ME232: Dynamics

Anshul Faye

afaye@iitbhilai.ac.in

Room # 106

Kinetics of particles

- According to Newton's second law, a particle will accelerate when it is subjected to unbalanced forces. Kinetics is the study of the relations between **unbalanced forces** and **the resulting changes in motion**.
- Study of the kinetics of particles will require the knowledge of the properties of forces and the kinematics of particle motion.
- The three general approaches to the solution of kinetics problems are:

  (a) direct application of Newton's second law (called the force-mass-acceleration method),

  (b) use of work and energy principles, and

  (c) solution by impulse and momentum methods.

Newton's second law

- The basic relation between force and acceleration is found in Newton's second law.
- Subject a mass particle to the action of a single force F_1, and we measure the acceleration a_1 of the particle in the primary inertial system. The ratio F_1/a_1 of the magnitudes of the force and the acceleration will be some number C_1 whose value depends on the units used for measurement of force and acceleration.
- Repeat the experiment by subjecting the same particle to a different force F_2 and measuring the corresponding acceleration a_2. The ratio F_2/a_2 of the magnitudes will again produce a number C_2 .
- Repeat the experiment as many times as desired.

- Two important conclusions can be drawn from the results of these experiments. First, the ratios of applied force to corresponding acceleration all equal the same number, provided the units used for measurement are not changed in the experiments. Thus,

$$F_1/a_1 = F_2/a_2 = \ldots\ldots = F/a = C, \quad \text{a constant}$$

- We conclude that the constant C is a measure of some invariable property of the particle. This property is the **inertia** of the particle, which is its resistance to rate of change of velocity. For a particle of high inertia (large C), the acceleration will be small for a given force F. On the other hand, if the inertia is small, the acceleration will be large.

- The mass m is used as a quantitative measure of inertia, and therefore, we may write the expression $C = km$, where k is a constant introduced to account for the units used. Thus, we may express the relation obtained from the experiments as

$$F = kma, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where F is the magnitude of the resultant force acting on the particle of mass m, and a is the magnitude of the resulting acceleration of the particle.

- The second conclusion is that the acceleration is always in the direction of the applied force. Thus, (1) becomes a vector relation and may be written as

$$\boldsymbol{F} = km\boldsymbol{a}. \qquad \text{........................(2)}$$

- It is customary to take k equal to unity in (2), thus putting the relation in the usual form of Newton's second law,

$$\boldsymbol{F} = m\boldsymbol{a}. \qquad \text{........................(3)}$$

- In SI units, the units of force (newtons, N). It is derived from Newton's second law. 1 N is the force required to accelerate 1 kg of mass by 1 m/s^2 of acceleration.

Equation of motion

When a particle of mass m is subjected to the action of concurrent forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$,.... whose vector sum is $\Sigma\boldsymbol{F}$, then (3) becomes,

$$\Sigma\boldsymbol{F} = m\boldsymbol{a}. \qquad \cdots\cdots\cdots\cdots(4)$$

Equation (4) is usually called **the equation of motion**. It gives the instantaneous value of the acceleration corresponding to the instantaneous values of the forces acting of mass m.

Rectilinear motion

If we choose the x-direction, as the direction of the rectilinear motion of a particle of mass m, the acceleration in the y- and z-directions will be zero and the scalar components of (4) become

$$\Sigma F_x = ma_x ,\ \Sigma F_y = 0, \text{ and } \Sigma F_z = 0. \qquad \cdots\cdots\cdots\cdots(5)$$

When are not free to choose a coordinate direction along the motion, then all three component of acceleration may be present.

$$\Sigma F_x = ma_x ,\ \Sigma F_y = ma_y ,\ \text{and } \Sigma F_z = ma_z . \qquad \cdots\cdots\cdots\cdots(6)$$

where the acceleration and resultant force are given by

$$\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}, \quad |\boldsymbol{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}, \qquad \cdots\cdots\cdots\cdots(7)$$

$$\sum \boldsymbol{F} = \sum F_x\boldsymbol{i} + \sum F_y\boldsymbol{j} + \sum F_z\boldsymbol{k}, \quad \left|\sum \boldsymbol{F}\right| = \sqrt{(\sum F_x)^2 + (\sum F_y)^2 + (\sum F_z)^2}. \qquad \cdots\cdots(8)$$

Curvilinear motion

For curvilinear motion, depending upon the choice of an appropriate coordinate system, equation of motion will be as follows.

Rectangular coordinate system

$\Sigma F_x = ma_x$,
$\Sigma F_y = ma_y$,

where, $a_x = \ddot{x}$, and $a_y = \ddot{y}$.

Normal and tangential coordinate system

$\Sigma F_n = ma_n$,
$\Sigma F_t = ma_t$,

where, $a_n = v^2/\rho = \rho\dot{\beta}^2 = v\dot{\beta}, a_t = \dot{v}$, and $v = \rho\dot{\beta}$.

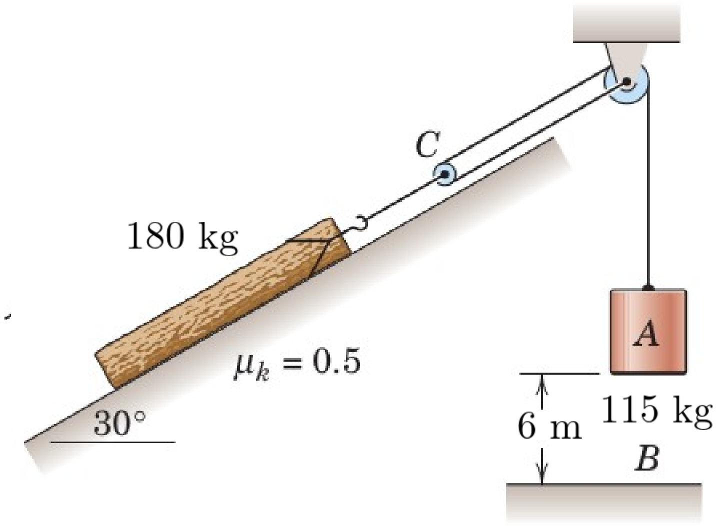
Polar coordinate system

$\Sigma F_r = ma_r$,
$\Sigma F_\theta = ma$,

where, $a_r = \ddot{r} - r\dot{\theta}^2$, and $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$.

Example 1

The 115 kg concrete block A is released from rest in the position shown and pulls the 180 kg log up the 30° ramp. If the coefficient of kinetic friction between the log and the ramp is 0.5, determine the velocity of the block as it hits the ground at B.



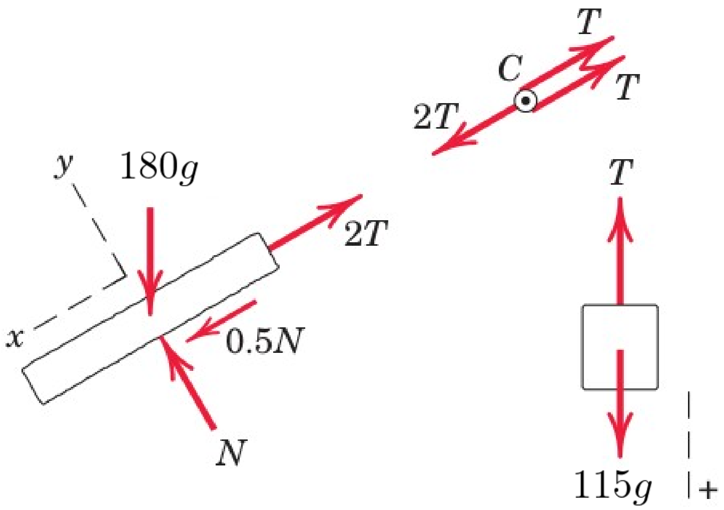
We first perform the kinematic analysis for the given system, accordingly,

$$L = s_A + 2s_C + \text{constant},$$

$$\dot{s}_A + 2\dot{s}_C = 0 \Rightarrow v_A + 2v_C = 0,$$

$$\ddot{s}_A + 2\ddot{s}_C = 0 \Rightarrow a_A + 2a_C = 0.$$

Next step is to draw the free body diagram of individual masses and write equation of motion for the same.



For block A,

$$\sum F = ma_A = 115g - T = 115a_A,$$

For wooden log,

$$\sum F_x = ma_x = 0.5N + 180g\sin 30° - 2T = 180a_C,$$

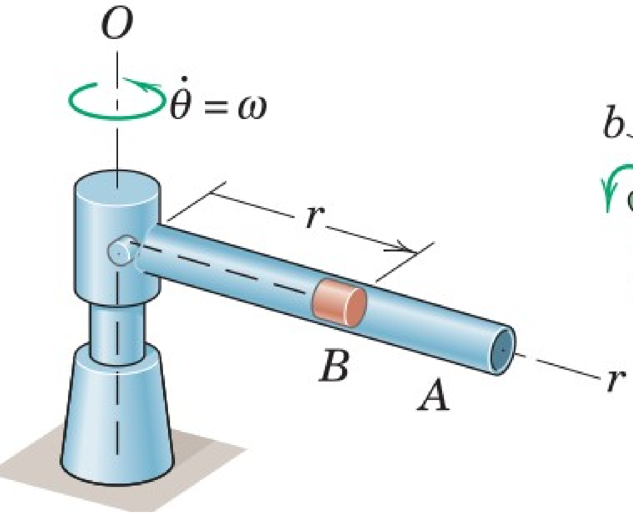
$$\sum F_y = ma_y = N - 180g\cos 30° = 0,$$

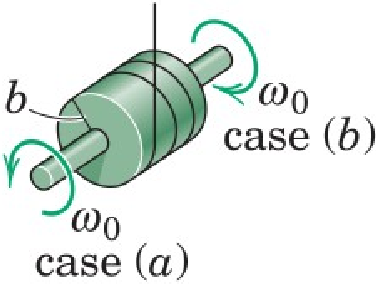
Four equations can be solved for 4 unknowns N, T, a_A, and a_C.

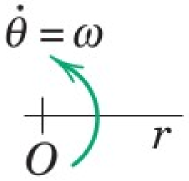
For the 6 m drop with constant acceleration, velocity of the block can be calculated using the kinematic relations for constant accelerations.

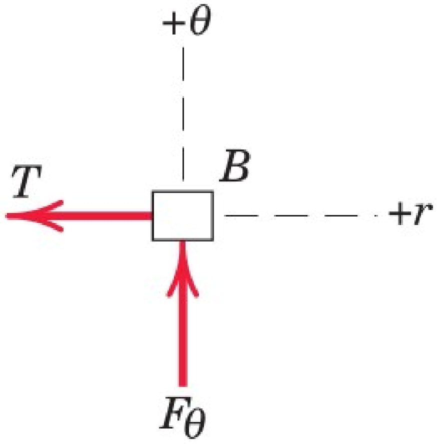
Example 2

Tube A rotates about the vertical O-axis with a constant angular rate ω and contains a small cylindrical plug B of mass m whose radial position is controlled by the cord which passes freely through the tube and shaft and is wound around the drum of radius b. Determine the tension T in the cord and the horizontal component F of force exerted by the tube on the plug if the constant angular rate of rotation of the drum is ω_0 first in the direction for case (a) and second in the direction for case (b). Neglect friction.



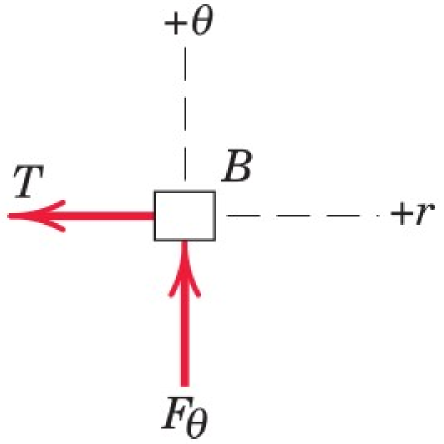






We use the polar-coordinate form of the equations of motion.

The free-body diagram of B is shown in the horizontal plane. The equations of motion are



$$\sum F_r = ma_r \Rightarrow -T = m(\ddot{r} - r\dot{\theta}^2) \Rightarrow T = -m(\ddot{r} - r\omega^2),$$

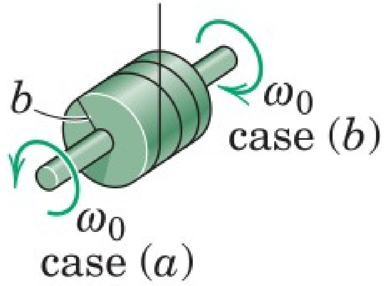
$$\sum F_\theta = ma_\theta \Rightarrow F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \Rightarrow F_\theta = 2m\dot{r}\omega.$$

For case (a), $\dot{r} = +b\omega_0, \ddot{r} = 0,$ thus,

$$T = mr\omega^2, \quad F_\theta = 2mb\omega_0\omega.$$

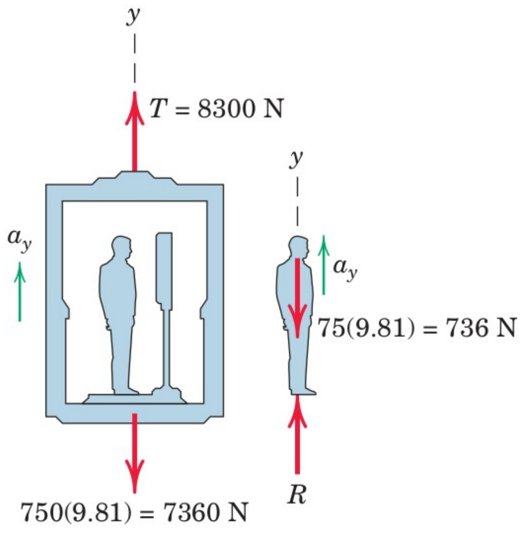
For case (b), $\dot{r} = -b\omega_0, \ddot{r} = 0,$ thus,

$$T = mr\omega^2, \quad F_\theta = -2mb\omega_0\omega.$$



Example 3

A 75-kg man stands on a spring scale in an elevator. During the first 3 seconds of motion from rest, the tension T in the hoisting cable is 8300 N. Find the reading R of the scale in newtons during this interval and the upward velocity v of the elevator at the end of the 3 seconds. The total mass of the elevator, man, and scale is 750 kg.



