ME231: Solid Mechanics-I

Deflections due to bending

The moment-curvature relationship

For a symmetrical, linearly elastic beam element subjected to pure bending, we have derived the following moment-curvature relationship.

$$\frac{1}{\rho} = \frac{d\phi}{ds} = \frac{M_b}{EI_{zz}} \qquad \cdots\cdots\cdots\cdots(1)$$

The curvature of the neutral axis completely defines the deformation of an element in pure bending.

We can extend this to the case of general bending where the bending moment varies along the length of the beam.

We assume that the shear forces which necessarily accompany a varying bending moment do not contribute significantly to the overall deformation.

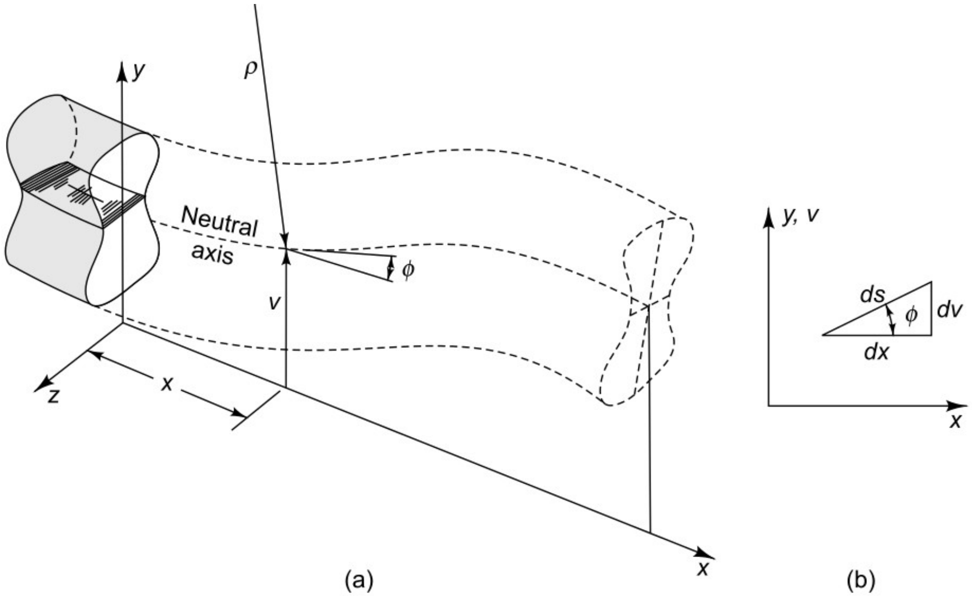
Thus we assume that the deformation is still defined by the curvature and that the curvature is still given by above relation.

Accordingly, if we know how the bending moment varies along the length of the beam, we will then know how the curvature varies.

To determine the bent shape of the beam, we thus need to be able to deduce the deflection of the neutral axis from a knowledge of its curvature. To facilitate this, we first derive a differential equation relating the curvature $d\phi/ds$ to the deflection $v(x)$.

We start with the definition of the slope of the neutral axis in

$$\frac{dv}{dx} = \tan\phi. \qquad \cdots\cdots\cdots\cdots(2)$$



Differentiating (2) w.r.t. s,

$$\frac{d^2v}{dxds} = \frac{d^2v}{dx^2}\frac{dx}{ds} = \sec^2\phi\frac{d\phi}{ds},$$

or the curvature,

$$\frac{d\phi}{ds} = \cos^2\phi\frac{d^2v}{dx^2}\frac{dx}{ds} = \frac{d^2v/dx^2}{[1+(dv/dx)^2]^{3/2}}, \qquad \left(\because \cos\phi = \frac{dx}{ds} = \frac{1}{[1+(dv/dx)^2]^{1/2}}\right)$$

$$\cdots\cdots\cdots\cdots(3)$$

From (1) and (3) we get,

$$\frac{M_b}{EI_{zz}} = \frac{d^2v/dx^2}{\left[1+(dv/dx)^2\right]^{3/2}}.$$

When the slope angle ϕ is small, then dv/dx is small compared to unity. If we neglect $(dv/dx)^2$ in the denominator, we obtain a simple approximation as

