Linear elasticity

Spacial cases:

(a) When wall of the spherical shell is very thin, i.e. $t << a$, which implies

$$b^3 - a^3 = (b-a)(b^2 + ab + a^2) \approx 3ta^2,$$
$$a^3 p^{(1)} - b^3 p^{(2)} \approx a^3 \left(p^{(1)} - p^{(2)}\right),$$
$$b^3 \approx a^3,$$
$$R^3 \approx a^3,$$

which leads to

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} \approx \frac{a\left(p^{(1)} - p^{(2)}\right)}{2t} \quad \text{and} \quad \sigma_{RR} \approx 0, \qquad \cdots\cdots\cdots\cdots(26)$$

This is the thin-walled pressure vessel approximation commonly studied in strength of materials.

Spacial cases:

(b) In the limit as $b \to \infty$, the problem reduces to that of a pressurized spherical void in an infinite medium under remote hydrostatic pressure. The solution to this problem then is found by taking the solution derived for hollow sphere in the limit as $b \to \infty$, as

$$\sigma_{RR} = -p^{(2)} + \frac{1}{2}\left(p^{(1)} - p^{(2)}\right)\frac{a^3}{R^3}, \qquad \text{.................}(27)$$

Far away from the void, as $R \to \infty$, the state of stress is one of pure hydrostatic pressure:

$$\sigma_{RR} = \sigma_{\theta\theta} = \sigma_{\phi\phi} = -p^{(2)} \qquad \text{.................}(28)$$

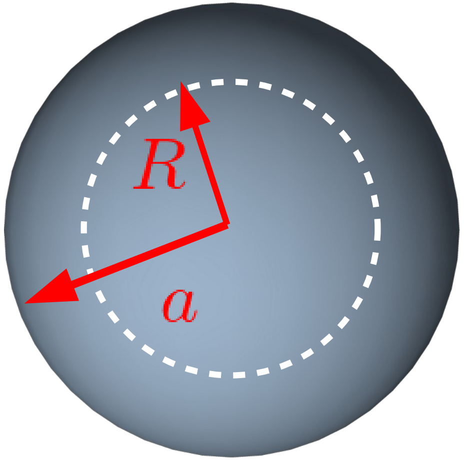
If the void is unpressurized (i.e. $p^{(1)} = 0$), then

$$\sigma_{RR} = -p^{(2)}\left(1 - \frac{a^3}{R^3}\right), \quad \sigma_{\theta\theta} = \sigma_{\phi\phi} = -p^{(2)}\left(1 + \frac{a^3}{2R^3}\right). \qquad \text{.................}(29)$$

At $R{=}a$, $\sigma_{RR} = 0$ and $\sigma_{\theta\theta} = \sigma_{\phi\phi} = -3p^{(2)}/2$.

Far away from the void the maximum absolute value of the principal stresses is $|p^{(2)}|$, whereas at $R = a$, it is $3p^{(2)}/2$. The ratio of the later over the former is the **stress concentration factor**, s.c.f $= 3/2$, for an unpressurized spherical void in an infinite medium.

Gravitating sphere



A planet under its own gravitational attraction may be idealized (rather crudely) as a solid sphere with radius a. The solid is subjected to the following loading:

- A body force $\boldsymbol{b} = -(gR/a)\boldsymbol{e}_{\mathrm{R}}$ per unit mass, where g is the acceleration attributable to gravity at the surface of the sphere
- A traction-free surface at $R{=}a$

We start with Navier's equation,

$$(\lambda + 2\mu)\frac{d}{dR}\left[\frac{1}{R^2}\frac{d}{dR}(R^2U)\right] - \rho_0\frac{gR}{a} = 0,$$

or

$$\frac{d}{dR}\left[\frac{1}{R^2}\frac{d}{dR}(R^2U)\right] = \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\rho_0\frac{gR}{a}. \qquad \text{..................(30)}$$

Similar to the previous problem, integrate the equation twice to get the displacement,

$$U(R) = \frac{(1+\nu)(1-2\nu)}{E(1-\nu)} \frac{\rho_0 g R^3}{10a} + AR + B/R^2, \qquad \cdots\cdots\cdots\cdots\cdots\cdots(31)$$

where A and B are constants of integration which can be found by applying boundary conditions.

