Cauchy-elastic materials

An alternate form of the elastic constitutive relation is terms of first PK stress $\boldsymbol{P}$ as,

$$\boldsymbol{P} = J^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T} = J^{-1}\mathfrak{g}(\boldsymbol{F})\boldsymbol{F}^{-T} = \mathfrak{G}(\boldsymbol{F}), \quad \cdots\cdots(3)$$

where $\mathfrak{G}$ is another tensor valued function corresponding to first PK stress tensor.

It can be shown that $\mathfrak{G}$ must follow the following constraint for relation (3) to be objective,

$$\boldsymbol{Q}\mathfrak{G}(\boldsymbol{F}) = \mathfrak{G}(\boldsymbol{Q}\boldsymbol{F}).$$

Another from of the constitutive relationship can be written in terms of second PK stress $\boldsymbol{S}$, as

$$\boldsymbol{S} = J\boldsymbol{F}^{-T}\boldsymbol{\sigma}\boldsymbol{F}^{-T} = \det\boldsymbol{U}\,\boldsymbol{U}^{-1}\mathfrak{g}(\boldsymbol{U})\boldsymbol{U}^{-1}.$$

Recall that $\boldsymbol{U}$ is the unique square root of the right Cauchy-Green tensor $\boldsymbol{C}$, hence we may write $\boldsymbol{C}^{1/2}$ in the place of $\boldsymbol{U}$ and we can define,

$$\boldsymbol{S} = \mathfrak{H}(\boldsymbol{C}). \quad \cdots\cdots(4)$$

where $\mathfrak{H}$ is a tensor valued function of $\boldsymbol{C}$. Since the reference configuration is unaffected by superimposed rigid-body motions, it can be shown that $\boldsymbol{S}^* = \boldsymbol{S}$ and $\boldsymbol{C}^* = \boldsymbol{C}$.

Hence, it can be concluded that the relation (4) is independent of the observer.

Exercise: Check if

$$\boldsymbol{\sigma} = \mathfrak{J}(\boldsymbol{E}).$$

follows the principle of material objectivity (or material frame indifference).

Isotropic Cauchy-elastic material

Let us assume now that,

$$\boldsymbol{\sigma} = \mathfrak{h}(\boldsymbol{F}), \qquad \text{.....(5)}$$

We can show that the restriction on $\mathfrak{h}$ because of material objectivity is the following,

$$\boldsymbol{Q}\mathfrak{h}(\boldsymbol{b})\boldsymbol{Q}^T = \mathfrak{g}(\boldsymbol{Q}\boldsymbol{b}\boldsymbol{Q}^T). \qquad \text{.....(6)}$$

A specific elastic material which may be described by the constitutive equation in the form (5), with property (6), is said to be isotropic.

A tensor-valued function such as $\mathfrak{h}(\boldsymbol{b})$ is said to be isotropic if it satisfies relations of type (6). and it is a tensor-valued isotropic tensor function of one variable $\boldsymbol{b}$.

From the physical point of view the condition of isotropy is expressed by the property that the material exhibits no preferred directions.

In fact, the stress response of an isotropic elastic material is not affected by the choice of the reference configuration.

The isotropic tensor function $\mathfrak{h}(\boldsymbol{b})$ satisfying (6) may be represented in the most general form as,

$$\boldsymbol{\sigma} = \mathfrak{h}(\boldsymbol{b}) = \alpha_0 \boldsymbol{I} + \alpha_1 \boldsymbol{b} + \alpha_2 \boldsymbol{b}^2, \alpha_a = \alpha_a[I_1(\boldsymbol{b}), I_2(\boldsymbol{b}), I_3(\boldsymbol{b})]. \quad \cdots\cdots\cdots\cdots(7)$$

Here, α_a = 0,1,2, are three scalar functions called response coefficients or material functions. Hence, in general, for an isotropic material only three parameters are needed in order to describe the stress state. The scalar functions α_a depend on the three invariants of tensor $\boldsymbol{b}$ and therefore on the current deformation state. These invariants are defined as,

$$I_1(\boldsymbol{b}) = \mathrm{tr}\boldsymbol{b} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2,$$

$$I_2(\boldsymbol{b}) = \frac{1}{2}\left[(\mathrm{tr}\boldsymbol{b})^2 - \mathrm{tr}\left(\boldsymbol{b}^2\right)\right] = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2,$$

$$I_3(\boldsymbol{b}) = \det\boldsymbol{b} = J^2 = \lambda_1^2\lambda_2^2\lambda_3^2,$$

where λ_a^2 are the three eigenvalues of the symmetric spatial tensor $\boldsymbol{b}$.

Alternate form of (7) may be obtained by using Cayley-Hamilton theorem as follows,

$$\boldsymbol{\sigma} = \mathfrak{h}(\boldsymbol{b}) = \beta_0 \boldsymbol{I} + \beta_1 \boldsymbol{b} + \beta_{-1} \boldsymbol{b}^{-1}, \beta_a = \beta_a[I_1(\boldsymbol{b}), I_2(\boldsymbol{b}), I_3(\boldsymbol{b})]. \quad \cdots\cdots\cdots\cdots(8)$$

where β_a = 0,1, -1, are $\beta_0 = \alpha_0 - I_2\alpha_2$, $\beta_1 = \alpha_1 + I_1\alpha_2$, and $\beta_{-1} = I_3\alpha_2$.

