APPENDIX

A Moments of Areas

A.1 First Moment of an Area and Centroid of an Area

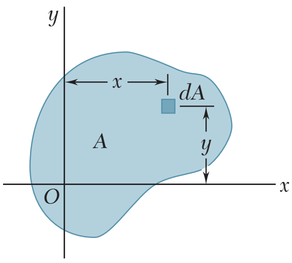


Fig. A.1 General area A with infinitesimal area dA referred to xy coordinate system.

Consider an area A located in the xy plane (Fig. A.1). Using x and y as the coordinates of an element of area dA, the *first moment of the area A with respect to the x axis* is the integral

$$Q_x = \int_A y\, dA \tag{A.1}$$

Similarly, the *first moment of the area A with respect to the y axis* is the integral

$$Q_y = \int_A x\, dA \tag{A.2}$$

Note that each of these integrals may be positive, negative, or zero, depending on the position of the coordinate axes. When SI units are used, the first moments Q_x and Q_y are given in m^3 or mm^3. When U.S. customary units are used, they are given in ft^3 or in^3.

The *centroid of the area A* is the point C of coordinates $\bar{x}$ and $\bar{y}$ (Fig. A.2), which satisfy the relationship

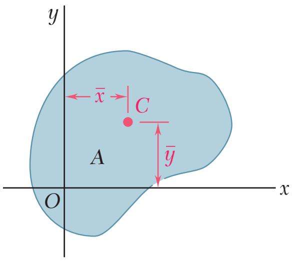


Fig. A.2 Centroid of area A.

$$\int_A x\, dA = A\bar{x} \qquad \int_A y\, dA = A\bar{y} \tag{A.3}$$

Comparing Eqs. (A.1) and (A.2) with Eqs. (A.3), the first moments of the area A can be expressed as the products of the area and the coordinates of its centroid:

$$Q_x = A\bar{y} \qquad Q_y = A\bar{x} \tag{A.4}$$

When an area possesses an *axis of symmetry,* the first moment of the area with respect to that axis is zero. Considering area A of Fig. A.3, which is symmetric with respect to the y axis, every element of area dA of

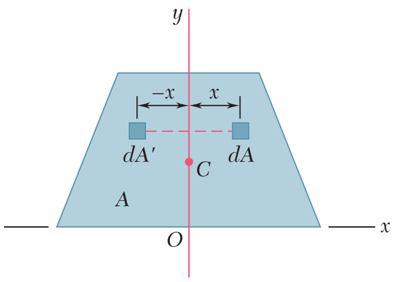


Fig. A.3 Area having axis of symmetry.

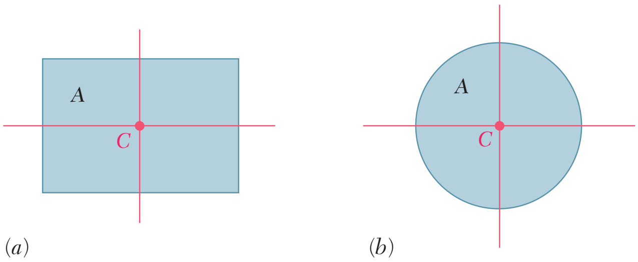


Fig. A.4 Areas having two axes of symmetry have the centroid at their intersection.

abscissa x corresponds to an element of area dA' of abscissa $-x$. Therefore, the integral in Eq. (A.2) is zero, and $Q_y = 0$. From the first of the relationships in Eq. (A.3), $\bar{x} = 0$. Thus, if an area A possesses an axis of symmetry, its centroid C is located on that axis.

Since a rectangle possesses two axes of symmetry (Fig. A.4*a*), the centroid C of a rectangular area coincides with its geometric center. Similarly, the centroid of a circular area coincides with the center of the circle (Fig. A.4*b*).

When an area possesses a *center of symmetry O*, the first moment of the area about any axis through O is zero. Considering the area A of Fig. A.5, every element of area dA with coordinates x and y corresponds to an element of area dA' with coordinates $-x$ and $-y$. It follows that the integrals in Eqs. (A.1) and (A.2) are both zero, and $Q_x = Q_y = 0$. From Eqs. (A.3), $\bar{x} = \bar{y} = 0$, so the centroid of the area coincides with its center of symmetry.

