ME531: Advanced Mechanics of Solids

Motion, Strain and Stress

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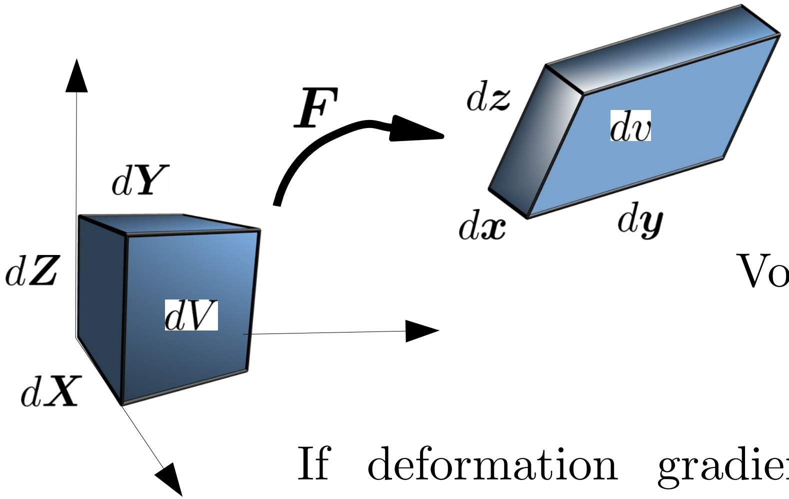


Consider an infinitesimally small volume element with reference volume dV going under deformation and the current volume is dv.

Volume of the element in current configuration can be given as,

$$dv = d\boldsymbol{z} \cdot (d\boldsymbol{x} \times d\boldsymbol{y}) \text{ or } dv = e_{ijk} dz_i dx_j dy_k$$

If deformation gradient for the motion is $\boldsymbol{F}$, then edges in the reference configuration can be mapped to the edges in undeformed configuration as,

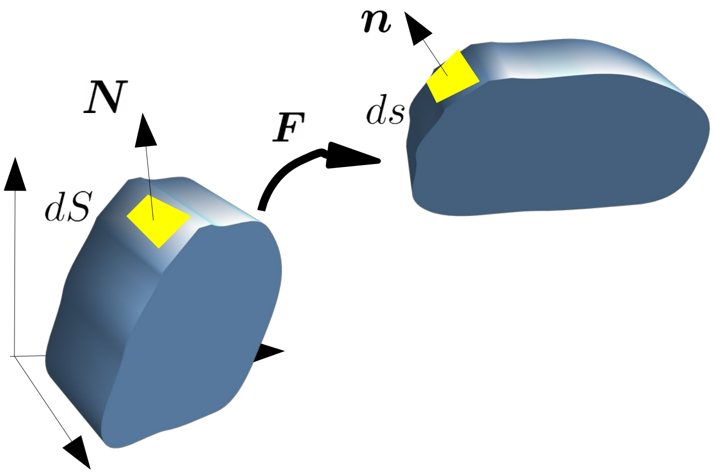
$$dx_i = F_{ij} dX_J, dy_i = F_{ij} dY_J, \text{ and } dz_i = F_{ij} dZ_J$$

Substituting above in the expression for volume, we get

$$dv = e_{ijk}(F_{ip} dZ_p)(F_{jq} dX_q)(F_{kr} dY_r)$$

It can be shown that, $e_{ijk} A_{ip} A_{jq} A_{kr} = e_{pqr} \det \boldsymbol{A}$.

Thus, $dv = e_{pqr} dZ_p dX_q dY_r \det \boldsymbol{F} \Rightarrow \boxed{dv = J dV,}$ where $J = \det \boldsymbol{F}$ is called the *Jacobian* of deformation gradient tensor. The Jacobian is the ratio of elemental volume deformed configuration to its volume in undeformed configuration.



Deformation gradient $\boldsymbol{F}$ is used to map a vector in undeformed configuration to the deformed configuration. However, the same does not apply to the unit normal vector $\boldsymbol{N}$ to an infinitesimal surface element dS. To find the relation between the normal vector $\boldsymbol{N}$ and $\boldsymbol{n}$ (deformed configuration of $\boldsymbol{N}$) we use the following relation.

$$dv = JdV$$

We now present the volume dV as $d\boldsymbol{S}\cdot d\boldsymbol{X}$, where $d\boldsymbol{X}$ is a line element in undeformed configuration, which deform to the line element $d\boldsymbol{x}$, and hence $dv = d\boldsymbol{s}\cdot d\boldsymbol{x}$.

