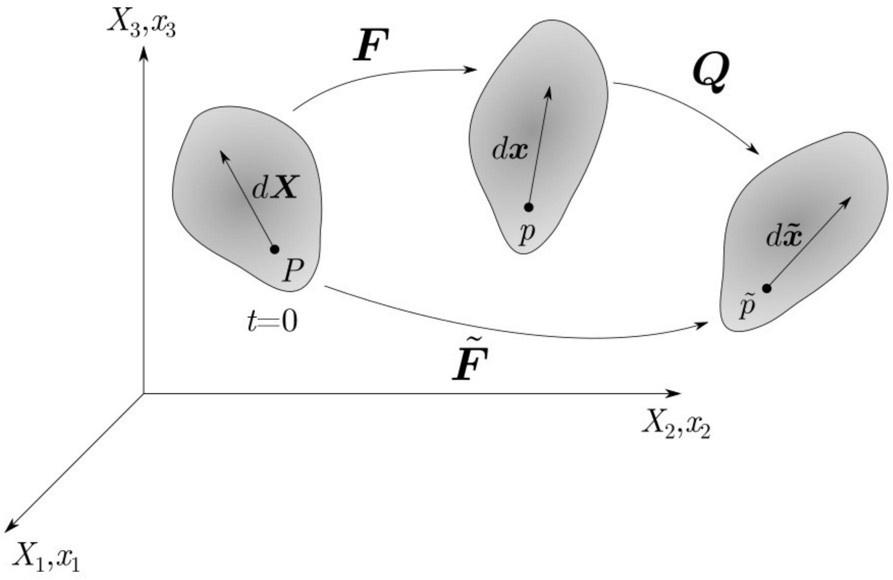
Material Objectivity

The natural/physical processes do not depend on the change of observer. Hence, The mathematical representation of physical phenomena must reflect this invariance.

That is, two observers even if in the realtive motion with respect to each other observe the same process.

An important concept in solid mechanics is the notion of objectivity. The concept of objectivity can be understood by studying the effect of a rigid body motion superimposed on the deformed configuration.

From the point of view of an observer attached to and rotating with the body many quantities describing the behavior of the solid will remain unchanged. Such quantities, for example the distance between any two particles and, among others, the state of stresses in the body, are said to be objective.

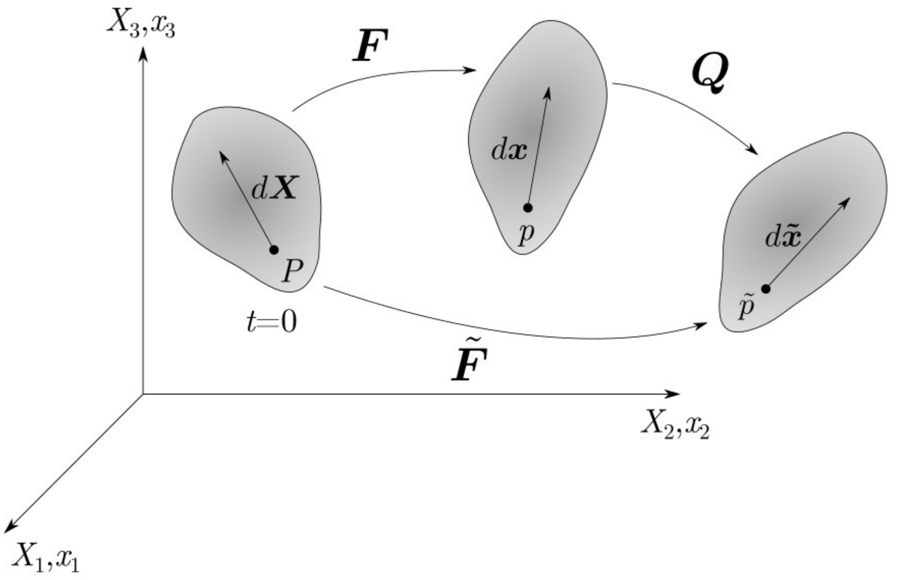


Although the intrinsic nature of these quantities remains unchanged, their spatial description may change.

To express these concepts in a mathematical framework, consider an elemental vector $d\boldsymbol{X}$ in the initial configuration that deforms to $d\boldsymbol{x}$ and is subsequently rotated to $d\tilde{\boldsymbol{x}}$.

Following relationships can be written,

$$d\tilde{\boldsymbol{x}} = \boldsymbol{Q}d\boldsymbol{x} = \boldsymbol{Q}\boldsymbol{F}d\boldsymbol{X}.$$



Although the vector $d\tilde{\boldsymbol{x}}$ is different from $d\boldsymbol{x}$, their magnitudes are equal. In this sense it can be said the $d\boldsymbol{x}$ is objective under rigid body motions.

