ME632: Fracture Mechanics

Timings	
Monday	10:00 to 11:20
Thursday	08:30 to 09:50

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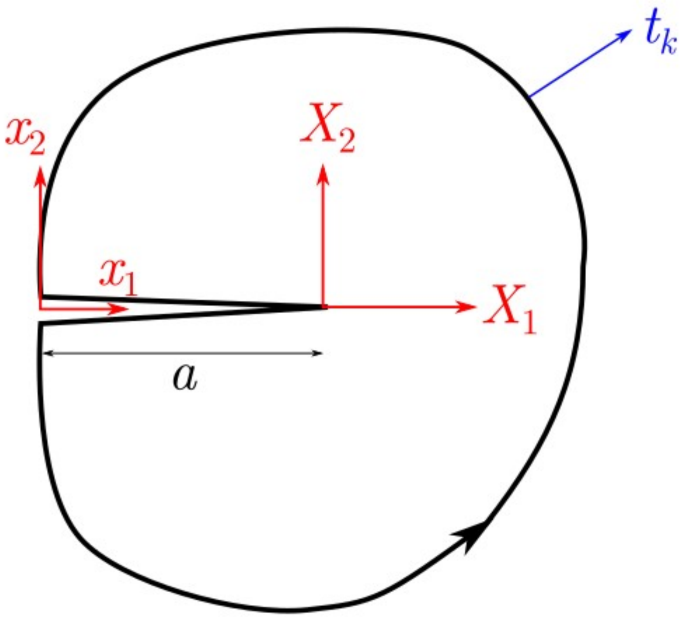
Relationship between J and G



Consider a linear or nonlinear elastic plane body, with a crack of length a subjected to prescribed tractions and displacements along parts of its boundary. Tractions and displacements are assumed to be independent of crack length. The body is referred to a fixed system of Cartesian coordinates x_1-x_2 with the x_1-axis parallel to the crack faces. It is also assumed that the crack extends in a self-similar manner.

The potential energy of the body is

$$\Pi(a) = \int_A w dA - \int_\Gamma t_k u_k ds,$$

where A is the area of the region and Γ is its boundary.

$$\frac{d\Pi(a)}{da} = \int_A \frac{dw}{da} dA - \int_\Gamma t_k \frac{du_k}{da} ds, \qquad \cdots\cdots\cdots\cdots\cdots\cdots(25)$$

We now use a new coordinate system X_1-X_2 attached to the crack tip is introduced,

$$X_1 = x_1 - a,\ X_2 = x_2.$$

Any function which depends upon a and X_1,

$$\frac{d}{da} = \frac{\partial}{\partial a} + \frac{\partial}{\partial X_1}\frac{\partial X_1}{\partial a} = \frac{\partial}{\partial a} - \frac{\partial}{\partial x_1} \qquad \because \left(\frac{\partial}{\partial X_1} = \frac{\partial}{\partial x_1} \quad \text{and} \quad \frac{\partial X_1}{\partial a} = -1 \right)$$

Thus from (25),

$$\frac{d\Pi(a)}{da} = \int_A \left(\frac{\partial w}{\partial a} - \frac{\partial w}{\partial x_1} \right) dA - \int_\Gamma t_k \left(\frac{\partial u_k}{\partial a} - \frac{\partial u_k}{\partial x_1} \right) ds, \qquad \cdots\cdots\cdots\cdots\cdots(26)$$

