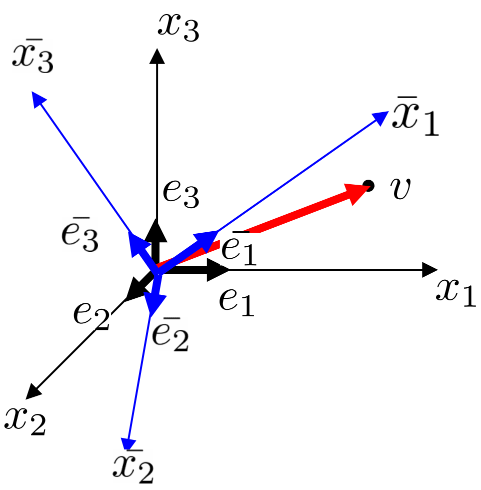
Introduction to Tensors

Transformation laws of tensors

- Through a tensor is invarient with respect to coordinate system, its components change if the coordinate system changes.
- Consider two coordinate system denoted by x_i and $\bar{x}_i$ with base vectors $\boldsymbol{e}_i$ and $\bar{e}_i$, respectively.
- As shown in the figure the components of a vector $\boldsymbol{v}$ are $v_i = \boldsymbol{e}_i \cdot \boldsymbol{v}$ in the first system, and $\bar{v}_i = \bar{e}_i \cdot \boldsymbol{v}$ in the second system.
- Similarly, the components of a tensor $\boldsymbol{A}$ is $A_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{A}\boldsymbol{e}_j$ in the first system and $\bar{A}$ is $\bar{A}_{ij} = \bar{e}_i \cdot \boldsymbol{A}\bar{e}_j$ in the second system. We will derive the transformation laws for a tensor, i.e. the relationship between the components A_{ij} and $\bar{A}_{ij}$.



Transformation laws of tensors

Let $\boldsymbol{Q}$ be an orthogonal tensor and the components Q_{ij} of tensor are defined as,

$$\boldsymbol{Q} = \cos(\boldsymbol{e}_i, \bar{\boldsymbol{e}}_j)$$

Tensor $\boldsymbol{Q}$ transforms the basis vectors $\boldsymbol{e}_i$ to $\bar{\boldsymbol{e}}_i$ i.e.,

$$\bar{\boldsymbol{e}}_i = \boldsymbol{Q}\boldsymbol{e}_i, \text{ and } \boldsymbol{e}_i = \boldsymbol{Q}^T\bar{\boldsymbol{e}}_i.$$

Consider a vector $\boldsymbol{u}$, which can be written as,

$$\boldsymbol{u} = u_i\boldsymbol{e}_i = \bar{u}_i\bar{\boldsymbol{e}}_i.$$

Using the relation between $\boldsymbol{e}_i$ and $\bar{\boldsymbol{e}}_i$, we can write,

$$\begin{aligned}
\bar{u}_i &= \boldsymbol{u}\cdot\bar{\boldsymbol{e}}_i = \boldsymbol{u}\cdot\boldsymbol{Q}\boldsymbol{e}_i \\
&\Rightarrow u_k\boldsymbol{e}_k\cdot(Q_{mn}\boldsymbol{e}_m\boldsymbol{e}_n)\,\boldsymbol{e}_i \\
&\Rightarrow u_k\boldsymbol{e}_k\cdot Q_{mn}\boldsymbol{e}_m\delta_{ni} = u_k\boldsymbol{e}_k\cdot Q_{mi}\boldsymbol{e}_m \\
&\Rightarrow Q_{mi}u_k\delta_{km} = Q_{ki}u_k = \bar{u}_i
\end{aligned}$$

