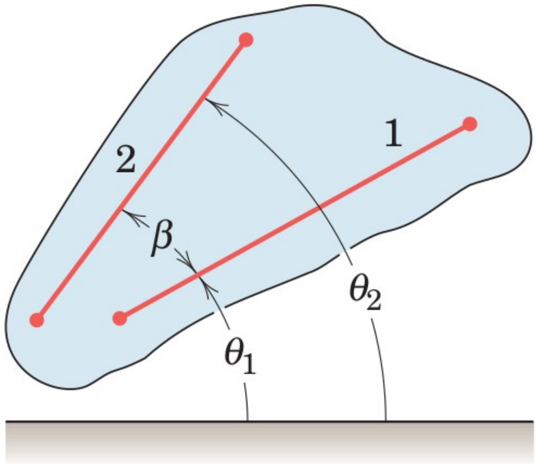
ME232: Dynamics

Anshul Faye
afaye@iitbhilai.ac.in
Room # 106

Rotation

The rotation of a rigid body is described by its angular motion. A rigid body is shown during the rotation in a plane. The angular positions of any two lines 1 and 2 attached to the body are specified by θ_1 and θ_2 measured from any convenient fixed reference direction. Because the angle β is invariant, the relation $\theta_2 = \theta_1 + \beta$ upon differentiation with respect to time gives,



$$\dot{\theta}_1 = \dot{\theta}_2, \text{ and } \ddot{\theta}_1 = \ddot{\theta}_2, \text{ or during a finite interval } \Delta\theta_2 = \Delta\theta_1.$$

Thus, all lines on a rigid body in its plane of motion have the same angular displacement, the same angular velocity, and the same angular acceleration.

Angular motion relations

The angular velocity ω and angular acceleration α of a rigid body in plane rotation are, respectively, the first and second time derivatives of the angular position coordinate θ of any line in the plane of motion of the body. These definitions give

$$\omega = \frac{d\theta}{dt} = \dot{\theta},$$

$$\alpha = \frac{d\omega}{dt} = \dot{\omega}, \quad \text{or} \quad \alpha = \frac{d^2\theta}{dt^2} = \ddot{\theta}, \qquad \text{.................(1)}$$

$$\omega d\omega = \alpha d\theta, \quad \text{or} \quad \dot{\theta} d\dot{\theta} = \ddot{\theta} d\theta.$$

For rotation with constant angular acceleration, the integrals of (1) becomes

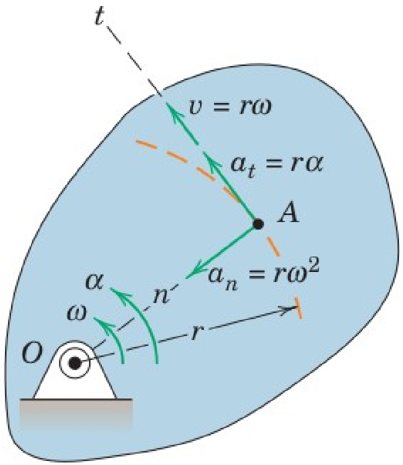
$$\omega = \omega_0 + \alpha t,$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2, \qquad \text{.................(2)}$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0).$$

Rotation about a fixed axis

When a rigid body rotates about a fixed axis, all points other than those on the axis move in concentric circles about the fixed axis. Thus, for the rigid body rotating about a fixed axis normal to the plane of the figure through O, any point such as A moves in a circle of radius r. Thus, the relations for the angular velocity and angular acceleration, respectively, of the body are



$$v = r\omega,$$

$$a_n = r\dot{\theta}^2 = r\omega^2 = v^2/r = v\omega, \qquad \cdots\cdots\cdots\cdots(3)$$

