ME232: Dynamics

Vibration

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Undamped force vibration:

We start with the case where damping is negligible ($c = 0$). The basic equation of motion becomes

$$\ddot{x} + \omega_n^2 x = \frac{F_0}{m} \sin \omega t. \qquad \cdots\cdots\cdots\cdots(22)$$

The complete solution to (22) is the sum of the complementary solution x_c, which is the general solution of (22) with the right side set to zero, and the particular solution x_p, which is any solution to the complete equation. Thus, $x = x_c + x_p$. The complementary solution in already available (i.e. Equation (5) or (7)). A particular solution is investigated by assuming that the form of the response to the force should resemble that of the force term. Hence, we assume

$$x_p = X \sin \omega t \qquad \cdots\cdots\cdots\cdots(23)$$

where X is the amplitude (in units of length) of the particular solution. Substituting this expression into (22) and solving for X yield

$$X = \frac{F_0/k}{1 - (\omega/\omega_n)^2}.$$

Thus the particular solution is,

$$x_p = \frac{F_0/k}{1-(\omega/\omega_n)^2}\sin\omega t. \qquad \text{.............(24)}$$

The complementary solution, known as the transient solution, is of no special interest here since, with time, it dies out with the small amount of damping which is always unavoidably present.

The particular solution x_p describes the continuing motion and is called the steady-state solution. Its period is $\tau = 2\pi/\omega$, the same as that of the forcing function.

We are primarily interested in amplitude X of the motion. If δ_{st} be the magnitude of the static deflection of the mass under a static load F_0, (i.e., $\delta_{st} = F_0/k$), then we can write,

$$M = \frac{x_p}{\delta_{st}} = \frac{1}{1-(\omega/\omega_n)^2}. \qquad \text{.............(25)}$$

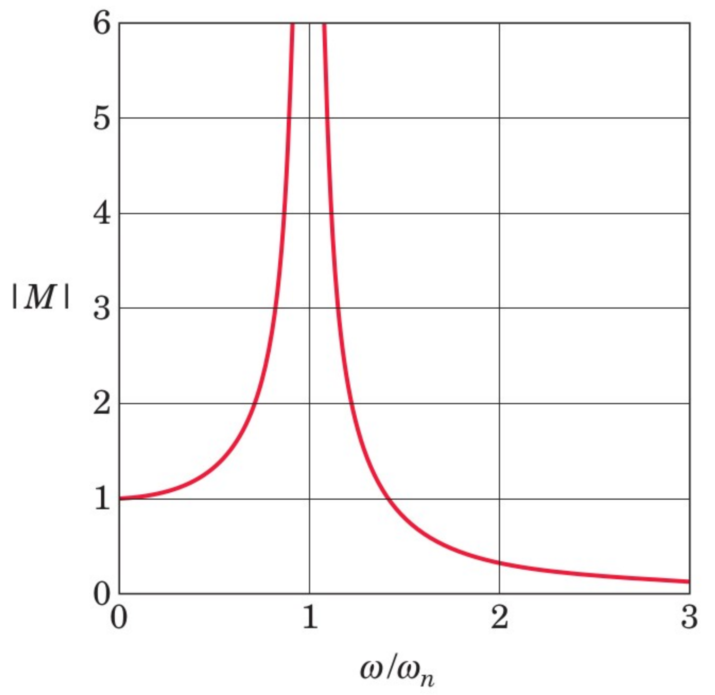
The ratio M is called the amplitude ratio or magnification factor and is a measure of the severity of the vibration.

M approaches infinity as ω approaches ω_n. Consequently, if the system possesses no damping and is excited by a harmonic force whose frequency ω approaches the natural frequency ω_n of the system, then M, and thus X, increase without limit. Physically, this means that the motion amplitude would reach the limits of the attached spring, which is a condition to be avoided.

The value ω_n is called the resonant or critical frequency of the system, and the condition of ω being close in value to ω_n with the resulting large displacement amplitude X is called resonance.

For $\omega<\omega_n$, the magnification factor M is positive, and the vibration is in phase with the force F. For $\omega>\omega_n$, the magnification factor is negative, and the vibration is 180° out of phase with F.

Figure shows a plot of the absolute value of M as a function of the driving-frequency ratio ω/ω_n.



Damped force vibration:

We now reintroduce damping in our expressions for forced vibration. Our basic differential equation of motion is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}\sin\omega t. \qquad \cdots\cdots\cdots\cdots(26)$$

Again, the complete solution is the sum of the complementary solution x_c, which is the general solution of (26) with the right side equal to zero, and the particular solution x_p, which is any solution to the complete equation. We have already developed the complementary solution x_c (i.e., Eq.(16)).

