ME232: Dynamics

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Room # 106

Conservation of energy and momentum

Conservation of energy:

A mass system is said to be conservative if it does not dissipate energy because of friction or inelasticity.

If no work is done on a conservative system during an interval of motion by external forces (other than gravity or other potential forces), then none of the energy of the system is lost.

For this case, $U'_{1-2} = 0$ and hence, $\Delta T + \Delta V = 0$ or $T_1 + V_1 = T_2 + V_2$.

This is the law of **conservation of dynamical energy**.

The total energy $E = T + V$ is a constant, so that $E_1 = E_2$.

Remember this law holds only in the ideal case where dissipative phenomenon are neglected.

Conservation of momentum:

If, for a certain interval of time, the resultant external force $\Sigma \boldsymbol{F}$ acting on a conservative or nonconservative mass system is zero, it requires that $\dot{\boldsymbol{G}} = 0$, so that during this interval $\boldsymbol{G}_1 = \boldsymbol{G}_2$, which expresses the principle of conservation of linear momentum. Thus, in the absence of an external impulse, the linear momentum of a system remains unchanged.

Similarly, if the resultant moment about a fixed point O or about the mass center G of all external forces on any mass system is zero, then

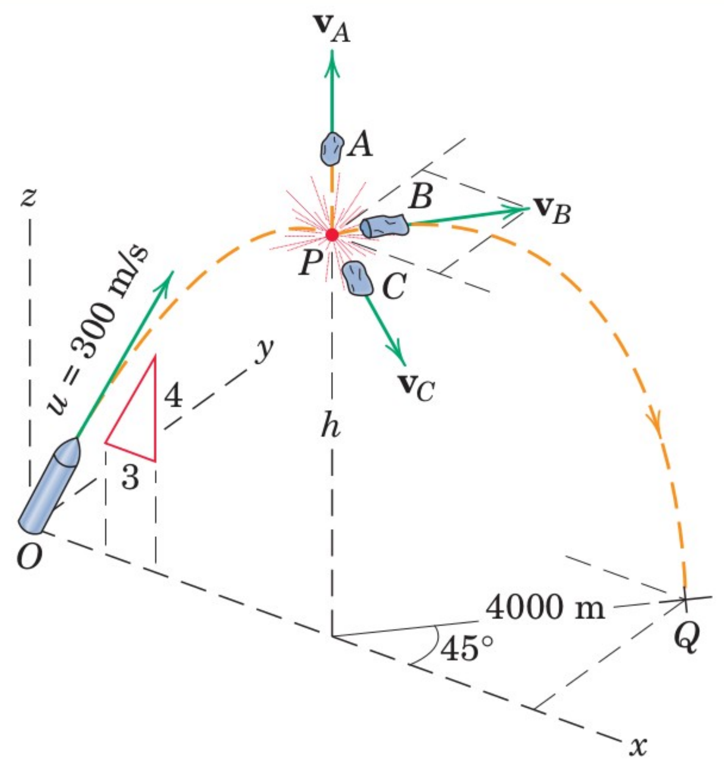
$$(\boldsymbol{H}_O)_1 = (\boldsymbol{H}_O)_2 \qquad \text{or} \qquad (\boldsymbol{H}_G)_1 = (\boldsymbol{H}_\mathrm{G})_2.$$

These relations express the principle of conservation of angular momentum for a general mass system in the absence of an angular impulse.

Thus, if there is no angular impulse about a fixed point (or about the mass center), the angular momentum of the system about the fixed point (or about the mass center) remains unchanged.

Example 1

A shell with a mass of 20 kg is fired from point O, with a velocity $u = 300$ m/s in the vertical x-z plane at the inclination shown. When it reaches the top of its trajectory at P, it explodes into three fragments A, B, and C. Immediately after the explosion, fragment A is observed to rise vertically a distance of 500 m above P, and fragment B is seen to have a horizontal velocity $\boldsymbol{v}_B$ and eventually lands at point Q. When recovered, the masses of the fragments A, B, and C are found to be 5, 9, and 6 kg, respectively. Calculate the velocity which fragment C has immediately after the explosion. Neglect atmospheric resistance.



From our knowledge of projectile motion, the time required for the shell to reach P and its vertical rise are

$$t = u_z / g = 24.5 \text{ s}$$

$$h = u_z^2 / 2g = 2940 \text{ m}$$

The velocity of A has the magnitude

$$v_A = (2gh)^{1/2} = 99 \text{ m/s}$$

With no z-component of velocity initially, fragment B requires 24.5 s to return to the ground. Thus, its horizontal velocity, which remains constant, is