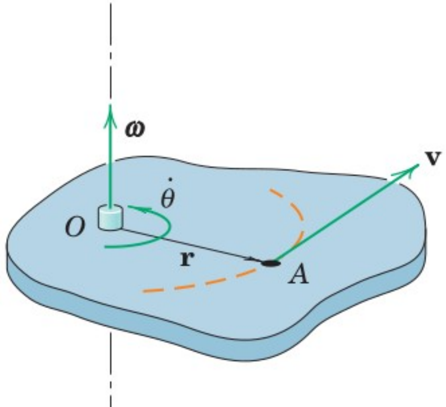
$$a_t = r\ddot{\theta} = r\alpha.$$

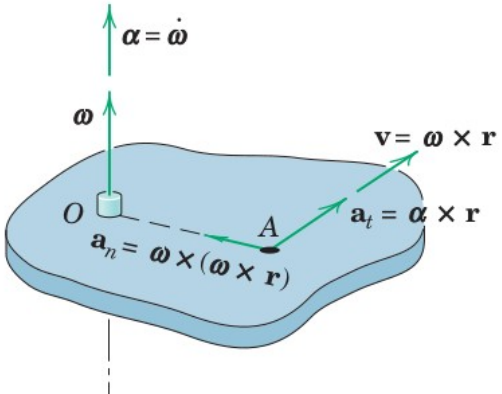
Vector form of (3) can be written as,

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \boldsymbol{\omega} \times \boldsymbol{r},$$

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times \dot{\boldsymbol{r}} = \boldsymbol{\alpha} \times \boldsymbol{r} + \boldsymbol{\omega} \times \dot{\boldsymbol{v}}$$

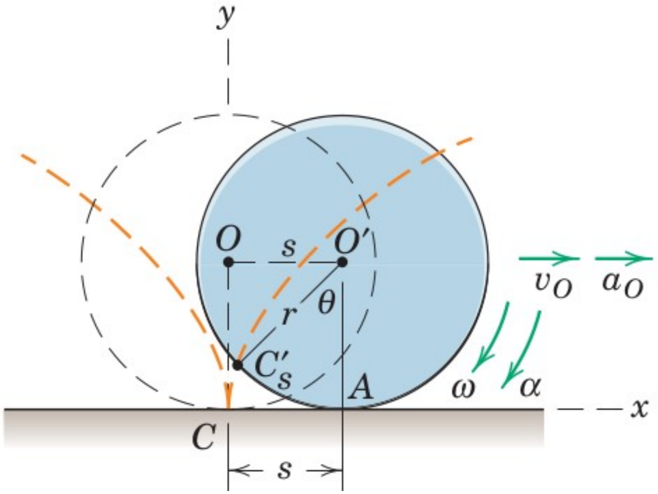
$$\cdots\cdots\cdots\cdots(4)$$





Example 1

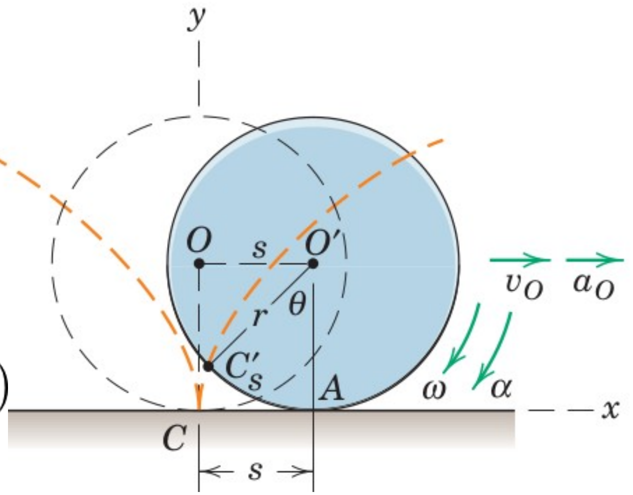
A wheel of radius r rolls on a flat surface without slipping. Determine the angular motion of the wheel in terms of the linear motion of its center O. Also determine the acceleration of a point on the rim of the wheel as the point comes into contact with the surface on which the wheel rolls.



