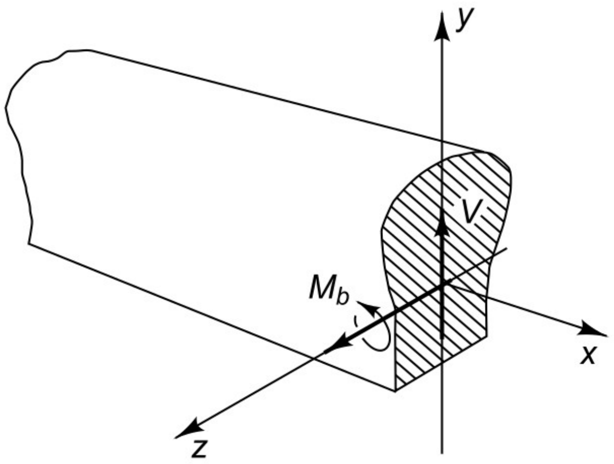
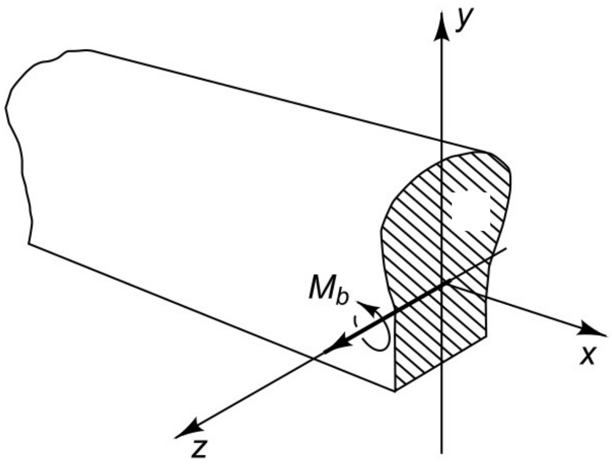
ME231: Solid Mechanics-I

Stresses due to bending

Introduction

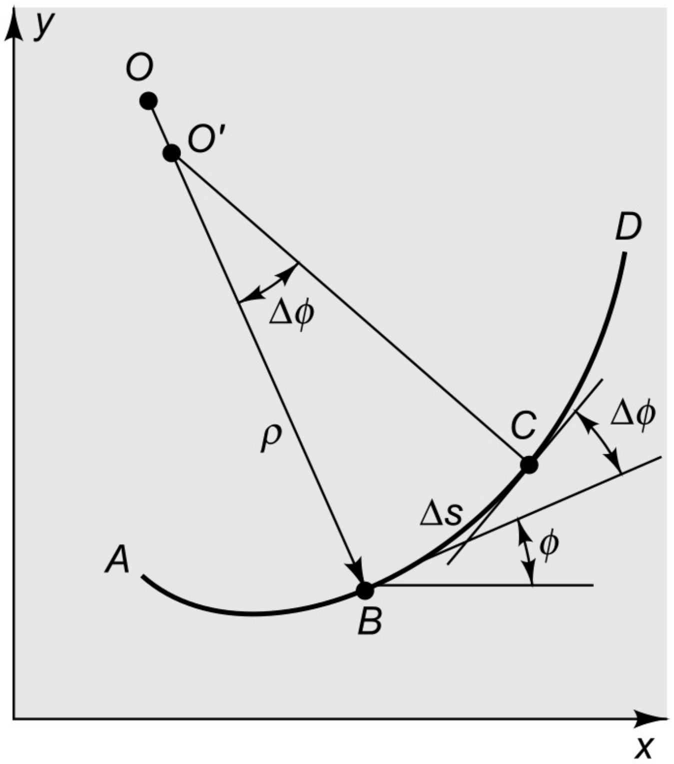


In general, both shear force and bending moment is transmitted through a slender beam



In case of pure bending, a constant bending moment is transmitted

Curvature



Curvature is the rate of change of *slope angle* of the curve w.r.t. the *distance along the curve*

