# ME632: Fracture Mechanics

| Timings | |
|---|---|
| Monday | 10:00 to 11:20 |
| Thursday | 08:30 to 09:50 |

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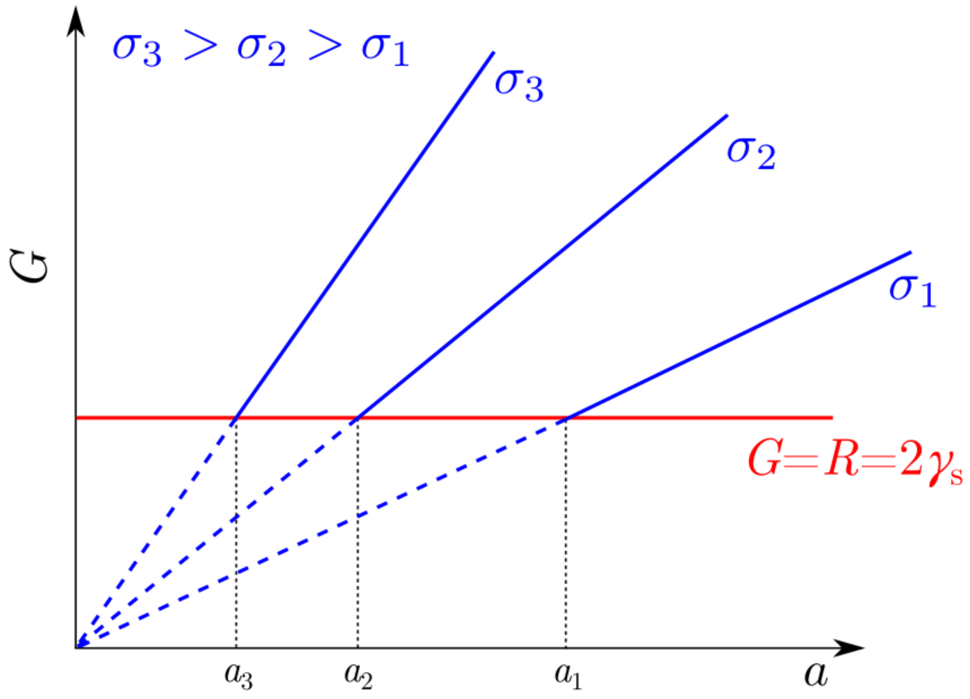
# Crack Resistance Curve (R – Curve)

Many times it is advantageous to use energy release rate ($G$) or crack growth resistance ($R$) vs. crack length coordinates for crack growth study with the load appearing as a parameter.

For e.g., we have already seen that for an infinite elastic plate with far field loading $\sigma$ energy release rate $G$ and crack resistance $R$ for brittle materials is,

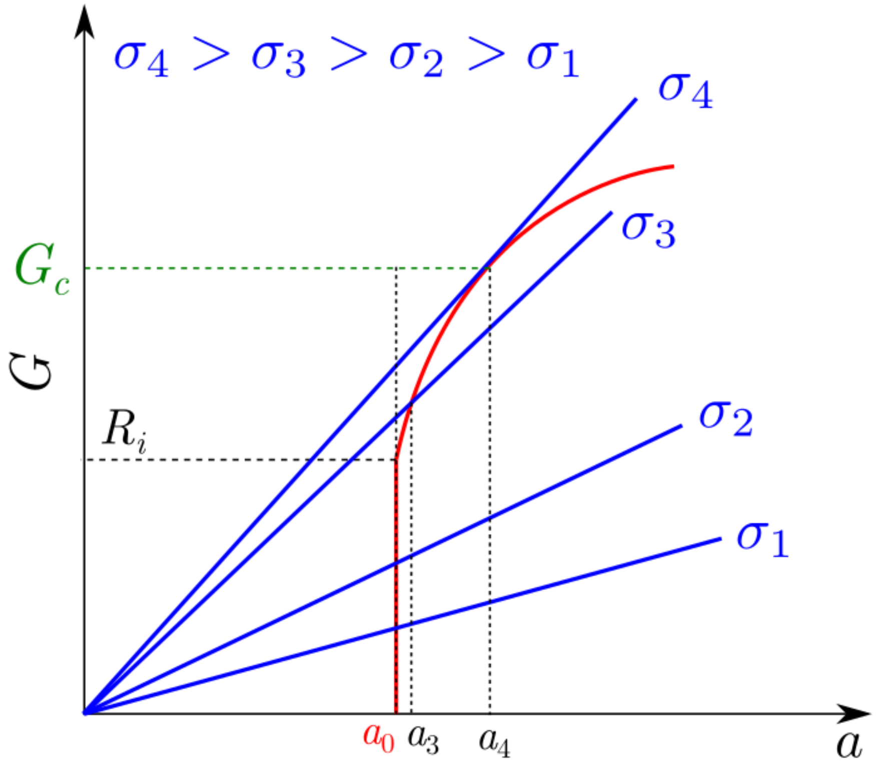
$$G = \frac{2a\pi B\sigma^2}{E} \quad \text{and} \quad R = 2\gamma_s. \qquad \cdots\cdots(52)$$