The figure shows the wheel rolling to the right from the dashed to the full position without slipping. The linear displacement of the center O is s, which is also the arc length $C'A$ along the rim on which the wheel rolls. The radial line CO rotates to the new position $C'O'$ through the angle θ which is measured from the vertical direction. If the wheel does not slip, the arc $C'A$ must equal the distance s. Thus, the displacement relationship and its two time derivatives give

$$s = r\theta, \quad \dot{s} = v_O = r\dot{\theta} = r\omega, \quad \dot{v}_O = a_O = r\dot{\omega} = r\alpha.$$

The origin of fixed coordinates is taken arbitrarily but conveniently at the point of contact between on the rim of the wheel and the ground (point C). When point C has moved along its cycloidal path to C', its new coordinates and their time derivatives become



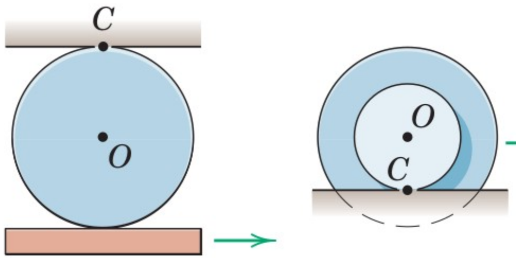
$$x = s - r\cos\theta = r(\theta - \sin\theta), \quad y = r - r\cos\theta = r(1 - \cos\theta)$$

$$\dot{x} = r\dot{\theta}(1 - \cos\theta) = v_O(1 - \cos\theta), \quad \dot{y} = r\dot{\theta}\sin\theta = v_O\sin\theta$$

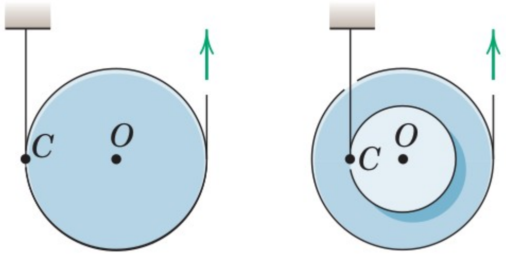
$$\ddot{x} = \dot{v}_O(1 - \cos\theta) + v_O\dot{\theta}\sin\theta = a_O(1 - \cos\theta) + r\omega^2\sin\theta$$

$$\ddot{y} = \dot{v}_O\sin\theta + v_O\dot{\theta}\cos\theta = a_O\sin\theta + r\omega^2\cos\theta.$$

At the given instant, $\theta = 0$, hence, $\ddot{x} = 0$ and $\ddot{y} = r\omega^2$.



Thus, the acceleration of the point C on the rim at the instant of contact with the ground depends only on r and ω and is directed toward the center of the wheel. If desired, the velocity and acceleration of C at any position θ may be obtained by writing the expressions $\boldsymbol{v} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j}$ and $\boldsymbol{a} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j}$.



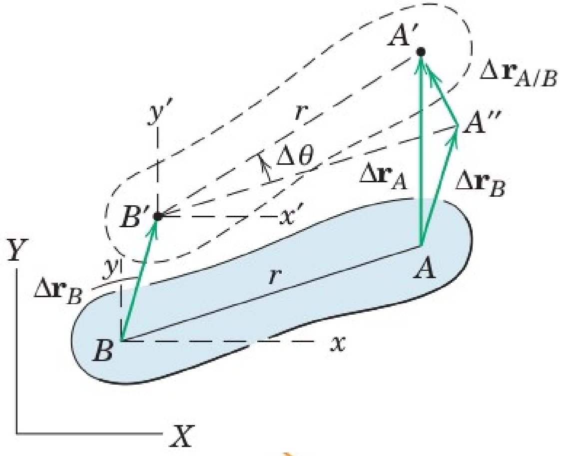
Relative velocity

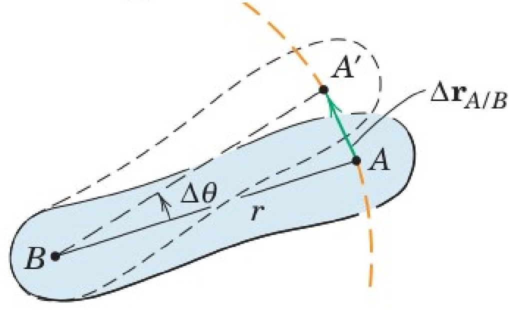
Figure shows a rigid body moving in the plane of the figure from position AB to $A'B'$ during time Δt. This movement may be visualized as occurring in two parts.

First, the body translates to the parallel position $A''B'$ with the displacement $\Delta \boldsymbol{r}_B$.

