ME232: Dynamics

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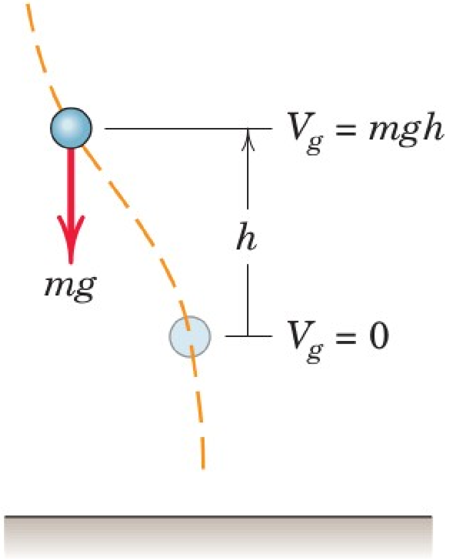
Room # 106

Gravitational potential energy



Consider the motion of a particle of mass m close to the surface of the earth, where the gravitational attraction (weight) mg is essentially constant.

The **gravitational potential energy** V_g of the particle is defined as the work mgh done against the gravitational field to elevate the particle a distance h above some arbitrary reference plane (called a datum), where V_g is taken to be zero.

Thus, we write the potential energy as $V_g = mgh$. ………(22)

This work is called potential energy because it may be converted into energy if the particle is allowed to do work on a supporting body while it returns to its lower original datum plane.

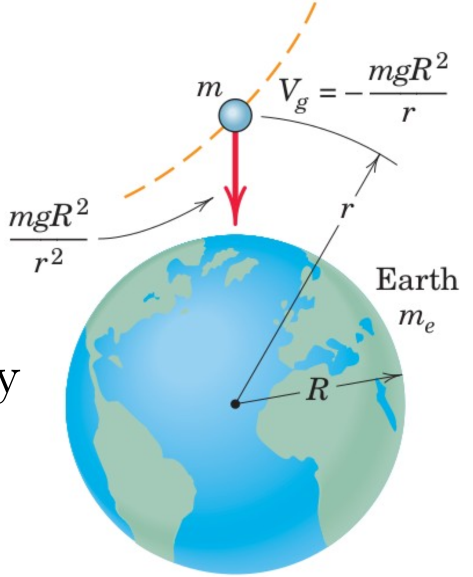
In going from one level at $h = h_1$ to a higher level at $h = h_2$, the change in potential energy becomes

$$\Delta V_g = mg(h_2 - h_1). \quad \text{………(23)}$$

The corresponding work done by the gravitational force on the particle is $-mg\Delta h$. Thus, the work done by the gravitational force is the negative of the change in potential energy.

When large changes in altitude in the field of the earth are encountered, the gravitational force $Gmm_e/r^2 = mgR^2/r^2$ is no longer constant. The work done against this force to change the radial position of the particle from r_1 to r_2 is the change $(V_g)_2 - (V_g)_1$ in gravitational potential energy, which is

$$\int_{r_1}^{r_2} mgR^2 \frac{dr}{r^2} = mgR^2\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = (V_g)_2 - (V_g)_1. \qquad \cdots\cdots\cdots(24)$$



It is customary to take $(V_g)_2 = 0$ when $r_2 = \infty$, so that with this datum we have,

$$V_g = -mgR^2/r. \qquad \cdots\cdots\cdots(25)$$

In going from r_1 to r_2, the corresponding change in potential energy is

$$\Delta V_g = mgR^2\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \qquad \cdots\cdots\cdots(26)$$

Elastic potential energy

Another example of potential energy occurs in the deformation of an elastic body, such as a spring.

The work done on the spring to deform it, is stored in the spring and is called its **elastic potential energy** V_e . This energy is recoverable in the form of work done by the spring on the body attached to its movable end during the release of the deformation of the spring.