Now, we can write,

$$d\boldsymbol{s} \cdot d\boldsymbol{x} = Jd\boldsymbol{S} \cdot d\boldsymbol{X}$$

Line element $d\boldsymbol{x}$ and $d\boldsymbol{X}$ are related as, $d\boldsymbol{x} = Fd\boldsymbol{X}$. Using this relation we can write,

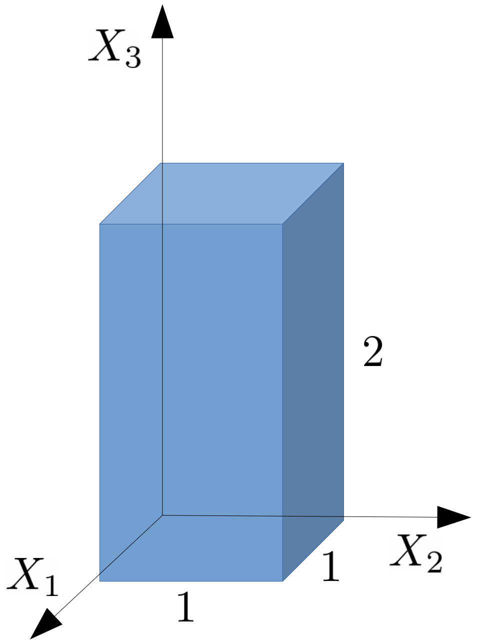
$$d\boldsymbol{s} \cdot \boldsymbol{F}d\boldsymbol{X} = Jd\boldsymbol{S} \cdot d\boldsymbol{X} \Rightarrow (\boldsymbol{F}^T d\boldsymbol{s} - Jd\boldsymbol{S}) \cdot d\boldsymbol{X} = 0$$

This relation holds for any arbitrary $d\boldsymbol{X}$, which implies $\boldsymbol{F}^T d\boldsymbol{s} - Jd\boldsymbol{S} = 0$.

Above gives us the ***Nanson's formula*** relating an area element in undeformed configuration to the area element in deformed configuration as $\boxed{d\boldsymbol{s} = J\boldsymbol{F}^{-T} d\boldsymbol{S}.}$

Exercise

Plot the deformed shape of the cube which goes under the following motion.



$$x_1 = X_1 \cos(\pi X_3/2) - X_2 \sin(\pi X_3/2)$$
$$x_2 = X_1 \sin(\pi X_3/2) + X_2 \cos(\pi X_3/2)$$
$$x_3 = X_3$$

Displacement gradient

Using displacements we can write, $\boldsymbol{x}(\boldsymbol{X},t) = \boldsymbol{X} + \boldsymbol{U}(\boldsymbol{X},t)$ or $x_i = X_i + U_i$.

From the definition of deformation gradient,

$$F_{iJ} = \frac{\partial x_i}{\partial X_J} = \frac{\partial}{\partial X_J}(X_i + U_i) = \frac{\partial X_i}{\partial X_J} + \frac{\partial U_i}{\partial X_J}$$

$$F_{iJ} = \delta_{ij} + \frac{\partial U_i}{\partial X_J}, \text{ or } \quad \boldsymbol{F} = \boldsymbol{I} + \boldsymbol{\nabla}_X \boldsymbol{U}$$

where, $\boldsymbol{\nabla}_X \boldsymbol{U} = \dfrac{\partial U_i}{\partial X_J}$ is called *displacement gradient tensor.*

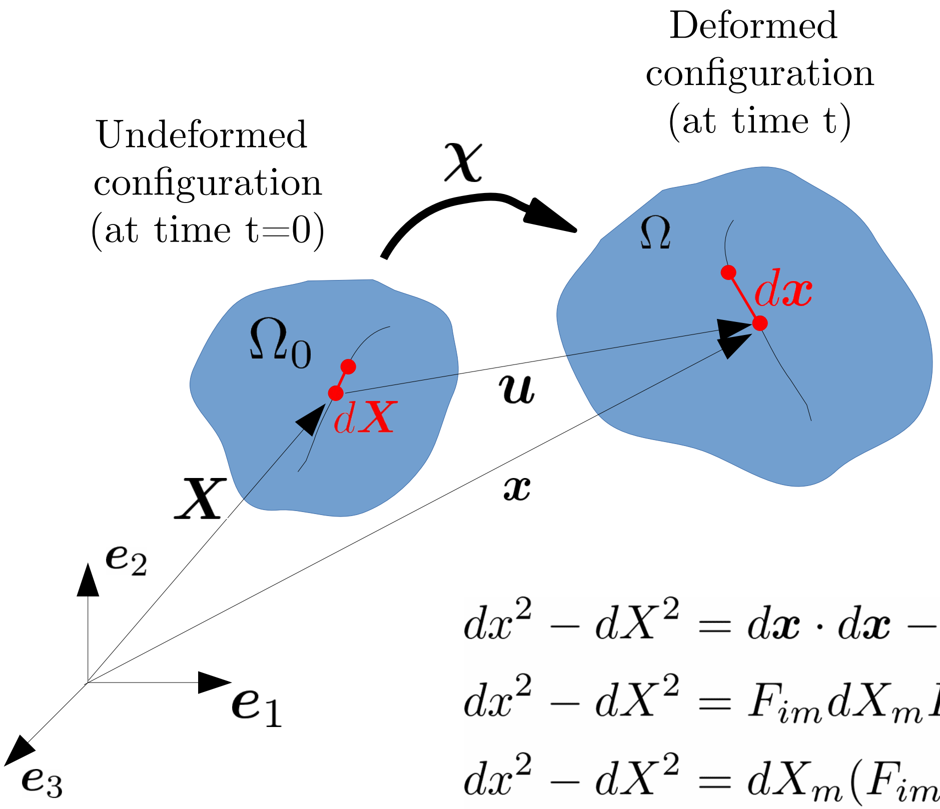
In material description, displacement gradient tensor is

$$\boldsymbol{\nabla}_X \boldsymbol{U}(\boldsymbol{X},t) = \boldsymbol{F}(\boldsymbol{X},t) - \boldsymbol{I} \text{ or } \quad \frac{\partial U_i}{\partial X_J} = F_{iJ} - \delta_{ij}.$$

