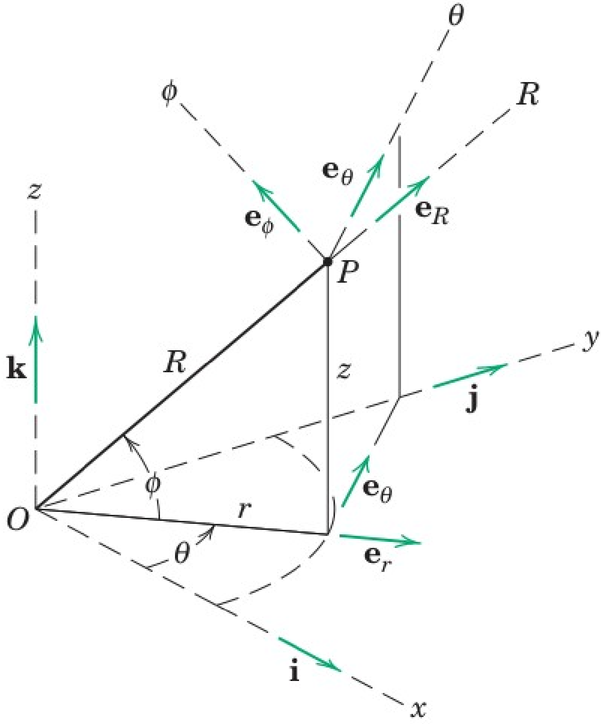
ME232: Dynamics

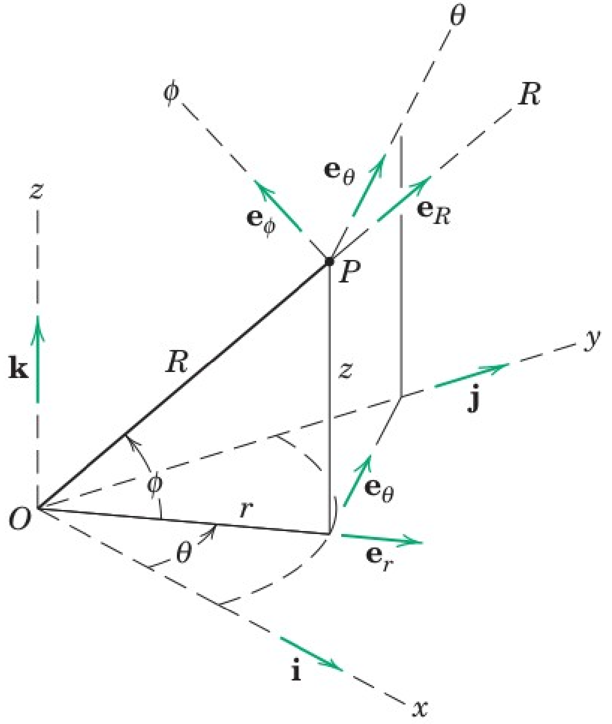
Anshul Faye
afaye@iitbhilai.ac.in
Room # 106

Space curvilinear motion



For the general case of three-dimensional motion of a particle along a space curve three coordinate systems, rectangular (x-y-z), cylindrical (r-–z), and spherical (R-–ϕ), are commonly used to describe this motion. These systems are indicated in Figure, which also shows the unit vectors for the three coordinate systems.

After understanding physical meaning of different terms in velocity and acceleration for plane motion, we simply extend the concept for the space curvilinear motion and just write the mathematical expressions.



Rectangular coordinates (x-y-z):

$$\boldsymbol{R} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

$$\dot{\boldsymbol{R}} = \boldsymbol{v} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k} \quad \text{.................}(27)$$

$$\ddot{\boldsymbol{R}} = \dot{\boldsymbol{v}} = \boldsymbol{a} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{z}\boldsymbol{k}$$

Cylindrical coordinates (r-θ-z):

$$\boldsymbol{R} = r\boldsymbol{e}_r + z\boldsymbol{k}$$

$$\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta + \dot{z}\boldsymbol{k} \quad \text{.................}(28)$$

$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e}_\theta + \ddot{z}\boldsymbol{k}$$