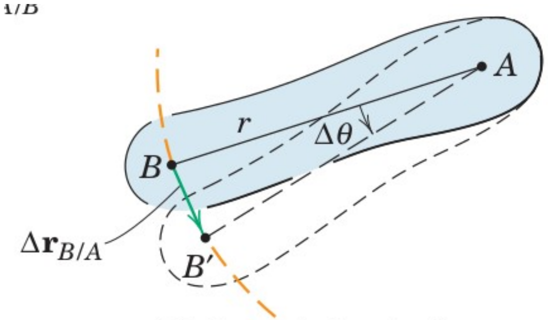
Second, the body rotates about B' through the angle $\Delta\theta$. From the nonrotating reference axes x'-y' attached to the reference point B', it is observed that the remaining motion of the body is one of simple rotation about B', giving rise to the displacement $\Delta \boldsymbol{r}_{A/B}$ of A with respect to B.

To the nonrotating observer attached to B, the body appears to undergo fixed-axis rotation about B with A executing circular motion. Therefore, the relationships developed for circular describe the relative portion of the motion of point A.



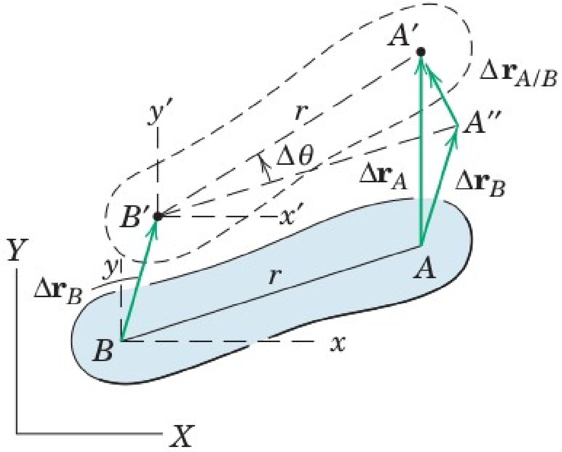


Motion relative to B



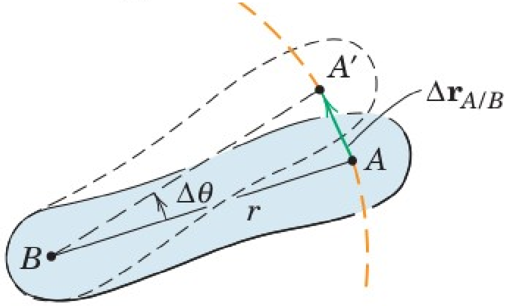
Motion relative to A

Point B was arbitrarily chosen as the reference point for attachment of our nonrotating reference axes x-y. Point A could have been used just as well, in which case we would observe B to have circular motion about A considered fixed. We see that the sense of the rotation is the same whether we choose A or B as the reference, and we see that



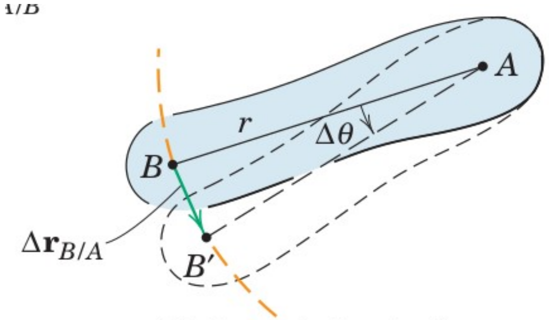
$$\Delta \boldsymbol{r}_{B/A} = -\Delta \boldsymbol{r}_{A/B} \,.$$

With B as the reference point, we see that the total displacement of A is $\Delta \boldsymbol{r}_A = \Delta \boldsymbol{r}_B + \Delta \boldsymbol{r}_{A/B}$,



Motion relative to B

where $\Delta \boldsymbol{r}_{A/B}$ has the magnitude $r\Delta\theta$ as $\Delta\theta$ approaches zero. We note that the relative linear motion $\Delta \boldsymbol{r}_{A/B}$ is accompanied by the absolute angular motion $\Delta\theta$, as seen from the translating axes x'-y'. Dividing the expression for $\Delta \boldsymbol{r}_A$ by the corresponding time interval, we get,



