ME632: Fracture Mechanics

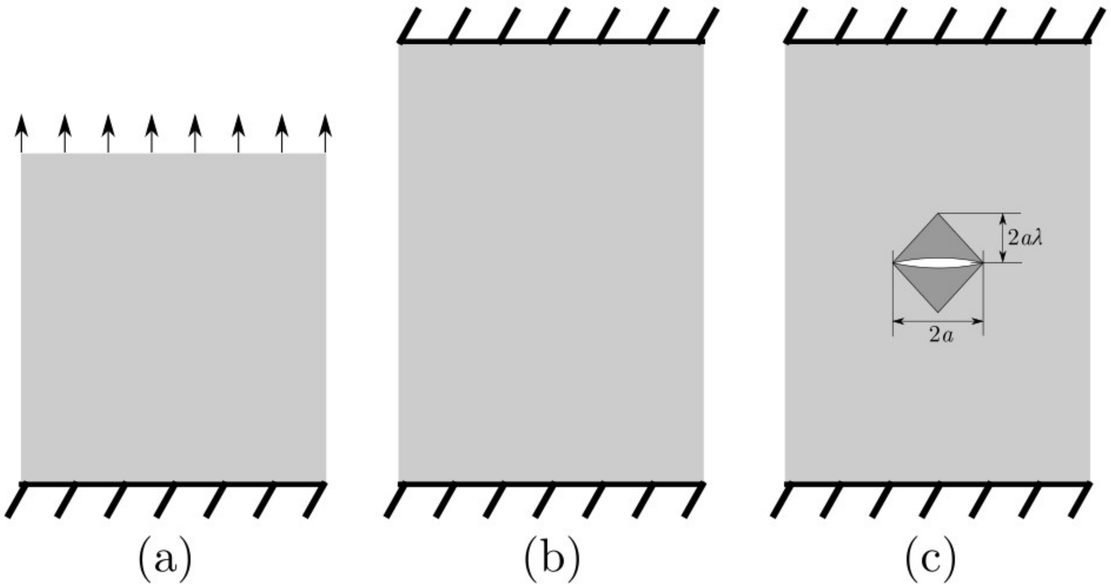
Timings	
Monday	10:00 to 11:20
Thursday	08:30 to 09:50

Anshul Faye
afaye@iitbhilai.ac.in
Room No. # 106

Griffith's theory

As we have already seen that the result obtained by Inglis suggest that the sharp crack must fail even at infinitesimal remote loading, i.e., the strength of the plate is independent of the flaw size. This paradox motivated Griffith to develop a fracture theory based on energy rather than local stress.

Think of the following experiment. Consider an elastic plate with no prior crack (Fig.a). The plate is pulled and then maintained in tension between supports (Fig.b). Then a crack normal to the direction of tension is cut with a knife at the center of the plate. The crack length is then gradually increased with the help of a knife. A critical stage is reached when the crack starts growing on its own; i.e., without any need of the knife.



Questions which we need to answer are following:

(a) What is the critical length after which crack grows on its own?

(b) How do we predict it?

To look for these answer note the following.

(i) Stiffness of the plate decreases with increasing crack length.

(ii) Stress near the crack surfaces (shaded region) decreases and hence stored strain energy also decreases. This strain energy is said to be released.

For answering the above questions and understand Griffith's analysis, we make following assumptions. The plate width is infinitely large compared to the largest crack size, which ensures that the stress far from the crack can be assumed to be constant.

As the crack advances, most of the energy release comes from region near the crack surfaces (shaded region), because they are traction free. For the sake of convenience of calculations, we consider this area as a triangle on each side of the crack plane. In fact, other shapes such as a parabola will serve the purpose. With the increase in crack length the base ($2a$) and the height, which is proportional to the base ($2a\lambda$), of triangles increase and, therefore, the area from which the strain energy is released is proportional to the square of the crack length.

So the total release of strain energy

$$E_R = \frac{\sigma^2}{2E} \times 2(\text{Area of triangle}) \times (\text{thickness}),$$

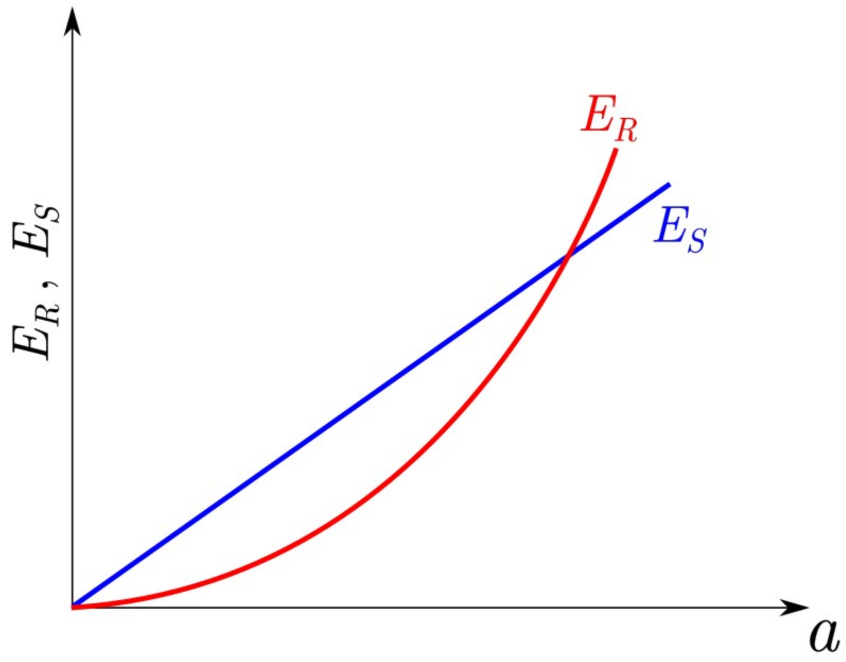
$$E_R = \frac{\sigma^2}{2E} \times 2\left[\frac{1}{2}(2a)(2a\lambda)\right] \times B = \frac{2\lambda a^2 B\sigma^2}{E}.$$