Straight lines for the three different values of the applied stress are shown. The intersection of these lines with the constant line $G = 2\gamma$ gives the critical crack length for crack growth.

Or, inversely, for a given crack length $a_3$ the applied stress should be increased to $\sigma_3$ for crack growth. For a larger crack length $a_2$ a lower stress $\sigma_2$ is required for crack growth.

For ductile materials crack resistance curve is shown. For most engineering materials the resistance increases with crack length. A minimum value $R_i$ is needed to make the crack grow. Crack resistance depends on the size of the plastic zone. A crack having large plastic zone size, requires high energy to grow as more material is subjected to plastic deformation. In this process a significant portion of the energy is lost to the surroundings.




Again if we consider a large plate with central crack subject to far field loading, the energy release rate $G$ is given by (52).

Now if stress $\sigma_1$ and $\sigma_2$ and the $G$ curve intersects the R-curve below $R_i$, the crack will not propagate because the energy release rate is not enough.

If the stress is further increased to $\sigma_3$, $G$ exceeds $R_i$ for crack having length $a_0$ and hence the crack grwos to a length of $a_3$ where the difference between $G$ and $R$ diminishes and then there will not be any further advancement of the crack.

If the applied stress increases gradually the crack grows slowly (stable crack growth). For stress $\sigma_4$, $G$-curve just touches the $R$-curve which means $G$ is equal to $R$. Corresponding energy release rate is called *critical energy release rate* $G_c$, which is the property of a material. For all the crack length higher than $a_4$ energy release rate is higher than $R$. In this situation even a slightly higher stress or a perturbation makes the crack grow. As soon as the crack length increases, the difference between $G$ and $R$ grows, which provides excess energy to the crack and the crack gains velocity, ending up in a catastrophic failure.

Thus, for a sharp crack to grow and become critical, two conditions are necessary