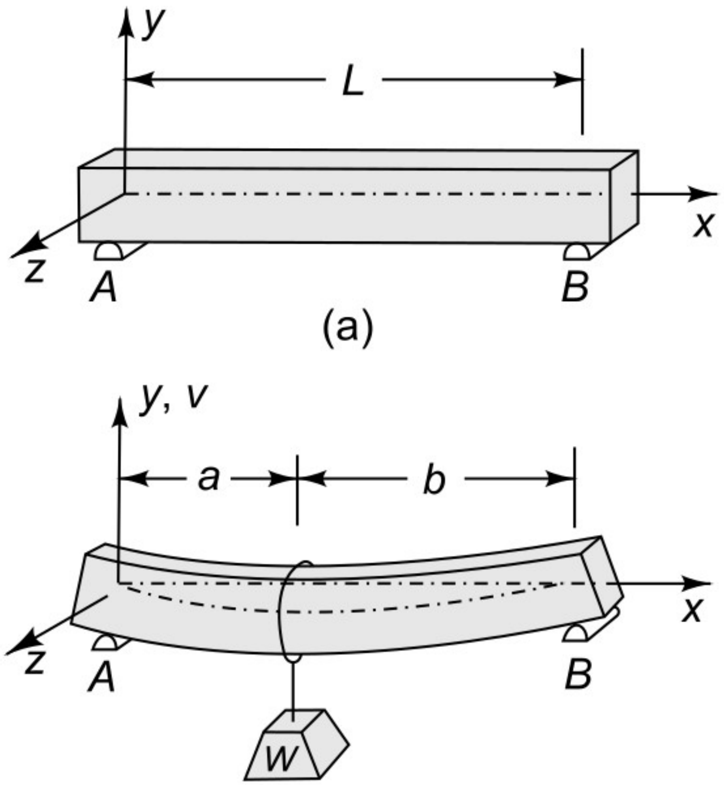
$$\frac{M_b}{EI_{zz}} \approx \frac{d^2v}{dx^2}. \qquad \cdots\cdots\cdots\cdots(4)$$

Equation (4) relates the bending moment to the transverse displacement. Although (4) involves an approximation to the curvature which is valid only for small bending angles, we shall henceforth call it the moment-curvature relation. It is essentially a “force-deformation” or “stress-strain” relation in which the bending moment is the “force” or “stress” and the approximate curvature is the resulting “deformation” or “strain.”

The relation is a linear one; the constant of proportionality EI is sometimes called the flexural rigidity or the bending modulus.

Example 1

The simply supported beam of uniform cross section shown in figure is subjected to a concentrated load W. It is desired to obtain the deflection curve of the deformed neutral axis.



From the bending moment analysis of the beam, it can be shown that the distribution of bending moment is as follows,

$$M_b = \frac{Wb}{L}x - W\left\langle x - a \right\rangle^1 .$$

..........(1.a)

For determining the displacement, we use the moment-curvature relationship (4) as

$$EI\frac{d^2v}{dx^2} = M_b = \frac{Wb}{L}x - W\langle x-a\rangle^1 \qquad \cdots\cdots\cdots\cdots(1.b)$$

Since EI is constant throughout the beam, integration of (1.b) gives following,

$$EI\frac{dv}{dx} = \frac{Wb}{L}\frac{x^2}{2} - W\frac{\langle x-a\rangle^2}{2} + c_1$$

$$EIv = \frac{Wb}{L}\frac{x^3}{6} - W\frac{\langle x-a\rangle^3}{6} + c_1x + c_2$$

Constants of integration c_1 and c_2 can be determined by applying the following boundary conditions, v=0 at x=a and at x=L. We get

