ME232: Dynamics

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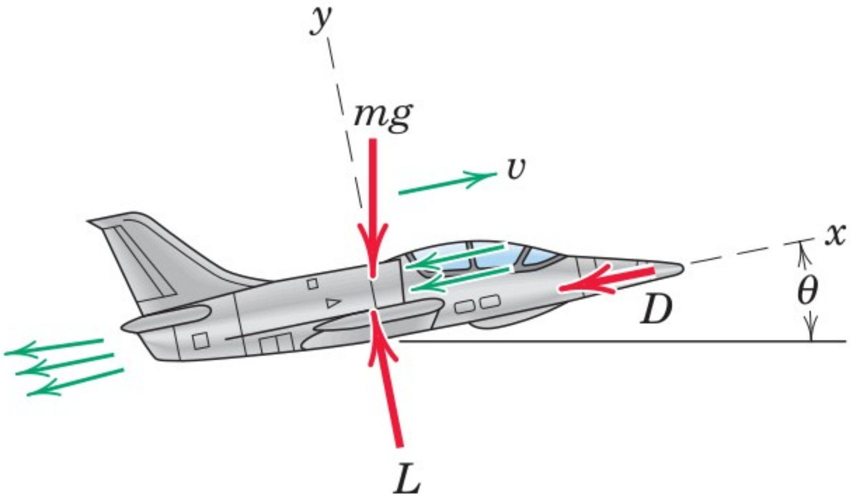
Room # 106

Example 4

An air-breathing jet aircraft of total mass m flying with a constant speed v consumes air at the mass rate m'_a and exhausts burned gas at the mass rate m'_g with a velocity u relative to the aircraft. Fuel is consumed at the constant rate m'_f. The total aerodynamic forces acting on the aircraft are the lift L, normal to the direction of flight, and the drag D, opposite to the direction of flight. Any force due to the static pressure across the inlet and exhaust surfaces is assumed to be included in D. Write the equation for the motion of the aircraft and identify the thrust T.



The free-body diagram of the aircraft together with the air, fuel, and exhaust gas within it is given and shows only the weight, lift, and drag forces as defined.

We attach axes x-y to the aircraft and apply our momentum equation relative to the moving system.

The fuel will be treated as a steady stream entering the aircraft with no velocity relative to the system and leaving with a relative velocity u in the exhaust stream.

We now treat the air and fuel flows separately. For the air flow, the change in velocity in the x-direction relative to the moving system is

$$\Delta v_a = -u - (-v) = -(u - v),$$

and for the fuel flow the x-change in velocity relative to x-y is

$$\Delta v_f = -u - (0) = -u.$$

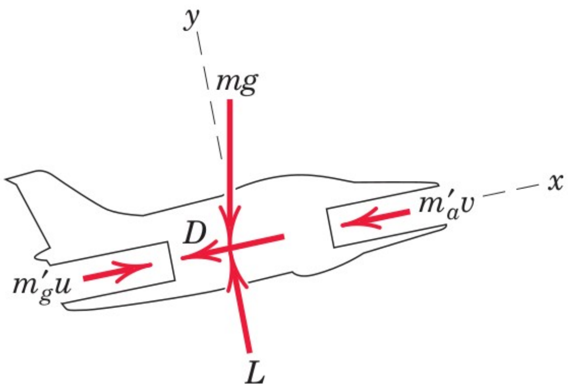
Thus, we have, $\Sigma F_x = m'\Delta v_x$,

$$-mg\sin\theta - D = -m'_a(u - v) - m'_f u = -m'_g u + m'_a v.$$

Now, $m'_g = m'_a + m'_f$,

Hence, we get

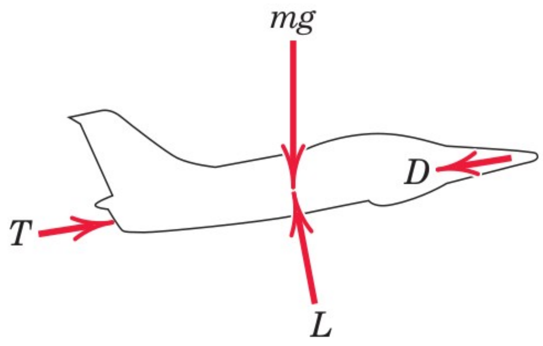
$$m'_g u - m'_a v = mg\sin\theta + D ,$$



which is the equation of motion of the system.