This is the transformation laws for vectors, which in vector notation can be written as,

$$\{\bar{\boldsymbol{u}}\}^T = [\boldsymbol{Q}]^T\{\boldsymbol{u}\}$$

Transformation laws of tensors

The transformation laws for vector can be used to derive the transformation laws for tensors. For a second order tensor, we can write,

$$\boldsymbol{A} = A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \bar{A}_{ij}\bar{\boldsymbol{e}}_i \otimes \bar{\boldsymbol{e}}_j.$$

which follows that,

$$\begin{aligned} \bar{A}_{ij} &= \bar{\boldsymbol{e}}_i \cdot \boldsymbol{A}\bar{\boldsymbol{e}}_j = \boldsymbol{Q}\boldsymbol{e}_i \cdot \boldsymbol{A}\left(\boldsymbol{Q}\boldsymbol{e}_j\right) \\ &\Rightarrow Q_{mi}\boldsymbol{e}_m \cdot \boldsymbol{A}Q_{nj}\boldsymbol{e}_n = Q_{mi}Q_{nj}\boldsymbol{e}_m \cdot \boldsymbol{A}\boldsymbol{e}_n \\ &\Rightarrow Q_{mi}Q_{nj}A_{mn} \end{aligned}$$

Thus, $\bar{A}_{ij} = Q_{mi}Q_{nj}A_{mn}$ which is the transformation law for second order tensor. In the matrix form it can be written as

$$[\bar{\boldsymbol{A}}] = [\boldsymbol{Q}]^T[\boldsymbol{A}][\boldsymbol{Q}]$$

The transformation law can be generalized to a n^{th} order tensor $\boldsymbol{\mathcal{A}}$ as,

$$\bar{\mathcal{A}}_{i_1 i_2 i_3 \ldots i_n} = Q_{m_1 i_1}Q_{m_2 i_2}Q_{m_3 i_3}\ldots Q_{m_n i_n}\mathcal{A}_{m_1 m_2 m_3 \ldots m_n}$$

Invariants of a tensor

For a second order tensor $\boldsymbol{A}$ following can be defined,

$$I_1(\boldsymbol{A}) = \mathrm{tr}\boldsymbol{A} = A_{ii}$$
$$I_2(\boldsymbol{A}) = \frac{1}{2}\left[(\mathrm{tr}\boldsymbol{A})^2 - \mathrm{tr}\boldsymbol{A}^2\right] = \frac{1}{2}(A_{ii}A_{jj} - A_{ij}A_{ji})$$
$$I_3(\boldsymbol{A}) = \det \boldsymbol{A} = e_{ijk}A_{1i}A_{2j}A_{3k}$$

I_1, I_2, I_3 are three invarients of tensor $\boldsymbol{A}$, as they remain constant with transformation of tensor.

Here, we can introduce Cayley-Hamilton equation, which states every second order tensor $\boldsymbol{A}$ will statisfy the following equation (called characterstic equation of $\boldsymbol{A}$)

$$\boldsymbol{A}^3 - I_1\boldsymbol{A}^2 + I_2\boldsymbol{A} - I_3\boldsymbol{I} = \boldsymbol{O}.$$

Tensor functions

Tensor functions have *one* or *more tensor variables* as argument and their values are *scalars*, *vectors* or *tensors*.

For example: $\varPhi(\boldsymbol{A})$, $\boldsymbol{u}(\boldsymbol{A})$, $\boldsymbol{F}(\boldsymbol{A})$ are scaler-valued, vector-valued and tensor-valued *tensor functions* of one tensor variable $\boldsymbol{A}$, respectively.

Similarly, $\varPhi(\boldsymbol{v})$, $\boldsymbol{u}(\boldsymbol{v})$, $\boldsymbol{F}(\boldsymbol{v})$ are scaler-valued, vector-valued and tensor-valued *vector functions* of one vector variable, respectively.

In general, we will call all those functions, whose arguments are tensors, vectors or scalers as *tensor functions*.