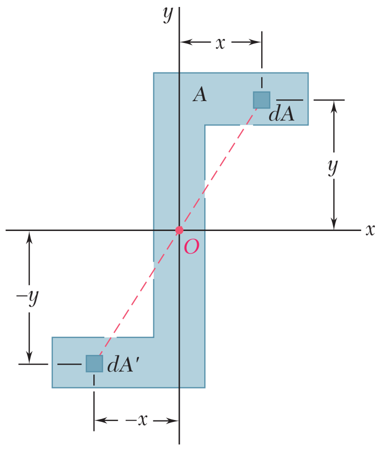


Fig. A.5 Area with center of symmetry has its centroid at the origin.

When the centroid C of an area can be located by symmetry, the first moment of that area with respect to any given axis can be obtained easily from Eqs. (A.4). For example, for the rectangular area of Fig. A.6,

$$Q_x = A\bar{y} = (bh)(\tfrac{1}{2}h) = \tfrac{1}{2}bh^2$$

and

$$Q_y = A\bar{x} = (bh)(\tfrac{1}{2}b) = \tfrac{1}{2}b^2h$$

In most cases, it is necessary to perform the integrations indicated in Eqs. (A.1) through (A.3) to determine the first moments and the centroid of a given area. While each of the integrals is actually a double integral, it is possible to select elements of area dA in the shape of thin horizontal or vertical strips and to reduce the equations to integrations in a single variable. This is illustrated in Concept Application A.1. Centroids of common geometric shapes are given in a table inside the back cover.

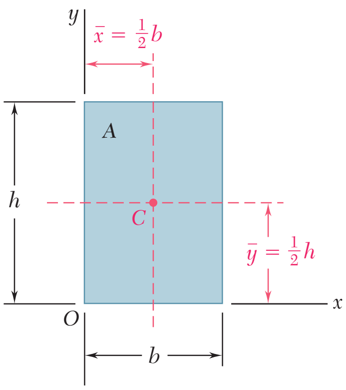


Fig. A.6 Centroid of a rectangular area.

Concept Application A.1

For the triangular area of Fig. A.7*a*, determine (*a*) the first moment Q_x of the area with respect to the *x* axis, (*b*) the ordinate $\bar{y}$ of the centroid of the area.

a. First Moment Q_x. We choose to select as an element of area a horizontal strip with a length of *u* and thickness *dy*. Note that all of the points within the element are at the same distance *y* from the *x* axis (Fig. A.7*b*). From similar triangles,

$$\frac{u}{b} = \frac{h - y}{h} \qquad u = b\frac{h - y}{h}$$

and

$$dA = u\,dy = b\frac{h - y}{h}\,dy$$

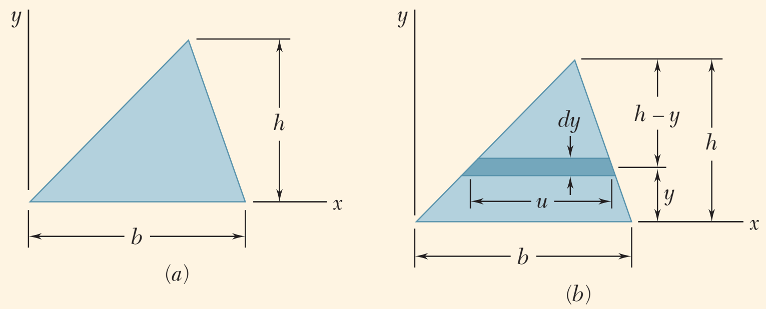


Fig. A.7 (*a*) Triangular area. (*b*) Horizontal element used in integration to find centroid.

