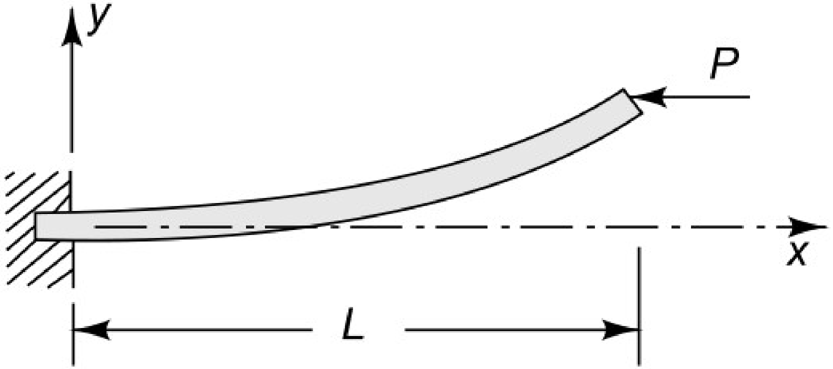
Stability of equilibrium: Buckling

Now we come to elastic-instability problem for a cantilever beam (column).



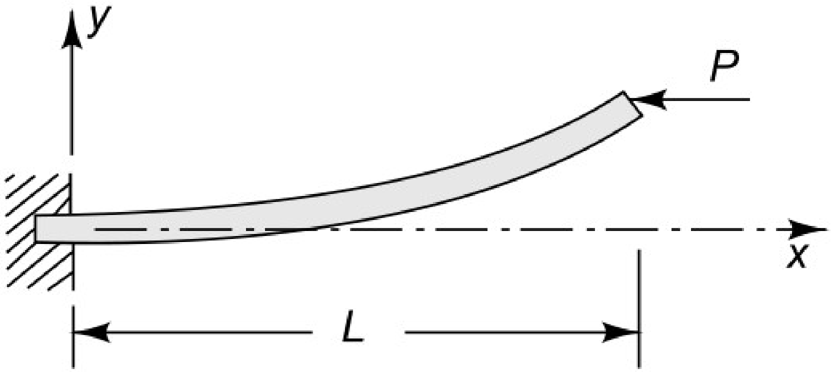
We consider a beam with constant flextural modulus (EI) and we assume that the buckling happens in the xy-plane.

If the beam should be accidentally displaced from a straight position, then force P will produce a bending moment along the beam which caused it to bend even further, which elastic forces in in the beam tend to restore it to original position.

For small load the straight beam is stable and the beam is subjected to uniform compression. For large loads the straight position is unstable and the beam buckles.

The most important result is the critical load which marks the border between stability and instability.

We obtain the critical load by arguing that when the critical load is acting, the restoring tendencies just balance the upsetting tendencies, and the system is in a state of neutral equilibrium. Thus the figure shows an equilibrium position.



The magnitude of the critical load and the shape of the bent beam are initially unknown. We find them by requiring that the governing equation (4) with q=0, and the following boundary conditions to be satisfied.

$$v = 0 \text{ and } \frac{dv}{dx} = 0 \quad \text{at} \quad x = 0, \qquad \cdots\cdots\cdots\cdots\cdots\cdots(5)$$

$$M_b = 0 \text{ and } V = 0 \quad \text{at} \quad x = L. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(6)$$

Conditions (6) can be represented in terms of displacement v as

$$M_b = EI\frac{d^2v}{dx^2} = 0 \quad \text{at} \quad x = L. \qquad \text{................(7a)}$$

$$-V = \frac{dM_b}{dx} + P\frac{dv}{dx} = \frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right) + P\frac{dv}{dx} \quad \text{at} \quad x = L. \qquad \text{................(7b)}$$

Now our job is to find the equilibrium configuration $v(x)$ and the critical load P which simultaneously satisfy (4a) and the boundary condition at x=0 and x=L.