$$v_B = s/t = 163.5 \text{ m/s.}$$

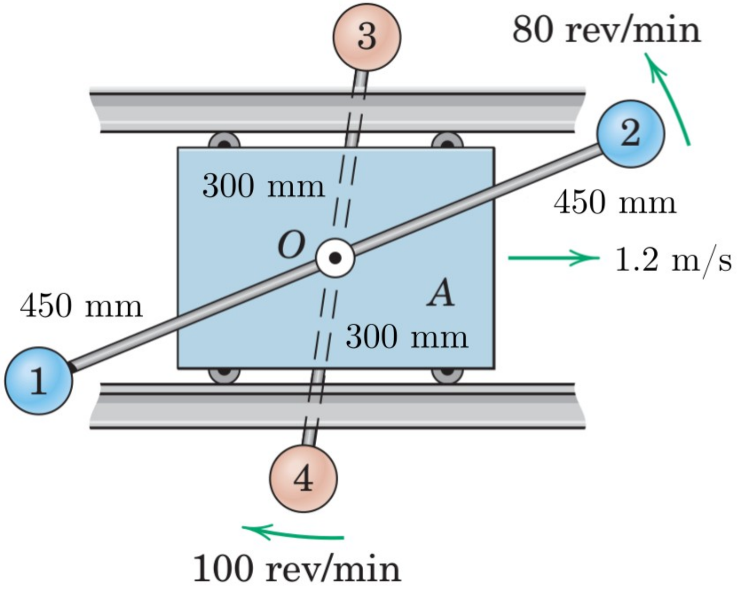
Since the force of the explosion is internal to the system of the shell and its three fragments, the linear momentum of the system remains unchanged during the explosion. Thus,

$$\boldsymbol{G}_1 = \boldsymbol{G}_2 \qquad \text{or} \qquad m\boldsymbol{v} = m_A \boldsymbol{v}_A + m_B \boldsymbol{v}_B + m_C \boldsymbol{v}_C.$$

$\boldsymbol{v}_c$ can now be obtained.

Example 2

The 16-kg carriage A moves horizontally in its guide with a speed of 1.2 m/sec and carries two assemblies of balls and light rods which rotate about a shaft at O in the carriage. Each of the four balls weighs 1.6-kg The assembly on the front face rotates counterclockwise at a speed of 80 rev/min, and the assembly on the back side rotates clockwise at a speed of 100 rev/min. For the entire system, calculate (a) the kinetic energy T, (b) the magnitude $\boldsymbol{G}$ of the linear momentum, and (c) the magnitude $\boldsymbol{H}_O$ of the angular momentum about point O.



The velocities of the balls with respect to O are

$(v_{\text{rel}})_{1,2} = (0.450) \cdot 80\ (2\pi/60) = 3.77$ m/s, $(v_{\text{rel}})_{3,4} = (0.3) \cdot 100\ (2\pi/60) = 3.14$ m/s

Transnational part of the KE of the the system is,

$$\frac{1}{2}m\bar{v}^2 = \frac{1}{2}\left[16 + 4(1.6)\right](1.2)^2 = 16.13 \text{ J}$$

Rotational part of the KE of the the system is,

$$\sum \frac{1}{2}m_i \left|\dot{\boldsymbol{\rho}}\right|^2 = 38.5 \text{ J}$$

Total kinetic energy is $16.13 + 38.5 = 54.7$ J

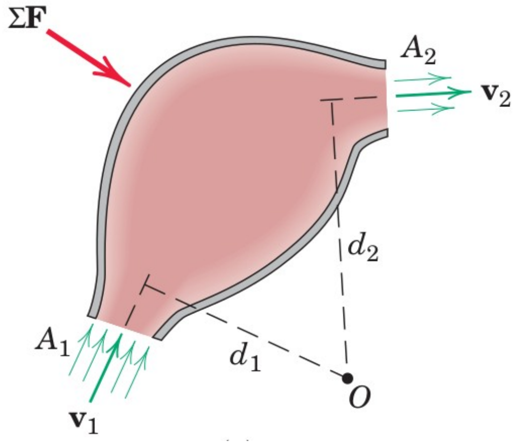
Linear momentum $G = m\bar{\boldsymbol{v}}$ =26.9 kg.m/s

Angular momentum about $O = H_O = \Sigma \mid \boldsymbol{r}_i \text{ x } m\boldsymbol{v}_i \mid = 2.41$ kg.m²/s

Steady mass flow

- The momentum relation developed for a general system of mass enable us to analyze the action of mass flow. This is a situation where a change in momentum occurs.
- The dynamics of mass flow is of importance in the description of fluid machinery of all types including turbines, pumps, nozzles, air-breathing jet engines, and rockets.
- An important cases of mass flow is the steady-flow where the **rate at which mass enters** a given volume **equals the rate at which mass leaves** the same volume. The volume may be enclosed by a rigid container, fixed or moving, such as the nozzle of a jet aircraft or rocket, the space between blades in a gas turbine, the volume within the casing of a centrifugal pump, or the volume within the bend of a pipe through which a fluid is flowing at a steady rate.
- The design of such fluid machines depends on the analysis of the forces and moments associated with the corresponding momentum changes of the flowing mass.