The first moment of the area with respect to the *x* axis is

$$Q_x = \int_A y\,dA = \int_0^h yb\frac{h - y}{h}\,dy = \frac{b}{h}\int_0^h (hy - y^2)\,dy$$

$$= \frac{b}{h}\left[h\frac{y^2}{2} - \frac{y^3}{3}\right]_0^h \qquad Q_x = \tfrac{1}{6}bh^2$$

b. Ordinate of Centroid. Recalling the first of Eqs. (A.4) and observing that $A = \tfrac{1}{2}bh$,

$$Q_x = A\bar{y} \qquad \tfrac{1}{6}bh^2 = (\tfrac{1}{2}bh)\bar{y}$$

$$\bar{y} = \tfrac{1}{3}h$$

A.2 The First Moment and Centroid of a Composite Area

Consider area A of the quadrilateral area shown in Fig. A.8, which can be divided into simple geometric shapes. The first moment Q_x of the area with respect to the x axis is represented by the integral $\int y\,dA$, which extends over the entire area A. Dividing A into its component parts A_1, A_2, A_3, write

$$Q_x = \int_A y\,dA = \int_{A_1} y\,dA + \int_{A_2} y\,dA + \int_{A_3} y\,dA$$

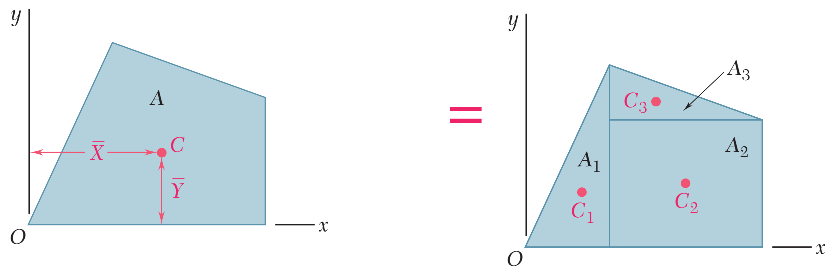


Fig. A.8 Quadrilateral area divided into simple geometric shapes.

or recalling the second of Eqs. (A.3),

$$Q_x = A_1\bar{y}_1 + A_2\bar{y}_2 + A_3\bar{y}_3$$

where $\bar{y}_1, \bar{y}_2$, and $\bar{y}_3$ represent the ordinates of the centroids of the component areas. Extending this to an arbitrary number of component areas and noting that a similar expression for Q_y may be obtained, write

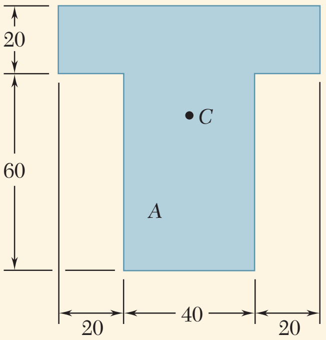
$$Q_x = \sum A_i\bar{y}_i \qquad Q_y = \sum A_i\bar{x}_i \tag{A.5}$$

To obtain the coordinates $\bar{X}$ and $\bar{Y}$ of the centroid C of the composite area A, substitute $Q_x = A\bar{Y}$ and $Q_y = A\bar{X}$ into Eqs. (A.5):

$$A\bar{Y} = \sum_i A_i\bar{y}_i \qquad A\bar{X} = \sum_i A_i\bar{x}_i$$

Solving for $\bar{X}$ and $\bar{Y}$ and recalling that the area A is the sum of the component areas A_i,

$$\bar{X} = \frac{\sum_i A_i\bar{x}_i}{\sum_i A_i} \qquad \bar{Y} = \frac{\sum_i A_i\bar{y}_i}{\sum_i A_i} \tag{A.6}$$



Concept Application A.2

Locate the centroid C of the area A shown in Fig. A.9*a*.

Selecting the coordinate axes shown in Fig. A.9*b*, note that the centroid C must be located on the y axis, since this is an axis of symmetry. Thus, $\overline{X} = 0$.