$$c_2 = 0,\ \ 0 = \frac{Wb}{6}L^2 - \frac{Wb^3}{6} + c_1L.$$

So finally,

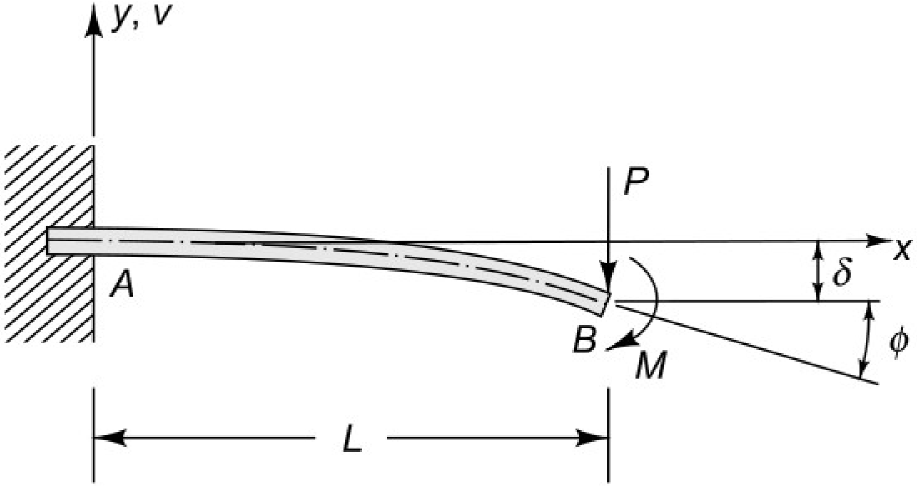
$$v = -\frac{W}{6EI}\left[\frac{bx}{L}\left(L^2 - b^2 - x^2\right) + \langle x-a\rangle^3\right].$$

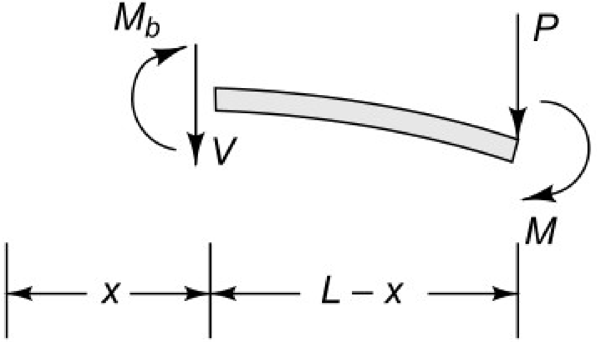
Check deflection and slope for the given data, which correspond to a very common case in small-house construction. Verify that the assumptions considered for the derivation are satisfied.

$$L = 3.70 \text{ m}$$
$$a = b = 1.85 \text{ m}$$
$$W = 1.8 \text{ kN}$$
$$E = 11 \text{ GN/m}^2$$
$$I = 3.33 \times 10^7 \text{ mm}^4$$

Example 2

A uniform cantilever beam has bending modulus EI and length L. It is built in at A and subjected to a concentrated force P and moment M applied at B, as shown in figure. We shall find the deflection δ and the slope angle ϕ at B due to these loads.





$$M_b = -P(L - x) - M$$

B.C.,

$v = 0$ at $x = 0$ and

$\phi = dv/dx = 0$ at $x = 0$.

Use moment-curvature relations, integrate and apply boundary conditions to find the expression of the deflection as follows.

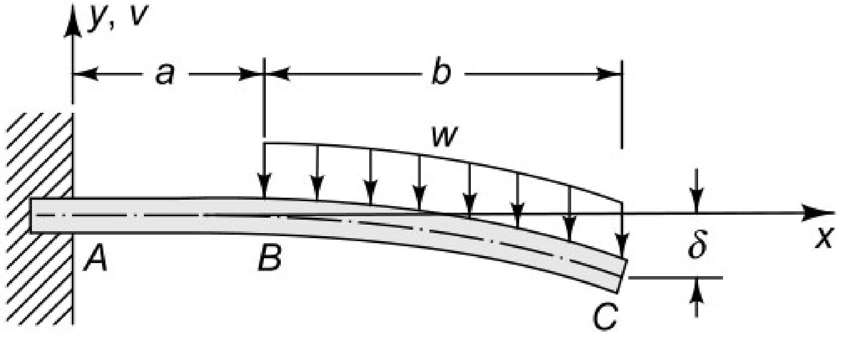
$$v = -\frac{1}{EI}\left[P\frac{x^2}{6}(3L - x) + M\frac{x^2}{2} \right]$$

$$\delta = -(v)_{x=L} = \frac{PL^3}{3EI} + \frac{ML^2}{2EI}$$

$$\phi = -\left(\frac{dv}{dx}\right)_{x=L} = \frac{PL^2}{2EI} + \frac{ML}{EI}$$

Example 3

Figure shows a cantilever beam built-in at A and subjected to a uniformly distributed load of intensity w per unit length acting on the segment BC. It is desired to obtain the deflection δ of the neutral axis at C due to the distributed load in terms of the constant bending modulus EI and the dimensions shown.