Tensor functions

Usual rules of differentiation apply to tensor function of one scaler variable
For e.g. to find the derivative of $\boldsymbol{A}^{-1}$, where $\boldsymbol{A}$ is a function of scaler variable t, we use the identity,

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$$

$$\frac{D}{Dt}\boldsymbol{A}\boldsymbol{A}^{-1} = \frac{D}{Dt}\boldsymbol{I}$$

$$\Rightarrow \dot{\boldsymbol{A}}\boldsymbol{A}^{-1} + \boldsymbol{A}\dot{\boldsymbol{A}}^{-1} = 0$$

$$\Rightarrow \boldsymbol{A}\dot{\boldsymbol{A}}^{-1} = -\dot{\boldsymbol{A}}\boldsymbol{A}^{-1}$$

$$\Rightarrow \dot{\boldsymbol{A}}^{-1} = -\boldsymbol{A}^{-1}\dot{\boldsymbol{A}}\boldsymbol{A}^{-1}$$

Notice that usual chain rule of differentiation is applied in the above derivation.

Gradient of a scaler function of tensor varialbe can be obtained by realizing that $\phi\ (\boldsymbol{A}) = \phi\ (A_{11}, A_{12}, A_{13} ...)$, so that the total derivation of ϕ is given as,

$$
\begin{aligned}
d\phi &= \frac{\partial \phi}{\partial A_{11}} dA_{11} + \frac{\partial \phi}{\partial A_{12}} dA_{12} + \frac{\partial \phi}{\partial A_{13}} dA_{13} \dots \\
&\Rightarrow \frac{\partial \phi}{\partial A_{ij}} dA_{ij} \\
&\Rightarrow \frac{\partial \phi}{\partial \boldsymbol{A}} : d\boldsymbol{A} \\
&\Rightarrow \dot{\phi} = \frac{\partial \phi}{\partial \boldsymbol{A}} : \dot{\boldsymbol{A}}
\end{aligned}
$$

where, $\dfrac{\partial \phi}{\partial \boldsymbol{A}} = \dfrac{\partial \phi}{\partial A_{ij}} \boldsymbol{e}_i \boldsymbol{e}_j$ is a second order tensor called the gradient of ϕ.

Similarly for a tensor function of tensor variable, we can write,

$$
\begin{aligned}
dF_{ij} &= \frac{\partial F_{ij}}{\partial A_{mn}} dA_{mn} \\
&\Rightarrow \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{A}} : d\boldsymbol{A} \\
&\Rightarrow \dot{\boldsymbol{F}} = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{A}} : \dot{\boldsymbol{A}}
\end{aligned}
$$

where, the gradient of $\boldsymbol{F}$, $\dfrac{\partial \boldsymbol{F}}{\partial \boldsymbol{A}} = \dfrac{\partial F_{ij}}{\partial A_{mn}} \boldsymbol{e}_i \boldsymbol{e}_j \boldsymbol{e}_m \boldsymbol{e}_m$ is a fourth order tensor.

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Examples

Example 10: If $\boldsymbol{A}$ is a second order invertible tensor then show that,

$$\frac{\partial \boldsymbol{A}^{-1}}{\partial \boldsymbol{A}} : \boldsymbol{B} = -\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{A}^{-1}.$$

We start from the fact that,

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$$

$$\Rightarrow \frac{\partial\left(\boldsymbol{A}\boldsymbol{A}^{-1}\right)}{\partial \boldsymbol{A}} = \frac{\partial\left(A_{im}A^{-1}_{mj}\right)}{\partial A_{kl}} = 0$$

$$\Rightarrow \frac{\partial A_{im}}{\partial A_{kl}} A^{-1}_{mj} + A_{im}\frac{\partial A^{-1}_{mj}}{\partial A_{kl}} = 0$$

$$\Rightarrow A_{im}\frac{\partial A^{-1}_{mj}}{\partial A_{kl}} = -\delta_{ik}\delta_{ml}A^{-1}_{mj}$$

Multiplying bothside by A^{-1}_{ni}

$$\Rightarrow A^{-1}_{ni}A_{im}\frac{\partial A^{-1}_{mj}}{\partial A_{kl}} = -A^{-1}_{ni}\delta_{ik}\delta_{ml}A^{-1}_{mj}$$

$$\Rightarrow \delta_{nm}\frac{\partial A^{-1}_{mj}}{\partial A_{kl}} = -A^{-1}_{nk}A^{-1}_{lj}$$

$$\Rightarrow \frac{\partial A^{-1}_{nj}}{\partial A_{kl}} = -A^{-1}_{nk}A^{-1}_{lj}$$

$$\Rightarrow \frac{\partial A^{-1}_{nj}}{\partial A_{kl}}B_{kl} = -A^{-1}_{nk}B_{kl}A^{-1}_{lj}$$

$$\Rightarrow \frac{\partial \boldsymbol{A}^{-1}}{\partial \boldsymbol{A}} : \boldsymbol{B} = -\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{A}^{-1}$$