Note that $v(x)$=0 is a solution for all values of P. This means that the straight position of the column is always a possible equilibrium position. This solution is a trivial solution, whereas we are interested in the nonstright equilibrium position. It can be shown that the general solution of (4a) is

$$v = c_1 + c_2 x + c_3 \sin\sqrt{\frac{P}{EI}}x + c_4 \cos\sqrt{\frac{P}{EI}}x. \qquad \text{................(8)}$$

The solution satisfy differential equation (4a) for arbitrary values of constants c_1 to c_4. Now we substitute (8) in boundary conditions (5) and (7) to obtain following algebric equations,

$$
\begin{aligned}
& c_1 + c_4 = 0 \\
& c_2 + c_3\sqrt{\frac{P}{EI}} = 0 \\
& -c_3\frac{P}{EI}\sin\sqrt{\frac{P}{EI}}L - c_4\frac{P}{EI}\cos\sqrt{\frac{P}{EI}}L = 0 \\
& c_2 P = 0
\end{aligned}
\qquad \text{................}(9)
$$

Observe that the obvious solution of equations (9) is $c_1 = c_2 = c_3 = c_4 = 0$, which is again a trivial solution which gives $v = 0$ (a straight beam). Our objective is to find another solution which can be obtained by using the first equation as $c_4 = -c_1$ and from seond and fourth equations we get $c_2 = c_3 = 0$, then the third equation gives,

$$
c_4\frac{P}{EI}\cos\sqrt{\frac{P}{EI}}L = 0 \Rightarrow \cos\sqrt{\frac{P}{EI}}L = 0. \qquad \text{................}(10)
$$

Equation (10) give the solution

$$P = \frac{(2n-1)^2\pi^2}{4}\frac{EI}{L^2}.$$

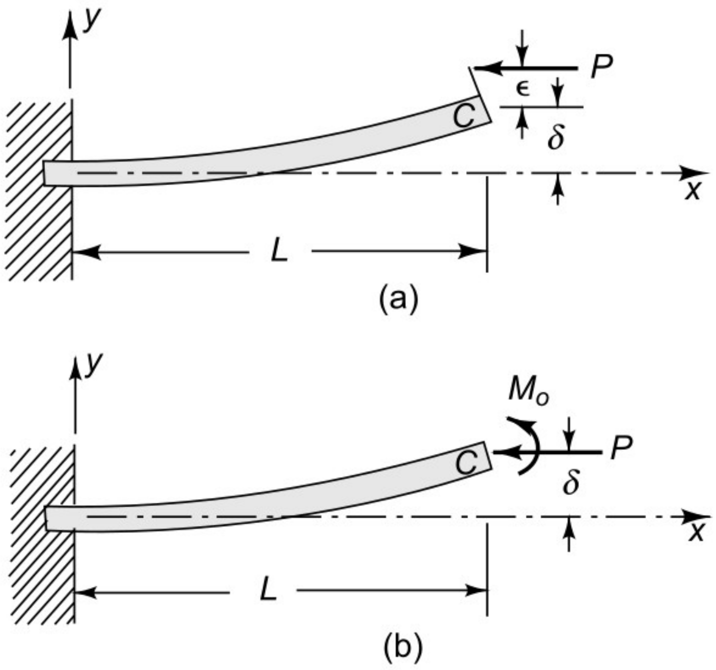
Smallest value of P is obtained for n=1 as $$P_{\mathrm{cr}} = \frac{\pi^2}{4}\frac{EI}{L^2}. \qquad \text{................(11)}$$

Substituting (11) back into (8), the corresponding deflection curve is

$$v = c_1\left(1 - \cos\frac{\pi x}{2L}\right). \qquad \text{................(12)}$$

Note that the deflection v is an equilibrium position for any arbitrary value of c_1 so long as the deflection is within the validity of small deformation theory. It can be shown that all neutral equilibrium position; i.e., the critical load will hold any of these deflections in equilibrium.

For smaller values of P the straight column is stable; i.e., if accidental bending occurs, the restoring tendencies overcome the upsetting tendencies. For large values of P straight position is no longer stable; any small disturbance will result in buckling.