For the one-dimensional linear spring of stiffness k, the force supported by the spring at any deformation x, tensile or compressive, from its undeformed position is $F = kx$. Thus, we define the elastic potential energy of the spring as the work done on it to deform it an amount x, and we have

$$V_e = \int_0^x kx dx = \frac{1}{2}kx^2. \qquad \cdots\cdots\cdots(27)$$

If the deformation, either tensile or compressive, of a spring increases from x_1 to x_2 during the motion, then the change in potential energy of the spring is,

$$\Delta V_e = \int_0^x kx dx = \frac{1}{2}k(x_2^2 - x_1^2). \qquad \cdots\cdots\cdots(28)$$

Work-Energy Equation

With the elastic member included in the system, we now modify the work-energy equation to account for the potential-energy terms. It stands for the work of all external forces **other than gravitational forces and spring forces**, we may write

$$U'_{1\text{-}2} + (-\Delta V_g) + (-\Delta V_e) = \Delta T \qquad \text{or} \qquad U'_{1\text{-}2} = \Delta T + \Delta V \qquad \cdots\cdots\cdots(29)$$

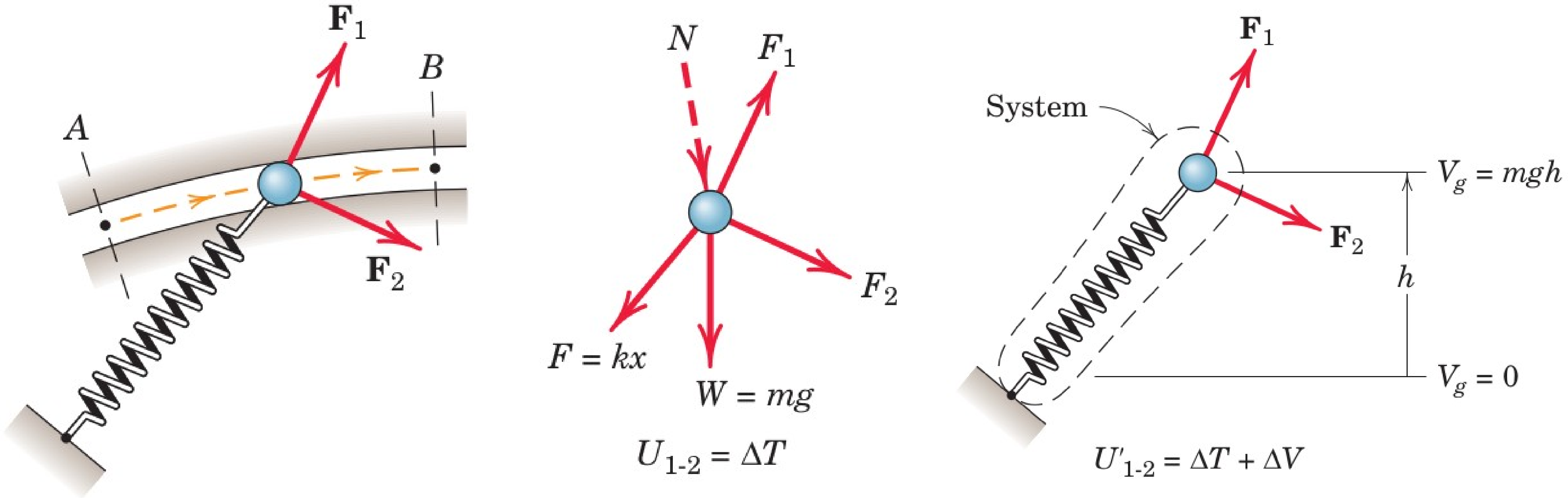
where ΔV is the change in total potential energy, i.e., gravitational plus elastic.

This alternate form of the work-energy equation is often far more convenient to use than the previous form since the work of both gravity and spring forces is accounted for only on the end-point positions of the particle and on the end-point lengths of the elastic spring. The path followed between these end-point positions is of no consequence in the evaluation of ΔV_g and ΔV_e. Equation (29) may also be written as,

$$T_1 + V_1 + U'_{1\text{-}2} = T_2 + V_2 \qquad \cdots\cdots\cdots(30)$$

The difference between two forms of work-energy equation can be understood as follows.

Consider a particle of mass m constrained to move along a fixed path under the action of forces F_1 and F_2, the gravitational force $W = mg$, the spring force F, and the normal reaction N.