From the free-body diagram of the elevator, scale, and man taken together, the acceleration is found to be

$$\sum F_y = ma_y \Rightarrow 8300 - 7360 = 750a_y \Rightarrow a_y = 1.257 \text{ m/s}^2.$$

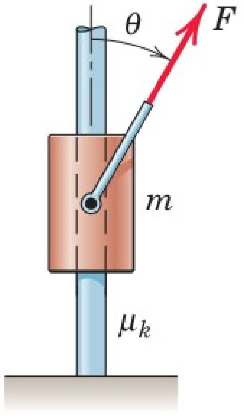
The scale reads the downward force exerted on it by the man's feet. The equal and opposite reaction R act on the man together with its weight, hence, equation of motion is,

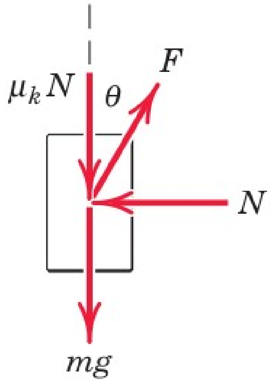
$$\sum F_y = ma_y \Rightarrow R - 736 = 75 \times 1.257 \Rightarrow R = 830 \text{ N}.$$

Velocity after 3 second can be calculated using the expression derived for constant acceleration.

Example 4

The collar of mass m slides up the vertical shaft under the action of a force F of constant magnitude but variable direction. If $\theta = kt$ where k is a constant and if the collar starts from rest with $\theta = 0$, determine the magnitude F of the force which will result in the collar coming to rest as θ reaches $\pi/2$. The coefficient of kinetic friction between the collar and shaft is μ_k .





$$\sum F_x = ma_x$$

$$F \sin\theta - N = 0 \Rightarrow N = F \sin\theta.$$

$$\sum F_y = ma_y$$

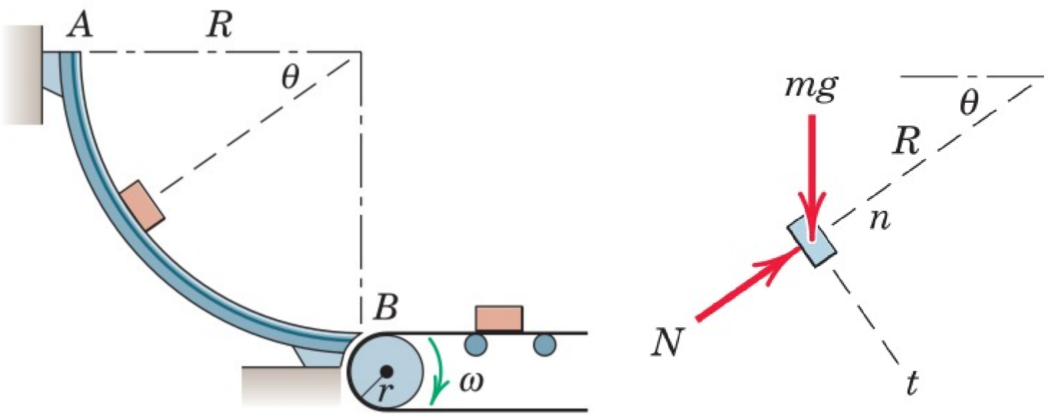
$$F \cos\theta - \mu_k N - mg = mdv/dt,$$

$$\Rightarrow F \cos kt - \mu_k F \sin kt - mg = mdv/dt,$$

Find the expression of force as a function of time and velocity by integrating above the equation. Substituting $t = \pi/2k$ and $v=0$ in that expression will give the required force.

Example 5

Small objects are released from rest at A and slide down the smooth circular surface of radius R to a conveyor B. Determine the expression for the normal contact force N between the guide and each object in terms of and specify the correct angular velocity ω of the conveyor pulley of radius r to prevent any sliding on the belt as the objects transfer to the conveyor.



The normal force N depends on the n-component of the acceleration which, in turn, depends on the velocity.

The velocity will be cumulative according to the tangential acceleration a_t. To find a_t we use,

$$\sum F_t = ma_t,$$

$$mg\cos\theta = ma_t \Rightarrow a_t = g\cos\theta. \quad \cdots\cdots\cdots\text{(a)}$$

$$\sum F_n = ma_n,$$

$$N - mg\sin\theta = ma_n,$$

$$\Rightarrow N = mv^2/R + mg\sin\theta. \quad \cdots\cdots\cdots\text{(b)}$$

To represent N as a function of θ, we first need to express v in terms of θ using (a) as,

$$a_t = \dot{v} = v dv/ds = g\cos\theta,$$

$$\Rightarrow v dv = g\cos\theta ds = g\cos\theta(R d\theta),$$

integrating the above relation as,

$$\Rightarrow \int_0^v v dv = \int_0^\theta gR\cos\theta d\theta, \qquad \Rightarrow v^2 = 2gR\sin\theta, \qquad \cdots\cdots\cdots(c)$$

substituting (c) in (b), we get,

$$\Rightarrow N = 3mg\sin\theta.$$

To find the required rotational velocity,

$$v|_{\theta=\pi/2} = \omega r = \sqrt{2gR}, \text{ hence, } \omega = \sqrt{2gR}/r.$$

Work and Energy

We have used Newton's second law $\boldsymbol{F} = m\boldsymbol{a}$ to to establish the instantaneous relationship between the net force acting on a particle and the resulting acceleration of the particle.

There are two general classes of problems in which the cumulative effects of unbalanced forces acting on a particle are of interest to us.

(1) integration of the forces with respect to the displacement of the particle, and

(2) integration of the forces with respect to the time they are applied for.

Integration with respect to displacement $\rightarrow$ the equations of work and energy

Integration with respect to time $\rightarrow$ the equations of impulse and momentum.

Work

A force $\boldsymbol{F}$ is acting on a particle at A. The position vector of A is $\boldsymbol{r}$, and $d\boldsymbol{r}$ is the differential displacement from A to A'.

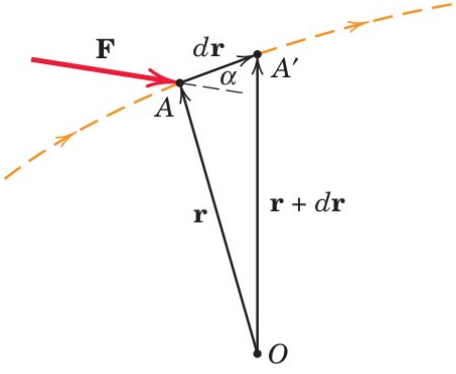
The work done by the force $\boldsymbol{F}$ during the displacement $d\boldsymbol{r}$ is

$$dU = \boldsymbol{F} \cdot d\boldsymbol{r} = F\, ds \cos\alpha, \qquad \cdots\cdots\cdots\cdots(9)$$

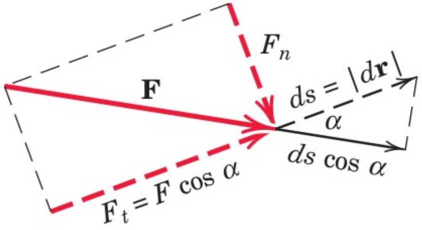
where α is the angle between $\boldsymbol{F}$ and $d\boldsymbol{r}$, and ds is the magnitude of $d\boldsymbol{r}$.

Equation (9) may be interpreted as the displacement multiplied by the force component $F_t = F \cos\alpha$ in the direction of the displacement.

Alternatively, the work dU may be interpreted as the force multiplied by the displacement component $ds \cos\alpha$ in the direction of the force.



(a)



(b)

With this definition of work, it should be noted that the component $F_n = F \sin a$ normal to the displacement does no work. Thus, the work dU may be written as

$$dU = F_t \, ds. \qquad \cdots\cdots\cdots\cdots(10)$$

Work is positive if the working component F_t is in the direction of the displacement and negative if it is in the opposite direction.

Forces which do work are termed **active forces**.

Constraint forces which do no work are termed **reactive forces**.

The SI units of work are those of force (N) times displacement (m) or N·m. This unit is given the special name **joule** (J), which is defined as the work done by a force of 1 N acting through a distance of 1 m in the direction of the force.

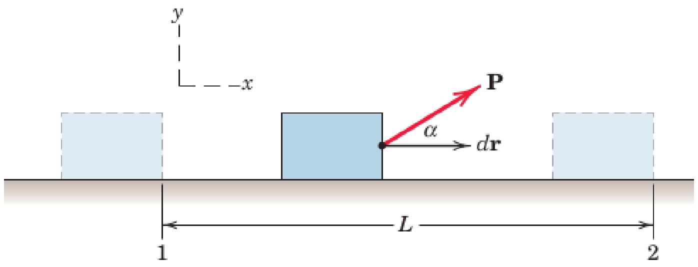
Calculation of work

The work done by a force $\boldsymbol{F}$ during a finite movement of the point of application of $\boldsymbol{F}$ is equal to

$$U = \int_1^2 \boldsymbol{F} \cdot d\boldsymbol{r} = \int_1^2 (F_x dx + F_y dy + F_z dz) \quad \text{or} \quad U = \int_1^2 F_t \, ds. \qquad \cdots\cdots\cdots\cdots(11)$$

1) Work associated with a constant external force:

Consider the constant force $\boldsymbol{P}$ applied to the body as it moves from position 1 to position 2. With the force $\boldsymbol{P}$ and the differential displacement $d\boldsymbol{r}$ written as vectors, the work done on the body by the force is



$$U_{1-2} = \int_1^2 \boldsymbol{F} \cdot d\boldsymbol{r} = \int_1^2 (P\cos\alpha\,\boldsymbol{i} + P\sin\alpha\,\boldsymbol{j})dx\,\boldsymbol{i},$$

$$U_{1-2} = \int_1^2 P\cos\alpha\,dx = P\cos\alpha(x_2 - x_1) = PL\cos\alpha. \qquad \cdots\cdots\cdots\cdots(12)$$

2) Work associated with a spring force:

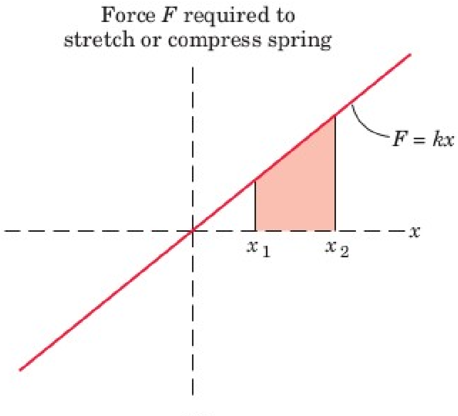
Consider the linear spring of stiffness k where the force required to stretch or compress the spring is proportional to the deformation x.

The force exerted by the spring on the body is $\boldsymbol{F} = -kx\,\boldsymbol{i}$.

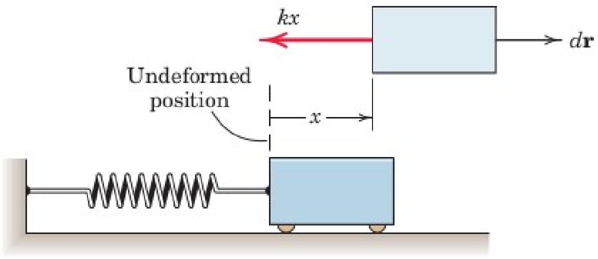
The work done on the body by the spring force during an arbitrary displacement from an initial position x_1 to a final position x_2 is

$$U_{1-2} = \int_1^2 \boldsymbol{F}\cdot d\boldsymbol{r} = \int_1^2 -kx\boldsymbol{i}\cdot dx\,\boldsymbol{i} = \int_{x_1}^{x_2} -kx\,dx,$$

$$U_{1-2} = \frac{1}{2}k(x_1^2 - x_2^2). \qquad \text{..........(13)}$$



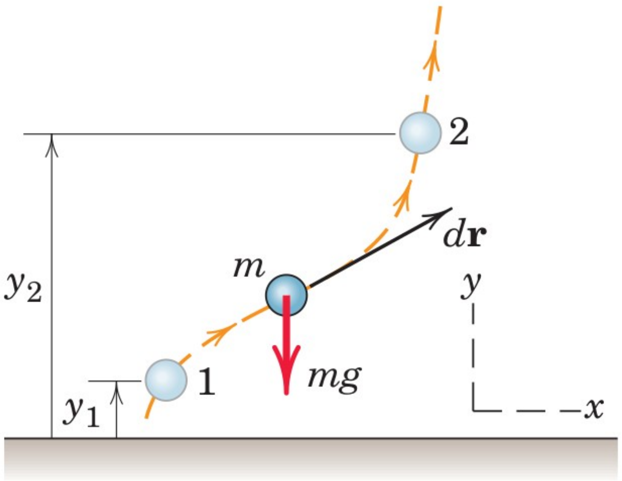
(a)



(b)

3a) Work associated with weight: ($\boldsymbol{g}$ = constant)

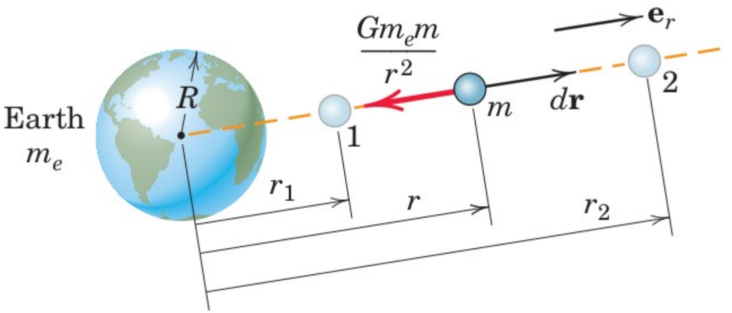
For the sufficiently low variation in altitude (so that the acceleration of gravity $\boldsymbol{g}$ may be considered constant), the work done by the weight $m\boldsymbol{g}$ of the body as the body is displaced from an arbitrary altitude y_1 to a final altitude y_2 is



$$U_{1-2} = \int_1^2 \boldsymbol{F} \cdot d\boldsymbol{r} = \int_1^2 -mg\boldsymbol{j} \cdot (dx\,\boldsymbol{i} + dy\,\boldsymbol{j}) = \int_{y_1}^{y_2} -mgdy = -mg(y_2 - y_1). \quad \cdots\cdots\cdots(14)$$