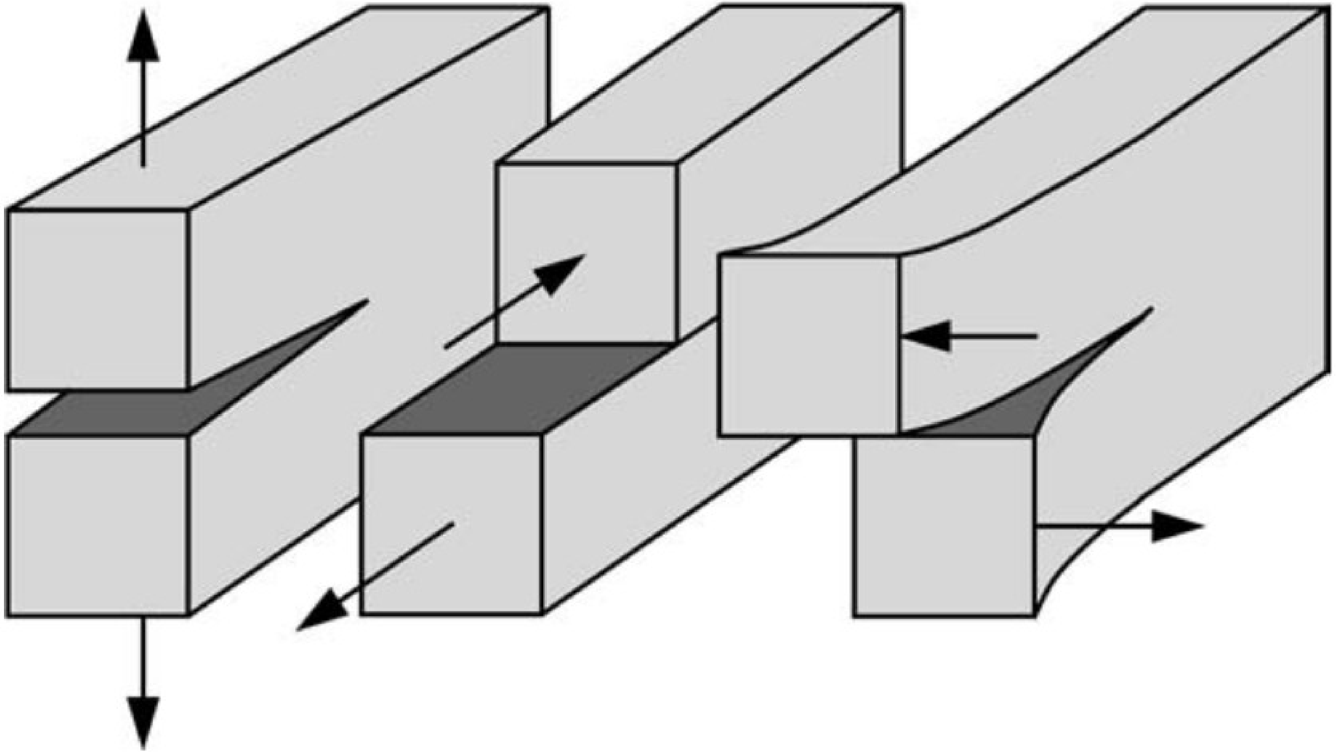
$$G \geq R, \qquad \cdots\cdots(53)$$

$$\frac{dG}{da} \geq \frac{dR}{da}. \qquad \cdots\cdots(54)$$

**Critical energy release rate for some common materials**

| **Material** | $G_c$ **(J/m²)** |
|---|---|
| Mild Steel | 250,000 |
| Alloy Steel (EN24) | 30,000 |
| Aluminum 7075-T6 | 8,000 |
| Titanium Ti-6Al-4V | 29,000 |
| Perspex (PMMA) | 800 |
| PVC | 4,500 |

# Modes of fracture



**Mode I**
**(Opening Mode)**

**Mode II**
**(Sliding Mode)**

**Mode III**
**(Tearing Mode)**

# Stresses near the crack-tip

- In previous classes we studied effects of far-field loading on a crack. We considered energy or change in energy of overall system due to applied load and we could correlated them with crack growth and stability of crack growth.
- Now, we will concentrate on the stresses near the crack-tip. In the vicinity of a crack-tip stresses are very high. Knowledge of the stress or displacement field near a crack-tip may be very useful.
- This knowledge may help material scientists to develop new materials which can diffuse high stresses at the crack-tip.
- It may help designers to modify features such as notches, cutouts, keyways, etc., to minimize stresses.
- Experimentalists can devise methods of characterizing cracks by measuring stresses or strains near the crack tip.
- One of the biggest advantages is that stress analysis leads to define a parameter, stress intensity factor (SIF) to characterize a crack. In comparison to energy release rate, SIF is simpler for a designer and easier for laboratory measurements, so as to determine material properties.

# Field equations of elasticity

**Equilibrium equations:**

For plane problems (plane stress or plane strain) in the absence of body forces equilibrium equations are,

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0, \qquad \cdots\cdots\cdots\cdots(1)$$
$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0.$$

**Strain-displacement and Compatibility relations:**

For plane problems there are three strain-displacement relations and only one compatibility equation.

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1},$$
$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2},$$
$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \qquad \cdots\cdots\cdots\cdots(2)$$

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0. \qquad \cdots\cdots\cdots\cdots(3)$$

.

**Stress-strain relations:**

For linear isotropic materials stress-strain relations are,

$$
\begin{aligned}
\varepsilon_{11} &= \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22} + \sigma_{33}\right)\right], \\
\varepsilon_{22} &= \frac{1}{E}\left[\sigma_{22} - \nu\left(\sigma_{11} + \sigma_{33}\right)\right], \\
\varepsilon_{33} &= \frac{1}{E}\left[\sigma_{33} - \nu\left(\sigma_{11} + \sigma_{22}\right)\right], \\
\varepsilon_{12} &= \frac{\sigma_{12}}{2\mu} = \frac{1+\nu}{E}\sigma_{12}.
\end{aligned}
\qquad \cdots\cdots\cdots\cdots(4)
$$

where $E$ is the Young's Modulus, $\mu$ is the shear modulus and $\nu$ is the Poisson's Ratio.

