ME232: Dynamics

Vibration

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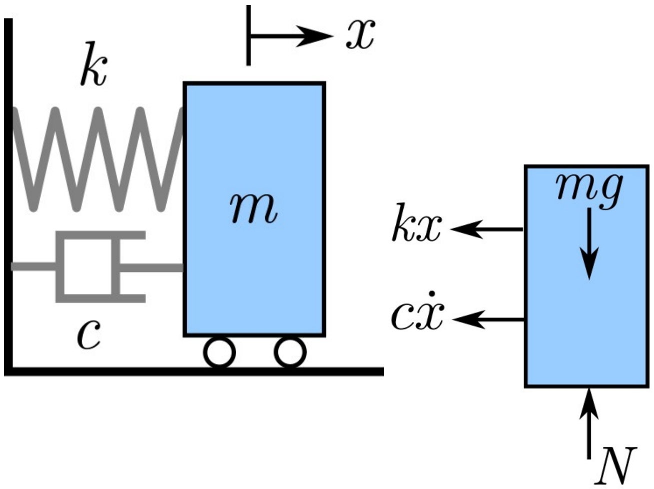
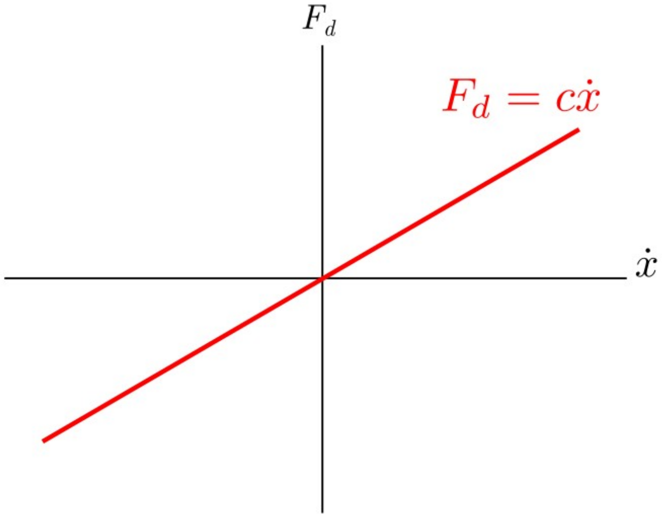
Room # 106

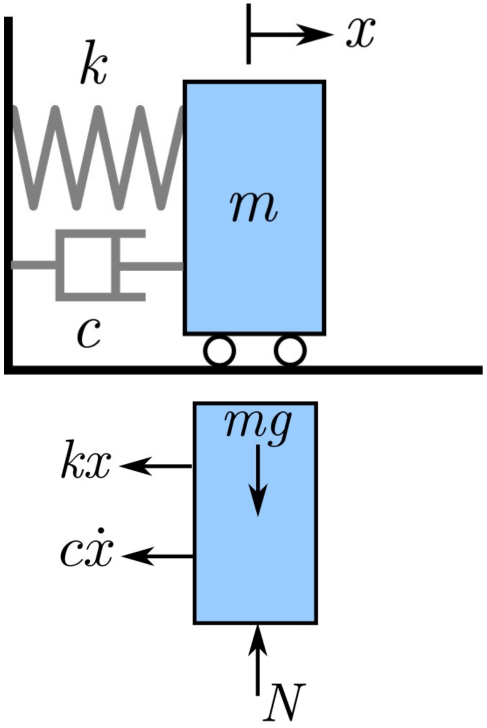
Damped free vibration





Every mechanical system possesses some forces which dissipates mechanical energy. A dashpot or viscous damper is a device added to systems for limiting or retarding vibration. It consists of a cylinder filled with a viscous fluid and a piston with holes or other passages by which the fluid can flow from one side of the piston to the other.

A simple linear dashpot is shown, which exert a force F_d whose magnitude is proportional to the velocity of the mass. The constant of proportionality c is called the viscous damping coefficient and has units of N·s/m. The direction of the damping force applied to the mass is opposite that of the velocity $\dot{x}$. Thus, the force on the mass is $-c\dot{x}$.



The equation of motion for the body with damping is given by Newton's second as

$$-kx - c\dot{x} = m\ddot{x} \quad \text{or} \quad m\ddot{x} + c\dot{x} + kx = 0. \qquad \text{........................(10)}$$

In addition to the substitution $\omega_n = \sqrt{k/m}$, it is convenient to introduce the combination of constants $\zeta = c/(2m\omega_n)$. The quantity ζ (zeta) is called the viscous damping factor or damping ratio and is a measure of the severity of the damping. It should be noted that ζ is non-dimensional. (10) may now be written as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 = 0. \qquad \text{........................(11)}$$

In order to solve the equation of motion (11) we assume solutions of the form

$$x = Ae^{\lambda t}. \qquad \text{........................(12)}$$

Substituting (12) in (11) yields, $\quad \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0. \qquad$ ....(13)

which is called the characteristic equation. Its roots are

$$\lambda_1 = \omega_n(-\zeta + \sqrt{\zeta^2 - 1}), \quad \lambda_2 = \omega_n(-\zeta - \sqrt{\zeta^2 - 1}). \qquad \text{........................(14)}$$