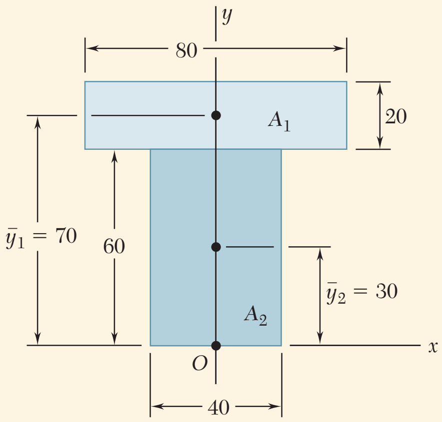


Fig. A.9 (*a*) Area *A*. (*b*) Composite areas A_1 and A_2 used to determine overall centroid.

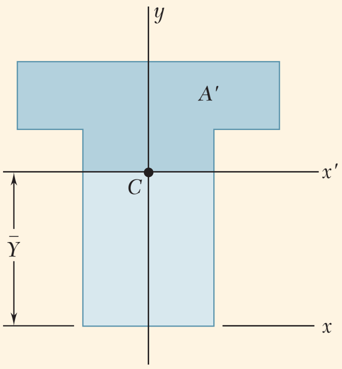
Divide A into its component parts A_1 and A_2 and use the second of Eqs. (A.6) to determine the ordinate $\overline{Y}$ of the centroid. The actual computation is best carried out in tabular form:

	Area, mm^2	$\bar{y}_i$, mm	$A_i\bar{y}_i$, mm^3
A_1	$(20)(80) = 1600$	70	112×10^3
A_2	$(40)(60) = 2400$	30	72×10^3
	$\sum_i A_i = 4000$		$\sum_i A_i\bar{y}_i = 184 \times 10^3$

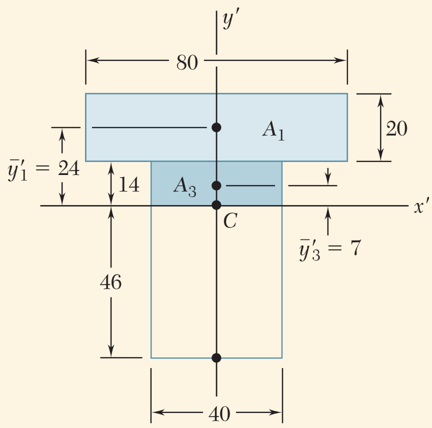
$$\overline{Y} = \frac{\sum_i A_i\bar{y}_i}{\sum_i A_i} = \frac{184 \times 10^3 \text{ mm}^3}{4 \times 10^3 \text{ mm}^2} = 46 \text{ mm}$$

Concept Application A.3

Referring to the area A of Concept Application A.2, consider the horizontal x' axis through its centroid C (called a *centroidal axis*). The portion of A located above that axis is A' (Fig. A.10*a*). Determine the first moment of A' with respect to the x' axis.



(*a*)



Dimensions in mm

(*b*)

Solution. Divide the area A' into its components A_1 and A_3 (Fig. A.10*b*). Recall from Concept Application A.2 that C is located 46 mm above the lower edge of A. The ordinates $\bar{y}'_1$ and $\bar{y}'_3$ of A_1 and A_3 and the first moment $Q'_{x'}$ of A' with respect to x' are

$$
\begin{aligned}
Q'_{x'} &= A_1\bar{y}'_1 + A_3\bar{y}'_3 \\
&= (20 \times 80)(24) + (14 \times 40)(7) = 42.3 \times 10^3 \text{ mm}^3
\end{aligned}
$$

Alternative Solution. Since the centroid C of A is located on the x' axis, the first moment $Q_{x'}$ of the *entire area* A with respect to that axis is zero:

$$Q_{x'} = A\bar{y}' = A(0) = 0$$

Using A'' as the portion of A located below the x' axis and $Q''_{x'}$ as its first moment with respect to that axis,