**Plane problems:**

For linear isotropic materials stress-strain relations are,

Plane stress

$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$$

$$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu\sigma_{22}\right],$$
$$\varepsilon_{22} = \frac{1}{E}\left[\sigma_{22} - \nu\sigma_{11}\right],$$
$$\varepsilon_{12} = \frac{\sigma_{12}}{2\mu} = \frac{1+\nu}{E}\sigma_{12},$$
$$\varepsilon_{33} = -\frac{\nu}{E}\left(\sigma_{11} + \sigma_{22}\right).$$

............(5)

Plane strain

$$\varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0$$

$$\varepsilon_{11} = \frac{1-\nu^2}{E}\left[\sigma_{11} - \frac{\nu}{1-\nu}\sigma_{22}\right] = \frac{1}{E'}\left[\sigma_{11} - \nu'\sigma_{22}\right],$$
$$\varepsilon_{22} = \frac{1-\nu^2}{E}\left[\sigma_{22} - \frac{\nu}{1-\nu}\sigma_{11}\right], = \frac{1}{E'}\left[\sigma_{22} - \nu'\sigma_{11}\right],$$
$$\varepsilon_{12} = \frac{\sigma_{12}}{2\mu} = \frac{1+\nu}{E}\sigma_{12} = \frac{1+\nu'}{E'}\sigma_{12},$$
$$\sigma_{33} = \nu\left(\sigma_{11} + \sigma_{22}\right).$$

............(6)

Observe that plane stress relations can be converted to plane strain relations by replacing $E$ with $E'$ and $\nu$ with $\nu'$ where

$$E' = \frac{E}{1-\nu^2} \quad \text{and} \quad \nu' = \frac{\nu}{1-\nu}. \qquad \text{............(7)}$$

To solve boundary value problems in elasticity all these equation need to be considered.

We can reduce the number of equations by combining some of them. We substitute (5) into (3) and obtain following equation,

$$\frac{\partial^2}{\partial x_2^2}(\sigma_{11} - \nu\sigma_{22}) + \frac{\partial^2}{\partial x_1^2}(\sigma_{22} - \nu\sigma_{11}) - 2(1+\nu)\frac{\partial^2 \sigma_{12}}{\partial x_1 \partial x_2} = 0. \qquad \text{............(8)}$$

Equation (61) has three dependent variables. To make the solution of (61) more convenient we represent all stress components in terms of a new function $\Phi$, which is called Airy's stress function, as follows,

$$\sigma_{11} = \frac{\partial^2 \Phi}{\partial x_2^2}, \quad \sigma_{22} = \frac{\partial^2 \Phi}{\partial x_1^2}, \quad \sigma_{12} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2}. \qquad \text{............(9)}$$

The reason to choose the particular form (9) is that it satisfy equilibrium equations (1).

Now we substitute (9) into (8) and get the following differential equation,

$$\frac{\partial^4 \Phi}{\partial x_1^2} + 2\frac{\partial^4 \Phi}{\partial^2 x_1 \partial^2 x_2} + \frac{\partial^4 \Phi}{\partial x_2^2} = 0. \qquad \text{............(10)}$$

We have now reduced the number of unknown function to one, i.e., $\Phi$.

Equation (63) can be written is a compact form as follows,

$$\nabla^2\left(\nabla^2\Phi\right)=0 \quad \text{or} \quad \nabla^4\Phi=0, \qquad \cdots\cdots\cdots\cdots(11)$$

where $\nabla^2=\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}$ is the Laplace operator. Equation (11) is called Biharmonic equation.

We derived (11) by using stress-strain relation (5) for plane stress case; however same equation can be derived for plane strain case using (6). Functions $\Phi$ satisfying (11) are called Biharmonic equation. We will use this equation to determine the stress field near crack-tip.

Note that in Polar coordinate system,

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial\theta^2}. \qquad \cdots\cdots\cdots\cdots(12)$$

and

$$\sigma_{rr}=\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}, \quad \sigma_{\theta\theta}=\frac{\partial^2\Phi}{\partial r^2}, \quad \sigma_{r\theta}=-\frac{\partial\Phi}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right). \qquad \cdots\cdots\cdots\cdots(13)$$