If we modify the boundaries of our system to expose the interior surfaces on which the air and gas act, we will have the simulated model shown, where the air exerts a force $m'_a\, v$ on the interior of the turbine and the exhaust gas reacts against the interior surfaces with the force $m'_g\, u$.

The commonly used model is shown in the final diagram, where the net effect of air and exhaust momentum changes is replaced by a simulated thrust



$$T = m'_g\, u - m'_a\, v$$

applied to the aircraft from a presumed external source.

Generally m'_f is only 2% or less of m'_{a} and we can approximate $m'_g \cong m'_a$ and express the thrust as

$$T \cong m'_g\,(u - v)$$

Variable mass

For derivation of kinetic equations for a system of particles, we assumed that the mass of the system remain constant.

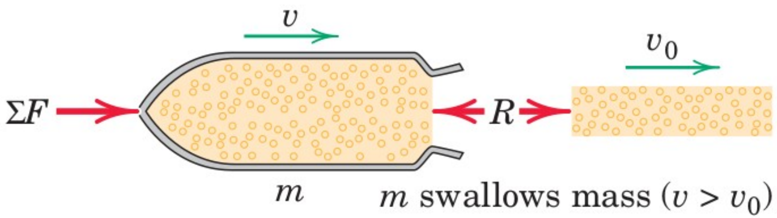
We also extended these principles to describe the action of forces on a system defined by a geometric volume through which a steady flow of mass passes. Therefore, the amount of mass within this volume was constant with respect to time.

When the mass within the boundary of a system under consideration is not constant, the foregoing relationships are no longer valid.

Equation of Motion:

We will now develop the equation for the linear motion of a system whose mass varies with time. Consider first a body which gains mass by overtaking and swallowing a stream of matter.

The mass of the body and its velocity at any instant are m and v, respectively.



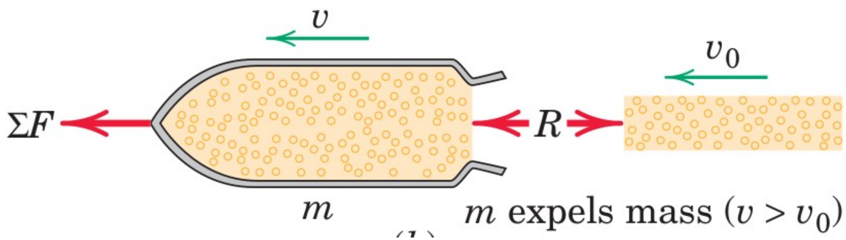
The stream of matter is assumed to be moving in the same direction as m with a constant velocity $v_0 < v$. The force exerted by m on the particles of the stream to accelerate them from a velocity v_0 to a greater velocity v is

$$R = m'(v - v_0) = \dot{m}u,$$

where the time rate of increase of m is $m' = \dot{m}$ and where u is the magnitude of the relative velocity with which the particles approach m. In addition to R, all other forces acting on m in the direction of its motion are denoted by ΣF. The equation of motion of m from Newton's second law is, therefore,

$$\Sigma F - R = m\dot{v} \qquad \text{or} \qquad \Sigma F = m\dot{v} + \dot{m}u. \qquad \text{..........(25)}$$

Similarly, if the body loses mass by expelling it rearward so that its velocity $v_0 < v$, the force R required to decelerate the particles from a velocity v to a lesser velocity v_0 is



$$R = m'\,(-v_0 - (-v)) = m'(v - v_0).$$

But $m' = -\dot{m}$ since m is decreasing. Also, the relative velocity with which the particles leave m is $u = v - v_0$. Thus, the force R becomes $R = -\dot{m}u$.

If ΣF denotes the resultant of all other forces acting on m in the direction of its motion, Newton's second law requires that

$$\Sigma F + R = m\dot{v} \qquad \text{or} \qquad \Sigma F = m\dot{v} + \dot{m}u.$$

which is the same relationship as in the case where m is gaining mass.