Analysis of flow through a rigid container:

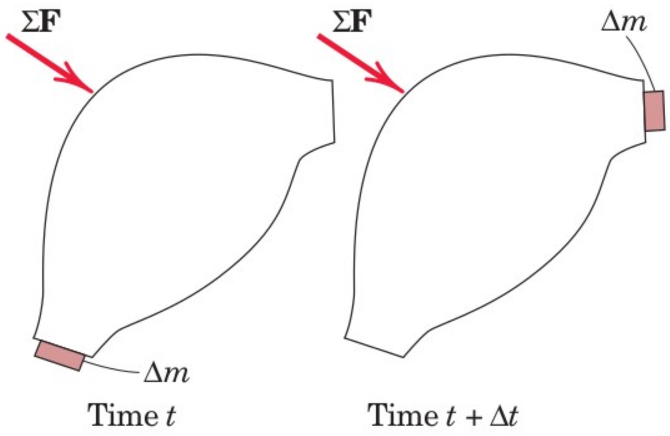


Consider a rigid container into which mass flows in a steady stream at the rate m' through the entrance section of area A_1. Mass leaves the container through the exit section of area A_2 at the same rate. This means that there is no accumulation or depletion of the total mass within the container during the period of observation.

The velocity of the entering stream is v_1 normal to A_1 and that of the leaving stream is v_2 normal to A_2. If 1 and 2 are the respective densities of the two streams, conservation of mass requires that

$$\rho_1 v_1 A_1 = \rho_2 v_2 A_2 = m'. \qquad \text{..........(22)}$$

- We isolate the mass of fluid within the container if the forces between the container and the fluid were to be described.
- The entire container and the fluid within it is isolated when the forces external to the container are desired.



Following the later approach, isolation of the system is described by a free-body diagram of the mass within a closed volume defined by the exterior surface of the container and the entrance and exit surfaces. All forces applied externally to this system are shown as the vector sum $\Sigma \boldsymbol{F}$, which includes:

1. the forces exerted on the container at points of its attachment to other structures, including attachments at A_1 and A_2 , if present,

2. the forces acting on the fluid within the container at A_1 and A_2 due to any static pressure which may exist in the fluid at these positions, and

3. the weight of the fluid and structure if appreciable.

The resultant $\Sigma \boldsymbol{F}$ of all of these external forces must equal $\dot{\boldsymbol{G}}$, the time rate of change of the linear momentum of the isolated system. The expression for $\dot{\boldsymbol{G}}$ may be obtained by an incremental analysis.

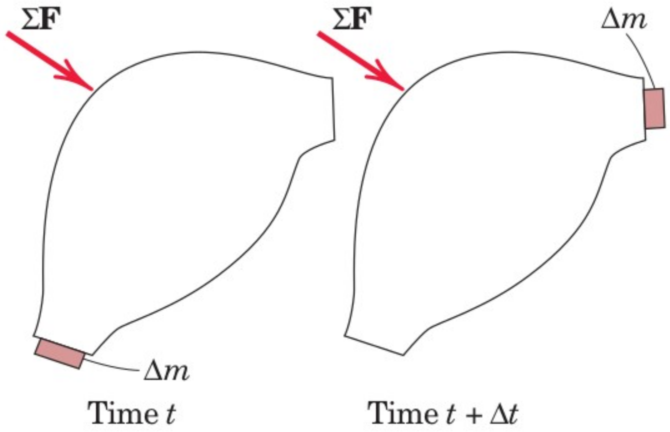


Figure shows the system at time t when the system mass is that of the container, the mass within it, and an increment Δm about to enter during time Δt.

At time $t + \Delta t$ the same total mass is that of the container, the mass within it, and an equal increment Δm which leaves the container in time Δt.

The linear momentum of the container and mass within it between the two sections A_1 and A_2 remains unchanged during Δt.

So the change in momentum of the system in time Δt is only because of mass Δm as

$$\Delta \boldsymbol{G} = (\Delta m)\boldsymbol{v}_2 - (\Delta m)\boldsymbol{v}_1 = \Delta m\,(\boldsymbol{v}_2 - \boldsymbol{v}_1)$$

Division by Δt and taking the limit yield

$$\dot{\boldsymbol{G}} = m'\Delta\boldsymbol{v},$$

where

$$m' = \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} = \frac{dm}{dt}.$$

Thus from the expression for linear momentum,

$$\Sigma\boldsymbol{F} = m'\Delta\boldsymbol{v}. \qquad \cdots\cdots\cdots\cdots(23)$$

Equation (23) establishes the relation between the resultant force on a steady-flow system and the corresponding mass flow rate and vector velocity increment.

We can now see one of the powerful applications of our general force-momentum equation which we derived for any mass system.

The system includes a body which is rigid (the structural container for the mass stream) and particles which are in motion (the flow of mass).