Spherical coordinates (R-θ-ϕ):

$$\boldsymbol{R} = R\boldsymbol{e}_R$$

$$\boldsymbol{v} = v_R\boldsymbol{e}_R + v_\theta\boldsymbol{e}_\theta + v_\phi\boldsymbol{e}_\phi$$

$$\boldsymbol{a} = a_R\boldsymbol{e}_R + a_\theta\boldsymbol{e}_\theta + a_\phi\boldsymbol{e}_\phi$$

$$v_R = \dot{R}, \quad v_\theta = R\dot{\theta}\cos\phi, \quad v_\phi = R\dot{\phi},$$

$$a_R = \ddot{R} - R\dot{\phi}^2 - R\dot{\theta}^2\cos^2\phi, \quad \text{.................}(29)$$

$$a_\theta = \frac{\cos\phi}{R}\frac{d}{dt}(R^2\dot{\theta}) - 2R\dot{\theta}\dot{\phi}\sin\phi,$$

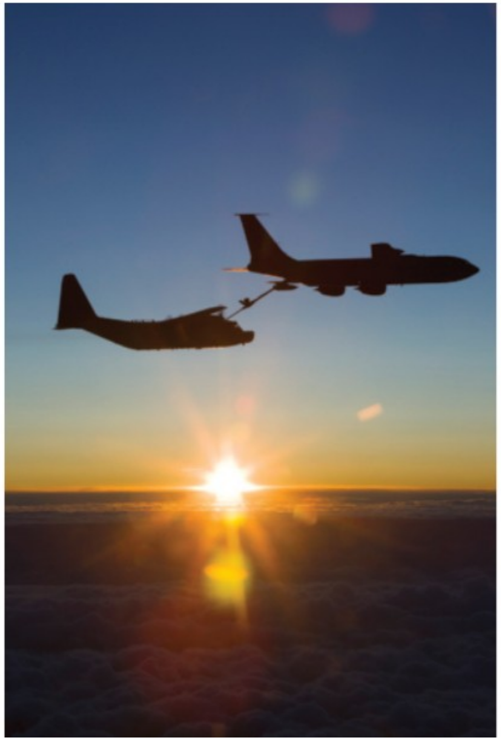
$$a_\phi = \frac{1}{R}\frac{d}{dt}(R^2\dot{\phi}) + R\dot{\theta}^2\sin\phi\cos\phi.$$

Relative motion (Translating axes)

We have already described particle motion w.r.t. to fixed reference axes, these are **absolute displacements, absolute velocities, and absolute accelerations.** It is not always possible or convenient, however, to use a fixed set of axes to describe or to measure motion.

In many engineering problems the analysis of motion become simple when measurements are made w.r.t. a **moving reference system.**

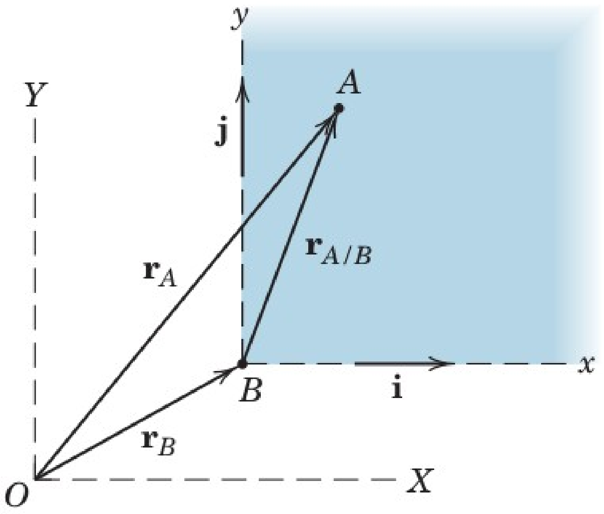
These measurements combined with the absolute motion of the moving coordinate system, enable us to determine the absolute motion in question. This approach is called a **relative-motion analysis.**



Choice of coordinate system

- For most engineering purpose, fixed coordinate system may be taken as any system whose absolute motion is negligible for the considered problem.
- For most earthbound problem it is sufficiently precise to take for the fixed coordinate system a set of axes attached to the earth.
- For the motion of a satellite around the earth, a non-rotating coordinate system is chosen with its origin on the axis of rotation of the earth
- For interplanetary travel, a non-rotating coordinate system fixed to the sun would be used.
- Currently, we will confine ourselves to moving reference system, which translate but do not rotate. We will also confine to relative motion analysis for plane motion only.

Vector representation



Consider two particles A and B which may have separate curvilinear motions in a given plane or in parallel planes.