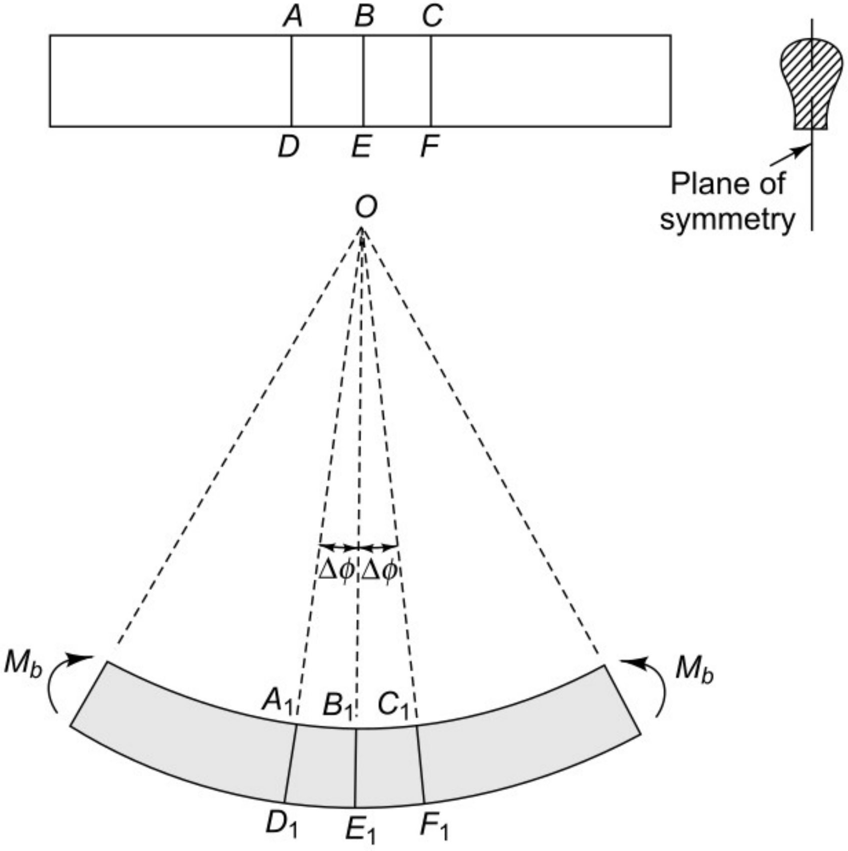
Curvature at point B is defined as

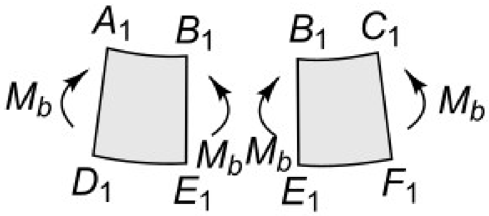
$$\frac{d\phi}{ds} = \lim_{\Delta s \to 0} \frac{\Delta \phi}{\Delta s} = \lim_{\Delta s \to 0} \frac{1}{O'B} = \frac{1}{\rho}$$

......................(1)