Expression for radial stress can be derived by first using strain-displacement relations, and then the constitutive relations (similar to the previous problem) as,

$$\sigma_{RR} = \frac{\rho_0 g(3-\nu)R^2}{10a(1-\nu)} + \frac{E}{(1+\nu)(1-2\nu)}\left[(1+\nu)A - 2(1-2\nu)\frac{B}{R^3}\right]. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(32)$$

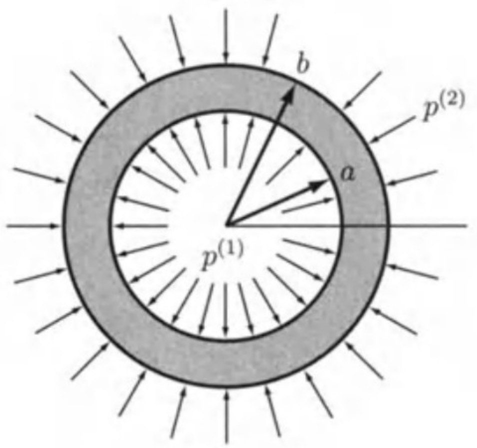
Constants A and B can be determined by using following:

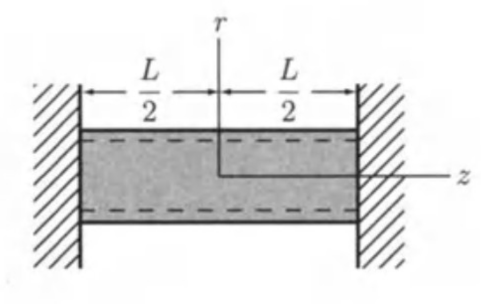
(i) the stress must be finite at $R \to 0$, which is only possible if $B = 0$;

(ii) the surface of the sphere is traction free, which requires $\sigma_{RR} = 0$ at $R = a$, which implies that

$$A = -\frac{(1-2\nu)(3-\nu)\rho_0 g a}{10E(1-\nu)}. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(33)$$

Pressurized long cylinder





- Position vector: $\boldsymbol{x} = r\boldsymbol{e}_r + z\boldsymbol{e}_z$
- Displacement vector: $\boldsymbol{u} = u(r)\,\boldsymbol{e}_r$

We assume that body forces are negligible. For cross sections sufficiently far from the ends, it is clear that $u_z = 0$ and that u_r and u are independent of z. Also, $u = 0$.

- The inner surface $R{=}a$ is subjected to pressure $p^{(1)}$, which implies

$$\sigma_{rr} = -p^{(1)}, \sigma_{r\theta} = 0 \quad \text{on} \quad r = a. \qquad \cdots\cdots\cdots\cdots(34)$$

- The outer surface $R{=}b$ is subjected to pressure $p^{(2)}$, which implies,

$$\sigma_{rr} = -p^{(2)}, \sigma_{r\theta} = 0 \quad \text{on} \quad r = b. \qquad \cdots\cdots\cdots\cdots(35)$$

This problem can be solved starting from Navier's equation, which will be very similar to that method we used for the pressurized hollow sphere. The only different here will be that we will be using equations corresponding to cylindrical coordinate system.