Additional understanding on the buckling of the columns can be obtained by considering a case of eccentric loading, which can be because of imperfection in the column or the loading mechanism.

An eccentric force P is statically equivalent to a central force P and a moment $P\epsilon$.

For this problem, the governing equation remain same as (4a). Boundary conditions are also similar expect (7a) which become

$$M_b = EI\frac{d^2v}{dx^2} = M_O \quad \text{at} \quad x = L. \qquad \qquad \text{................}(13)$$

Hence, the general solution (8) is still applicable and the boundary conditions now provide the following equations:

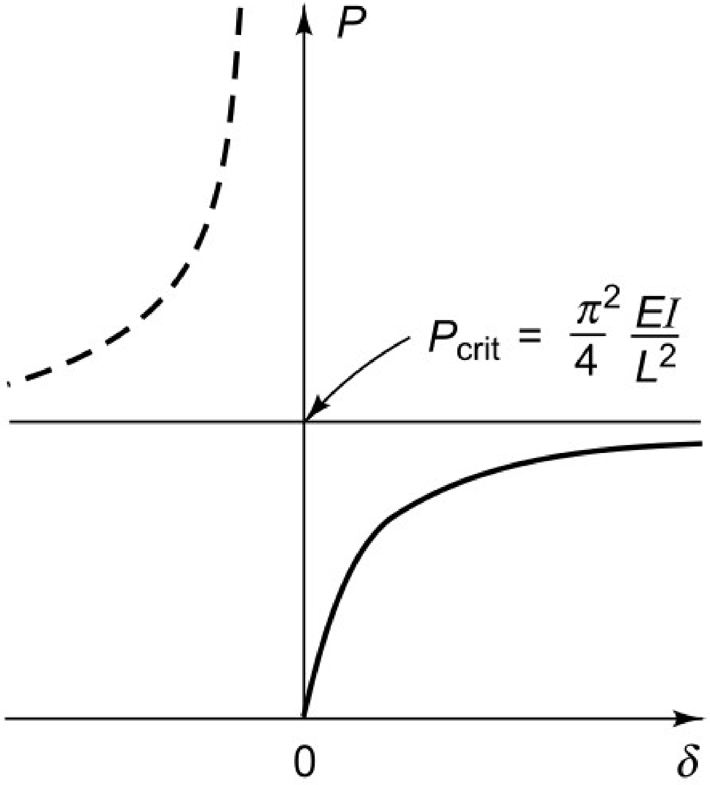
$$
\begin{aligned}
&c_1 + c_4 = 0 \\
&c_2 + c_3\sqrt{\frac{P}{EI}} = 0 \\
&-c_3\frac{P}{EI}\sin\sqrt{\frac{P}{EI}}L - c_4\frac{P}{EI}\cos\sqrt{\frac{P}{EI}}L = \frac{M_O}{EI} \qquad \cdots\cdots\cdots\cdots\cdots(14) \\
&c_2 P = 0
\end{aligned}
$$