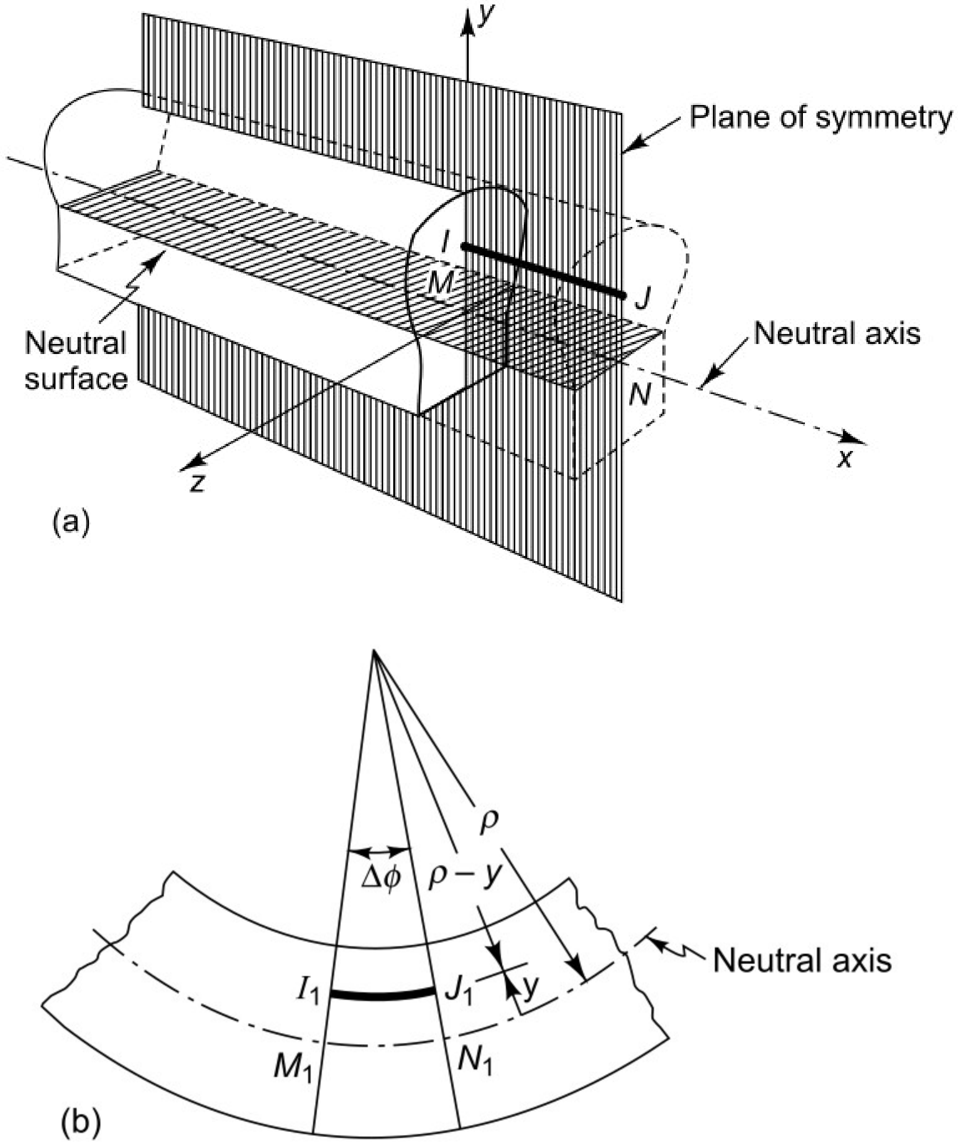
$\rho = OB$ is the *radius of curvature* at point B.

Deformation





Using symmetry arguments, it can be established that in pure bending in a **plane of symmetry plane cross sections remain plane**.



Assume that, deformation within the planes are sufficiently small

$$IJ = MN = M_1N_1$$

Strain of I_1J_1 is

$$\varepsilon_x = \frac{I_1J_1 - IJ}{IJ} = \frac{I_1J_1 - M_1N_1}{M_1N_1}$$

$$\varepsilon_x = \frac{(\rho - y)\Delta\phi - \rho\Delta\phi}{\rho\Delta\phi} = -\frac{y}{\rho} = -\frac{d\phi}{ds}y$$

......................(2)

- Strain is **linearly proportional** to the distance from the neural axis
- Derivation is strictly applicable to **the plane of symmetry**, however we **assume** that the longitudinal strain at all points in the c.s. of the beam is given by the same equation.
- As plane sections remain plane

$$\gamma_{xy} = \gamma_{xz} = 0 \Rightarrow \tau_{xy} = \tau_{xz} = 0. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(3)$$

- No quantitative statement about $\varepsilon_y, \varepsilon_z$ and γ_{yz} beyond the remark that they must be symmetrical w.r.t to the *xy*-plane.

Stresses from stress-strain relation

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] = -\frac{y}{\rho},$$
$$\gamma_{xy} = \frac{\tau_{xy}}{G} = 0,$$
$$\gamma_{xz} = \frac{\tau_{xz}}{G} = 0. \qquad \cdots\cdots\cdots\cdots(4)$$

Thus the shear-stress components τ_{xy} and τ_{xz} must vanish in pure bending.