$$u_r = \frac{r}{2(\mu+\lambda)}\left(\frac{a^2p^{(1)} - b^2p^{(2)}}{b^2-a^2}\right) + \frac{a^2b^2}{2\mu r}\left(\frac{p^{(1)}-p^{(2)}}{b^2-a^2}\right), \qquad \text{.................}(36)$$

$$u_\theta = 0.$$

$$\sigma_{rr} = \left(\frac{a^2p^{(1)} - b^2p^{(2)}}{b^2-a^2}\right) - \frac{a^2b^2}{r^2}\frac{p^{(1)}-p^{(2)}}{b^2-a^2},$$

$$\sigma_{\theta\theta} = \left(\frac{a^2p^{(1)} - b^2p^{(2)}}{b^2-a^2}\right) + \frac{a^2b^2}{r^2}\frac{p^{(1)}-p^{(2)}}{b^2-a^2},$$

$$\sigma_{zz} = 2\nu\left(\frac{a^2p^{(1)} - b^2p^{(2)}}{b^2-a^2}\right). \qquad \text{.................}(37)$$

Remember that this solution is only valid for cross sections sufficiently far from the ends of the pressure vessel.

Now, consider a problem where, instead of rigid supports at $z = \pm L/2$, there are end caps, such as on an actual cylindrical pressure vessel. The internal pressure has a resultant force on each end cap of $F^{(1)}=\pi a^2 p^{(1)}$ along the z-direction and the external pressure has a resultant force of $F^{(2)} = \pi b^2 p^{(2)}$. The cross-sectional area of the vessel wall $A = \pi(b^2\text{-}a^2)$. For equilibrium with the pressure on the end caps, therefore, the axial stress should be

$$\sigma_{zz} = \frac{F^{(1)} - F^{(2)}}{A} = \frac{a^2 p^{(1)} - b^2 p^{(2)}}{a^2 - b^2} \qquad \text{.................(37)}$$

Solution (37) can be generalized for a case of cylindrical pressure vessel with end caps by adding the stress given by (38) to (37) which gives,

$$\sigma_{zz} = (1 - 2\nu)\frac{a^2 p^{(1)} - b^2 p^{(2)}}{a^2 - b^2}. \qquad \text{.................(38)}$$

Corresponding strains and displacements can also be derived and superposed to (37).

Spacial cases:

(a) If the cylindrical vessel is thin-walled i.e. $t = b - a,\ t << a$, then

$$
\begin{aligned}
&b^2 - a^2 = (b-a)(b+a) \approx 2ta, \\
&a^2 p^{(1)} - b^2 p^{(2)} \approx a^2 \left(p^{(1)} - p^{(2)}\right), \\
&b^2 \approx a^2, \\
&r^2 \approx a^2,
\end{aligned}
\qquad \cdots\cdots\cdots\cdots(39)
$$

which leads to

$$
\sigma_{\theta\theta} \approx \frac{a\left(p^{(1)} - p^{(2)}\right)}{t} \quad \text{and} \quad \sigma_{rr} \approx 0, \qquad \cdots\cdots\cdots\cdots(40)
$$

Equation (39) is the thin-walled pressure vessel approximation from strength of materials.

Spacial cases:

(b) In limit $b \to \infty$, the problem reduces to that of a pressurized cylindrical hole in an infinite body under remote "plain strain pressure" (as $r \to \infty$, $\sigma_{rr} = \sigma_{\theta\theta} = -p^{(2)}$ and $\sigma_{zz} = -2\nu p^{(2)}$).

Now when $b \to \infty$, as

$$\sigma_{RR} = -p^{(2)} - \frac{1}{2}\left(p^{(1)} - p^{(2)}\right)\frac{a^2}{R^2},$$

$$\sigma_{\theta\theta} = -p^{(2)} + \frac{1}{2}\left(p^{(1)} - p^{(2)}\right)\frac{a^2}{R^2},$$

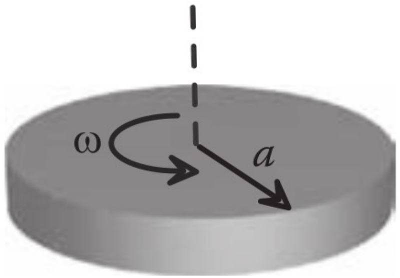
$$\sigma_{zz} = -2\nu p^{(2)} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(41)$$