Now these equations will have a unique solution for constants which gives

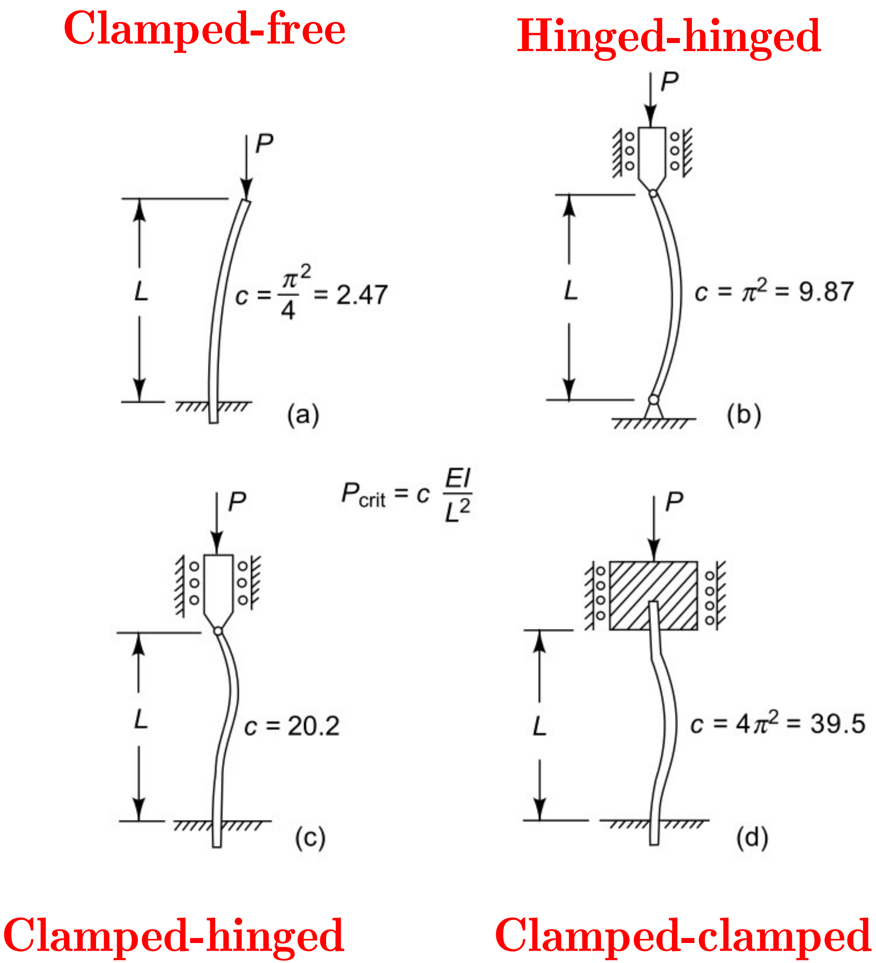
$$
v = \frac{M_O}{P}\left(\frac{1-\cos\sqrt{P/EI}x}{\cos\sqrt{P/EI}L}\right). \qquad \cdots\cdots\cdots\cdots\cdots(15)
$$