We will arbitrarily attach the origin of a set of translating (nonrotating) axes x-y to particle B and observe the motion of A from our moving position on B.

The position vector of A as measured relative to the frame x-y, i.e., position vector of A relative to B is

$$\boldsymbol{r}_{A/B} = x\boldsymbol{i} + y\boldsymbol{j}.$$

The unit vectors along the x- and y-axes are $\boldsymbol{i}$ and $\boldsymbol{j}$, and x and y are the coordinates of A measured in the x-y frame.

The absolute position of B is defined by the vector $\boldsymbol{r}_B$ measured from the origin of the fixed axes X-Y. The absolute position of A is seen, therefore, to be determined by the vector

$$\boldsymbol{r}_A = \boldsymbol{r}_B + \boldsymbol{r}_{A/B}. \quad \text{.................}(30)$$

Differentiating (30) w.r.t. time twice, we get,

$$\dot{\boldsymbol{r}}_A = \dot{\boldsymbol{r}}_B + \dot{\boldsymbol{r}}_{A/B} \quad \text{or} \quad \boldsymbol{v}_A = \boldsymbol{v}_B + \boldsymbol{v}_{A/B}. \qquad \cdots\cdots\cdots\cdots(31)$$

$$\ddot{\boldsymbol{r}}_A = \ddot{\boldsymbol{r}}_B + \ddot{\boldsymbol{r}}_{A/B} \quad \text{or} \quad \boldsymbol{a}_A = \boldsymbol{a}_B + \boldsymbol{a}_{A/B}. \qquad \cdots\cdots\cdots\cdots(32)$$

In (31) the velocity of A w.r.t. the moving axes x-y (or point B) is

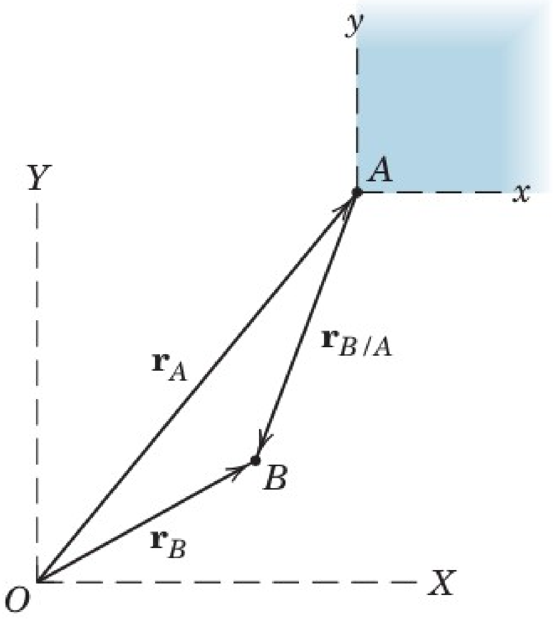
$$\dot{\boldsymbol{r}}_{A/B} = \boldsymbol{v}_{A/B} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j}.$$

Similarly, (32) gives the acceleration of A w.r.t. B as

$$\ddot{\boldsymbol{r}}_{A/B} = \boldsymbol{a}_{A/B} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j}.$$

(31) and (32) states that the absolute velocity (or acceleration) of A equals the absolute velocity (or acceleration) of B plus, vectorially, the velocity (or acceleration) of A relative to B.

The relative term is the velocity (or acceleration) measurement made by an observer attached to the moving coordinate system x-y.



The point A can also be used for the attachment of the moving system, in which case the three corresponding relative-motion equations for position, velocity, and acceleration are

$$\boldsymbol{r}_B = \boldsymbol{r}_A + \boldsymbol{r}_{B/A},$$

$$\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{v}_{B/A}, \qquad \cdots\cdots\cdots\cdots\cdots\cdots(33)$$

$$\text{and} \qquad \boldsymbol{a}_B = \boldsymbol{a}_A + \boldsymbol{a}_{B/A}.$$