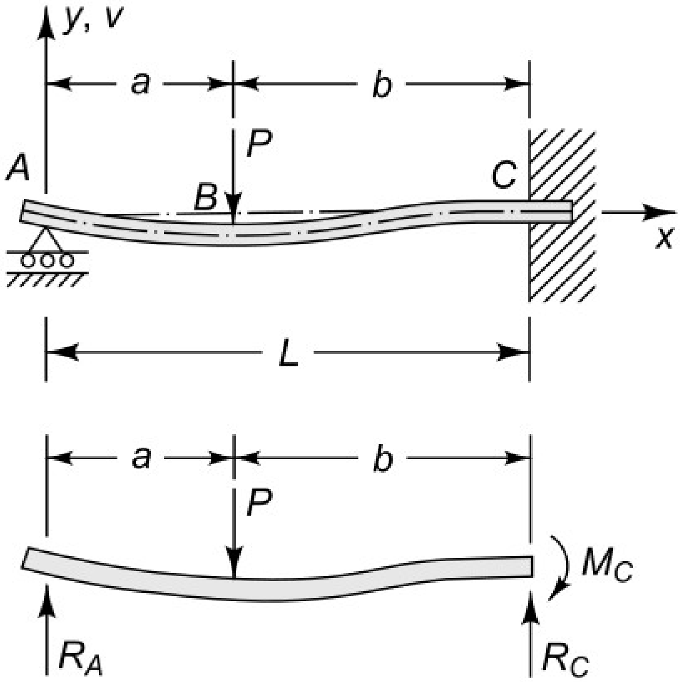


$$EI\frac{d^2v}{dx^2} = M_b = wbx - wb\left(a+\frac{b}{2}\right) - \frac{w\langle x-a\rangle^2}{2}$$

$$\left(\frac{dv}{dx}\right)_{x=0} = 0 \qquad (v)_{x=0} = 0$$

Example 4

Figure shows a beam whose neutral axis coincided with the x axis before the load P was applied. The beam has a simple support at A and a clamped or built-in support at C. The bending modulus EI is constant along the length of the beam.



$$R_C = P - R_A$$

$$M_C = Pb - R_A L$$

$$M_b = R_A x - P\langle x - a\rangle^1$$

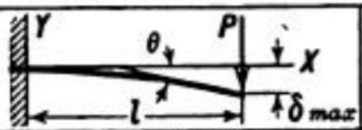
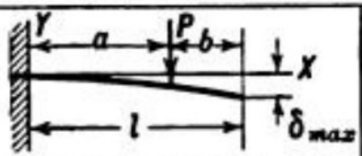
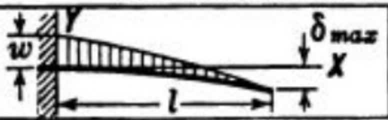
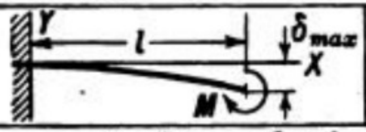
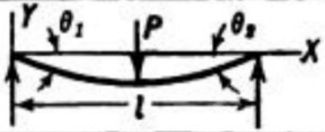
$v = 0$ at $x = 0$

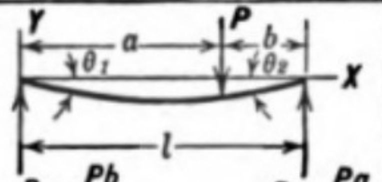
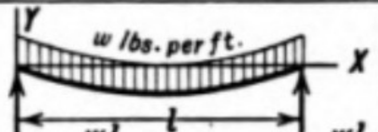
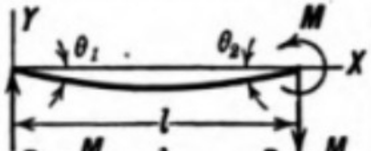
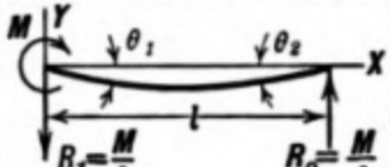
$v = 0$ at $x = L$