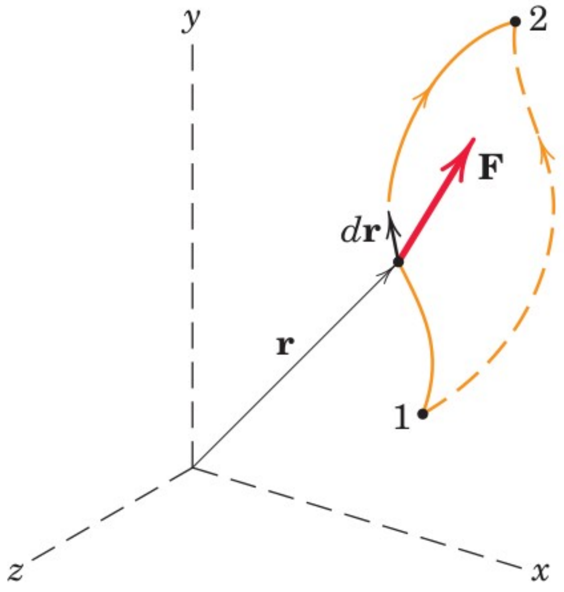
For problems where the only forces are gravitational, elastic, and nonworking constraint forces, the U' term of (29) is zero, and the energy equation becomes

$$T_1 + V_1 = T_2 + V_2 \qquad \text{or} \qquad E_1 = E_2 \, . \qquad \cdots\cdots\cdots(31)$$

where $E = T + V$ is the total mechanical energy of the particle and its attached spring. When E is constant, transfers of energy between kinetic and potential may take place as long as the total mechanical energy $T + V$ does not change. Equation (31) expresses the **law of conservation of dynamical energy**.

Conservative Force Fields

We have observed that the work done against a gravitational or an elastic force depends only on the net change of position and not on the particular path followed in reaching the new position. Forces with this characteristic are associated with conservative force fields, which possess an important mathematical property.



Consider a force field where the force $\boldsymbol{F}$ is a function of the coordinates. The work done by $\boldsymbol{F}$ during a displacement d$\boldsymbol{r}$ of its point of application is $dU = \boldsymbol{F} \cdot d\boldsymbol{r}$. The total work done along its path from 1 to 2 is

$$U = \int \boldsymbol{F} \cdot d\boldsymbol{r} = \int (F_x dx + F_y dy + F_z dz).$$

The integral $\int \boldsymbol{F} \cdot d\boldsymbol{r}$ is a line integral which depends, in general, on the particular path followed between any two points 1 and 2 in space.

Now, if there exists a scalar potential V such that, $\boldsymbol{F} = -\boldsymbol{\nabla V}$, ...(32)

where, $\boldsymbol{\nabla} = \boldsymbol{i}\dfrac{\partial}{\partial x} + \boldsymbol{j}\dfrac{\partial}{\partial y} + \boldsymbol{k}\dfrac{\partial}{\partial z}$, the vector operator *del.*

Now,

$$\boldsymbol{F}\cdot d\boldsymbol{r} = -\left(\boldsymbol{i}\frac{\partial V}{\partial x} + \boldsymbol{j}\frac{\partial V}{\partial y} + \boldsymbol{k}\frac{\partial V}{\partial z}\right)\cdot(dx\boldsymbol{i} + dy\boldsymbol{j} + dz\boldsymbol{k}) = -\left(\frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \frac{\partial V}{\partial z}\,dz\right),$$

$$\boldsymbol{F}\cdot d\boldsymbol{r} = -dV. \qquad \text{.........(33)}$$

Thus, the total work done along its path from 1 to 2 is

$$U_{1-2} = \int_1^2 -dV = -(V_2 - V_1). \qquad \text{.........(34)}$$

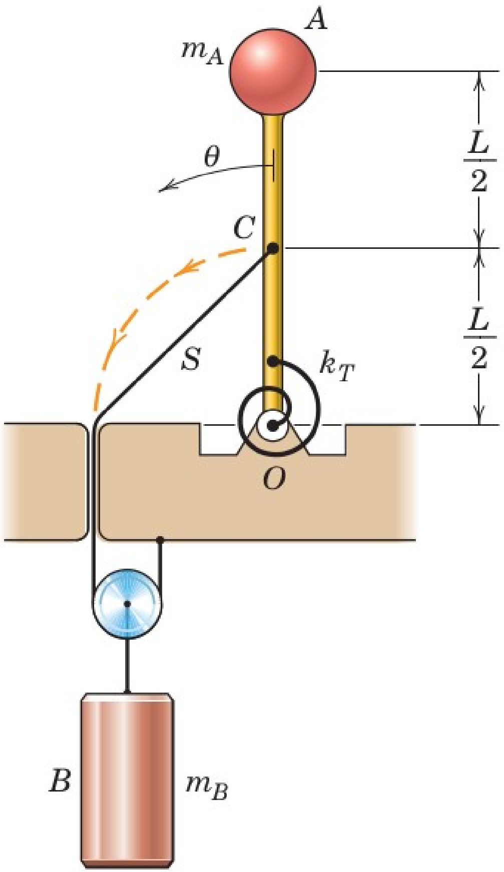
which depends only on the end points of the motion and which is thus independent of the path followed. The minus sign before dV is arbitrary but is chosen to agree with the customary designation of the sign of potential energy change in the gravity field of the earth.