We will show it later that $\lambda = \pi/2$ for a thin plate (i.e., plane stress case), hence,

$$E_R = \frac{\pi a^2 B\sigma^2}{E}. \qquad \cdots\cdots\cdots\cdots\cdots(18)$$

Now to create new surfaces energy is required. If γ is the surfaces energy per unit area of the surface, then the surface energy required to to create a crack of length is,

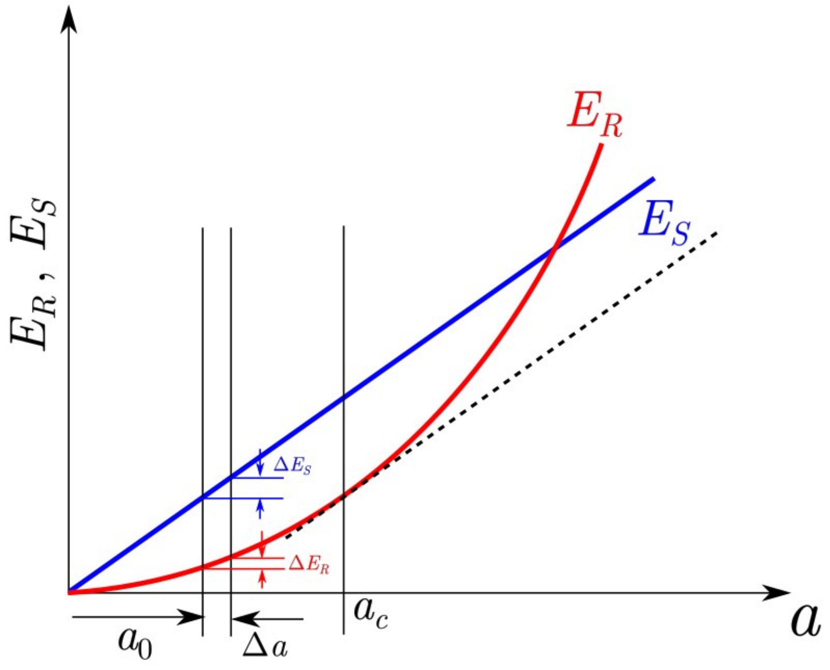
$$E_s = 2(2aB)\gamma = 4aB\gamma. \qquad \cdots\cdots\cdots\cdots\cdots(19)$$



Now consider a initial crack of length $2a_0$. If the crack length increases by an increment Δa, then the energy released during the incremental growth Δa is ΔE_R, whereas energy required for the increment is ΔE_S.

The crack will advance only when requirement of energy ΔE_S is fulfilled by the released strain energy ΔE_R.

If $\Delta E_S > \Delta E_R$ during the increment Δa, then the crack would not grow or would remain sub-critical unless additional energy is supplied through external sources (for e.g., through knife).



Now if the crack is slowly advanced by cutting through knife, and at certain crack length $\Delta E_S = \Delta E_R$ for advancing the crack by Δa length, then the crack becomes critical. Thus the condition for the crack to become critical is,

$$\frac{dE_R}{da} \geq \frac{dE_S}{da}. \qquad \text{...............(20)}$$

For the plate under tension, critical length of the crack can be determined using (18)-(20) as,

$$\frac{2a_c \pi B \sigma^2}{E} \geq 4B\gamma \quad \Rightarrow \quad a_c \geq \frac{2E\gamma}{\pi\sigma^2}. \qquad \cdots\cdots\cdots\cdots\cdots(21)$$

Crack will be safe when $a \leq \dfrac{2E\gamma}{\pi\sigma^2}$. ....(22)

Similarly, critical load for a given crack length a to be critical (in plane stress) can be determined from (21) as,

$$\sigma_c \geq \sqrt{\frac{2E\gamma}{\pi a}}. \qquad \cdots\cdots\cdots\cdots\cdots(23)$$

For the plane strain case, it can expression (21) and (22) can be obtained as,

$$a_c \geq \frac{2E\gamma}{(1-\nu^2)\pi\sigma^2}, \qquad \sigma_c \geq \sqrt{\frac{2E\gamma}{(1-\nu^2)\pi a}}. \qquad \cdots\cdots\cdots\cdots\cdots(24)$$

So from Griffith's analysis we conclude followings:

- The critical stress depends on modulus E, surface energy γ and crack length a.
- Higher the value of the surface energy of a material higher the critical stress.
- Longer crack reduces the critical stress.
- Larger modulus means that the plate is capable of storing less energy, thereby resulting into smaller energy release which, in turn requires higher stress for making the crack critical.
- Also note from the product $\sigma\sqrt{a}$ depends only on material properties (E, ν and γ). Therefore, it may be treated as a new variable in fracture mechanics.
- Results obtained from Griffith's analysis [Equation (23)] is consistent with (16) and (17).
- The change in the stored energy with crack formation [Equation (18)] is insensitive to the notch radius as long as $a >> b$; thus, the Griffith model implies that the fracture stress is insensitive to ρ. It is contradictory to the solution of Inglis.
- The actual material behavior is somewhere between these extremes; fracture stress does depend on notch root radius, but not to the extent implied by the Inglis stress analysis.

Energy release rate and Crack resistance

In last few slides, we have seen that during crack advance strain energy is released. This is measured with a parameter named Energy Release Rate, which is denoted by symbol G (after Griffith). It is defined as energy released per unit increase in area during crack growth. Here rate is w.r.t. change in crack area.