Examples

Example 11: Show that,

$$\frac{\partial \det \boldsymbol{A}}{\partial \boldsymbol{A}} = \left(\boldsymbol{A}^T\right)^2 - I_1 \boldsymbol{A}^T + I_2 \boldsymbol{I}.$$

We first find,

$$I_1 = \operatorname{tr} \boldsymbol{A} = A_{ii}$$

$$\frac{\partial I_1}{\partial A_{mn}} = \frac{\partial A_{ii}}{\partial A_{mn}} = \delta_{im}\delta_{in} = \delta_{mn} = \boldsymbol{I}$$

also,

$$I_2 = \frac{1}{2}\left[(\operatorname{tr} \boldsymbol{A})^2 - \operatorname{tr} \boldsymbol{A}^2\right] = \frac{1}{2}(A_{ii}A_{jj} - A_{ij}A_{ji})$$

$$\frac{\partial I_2}{\partial A_{mn}} = \frac{1}{2}\left(\frac{\partial A_{ii}}{\partial A_{mn}} A_{jj} + A_{ii}\frac{\partial A_{jj}}{\partial A_{mn}} - \frac{\partial A_{ij}}{\partial A_{mn}} A_{ji} - \frac{\partial A_{ji}}{\partial A_{mn}} A_{ij}\right)$$

$$\Rightarrow \frac{1}{2}\left(\delta_{im}\delta_{in}A_{jj} + A_{ii}\delta_{jm}\delta_{jn} - \delta_{im}\delta_{jn}A_{ji} - \delta_{jm}\delta_{in}A_{ij}\right)$$

$$\Rightarrow \frac{1}{2}\left(\delta_{mn}A_{jj} + A_{ii}\delta_{mn} - A_{nm} - A_{nm}\right) = A_{ii}\delta_{mn} - A_{nm} = I_1\boldsymbol{I} - \boldsymbol{A}^T$$

Examples

Now, We start with the Cayley- Hamilton equation,

$$\boldsymbol{A}^3 - I_1\boldsymbol{A}^2 + I_2\boldsymbol{A} - I_3\boldsymbol{I} = \boldsymbol{O}$$

$$A_{ip}A_{pq}A_{qj} - I_1A_{ip}A_{pj} + I_2A_{ij} - I_3\delta_{ij} = 0$$

$$\Rightarrow (A_{ip}A_{pq}A_{qj} - I_1A_{ip}A_{pj} + I_2A_{ij} - I_3\delta_{ij})\,\delta_{ij} = 0$$

$$\Rightarrow A_{ip}A_{pq}A_{qi} - I_1A_{ip}A_{pi} + I_2A_{ii} - I_3\delta_{ii} = 0$$

$$\Rightarrow 3I_3 = A_{ip}A_{pq}A_{qi} - I_1A_{ip}A_{pi} + I_2I_1$$

$$\Rightarrow 3\frac{\partial I_3}{\partial A_{mn}} = \delta_{im}\delta_{pn}A_{pq}A_{qi} + \delta_{pm}\delta_{qn}A_{ip}A_{qi} + \delta_{qm}\delta_{in}A_{ip}A_{pq} - \frac{\partial I_1}{\partial A_{mn}}A_{ip}A_{pi} - I_1\delta_{im}\delta_{pn}A_{pi} - I_1\delta_{pm}\delta_{in}A_{ip} + \frac{I_2}{\partial A_{mn}}I_1 + I_2\frac{\partial I_1}{\partial A_{mn}}$$