$\dfrac{dv}{dx} = 0$ at $x = L$

$$EI\frac{d^2v}{dx^2} = M_b = R_A x - P\langle x - a\rangle^1$$

$$R_A = \frac{Pb^2}{2L^3}(3L - b)$$

ART. 49	Slope at free end.	Deflection at any section in terms of x: δ is positive downward.	Maximum deflection.
1. Cantilever Beam — Concentrated load P at the free end.			
	$\theta = \frac{Pl^2}{2EI}$	$\delta = \frac{Px^2}{6EI}(3l - x)$	$\delta_{max} = \frac{Pl^3}{3EI}$
2. Cantilever Beam — Concentrated load P at any point.			
	$\theta = \frac{Pa^2}{2EI}$	$\delta = \frac{Px^2}{6EI}(3a - x)$ for $0 < x < a$ $\delta = \frac{Pa^2}{6EI}(3x - a)$ for $a < x < l$	$\delta_{max} = \frac{Pa^2}{6EI}(3l - a)$
3. Cantilever Beam — Uniformly distributed load of w lbs. per ft. over entire length.			
	$\theta = \frac{wl^3}{6EI}$	$\delta = \frac{wx^2}{24EI}(x^2 + 6l^2 - 4lx)$	$\delta_{max} = \frac{wl^4}{8EI}$
4. Cantilever Beam — Uniformly varying load; maximum intensity w lbs. per ft.			
	$\theta = \frac{wl^3}{24EI}$	$\delta = \frac{wx^2}{120\,lEI}(10l^3 - 10l^2x + 5lx^2 - x^3)$	$\delta_{max} = \frac{wl^4}{30EI}$
5. Cantilever Beam — Couple M applied at the free end.			
	$\theta = \frac{Ml}{EI}$	$\delta = \frac{Mx^2}{2EI}$	$\delta_{max} = \frac{Ml^2}{2EI}$
6. Beam freely supported at ends — Concentrated load P at the center.			
	$\theta_1 = \theta_2 = \frac{Pl^2}{16EI}$	$\delta = \frac{Px}{12EI}\left(\frac{3l^2}{4} - x^2\right)$ for $0 < x < \frac{l}{2}$	$\delta_{max} = \frac{Pl^3}{48EI}$

ART. 49	Slope at ends.	Deflection at any section in terms of x: δ is positive downward.	Maximum and center deflections.
7. Beam freely supported at the ends — Concentrated load at any point.			
 $R_1=\frac{Pb}{l}$ $R_2=\frac{Pa}{l}$	Left End. $\theta_1=\frac{Pb(l^2-b^2)}{6lEI}$ Right End. $\theta_2=\frac{Pab(2l-b)}{6lEI}$	To the left of load P: $\delta=\frac{Pbx}{6lEI}(l^2-x^2-b^2)$ To the right of load P: $\delta=\frac{Pb}{6lEI}\left[\frac{l}{b}(x-a)^3+(l^2-b^2)x-x^3\right]$	$\delta_{max}=\frac{Pb(l^2-b^2)^{3/2}}{9\sqrt{3}\,l\,EI}$ at $x=\sqrt{\frac{l^2-b^2}{3}}$ At center, if $a>b$ $\delta=\frac{Pb}{48EI}(3l^2-4b^2)$
8. Beam freely supported at the ends—Uniformly distributed load of w lbs. per ft.			
 $R_1=\frac{wl}{2}$ $R_2=\frac{wl}{2}$	$\theta_1=\theta_2=\frac{wl^3}{24EI}$	$\delta=\frac{wx}{24EI}(l^3-2lx^2+x^3)$	$\delta_{max}=\frac{5wl^4}{384EI}$
9. Beam freely supported at the ends — Couple M at the right end.			
 $R_1=\frac{M}{l}$ $R_2=\frac{M}{l}$	$\theta_1=\frac{Ml}{6EI}$ $\theta_2=\frac{Ml}{3EI}$	$\delta=\frac{Mlx}{6EI}\left(1-\frac{x^2}{l^2}\right)$	$\delta_{max}=\frac{Ml^2}{9\sqrt{3}\,EI}$ at $x=\frac{l}{\sqrt{3}}$ At center $\delta=\frac{Ml^2}{16EI}$
10. Beam freely supported at the ends—Couple M at the left end.			
 $R_1=\frac{M}{l}$ $R_2=\frac{M}{l}$	$\theta_1=\frac{Ml}{3EI}$ $\theta_2=\frac{Ml}{6EI}$	$\delta=\frac{Mx}{6lEI}(l-x)(2l-x)$	$\delta_{max}=\frac{Ml^2}{9\sqrt{3}\,EI}$ at $x=\left(1-\frac{1}{\sqrt{3}}\right)l$ At center $\delta=\frac{Ml^2}{16EI}$