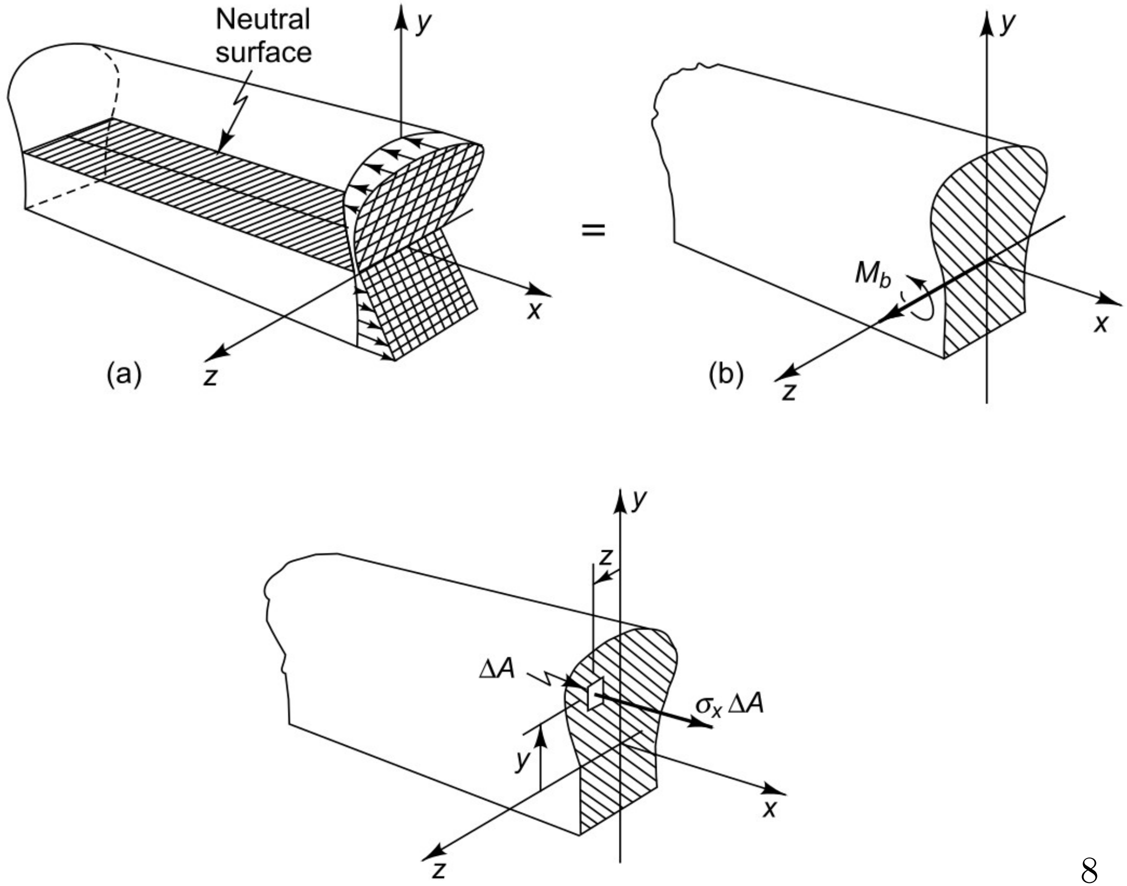
Equilibrium requirements

$$\sum F_x = \int_A \sigma_x dA = 0$$

$$\sum M_y = \int_A z\sigma_x dA = 0$$

$$\sum M_z = -\int_A y\sigma_x dA = M_b$$

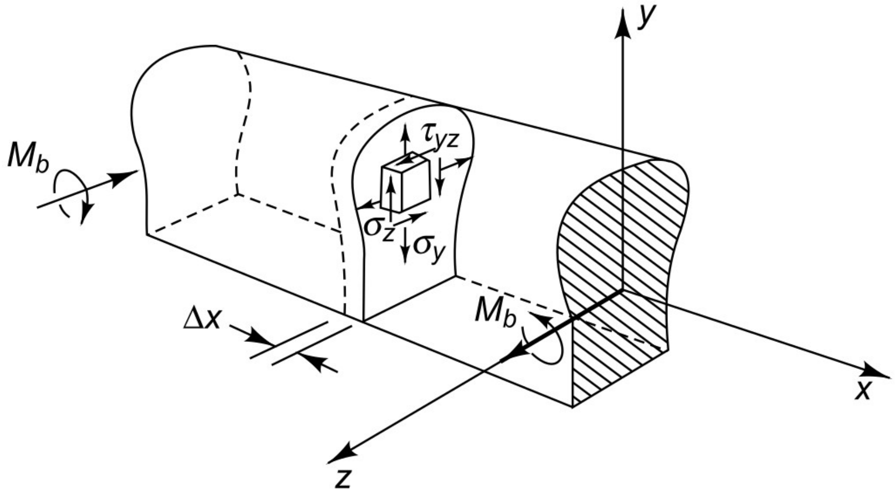
$$\cdots\cdots\cdots\cdots\cdots\cdots(5)$$



We can not say anything about ε_y and ε_z at this stage; hence also for σ_y and σ_z.