Putting x=L gives,

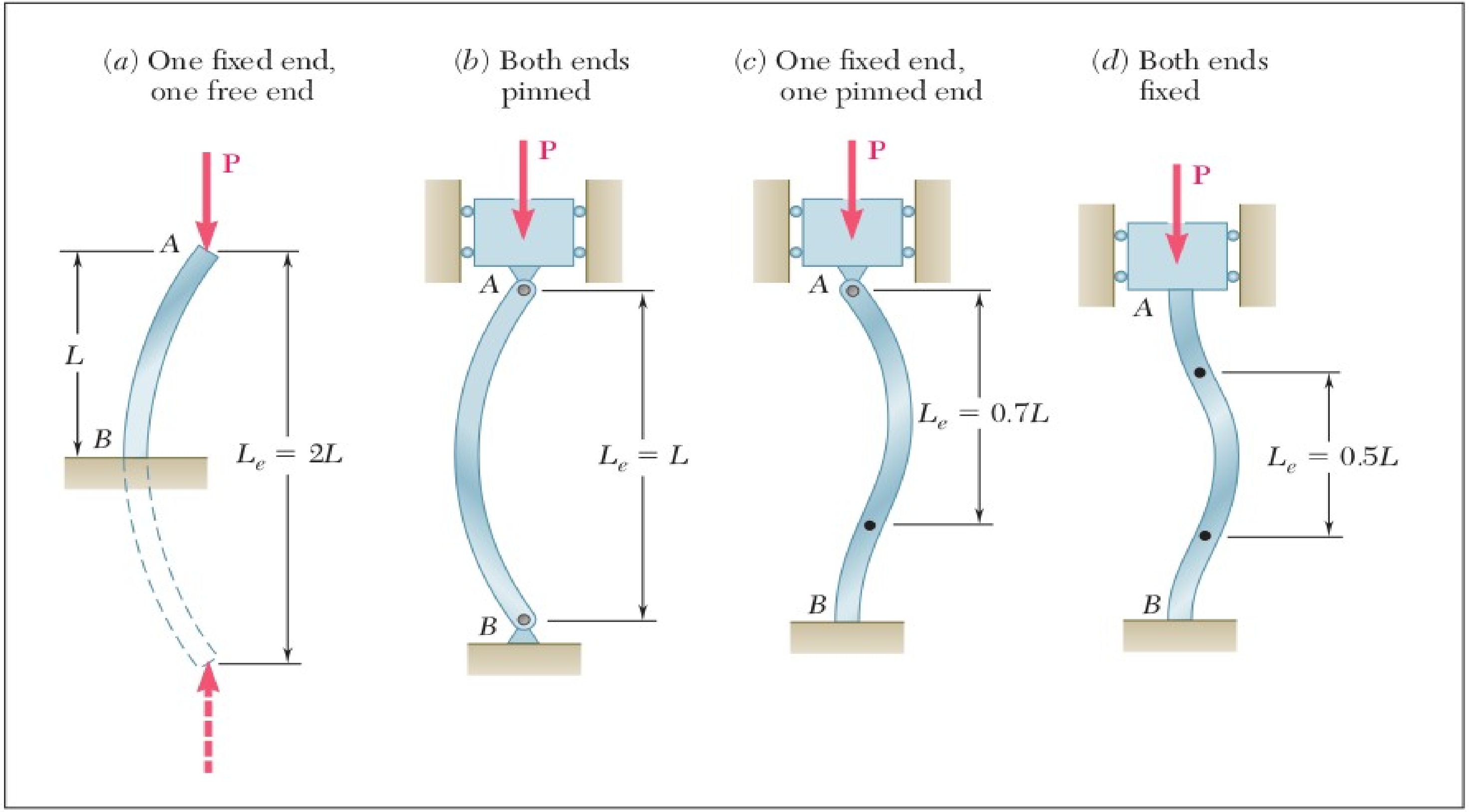
$$
\delta = \frac{M_O}{P}\left(\sec\sqrt{\frac{P}{EI}}L - 1\right) = \epsilon\left(\sec\sqrt{\frac{P}{EI}}L - 1\right). \qquad \cdots\cdots\cdots\cdots\cdots(16)
$$


P
$P_{crit} = \frac{\pi^2}{4}\frac{EI}{L^2}$
0
δ

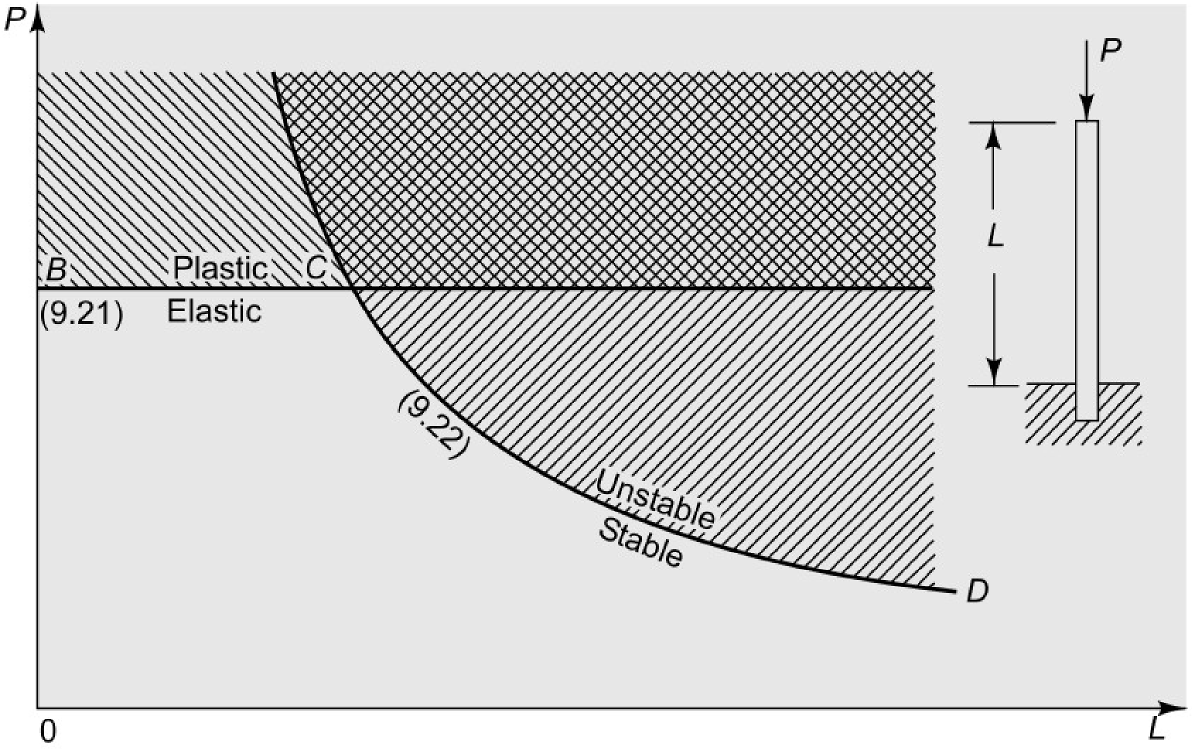
The critical load of a column is quite sensitive to the nature of the supports at the ends of the column.



