r_i\dot{v}_j dv = \int_{\partial\Omega} e_{ijk}r_i t_j ds + \int_\Omega e_{ijk}r_i b_j dv,$$

With the application of Cauchy's stress theorem and then Gauss-divergence theorem,

$$\int_\Omega e_{ijk}\rho r_i\dot{v}_j dv = \int_\Omega \left[e_{ijk}(r_i\sigma_{pj})_{,p} + e_{ijk}r_i b_j\right] dv, \qquad (r_i = x_i - x_{0i})$$

$$\Rightarrow \int_\Omega e_{ijk}\rho r_i\dot{v}_j dv = \int_\Omega \left[e_{ijk}\left(r_i\sigma_{pj,p} + r_{i,p}\sigma_{pj}\right) + e_{ijk}r_i b_j\right] dv,$$

$$\Rightarrow \int_\Omega e_{ijk}\rho r_i\dot{v}_j dv = \int_\Omega \left[e_{ijk}\left(r_i\sigma_{pj,p} + \sigma_{ij}\right) + e_{ijk}r_i b_j\right] dv, \qquad (r_{i,p} = x_{i,p} = \delta_{ip})$$

$$\Rightarrow \int_\Omega e_{ijk}r_i\left(\sigma_{pj,p} + b_j - \rho\dot{v}_j\right) + e_{ijk}\sigma_{ij}dv = 0,$$

From the balance of linear momentum,

$$\sigma_{pj,p} + b_j - \rho \dot{v}_j = 0,$$

which implies

$$\int_{\Omega} e_{ijk}\sigma_{ij} dv = 0,$$

where v is an arbitrary volume, hence, $e_{ijk}\sigma_{ij} = 0$, which results in following equations,

$$\sigma_{12} - \sigma_{21} = 0,\ \sigma_{23} - \sigma_{32} = 0, \text{ and } \sigma_{13} - \sigma_{31} = 0.$$

This is only possible when, $\boxed{\boldsymbol{\sigma} = \boldsymbol{\sigma}^T, \text{ or } \sigma_{ij} = \sigma_{ji}.}$

This is an important result from the local form of balance of angular momentum, often refereed as **Cauchy's second equation of motion.**

Symmetry of Cauchy stress implied the symmetry of Kirchoff stress and Second Piola-Kirchoff stress; whereas First Piola-Kirchoff is not symmetric.

Note that symmetric property of Cauchy stress tensor does not hold if distributed resultant couples are not neglected while writing balance of angular momentum.

System of forces

- The Cauchy traction vector $\boldsymbol{t}$ and the body force $\boldsymbol{b}$ acting on $\partial\Omega$ and Ω during a motion $\boldsymbol{\chi}$ consistent with the momentum balance equations are known as **system of forces.**
- Corresponding to each system of forces $(\boldsymbol{t}, \boldsymbol{b})$ their exist only one stress field $\boldsymbol{\sigma}$ satisfying the equation of motion and Cauchy's stress theorem.
- In other words, for a given Cauchy stress tensor $\boldsymbol{\sigma}$ and a motion $\boldsymbol{\chi}$, a system of forces $(\boldsymbol{t}, \boldsymbol{b})$ can be uniquely determined. The Cauchy traction vector $\boldsymbol{t}$ thus obtained satisfy the Cauchy's stress theorem. The pair $(\boldsymbol{\sigma}, \boldsymbol{\chi})$ defines a *dynamic process* or just a *process*.

Exercise

A process is given by Cauchy stress tensor,

$$\boldsymbol{\sigma} = x_1^2 \boldsymbol{e}_1 \boldsymbol{e}_1 + \alpha x_2 x_3^2 (\boldsymbol{e}_1 \boldsymbol{e}_2 + \boldsymbol{e}_2 \boldsymbol{e}_1) + x_2^2 \boldsymbol{e}_2 \boldsymbol{e}_2 + \beta x_1^3 \boldsymbol{e}_3 \boldsymbol{e}_3,$$

where α and β are scalar constants, and a motion,

$$x_1 = e^t X_1 - e^{-t} X_2, \quad x_2 = e^t X_1 + e^{-t} X_2, x_3 = X_3 \text{ for all } t > 0.$$

Find the system of forces so that the Cauchy's first equation of motion and continuity mass equation are satisfied. The Cauchy traction vector $\boldsymbol{t}$ is assumed to act at a point $\boldsymbol{x}$ of plane tangential to a sphere given by $\Phi = x_1^2 + x_2^2 + x_3^2$.

Balance of mechanical energy

We will be considering only the balance of mechanical energy. Other forms of energy are neglected in the present context.

The **external mechanical power** or the **rate of external work** is defined as power input on a region Ω at time t done by the system of forces $(\boldsymbol{t}, \boldsymbol{b})$, i.e.,

$$\mathcal{P}_{\text{ext}} = \int_{\partial\Omega} \boldsymbol{t} \cdot \boldsymbol{v} ds + \int_{\Omega} \boldsymbol{b} \cdot \boldsymbol{v} dv, \quad \text{or} \quad \mathcal{P}_{\text{ext}} = \int_{\partial\Omega} t_i v_i ds + \int_{\Omega} b_i v_i dv.$$

Here $\boldsymbol{v}$ is the spatial velocity field. The scalar quantities $t_i v_i$ and $b_i v_i$ give the external mechanical power per unit current surface s and, current volume v, respectively.