The quantity V is known as the **potential function**, and the expression $\boldsymbol{\nabla V}$ is known as the gradient of the potential function.

When force components are derivable from a potential as described, the force is said to be **conservative**, and the work done by $\boldsymbol{F}$ between any two points is independent of the path followed.

Example 8

The system shown is released from rest with the lightweight slender bar OA in the vertical position shown. The torsional spring at O is undeflected in the initial position and exerts a restoring moment of magnitude $k_T\theta$ on the bar, where θ is the counterclockwise angular deflection of the bar. The string S is attached to point C of the bar and slips without friction through a vertical hole in the support surface. For the values $m_A = 2$ kg, $m_B = 4$ kg, $L = 0.5$ m, and $k_T = 13$ N·m/rad. Determine the speed v_A of particle A when θ reaches 90°.



The potential energy associated with the deflection of a torsional spring can be obtained by calculating the work done on the spring to deform it, thus

$$V_e = \int_0^\theta k_T \theta d\theta = \frac{1}{2} k_T \theta^2.$$

It should be noted that $v_C = v_A/2$, and further noting that the speed of cylinder $v_B = v_C/2$ at $\theta = 90°$, thus we conclude that at $\theta = 90°$, $v_B = v_A/4$.

Establishing datums at the initial altitudes of bodies A and B, and with state 1 at $\theta = 0$ and state 2 at $\theta = 90°$, we write

$$T_1 + V_1 + U'_{1\text{-}2} = T_2 + V_2,$$

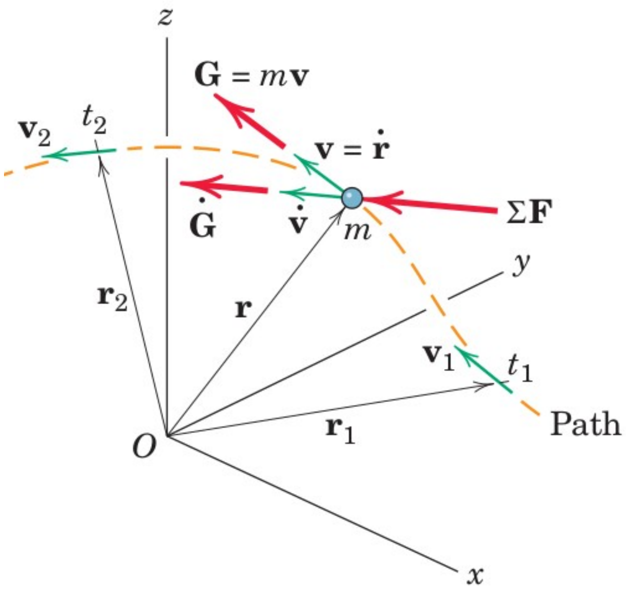
$$0 + 0 + 0 = \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2 - m_A g L - m_B g \frac{L\sqrt{2}}{4} + \frac{1}{2} k_T (\pi/2)^2,$$

Upon substituting values and solving, we get v_A at $\theta = 90°$.

Impulse and Momentum

As discussed earlier that integrating the equation of motion with respect to time leads to the equations of impulse and momentum.

These equations facilitate the solution of many problems in which the **applied forces act during extremely short periods of time** (as in impact problems) or over specified intervals of time.



Consider the general curvilinear motion in space of a particle of mass m, with a position vector $\boldsymbol{r}$. The velocity of the particle is $\boldsymbol{v} = \dot{\boldsymbol{r}}$. The resultant $\Sigma \boldsymbol{F}$ of all forces on m is in the direction of its acceleration $\dot{\boldsymbol{v}}$. We may now write the basic equation of motion for the particle as

$$\sum \boldsymbol{F} = m\dot{\boldsymbol{v}} = \frac{d}{dt}(m\boldsymbol{v}) \quad \text{or} \quad \sum \boldsymbol{F} = \dot{\boldsymbol{G}}. \qquad \cdots\cdots\cdots(35)$$

where the product of the mass and velocity is defined as the **linear momentum** $\boldsymbol{G} = m\boldsymbol{v}$ of the particle.