Hence, we make some assumptions about the transverse behaviour.

Basis of the assumption comes from the slenderness of the beam.



Slenderness of the beam suggests a plausibility of $\sigma_y = \sigma_z = \tau_{yz} = 0.$ ....(6)

With this assumption, now we are ready to find the following relations.

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] = \frac{\sigma_x}{E} = -\frac{y}{\rho}$$

$$\sigma_x = -E\frac{y}{\rho} = -E\frac{d\phi}{ds}y$$

First moment of c.s. area

$$\sum F_x = \int_A \sigma_x dA = -\int_A E\frac{y}{\rho}dA = -\frac{E}{\rho}\int_A y dA = 0 \quad \cdots\cdots\cdots\cdots\cdots\cdots(7)$$

- First moment of cross-sectional area about the neutral surface must be zero.
- The neutral surface must pass through the centroid of the cross-sectional area

$$\sum M_y = \int_A z\sigma_x dA = -\int_A E\frac{y}{\rho}z dA = -\frac{E}{\rho}\int_A yz dA = 0 \quad \cdots\cdots\cdots\cdots\cdots\cdots(8)$$

- Symmetry of the cross-section about xy-plane will ensure $\int_A yz dA = 0$.

$$\sum M_z = -\int_A y\sigma_x dA = \int_A E\frac{y^2}{\rho}dA = \frac{E}{\rho}\underbrace{\int_A y^2 dA}_{I_{zz}} = M_b \qquad \text{.....................(9)}$$

Second moment of c.s. area

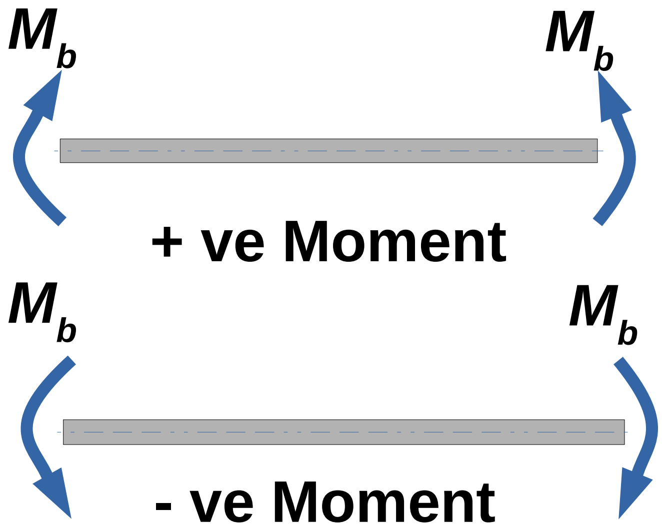
or

Moment of Inertia of the area about the neutral axis

$$M_b = \frac{EI_{zz}}{\rho} \Rightarrow \frac{M_b}{EI_{zz}} = \frac{1}{\rho}.$$

$$\varepsilon_x = -\frac{y}{\rho} = -\frac{d\phi}{ds}y \Rightarrow \frac{1}{\rho} = \frac{d\phi}{ds}.$$

$$\boxed{\frac{1}{\rho} = \frac{d\phi}{ds} = \frac{M_b}{EI_{zz}}} \qquad \text{.....................(10)}$$