We can extend this definition to any vector $\boldsymbol{a}$ that transforms according to $\tilde{\boldsymbol{a}} = \boldsymbol{Q}\boldsymbol{a}$.

Following relation is also valid for position vector of any points p and $\tilde{p}$,

$$\tilde{\boldsymbol{x}} = \boldsymbol{Q}\boldsymbol{x}.$$

Now, let us check the objectivity of velocity vector by differentiating the previous relation w.r.t. time,

$$\tilde{\boldsymbol{v}} = \frac{d\tilde{\boldsymbol{x}}}{dt} = \frac{d}{dt}(\boldsymbol{Q}\boldsymbol{x}) = \boldsymbol{Q}\boldsymbol{v} + \dot{\boldsymbol{Q}}\boldsymbol{x}.$$

Observe that the magnitudes of $\boldsymbol{v}$ and $\tilde{\boldsymbol{v}}$ are not equal because of the presence of the term $\dot{\boldsymbol{Q}}\boldsymbol{x}$, which violates the objectivity criteria. Hence, velocity is an example of a non-objective vector. Similar it can be shown that the acceleration is also a non-objective quantitiy.

Now, we extend the definition of objectivity to second-order tensors. Note that for the deformation gradients we can write,

$$\tilde{\boldsymbol{F}} = \boldsymbol{Q}\boldsymbol{F}.$$

Now, consider a tensor $\boldsymbol{A}$, which maps one elemental vector $d\boldsymbol{x}$ to another elemental vector d$\boldsymbol{y}$ in current configuration.

Hence, $$d\boldsymbol{y} = \boldsymbol{A}\, d\boldsymbol{x}.$$

Similarly, a tensor $\tilde{\boldsymbol{A}}$ maps one elemental vector $d\tilde{\boldsymbol{x}}$ to another elemental vector $d\tilde{\boldsymbol{y}}$ in the rotated configuration as,

$$d\tilde{\boldsymbol{y}} = \tilde{\boldsymbol{A}} d\tilde{\boldsymbol{x}}.$$

A relation between $\boldsymbol{A}$ and $\tilde{\boldsymbol{A}}$ can be eshtablished as $\tilde{\boldsymbol{A}} = \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T$, which is the transformation rule for the objective second-order tensor; i.e., all sencond-order tensors following the above transformation rule will be an objective tensor.

However, not all second-order tensors are objective. Let us examine objectivity of the velocity gradient tensor $\boldsymbol{l}$. We know that,

$$\boldsymbol{l} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}.$$

Thus,

$$\tilde{\boldsymbol{l}} = \dot{\tilde{\boldsymbol{F}}}\tilde{\boldsymbol{F}}^{-1} = \left(\overline{\boldsymbol{Q}\boldsymbol{F}}^{\boldsymbol{\cdot}}\right)(\boldsymbol{Q}\boldsymbol{F})^{-1},$$

$$\tilde{\boldsymbol{l}} = \dot{\boldsymbol{Q}}\boldsymbol{Q}^{-1} + \boldsymbol{Q}\dot{\boldsymbol{F}}\boldsymbol{F}^{-1}\boldsymbol{Q}^{-1} = \boldsymbol{\Omega} + \boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^T,$$

Hence, this tensor is affected by the superimposed rigid body motion and it is not objective.

Check the objectivity of symmetric part of $\boldsymbol{l}$, i.e., $\boldsymbol{d}$ and show that it is an objective tensor.

Note that in general, the rate of an objective second-order tensor is not objective.

$$\dot{\tilde{\boldsymbol{A}}} = \overline{\boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T}^{\boldsymbol{\cdot}} = \dot{\boldsymbol{Q}}\boldsymbol{A}\boldsymbol{Q}^T + \boldsymbol{Q}\dot{\boldsymbol{A}}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{A}\dot{\boldsymbol{Q}}^T.$$

Considering the fact that the rate of second order tensor is not objecitve, several definitions for an objective rate of an objective tensor are proposed. For example,

Jaumann-Zaremba rate – $\overset{\triangledown}{\boldsymbol{A}} = \dot{\boldsymbol{A}} - \boldsymbol{W}\boldsymbol{A} - \boldsymbol{A}\boldsymbol{W}^T$

Truesdell rate – $\overset{\circ}{\boldsymbol{A}} = \dot{\boldsymbol{A}} - \boldsymbol{L}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{L}^T - \mathrm{tr}(\boldsymbol{L})\boldsymbol{A}$

Oldroyd rates – $\overset{\diamond}{\boldsymbol{A}} = \dot{\boldsymbol{A}} - \boldsymbol{L}\boldsymbol{A} - \boldsymbol{A}\boldsymbol{L}^T$

It is left for your exercise to verify that these rate tensors are objective.

Push-forward and pull-back operation

The transformations between material and spatial quantities are typically called a push-forward operation and a pull-back operation.