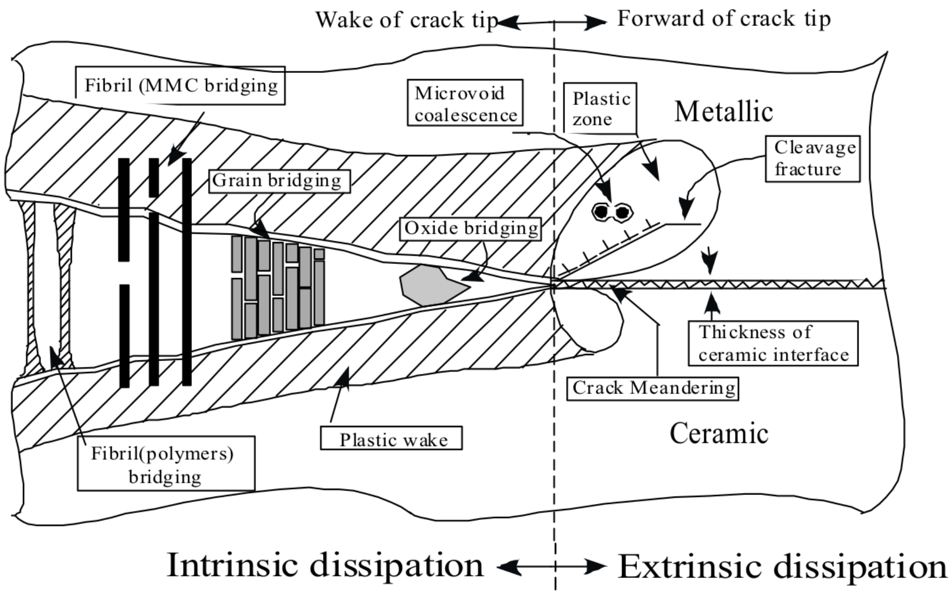
The energy release rate can be calculated even for the cracks which cannot grow under a given load condition. If there is a virtual crack growth, energy equal to G would be released from the system per unit extension of area.

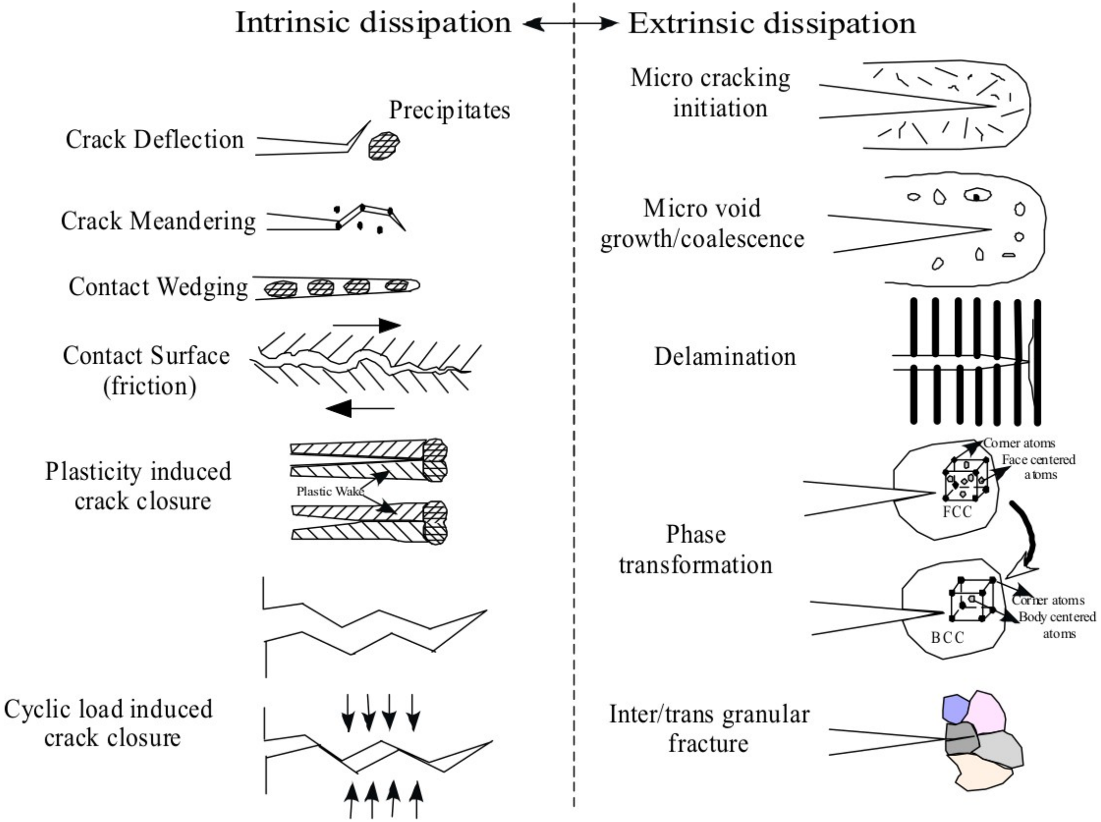
The energy requirement for a crack to grow per unit area extension is called Crack Resistance, denoted by the symbol R.

Symbol R is used in place of surface energy γ because during crack growth an anelastic deformation (e.g., plastic deformation in metals, craze growth in polymers etc.) also occurs up to a certain depth of the cracked surfaces. R is the sum of the energies required, (i) to form two new surfaces and (ii) to cause anelastic deformation. Like energy release rate, the crack resistance is also a rate.

For crack growth the condition is $G \geq R$.

Energy dissipating micro-mechanism in a crack





Source: Chandra and Shet, CMES, vol.5, no.1, pp.21-33, 2004

Energy dissipation during plastic deformation

Material	γ_s (J/m²)	γ_p (J/m²)
Mild Steel	1.20	120,000
Alloy Steel	1.20	15,000
Aluminum Alloy	0.60	4,000

It can be observed that for most of the engineering material the anelastic deformation in front of a crack-tip demands a large energy release rate. Therefore, for most of the cracks in a body, the energy release rate is not high enough to make them critical. Consequently, the cracks remain sleeping or dormant.

Mathematical Formulation

With an advancing crack following things may happen in general:

- Strain energy in the component decreases or increases.
- Stiffness of the component decreases.
- The points of the component, at which external loads are applied, may or may not move. Work is being done on the component by these forces if the points move.
- Energy is being consumed to create two new surfaces.

We apply the conservation of energy. Consider the case of an incremental increase in the crack area ΔA. During crack growth, the incremental external work ΔW_{ext} done by the external forces equals the increase in strain energy ΔU within the body of the component and the energy dissipated during crack growth $G\Delta A$, i.e.,

$$\varDelta W_{\text{ext}} = \Delta U + G\Delta A \qquad \cdots\cdots\cdots\cdots\cdots(25)$$

Dividing (25) by ΔA and talking the limit $\Delta A \to 0$, we get,

$$\frac{dW_{\text{ext}}}{dA} = \frac{dU}{dA} + G \quad \Rightarrow G = -\frac{d}{dA}(U - W_{\text{ext}}) = -\frac{d\Pi}{dA}. \qquad \cdots\cdots\cdots\cdots\cdots(26)$$

Here $\Pi = U - W_{\text{ext}}$ is known as total potential energy of the system.