Finally from the expressions of stress and strain, we get

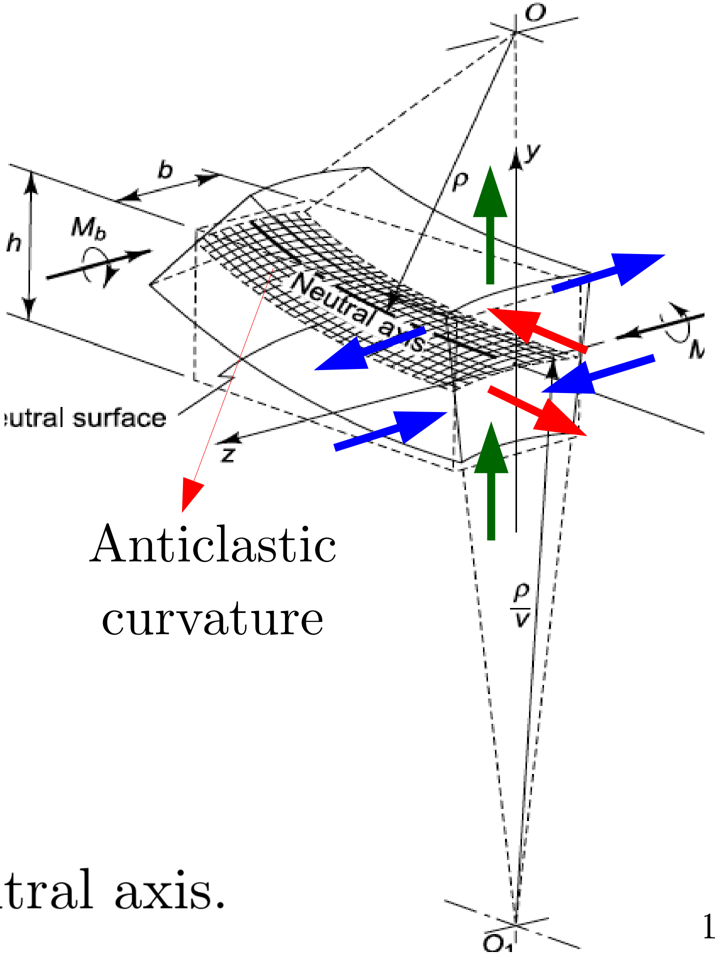
$$\varepsilon_x = -\frac{M_b y}{E I_{zz}}$$

$$\sigma_x = -\frac{M_b y}{I_{zz}}$$

$$\varepsilon_y = -\nu \varepsilon_x \quad \cdots\cdots\cdots\cdots\cdots\cdots(11)$$