In spatial description, displacement gradient tensor is

$$\boldsymbol{\nabla} \boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{I} - \boldsymbol{F}^{-1}(\boldsymbol{x},t) \text{ or } \quad \frac{\partial u_i}{\partial x_J} = \delta_{ij} - F^{-1}_{iJ}.$$

Strain



Squared length of the line element $d\boldsymbol{X}$ is $dX^2 = d\boldsymbol{X} \cdot d\boldsymbol{X}$.

Similarly, squared length of the line element $d\boldsymbol{x}$ is $dx^2 = d\boldsymbol{x} \cdot d\boldsymbol{x}$.

Let's examine the difference between the squared length of $d\boldsymbol{x}$ and $d\boldsymbol{X}$ as,

$$dx^2 - dX^2 = d\boldsymbol{x} \cdot d\boldsymbol{x} - d\boldsymbol{X} \cdot d\boldsymbol{X} = dx_i dx_i - dX_j dX_j,$$

$$dx^2 - dX^2 = F_{im} dX_m F_{in} dX_n - dX_m \delta_{mn} dX_n,$$

$$dx^2 - dX^2 = dX_m (F_{im} F_{in} - \delta_{mn}) dX_n = d\boldsymbol{X} \cdot (\boldsymbol{F}^T \boldsymbol{F} - \boldsymbol{I}) d\boldsymbol{X},$$

$$dx^2 - dX^2 = dX_m (F_{im} F_{in} - \delta_{mn}) dX_n = d\boldsymbol{X} \cdot 2\boldsymbol{E} d\boldsymbol{X}.$$

where, $\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^T \boldsymbol{F} - \boldsymbol{I}\right)$ or $E_{ij} = \frac{1}{2}\left(F_{ki} F_{kj} - \delta_{ij}\right)$ is called *Green-Lagrange* strain tensor.

Using the relaton, $\boldsymbol{F} = \boldsymbol{I} + \boldsymbol{\nabla}_X \boldsymbol{U}$ or $F_{ij} = \delta_{ij} + \dfrac{\partial U_i}{\partial X_j}$, we write,

$$E_{ij} = \frac{1}{2}\left[\left(\delta_{ki} + \frac{\partial U_k}{\partial X_i}\right)\left(\delta_{kj} + \frac{\partial U_k}{\partial X_j}\right) - \delta_{ij}\right]$$

$$E_{ij} = \frac{1}{2}\left[\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} + \frac{\partial U_k}{\partial X_i}\frac{\partial U_k}{\partial X_j}\right] \quad \text{or} \quad \boldsymbol{E} = \frac{1}{2}\left[\boldsymbol{\nabla}_X \boldsymbol{U} + \left(\boldsymbol{\nabla}_X \boldsymbol{U}\right)^T + \left(\boldsymbol{\nabla}_X \boldsymbol{U}\right)^T \boldsymbol{\nabla}_X \boldsymbol{U}\right]$$

Eulerian description of the Green-Lagrange strain tensor is called *Euler-Almansi strain tensor* and defined as,

$$\boldsymbol{e} = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{F}^{-T}\boldsymbol{F}^{-1}\right) \quad \text{or} \quad e_{ij} = \frac{1}{2}\left(\delta_{ij} - F^{-1}_{ki} F_{kj}\right)$$

$$\boldsymbol{e} = \frac{1}{2}\left[\boldsymbol{\nabla}\boldsymbol{u} + \left(\boldsymbol{\nabla}\boldsymbol{u}\right)^T - \left(\boldsymbol{\nabla}\boldsymbol{u}\right)^T \boldsymbol{\nabla}\boldsymbol{u}\right] \quad \text{or} \quad e_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right]$$

Strains in case of small deformations

When deformations are small, $\partial u_i / \partial x_j \ll 1$. Also, difference between reference and current coordinates become negligible, thus strain tensor become

$$E_{ij} \approx e_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right] \approx \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] = \varepsilon_{ij}$$

or

$\boldsymbol{\varepsilon} = \frac{1}{2}\left[\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T\right]$, which is known as *small strain tensor.*

It can be noted that the small strain tensor is symmetric part of the displacement gradient tensor. Antisymmetric part of the displacement gradient tensor is known as infinitesimal rotation tensor ($\boldsymbol{\omega}$). Thus,

$\boldsymbol{\nabla u} = \boldsymbol{\varepsilon} + \boldsymbol{\omega}$, where

$\boldsymbol{\varepsilon} = \frac{1}{2}\left[\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T\right]$ and $\boldsymbol{\omega} = \frac{1}{2}\left[\boldsymbol{\nabla u} - (\boldsymbol{\nabla u})^T\right]$.