Kinetic energy of the body is defined as,

$$\mathcal{K}(t) = \int_{\Omega} \frac{1}{2} \rho \boldsymbol{v}^2 dv = \int_{\Omega} \frac{1}{2} \rho \boldsymbol{v} \cdot \boldsymbol{v} dv, \quad \text{or} \quad \mathcal{K}(t) = \int_{\Omega} \frac{1}{2} \rho v_i v_i dv.$$

The **stress power** or the **rate of internal mechanical work** by the stress field is defined as,

$$\mathcal{P}_{\text{int}} = \int_{\partial\Omega} \boldsymbol{\sigma} : \boldsymbol{d} dv \quad \text{or} \quad \mathcal{P}_{\text{int}} = \int_{\partial\Omega} \sigma_{ij} d_{ij} dv.$$

For the **rigid body motion stress power is zero**, since the rate of deformation tensor vanishes.

Balance of mechanical energy (or theorem of power expended) states,

$$\frac{D}{Dt}\mathcal{K}(t) + \mathcal{P}_{\text{int}}(t) = \mathcal{P}_{\text{ext}}(t), \quad \text{or}$$
$$\frac{D}{Dt}\int_{\Omega} \frac{1}{2}\rho \boldsymbol{v}^2 dv + \int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{d} dv = \int_{\partial\Omega} \boldsymbol{t} \cdot \boldsymbol{v} ds + \int_{\Omega} \boldsymbol{b} \cdot \boldsymbol{v} dv.$$

the rate of change of kinetic energy of a mechanical system $+$

the rate of internal mechanical work (stress-power) done by internal stresses $=$

the rate of external mechanical work done on the system by surface tractions and body forces.

Hence, the rate of change of kinetic energy will have contributions from both internal as well as external sources. Note that in general, kinetic energy is not a conserved quantity.

If $\mathcal{P}_{\text{ext}}$ is zero, then we have a problem of **free vibration**, while if $D\mathcal{K}/Dt$ is zero, the problem is called **quasi-static**.

Proof:

We starts with the term, $\displaystyle\int_{\partial\Omega} \boldsymbol{t}\cdot\boldsymbol{v} ds$ or $\displaystyle\int_{\partial\Omega} t_i v_i ds.$

Using Cauchy's equation and applying Gauss-divergence theorem we can write,

$$\int_{\partial\Omega} t_i v_i ds = \int_{\partial\Omega} \sigma_{ij} n_j v_i ds = \int_{\Omega} \left(\sigma_{ij} v_i\right)_{,j} dv$$

$$\Rightarrow \int_{\Omega} \left(\sigma_{ij} v_i\right)_{,j} dv = \int_{\Omega} \sigma_{ij,j} v_i + \sigma_{ij} v_{i,j} dv.$$

Now, external power can be written as,

$$\mathcal{P}_{\text{ext}} = \int_{\partial\Omega} t_i v_i ds + \int_{\Omega} b_i v_i dv = \int_{\Omega} \left(\sigma_{ij,j} v_i + \sigma_{ij} v_{i,j} + b_i v_i\right) dv,$$

$$\Rightarrow \int_{\Omega} \left[\left(\sigma_{ij,j} + b_i\right) v_i + \sigma_{ij} v_{i,j}\right] dv,$$

$$\Rightarrow \int_{\Omega} \left[\rho \dot{v}_i v_i + \sigma_{ij} v_{i,j}\right] dv, \qquad \text{(From linear momentum balance)}$$

$$\mathcal{P}_{\text{ext}} = \int_\Omega [\rho\dot{v}_i v_i + \sigma_{ij} v_{i,j}]\, dv = \int_\Omega [\rho\dot{v}_i v_i + \sigma_{ij} l_{ij}]\, dv = \int_\Omega [\rho\dot{v}_i v_i + \sigma_{ij}(d_{ij} + w_{ij})]\, dv,$$

$\boldsymbol{\sigma}$ being a symmetric tensor and $\boldsymbol{w}$ being an anti-symmetric tensor, $\sigma_{ij} w_{ij} = 0$. Thus,

$$\mathcal{P}_{\text{ext}} = \int_\Omega [\rho\dot{v}_i v_i + \sigma_{ij} d_{ij}]\, dv,$$

which means that spin tensor $\boldsymbol{w}$ does not contribute to the rate of work. Above can be written in the form,

$$\mathcal{P}_{\text{ext}} = \frac{D}{Dt}\int_\Omega \frac{\rho}{2} v_i v_i dv + \int_\Omega \sigma_{ij} d_{ij} dv = \frac{D}{Dt}\underbrace{\boxed{\int_\Omega \frac{\rho}{2}\boldsymbol{v}\cdot\boldsymbol{v}_i dv}}_{\mathcal{K}(t)} + \underbrace{\boxed{\int_\Omega \boldsymbol{\sigma} : \boldsymbol{d}\, dv,}}_{\mathcal{P}_{\text{int}}}$$

which is the RHS of the theorem.