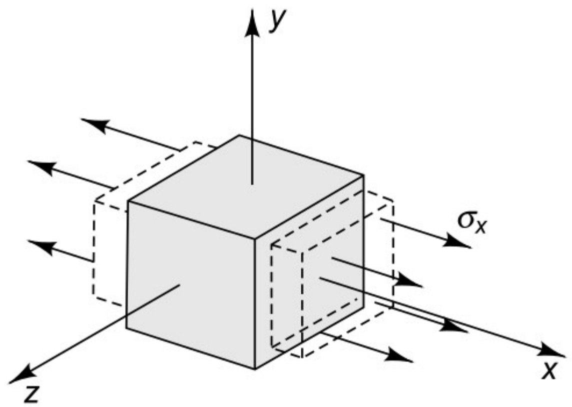
ME231: Solid Mechanics-I

Stress, Strain and Temperature relationship

First, consider an element on which only one component of normal stress is acting. This normal component of stress will produce a corresponding normal component of strain. Relation between the normal stress and normal strain produced is,



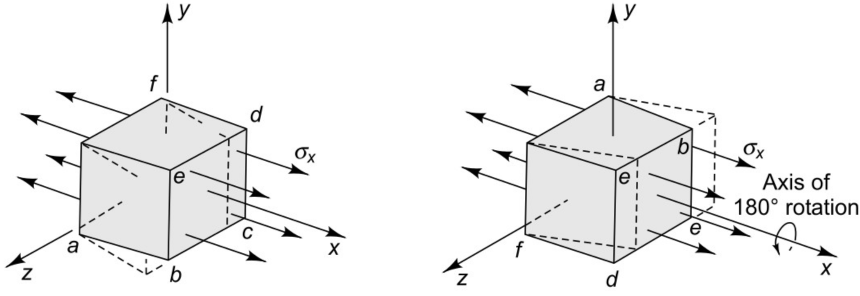
$$\epsilon_x = \frac{\sigma_x}{E} \qquad \text{............(6)}$$

From the measurement made during the uniaxial tensile test, it is observed that there are deformations in the lateral directions also. It is found that lateral strain is a fixed fraction of the normal strain. This fixed fracture is called Poisson's ratio and is denoted by the symbol ν. Thus, lateral strain can be defined as,

$$\epsilon_y = \epsilon_z = -\nu\epsilon_x = -\nu\frac{\sigma_x}{E}. \qquad \text{............(7)}$$

The possibility of occurrence of shear strain because of normal stress σ_x can be discarded because of isotropy.

Thus normal stress will produce only normal strains.



(a)

(b)
After 180° rotation

Now, if normal stress σ_y is considered then, normal strain in y-direction will be

$$\epsilon_y = \frac{\sigma_y}{E}, \qquad \cdots\cdots\cdots\cdots(8)$$

and corresponding lateral strains will be, $\epsilon_x = \epsilon_z = -\nu\epsilon_y = -\nu\frac{\sigma_y}{E}$. $\qquad \cdots\cdots\cdots\cdots(9)$

Similarly for normal stress σ_z corresponding strains are,

$$\epsilon_z = \frac{\sigma_z}{E}, \quad \text{and} \quad \epsilon_x = \epsilon_y = -\nu\epsilon_z = -\nu\frac{\sigma_z}{E}. \qquad \cdots\cdots\cdots\cdots(10)$$

Under the most general loading condition, shear stresses does not affect the normal strains directly when deformations are small. Also shear stresses in a direction does not affect shear strains in other directions. Hence, Hooke's law for shear stresses is

$$\gamma_{xy} = \frac{\tau_{xy}}{G}, \quad \gamma_{xz} = \frac{\tau_{xz}}{G}, \quad \text{and} \quad \gamma_{yz} = \frac{\tau_{yz}}{G}. \qquad \cdots\cdots\cdots\cdots(11)$$

where G is called the shear modulus.

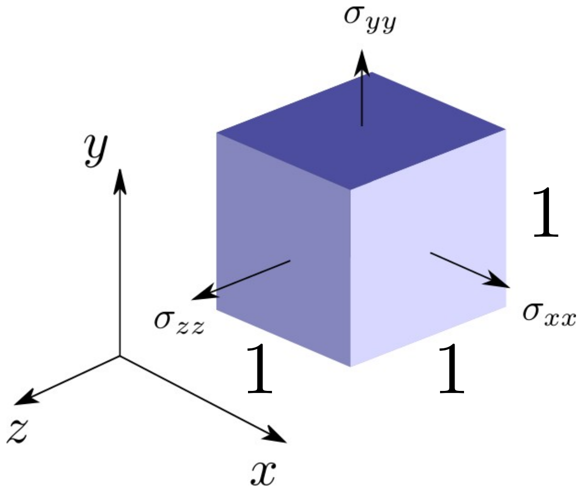
Multi-axial loading: Generalized Hooke's Law