Motion relative to A

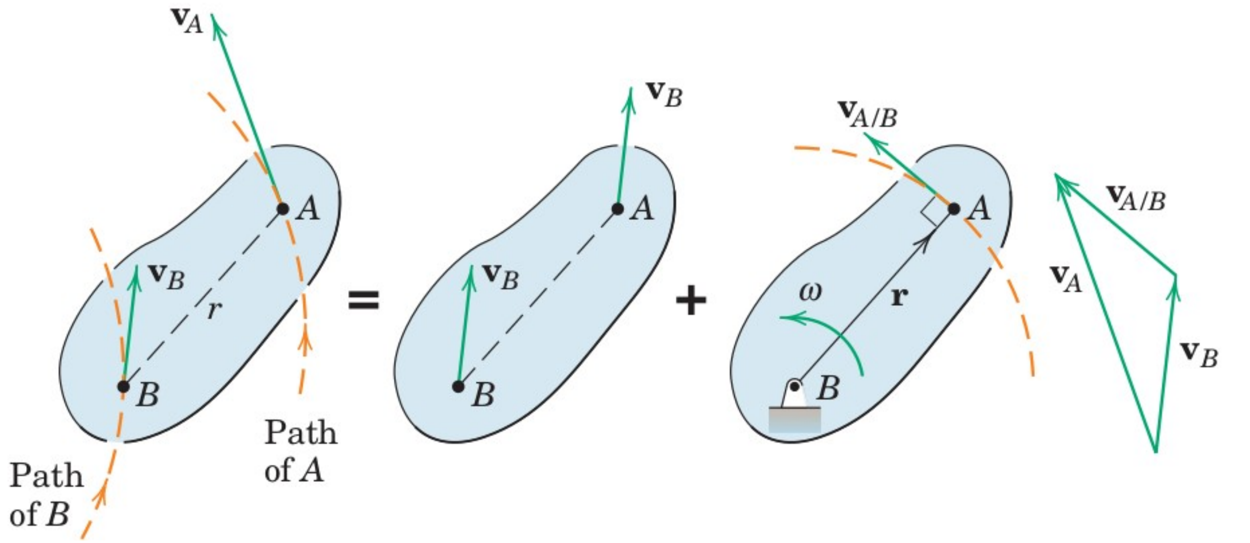
$$\Delta \boldsymbol{v}_A = \Delta \boldsymbol{v}_B + \Delta \boldsymbol{v}_{A/B} \,. \qquad \text{.................}(4)$$

Expression (4) is the same as the one we derived earlier for the motion of particle with the one restriction that the distance r between A and B remains constant. The magnitude of the relative velocity is thus seen to be

$$v_{A/B} = \lim_{\Delta t \to 0} (|\Delta \boldsymbol{r}_{A/B}|)/\Delta t = \lim_{\Delta t \to 0} r\Delta\theta/\Delta t = r\omega, \quad \text{where} \quad \omega = \dot{\theta}.$$

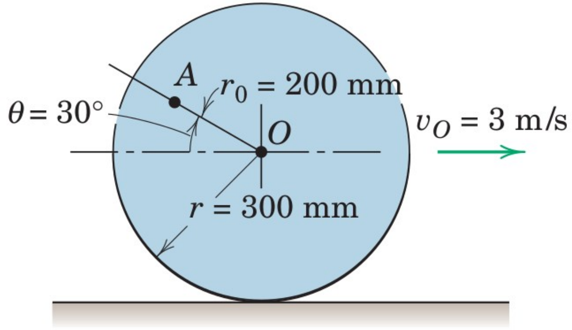
..................(5)

In the vector form, $\boldsymbol{v}_{A/B} = \boldsymbol{\omega} \times \boldsymbol{r}_{A/B}$. ....(6)



Example 2

The wheel of radius $r = 300$ mm rolls to the right without slipping and has a velocity $v_O = 3$ m/s of its center O. Calculate the velocity of point A on the wheel for the instant represented.