Look at the term,

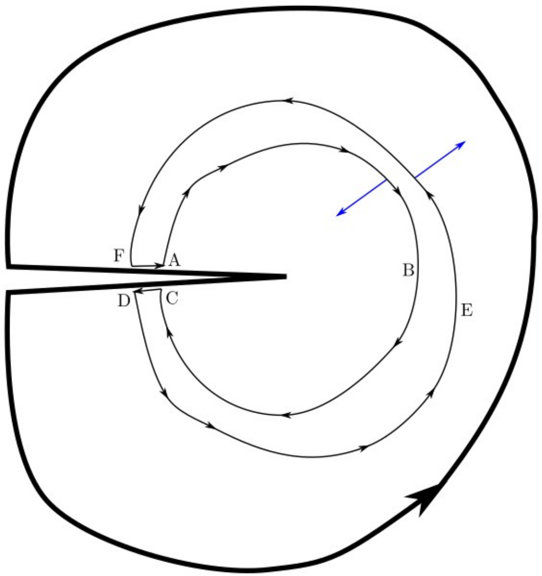
$$\begin{aligned}
\int_A \frac{\partial w}{\partial a} dA &= \int_A \frac{\partial w}{\partial \varepsilon_{ij}} \frac{\partial \varepsilon_{ij}}{\partial a} dA = \int_A \sigma_{ij} \frac{\partial \varepsilon_{ij}}{\partial a} dA = \frac{1}{2} \int_A \left(\sigma_{ij} \frac{\partial u_i}{\partial x_j \partial a} + \sigma_{ij} \frac{\partial u_j}{\partial x_i \partial a} \right) dA \\
&= \frac{1}{2} \int_A \left(\sigma_{ij} \frac{\partial u_i}{\partial x_j \partial a} + \sigma_{ji} \frac{\partial u_j}{\partial x_i \partial a} \right) dA = \int_A \sigma_{ij} \frac{\partial u_i}{\partial x_j \partial a} dA \\
&= \int_A \frac{\partial}{\partial x_j} \left(\sigma_{ij} \frac{\partial u_i}{\partial a} \right) - \cancelto{0}{\frac{\partial \sigma_{ij}}{\partial x_j}} \frac{\partial u_i}{\partial a} dA = \int_\Gamma \sigma_{ij} \frac{\partial u_i}{\partial a} n_j ds = \int_\Gamma t_i \frac{\partial u_i}{\partial a} ds
\end{aligned}$$

Hence from (26),

$$-\frac{d\Pi(a)}{da} = \int_A \frac{\partial w}{\partial x_1} dA - \int_\Gamma t_k \frac{\partial u_k}{\partial x_1} ds = \int_\Gamma \left(w n_1 - t_k \frac{\partial u_k}{\partial x_1} \right) ds = J \quad \text{from (24)}.$$

Thus, for any contour surrounding the crack-tip

$$J = -\frac{d\Pi(a)}{da} = G. \qquad \text{.................(27)}$$



Now consider another contour ABCDEF as shown in the figure. This contour is draws such that it does not include the crack-tip (hence, no singularities inside the contour). For such case we have already derived that $J = 0$. Thus,

$$J_{\text{ABCDEF}} = J_{\text{ABC}} + J_{\text{CD}} + J_{\text{DEF}} + J_{\text{FA}} = 0.$$

Note that J_{FA} = - J_{DC}.
Hence,

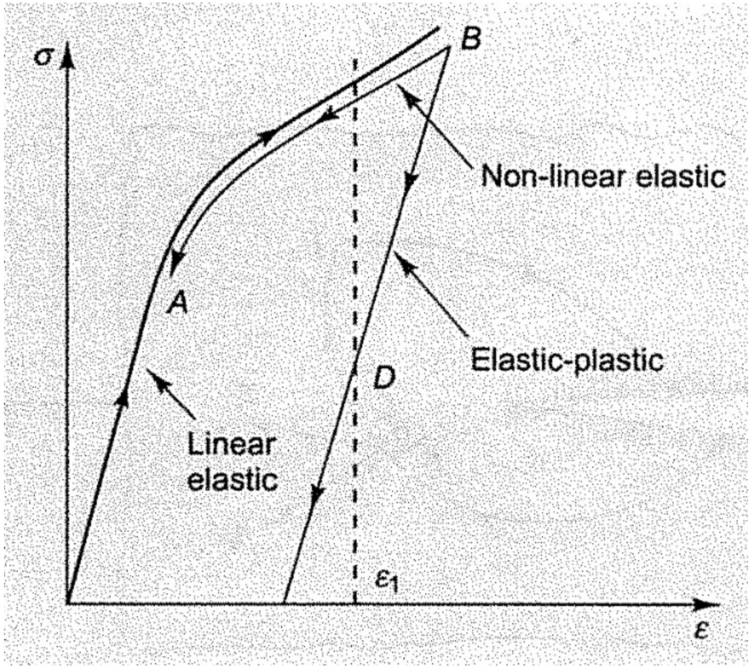
$$J_{\text{ABC}} + J_{\text{DEF}} = 0 \text{ or } J_{\text{ABC}} = J_{\text{FED}}.$$

The J-integral is independent of the contour, and it can be calculated for any contour taken inside the material.

Since J is equal to G, hence J is very popular candidate for fracture criteria. For crack initiation $J \geq J_C$ $(= G_C)$.

Note that the J-integral is derived for elastic (linear or nonlinear) material response, for which strain energy function w is defined.

Attempts have been made to extend the realm of applicability of the J-integral fracture criterion to ductile fracture where extensive plastic deformation and possibly stable crack growth precede fracture instability.