Push-forward operation of a second order tensor is written as

$$\phi_*(\bullet) = \boldsymbol{F}^{-T}(\bullet)\boldsymbol{F}^{-1}.$$

Pull-back operation of a second order tensor is written as

$$\phi_*^{-1}(\bullet) = \boldsymbol{F}^{T}(\bullet)\boldsymbol{F}.$$

For example,

$\boldsymbol{E}$	Push forward $\longrightarrow$	$\boldsymbol{e}$
$\boldsymbol{S}$	$\longleftarrow$ Pull back	$\boldsymbol{\tau}$

Problem Set

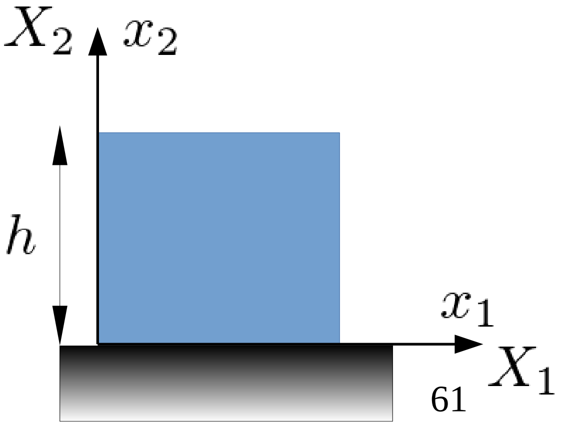
Problem 1: Show that Cauchy's first equation of motion may also be written in the following equivalent form,

$$\frac{\partial(\rho \boldsymbol{v})}{\partial t} = \operatorname{div}\left(\boldsymbol{\sigma} - \rho \boldsymbol{v} \otimes \boldsymbol{v}\right) + \boldsymbol{b}.$$

Problem 2: For a pure expansion the deformation gradient is $\boldsymbol{F} = a\boldsymbol{I}$, where a is a scalar. Show that the rate of deformation is

$$\dot{\boldsymbol{d}} = \frac{\dot{\alpha}}{\alpha}\boldsymbol{I}.$$

Problem 3: Derive a two-dimensional stress tensor for a block under the self-weight. The block has initial density ρ_0 resting on a frictionless surface as shown in the figure. For simplicity assume that there is no lateral deformation (in linear elasticity this would imply that the Poisson ratio $\nu = 0$).

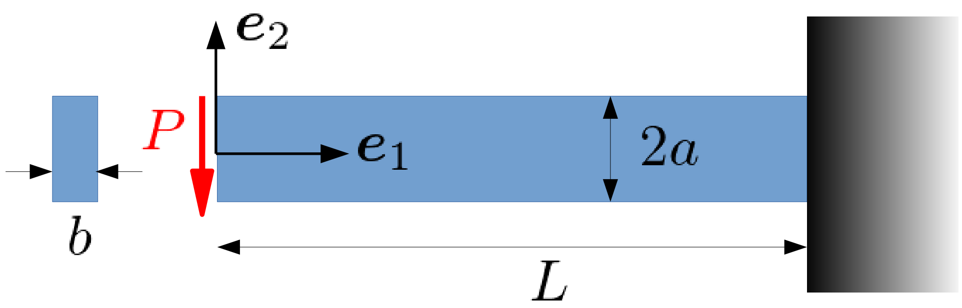


Problem 4: For a two-dimensional cantilever beam with rectangular cross-section (shown in fig.) infinitesimal strain field is given as,

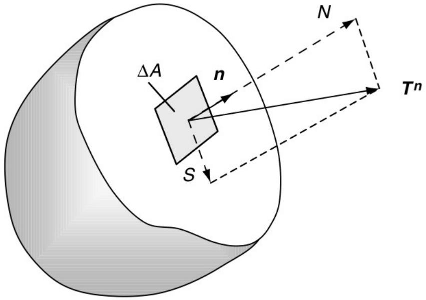
$$\varepsilon_{11} = 2Cx_1x_2, \quad \varepsilon_{22} = -2\nu Cx_1x_2,$$

$$\varepsilon_{12} = (1+\nu)C(a^2 - x_2^2), \quad C = \frac{3P}{4Ea^3b}.$$

Use the compatibility equation to determine the displacement field u_1 and u_2.



Problem 5: Show that shear stress S acting on a plane defined by the unit normal vector $\boldsymbol{n}$ can be written as,



$$S = \left[n_1^2 n_2^2(\sigma_1 - \sigma_2)^2 + n_2^2 n_3^2(\sigma_2 - \sigma_3)^2 + n_3^2 n_1^2(\sigma_3 - \sigma_1)^2\right].$$

Problem 6: Show that the principal directions of the deviatoric stress tensor coincide with the principal directions of the stress tensor. Also determine the principal values of the deviatoric stress in terms of principal values of the total stress.