3b) Work associated with weight: ($g \neq$ constant)

If large changes in altitude occur, then the weight (gravitational force) is no longer constant. We must therefore use the gravitational law and express the weight as a variable force of magnitude



$$F = G\frac{m_e m}{r^2}. \qquad \cdots\cdots\cdots(15)$$

Now, the work can be expressed as,

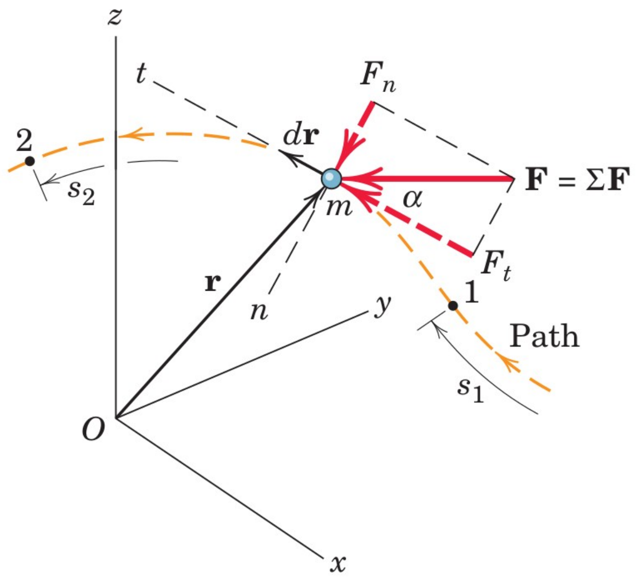
$$U_{1-2} = \int_1^2 \boldsymbol{F} \cdot d\boldsymbol{r} = \int_1^2 -\frac{Gm_e m}{r^2}\boldsymbol{e}_r \cdot dr\,\boldsymbol{e}_r = -Gm_e m\int_{r_1}^{r_2}\frac{dr}{r^2}$$

$$= Gm_e m\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = mgR^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right). \qquad \cdots\cdots\cdots(16)$$

where the equivalence $Gm_e = gR^2$ is used.

Work and curvilinear motion:

Let us consider the work done on a particle of mass m, moving along a curved path under the action of the force $\boldsymbol{F}$. The position of m is specified by $\boldsymbol{r}$, and its displacement along its path during the time dt is represented by the change $d\boldsymbol{r}$ in its position vector. The work done by $\boldsymbol{F}$ during a finite movement of the particle from point 1 to point 2 is



$$U = \int_1^2 \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{s_1}^{s_2} F_t \, ds.$$

where the limits specify the initial and final end points of the motion. Substituting Newton's second law $\boldsymbol{F} = m\boldsymbol{a}$, the expression for the work of all forces becomes

$$U = \int_1^2 m\boldsymbol{a} \cdot d\boldsymbol{r} = \int_{s_1}^{s_2} ma_t \, ds = \int_{s_1}^{s_2} m\dot{v} \, ds = \int_{v_1}^{v_2} mv \, dv = \frac{1}{2}m(v_2^2 - v_1^2).$$

$\cdots\cdots\cdots(17)$

Work and kinetic energy:

The **kinetic energy** ***T*** of the particle is defined as, $T = \frac{1}{2}mv^2.$ ....(18)

It is the total work which must be done on the particle to bring it from a state of rest to a velocity v. Kinetic energy T is a scalar quantity with the units of N·m or joules (J) in SI units. Kinetic energy is always positive, regardless of the direction of the velocity.

Accordingly, (17) can now be written as, $U_{1\text{-}2} = T_2 - T_1 = \Delta T.$ ....(19)

which is **the work-energy equation** for a particle.

The equation states that the total work done by all forces acting on a particle as it moves from point 1 to point 2 equals the corresponding change in kinetic energy of the particle. Although T is always positive, the change ΔT may be be positive, negative, or zero.

Alternatively, the work-energy relation may be expressed as, $T_1 + U_{1\text{-}2} = T_2$ ....(19a)

A major advantage of the method of work and energy is that it **avoids the necessity of computing the acceleration and leads directly to the velocity changes** as functions of the forces which do work.

Further, the work-energy equation **involves only those forces which do work** and thus give rise to changes in the magnitude of the velocities.

Consider now a system of two particles joined together by a rigid connection. The forces in the connection are equal and opposite, and their points of application necessarily have identical displacement components in the direction of the forces. Therefore, the net work done by these internal forces is zero during any movement of the system. Thus, (19) is applicable to the entire system, where $U_{1\text{-}2}$ is the total or net work done on the system by forces external to it and ΔT is the change, $T_2 - T_1$, in the total kinetic energy of the system. The total kinetic energy is the sum of the kinetic energies of both elements of the system. Thus, another advantage of the work-energy method is that it **enables us to analyze a system of particles joined in the manner described without dismembering the system**.

Power

- The capacity of a machine is measured by the time rate at which it can do work or deliver energy.
- The total work or energy output is not a measure of this capacity since a motor, no matter how small, can deliver a large amount of energy if given sufficient time.
- On the other hand, a large and powerful machine is required to deliver a large amount of energy in a short period of time.
- Thus, the capacity of a machine is rated by its power, which is defined as the time rate of doing work.
- Accordingly, the power P developed by a force $\boldsymbol{F}$ which does an amount of work U is $P = dU/dt = \boldsymbol{F}\cdot d\boldsymbol{r}/dt$. Because $d\boldsymbol{r}/\mathrm{dt}$ is the velocity $\boldsymbol{v}$ of the point of application of the force, we have

$$P = \boldsymbol{F}\cdot\boldsymbol{v}\ . \qquad \cdots\cdots\cdots(20)$$

- Power is a scalar quantity, and in SI it has the units of N·m/s = J/s. The special unit for power is the watt (W), which equals one joule per second (J/s).
- In U.S. customary units, the unit for mechanical power is the horsepower (hp). These units and their numerical equivalences are

$$1 \text{ W} = 1 \text{ J/s}$$

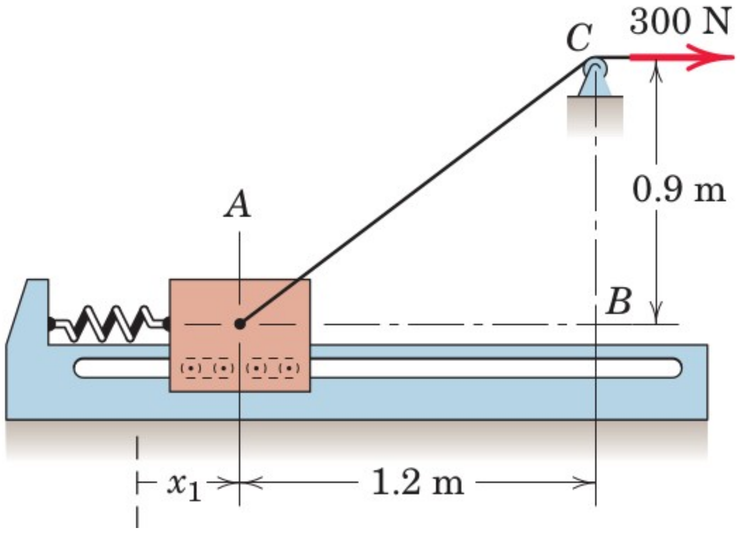
$$1 \text{ hp} = 746 \text{ W} = 0.746 \text{ kW}$$

- The ratio of the work done by a machine to the work done on the machine during the same time interval is called the mechanical efficiency e_m of the machine. Efficiency is always less than unity since every device operates with some loss of energy and since energy cannot be created within the machine. The mechanical efficiency at any instant of time may be expressed in terms of mechanical power P by

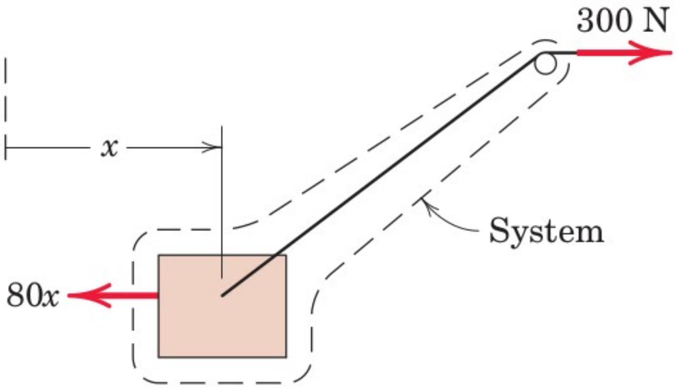
$$e_m = P_{\text{output}}/P_{\text{input}}. \qquad \cdots\cdots\cdots(21)$$

Example 6

The 50 kg block at A is mounted on rollers so that it moves along the fixed horizontal rail with negligible friction under the action of the constant 300 N force in the cable. The block is released from rest at A, with the spring to which it is attached extended an initial amount $x_1 = 0.233$ m. The spring has a stiffness $k = 80$ N/m. Calculate the velocity v of the block as it reaches position B.



We assume that the stiffness of the spring is small enough to allow the block to reach position B.
The active-force diagram for the system composed of both block and cable is shown for a general position.



As the block moves from $x_1 = 0.233$ m to $x_2 = 0.233 + 1.2 = 1.433$ m, the work done by the spring force acting on the block is

$$U_{1-2} = \frac{1}{2}k(x_1^2 - x_2^2) = \frac{1}{2}80 \times (0.233^2 - 1.433^2) = -80\,\mathrm{J}.$$

The work done on the system by the constant 300-N force in the cable is the force times the net horizontal movement of the cable over pulley C, which is $\sqrt{1.2^2 + 0.9^2} - 0.9 = 0.6$ m. Thus, the work done is $300 \times 0.6 = 180$ J.

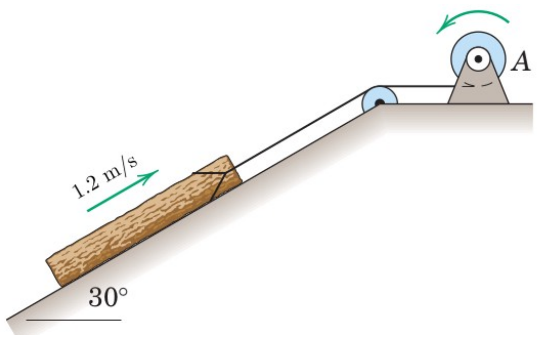
Now apply the work-energy equation to the system as

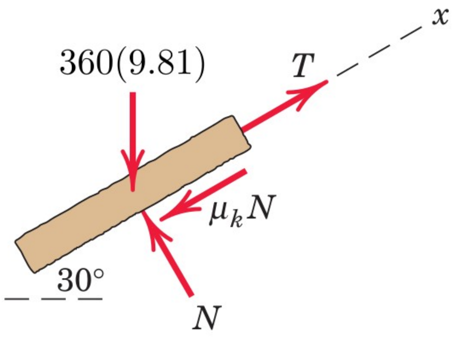
$T_1 + U_{1\text{-}2} = T_2$.

$0 - 80 + 180 = 1/2 \times 50 \times v^2$, which gives $v = 2$ m/s.

Example 7

The power winch A hoists the 360 kg log up the 30° incline at a constant speed of 1.2 m/sec. If the power output of the winch is 4 kW, compute the coefficient of kinetic friction μ_k between the log and the incline. If the power is suddenly increased to 6 kW, what is the corresponding instantaneous acceleration a of the log?





From the free-body diagram of the log,

$$\sum F_x = T - 3531.6\cos 30°\mu_k - 3531.6\sin 30° = 0$$

$$\Rightarrow T = 3058.6\mu_k + 1765.8 \qquad \cdots\cdots\cdots\text{(a)}$$

Power out of the winch gives the tension in the cable,

$P = Tv$, which gives, $T = 4000/1.2 = 3330$ N

Now, μ_k can be calculated from (a) as $\mu_k = 0.513$.

When the power is increased, the tension momentarily becomes,

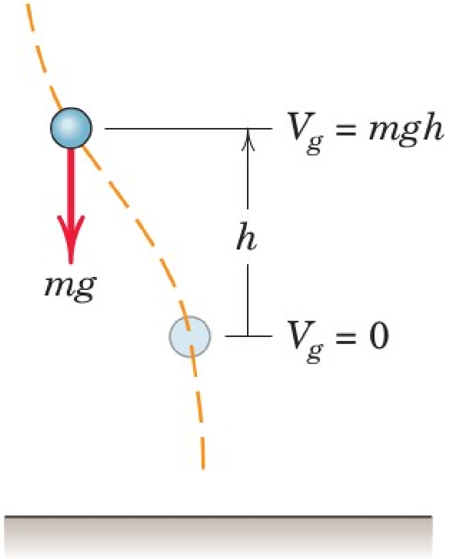
$T = 6000/1.2 = 5000$ N,

Then the corresponding acceleration is given by

$\Sigma F_x = ma_x$, $\quad 5000 - 3058.6\times 0.513 + 1765.8 = 360a$

Hence, $a = 4.63$ m/s^2

Gravitational potential energy



Consider the motion of a particle of mass m close to the surface of the earth, where the gravitational attraction (weight) mg is essentially constant.

The **gravitational potential energy** V_g of the particle is defined as the work mgh done against the gravitational field to elevate the particle a distance h above some arbitrary reference plane (called a datum), where V_g is taken to be zero.