$$\Rightarrow 3\frac{\partial I_3}{\partial A_{mn}} = A_{nq}A_{qm} + A_{im}A_{ni} + A_{np}A_{pm} - \delta_{mn}A_{ip}A_{pi} - I_1A_{nm} - I_1A_{nm} + \frac{I_2}{\partial A_{mn}}I_1 + I_2\delta_{mn}$$

$$\Rightarrow 3\frac{\partial I_3}{\partial A_{mn}} = A_{qm}A_{nq} + A_{im}A_{ni} + A_{pm}A_{np} - \delta_{mn}A_{ip}A_{pi} - 2I_1A_{nm} + \frac{\partial I_2}{\partial A_{mn}}I_1 + I_2\delta_{mn}$$

$$\Rightarrow 3\frac{\partial I_3}{\partial A_{mn}} = 3A_{qm}A_{nq} - \delta_{mn}\mathrm{tr}\boldsymbol{A}^2 - 2I_1A_{nm} + (I_1\delta_{mn} - A_{nm})I_1 + I_2\delta_{mn}$$

$$\Rightarrow 3\frac{\partial I_3}{\partial A_{mn}} = 3A_{qm}A_{nq} + \delta_{mn}(I_1^2 - \mathrm{tr}\boldsymbol{A}^2) - 2I_1A_{nm} - A_{nm}I_1 + I_2\delta_{mn}$$

$$\Rightarrow 3\frac{\partial I_3}{\partial A_{mn}} = 3A_{qm}A_{nq} + 2I_2\delta_{mn} - 3I_1A_{nm} + I_2\delta_{mn} = 3(A_{qm}A_{nq} + I_2\delta_{mn} - I_1A_{nm})$$

$$\Rightarrow \frac{\partial I_3}{\partial \boldsymbol{A}} = \left(\boldsymbol{A}^T\right)^2 + 2I_2\boldsymbol{I} - I_1\boldsymbol{A}^T$$

Gradient, Curl and Divergence of a vector field

Different operation of $\boldsymbol{\nabla}$ operator are governed by following rules:

$$\boldsymbol{\nabla} \cdot (\bullet) = \frac{\partial(\bullet)}{\partial x_i} \cdot \boldsymbol{e}_i, \quad \boldsymbol{\nabla} \times (\bullet) = \boldsymbol{e}_i \times \frac{\partial(\bullet)}{\partial x_i}, \quad \boldsymbol{\nabla} \otimes (\bullet) = \frac{\partial(\bullet)}{\partial x_i} \otimes \boldsymbol{e}_i.$$

Following above rules, following are defined.

Divergence of a vector field $\boldsymbol{u}$,

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial x_i} \cdot \boldsymbol{e}_i = \frac{\partial u_m}{\partial x_i} \boldsymbol{e}_m \cdot \boldsymbol{e}_i = \frac{\partial u_m}{\partial x_i} \delta_{mi} = \frac{\partial u_i}{\partial x_i}.$$

Curl of a vector field $\boldsymbol{u}$,

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{e}_i \times \frac{\partial \boldsymbol{u}}{\partial x_i} = \boldsymbol{e}_i \times \frac{\partial u_m}{\partial x_i} \boldsymbol{e}_m = \frac{\partial u_m}{\partial x_i} \boldsymbol{e}_i \times \boldsymbol{e}_m.$$

Gradient, Curl and Divergence of a vector field

Gradient of a vector field $\boldsymbol{u}$,

$$\boldsymbol{\nabla} \otimes \boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial x_i} \otimes \boldsymbol{e}_i = \frac{\partial u_m}{\partial x_i} \boldsymbol{e}_m \otimes \boldsymbol{e}_i.$$

In matrix notations,

$$[\boldsymbol{\nabla} \otimes \boldsymbol{u}] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

Transposed gradient of a vector field $\boldsymbol{u}$,

$$\boldsymbol{u} \otimes \boldsymbol{\nabla} = \boldsymbol{e}_i \otimes \frac{\partial \boldsymbol{u}}{\partial x_i} = \frac{\partial u_m}{\partial x_i} \boldsymbol{e}_i \otimes \boldsymbol{e}_m.$$