$$\varepsilon_z = -\nu \varepsilon_x$$

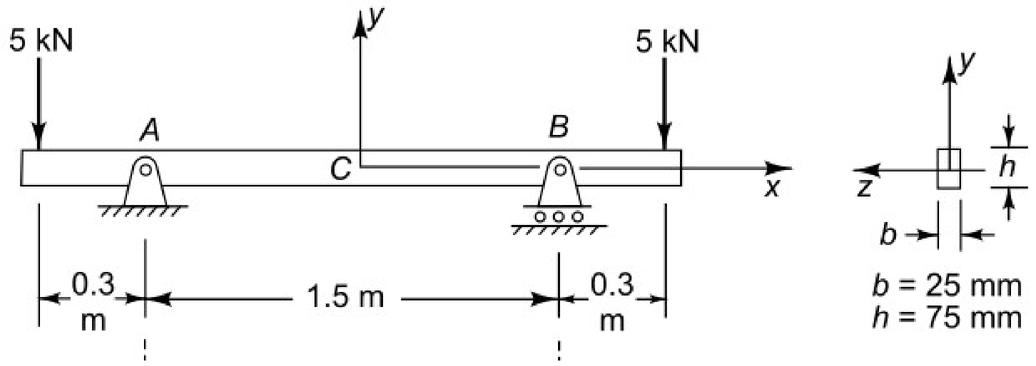
$$\gamma_{yz} = 0$$

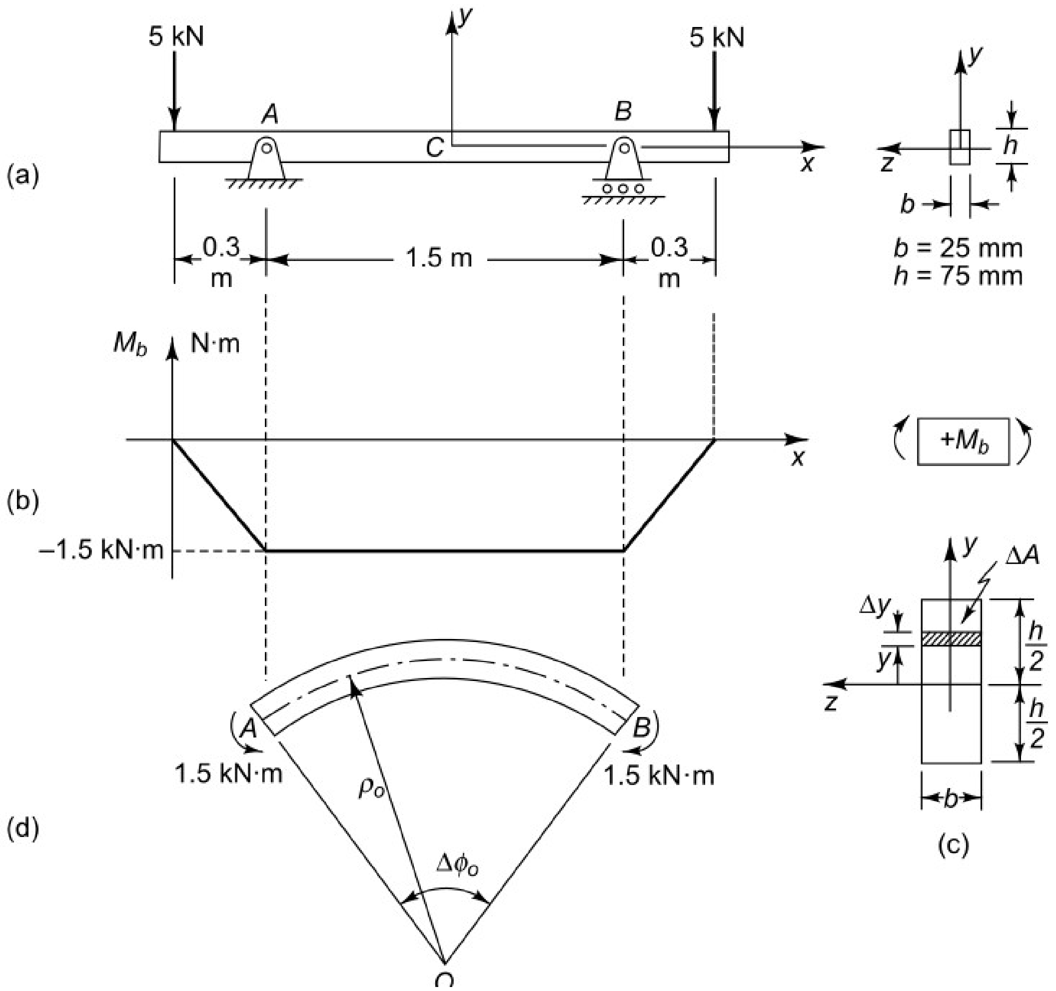


Section Modulus $S = \frac{I}{c}$
where, c is the maximum distance from the neutral axis.

Example 1

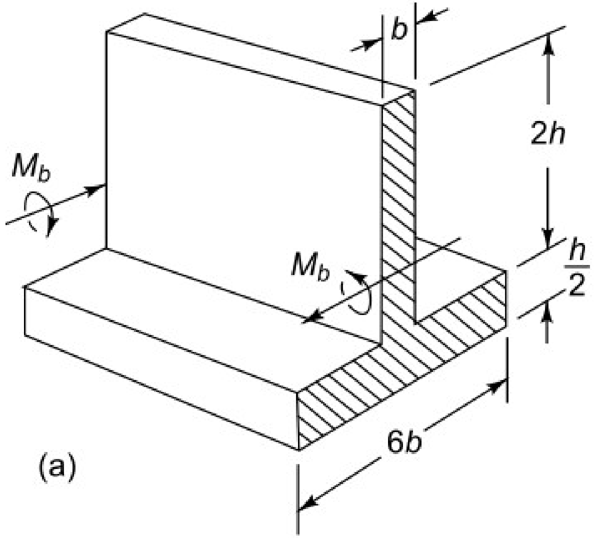
A steel beam 25 mm wide and 75 mm deep is pinned to supports at points A and B, as shown in figure, where the support B is on rollers and free to move horizontally. When the ends of the beam are loaded with 5-kN loads, we wish to find the maximum bending stress at the mid-span of the beam and also the angle $\Delta\phi_O$ subtended by the cross sections at A and B in the deformed beam.