Thus, we write the potential energy as $V_g = mgh.$ ·········(22)

This work is called potential energy because it may be converted into energy if the particle is allowed to do work on a supporting body while it returns to its lower original datum plane.

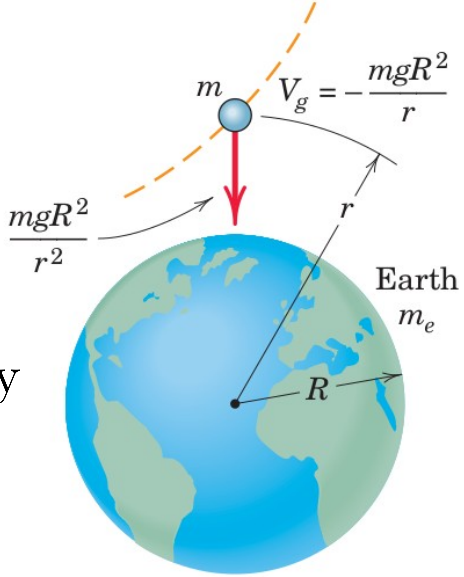
In going from one level at $h = h_1$ to a higher level at $h = h_2$, the change in potential energy becomes

$$\Delta V_g = mg(h_2 - h_1). \qquad \cdots\cdots\cdots(23)$$

The corresponding work done by the gravitational force on the particle is $-mg\Delta h$. Thus, the work done by the gravitational force is the negative of the change in potential energy.

When large changes in altitude in the field of the earth are encountered, the gravitational force $Gmm_e/r^2 = mgR^2/r^2$ is no longer constant. The work done against this force to change the radial position of the particle from r_1 to r_2 is the change $(V_g)_2 - (V_g)_1$ in gravitational potential energy, which is

$$\int_{r_1}^{r_2} mgR^2 \frac{dr}{r^2} = mgR^2\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = (V_g)_2 - (V_g)_1. \qquad \cdots\cdots\cdots(24)$$



It is customary to take $(V_g)_2 = 0$ when $r_2 = \infty$, so that with this datum we have,

$$V_g = -mgR^2/r. \qquad \cdots\cdots\cdots(25)$$

In going from r_1 to r_2, the corresponding change in potential energy is

$$\Delta V_g = mgR^2\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \qquad \cdots\cdots\cdots(26)$$

Elastic potential energy

Another example of potential energy occurs in the deformation of an elastic body, such as a spring.

The work done on the spring to deform it, is stored in the spring and is called its **elastic potential energy** V_e. This energy is recoverable in the form of work done by the spring on the body attached to its movable end during the release of the deformation of the spring.

For the one-dimensional linear spring of stiffness k, the force supported by the spring at any deformation x, tensile or compressive, from its undeformed position is $F = kx$. Thus, we define the elastic potential energy of the spring as the work done on it to deform it an amount x, and we have

$$V_e = \int_0^x kx dx = \frac{1}{2}kx^2. \qquad \cdots\cdots\cdots(27)$$

If the deformation, either tensile or compressive, of a spring increases from x_1 to x_2 during the motion, then the change in potential energy of the spring is,

$$\Delta V_e = \int_0^x kx dx = \frac{1}{2}k(x_2^2 - x_1^2). \qquad \cdots\cdots\cdots(28)$$

Work-Energy Equation

With the elastic member included in the system, we now modify the work-energy equation to account for the potential-energy terms. It stands for the work of all external forces **other than gravitational forces and spring forces**, we may write

$$U'_{1\text{-}2} + (-\Delta V_g) + (-\Delta V_e) = \Delta T \qquad \text{or} \qquad U'_{1\text{-}2} = \Delta T + \Delta V \qquad \cdots\cdots\cdots(29)$$

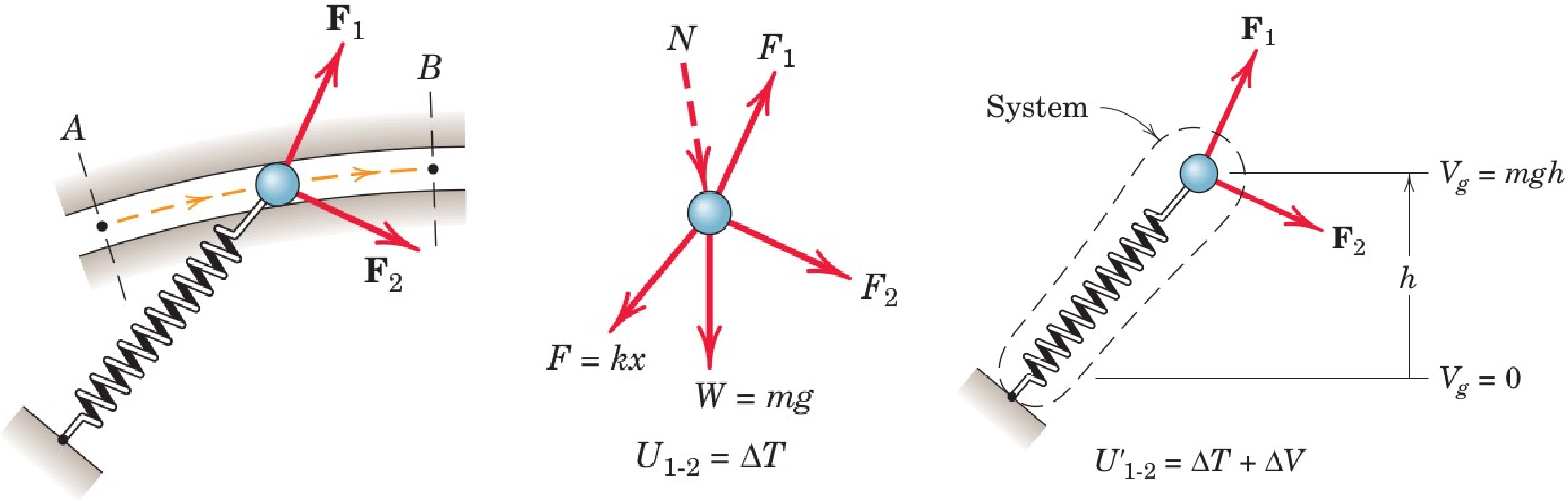
where ΔV is the change in total potential energy, i.e., gravitational plus elastic.

This alternate form of the work-energy equation is often far more convenient to use than the previous form since the work of both gravity and spring forces is accounted for only on the end-point positions of the particle and on the end-point lengths of the elastic spring. The path followed between these end-point positions is of no consequence in the evaluation of ΔV_g and ΔV_e. Equation (29) may also be written as,

$$T_1 + V_1 + U'_{1\text{-}2} = T_2 + V_2 \qquad \cdots\cdots\cdots(30)$$

The difference between two forms of work-energy equation can be understood as follows.

Consider a particle of mass m constrained to move along a fixed path under the action of forces F_1 and F_2, the gravitational force $W = mg$, the spring force F, and the normal reaction N.



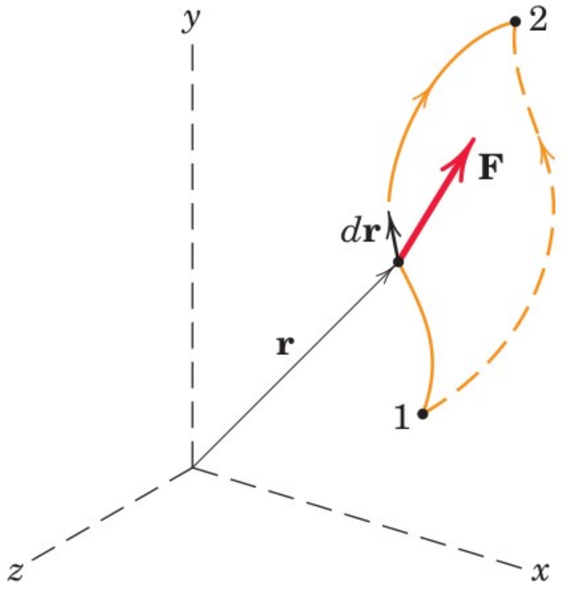
For problems where the only forces are gravitational, elastic, and nonworking constraint forces, the U' term of (29) is zero, and the energy equation becomes

$$T_1 + V_1 = T_2 + V_2 \qquad \text{or} \qquad E_1 = E_2 \,. \qquad \cdots\cdots\cdots(31)$$

where $E = T + V$ is the total mechanical energy of the particle and its attached spring. When E is constant, transfers of energy between kinetic and potential may take place as long as the total mechanical energy $T + V$ does not change. Equation (31) expresses the **law of conservation of dynamical energy**.

Conservative Force Fields

We have observed that the work done against a gravitational or an elastic force depends only on the net change of position and not on the particular path followed in reaching the new position. Forces with this characteristic are associated with **conservative force fields**, which possess an important mathematical property.



Consider a force field where the force $\boldsymbol{F}$ is a function of the coordinates. The work done by $\boldsymbol{F}$ during a displacement $d\boldsymbol{r}$ of its point of application is $dU = \boldsymbol{F} \cdot d\boldsymbol{r}$. The total work done along its path from 1 to 2 is

$$U = \int \boldsymbol{F} \cdot d\boldsymbol{r} = \int (F_x dx + F_y dy + F_z dz).$$

The integral $\int \boldsymbol{F} \cdot d\boldsymbol{r}$ is a line integral which depends, in general, on the particular path followed between any two points 1 and 2 in space.

Now, if there exists a scalar potential V such that, $\boldsymbol{F} = -\boldsymbol{\nabla V}$, ...(32)

where, $\boldsymbol{\nabla} = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$, the vector operator *del*.

Now,

$$\boldsymbol{F}\cdot d\boldsymbol{r} = -\left(\boldsymbol{i}\frac{\partial V}{\partial x} + \boldsymbol{j}\frac{\partial V}{\partial y} + \boldsymbol{k}\frac{\partial V}{\partial z}\right)\cdot(dx\boldsymbol{i} + dy\boldsymbol{j} + dz\boldsymbol{k}) = -\left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right),$$

$$\boldsymbol{F}\cdot d\boldsymbol{r} = -dV. \qquad \text{.........(33)}$$

Thus, the total work done along its path from 1 to 2 is

$$U_{1-2} = \int_1^2 -dV = -(V_2 - V_1). \qquad \text{.........(34)}$$

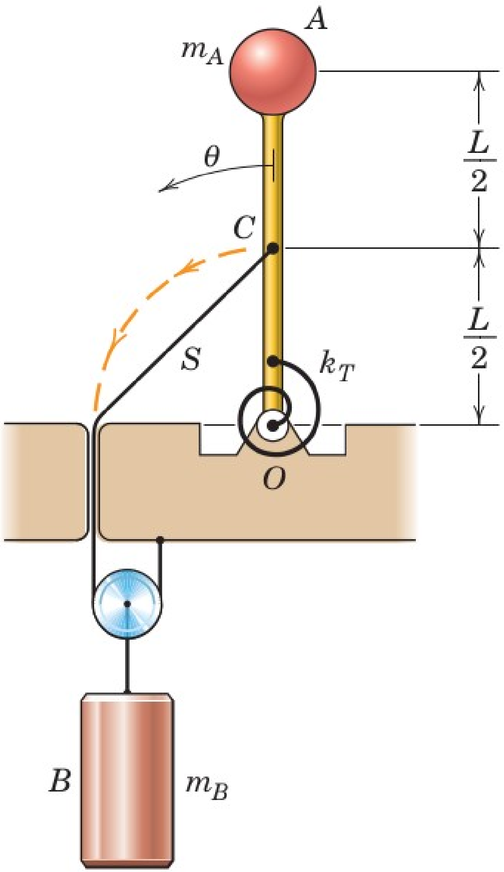
which depends only on the end points of the motion and which is thus **independent of the path followed**. The minus sign before dV is arbitrary but is chosen to agree with the customary designation of the sign of potential energy change in the gravity field of the earth.

The quantity V is known as the **potential function**, and the expression $\boldsymbol{\nabla V}$ is known as the gradient of the potential function.

When force components are derivable from a potential as described, the force is said to be **conservative**, and the work done by $\boldsymbol{F}$ between any two points is independent of the path followed.

Example 8

The system shown is released from rest with the lightweight slender bar OA in the vertical position shown. The torsional spring at O is undeflected in the initial position and exerts a restoring moment of magnitude $k_T\theta$ on the bar, where θ is the counterclockwise angular deflection of the bar. The string S is attached to point C of the bar and slips without friction through a vertical hole in the support surface. For the values $m_A = 2$ kg, $m_B = 4$ kg, $L = 0.5$ m, and $k_T = 13$ N·m/rad. Determine the speed v_A of particle A when θ reaches 90°.



The potential energy associated with the deflection of a torsional spring can be obtained by calculating the work done on the spring to deform it, thus

$$V_e = \int_0^\theta k_T \theta d\theta = \frac{1}{2} k_T \theta^2.$$

It should be noted that $v_C = v_A/2$, and further noting that the speed of cylinder $v_B = v_C/2$ at $\theta = 90°$, thus we conclude that at $\theta = 90°$, $v_B = v_A/4$.

Establishing datums at the initial altitudes of bodies A and B, and with state 1 at $\theta = 0$ and state 2 at $\theta = 90°$, we write

$$T_1 + V_1 + U'_{1\text{-}2} = T_2 + V_2,$$

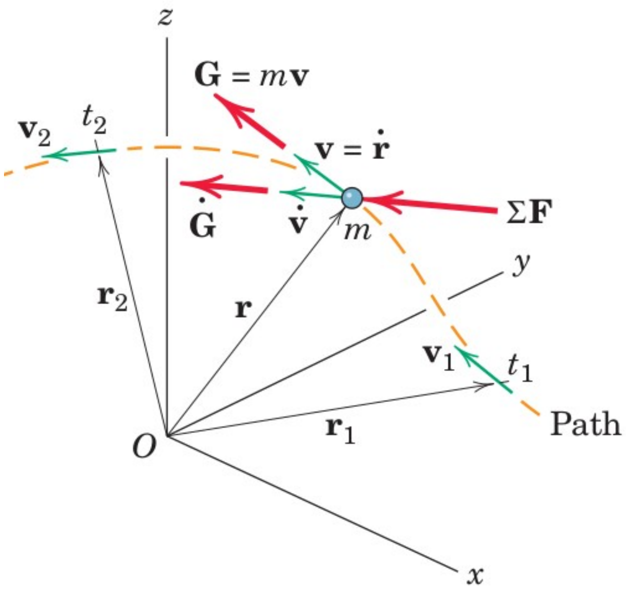
$$0 + 0 + 0 = \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2 - m_A g L - m_B g \frac{L\sqrt{2}}{4} + \frac{1}{2} k_T (\pi/2)^2,$$

Upon substituting values and solving, we get v_A at $\theta = 90°$.