It can be shown that in the present of damping a single sine or cosine term, such as we were able to use for the undamped case, is not sufficiently general for the particular solution. So we try

$$x_p = X_1\cos\omega t + X_2\sin\omega t \quad \text{or} \quad x_p = X\sin(\omega t - \phi).$$

Substituting in (26), match coefficients of $\sin\omega t$ and $\cos\omega t$, and solve the resulting two equations to obtain

$$X = \frac{F_0/k}{\sqrt{[1-(\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}} \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2}\right) \qquad \cdots\cdots\cdots\cdots(27)$$

The complete solution is now known, and for underdamped systems it can be written as

$$x = Ce^{-\zeta\omega_n t}\sin(\omega_d t + \psi) + X\sin(\omega t - \phi). \qquad \cdots\cdots\cdots\cdots(28)$$

Because the first term on the right side diminishes with time, it is known as the transient solution.

The particular solution x_p is the steady-state solution and is the part of the solution in which we are primarily interested.

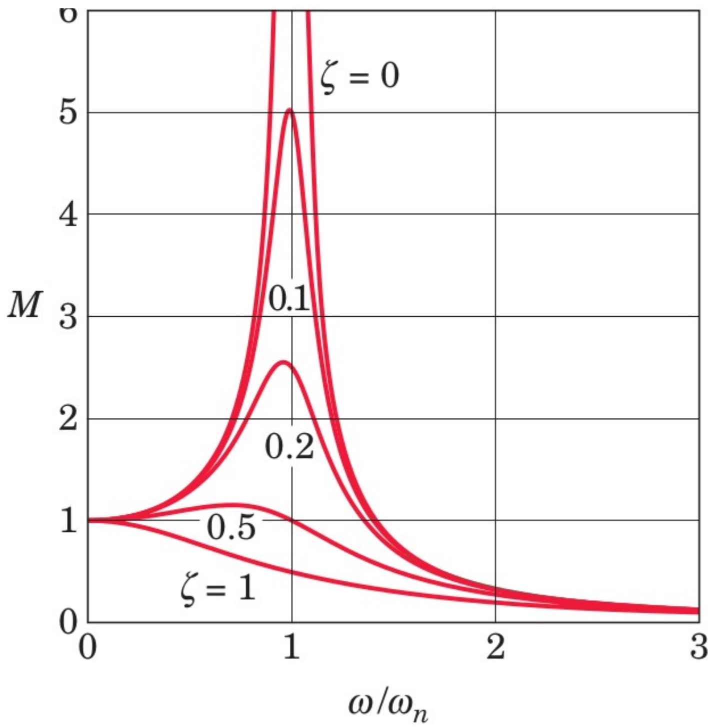
All quantities on the right side of (28) are properties of the system and the applied force, except for C and ψ (which are determinable from initial conditions) and the running time variable t.

Magnification factor and phase angle:

Near resonance the magnitude X of the steady-state solution is a strong function of the damping ratio ζ and the nondimensional frequency ratio ω/ω_n. It is again convenient to form the nondimensional ratio $M = X/(F_0/k)$, which is called the amplitude ratio or magnification factor

$$M = \frac{1}{\sqrt{[1-(\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}}. \quad \cdots\cdots\cdots\cdots(29)$$

Figure shows a plot of M vs. ω/ω_n for various values of the damping ratio ζ.



We can conclude that if a motion amplitude is excessive, two possible solutions would be to

(a) increase the damping (to obtain a larger value of ζ) or

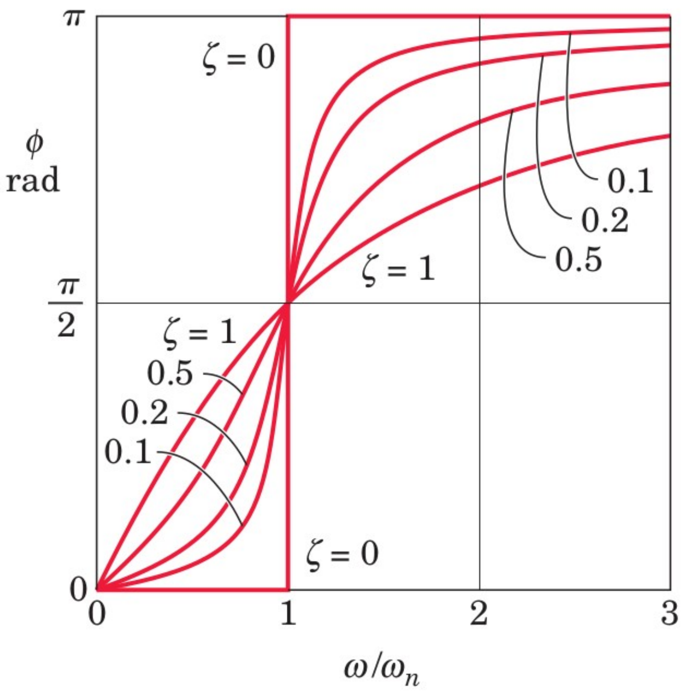
(b) alter the driving frequency so that ω is farther from the resonant frequency ω_n.