Equation (26) is used to evaluate the energy release rate of any system. It says that energy is available from the system if the potential energy decreases. Note that G is always positive for a crack studied for its probable growth.

In many engineering applications, we apply fracture mechanics to plates of uniform thickness, then ΔA can be expressed as $B\Delta a$, where B is the thickness and Δa is the increment in crack length. Equation (26) can be rewritten as

$$G = -\frac{1}{B}\frac{d\Pi}{da}. \qquad \cdots\cdots\cdots\cdots\cdots(27)$$

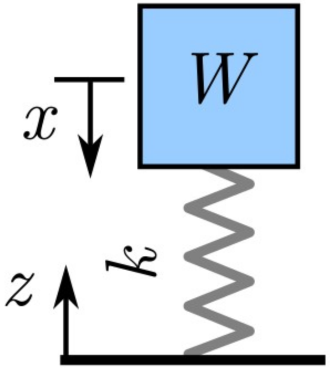
Equation (27) can be modified for dynamic crack propagation. As a crack moves rapidly, some energy is being consumed to the impart kinetic energy to cracked portions of the body and to generate stress waves. Therefore, (25) is modified to

$$\Delta W_{\text{ext}} = \Delta U + \Delta T + G\Delta A, \qquad \cdots\cdots\cdots\cdots\cdots(28)$$

where ΔT is the incremental increase in kinetic energy in the body. On taking the limit, the equation becomes

$$G = -\frac{d}{dA}(U - W_{\text{ext}}) - \frac{dT}{dA}. \qquad \cdots\cdots\cdots\cdots\cdots(29)$$

Total potential energy



Consider a mass with weight W resting on top of a linear spring with spring constant k. Undeformed length of the spring is z_0. After the mass is placed, the spring gets compressed by an amount x and the mass comes to a static equilibrium at $z = z_0 - x$.

For static equilibrium of the mass we know that

$$kx - W = 0. \quad \cdots\cdots(30)$$

Both of the forces acting on the mass are conservative. We represent the gravitational force as the potential

$$\Pi_g = Pz = P(z_0 - x), \quad \cdots\cdots(31)$$

where gravity is acting downwards and thus the gravity force acting on the mass is

$$F_g = -\frac{d\Pi}{dz} = -P. \quad \cdots\cdots(32)$$

The spring force is $F_s = kx = k(z_0 - z)$. $\cdots\cdots(33)$

The spring can also be expressed in terms of the potential as $\Pi_s = \frac{1}{2}k(z_0 - z)^2 = \frac{1}{2}kx^2$. $\cdots(34)$

Note that

$$F_s = -\frac{d\Pi_s}{dz} = k(z_0 - z). \quad \cdots\cdots(35)$$

The total potential energy of the system is given by

$$\Pi(z) = \Pi_g(z) + \Pi_s(z) = \frac{1}{2}k(z_0 - z)^2 + Pz, \qquad \cdots\cdots\cdots\cdots\cdots(36)$$

and the force balance is given by

$$F_g + F_s = -\frac{d\Pi_g}{dz} - \frac{d\Pi_s}{dz} = -\frac{d\Pi}{dz} = 0. \qquad \cdots\cdots\cdots\cdots\cdots(37)$$

Equation (37) says that for a static equilibrium, the total potential energy of a system must be stationary. It is equivalent to the condition that the sum of the forces must be zero. This is known as the principle of stationary potential energy.

It is usually more convenient to express the potentials in terms of the displacement of the system as

$$\Pi(\Delta) = \Pi_g(\Delta) + \Pi_s(\Delta) = \frac{1}{2}k\Delta^2 + P\Delta, \qquad \cdots\cdots\cdots\cdots\cdots(38)$$

In (38) the constant term Pz_0 is omitted. It is also equivalent to saying that at $z = z_0$ gravitational potential energy is zero. Requirement of stationarity with respect to Δ gives

$$\frac{d\Pi}{d\Delta} = k\Delta - P = 0. \qquad \cdots\cdots\cdots\cdots\cdots(39)$$

Following points to be noted:

- The potential energy of an elastic system will always be its stored elastic energy or strain energy. The strain energy of a linear elastic systems is given as

$$U = \int_V \frac{1}{2} \left(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \sigma_{xy}\gamma_{xy} + \sigma_{xz}\gamma_{xz} + \sigma_{yz}\gamma_{yz}\right) dV. \quad \text{...............(40)}$$

- All external loads which are constant can always be modeled as being provided by gravitational weights. For an arbitrary constant load F, the potential of the load can always be expressed as

$$\Pi_{\text{load}} = -F\Delta, \quad \text{...............(41)}$$

  where Δ is the motion at the point of application of the load in the direction in which the load is applied. Potential of the externally applied load is also equal to the negative of the work done by the load.

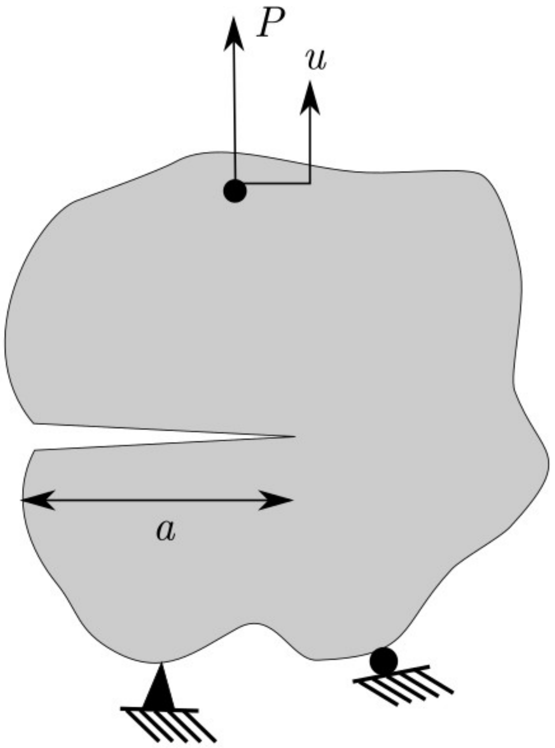
- Hence, total potential energy can also be written as $\Pi = \Pi_s + \Pi_F = U - W_{ext}$. ....(42)

- It is important that to realize that the total potential energy of a conservative force has no particular meaning. It is the change in the potential energy as the force moves from one point to another that is significant.