Equation (35) states that the resultant of all forces acting on a particle equals its time rate of change of linear momentum.

In SI the units of linear momentum mv are seen to be kg·m/s, which also equals N·s.

Equation (35) is a vector equation. Note that the direction of the resultant force coincides with the direction of the rate of change in linear momentum, which is the direction of the rate of change in velocity.

Note that (35) is valid as long as the mass m of the particle is not changing with time.

The three scalar components of (35) can be written as

$$\sum F_x = \dot{G}_x, \quad \sum F_y = \dot{G}_y, \quad \sum F_z = \dot{G}_z. \qquad \cdots\cdots\cdots(36)$$

The Linear Impulse-Momentum Principle

We can describe the effect of the resultant force $\Sigma \boldsymbol{F}$ on the linear momentum of the particle over a finite period of time simply by integrating (35) with respect to the time t. Multiplying the equation by dt gives $\Sigma \boldsymbol{F}\, dt = d\boldsymbol{G}$, and integrating from time t_1 to time t_2 , we get,

$$\int_{t_1}^{t_2} \sum \boldsymbol{F} dt = \boldsymbol{G}_2 - \boldsymbol{G}_1 = \Delta \boldsymbol{G}. \qquad \cdots\cdots\cdots(37)$$

The product of force and time is defined as the **linear impulse of the force**, and (37) states that the total linear impulse on m equals the corresponding change in linear momentum of m.

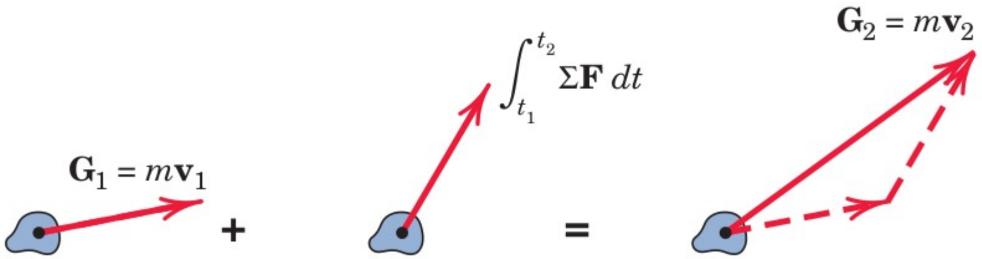
Alternatively, we may write (37) as $\boldsymbol{G}_1 + \int_{t_1}^{t_2} \sum \boldsymbol{F} dt = \boldsymbol{G}_2.$ $\cdots\cdots\cdots$(37a)

which says that the initial linear momentum of the body plus the linear impulse applied to it equals its final linear momentum.

The components of (37) are the scalar equations

$$mv_{1x} + \int_{t_1}^{t_2} \sum F_x dt = mv_{2x}, \quad mv_{1y} + \int_{t_1}^{t_2} \sum F_y dt = mv_{2y}, \qquad mv_{1z} + \int_{t_1}^{t_2} \sum F_z dt = mv_{2z}$$

We introduce the concept of the **impulse-momentum diagram**. Once the body to be analyzed has been clearly identified and isolated, we construct three drawings of the body as shown. In the first drawing the initial momentum $m\boldsymbol{v}_1$ is shown, in the second one, we show all the external linear impulses. In the final drawing, the final linear momentum $m\boldsymbol{v}_2$ is shown. The writing of the impulse- momentum equations then follows directly from these drawings, with a clear one-to-one correspondence between diagrams and equation terms.



Note that the center diagram is very much like a free-body diagram, except that the impulses of the forces appear rather than the forces themselves.

- In some cases, certain forces are very large and of short duration. Such forces are called **impulsive forces**. An example is a force of sharp impact.
- It is frequently assumed that **impulsive forces are constant over their time** of duration, so that they can be brought outside the linear-impulse integral.
- In addition, we frequently assume that **nonimpulsive forces can be neglected** in comparison with impulsive forces. An example of a nonimpulsive force is the weight of a ball during its collision with a bat-the weight of the ball is small compared with the force (which could be about several hundred times more in magnitude) exerted on the ball by the bat.
- There are cases where a force acting on a particle varies with the time in a manner determined by experimental measurements or by other approximate means. In this case a graphical or numerical integration must be performed.