Laplacian and Hessian

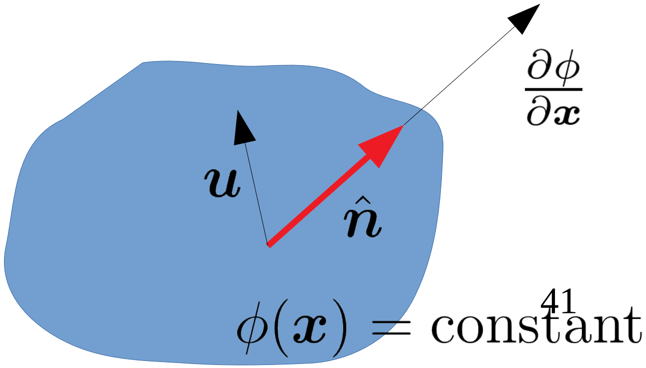
$\boldsymbol{\nabla}$ operator dotted with itself gives *Laplacian* as,

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}(\bullet) = \frac{\partial}{\partial x_i}\boldsymbol{e}_i \cdot \frac{\partial(\bullet)}{\partial x_j}\boldsymbol{e}_j = \frac{\partial}{\partial x_i}\frac{\partial(\bullet)}{\partial x_j}\delta_{ij} = \frac{\partial^2(\bullet)}{\partial x_i^2} = \nabla^2(\bullet).$$

Similarly, $\boldsymbol{\nabla}\otimes\boldsymbol{\nabla}$ gives *Hessian* as,

$$\boldsymbol{\nabla}\otimes\boldsymbol{\nabla}(\bullet) = \frac{\partial}{\partial x_i}\boldsymbol{e}_i \otimes \frac{\partial(\bullet)}{\partial x_j}\boldsymbol{e}_j = \frac{\partial}{\partial x_i}\frac{\partial(\bullet)}{\partial x_j}\boldsymbol{e}_i\otimes\boldsymbol{e}_j = \frac{\partial^2(\bullet)}{\partial x_i \partial x_j}\boldsymbol{e}_i\otimes\boldsymbol{e}_j.$$

An important concept of *directinal derivative* can be introduced here. $\boldsymbol{\nabla}\phi\cdot\boldsymbol{u}$ is the directinal derivative of ϕ with respect to $\boldsymbol{x}$ in the direction of vector $\boldsymbol{u}$.



Examples

Example 12:

If $\boldsymbol{u}(\boldsymbol{x}) = x_1x_2x_3\boldsymbol{e}_1 + x_1x_2\boldsymbol{e}_2 + x_1\boldsymbol{e}_3$, then determine $\boldsymbol{\nabla}\boldsymbol{u}, \boldsymbol{\nabla}\cdot\boldsymbol{u}$, and $\nabla^2\boldsymbol{u}$.

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial x_i}\cdot\boldsymbol{e}_i$$

$$\Rightarrow (x_2x_3\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + \boldsymbol{e}_3)\cdot\boldsymbol{e}_1 + (x_1x_3\boldsymbol{e}_1 + x_1\boldsymbol{e}_2)\cdot\boldsymbol{e}_2 + (x_1x_2\boldsymbol{e}_1)\cdot\boldsymbol{e}_3$$

$$\Rightarrow x_2x_3 + x_1$$

$$\boldsymbol{\nabla}\boldsymbol{u} = \frac{\partial u_i}{\partial x_j}\boldsymbol{e}_i\boldsymbol{e}_j$$

$$\Rightarrow x_2x_3\boldsymbol{e}_1\boldsymbol{e}_1 + x_1x_3\boldsymbol{e}_1\boldsymbol{e}_2 + x_1x_2\boldsymbol{e}_1\boldsymbol{e}_3$$

$$+ x_2\boldsymbol{e}_2\boldsymbol{e}_1 + x_1\boldsymbol{e}_2\boldsymbol{e}_2 + \boldsymbol{e}_3\boldsymbol{e}_1$$