Determination of G

We can use (27) to determine the energy release rate of any system. There are two approaches.

The first method is based on the fact that with the change in crack length, stiffness of a body changes.



Consider a body having a crack of length a. It is fixed at some point and a load P is applied. The displacement u at the point of application of load can be given as,

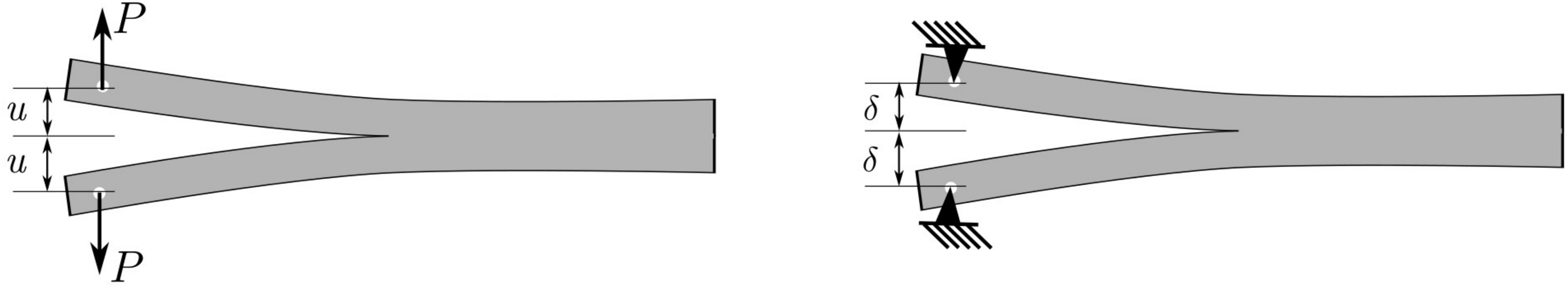
$$u = CP, \qquad \cdots\cdots\cdots\cdots\cdots(43)$$

where, C is the compliance of the body. The objective now is to find the energy release rate G in terms of change of compliance with respect to the crack length a.

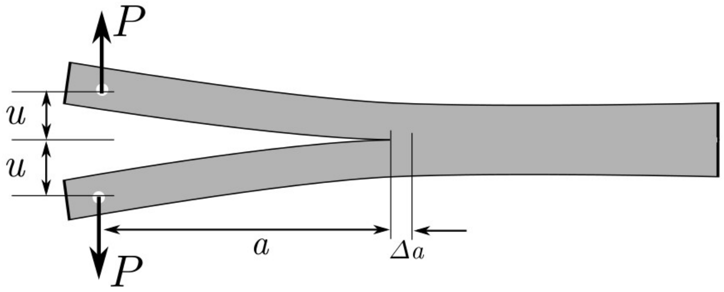
We consider a double cantilever beam (DCB), which is made by splitting a beam on one end. We will look at the change in compliance of this beam because of change in crack length. However, the derivation will be general and applicable to any specimen. We have chosen the case in which the crack is at mid-plane of the beam, with both cantilevers having identical geometry. We solve this problem for two extreme cases:

(i) constant load P, in which the displacement of load point increases as the crack grows, and

(ii) constant displacement δ, where load decreases with crack growth.



Constant Load

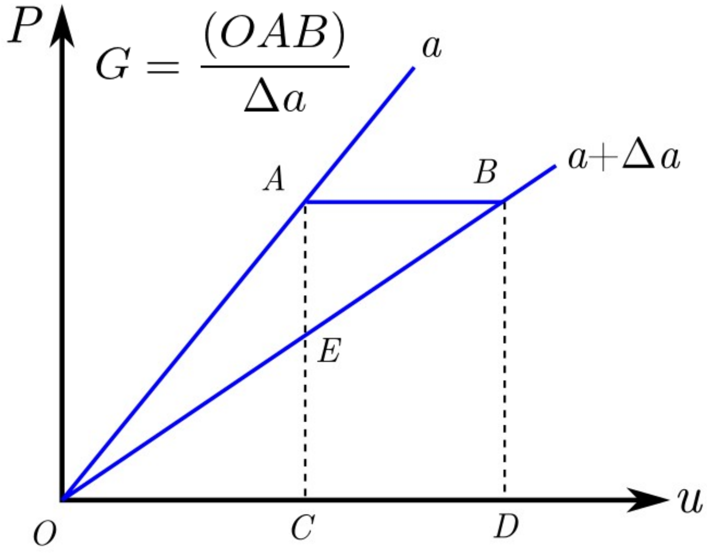


First write the total potential energy of the system, which is

$$\Pi = U - W_{ext}$$

During deflection the cantilevers are flexed and the energy absorbed by them are which is given as,

$$U = 2\left(\frac{1}{2}Pu\right) = Pu, \qquad \cdots\cdots\cdots\cdots\cdots(44)$$



and

$$W_{\text{ext}} = 2(Pu) = 2Pu. \qquad \cdots\cdots\cdots\cdots\cdots(45)$$

Thus,

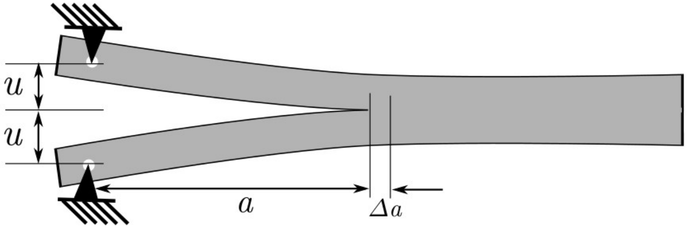
$$\Pi = Pu - 2Pu = -Pu.$$

Hence,

$$G = -\frac{1}{B}\frac{d\Pi}{da} = \frac{P}{B}\frac{du}{da} = \frac{P^2}{B}\frac{dC}{da}. \qquad \cdots\cdots\cdots\cdots\cdots(46)$$

[using (27) and (43)]

Constant Displacement



Note that, the case when applied displacement is constant then there is no external work done when crack advances by amount Δa, hence $W_{\text{ext}} = 0$.