Comparing with (30, 31 and 32) we see that,

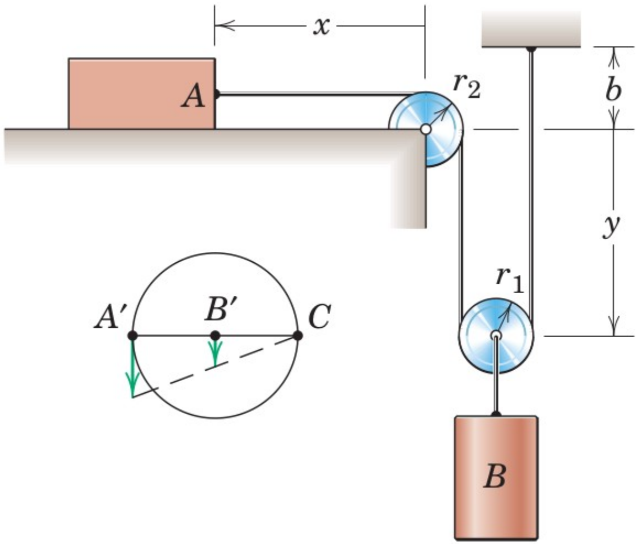
$$\boldsymbol{r}_{B/A} = -\boldsymbol{r}_{A/B},\ \boldsymbol{v}_{B/A} = -\boldsymbol{v}_{A/B},\ \text{and}\ \boldsymbol{a}_{B/A} = -\boldsymbol{a}_{A/B}.$$

Note that the acceleration of a particle as observed in a translating system x-y is the same as that observed in a fixed system X-Y if the **moving system has a constant velocity.**

This conclusion widen the area of application of Newton's second law of motion. These laws can be applied to **a set of axes which has a constant absolute velocity** in addition to a "fixed" system for the determination of accelerations.

A translating reference system which has no acceleration is called an inertial system.

Constrained motion of connected particles



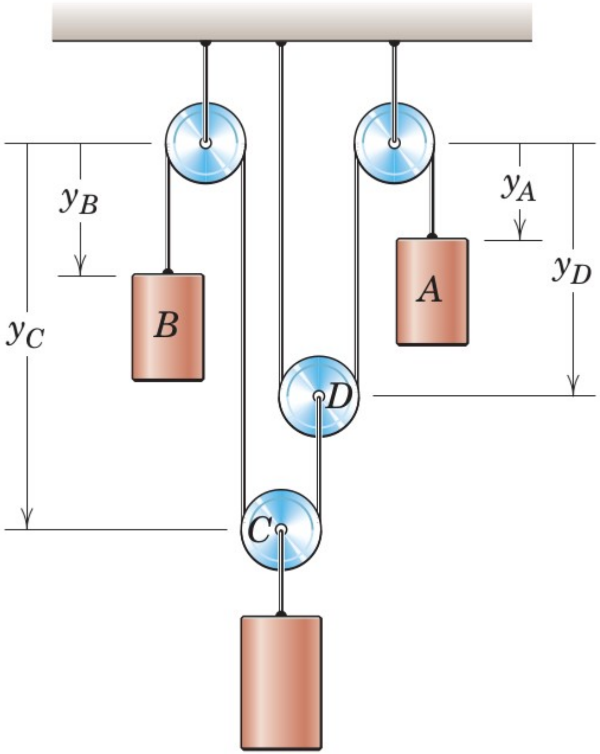
Consider a very simple system of two interconnected particles A and B shown.

Position coordinates y and x measured from a convenient fixed datum. The total length of the cable is

$$L = x + \frac{\pi r_2}{2} + 2y + \pi r_1 + b. \qquad \text{.................(34)}$$

Note that L, r_2, r_1 and b all are constant. So the first and second derivative of (34) will be given as,

$$\begin{aligned} 0 &= \dot{x} + 2\dot{y} \quad \text{or} \quad 0 = v_A + 2v_B, \\ 0 &= \ddot{x} + 2\ddot{y} \quad \text{or} \quad 0 = a_A + 2a_B. \end{aligned} \qquad \text{.................(35)}$$



Consider another system as shown in the figure. The positions of the lower cylinder and pulley C depend on the separate specifications of the two coordinates y_A and y_B. The lengths of the cables attached to cylinders A and B can be written, respectively, as

$$L_A = y_A + 2y_D + \text{constant},$$
$$L_B = y_B + y_C + (y_C - y_D) + \text{constant}. \quad \text{.................(36)}$$