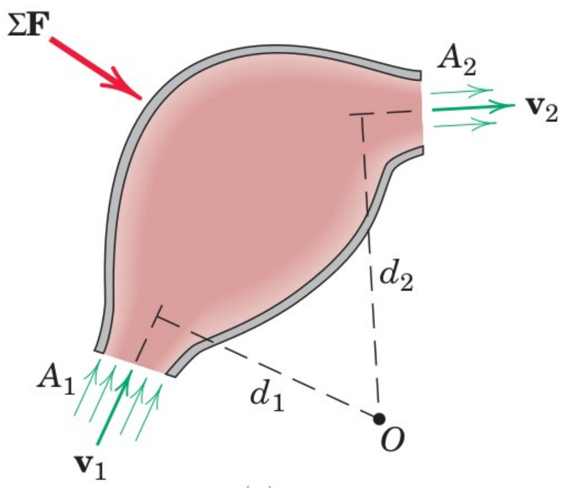
By defining the boundary of the system, the mass within which is constant for steady-flow conditions, we are able to utilize the generality of Newton's second law.

However, care must be taken while accounting for all external forces acting on the system, and they become clear if our free-body diagram is correct.

The blades of a helicopter impart downward momentum to a column of air, thereby creating the forces necessary for hovering and maneuvering.

Analysis of flow through a rigid container:

A similar formulation is obtained for the case of angular momentum in steady-flow systems. The resultant moment of all external forces about some fixed point O on or off the system, equals the time rate of change of angular momentum of the system about O (Equation 13).



For the case of steady flow in a single plane, becomes

$$\Sigma M_O = m' \left(v_2 \, d_2 - v_1 \, d_1 \right) \qquad \cdots\cdots\cdots\cdots(24)$$

When the velocities of the incoming and outgoing flows are not in the same plane, the equation may be written in vector form as

$$\Sigma \boldsymbol{M}_O = m' \left(\boldsymbol{d}_2 \times \boldsymbol{v}_2 - \boldsymbol{d}_1 \times \boldsymbol{v}_1 \right) \qquad \cdots\cdots\cdots\cdots(24a)$$

where $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are the position vectors to the centers of A_1 and A_2 from the fixed reference O. As discussed earlier, the mass center G may be used alternatively as a moment center.

Equations (23) and (24a) are very simple relations having important use in describing relatively complex fluid actions. Note that these equations relate external forces to the resultant changes in momentum and are independent of the flow path and momentum changes internal to the system.

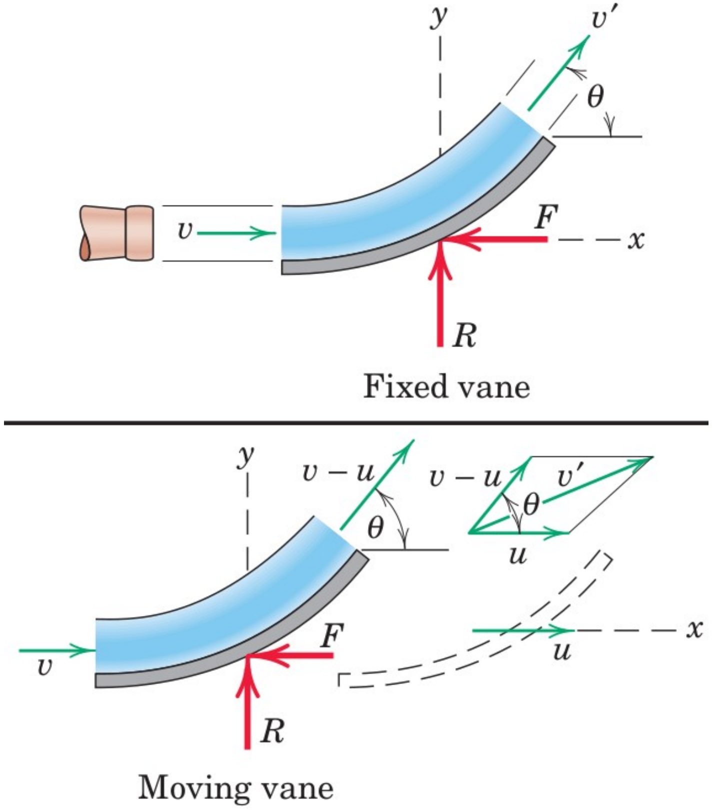
Above analysis may also be applied to systems which move with constant velocity (i.e. for inertial frame). The only restriction is that the mass within the system remain constant with respect to time.

Example 3

The smooth vane shown diverts the open stream of fluid of cross-sectional area A, mass density ρ, and velocity v.