Impulse and Momentum

As discussed earlier that integrating the equation of motion with respect to time leads to the equations of impulse and momentum.

These equations facilitate the solution of many problems in which the **applied forces act during extremely short periods of time** (as in impact problems) or over specified intervals of time.



Consider the general curvilinear motion in space of a particle of mass m, with a position vector $\boldsymbol{r}$. The velocity of the particle is $\boldsymbol{v} = \dot{\boldsymbol{r}}$. The resultant $\Sigma \boldsymbol{F}$ of all forces on m is in the direction of its acceleration $\dot{\boldsymbol{v}}$. We may now write the basic equation of motion for the particle as

$$\sum \boldsymbol{F} = m\dot{\boldsymbol{v}} = \frac{d}{dt}(m\boldsymbol{v}) \quad \text{or} \quad \sum \boldsymbol{F} = \dot{\boldsymbol{G}}. \qquad \cdots\cdots\cdots(35)$$

where the product of the mass and velocity is defined as the **linear momentum** $\boldsymbol{G} = m\boldsymbol{v}$ of the particle.

Equation (35) states that the resultant of all forces acting on a particle equals its time rate of change of linear momentum.

In SI the units of linear momentum mv are seen to be kg·m/s, which also equals N·s.

Equation (35) is a vector equation. Note that the direction of the resultant force coincides with the direction of the rate of change in linear momentum, which is the direction of the rate of change in velocity.

Note that (35) is valid as long as the mass m of the particle is not changing with time.

The three scalar components of (35) can be written as

$$\sum F_x = \dot{G}_x, \quad \sum F_y = \dot{G}_y, \quad \sum F_z = \dot{G}_z. \qquad \cdots\cdots\cdots(36)$$

The Linear Impulse-Momentum Principle

We can describe the effect of the resultant force $\Sigma \boldsymbol{F}$ on the linear momentum of the particle over a finite period of time simply by integrating (35) with respect to the time t. Multiplying the equation by dt gives $\Sigma \boldsymbol{F}\, dt = d\boldsymbol{G}$, and integrating from time t_1 to time t_2 , we get,

$$\int_{t_1}^{t_2} \sum \boldsymbol{F} dt = \boldsymbol{G}_2 - \boldsymbol{G}_1 = \Delta \boldsymbol{G}. \qquad \cdots\cdots\cdots(37)$$

The product of force and time is defined as the **linear impulse of the force**, and (37) states that the total linear impulse on m equals the corresponding change in linear momentum of m.

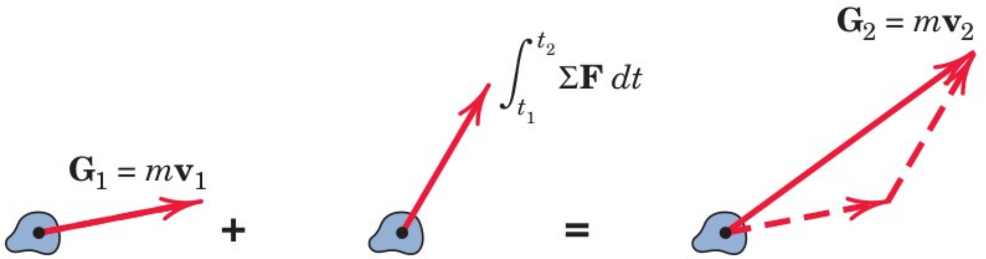
Alternatively, we may write (37) as $\boldsymbol{G}_1 + \int_{t_1}^{t_2} \sum \boldsymbol{F} dt = \boldsymbol{G}_2.$ $\cdots\cdots\cdots$(37a)

which says that the initial linear momentum of the body plus the linear impulse applied to it equals its final linear momentum.

The components of (37) are the scalar equations

$$mv_{1x} + \int_{t_1}^{t_2} \sum F_x dt = mv_{2x}, \quad mv_{1y} + \int_{t_1}^{t_2} \sum F_y dt = mv_{2y}, \qquad mv_{1z} + \int_{t_1}^{t_2} \sum F_z dt = mv_{2z}$$

We introduce the concept of the **impulse-momentum diagram**. Once the body to be analyzed has been clearly identified and isolated, we construct three drawings of the body as shown. In the first drawing the initial momentum $m\boldsymbol{v}_1$ is shown, in the second one, we show all the external linear impulses. In the final drawing, the final linear momentum $m\boldsymbol{v}_2$ is shown. The writing of the impulse- momentum equations then follows directly from these drawings, with a clear one-to-one correspondence between diagrams and equation terms.



Note that the center diagram is very much like a free-body diagram, except that the impulses of the forces appear rather than the forces themselves.

- In some cases, certain forces are very large and of short duration. Such forces are called **impulsive forces**. An example is a force of sharp impact.
- It is frequently assumed that **impulsive forces are constant over their time** of duration, so that they can be brought outside the linear-impulse integral.
- In addition, we frequently assume that **nonimpulsive forces can be neglected** in comparison with impulsive forces. An example of a nonimpulsive force is the weight of a ball during its collision with a bat-the weight of the ball is small compared with the force (which could be about several hundred times more in magnitude) exerted on the ball by the bat.
- There are cases where a force acting on a particle varies with the time in a manner determined by experimental measurements or by other approximate means. In this case a graphical or numerical integration must be performed.

Conservation of Linear Momentum

If the resultant force on a particle is zero during an interval of time then its linear momentum $\boldsymbol{G}$ remain constant and we say that the linear momentum of the particle is **conserved**.

Linear momentum may be conserved in one coordinate direction, such as x, but not necessarily in the y- or z-direction. Analysis of the impulse-momentum diagram of the particle will disclose whether the total linear impulse on the particle in a particular direction is zero. If it is, the corresponding linear momentum is unchanged (conserved) in that direction.

Consider, the motion of two particles a and b which interact during an interval of time. If the interactive forces $\boldsymbol{F}$ and $-\boldsymbol{F}$ between them are the only unbalanced forces acting on the particles during the interval, it follows that the linear impulse on particle a is the negative of the linear impulse on particle b. Therefore, from (37), the change in linear momentum of a is negative of the change in linear momentum of particle b. So we have,

$$\Delta \boldsymbol{G}_a = -\Delta \boldsymbol{G}_b \text{ or } \Delta(\boldsymbol{G}_a + \boldsymbol{G}_b) = 0.$$

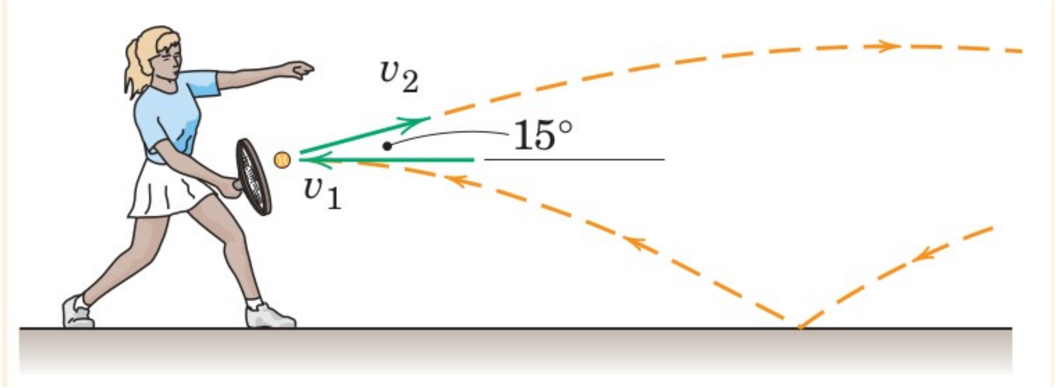
Thus, the total linear momentum $\boldsymbol{G} = \boldsymbol{G}_a + \boldsymbol{G}_b$ for the system of the two particles remains constant during the interval, and we write

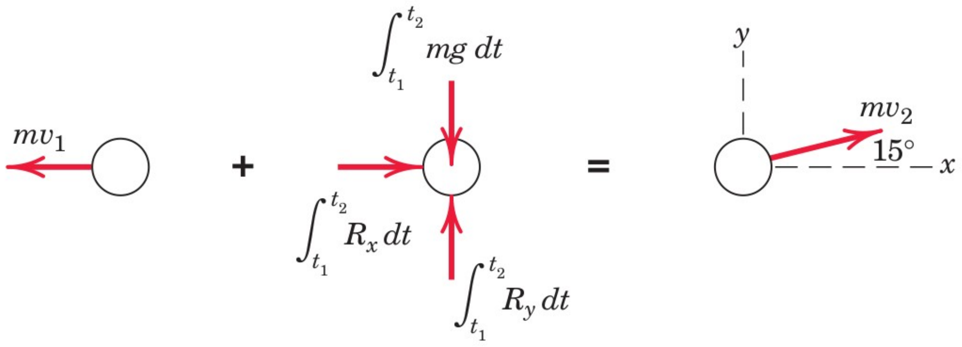
$$\Delta \boldsymbol{G} = \boldsymbol{0} \qquad \text{or} \qquad \boldsymbol{G}_1 = \boldsymbol{G}_2 . \qquad \cdots\cdots\cdots(38)$$

Equation (38) expresses the principle of **conservation of linear momentum.**

Example 9

A tennis player strikes the tennis ball with her racket when the ball is at the uppermost point of its trajectory as shown. The horizontal velocity of the ball just before impact with the racket is $v_1 = 15$ m/sec, and just after impact its velocity is $v_2 = 21$ m/sec directed at the 15° angle as shown. If the 60 g ball is in contact with the racket for 0.02 sec, determine the magnitude of the average force $\boldsymbol{R}$ exerted by the racket on the ball. Also determine the angle β made by $\boldsymbol{R}$ with the horizontal.



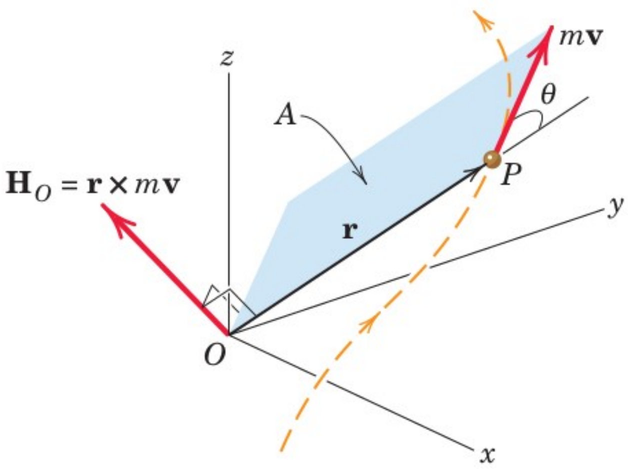


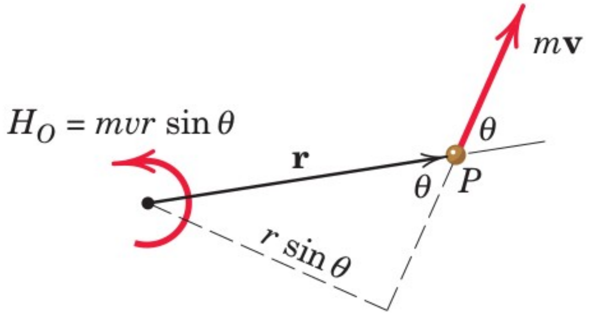
$$mv_{1x} + \int_{t_1}^{t_2} \sum F_x dt = mv_{2x},$$

$$mv_{1y} + \int_{t_1}^{t_2} \sum F_y dt = mv_{2y},$$

$R_x = 105.9$ N
$R_y = 16.89$ N
$R = 107.2$ N, angle $9.07°$

Angular momentum





A particle P of mass m moves along a curve in space. The particle is located by its position vector $\boldsymbol{r}$ as shown. The velocity of the particle is $\boldsymbol{v} = \dot{\boldsymbol{r}}$, and its linear momentum is $\boldsymbol{G} = m\boldsymbol{v}$.

The **moment of the linear momentum vector** $m\boldsymbol{v}$ about the origin O is defined as the **angular momentum** $\boldsymbol{H}_O$ of P about O and is given by the following cross-product

$$\boldsymbol{H}_0 = \boldsymbol{r} \times m\boldsymbol{v}. \qquad \cdots\cdots\cdots(39)$$

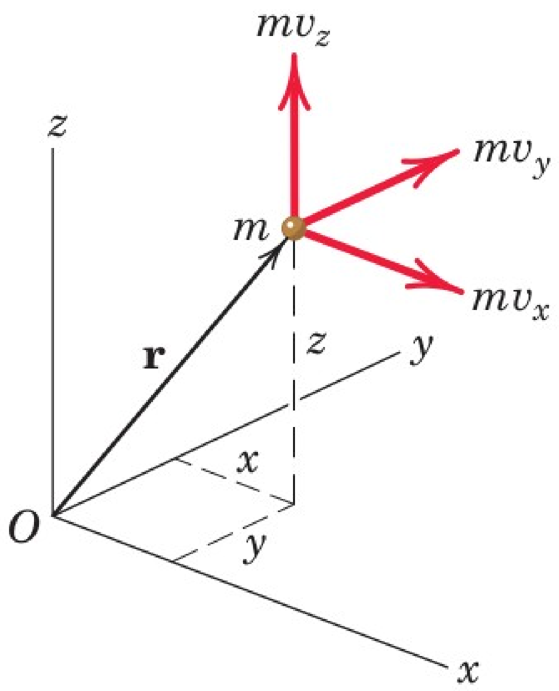
The angular momentum is a vector perpendicular to the plane A defined by $\boldsymbol{r}$ and $\boldsymbol{v}$. The sense of $\boldsymbol{H}_O$ is defined by the right-hand rule for cross products.