If the hole is unpressurized (i.e., $p^{(1)} = 0$), then

$$\sigma_{RR} = -p^{(2)}\left(1 - \frac{a^2}{R^2}\right), \ \sigma_{\theta\theta} = -p^{(2)}\left(1 + \frac{a^2}{R^2}\right), \quad \text{and} \quad \sigma_{zz} = -2\nu p^{(2)}. \qquad \cdots\cdots\cdots\cdots(42)$$

Far away from the hole the maximum absolute value of the principal stresses is $|p^{(2)}|$, whereas at R $=$ a, it is $2|p^{(2)}|$. Thus a cylindrical hole in an infinite medium under far-field plane strain pressure has a stress concentration factor of 2.

Spinning circular plate



Consider a thin solid plate with radius a that spins with angular speed ω about its axis. Following are our assumptions:

- No body forces act on the disk.
- The disk has constant angular velocity.
- The outer surface $r = a$ and the top and bottom faces of the disk are free of traction.
- The disk is sufficiently thin to ensure a state of plane stress in the disk.

It can be easily observed that geometry is symmetric; hence the solution also exhibits the cylindrical symmetry. It also implies that the problem is independent of and ϕ direction.

Then, the Navier's equation become,

$$(\lambda + 2\mu)\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru)\right] = -\rho_0\omega^2 r \qquad \cdots\cdots\cdots\cdots(43)$$

Equation (43) can be integrated to solve for displacement as,

$$u = Ar + \frac{B}{r} - \frac{1-\nu^2}{8E}\rho_0\omega^2 r^3. \qquad \cdots\cdots\cdots\cdots(44)$$

The radial stress follows as,

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right) = \frac{E(1+\nu)}{1-\nu^2}\left[A - \frac{1-\nu}{1+\nu}\frac{B}{r^2} + \frac{(1-\nu)(3+\nu)}{8E}\rho_0\omega^2 r^2\right] \qquad \cdots\cdots\cdots\cdots(45)$$

Note that the radial stress must be bounded at r=0, which is possible only when B=0. Another constant A can be determined by apply traction free boundary condition at r=a, as

$$A = \frac{3+\nu}{8E(1+\nu)}\rho_0\omega^2 a^2. \qquad \cdots\cdots\cdots\cdots(46)$$

The complete solution of the problem is:

$$u_r = \frac{1-\nu}{8E}\rho_0\omega^2 r\left[(3+\nu)a^2-(1+\nu)r^2\right], \quad u_\theta = 0,$$
$$u_z = -\frac{\nu}{8E}\rho_0\omega^2 z\left[2(3+\nu)a^2-(3\nu+2)r^2\right], \qquad \cdots\cdots\cdots\cdots(47)$$

$$\varepsilon_{rr} = \frac{1-\nu}{8E}\rho_0\omega^2\left[(3+\nu)a^2-3(1+\nu)r^2\right],$$
$$\varepsilon_{\theta\theta} = \frac{1-\nu}{8E}\rho_0\omega^2\left[(3+\nu)a^2-(1+\nu)r^2\right],$$
$$\varepsilon_{zz} = -\frac{\nu}{8E}\rho_0\omega^2\left[2(3+\nu)a^2-(3\nu+2)r^2\right], \qquad \cdots\cdots\cdots\cdots(48)$$

$$\sigma_{rr} = \frac{3+\nu}{8}\rho_0\omega^2\left[a^2-r^2\right],$$
$$\sigma_{\theta\theta} = \frac{\rho_0\omega^2}{8}\left[(3+\nu)a^2-(3\nu+1)r^2\right], \quad \sigma_{zz} = 0. \qquad \cdots\cdots\cdots\cdots(49)$$