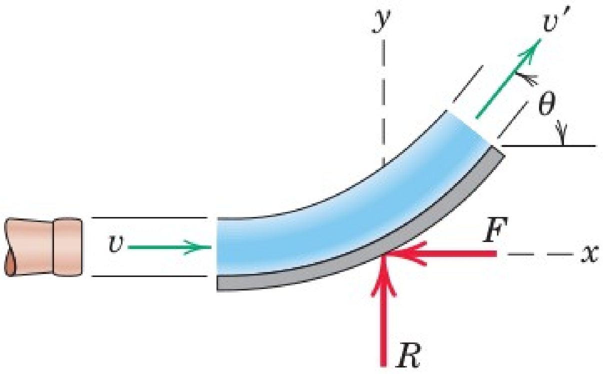
(a) Determine the force components R and F required to hold the vane in a fixed position.

(b) Find the forces when the vane is given a constant velocity u less than v and in the direction of v.



Part a:

The momentum equation may be applied to the isolated system for the change in motion in both the x- and y-directions.



Fixed vane

With the vane stationary, the magnitude of the exit velocity v' equals that of the entering velocity v with fluid friction neglected.

The changes in the velocity components are then

$$\Delta v_x = v'\cos\theta - v = -v\,(1 - \cos\theta\,) \text{ and } \Delta v_y = v'\sin\theta - 0 = v\sin\theta.$$

The mass rate of flow is $m' = Av$, thus,

$\Sigma F_x = m'\Delta v_x$, $\quad -F = \rho Av\,[-v\,(1\text{-}\cos\,\theta\,)]$ $\quad$ or $\quad F = \rho Av^2\,(1\text{-}\cos\,\theta\,)$.

$\Sigma F_y = m'\Delta v_y$, $\quad R = \rho Av\,[v\sin\,\theta\,]$ $\quad$ or $\quad R = \rho Av^2 \sin\,\theta$.

Part b:

In the case of the moving vane, the magnitude of velocity of the fluid during entry and exit is the velocity of the fluid relative to the vane v - u.

Thus, the change in x-velocity of the stream is

$$\Delta v_x = (v - u)\ \cos\theta - (v - u) = -(v - u)(1 - \cos\theta\ )$$

The y-component of v' is $(v - u)\ \sin\theta$, so that the change in the y - velocity of the stream is

$$\Delta v_y = (v - u)\ \sin\theta\ .$$

The mass rate of flow m' is the mass undergoing momentum change per unit of time. This rate is the mass flowing over the vane per unit time and not the rate of issuance from the nozzle. Thus, $m' = \rho A(v - u)$.

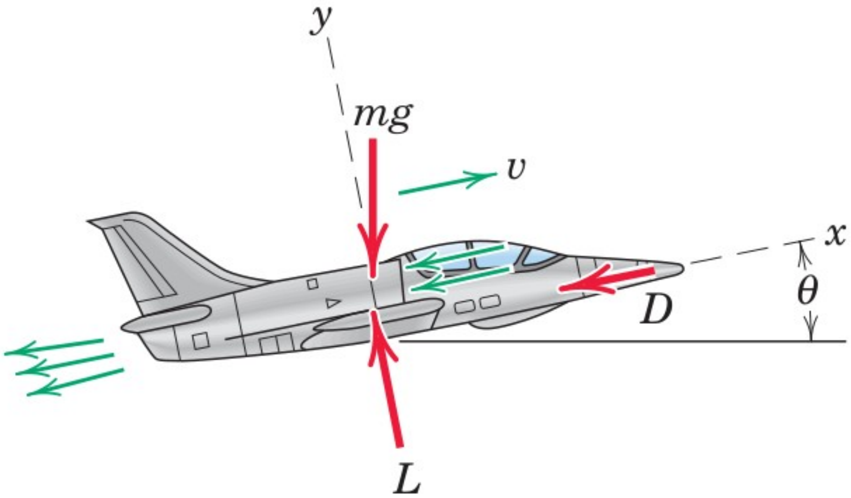
The mass rate of flow is $m' = A(v\text{-}u)$, thus,

$\Sigma F_x = m'\Delta v_x$, $\quad$ $-F = \rho A\ (v\text{-}u)\ [-(v\text{-}u)\ (1\text{-}\cos\theta)]$ or $F = \rho A(v\text{-}u)^2 (1\text{-}\cos\theta)$.

$\Sigma F_y = m'\Delta v_y$, $\quad$ $R = \rho A(v\text{-}u)^2 \sin\theta$.

Example 4

An air-breathing jet aircraft of total mass m flying with a constant speed v consumes air at the mass rate m'_a and exhausts burned gas at the mass rate m'_g with a velocity u relative to the aircraft. Fuel is consumed at the constant rate m'_f. The total aerodynamic forces acting on the aircraft are the lift L, normal to the direction of flight, and the drag D, opposite to the direction of flight. Any force due to the static pressure across the inlet and exhaust surfaces is assumed to be included in D. Write the equation for the motion of the aircraft and identify the thrust T.



The free-body diagram of the aircraft together with the air, fuel, and exhaust gas within it is given and shows only the weight, lift, and drag forces as defined.