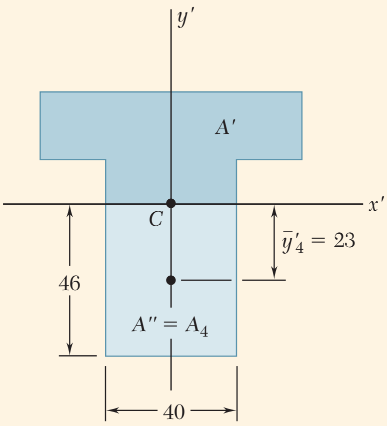
$$Q_{x'} = Q'_{x'} + Q''_{x'} = 0 \quad \text{or} \quad Q'_{x'} = -Q''_{x'}$$

This shows that the first moments of A' and A'' have the same magnitude and opposite signs. Referring to Fig. A.10*c*, write

$$Q''_{x'} = A_4\bar{y}'_4 = (40 \times 46)(-23) = -42.3 \times 10^3 \text{ mm}^3$$

and

$$Q'_{x'} = -Q''_{x'} = +42.3 \times 10^3 \text{ mm}^3$$



Dimensions in mm

(*c*)

Fig. A.10 (*a*) Area A with centroidal $x'y'$ axes, highlighting portion A'. (*b*) Areas used to determine the first moment of area A' with respect to the x' axis. (*c*) Alternative solution using the other portion A'' of the total area A.

A.3 Second Moment, or Moment of Inertia of an Area, and Radius of Gyration

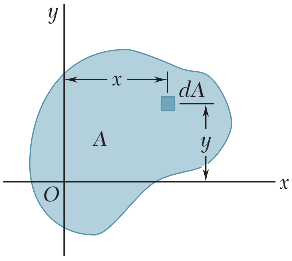


Fig. A.1 *(repeated)*

Consider an area A located in the xy plane (Fig. A.1) and the element of area dA of coordinates x and y. The *second moment,* or *moment of inertia,* of area A with respect to the x and y axes is

$$I_x = \int_A y^2\,dA \quad I_y = \int_A x^2\,dA \tag{A.7}$$

These integrals are called *rectangular moments of inertia,* since they are found from the rectangular coordinates of element dA. While each integral is actually a double integral, it is possible to select elements of area dA in the shape of thin horizontal or vertical strips and to reduce the equations to integrations in a single variable. This is illustrated in Concept Application A.4.

The *polar moment of inertia* of area A with respect to point O (Fig. A.11) is the integral

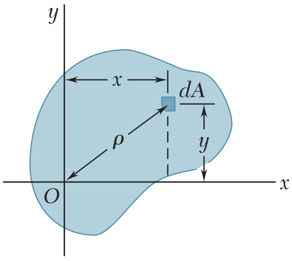


Fig. A.11 Area *dA* located by distance *ρ* from point *O*.

$$J_O = \int_A \rho^2\,dA \tag{A.8}$$

where ρ is the distance from O to the element dA. While this integral is also a double integral, for circular areas it is possible to select elements of area dA in the shape of thin circular rings and to reduce the equation of J_O to a single integration (see Concept Application A.5).

Note from Eqs. (A.7) and (A.8) that the moments of inertia of an area are positive quantities. When SI units are used, moments of inertia are given in m^4 or mm^4. When U.S. customary units are used, they are given in ft^4 or in^4.

An important relationship can be established between the polar moment of inertia J_O of a given area and the rectangular moments of inertia I_x and I_y. Noting that $\rho^2 = x^2 + y^2$,

$$J_O = \int_A \rho^2\,dA = \int_A (x^2 + y^2)\,dA = \int_A y^2\,dA + \int_A x^2\,dA$$

or

$$J_O = I_x + I_y \tag{A.9}$$

The *radius of gyration* of an area A with respect to the x axis is r_x, which satisfies the relationship

$$I_x = r_x^2 A \tag{A.10}$$

where I_x is the moment of inertia of A with respect to the x axis. Solving Eq. (A.10) for r_x,

$$r_x = \sqrt{\frac{I_x}{A}} \tag{A.11}$$

The radii of gyration with respect to the y axis and the origin O are

$$I_y = r_y^2 A \qquad r_y = \sqrt{\frac{I_y}{A}} \tag{A.12}$$

$$J_O = r_O^2 A \qquad r_O = \sqrt{\frac{J_O}{A}} \tag{A.13}$$

Substituting for J_O, I_x, and I_y in terms of the corresponding radii of gyration in Eq. (A.9),

$$r_O^2 = r_x^2 + r_y^2 \tag{A.14}$$

The results obtained in the following two Concept Applications are included in the table for moments of inertias of common geometric shapes, located inside the back cover of this book.