A frequent error in the use of the force-momentum equation is to express the partial force sum ΣF as

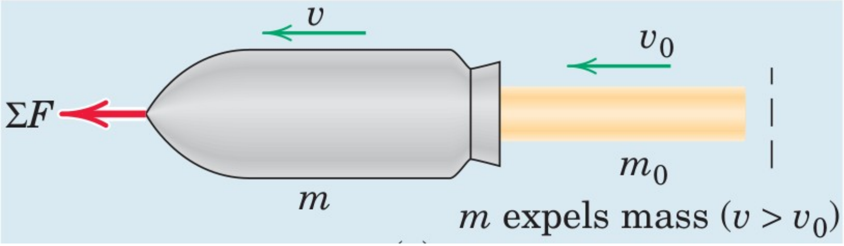
$$\Sigma F = d\,(mv)/dt = m\dot{v} + \dot{m}v.$$

From this expansion we see that the direct differentiation of the linear momentum gives the correct force ΣF only when the body picks up mass initially at rest or when it expels mass which is left with zero absolute velocity. In both instances, $v_0 = 0$ and $u = v$.

The Super Scooper is a firefighting airplane which can quickly ingest water from a lake by skimming across the surface with just a bottom-mounted scoop entering the water. The mass within the aircraft boundary varies during the scooping operation as well as during the dumping operation shown.

Alternate approach:

Equation (25) may also be obtained by a direct differentiation of the momentum from the basic relation $\Sigma F = \dot{G}$, provided a proper system of constant total mass is chosen.



Consider the case where m is losing mass. Figure shows the system of m and an arbitrary portion m_0 of the stream of ejected mass. The mass of this system is $m + m_0$ and is constant.

The ejected stream of mass is assumed to move undisturbed once separated from m, and the only force external to the entire system is ΣF which is applied directly to m as before. The reaction $R = -\dot{m}u$ is now an internal to the system. With constant total mass, the momentum principle $\Sigma F = \dot{G}$ is applicable and we have

$$\Sigma F = d(mv + m_0 v_0)/dt = m\dot{v} + \dot{m}v + m_0\dot{v}_0 + \dot{m}_0 v_0$$

Here, $\dot{m}_0 = -\dot{m}$, and the velocity of the ejected mass with respect to m is $u = v - v_0$. Also $\dot{v}_0 = 0$ since m_0 moves undisturbed with no acceleration once free of m. Thus, the relation becomes $\Sigma F = m\dot{v} + \dot{m}u$ which is identical to (25).

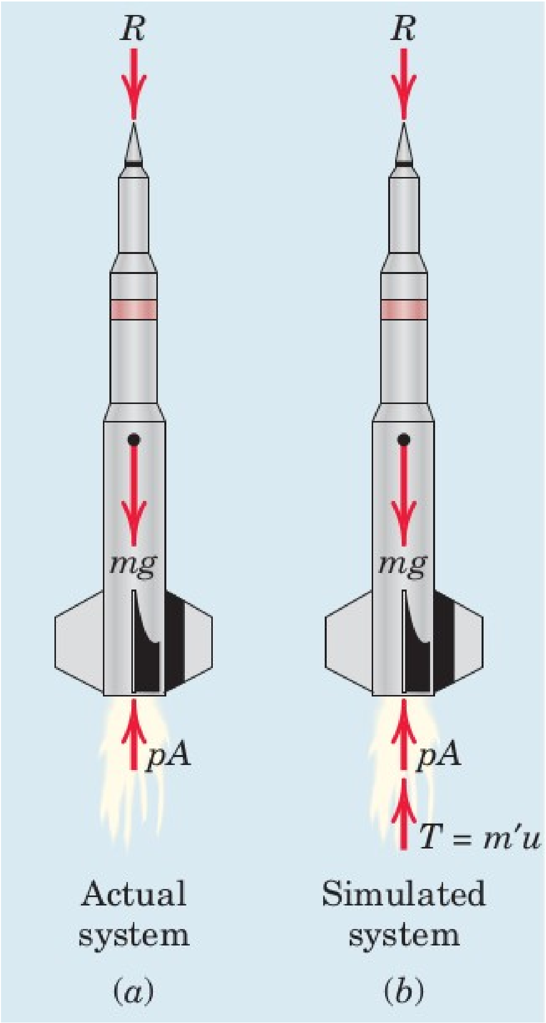
Application: Rocket propulsion

The case of m losing mass is clearly descriptive of rocket propulsion. Figure shows a vertically ascending rocket, the system for which is the mass within the volume defined by the exterior surface of the rocket and the exit plane across the nozzle. External to this system, the free-body diagram discloses the instantaneous values of gravitational attraction mg, aerodynamic resistance R, and the force pA due to the average static pressure p across the nozzle exit plane of area A. The rate of mass flow is $m' = -\dot{m}$. Thus, we may write the equation of motion of the rocket,