The scalar components of angular momentum may be obtained from the expansion

$$\boldsymbol{H}_0 = \boldsymbol{r} \times m\boldsymbol{v} = m(v_z y - v_y z)\boldsymbol{i} + m(v_x z - v_z x)\boldsymbol{j} + m(v_y x - v_x y)\boldsymbol{k}, \qquad \cdots\cdots\cdots(40)$$

so that,

$$H_x = m(v_z y - v_y z), \quad H_y = m(v_x z - v_z x), \quad H_z = m(v_y x - v_x y). \qquad \cdots\cdots\cdots(41)$$

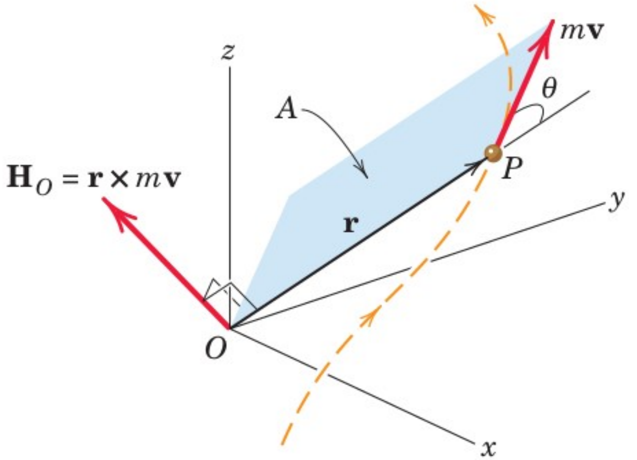


Expressions (41) for angular momentum may be visualized from the figure as the moments of three linear-momentum components components about the respective axes.

In SI units, angular momentum has the units

kg $\cdot$(m/s)$\cdot$m $=$ kg$\cdot$m^2/s $=$ N$\cdot$m$\cdot$s.

Rate of angular momentum



The moment of the forces acting on the particle P can be related to its angular momentum. If $\Sigma \boldsymbol{F}$ represents the resultant of all forces acting on the particle P, the moment $\boldsymbol{M}_O$ about the origin O is the vector cross product

$$\sum \boldsymbol{M}_0 = \boldsymbol{r} \times \sum \boldsymbol{F} = \boldsymbol{r} \times m\dot{\boldsymbol{v}}, \qquad \cdots\cdots\cdots(42)$$

Now differentiate (39) with time, using the rule for the differentiation of a cross product,

$$\dot{\boldsymbol{H}}_O = \dot{\boldsymbol{r}} \times m\boldsymbol{v} + \boldsymbol{r} \times m\dot{\boldsymbol{v}} = \cancelto{0}{\boldsymbol{v} \times m\boldsymbol{v}} + \boldsymbol{r} \times m\dot{\boldsymbol{v}},$$

Thus,

$$\sum \boldsymbol{M}_0 = \dot{\boldsymbol{H}}_0. \qquad \cdots\cdots\cdots(43)$$

Equation (43) states that **the moment about the fixed point O of all forces acting on m equals the time rate of change of angular momentum of m about O**. Scalar components of (43) are,

$$\sum M_{0_x} = \dot{H}_{0_x}, \quad \sum M_{0_y} = \dot{H}_{0_y}, \quad \sum M_{0_z} = \dot{H}_{0_z}. \qquad \cdots\cdots\cdots(44)$$

The Angular Impulse-Momentum Principle

Equation (43) gives the instantaneous relation between the moment and the time rate of change of angular momentum. To obtain the effect of the moment $\Sigma \boldsymbol{M}_O$ on the angular momentum of the particle over a finite period of time, we integrate (43) from time t_1 to time t_2. Multiplying (43) by dt gives $\Sigma \boldsymbol{M}_O\, dt = d\boldsymbol{H}_O$ and integrating,

$$\int_{t_1}^{t_2} \sum \boldsymbol{M}_0 dt = (\boldsymbol{H}_0)_2 - (\boldsymbol{H}_0)_1 = \Delta \boldsymbol{H}_0. \qquad \cdots\cdots\cdots(45)$$

where $(\boldsymbol{H}_O)_2 = \boldsymbol{r}_2 \times \mathrm{m}\boldsymbol{v}_2$ and $(\boldsymbol{H}_O)_1 = \boldsymbol{r}_1 \times m\boldsymbol{v}_1$. The product of moment and time is defined as **angular impulse** and (45) states that **the total angular impulse on *m* about the fixed point *O* equals the corresponding change in angular momentum of *m* about *O*.** (45) can also be written as,

$$(\boldsymbol{H}_0)_1 + \int_{t_1}^{t_2} \sum \boldsymbol{M}_0 dt = (\boldsymbol{H}_0)_2. \qquad \cdots\cdots\cdots(45\text{a})$$

The SI units of angular impulse are same as that of angular momentum, which are N·m·s or kg·m^2/s.

The equation of angular impulse and angular momentum is a vector equation where changes in direction as well as magnite may occur during the interval of integration. Component form of (45) in x-direction is,

$$(H_{0x})_1 + \int_{t_1}^{t_2} \sum M_{0x} dt = (H_{0x})_2, \quad \text{or,}$$
$$m(v_z y - v_y z)_1 + \int_{t_1}^{t_2} \sum M_{0x} dt = m(v_z y - v_y z)_2. \qquad \cdots\cdots\cdots(46)$$

where the subscripts 1 and 2 refer to the values of the respective quantities at times t_1 and t_2. Similar expressions can be written for the y- and z-components of the angular impulse-momentum equation.

We consider the application of angular impulse-momentum principal for plane-motion problems where moments are taken about a single axis normal to the plane of motion. In this case, the angular momentum may change magnitude and sense, but the direction of the vector remains unaltered.

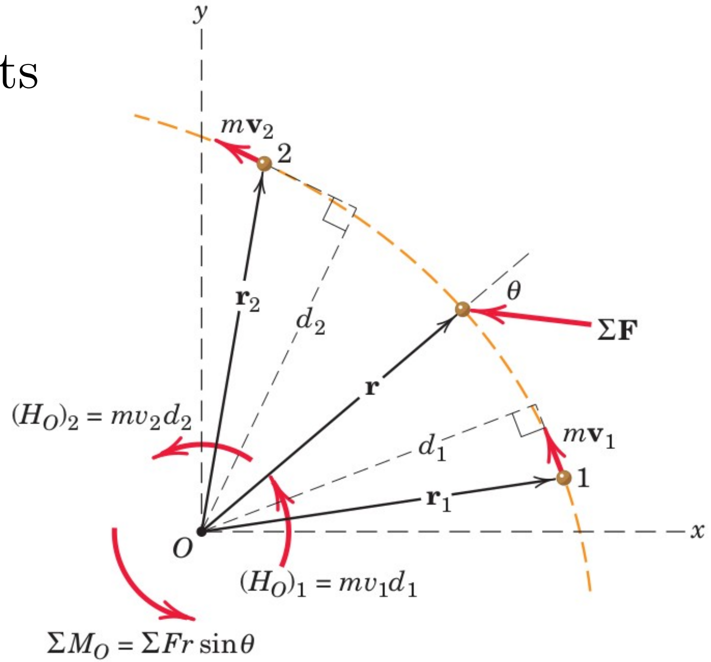
Thus, for a particle of mass m moving along a curved path in the x-y plane, the angular momenta about O at points 1 and 2 have the magnitudes $(H_O)_1 = |\boldsymbol{r}_1 \times m\boldsymbol{v}_1| = mv_1 d_1$ and $(H_O)_2 = |\boldsymbol{r}_2 \times m\boldsymbol{v}_2| = mv_2 d_2$, respectively.

The scalar form (46) applied to the motion between points 1 and 2 during the time interval t_1 to t_2 becomes,

$$(H_0)_1 + \int_{t_1}^{t_2} \sum M_0 dt = (H_0)_2,$$

$$\Rightarrow m_1 v_1 d_1 + \int_{t_1}^{t_2} \sum F r \sin\theta dt = m_2 v_2 d_2,$$

$$\cdots\cdots\cdots(47)$$



Conservation of Angular Momentum

If the resultant moment about a fixed point O of all forces acting on a particle is zero during an interval of time, then (45) requires that its angular momentum $\boldsymbol{H}_O$ about that point remain constant. In this case, the angular momentum of the particle is said to be conserved. Angular momentum may be conserved about one axis but not about another axis.

Consider the motion of two particles a and b which interact during an interval of time. If the interactive forces $\boldsymbol{F}$ and -$\boldsymbol{F}$ between them are the only unbalanced forces acting on the particles during the interval, then the moments of the equal and opposite forces about any fixed point O not on their line of action are equal and opposite. If we apply (45) to particle a and then to particle b and add the two equations, we obtain $\Delta H_a + \Delta H_b = 0$ (where all angular momenta are referred to point O). Thus, the total angular momentum for the system of the two particles remains constant during the interval, and we write

$$\Delta \boldsymbol{H}_O = \boldsymbol{0} \quad \text{or} \quad (\boldsymbol{H}_O)_1 = (\boldsymbol{H}_O)_2. \qquad \cdots\cdots\cdots(48)$$

(48) expresses the principle of **conservation of angular momentum**.

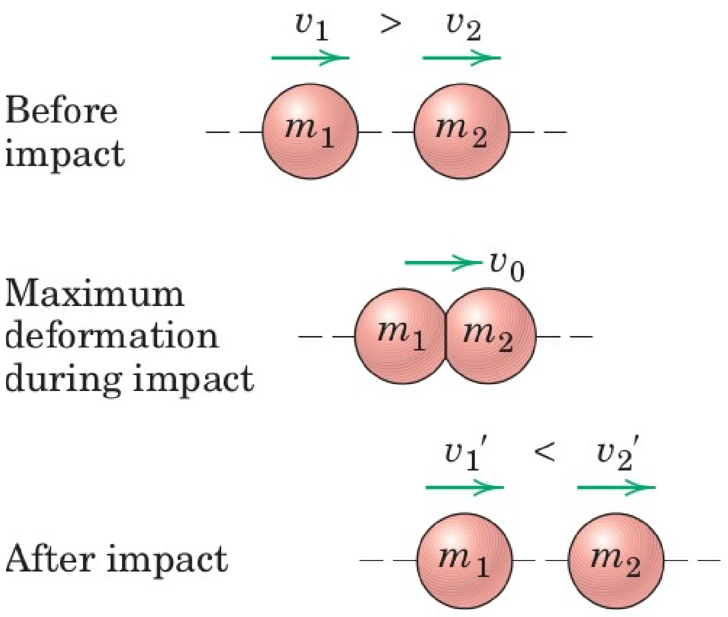
Application: Impact

- Impact refers to the **collision between two bodies** and is characterized by the generation of relatively large contact forces which act over a very short interval of time.
- The principles of impulse and momentum have important use in describing the impact behaviour.
- It is important to realize that an impact is a very complex event involving material deformation and recovery and the generation of heat and sound. Small changes in the impact conditions may cause large changes in the impact process and thus in the conditions immediately following the impact.

Direct Central Impact:

Consider the collinear motion of two spheres of masses m_1 and m_2, traveling with velocities v_1 and v_2. If $v_1 > v_2$, collision occurs with the contact forces directed along the line of centers. This condition is called **direct central impact**.

Following initial contact, a short period of increasing deformation takes place until the contact area between the spheres ceases to increase. At this instant, both spheres, are moving with the same velocity v_0.



During the remainder of contact, a period of restoration occurs during which the contact area decreases to zero. In the final condition the spheres now have new velocities v'_1 and v'_2, where v'_1 must be less than v'_2. All velocities are arbitrarily assumed positive to the right, so that with this scalar notation a velocity to the left would carry a negative sign.

If the impact is not overly severe and if the spheres are highly elastic, they will regain their original shape following the restoration.

With a more severe impact and with less elastic bodies, a permanent deformation may result.

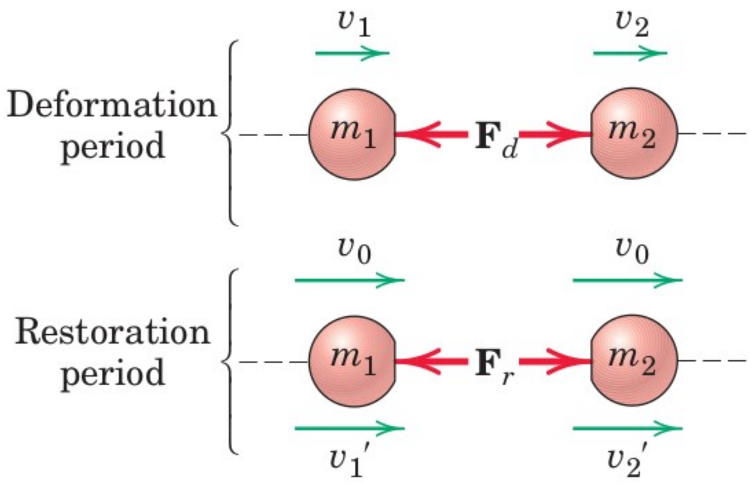
Because the contact forces are equal and opposite during impact, the linear momentum of the system remains unchanged. Thus, we apply the law of conservation of linear momentum and write

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2. \qquad \cdots\cdots\cdots(49)$$

While writing (49) we assume that any forces acting on the spheres during impact, other than the large internal forces of contact, are relatively small and produce negligible impulses compared with the impulse associated with each internal impact force.