Time derivatives are,

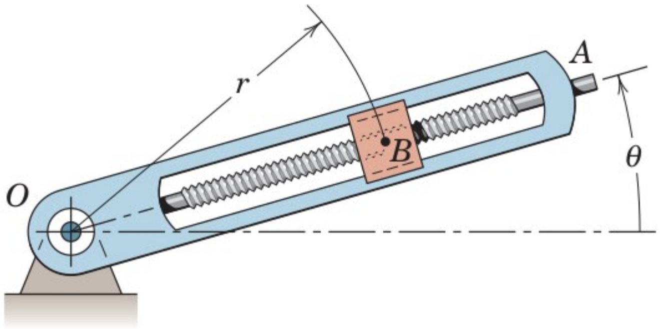
$$0 = \dot{y}_A + 2\dot{y}_D \text{ and } 0 = \dot{y}_B + 2\dot{y}_C - \dot{y}_D,$$
$$0 = \ddot{y}_A + 2\ddot{y}_D \text{ and } 0 = \ddot{y}_B + 2\ddot{y}_C - \ddot{y}_D. \quad \text{.................(37)}$$

Eliminating $\dot{y}_D$ and $\ddot{y}_D$,

$$\dot{y}_A + 2\dot{y}_B + 4\dot{y}_C = 0 \text{ or } v_A + 2v_B + 4v_C = 0,$$
$$\ddot{y}_A + 2\ddot{y}_B + 4\ddot{y}_C = 0 \text{ or } a_A + 2a_B + 4a_C = 0. \quad \text{.................(38)}$$

Example 1

Rotation of the radially slotted arm is governed by $= 0.2t + 0.02t^{3}$, where is in radians and t is in seconds. Simultaneously, the power screw in the arm engages the slider B and controls its distance from O according to $r = 0.2 + 0.04t^{2}$, where r is in meters and t is in seconds. Calculate the magnitudes of the velocity and acceleration of the slider for the instant when $t = 3$ s.



For given expression of r and ,

$\dot{r} = 0.08\,t$ and $\ddot{r} = 0.08.$

$\dot{\theta} = 0.2\,t + 0.06\,t^2$ and $\ddot{\theta} = 0.2 + 0.12\,t.$

$$v_r = \dot{r},\, v_\theta = r\dot{\theta} \text{ and } v = \sqrt{v_r^2 + v_\theta^2}.$$

$$a_r = (\ddot{r} - r\dot{\theta}^2),\, a_\theta = (r\ddot{\theta} + 2\dot{r}\dot{\theta}), \text{ and } a = \sqrt{a_r^2 + a_\theta^2}.$$

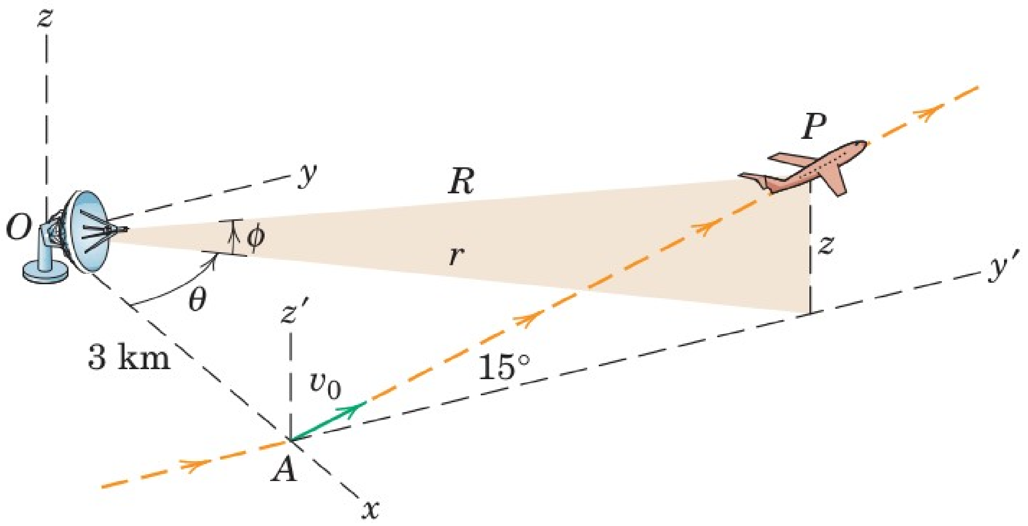
By substituting $t = 3$ s in above expressions, magnitude of velocity and acceleration can be obtained at given time.