In matrix notations,

$$[\boldsymbol{\nabla}\boldsymbol{u}] = \begin{bmatrix} x_2x_3 & x_1x_3 & x_1x_2 \\ x_2 & x_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\nabla^2 \boldsymbol{u} = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \boldsymbol{u} = \frac{\partial \boldsymbol{\nabla} \boldsymbol{u}}{\partial x_i} \cdot \boldsymbol{e}_i$$

$$\Rightarrow (x_3 \boldsymbol{e}_1 \boldsymbol{e}_2 + x_2 \boldsymbol{e}_1 \boldsymbol{e}_3 + \boldsymbol{e}_2 \boldsymbol{e}_2) \cdot \boldsymbol{e}_1 + (x_3 \boldsymbol{e}_1 \boldsymbol{e}_1 + x_1 \boldsymbol{e}_1 \boldsymbol{e}_3 + \boldsymbol{e}_2 \boldsymbol{e}_1) \cdot \boldsymbol{e}_2$$

$$+ (x_2 \boldsymbol{e}_1 \boldsymbol{e}_1 + x_1 \boldsymbol{e}_1 \boldsymbol{e}_2) \cdot \boldsymbol{e}_3$$

$$\Rightarrow 0$$

Problem Set

Problem 1: Given that $T_{ij} = 2\mu E_{ij} + \delta_{ij} E_{kk}$. Find $T_{mn} E_{mn}$.

Problem 2: Show that (i) $e_{ijk} e_{ijk} = 6$, (ii) $e_{ijp} e_{ijq} = 2\delta_{pq}$.

Problem 3: Using the properties of $\boldsymbol{\nabla}$ operator, prove that

(i) $\boldsymbol{\nabla} \cdot (\boldsymbol{A}^T \boldsymbol{u}) = \boldsymbol{\nabla} \cdot \boldsymbol{A} \cdot \boldsymbol{u} + \boldsymbol{A} : \boldsymbol{\nabla} \boldsymbol{u}$

(ii) $\boldsymbol{\nabla} (\phi \boldsymbol{u}) = \boldsymbol{u} \otimes \boldsymbol{\nabla} \phi + \phi \boldsymbol{\nabla} u$

Problem 4: Show that

$$\int_S \boldsymbol{u} \cdot \boldsymbol{A} \boldsymbol{n} \, dS = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{A}^T \boldsymbol{u} \, dV$$

Problem 5: The most general form of a fourth-order isotropic tensor can be expressed by

$$\alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk}$$

where α, β, and γ are arbitrary constants. Verify that this form remains the same under the general transformation.

Problem Set

Problem 6: Provided that $\boldsymbol{T}$ is symmetric, show that $\mathrm{tr}(\boldsymbol{\nabla} \times \boldsymbol{T}) = 0.$

Problem 7: Let a new right-handed Cartesian coordinate system be represented by the set $\{\bar{\boldsymbol{e}}_i\}$ of basis vectors with transformation law, $\bar{\boldsymbol{e}}_2 = -\sin\theta \boldsymbol{e}_1 + \cos\theta \boldsymbol{e}_2$ and $\bar{\boldsymbol{e}}_3 = \boldsymbol{e}_3$.
The origin of the new coordinate system coincides with the old origin.
(a) Find in terms of the old set $\{\bar{\boldsymbol{e}}_i\}$ of basis vectors.
(b) Find the orthogonal matrix $[\boldsymbol{Q}\ ]$ and express the new coordinates in terms of the old one.
(c) Express the vector $\boldsymbol{u} = -6\boldsymbol{e}_1 - 3\boldsymbol{e}_2 + \boldsymbol{e}_3$ in terms of the new set $\{\bar{\boldsymbol{e}}_i\}$ of basis vectors.

Problem 8: Proof that $\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{v}) = u(\boldsymbol{\nabla} \cdot \boldsymbol{v}) - v(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + (\boldsymbol{\nabla}\boldsymbol{u})\boldsymbol{v} - (\boldsymbol{\nabla}\boldsymbol{v})\boldsymbol{u}.$