We attach axes x-y to the aircraft and apply our momentum equation relative to the moving system.

The fuel will be treated as a steady stream entering the aircraft with no velocity relative to the system and leaving with a relative velocity u in the exhaust stream.

We now treat the air and fuel flows separately. For the air flow, the change in velocity in the x-direction relative to the moving system is

$$\Delta v_a = -u - (-v) = -(u - v),$$

and for the fuel flow the x-change in velocity relative to x-y is

$$\Delta v_f = -u - (0) = -u.$$

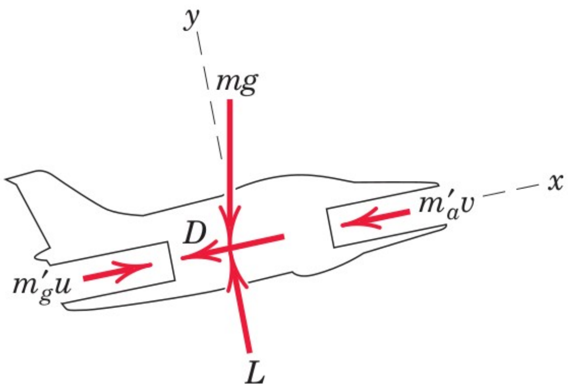
Thus, we have, $\Sigma F_x = m'\Delta v_x$,

$$-mg\sin\theta - D = -m'_a(u - v) - m'_f u = -m'_g u + m'_a v.$$

Now, $m'_g = m'_a + m'_f$,

Hence, we get

$$m'_g u - m'_a v = mg\sin\theta + D ,$$



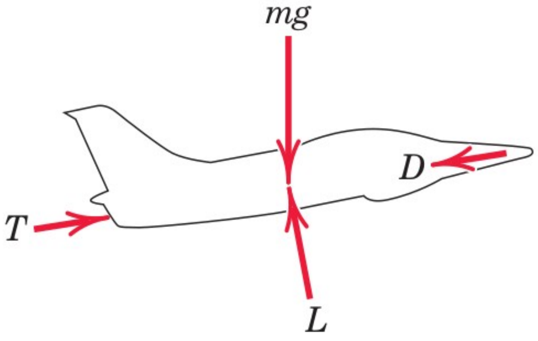
which is the equation of motion of the system.

If we modify the boundaries of our system to expose the interior surfaces on which the air and gas act, we will have the simulated model shown, where the air exerts a force $m'_a\, v$ on the interior of the turbine and the exhaust gas reacts against the interior surfaces with the force $m'_g\, u$.

The commonly used model is shown in the final diagram, where the net effect of air and exhaust momentum changes is replaced by a simulated thrust

$$T = m'_g\, u - m'_a\, v$$

applied to the aircraft from a presumed external source.



Generally m'_f is only 2% or less of m'_{a} and we can approximate $m'_g \cong m'_a$ and express the thrust as

$$T \cong m'_g\, (u - v)$$

Variable mass

For derivation of kinetic equations for a system of particles, we assumed that the mass of the system remain constant.

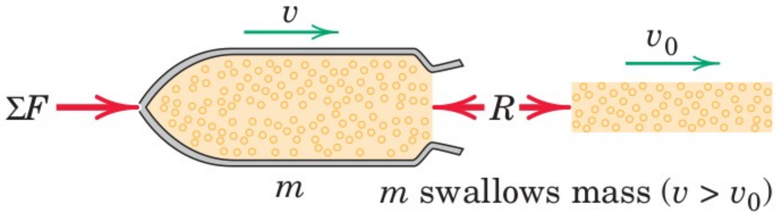
We also extended these principles to describe the action of forces on a system defined by a geometric volume through which a steady flow of mass passes. Therefore, the amount of mass within this volume was constant with respect to time.

When the mass within the boundary of a system under consideration is not constant, the foregoing relationships are no longer valid.

Equation of Motion:

We will now develop the equation for the linear motion of a system whose mass varies with time. Consider first a body which gains mass by overtaking and swallowing a stream of matter.

The mass of the body and its velocity at any instant are m and v, respectively.



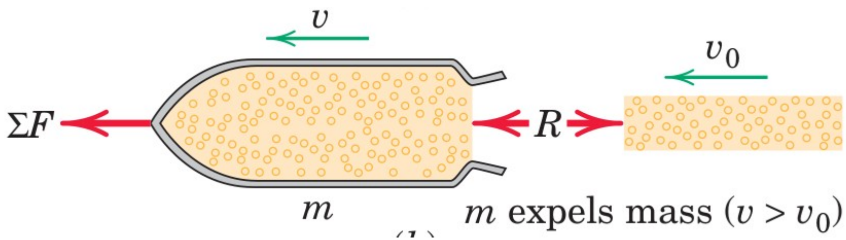
The stream of matter is assumed to be moving in the same direction as m with a constant velocity $v_0 < v$. The force exerted by m on the particles of the stream to accelerate them from a velocity v_0 to a greater velocity v is

$$R = m'(v - v_0) = \dot{m}u,$$

where the time rate of increase of m is $m' = \dot{m}$ and where u is the magnitude of the relative velocity with which the particles approach m. In addition to R, all other forces acting on m in the direction of its motion are denoted by ΣF. The equation of motion of m from Newton's second law is, therefore,

$$\Sigma F - R = m\dot{v} \qquad \text{or} \qquad \Sigma F = m\dot{v} + \dot{m}u. \qquad \cdots\cdots\cdots(25)$$