Example 2

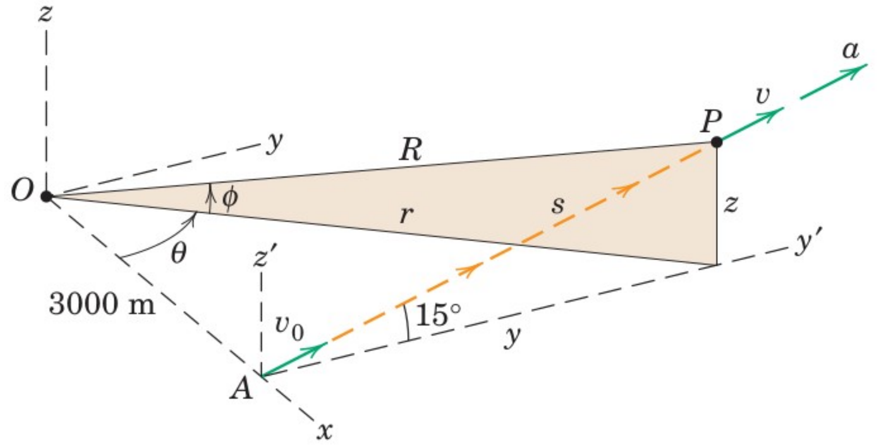
An aircraft P takes off at A with a velocity v_0 of 250 km/h and climbs in the vertical y'-z' plane at the constant 15° angle with an acceleration along its flight path of 0.8 m/s^2. Flight progress is monitored by radar at point O.

(a) Resolve the velocity of P into cylindrical-coordinate components 60 seconds after take-off and find $\dot{r}, \dot{\theta}$ and $\dot{z}$ for that instant.

(b) Resolve the velocity of the aircraft P into spherical-coordinate components 60 seconds after take-off and find $\dot{R}, \dot{\theta}$ and $\dot{\phi}$ for that instant.



The velocity and acceleration vectors in the y'-z' plane is shown.



The take-off speed $v_0 = 250/3.6 = 69.4$ m/s.

The speed after 60 seconds can be calculated as,

$v = v_0 + at = 69.4 + 0.8 \times 60 = 117.4$ m/s.

The distance s traveled after 60 sec. is,

$s = s_0 + v_0 t + \frac{1}{2}\, at^2 = 0 + (69.4 \times 60) + (\frac{1}{2} \times 0.8 \times 60^2) = 5610$ m.

The y-coordinate and associated angle are,

$y = 5610 \cos 15° = 5420$ m, $= \tan^{-1}(5420/3000) = 61°$.

Now, $r^2 = y^2 + 3000^2$, hence $r = 6190$ m.

$v_{xy} = v \cos 15° = 117.4 \cos 15° = 113.4$ m/s

$v_r = \dot{r} = v_{xy} \sin\theta = 113.4 \sin 61° = 99.2$ m/s

$v_\theta = r\dot{\theta} = v_{xy} \cos\theta = 113.4 \cos 61° = 55$ m/s $\Rightarrow \dot{\theta} = 8.88 \times 10^{-3}$ rad/s

Finally, $v_z = \dot{z} = v \sin 15° = 30.4$ m/s.

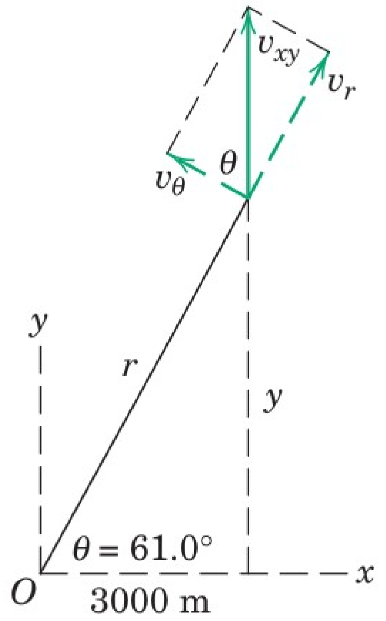
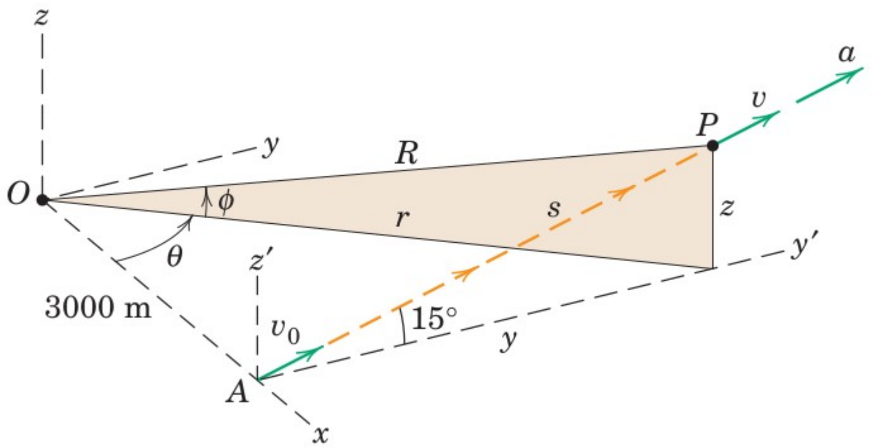