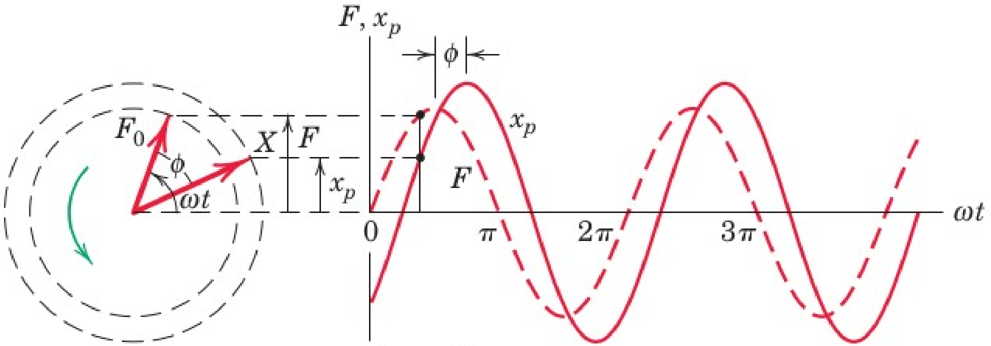
(c)The addition of damping is most effective near resonance.

The phase angle ϕ, given by (27), can vary from 0 to π and represents the part of a cycle (and thus the time) by which the response x_p lags the forcing function F.

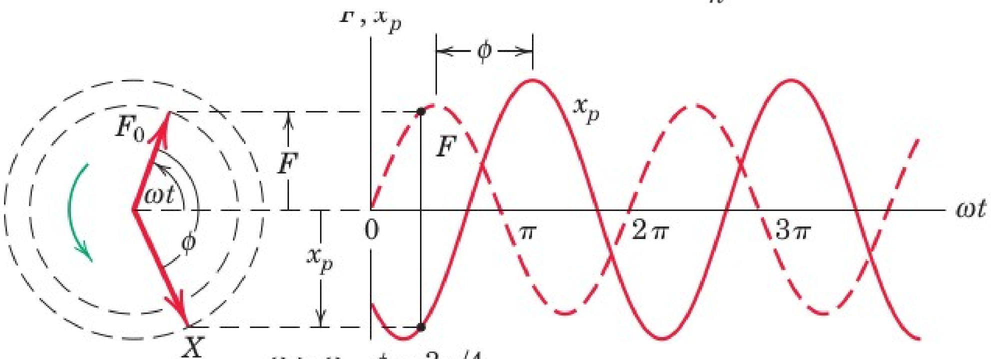
The phase angle ϕ varies with the frequency ratio is plotted for various values of the damping ratio ζ. Note that the value of ϕ, when $\omega/\omega_n = 1$, is 90° for all values of ζ.

Two examples of the variation of F and x_p with ωt are shown. They show the phase difference between the response and the forcing function.





$\omega < \omega_n$, $\phi = \pi/4$



$\omega > \omega_n$, $\phi = 3\pi/4$

Applications:

Vibration-measuring instruments such as seismometers and accelerometers are frequently encountered applications of harmonic excitation. The elements of this class of instruments are shown in figure.

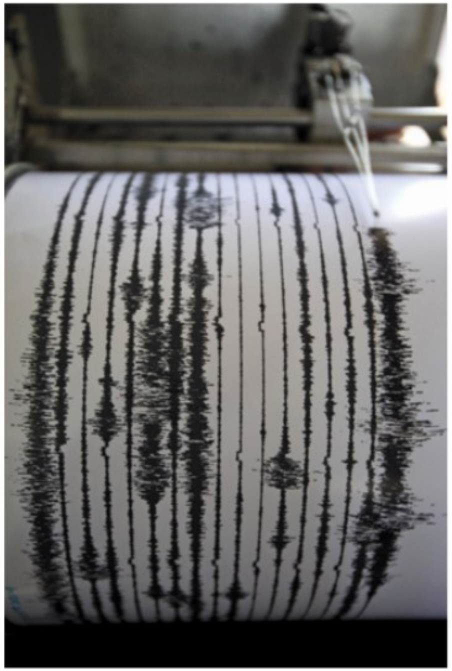
The entire system is subjected to the motion x_B of the frame. Letting x denote the position of the mass relative to the frame, apply Newton's second law to obtain

$$-kx - c\dot{x} = m(\ddot{x} + \ddot{x}_B) \quad \text{or} \quad m\ddot{x} + c\dot{x} + kx = -m\ddot{x}_B.$$

where $(x+x_B)$ is the inertial displacement of the mass. If $x_B = b \sin \omega t$, then our equation of motion with the usual notation is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = b\omega^2 \sin \omega t, \quad \cdots\cdots\cdots\cdots(30)$$

which is similar to (26).



The seismograph is a useful application of the principles of this article.