Strain energy stored in the cantilevers remain same as that for the constant load case, i.e.,

$$U = 2\left(\frac{1}{2}Pu\right) = Pu.$$

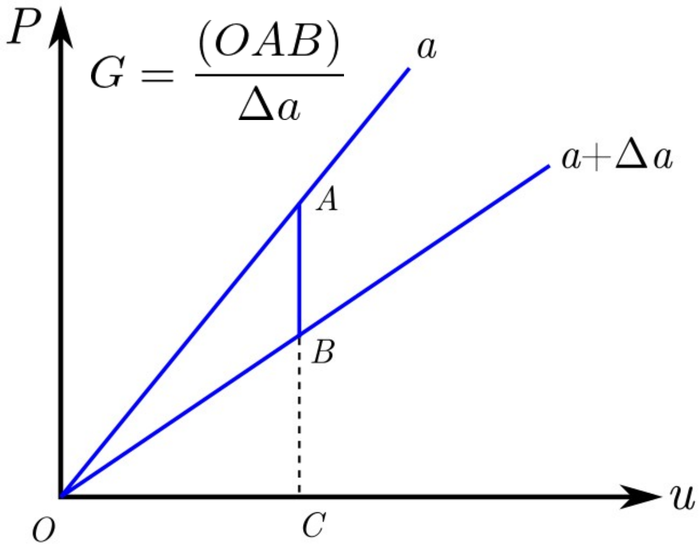
Hence, $\Pi = Pu.$

Thus, $$G = -\frac{1}{B}\frac{d\Pi}{da} = -\frac{u}{B}\frac{dP}{da} = \frac{u^2}{BC^2}\frac{dC}{da}. \quad \cdots\cdots\cdots\cdots\cdots(47)$$

[using (27) and (43)]

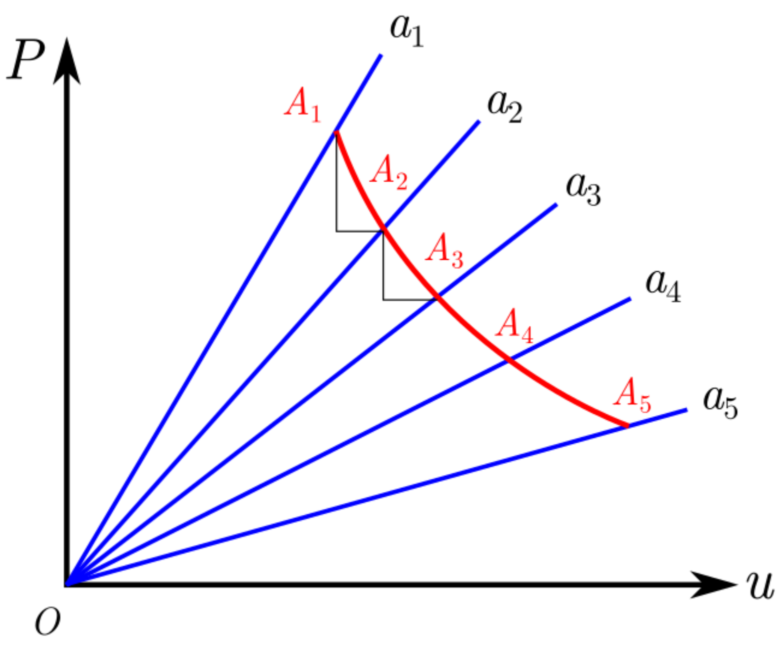
or in terms of load P, $$G = \frac{P^2}{B}\frac{dC}{da}, \quad \cdots\cdots\cdots\cdots\cdots(48)$$

which is same as that for the constant load.



- From the two analysis we get to know that the energy required to propagate a crack may come from different sources.
- In the case of a constant load, energy requirements of the crack surfaces are met by the external work done on the body. In fact, the external work done is split into two parts, first part (50%) increases the strain energy, as the cantilever deforms to higher curvature and the second part of the work is released for the crack growth.
- In the case of a fixed grip, the entire energy needed for the advancement of the crack is met by the decrease in existing strain energy.
- The work supplied for crack growth under constant load loading differs from that necessary for crack growth under fixed-grips loading by the amount (ABE) which disappears as the crack growth increment Δa tends to zero.

In practice both load and displacement change during crack growth. The load-displacement response depends mainly on the form of the specimen and the type of testing machine. In this case there is no mathematical relation between the energy release rate and the change in elastic strain energy.

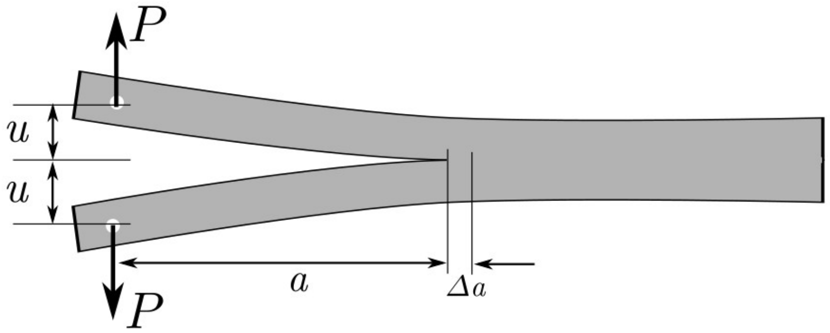


In this situation we can assume that for a short while, P is changed by ΔP at constant u and, then, u is changed by Δu at constant P. Such successive changes approximately follow the actual loading path. Since there is no constraint on the magnitude of Δu and ΔP, they can be made as small as we like; so much so that the actual load-displacement curve is approached exactly. Since the expressions for G is same for either case, (48) is valid for any general kind of loading.

Graphically,

$$G = \sum \frac{(OA_iA_j)}{a_j - a_i} \qquad \text{...............(49)}$$

Energy release rate for DCB specimen



Deflection of cantilever beams of length a under the end load P is given as

$$u = \frac{Pa^3}{3EI}. \qquad \cdots\cdots\cdots\cdots\cdots(50)$$

Hence compliance of the beam is $C = \dfrac{u}{P} = \dfrac{a^3}{3EI}$.