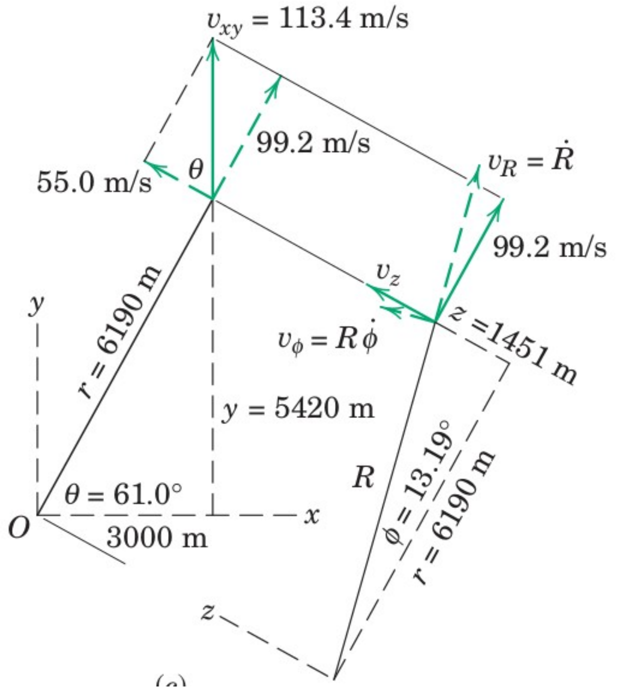
Figure shows the x-y plane and various velocity components projected into the vertical plane containing r and R.



$z = y \tan 15° = 5420 \tan 15° = 1451$ m

$\phi = \tan\text{-}1(z/r) = 13.19°$

$R = (r^2 + z^2)^{1/2} = 6360$ m/s



Now, from figure,

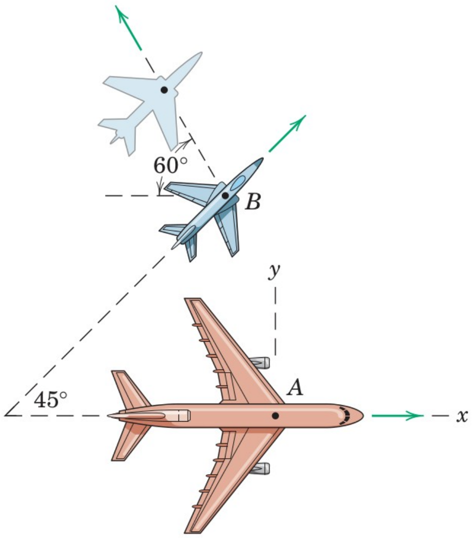
$$v_R = \dot{R} = 99.2 \cos 13.19° + 30.4 \sin 13.19° = 103.6 \text{ m/s}$$

$$\dot{\theta} = 8.88 \times 10^{-3} \text{ rad/s (already obtained)}$$

$$v_\phi = r\dot{\phi} = 30.4 \cos 13.19° - 99.2 \sin 13.19° = 6.95 \text{ m/s}$$
$$\Rightarrow \dot{\phi} = 1.093 \times 10^{-3} \text{ rad/s}$$

Example 3

Passengers in the jet transport A flying east at a speed of 800 km/h observe a second jet plane B that passes under the transport in horizontal flight. Although the nose of B is pointed in the 45° northeast direction, plane B appears to the passengers in A to be moving away from the transport at the 60° angle as shown. Determine the true velocity of B.