Similarly, if the body loses mass by expelling it rearward so that its velocity $v_0 < v$, the force R required to decelerate the particles from a velocity v to a lesser velocity v_0 is



$$R = m'(-v_0 - (-v)) = m'(v - v_0).$$

But $m' = -\dot{m}$ since m is decreasing. Also, the relative velocity with which the particles leave m is $u = v - v_0$. Thus, the force R becomes $R = -\dot{m}u$.

If ΣF denotes the resultant of all other forces acting on m in the direction of its motion, Newton's second law requires that

$$\Sigma F + R = m\dot{v} \qquad \text{or} \qquad \Sigma F = m\dot{v} + \dot{m}u.$$

which is the same relationship as in the case where m is gaining mass.

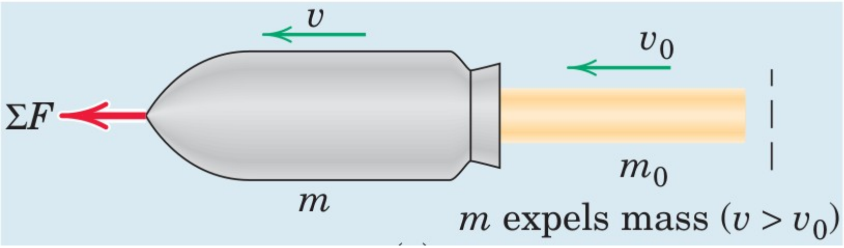
A frequent error in the use of the force-momentum equation is to express the partial force sum ΣF as

$$\Sigma F = d\,(mv)/dt = m\dot{v} + \dot{m}v.$$

From this expansion we see that the direct differentiation of the linear momentum gives the correct force ΣF only when the body picks up mass initially at rest or when it expels mass which is left with zero absolute velocity. In both instances, $v_0 = 0$ and $u = v$.

The Super Scooper is a firefighting airplane which can quickly ingest water from a lake by skimming across the surface with just a bottom-mounted scoop entering the water. The mass within the aircraft boundary varies during the scooping operation as well as during the dumping operation shown.

Alternate approach:



Equation (25) may also be obtained by a direct differentiation of the momentum from the basic relation $\Sigma F = \dot{G}$, provided a proper system of constant total mass is chosen.

Consider the case where m is losing mass. Figure shows the system of m and an arbitrary portion m_0 of the stream of ejected mass. The mass of this system is $m + m_0$ and is constant.

The ejected stream of mass is assumed to move undisturbed once separated from m, and the only force external to the entire system is ΣF which is applied directly to m as before. The reaction $R = -\dot{m}u$ is now an internal to the system. With constant total mass, the momentum principle $\Sigma F = \dot{G}$ is applicable and we have

$$\Sigma F = d(mv + m_0 v_0)/dt = m\dot{v} + \dot{m}v + m_0\dot{v}_0 + \dot{m}_0 v_0$$

Here, $\dot{m}_0 = -\dot{m}$, and the velocity of the ejected mass with respect to m is $u = v - v_0$. Also $\dot{v}_0 = 0$ since m_0 moves undisturbed with no acceleration once free of m. Thus, the relation becomes $\Sigma F = m\dot{v} + \dot{m}u$ which is identical to (25).