If height of the beam is h then, $I = Bh^3/12$. Hence, $C = \dfrac{4a^3}{EBh^3}$.

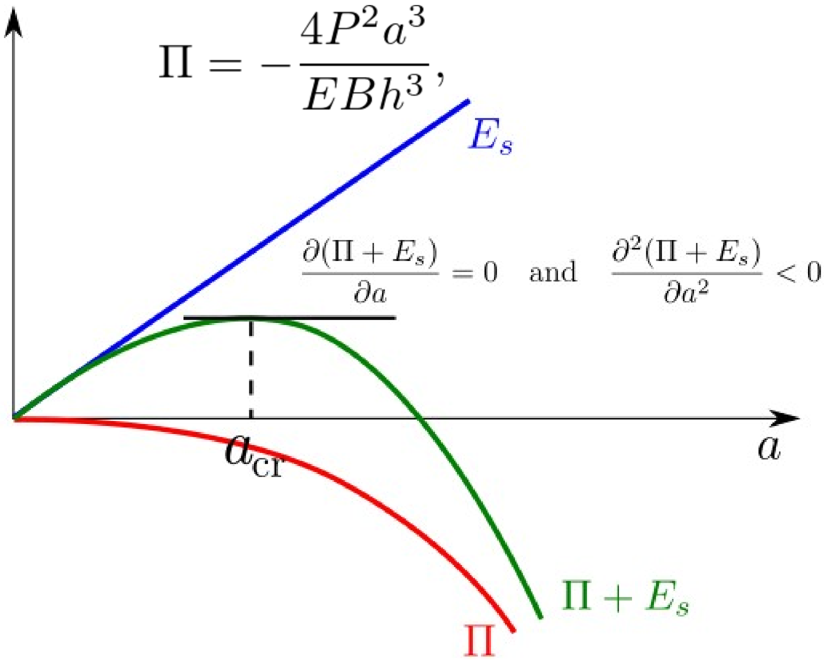
Using (48) we can write,

$$G = \frac{12P^2a^2}{EB^2h^3} \qquad \cdots\cdots\cdots\cdots\cdots(51)$$

- The depth h plays a dominant role because deflection of the cantilever depends prominently upon its depth. A cantilever of high depth has a poor capability of storing strain energy and hence a lower energy release rate.
- When the crack extends with fixed grip condition, the energy release comes from the decrease in strain energy and if the capability to store energy is small, the energy release rate is also small.
- In case of a constant load, the release of energy comes from the external work, but it is equal to increase in strain energy. Therefore, a body with low capability of storing strain energy provides small values of energy release rate.
- The thickness B also controls the deflection of the cantilever and therefore a beam of larger thickness would make the beam less flexible and provide a smaller energy release rate.
- Similarly, material property, modulus E, also governs the deflection; a stiff material like steel does not allow large deflection and, therefore, releases less energy in comparison to low modulus materials like glass fiber composites or aluminum.

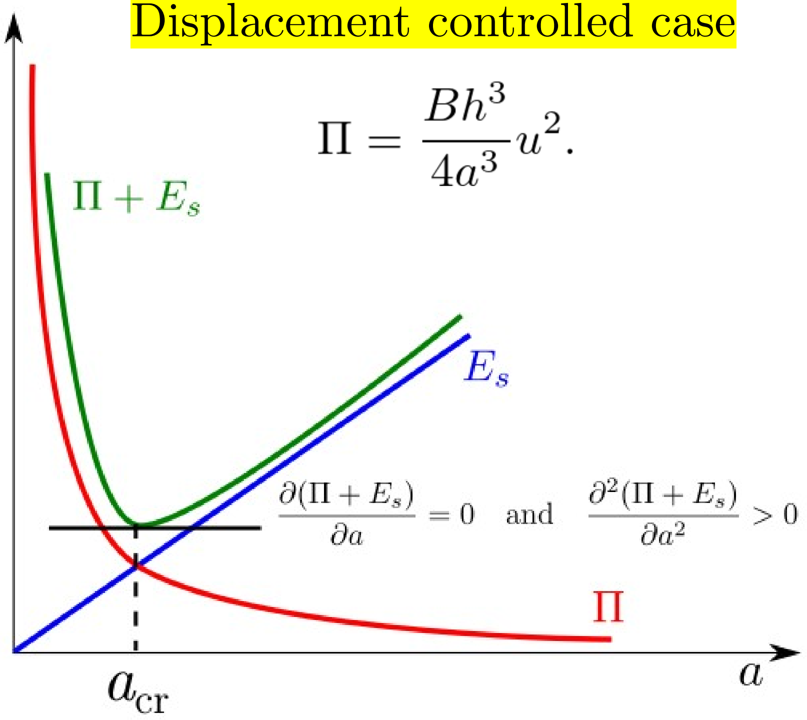
Crack stability

Load controlled case



For $a < a_{\text{cr}}$ growth is stable with increase in load; whereas for crack length $a > a_{\text{cr}}$ crack grows unstably with high velocities and lead to catastrophic failure .