Hyperelastic material

- The nonlinear constitutive theory is suitable to describe a wide variety of physical phenomena in which the strains may be large, i.e. finite.
- We will discuss the so-called phenomenological approach, describing the macroscopic nature of materials as continua. The phenomenological approach is mainly concerned with fitting mathematical equations to experimental data and is particularly successful in solid mechanics. However, phenomenological modeling is not capable of relating the mechanism of deformation to the underlying physical (microscopic) structure of the material.
- For the case of a (hyper)elastic material the nonlinear constitutive theory is called finite (hyper)elasticity theory or just finite (hyper)elasticity.
- A so-called hyperelastic material (or a Green-elastic material) postulates the existence of a Helmholtz free-energy function Ψ , which is defined per unit reference volume.
- When $\Psi=\Psi(\boldsymbol{F})$ is solely a function of $\boldsymbol{F}$ or some strain tensor, the Helmholtz free-energy function is referred to as the strain-energy function or stored-energy function.
- We will restrict ourselves to homogeneous materials.

A hyperelastic material is defined as a subclass of an elastic material, whose response function has the following form,

$$\boldsymbol{P} = \mathfrak{G}(\boldsymbol{F}) = \frac{\partial \Psi(\boldsymbol{F})}{\partial \boldsymbol{F}}, \text{ or} \qquad \text{.....(9)}$$

$$\boldsymbol{\sigma} = \mathfrak{g}(\boldsymbol{F}) = J^{-1}\frac{\partial \Psi(\boldsymbol{F})}{\partial \boldsymbol{F}}\boldsymbol{F}^T = J^{-1}\boldsymbol{F}\left(\frac{\partial \Psi(\boldsymbol{F})}{\partial \boldsymbol{F}}\right)^T. \qquad (\because \boldsymbol{\sigma} = J^{-1}\boldsymbol{P}\boldsymbol{F}^T = \boldsymbol{\sigma}^T) \qquad \text{.....(10)}$$

As stress response of hyperelastic materials is derived from a given scalar-valued energy function, it implies that hyperelasticity has a conservative structure, i.e., there is no internal dissipation.

Above relations can also be derived directly from the Clausius-Planck inequality, which turns to an equality when there is no internal dissipation. Thus,

$$\mathcal{D}_{\text{int}} = \boldsymbol{P} : \dot{\boldsymbol{F}} - \dot{\Psi} = \left(\boldsymbol{P} - \frac{\partial \Psi}{\partial \boldsymbol{F}}\right) : \dot{\boldsymbol{F}} = 0 \qquad \text{.....(11)}$$

As $\boldsymbol{F}$ and hence $\dot{\boldsymbol{F}}$ can be chosen arbitrarily, the expressions in parentheses must be zero. Therefore, as a consequence of the 2$^{\text{nd}}$ law of thermodynamics, the physical expression (9) holds.

For convenience, we require that the strain-energy function vanishes in the reference configuration, i.e. where $\boldsymbol{F} = \boldsymbol{I}$, hence

$$\Psi = \Psi(\boldsymbol{I}) = 0.$$

Strain energy function Ψ increases with deformation, hence, $\Psi = \Psi(\boldsymbol{F}) \geq 0$.
Thus,

$$\Psi(\boldsymbol{F}) = 0 \quad \text{as} \quad \boldsymbol{F} = \boldsymbol{I}$$

$$\Psi(\boldsymbol{F}) \to +\infty \quad \text{as} \quad \det \boldsymbol{F} \to \infty$$

$$\Psi(\boldsymbol{F}) \to +\infty \quad \text{as} \quad \det \boldsymbol{F} = 0^{+}$$

Following the requirements of material objectivity, it can be shown that strain energy function is independent of rotation and only depends upon stretch part of $\boldsymbol{F}$. Thus, equivalent forms of strain energy functions are following:

$$\Psi(\boldsymbol{F}) = \Psi(\boldsymbol{U}) = \Psi(\boldsymbol{C}) = \Psi(\boldsymbol{E}).$$

Using these forms of strain energy function, alternate forms of constitutive equations are following:

$$\boldsymbol{\sigma} = J^{-1}\boldsymbol{F}\frac{\partial \Psi(\boldsymbol{F})}{\partial \boldsymbol{F}}\boldsymbol{F}^{T}, \quad \boldsymbol{P} = 2\boldsymbol{F}\frac{\partial \Psi(\boldsymbol{C})}{\partial \boldsymbol{C}}, \quad \boldsymbol{S} = 2\frac{\partial \Psi(\boldsymbol{C})}{\partial \boldsymbol{C}} = \frac{\partial \Psi(\boldsymbol{E})}{\partial \boldsymbol{E}}.$$

Isotropic Hyperelastic material

As discussed earlier that the isotropy requires that the strain energy function to be expressed as a function of invariants of the symmetric Cauchy-Green tensors $\boldsymbol{C}$ or $\boldsymbol{b}$.

$$\Psi = \Psi[I_1(\boldsymbol{C}), I_2(\boldsymbol{C}), I_3(\boldsymbol{C})] = \Psi[I_1(\boldsymbol{b}), I_2(\boldsymbol{b}), I_3(\boldsymbol{b})].$$