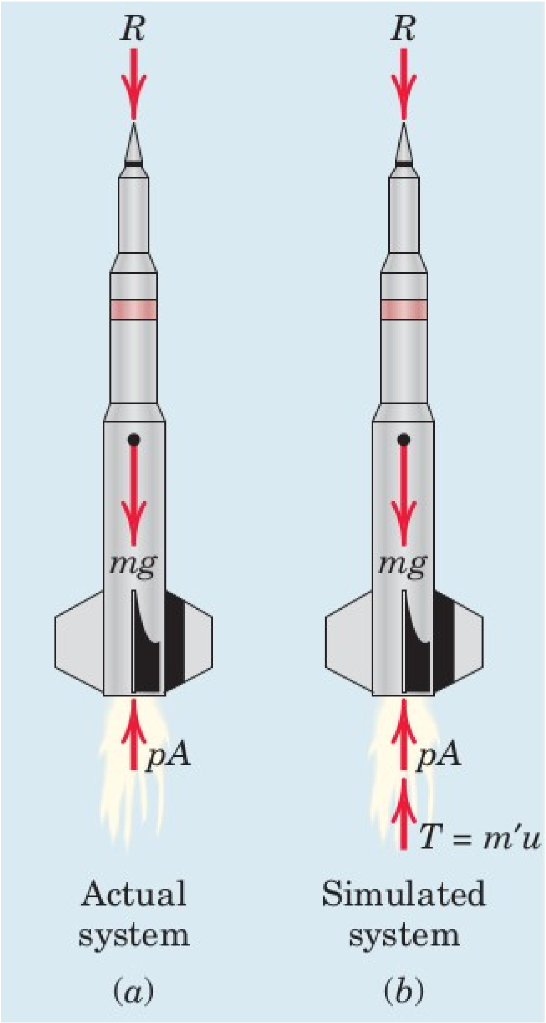
Application: Rocket propulsion

The case of m losing mass is clearly descriptive of rocket propulsion. Figure shows a vertically ascending rocket, the system for which is the mass within the volume defined by the exterior surface of the rocket and the exit plane across the nozzle. External to this system, the free-body diagram discloses the instantaneous values of gravitational attraction mg, aerodynamic resistance R, and the force pA due to the average static pressure p across the nozzle exit plane of area A. The rate of mass flow is $m' = -\dot{m}$. Thus, we may write the equation of motion of the rocket,

$$\Sigma F = m\dot{v} + \dot{m}u,$$

$$pA - mg - R = m\dot{v} + \dot{m}u, \quad \text{or}$$

$$m'u + pA - mg - R = m\dot{v}. \qquad \cdots\cdots\cdots\cdots(26)$$

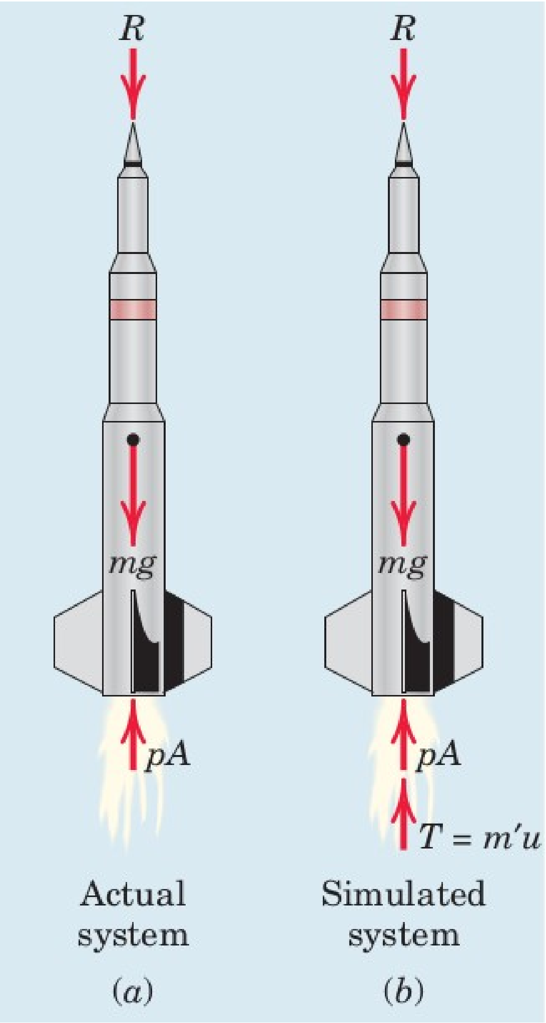


Equation (26) is of the form "$\Sigma F = ma$" where the first term in "ΣF " is the thrust $T = m'u$. Thus, the rocket may be simulated as a body to which an external thrust T is applied, and the problem may then be analyzed like any other problem, except that m is a function of time.

During the initial stages of motion

- velocity v of the rocket $<$ relative exhaust velocity u, Hence, the absolute velocity v_0 of the exhaust gases will be directed rearward.
- when rocket velocity $v > u$, the absolute velocity v_0 of the exhaust gases will be directed forward.

For a given mass rate of flow, the rocket thrust T depends only on the relative exhaust velocity u and not on the magnitude or on the direction of the absolute velocity v_0 of the exhaust gases.



In the foregoing treatment of bodies whose mass changes with time, we have assumed that **all elements of the mass m of the body were moving with the same velocity v at any instant of time** and that the particles of mass added to or expelled from the body underwent **an abrupt transition of velocity upon entering or leaving the body**. Thus, this **velocity change has been modeled as a mathematical discontinuity**.

In reality, this change in velocity cannot be discontinuous even though the transition may be rapid. In the case of a rocket, for example, the velocity change occurs continuously in the space between the combustion zone and the exit plane of the exhaust nozzle. A more general analysis of variable-mass dynamics removes this restriction of discontinuous velocity change and introduces a slight correction to (25).