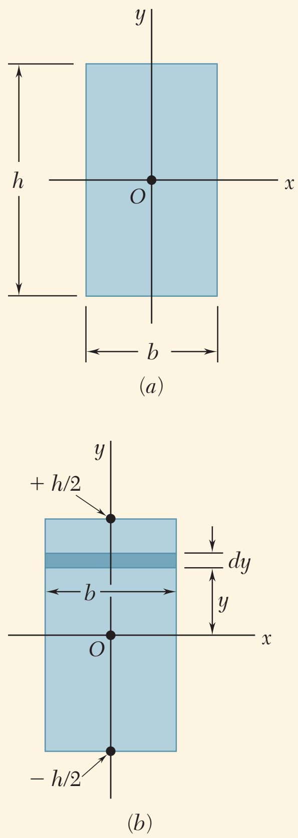


Fig. A.12 (*a*) Rectangular area. (*b*) Horizontal strip used to determine moment of inertia I_x.

Concept Application A.4

For the rectangular area of Fig. A.12*a*, determine (*a*) the moment of inertia I_x of the area with respect to the centroidal x axis, (*b*) the corresponding radius of gyration r_x.

a. Moment of Inertia I_x. We choose to select a horizontal strip of length b and thickness dy (Fig. A.12*b*). Since all of the points within the strip are at the same distance y from the x axis, the moment of inertia of the strip with respect to that axis is

$$dI_x = y^2\, dA = y^2(b\, dy)$$

Integrating from $y = -h/2$ to $y = +h/2$,

$$I_x = \int_A y^2\, dA = \int_{-h/2}^{+h/2} y^2(b\, dy) = \tfrac{1}{3}b\left[y^3\right]_{-h/2}^{+h/2}$$

$$= \tfrac{1}{3}b\left(\frac{h^3}{8} + \frac{h^3}{8}\right)$$

or

$$I_x = \tfrac{1}{12}bh^3$$

b. Radius of Gyration r_x. From Eq. (A.10),

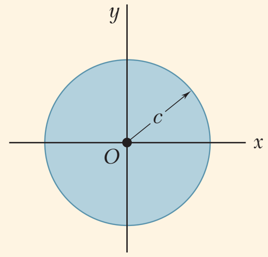
$$I_x = r_x^2 A \qquad \tfrac{1}{12}bh^3 = r_x^2(bh)$$

and solving for r_x gives

$$r_x = h/\sqrt{12}$$

Concept Application A.5

For the circular area of Fig. A.13*a*, determine (*a*) the polar moment of inertia J_O, (*b*) the rectangular moments of inertia I_x and I_y.



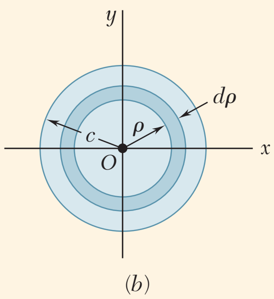


Fig. A.13 (*a*) Circular area. (*b*) Annular strip used to determine polar moment of inertia J_O.

a. Polar Moment of Inertia. We choose to select as an element of area a ring of radius ρ with a thickness $d\rho$ (Fig. A.13*b*). Since all of the points within the ring are at the same distance ρ from the origin O, the polar moment of inertia of the ring is

$$dJ_O = \rho^2\, dA = \rho^2(2\pi\rho\, d\rho)$$

Integrating in ρ from 0 to c,

$$J_O = \int_A \rho^2\, dA = \int_0^c \rho^2(2\pi\rho\, d\rho) = 2\pi \int_0^c \rho^3\, d\rho$$

