Hyperelasticity

Constitutive equation

- We already derived the equilibrium equations, which are written in terms of the stresses. These stresses result from the deformation of the material, and it must be expressed in terms of some deformation measure. The relationship, between the stress and the deformation measure is known as **constitutive equations**.
- Obviously constitutive equations will depend on the type of material under consideration and may be dependent on or independent of time.
- For example, the small strain linear elasticity equations involving Young modulus and Poisson ratio are time-independent, whereas viscous fluids are clearly entirely dependent on strain rate.
- Generally, constitutive equations must satisfy certain physical principles, such as material objectivity.
- Materials for which the constitutive behavior is only a function of the current state of deformation (and not the history of deformation) are generally known as **elastic materials**.

Hyperelastic material

- When materials respond elastically even when they are subjected to **very large strains**, they are called **hyperelastic materials**. They account for both nonlinear material behavior and large shape changes.
- The nonlinear constitutive theory is suitable to describe the behaviour of hyperelastic material is known as **hyperelasticity**.
- The main applications of the theory are
  - to model the rubbery behavior of a polymeric material and
  - to model polymeric foams that can be subjected to large reversible shape changes (e.g., a sponge).
- We are particularly discussing the constitutive equations of hyperelastic material because of their simplicity, and because it constitutes the basis for more complex material models such as elastoplasticity, viscoplasticity, and viscoelasticity.

Hyperelastic material

- We will discuss the so-called **phenomenological approach**, which describe the macroscopic nature of materials as a continuous medium. The phenomenological approach is mainly concerned with fitting mathematical equations to experimental data and is particularly successful in solid mechanics. However, phenomenological modeling is **not capable of relating the mechanism of deformation to the underlying physical (microscopic) structure of the material.**

- Since the material is elastic, it **does not dissipate energy** during loading-unloading cycles.

- Recall the Clausius-Planck inequality for **a reversible elastic material**,

$$\mathcal{D}_{\text{int}} = \boldsymbol{P} : \dot{\boldsymbol{F}} - \dot{\Psi} = 0 \quad \Rightarrow \quad \dot{\Psi} = \boldsymbol{P} : \dot{\boldsymbol{F}}. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(1)$$

- Presuming that from physical experiments it is possible to construct the function $\Psi(\boldsymbol{F},\boldsymbol{X})$, which defines a given material, then the rate of change of the potential can be expressed as

$$\dot{\Psi} = \frac{\partial \Psi}{\partial \boldsymbol{F}} : \dot{\boldsymbol{F}}, \quad \text{or} \quad \dot{\Psi} = \frac{\partial \Psi}{\partial F_{ij}} \dot{F}_{ij}. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(2)$$

Comparing (1) and (2), we get

$$\boldsymbol{P} = \frac{\partial \Psi}{\partial \boldsymbol{F}} \quad \text{or} \quad \boldsymbol{P}(\boldsymbol{F}, \boldsymbol{X}) = \frac{\partial \Psi(\boldsymbol{F}, \boldsymbol{X})}{\partial \boldsymbol{F}}. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(3)$$

- The general constitutive equation (3) can be further developed by imposing the **restrictions of material objectivity**, which requires Ψ to be remain invariant when the current configuration undergoes a rigid body rotation. This implies that Ψ depends on $\boldsymbol{F}$ only via the stretch component $\boldsymbol{U}$ and is **independent of the rotation component $\boldsymbol{R}$** . Hence, for convenience, Ψ is often expressed as a function of $\boldsymbol{C} = \boldsymbol{U}^2 = \boldsymbol{F}^T\boldsymbol{F}$ as,

$$\Psi(\boldsymbol{F}, \boldsymbol{X}) = \Psi(\boldsymbol{C}, \boldsymbol{X}). \qquad \cdots\cdots\cdots\cdots\cdots\cdots(4)$$

Hence, (2) become,

$$\dot{\Psi} = \frac{\partial \Psi}{\partial \boldsymbol{C}} : \dot{\boldsymbol{C}} = \frac{1}{2}\boldsymbol{S} : \dot{\boldsymbol{C}}, \quad \boldsymbol{S} = 2\frac{\partial \Psi}{\partial \boldsymbol{C}} = \frac{\partial \Psi}{\partial \boldsymbol{E}}. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(5)$$

Equation (5) is the Lagrangian form of constitutive equation.

- The hyperelastic constitutive equation (5) is unrestricted in their application. We are now going to restrict these equations to the common and important isotropic case.

- **Isotropy** is defined as the requirement of the constitutive behavior to be **independent of any material direction**, which implies that the relationship between Ψ and $\boldsymbol{C}$ must be independent of the material axes chosen and, consequently, Ψ must only be **a function of the invariants of $\boldsymbol{C}$** as

$$\Psi(\boldsymbol{C}, \boldsymbol{X}) = \Psi(I_{1C}, I_{2C}, I_{3C}, \boldsymbol{X}). \qquad \text{.....(6)}$$

- With this restriction,

$$\boldsymbol{S} = 2\frac{\partial \Psi}{\partial \boldsymbol{C}} = 2\left[\frac{\partial \Psi}{\partial I_{1C}}\frac{\partial I_{1C}}{\partial \boldsymbol{C}} + \frac{\partial \Psi}{\partial I_{2C}}\frac{\partial I_{2C}}{\partial \boldsymbol{C}} + \frac{\partial \Psi}{\partial I_{3C}}\frac{\partial I_{3C}}{\partial \boldsymbol{C}}\right]$$

$$\boldsymbol{S} = 2\left[\frac{\partial \Psi}{\partial I_{1C}}\boldsymbol{I} + 2\frac{\partial \Psi}{\partial I_{2C}}\boldsymbol{C} + J^2\frac{\partial \Psi}{\partial I_{3C}}\boldsymbol{C}^{-1}\right] \qquad \text{.....(7)}$$

- In design practice, it is the Cauchy stresses that are of engineering significance, which can be obtained from the 2nd PK stress as

$$\boldsymbol{\sigma} = \frac{1}{J}\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^T = \frac{2}{J}\left[\frac{\partial \Psi}{\partial I_{1b}}\boldsymbol{b} + 2\frac{\partial \Psi}{\partial I_{2b}}\boldsymbol{b}^2 + J^2\frac{\partial \Psi}{\partial I_{3b}}\boldsymbol{I}\right]. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(8)$$

St. Venant–Kirchhoff Material:

- It is one of the simplest example of a hyperelastic material. The strain energy function Ψ for this material is defined as,

$$\Psi(\boldsymbol{E}) = \frac{1}{2}\lambda(\mathrm{tr}\boldsymbol{E})^2 + \mu\boldsymbol{E}\colon\boldsymbol{E}, \qquad \cdots\cdots\cdots\cdots\cdots\cdots(9)$$

where λ and μ are material coefficients. Using (5) following expression for 2nd PK stress can be derived.

$$\boldsymbol{S} = \lambda(\mathrm{tr}\boldsymbol{E})\boldsymbol{I} + 2\mu\boldsymbol{E}. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(10)$$

Does equation (10) look familiar to you?