In addition, we assume that no appreciable change in the positions of the mass centers occurs during the short duration of the impact.

For given masses and initial conditions, the momentum equation contains two unknowns, v'_1 and v'_2. Thus, need an additional relationship to find the final velocities. This relationship must reflect the capacity of the contacting bodies to recover from the impact and can be expressed by the ratio e of the magnitude of the restoration impulse to the magnitude of the deformation impulse. This ratio is called **the coefficient of restitution**.



Let $\boldsymbol{F}_r$ and $\boldsymbol{F}_d$ represent the magnitudes of the contact forces during the restoration and deformation periods, respectively. For particle 1 the definition of e together with the impulse-momentum equation give

$$e = \frac{\int_{t_0}^{t} F_r \, dt}{\int_{0}^{t_0} F_d \, dt} = \frac{-m_1(v_1' - v_0)}{-m_1(v_0 - v_1)} = \frac{(v_0 - v_1')}{(v_1 - v_0)}, \qquad \cdots\cdots\cdots(50)$$

Similarly for particle 2,

$$e = \frac{\int_{t_0}^{t} F_r\, dt}{\int_0^{t_0} F_d\, dt} = \frac{m_2(v_2' - v_0)}{m_2(v_0 - v_2)} = \frac{(v_2' - v_0)}{(v_0 - v_2)}, \qquad \cdots\cdots\cdots(51)$$

In these equations the change of momentum (and therefore Δv) is in the same direction as the impulse (and thus the force).

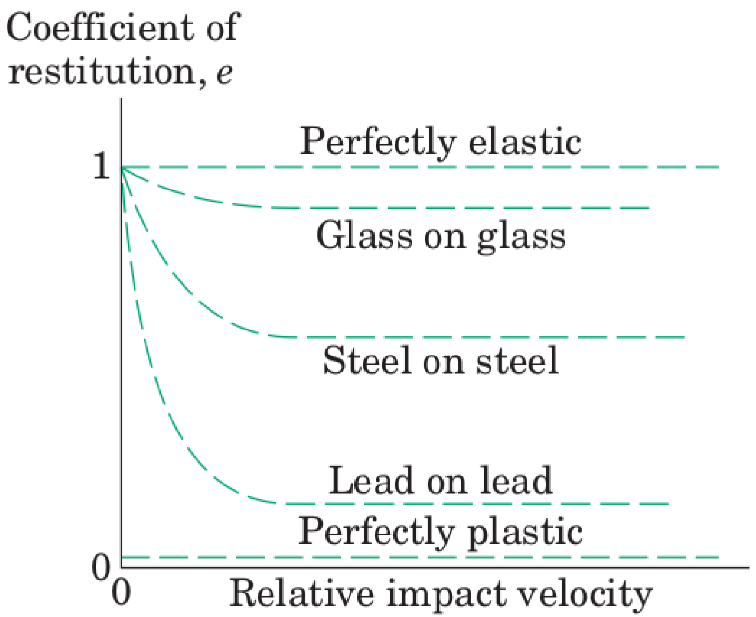
The time for the deformation is taken as t_0 and the total time of contact is t. Eliminating v_0 between the two expressions for e gives us

$$e = \frac{(v_2' - v_1')}{(v_1 - v_2)} = \frac{|\text{relative velocity of seperation}|}{|\text{relative velocity of approach}|}, \qquad \cdots\cdots\cdots(53)$$

If the two initial velocities v_1 and v_2 and the coefficient of restitution e are known, then (49) and (53) give us two equations in the two unknown final velocities v'_1 and v'_2.

Energy loss during impact:

Energy loss during the impact may be calculated by subtracting the kinetic energy of the system just after impact from that just before impact. Energy is lost through the **generation of heat** during the **localized inelastic deformation** of the material, through the **generation and dissipation of elastic stress waves** within the bodies, and through the **generation of sound energy**.

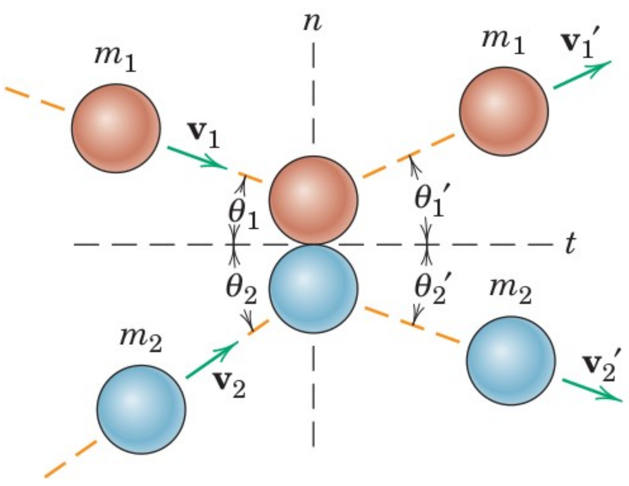


According to this classical theory of impact,

- the value $e = 1$ means that the capacity of the two particles to recover equals their tendency to deform. This condition is one of elastic impact with no energy loss.
- the value $e = 0$, on the other hand, describes inelastic or plastic impact where the particles cling together after collision and the loss of energy is a maximum.
- all impact conditions lie somewhere between these two extremes.

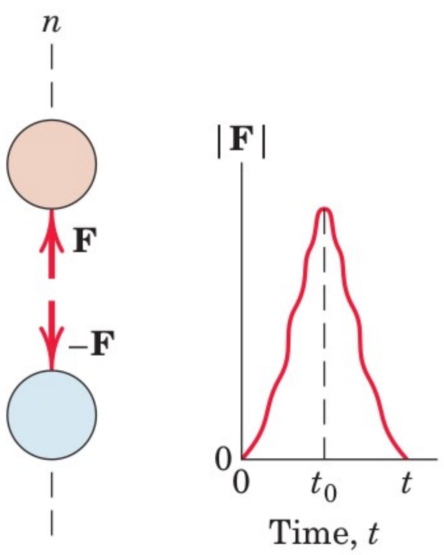
Oblique Central Impact:

Now we discuss the case where the initial and final velocities are not parallel.



Spherical particles of mass m_1 and m_2 have initial velocities v_1 and v_2 in the same plane and approach each other on a collision course. The directions of the velocity vectors are measured from the direction tangent to the contacting surfaces.

Thus, the initial velocity components along the t- and n-axes are $(v_1)_n = -v_1 \sin\theta_1$, $(v_1)_t = v_1 \cos\theta_1$, $(v_2)_n = v_2 \sin\theta_2$, and $(v_2)_t = v_2 \cos\theta_2$.



The final rebound conditions are also shown. The impact forces are $\boldsymbol{F}$ and -$\boldsymbol{F}$. They vary from zero to their peak value during the deformation portion of the impact and back again to zero during the restoration period.

For given initial conditions of m_1, m_2, $(v_1)_n$, $(v_1)_t$, $(v_2)_n$, and $(v_2)_t$, there will be four unknowns, namely, $(v'_1)_n$, $(v'_1)_t$, $(v'_2)_n$, and $(v'_2)_t$. The four equations are obtained as follows:

(1) Momentum of the system is conserved in the n-direction. This gives

$$m_1(v_1)_n + m_2(v_2)_n = m_1(v_1')_n + m_2(v_2')_n.$$

(2) and (3) The momentum for each particle is conserved in the t-direction since there is no impulse on either particle in the t-direction. Thus,

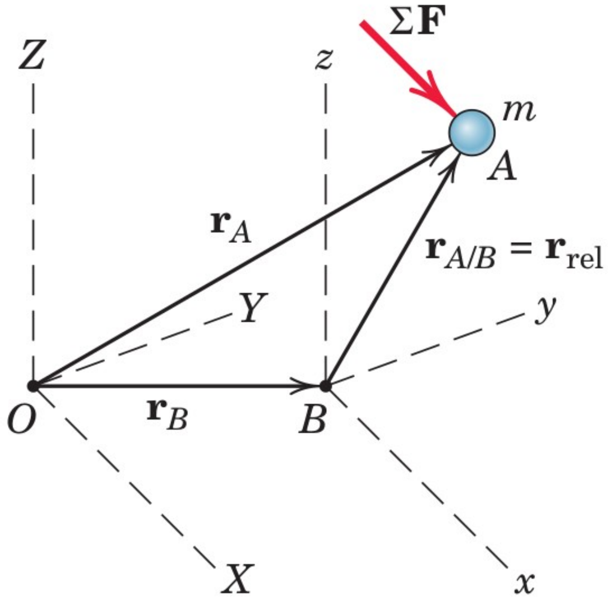
$$m_1(v_1)_t = m_1(v_1')_t, \quad m_2(v_2)_t = m_2(v_2')_t.$$

(4) The coefficient of restitution, as in the case of direct central impact, is the positive ratio of the recovery impulse to the deformation impulse. Equation (53) applies, then, to the velocity components in the n-direction. Thus,

$$e = \frac{|\text{relative velocity of seperation}|}{|\text{relative velocity of approach}|} = \frac{(v_2')_n - (v_1')_n}{(v_1)_n - (v_2)_n}. \qquad \cdots\cdots\cdots(54)$$

Once the four final velocity components are found, the angles θ'_1 and θ'_2 may be easily determined.

Application: Relative motion



We now consider a particle A of mass m, whose motion is observed from a set of axes x-y-z which translate with respect to a fixed reference frame X-Y-Z.

The acceleration of the origin B of x-y-z is $\boldsymbol{a}_B$. The acceleration of A as observed from or relative to x-y-z is $\boldsymbol{a}_{\text{rel}} = \boldsymbol{a}_{A/B} = \ddot{\boldsymbol{r}}_{A/B}$, and by the relative-motion principle, the absolute acceleration of A is

$$\boldsymbol{a}_A = \boldsymbol{a}_B + \boldsymbol{a}_{A/B} = \boldsymbol{a}_B + \boldsymbol{a}_{\text{rel}}.$$

Thus, Newton's second law $\Sigma \boldsymbol{F} = m\boldsymbol{a}_A$ becomes

$$\sum \boldsymbol{F} = m\left(\boldsymbol{a}_B + \boldsymbol{a}_{\text{rel}}\right). \qquad \cdots\cdots\cdots(66)$$

$\Sigma \boldsymbol{F}$ can be identified by a complete free-body diagram. This diagram will appear the same to an observer in x-y-z or to one in X-Y-Z as long as only the real forces acting on the particle are represented. Then we can conclude that Newton's second law does not hold with respect to an accelerating system, as $\Sigma \boldsymbol{F} \neq m\boldsymbol{a}_{\text{rel}}$.

Reg. D'Alembert's Principle:

Opinion differs concerning the original interpretation of D'Alembert's principle, but the principle in the form in which it is generally known is regarded in this book as being mainly of historical interest. It evolved when understanding and experience with dynamics were extremely limited and was a means of explaining dynamics in terms of the principles of statics, which were more fully understood. This excuse for using an artificial situation to describe a real one is no longer justified, as today a wealth of knowledge and experience with dynamics strongly supports the direct approach of thinking in terms of dynamics rather than statics. It is somewhat difficult to understand the long persistence in the acceptance of statics as a way of understanding dynamics, particularly in view of the continued search for the understanding and description of physical phenomena in their most direct form.

Engineering Mechanics:Dynamics
By J. L. Meriam

†In the 1700s, Jean-Baptiste le Rond d'Alembert expressed Newton's second law as $\Sigma \mathbf{F} - m\mathbf{a} = 0$ so he could solve dynamics problems using the principles of statics. The $-ma$ term has been called a fictitious *inertial force*, but it is important for you to realize that there is no such thing as inertial forces (or centrifugal forces that "push" you outward when going around a curve). D'Alembert's principle (also called dynamic equilibrium) is seldom used in modern engineering.

Vector Mechanics for Engineers:
Statics and Dynamics
By Beer and Johnston

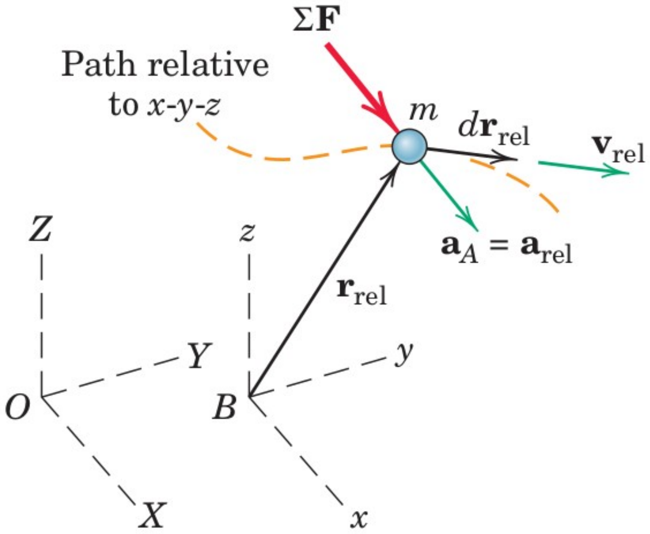
Constant-Velocity, Nonrotating Systems:

In discussing particle motion relative to moving reference systems, we should note the special case where the reference system has a constant velocity and no rotation. If the x-y-z axes have a constant velocity, then $\boldsymbol{a}_B = 0$ and the acceleration of the particle is $\boldsymbol{a}_{\text{A}} = \boldsymbol{a}_{\text{rel}}$. Therefore, we write (66) as

$$\sum \boldsymbol{F} = m\boldsymbol{a}_{\text{rel}}, \qquad \cdots\cdots\cdots(67)$$

which tells us that Newton's second law holds for measurements made in a system moving with a constant velocity. Such a system is known as **an inertial system** or as **a Newtonian frame of reference**. Observers in the moving system and in the fixed system will also agree on the designation of the resultant force acting on the particle from their identical free-body diagrams, provided they avoid the use of any so-called "inertia forces."

We will also check the validity of the work-energy equation and the impulse-momentum equation relative to a constant-velocity, nonrotating system.