$$P_{\mathrm{cr}} = \frac{\pi^2 EI}{L_e^2}.$$

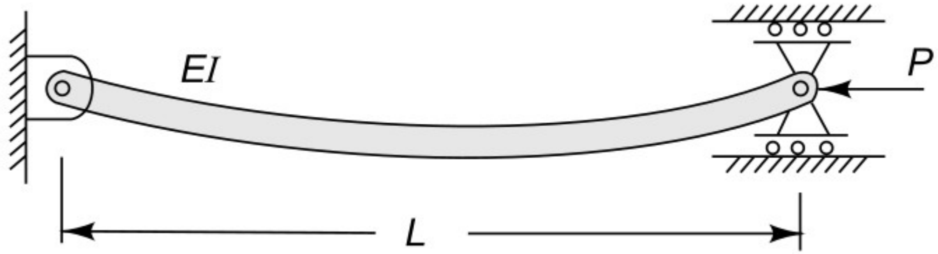


Instability as a mode of failure



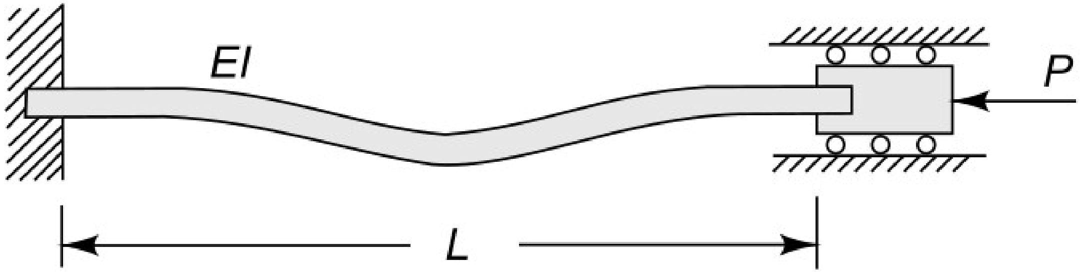
Example 1

Find the critical elastic compressive load for a uniform flexible beam which is hinged at both ends.



Example 2

Find the critical elastic compressive load for a uniform ﬂ exible beam which is clamped at both ends.



Example 3

An aluminum column with a length of L and a rectangular cross section has a fixed end B and supports a centric load at A. Two smooth and rounded fixed plates restrain end A from moving in one of the vertical planes of symmetry of the column but allow it to move in the other plane. (a) Determine the ratio a/b of the two sides of the cross section corresponding to the most efficient design against buckling. (b) Design the most efficient cross section for the column, knowing that L=0.5 m, E=70 GPa, P=20 kN.