$$J_O = \tfrac{1}{2}\pi c^4$$

b. Rectangular Moments of Inertia. Because of the symmetry of the circular area, $I_x = I_y$. Recalling Eq. (A.9), write

$$J_O = I_x + I_y = 2I_x \quad \tfrac{1}{2}\pi c^4 = 2I_x$$

and,

$$I_x = I_y = \tfrac{1}{4}\pi c^4$$

A.4 Parallel-Axis Theorem

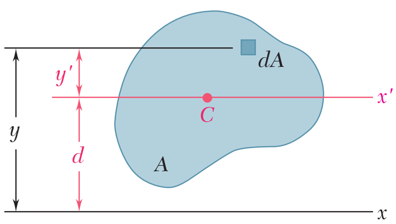


Fig. A.14 General area with centroidal x' axis, parallel to arbitrary x axis.

Consider the moment of inertia I_x of an area A with respect to an arbitrary x axis (Fig. A.14). Using y as the distance from an element of area dA to that axis, recall from Sec. A.3 that

$$I_x = \int_A y^2\, dA$$

We now draw the *centroidal x' axis,* which is the axis parallel to the x axis that passes through the centroid C. Using y' as the distance from element dA

to that axis, $y = y' + d$, where d is the distance between the two axes. Substituting for y in the integral representing I_x gives

$$I_x = \int_A y^2\, dA = \int_A (y' + d)^2 dA$$

$$I_x = \int_A y'^2\, dA + 2d \int_A y'\, dA + d^2 \int_A dA \tag{A.15}$$

The first integral in Eq. (A.15) represents the moment of inertia $\bar{I}_{x'}$ of the area with respect to the centroidal x' axis. The second integral represents the first moment $Q_{x'}$ of the area with respect to the x' axis and is equal to zero, since the centroid C of the area is located on that axis. In other words, recalling from Sec. A.1 we write

$$Q_{x'} = A\bar{y}' = A(0) = 0$$

The last integral in Eq. (A.15) is equal to the total area A. Therefore,

$$\boxed{I_x = \bar{I}_{x'} + Ad^2} \tag{A.16}$$

This equation shows that the moment of inertia I_x of an area with respect to an arbitrary x axis is equal to the moment of inertia $\bar{I}_{x'}$ of the area with respect to the centroidal x' axis parallel to the x axis *plus* the product Ad^2 of the area A and of the square of the distance d between the two axes. This result is known as the *parallel-axis theorem.* With this theorem, the moment of inertia of an area with respect to a given axis can be determined when its moment of inertia with respect to a centroidal axis of the same direction is known. Conversely, it makes it possible to determine the moment of inertia $\bar{I}_{x'}$ of an area A with respect to a centroidal axis x' when the moment of inertia I_x of A with respect to a parallel axis is known. This is done by *subtracting* from I_x the product Ad^2. Note that the parallel-axis theorem may be used *only if one of the two axes involved is a centroidal axis.*

A similar formula relates the polar moment of inertia J_O of an area with respect to an arbitrary point O and the polar moment of inertia $\bar{J}_C$ of the same area with respect to its centroid C. Using d as the distance between O and C,

$$J_O = \bar{J}_C + Ad^2 \tag{A.17}$$

A.5 Moment of Inertia of a Composite Area

Consider a composite area A made of several component parts A_1, A_2, and so forth. Since the integral for the moment of inertia of A can be subdivided into integrals extending over A_1, A_2, etc. the moment of inertia of A with respect to a given axis is obtained by adding the moments of inertia of the areas A_1, A_2, etc. with respect to the same axis. The moment of inertia of an area made of several common shapes may be found by using the formulas shown in the inside back cover of this book. Before adding the moments of inertia of the component areas, the parallel-axis theorem should be used to transfer each moment of inertia to the desired axis. This is shown in Concept Application A.6.