If the moving reference axes x-y are attached to A, then we can write,

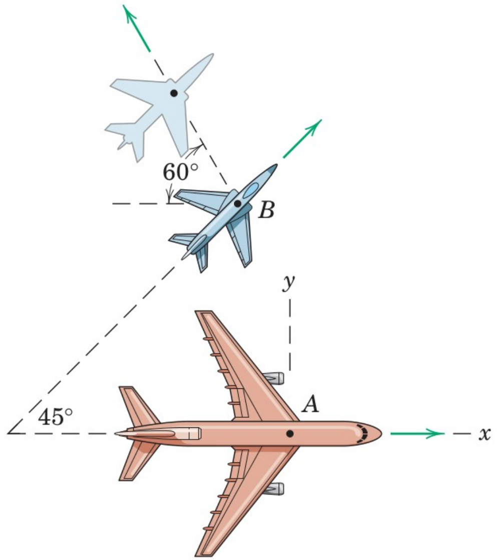
$$\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{v}_{B/A}.$$

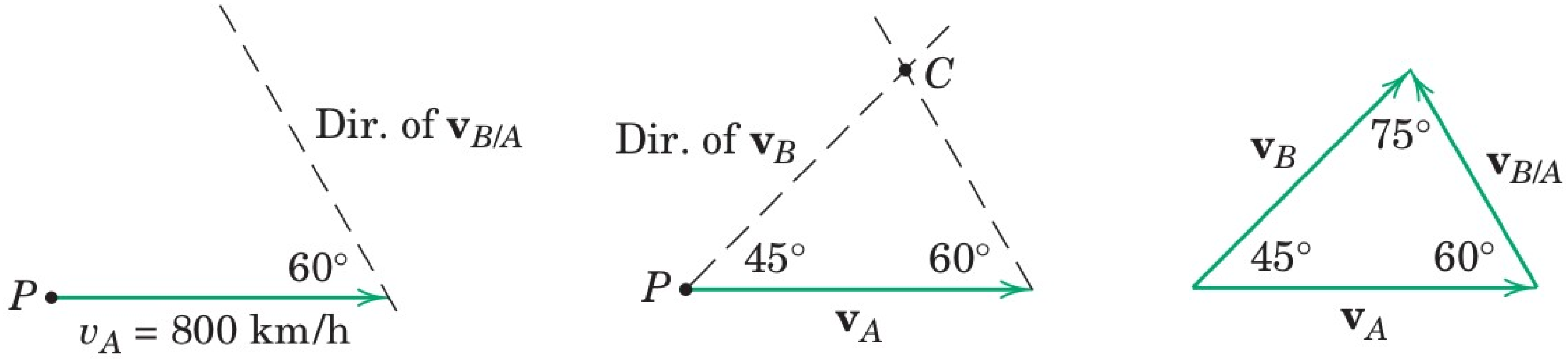
Next we identify the knowns and unknowns.

- The velocity v_A is given in both magnitude and direction.
- The 60° direction of $v_{B/A}$, the velocity which B appears to have to the moving observers in A, is known
- The true velocity of B is in the 45° direction in which it is heading.
- The two remaining unknowns are the magnitudes of v_B and $v_{B/A}$.

Graphical Approach:

- Start the vector sum at some point P by drawing v_A to a convenient scale.
- Construct a line through the tip of v_A with the known direction of $v_{B/A}$.
- The known direction of v_B is then drawn through P, and the intersection C yields the unique solution.
- Complete the vector triangle and scale off the unknown magnitudes.





Vector Algebra:

$$\boldsymbol{v}_A = 800\,\boldsymbol{i}\ \text{km/h},$$

$$\boldsymbol{v}_B = (v_B \cos 45°)\,\boldsymbol{i} + (v_B \sin 45°)\,\boldsymbol{j},$$

$$\boldsymbol{v}_{B/A} = (v_{B/A} \cos 120°)\,\boldsymbol{i} + (v_{B/A} \sin 120°)\,\boldsymbol{j} = -(v_{B/A} \cos 60°)\,\boldsymbol{i} + (v_{B/A} \sin 60°)\,\boldsymbol{j}.$$

From the expression of relative velocity,

$$\boldsymbol{v}_B = \boldsymbol{v}_A + \boldsymbol{v}_{B/A}.$$

Solving separately for the $\boldsymbol{i}$ and $\boldsymbol{j}$ terms give,

$$v_B \cos 45° = 800 - v_{B/A} \cos 60°,$$

$$v_B \sin 45° = v_{B/A} \sin 60°.$$

Above two equation can be solved for two unknowns v_B and $v_{B/A}$.