$$\Sigma F = m\dot{v} + \dot{m}u,$$

$$pA - mg - R = m\dot{v} + \dot{m}u, \quad \text{or}$$

$$m'u + pA - mg - R = m\dot{v}. \qquad \text{..........(26)}$$

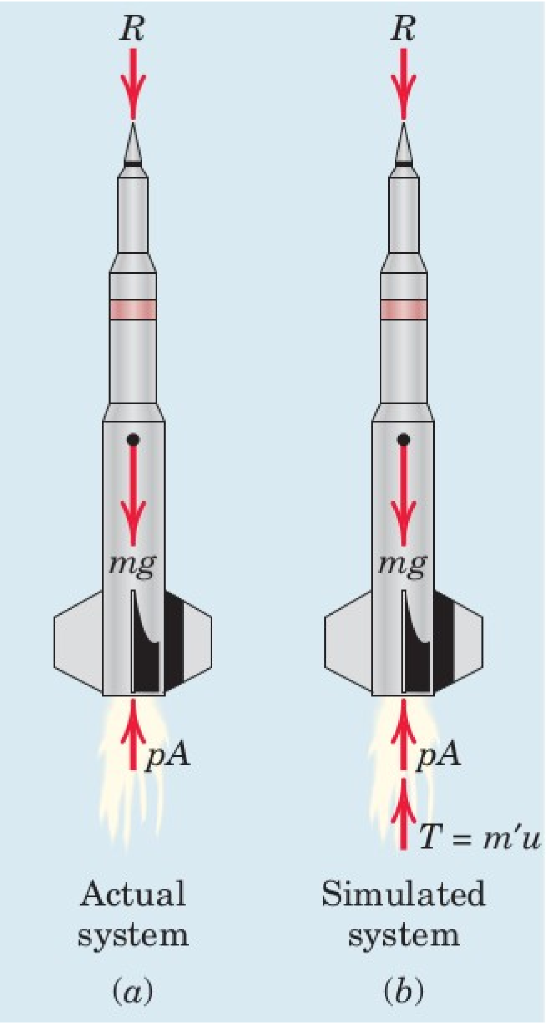


Equation (26) is of the form "$\Sigma F = ma$" where the first term in "ΣF " is the thrust $T = m'u$. Thus, the rocket may be simulated as a body to which an external thrust T is applied, and the problem may then be analyzed like any other problem, except that m is a function of time.

During the initial stages of motion

- velocity v of the rocket $<$ relative exhaust velocity u, Hence, the absolute velocity v_0 of the exhaust gases will be directed rearward.
- when rocket velocity $v > u$, the absolute velocity v_0 of the exhaust gases will be directed forward.

For a given mass rate of flow, the rocket thrust T depends only on the relative exhaust velocity u and not on the magnitude or on the direction of the absolute velocity v_0 of the exhaust gases.

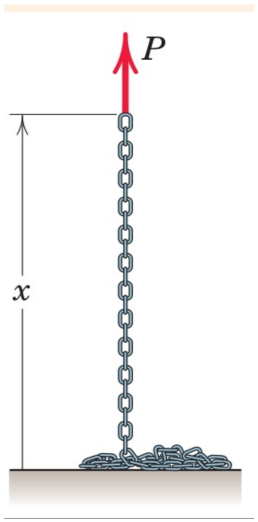


In the foregoing treatment of bodies whose mass changes with time, we have assumed that **all elements of the mass m of the body were moving with the same velocity v at any instant of time** and that the particles of mass added to or expelled from the body underwent **an abrupt transition of velocity upon entering or leaving the body**. Thus, this **velocity change has been modeled as a mathematical discontinuity**.

In reality, this change in velocity cannot be discontinuous even though the transition may be rapid. In the case of a rocket, for example, the velocity change occurs continuously in the space between the combustion zone and the exit plane of the exhaust nozzle. A more general analysis of variable-mass dynamics removes this restriction of discontinuous velocity change and introduces a slight correction to (25).