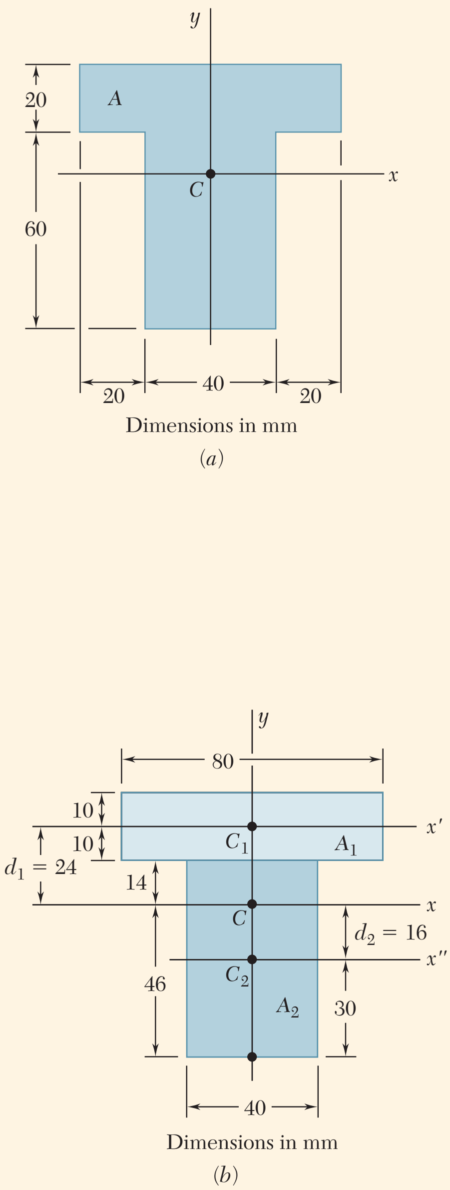


Fig. A.15 (*a*) Area *A*. (*b*) Composite areas and centroids.

Concept Application A.6

Determine the moment of inertia $\bar{I}_x$ of the area shown with respect to the centroidal x axis (Fig. A.15*a*).

Location of Centroid. The centroid C of the area has been located in Concept Application A.2 for the given area. From this, C is located 46 mm above the lower edge of area A.

Computation of Moment of Inertia. Area A is divided into two rectangular areas A_1 and A_2 (Fig. A.15*b*), and the moment of inertia of each area is found with respect to the x axis.

Rectangular Area A_1. To obtain the moment of inertia $(I_x)_1$ of A_1 with respect to the x axis, first compute the moment of inertia of A_1 *with respect to its own centroidal axis* x'. Recalling the equation in part *a* of Concept Application A.4 for the centroidal moment of inertia of a rectangular area,

$$(\bar{I}_{x'})_1 = \tfrac{1}{12}bh^3 = \tfrac{1}{12}(80\text{ mm})(20\text{ mm})^3 = 53.3 \times 10^3\text{ mm}^4$$

Using the parallel-axis theorem, transfer the moment of inertia of A_1 from its centroidal axis x' to the parallel axis x:

$$\begin{aligned}(I_x)_1 &= (\bar{I}_{x'})_1 + A_1 d_1^2 = 53.3 \times 10^3 + (80 \times 20)(24)^2 \\ &= 975 \times 10^3\text{ mm}^4\end{aligned}$$

Rectangular Area A_2. Calculate the moment of inertia of A_2 with respect to its centroidal axis x'' and use the parallel-axis theorem to transfer it to the x axis to obtain

$$(\bar{I}_{x''})_2 = \tfrac{1}{12}bh^3 = \tfrac{1}{12}(40)(60)^3 = 720 \times 10^3\text{ mm}^4$$

$$(I_x)_2 = (\bar{I}_{x''})_2 + A_2 d_2^2 = 720 \times 10^3 + (40 \times 60)(16)^2$$

$$= 1334 \times 10^3\text{ mm}^4$$

Entire Area A. Add the values for the moments of inertia of A_1 and A_2 with respect to the x axis to obtain the moment of inertia $\bar{I}_x$ of the entire area:

$$\bar{I}_x = (I_x)_1 + (I_x)_2 = 975 \times 10^3 + 1334 \times 10^3$$

$$\bar{I}_x = 2.31 \times 10^6\text{ mm}^4$$