Again, we take the x-y-z axes to be moving with a constant velocity $\boldsymbol{v}_B = \dot{\boldsymbol{r}}_B$ relative to the fixed axes X-Y-Z. The path of the particle A relative to x-y-z is governed by $\boldsymbol{r}_{\text{rel}}$.

The work done by $\Sigma\boldsymbol{F}$ relative to x-y-z is

$$dU_{\text{rel}} = \Sigma\boldsymbol{F}\cdot d\boldsymbol{r}_{\text{rel}}\text{. But } \Sigma\boldsymbol{F} = m\boldsymbol{a}_A = m\boldsymbol{a}_{\text{rel}} \text{ since } \boldsymbol{a}_B = 0.$$

Also $\boldsymbol{a}_{\text{rel}}\cdot d\boldsymbol{r}_{\text{rel}} = \boldsymbol{v}_{\text{rel}} \cdot d\boldsymbol{v}_{\text{rel}}$ (similar to $\boldsymbol{a}_t$ ds $= vdv$ for curvilinear motion). Thus, we have

$$dU_{\text{rel}} = m\boldsymbol{a}_{\text{rel}} \cdot d\boldsymbol{r}_{\text{rel}} = mv_{\text{rel}}dv_{\text{rel}} = d\left(\tfrac{1}{2}mv_{\text{rel}}^2\right).$$

We define the kinetic energy relative to x-y-z as

$$T_{\text{rel}} = \tfrac{1}{2} m v_{\text{rel}}^2,$$

so now we have,

$$dU_{\text{rel}} = dT_{\text{rel}} \qquad \text{or} \qquad \Delta U_{\text{rel}} = \Delta T_{\text{rel}}. \qquad \text{.........(68)}$$

which shows that the work-energy equation holds for measurements made relative to a constant-velocity, nonrotating system.

Relative to x-y-z, the impulse on the particle during time dt is

$$\Sigma \boldsymbol{F}\, dt = m\boldsymbol{a}_A\, dt = m\boldsymbol{a}_{\text{rel}}\, dt.$$

But $m\boldsymbol{a}_{\text{rel}}\, dt = m\, d\boldsymbol{v}_{\text{rel}} = d(m\boldsymbol{v}_{\text{rel}})$ so $\Sigma \boldsymbol{F}\, dt = d(m\boldsymbol{v}_{\text{rel}})$.

We define the linear momentum of the particle relative to x-y-z as $\boldsymbol{G}_{\text{rel}} = m\boldsymbol{v}_{\text{rel}}$, which gives us $\Sigma \boldsymbol{F}\, dt = d\boldsymbol{G}_{\text{rel}}$. Dividing by dt and integrating give,

$$\sum \boldsymbol{F} = \dot{\boldsymbol{G}}_{\text{rel}} \quad \text{and} \quad \int \sum \boldsymbol{F} dt = \Delta \boldsymbol{G}_{\text{rel}}. \qquad \cdots\cdots\cdots(69)$$

Thus, the impulse-momentum equations for a fixed reference system also hold for measurements made relative to a constant-velocity, nonrotating system.

Finally, we define the relative angular momentum of the particle about a point in x-y-z, such as the origin B, as the moment of the relative linear momentum.

Thus, $(\boldsymbol{H}_B)_{\text{rel}} = \boldsymbol{r}_{\text{rel}} \times \boldsymbol{G}_{\text{rel}}$. The time derivative gives $(\dot{\boldsymbol{H}}_B)_{\text{rel}} = \dot{\boldsymbol{r}}_{\text{rel}} \times \boldsymbol{G}_{\text{rel}} + \boldsymbol{r}_{\text{rel}} \times \dot{\boldsymbol{G}}_{\text{rel}}$. The first term become 0, and the second term becomes $\boldsymbol{r}_{\text{rel}} \times \Sigma\boldsymbol{F} = \Sigma\boldsymbol{M}_B$, the sum of the moments about B of all forces on m. Thus, we have

$$\sum \boldsymbol{M}_B = (\dot{\boldsymbol{H}}_B)_{\text{rel}}. \qquad \cdots\cdots\cdots(70)$$

Equation (70) shows that the moment-angular momentum relation holds with respect to a constant-velocity, nonrotating system.

Although the work-energy and impulse-momentum equations hold relative to a system translating with a constant velocity, the individual expressions for work, kinetic energy, and momentum differ between the fixed and the moving systems. Thus,

$$\left(dU = \sum \boldsymbol{F} \cdot d\boldsymbol{r}_A\right) \neq \left(dU_{\text{rel}} = \sum \boldsymbol{F} \cdot d\boldsymbol{r}_{\text{rel}}\right)$$

$$\left(T = \frac{1}{2}mv_A^2\right) \neq \left(T_{\text{rel}} = \frac{1}{2}mv_{\text{rel}}^2\right)$$

$$(\boldsymbol{G} = m\boldsymbol{v}_A) \neq (\boldsymbol{G}_{\text{rel}} = mv_{\text{rel}})$$

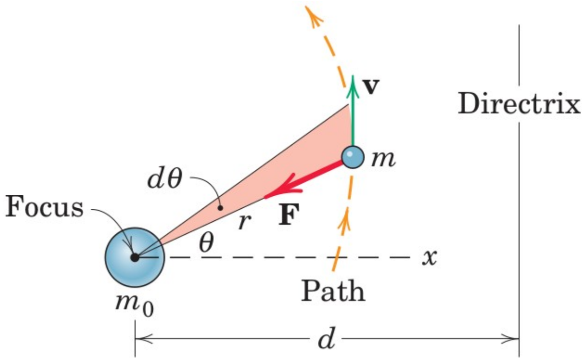
Application: Central force motion

Motion of a particle under the influence of a force directed toward a fixed center of attraction is called **central-force motion**. The most common example of central-force motion is the **orbital movement of planets and satellites**. The laws which govern this motion were deduced from observation of the motions of the planets by J. Kepler (1571–1630). An understanding of such motion is required to design high-altitude rockets, earth satellites, and space vehicles.

Consider a particle of mass m moving under the action of the central gravitational attraction

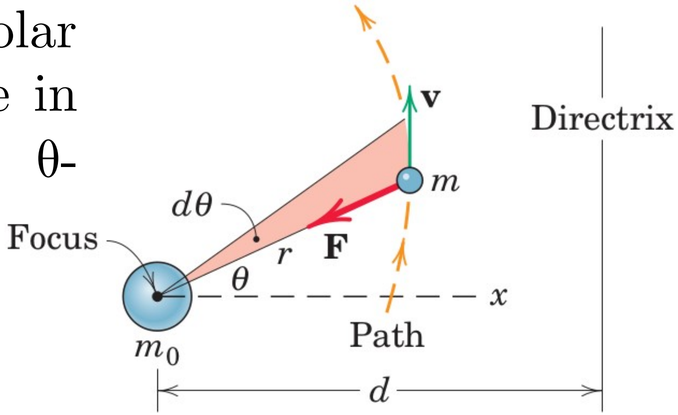
$$F = G\frac{mm_0}{r^2},$$

where m_0 is the mass of the attracting body, which is assumed to be fixed, G is the universal gravitational constant, and r is the distance between the centers of the masses.



The particle of mass m could represent the earth moving about the sun, the moon moving about the earth, or a satellite in its orbital motion about the earth above the atmosphere.

The most convenient coordinate system to use is polar coordinates in the plane of motion since $\boldsymbol{F}$ will always be in the negative r-direction and there is no force in the θ-direction. Thus, we can write,



$$-G\frac{mm_0}{r^2} = m\boldsymbol{a}_n = m(\ddot{r} - r\dot{\theta}^2),$$
$$0 = m\boldsymbol{a}_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}). \quad \cdots\cdots\cdots(55)$$

By multiplying second equation of (55) with r/m, it can be rewritten as

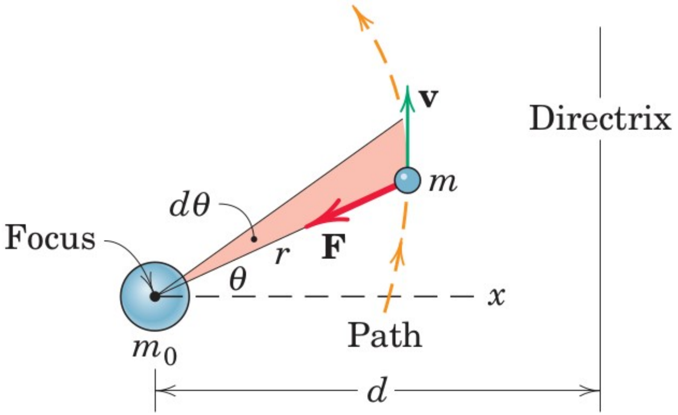
$$\frac{d}{dt}(r^2\dot{\theta}) = 0 \quad \text{or} \quad r^2\dot{\theta} = h \quad \text{(a constant)}. \qquad \cdots\cdots\cdots(56)$$

The physical significance of (56) become clear when we note that the angular momentum $\boldsymbol{r} \times m\boldsymbol{v}$ of m about m_0 has the magnitude $mr^2\dot{\theta}$. Thus, (56) merely states that **the angular momentum of m about m_0 remains constant (is conserved)**. This statement is easily deduced from (48), which shows that the angular momentum $\boldsymbol{H}_O$ is conserved if there is no moment acting on the particle about a fixed point O.

We observe that during time dt, the radius vector sweeps out an area, shaded in figure, equal to

$dA = (\frac{1}{2}\ r)(rd\theta)$. Therefore, the rate at which area is swept by the radius vector is $\dot{A} = 1/2r^2\dot{\theta}$,

which is constant according (56). This conclusion is expressed in **Kepler's second law** of planetary motion, which states that **the areas swept through in equal times are equal.**



The shape of the path followed by m may be obtained by solving the first of (55), and (56) and eliminating the time t.

To that end substitute $r = 1/u$. Thus, $\dot{r} = -(1/u^2)\dot{u}$, which from (56) becomes $\dot{r} = -h(\dot{u}/\dot{\theta})$ or $\dot{r} = -h(du/d\theta)$. The second time derivative is $\ddot{r} = -h(d^2u/d\theta^2)\dot{\theta}$, which by combining with (56), become $\ddot{r} = -h^2u^2(d^2u/d\theta^2)$.

Substitution into the first of (55) now gives,

$$-Gm_0u^2 = -h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}h^2u^4, \quad \text{or} \quad \frac{d^2u}{d\theta^2} + u = \frac{Gm_0}{h^2}, \qquad \cdots\cdots\cdots(57)$$

which is a nonhomogeneous linear differential equation.

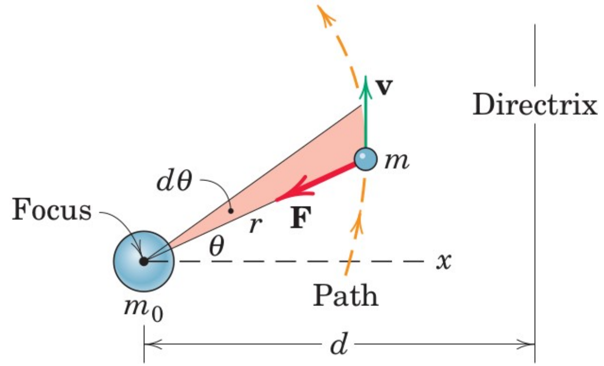
The solution of (57) may be verified by direct substitution and is

$$u = \frac{1}{r} = C\cos(\theta + \delta) + \frac{Gm_0}{h^2}, \qquad \cdots\cdots\cdots(58)$$

where C and δ are the two integration constants. The phase angle δ may be eliminated by choosing the x-axis so that r is a minimum when $\theta = 0$. Thus,

$$\frac{1}{r} = C\cos\theta + \frac{Gm_0}{h^2}. \qquad \cdots\cdots\cdots(59)$$

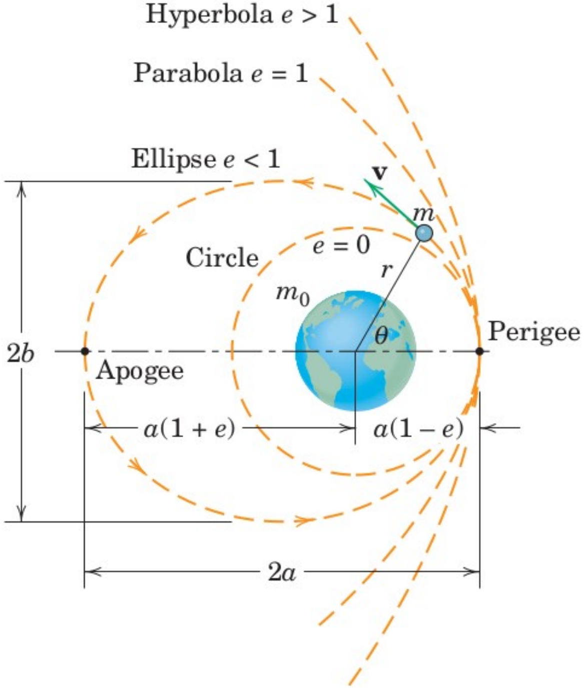
The interpretation of (59) requires a knowledge of the equations for conic sections. A conic section is formed by the locus of a point which moves so that the ratio e of its distance from a point (focus) to a line (directrix) is constant. Thus, $e = r/(d - r\cos\theta)$, which may be rewritten as



$$\frac{1}{r} = \frac{1}{d}\cos\theta + \frac{1}{ed}. \qquad \cdots\cdots\cdots(60)$$

It can be observed that (59) and (60) have same forms. Thus, we see that the motion of m is along a conic section with $d = 1/C$ and $ed = h^2/(Gm_0)$, or

$$e = h^2C/Gm_0. \qquad \cdots\cdots\cdots(61)$$



The three cases to be investigated correspond to $e < 1$ (ellipse), $e = 1$ (parabola), and $e > 1$ (hyperbola). The trajectory for each of these cases is shown in figure.