Example 5

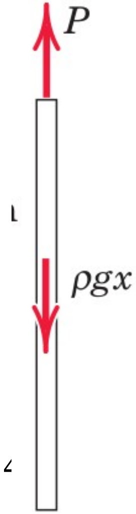
The end of a chain of length L and mass per unit length which is piled on a platform is lifted vertically with a constant velocity v by a variable force P. Find P as a function of the height x of the end above the platform.



Variable-Mass Approach:

We apply the equation of motion to the moving part of the chain of length x which is gaining mass. The force summation ΣF includes all forces acting on the moving part except the force exerted by the particles which are being attached. From the diagram we have

$$\Sigma F_x = P - \rho g x$$



The velocity is constant so that $\dot{v} = 0$. The rate of increase of mass is $\dot{m} = \rho v$, and the relative velocity with which the attaching particles approach the moving part is $u = v - 0 = v$. Thus,

$$[\Sigma F = m\dot{v} + \dot{m}u], \qquad P - \rho g x = 0 + \rho v(v), \qquad P = \rho(gx + \mathrm{v}^2)$$

The force P consists of the two parts, (i) $\rho g x$, the weight of the moving part of the chain, and (ii) ρv^2, the force required to change the momentum of the links on the platform from a condition at rest to a velocity v.

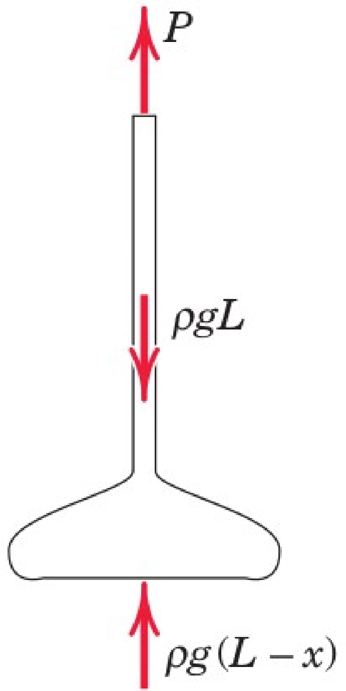
Constant-Mass Approach:

The principle of impulse and momentum for a system of particles can be applied to the entire chain considered as the system of constant mass. The free-body diagram of the system shows the unknown force P, the total weight of all links $\rho g L$, and the force $\rho g(L - x)$ exerted by the platform on those links which are at rest on it. The momentum of the system at any position is $G_x = \rho x v$ and the momentum equation gives

$$\Sigma F_x = dG_x/dt, \qquad P + \rho g(L - x) - \rho g L = d(\rho x v)/dt,$$

$$P = \rho(gx + v^2)$$

Again the force P is seen to be equal to the weight of the portion of the chain which is off the platform plus the added term which accounts for the time rate of increase of momentum of the chain.



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Introduction

- In rigid-body kinematics we use all the relationships derived for kinematics of particles and system of particles, in addition to accounting for the **rotational motion of the body**. Thus rigid-body kinematics **involves both linear and angular displacements, velocities, and accelerations**.
- The motion of rigid bodies need to be described for two important reasons.
  - First, we frequently need to generate, transmit, or control certain motions by the use of cams, gears, and linkages of various types. Here we must analyze the displacement, velocity, and acceleration of the motion to determine the design geometry of the mechanical parts. Furthermore, as a result of the motion generated, forces may be developed which must be accounted for in the design of the parts.
  - Second, we must determine the motion of a rigid body caused by the forces applied to it. Calculation of the motion of a rocket under the influence of its thrust and gravitational attraction is an example of such a problem.

Rigid body assumption and Plane motion

- A rigid body is a system of particles for which the **distances between the particles remain unchanged**. Thus, if each particle of such a body is located by a position vector from reference axes attached to and rotating with the body, there will be no change in any position vector as measured from these axes.
- A rigid body executes plane motion when all parts of the body move in parallel planes. For convenience, we generally consider the plane of motion to be the **plane which contains the mass center**, and the body is treated as a thin slab whose motion is confined to the plane of the slab. This idealization adequately describes a very large category of rigid-body motions encountered in engineering.

Plane motion