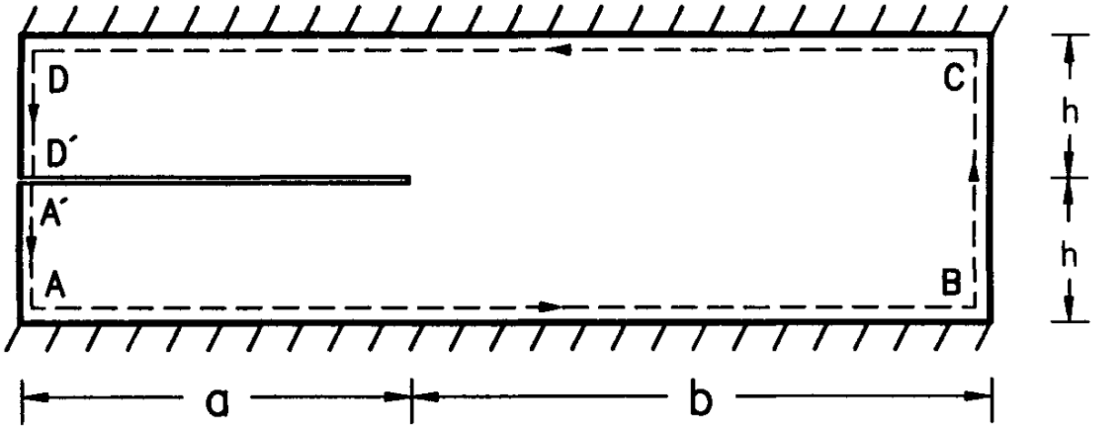


In fact, the presence of plastic zone nullifies the path independence property of the J-integral. For any closed path surrounding the crack tip and taken entirely within the plastic zone or within the elastic zone, the necessary requirements for path independence are not satisfied.

In an effort to establish path independence for the J-integral the deformation theory of plasticity has been invoked. This theory is a nonlinear elasticity theory, and no unloading is permitted. Any solution based on the this theory remains valid under proportional loading (the stress components change in fixed proportion to one another).[33]

No unloading is permitted at any point of the plastic zone. Although, strictly speaking, the condition of proportional loading is not satisfied in practice.

It is argued that in a number of stationary problems, under a single monotonically applied load, the loading condition is close to proportionality. This argument has been supported by finite element simulations.



Consider an infinite strip of height $2h$ with a semi-infinite crack is rigidly clamped along its upper and lower faces at $y = \pm h$. The upper and lower faces are moved in the positive and negative y-direction over distances u_0, respectively.

Let us determine the value of the J-integral and the stress intensity factor.

Consider the path A'ABCDD' along the upper and lower surfaces of the strip up to infinity and traversing the strip perpendicularly to the crack. J is calculated from

$$J_{\mathrm{A'ABCDD'}} = J_{\mathrm{A'A}} + J_{\mathrm{AB}} + J_{\mathrm{BC}} + J_{\mathrm{CD}} + J_{\mathrm{DD'}}$$

Remember the expression for J

$$J = \int_\Gamma w dy - t_1 \frac{\partial u_1}{\partial x} - t_2 \frac{\partial u_2}{\partial x} dS.$$

Let us evaluate J for every segment of this contour.

For segment AB and CD, we have

$$dy = 0 \quad \text{and } du_{1,2}/dx = 0,$$

Hence, $J_{\text{AB}} = J_{\text{CD}} = 0$.

For segment DD' and AA',

$$w = 0 \quad \text{and } du_{1,2}/dx = 0,$$

Hence, $J_{\text{DD'}} = J_{\text{AA'}} = 0$.

For segment BC,

$$\varepsilon_y = u_0/h, \sigma_y = \beta E epsilon_y = \beta E u_0/h,$$

$$u_{1,2}/dx = 0,$$

Hence,

$$J = \int_{-h}^{h} w|_{x\to\infty} dy = \int_{-h}^{h} \frac{1}{2}\sigma_y \varepsilon_y dy.$$

where, $\beta = 1/(1-\nu^2)$ (for plane stress), and $\beta = (1-\nu)/[(1+\nu)(1-2\nu)]$ (for plane strain).

Thus, $J = \dfrac{\beta E u_0^2}{h}$ $K_I = \sqrt{\dfrac{JE}{\eta}}$. ($\eta = 1$ for plane stress and $\eta = (1-\nu^2)$ for plane strain)