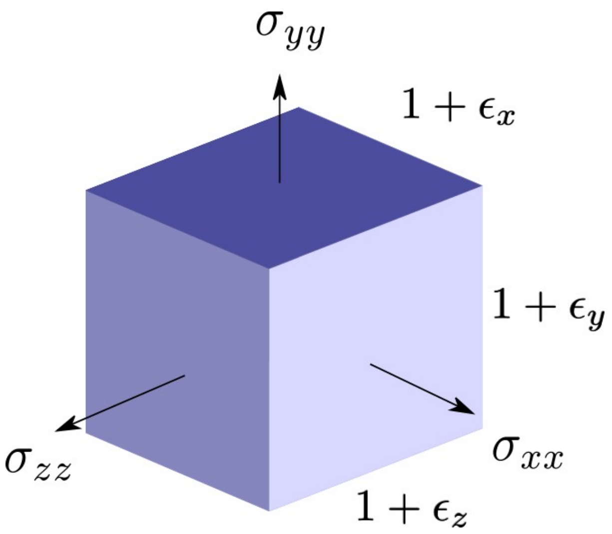
Consider a case where all stress components are $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{xz}$ and τ_{yz} acting simultaneously, then within the limits of linear elasticity and small deformations stresses and strains can be related as,

$$
\begin{aligned}
\epsilon_{xx} &= \frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{yy}}{E} - \nu\frac{\sigma_{zz}}{E}, & \gamma_{xy} &= \frac{\tau_{xy}}{G}, \\
\epsilon_{yy} &= \frac{\sigma_{yy}}{E} - \nu\frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{zz}}{E}, & \gamma_{xz} &= \frac{\tau_{xz}}{G}, \\
\epsilon_{zz} &= \frac{\sigma_{zz}}{E} - \nu\frac{\sigma_{yy}}{E} - \nu\frac{\sigma_{xx}}{E}, & \gamma_{yz} &= \frac{\tau_{yz}}{G}.
\end{aligned}
\qquad \cdots\cdots\cdots\cdots(12)
$$

These equations are known as the generalized Hooke's law. These equations involves three constants E, G and ν.

Dilatation and Bulk Modulus





Consider a cubic material element having unit volume shown in its unstressed state. Under the stresses σ_{xx}, σ_{yy} and σ_{zz} it deforms into a rectangular parallelepiped of volume v, where

$$v = (1+\epsilon_x)(1+\epsilon_y)(1+\epsilon_z).$$

As strains are smaller than unity, we can write,

$$v \approx 1 + \epsilon_x + \epsilon_y + \epsilon_z.$$

Now the change in volume is

$$e = v - 1 = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} \qquad \text{............(13)}$$

Here, e represents the change in volume per unit volume which is called dilatation of the material. Using (12) we can rewrite (13) as,

$$e = \frac{1-2\nu}{E}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right). \qquad \text{............(14)}$$

If a body is subjected to uniform hydrostatic pressure, i.e., $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$, then (14) yields

$$e = -\frac{3p(1-2\nu)}{E} = -\frac{p}{k}, \qquad \cdots\cdots\cdots\cdots(15)$$

where $k = \dfrac{E}{3(1-2\nu)}$ is a material constant, known as bulk modulus of the material.

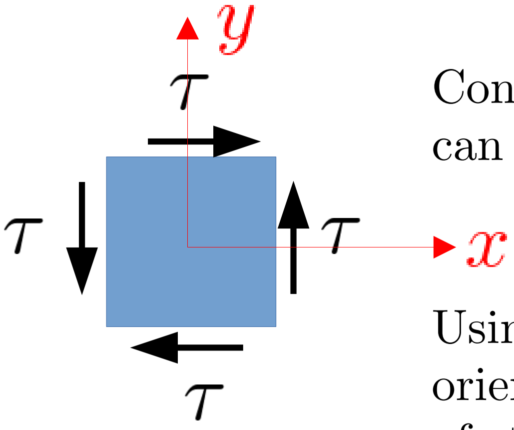
Bulk modulus is defined as the ratio of pressure to dilatation/volumetric strain (e). Note that k is always positive, as hydrostatic pressure will always decrease the volume.

Hence, $(1-2\nu)>0$ or $\nu < 0.5$. ν is also positive, hence for any engineering material

$$0 < \nu < 0.5.$$

- $\nu=0$ – Stretching is one directional without contraction in lateral direction.
- $\nu=0.5$, i.e., $k=\infty$, which means, zero dilatation or no change is volume when pressure is applied. i.e., perfectly incompressible materials.

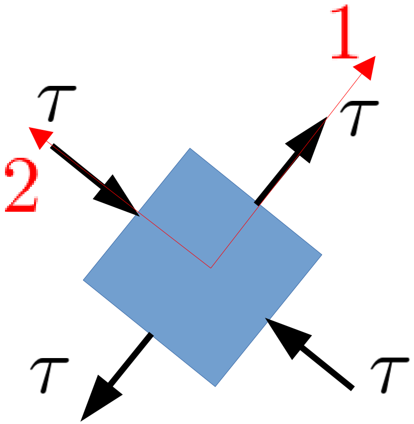
Relationship between E, ν and G



Consider a material element in pure shear loading. Using Hooke's law, we can write shear strain in the element as,

$$\gamma_{xy} = \frac{\tau}{G}. \quad \cdots\cdots\cdots\cdots(16)$$

Using stress transformation, let us determine the state of stress at angle orientation of 45°. We already did this as exercise and shown that the state of stress at 45° orientation of the element will be as follows.



For this element, applying generalized Hooke's law yields,

$$\begin{aligned} \epsilon_1 &= \frac{\tau}{E} - \nu\frac{-\tau}{E} = \frac{(1+\nu)\tau}{E} \\ \epsilon_2 &= \frac{-\tau}{E} - \nu\frac{\tau}{E} = -\frac{(1+\nu)\tau}{E} \end{aligned} \quad \cdots\cdots\cdots\cdots(17)$$

Maximum shear strain is nothing but γ_{xy}, which can be determined as

$$\gamma_{xy} = \epsilon_1 - \epsilon_2 = \frac{2(1+\nu)}{E}\tau. \quad \cdots\cdots\cdots\cdots(18)$$

Now equating (16) and (18) we can write,

$$G = \frac{E}{2(1+\nu)}. \qquad \cdots\cdots\cdots\cdots(19)$$

Thus for an isotropic elastic material there are just two independent elastic constants.