The center O is chosen as the reference point for the relative-velocity equation since its motion is given. Thus,

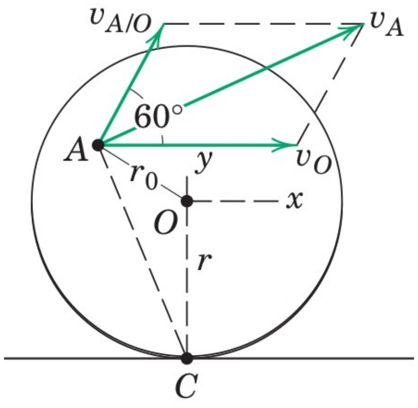
$$\boldsymbol{v}_A = \boldsymbol{v}_O + \boldsymbol{v}_{A/O},$$

$$\boldsymbol{v}_A = \boldsymbol{v}_O + \boldsymbol{\omega} \times \boldsymbol{r}_0,$$

where $\boldsymbol{\omega} = -10\boldsymbol{k}$ rad/s

$$\boldsymbol{r}_0 = 0.2(-\boldsymbol{i}\cos 30^\circ + \boldsymbol{j}\sin 30^\circ) = -0.1732\boldsymbol{i} + 0.1\boldsymbol{j} \text{ m}$$

$$\boldsymbol{v}_O = 3\boldsymbol{i} \text{ m/s}$$

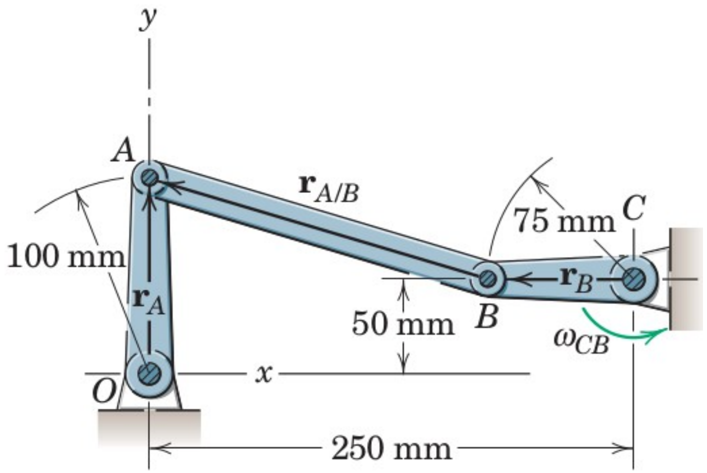


Solution gives,

$$\boldsymbol{v}_A = 4\boldsymbol{i} + 1.732\boldsymbol{j} \text{ m/s}$$

Example 3

Crank CB oscillates about C through a limited arc, causing crank OA to oscillate about O. When the linkage passes the position shown with CB horizontal and OA vertical, the angular velocity of CB is 2 rad/s counterclockwise. For this instant, determine the angular velocities of OA and AB.



The relative-velocity equation

$$\boldsymbol{v}_A = \boldsymbol{v}_B + \boldsymbol{v}_{A/B}$$

$$\boldsymbol{\omega}_{OA} \times \boldsymbol{r}_A = \boldsymbol{\omega}_{OB} \times \boldsymbol{r}_B + \boldsymbol{\omega}_{AB} \times \boldsymbol{r}_{AB}$$

Everything except $\boldsymbol{\omega}_{AB}$ and $\boldsymbol{\omega}_{OA}$ is unknown, which can be found by substituting all known quantities.

$\boldsymbol{\omega}_{AB} =$ -6/7 rad/s and $\boldsymbol{\omega}_{OA} =$ -3/7 rad/s

Another approach of solution is geometric.

Velocity of point B is $v_B = \omega_{CB} \cdot r_B = 150$ mm/s, direction is perpendicular to CB in downward direction.

For velocity of point A, magnitude is $\omega_{OA} \cdot r_A$, which is unknown. However, the direction is known, which is perpendicular to OA.

The vector $\boldsymbol{v}_{A/B}$ must be perpendicular to AB.

the angle θ between $\boldsymbol{v}_{A/B}$ and $\boldsymbol{v}_B$ is also the angle made by AB with the horizontal direction.

Now all unknown quantities can be calculated simply by applying geometric relations.