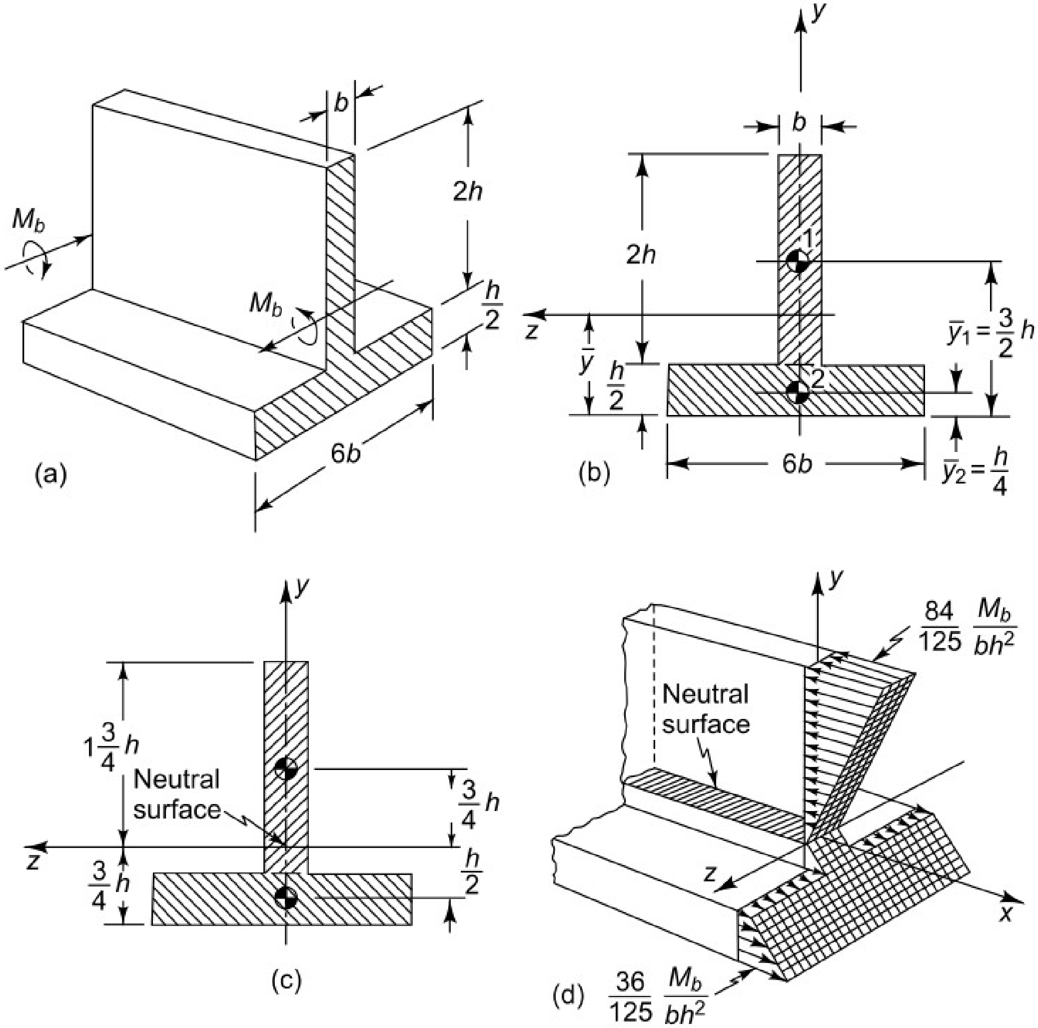



5 kN
y
5 kN
A
B
C
x
(a)
z
h
b
0.3 m
1.5 m
0.3 m
b = 25 mm
h = 75 mm
M_b
N·m
x
(b)
$+M_b$
–1.5 kN·m
y
ΔA
Δy
y
h/2
z
h/2
b
(c)
A
B
1.5 kN·m
1.5 kN·m
ρ_o
(d)
$\Delta\phi_o$
O

Example 2

We wish to find the maximum tensile and compressive bending stresses in the symmetrical T beam of under the action of a constant bending moment M_b .





b
2h
M_b
M_b
$\frac{h}{2}$
6b
(a)
y
b
2h
1
z
$\bar{y}_1 = \frac{3}{2}h$
$\bar{y}$
$\frac{h}{2}$
2
6b
(b)
$\bar{y}_2 = \frac{h}{4}$
y
$1\frac{3}{4}h$
Neutral
surface
$\frac{3}{4}h$
z
$\frac{3}{4}h$
$\frac{h}{2}$
(c)
y
$\frac{84}{125}\frac{M_b}{bh^2}$
Neutral
surface
z
x
(d)
$\frac{36}{125}\frac{M_b}{bh^2}$