Displacement controlled case

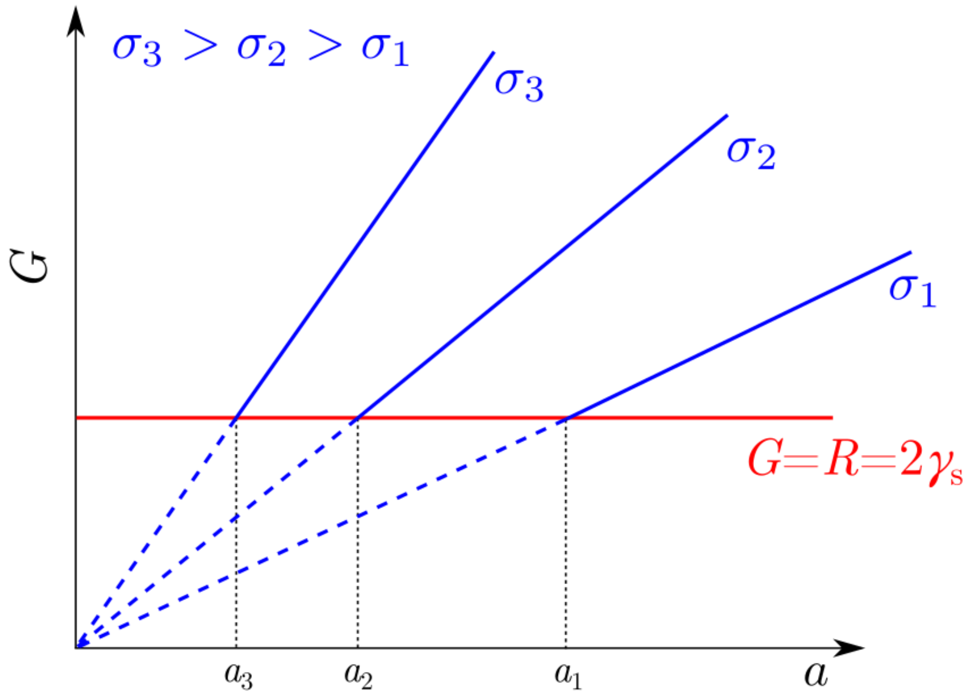


For $a < a_{\text{cr}}$ growth is unstable, whereas for crack length $a > a_{\text{cr}}$ crack growth is stable and additional displacement need to be provided for crack growth.

Crack Resistance Curve (R – Curve)

Many times it is advantageous to use energy release rate (G) or crack growth resistance (R) vs. crack length coordinates for crack growth study with the load appearing as a parameter.

For e.g., we have already seen that for an infinite elastic plate with far field loading σ energy release rate G and crack resistance R for brittle materials is,

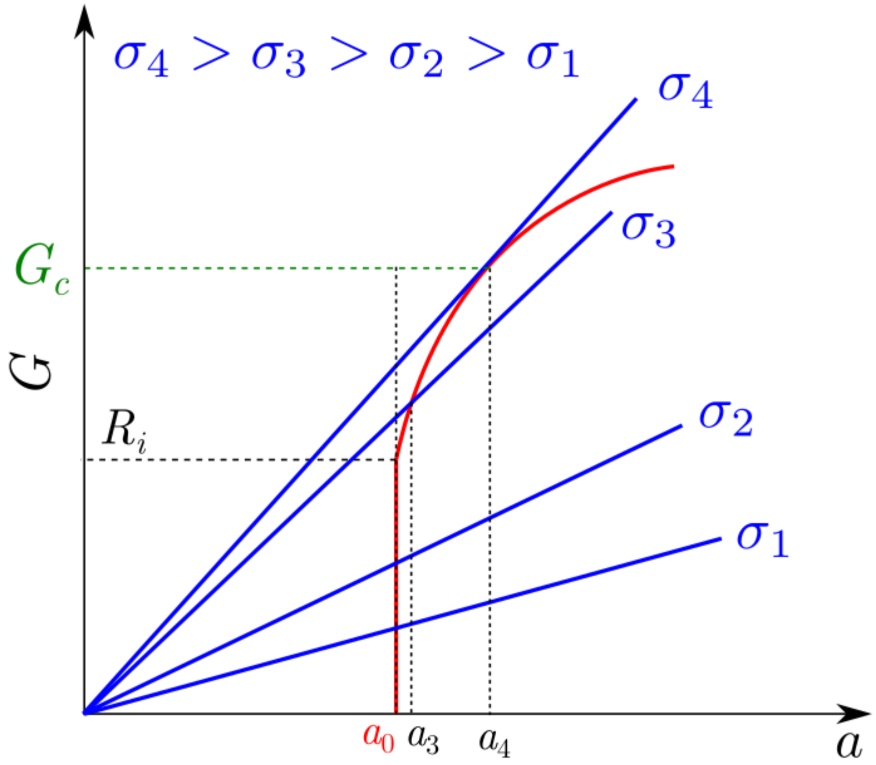


$$G = \frac{2a\pi B\sigma^2}{E} \quad \text{and} \quad R = 2\gamma_s. \qquad \cdots\cdots(52)$$

Straight lines for the three different values of the applied stress are shown. The intersection of these lines with the constant line $G = 2\gamma$ gives the critical crack length for crack growth.

Or, inversely, for a given crack length a_3 the applied stress should be increased to σ_3 for crack growth. For a larger crack length a_2 a lower stress σ_2 is required for crack growth.

For ductile materials crack resistance curve is shown. For most engineering materials the resistance increases with crack length. A minimum value R_i is needed to make the crack grow. Crack resistance depends on the size of the plastic zone. A crack having large plastic zone size, requires high energy to grow as more material is subjected to plastic deformation. In this process a significant portion of the energy is lost to the surroundings.



Again if we consider a large plate with central crack subject to far field loading, the energy release rate G is given by (52).

Now if stress σ_1 and σ_2 and the G curve intersects the R-curve below R_i, the crack will not propagate because the energy release rate is not enough.

If the stress is further increased to σ_3, G exceeds R_i for crack having length a_0 and hence the crack grwos to a length of a_3 where the difference between G and R diminishes and then there will not be any further advancement of the crack.

If the applied stress increases gradually the crack grows slowly (stable crack growth). For stress σ_4, G-curve just touches the R-curve which means G is equal to R. Corresponding energy release rate is called *critical energy release rate* G_c, which is the property of a material. For all the crack length higher than a_4 energy release rate is higher than R. In this situation even a slightly higher stress or a perturbation makes the crack grow. As soon as the crack length increases, the difference between G and R grows, which provides excess energy to the crack and the crack gains velocity, ending up in a catastrophic failure.

Thus, for a sharp crack to grow and become critical, two conditions are necessary

$$G \geq R, \qquad \cdots\cdots(53)$$

$$\frac{dG}{da} \geq \frac{dR}{da}. \qquad \cdots\cdots(54)$$

Critical energy release rate for some common materials

Material	G_c **(J/m²)**
Mild Steel	250,000
Alloy Steel (EN24)	30,000
Aluminum 7075-T6	8,000
Titanium Ti-6Al-4V	29,000
Perspex (PMMA)	800
PVC	4,500