Linear systems have the property of superposition, which means that the general solution is the sum of the individual solutions each of which corresponds to one root of the characteristic equation. Thus, the general solution is

$$x = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} = A_1 e^{(-\zeta+\sqrt{\zeta^2-1})\omega_n t} + A_2 e^{(-\zeta-\sqrt{\zeta^2-1})\omega_n t}. \qquad \text{..................(14)}$$

Categories of damped motion:

Because $0 \le \zeta \le \infty$, the radicand $(\zeta^2 - 1)$ may be positive, negative, or even zero, giving rise to the following three categories of damped motion:

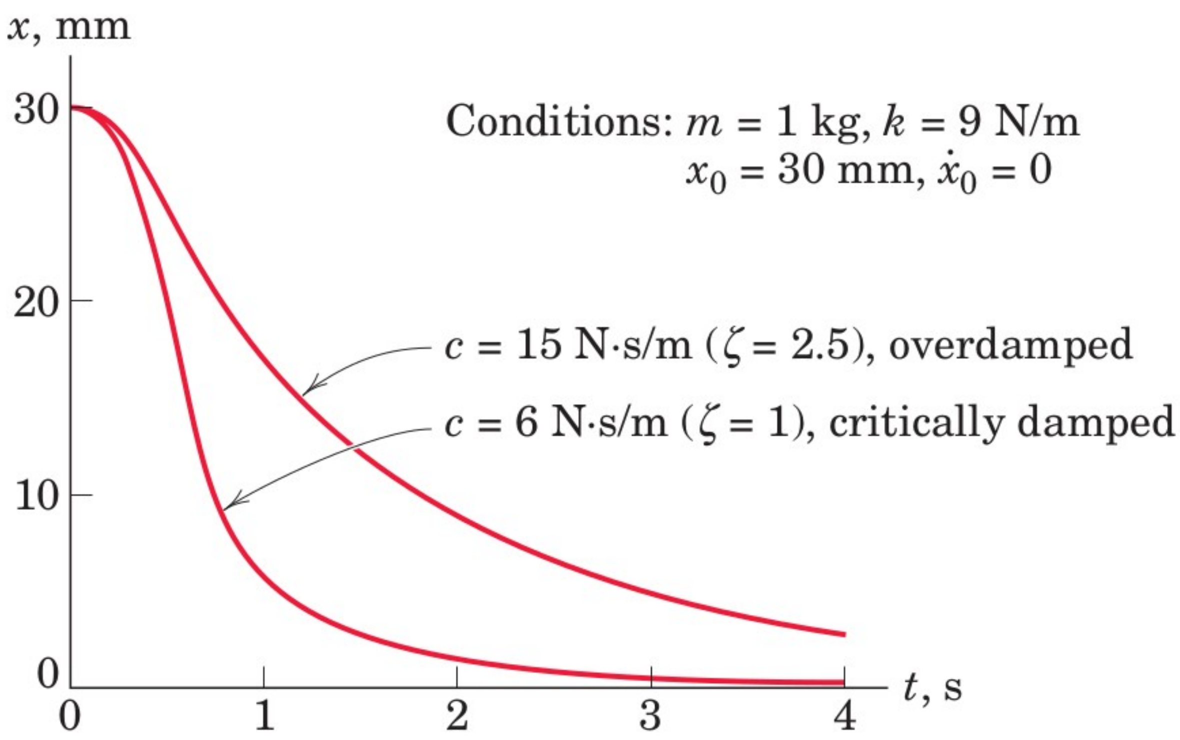
I. $\zeta > 1$ (overdamped): The roots λ_1 and λ_2 are distinct, real, and negative numbers. The motion as given by (14) decays so that x approaches zero for large values of time t. There is no oscillation and therefore no period associated with the motion.

II. $\zeta = 1$ (critically damped): The roots λ_1 and λ_2 are equal, real, and negative numbers ($\lambda_1 = \lambda_2 = -\omega_n$). The solution to the differential equation for the special case of equal roots is given by

$$x = (A_1 + A_2 t)e^{-\omega_n t}. \qquad \text{..................(15)}$$

Again, the motion decays with x approaching zero for large t, and the motion is nonperiodic.

A critically damped system, when excited with an initial velocity or displacement (or both), will approach equilibrium faster than will an overdamped system.



III. $\zeta < 1$ (underdamped): So the radicand $(\zeta^2 - 1)$ is negative and we may rewrite (14) as

$$x = e^{-\zeta\omega_n t}[A_1 e^{i\sqrt{1-\zeta^2}\omega_n t} + A_2 e^{-i\sqrt{1-\zeta^2}\omega_n t}].$$

It is convenient to let a new variable ω_d represent the combination $\omega_n\sqrt{1-\zeta^2}$. Thus,

$$x = e^{-\zeta\omega_n t}[A_1 e^{i\omega_d t} + A_2 e^{-i\omega_d t}]. \qquad \text{................(16a)}$$

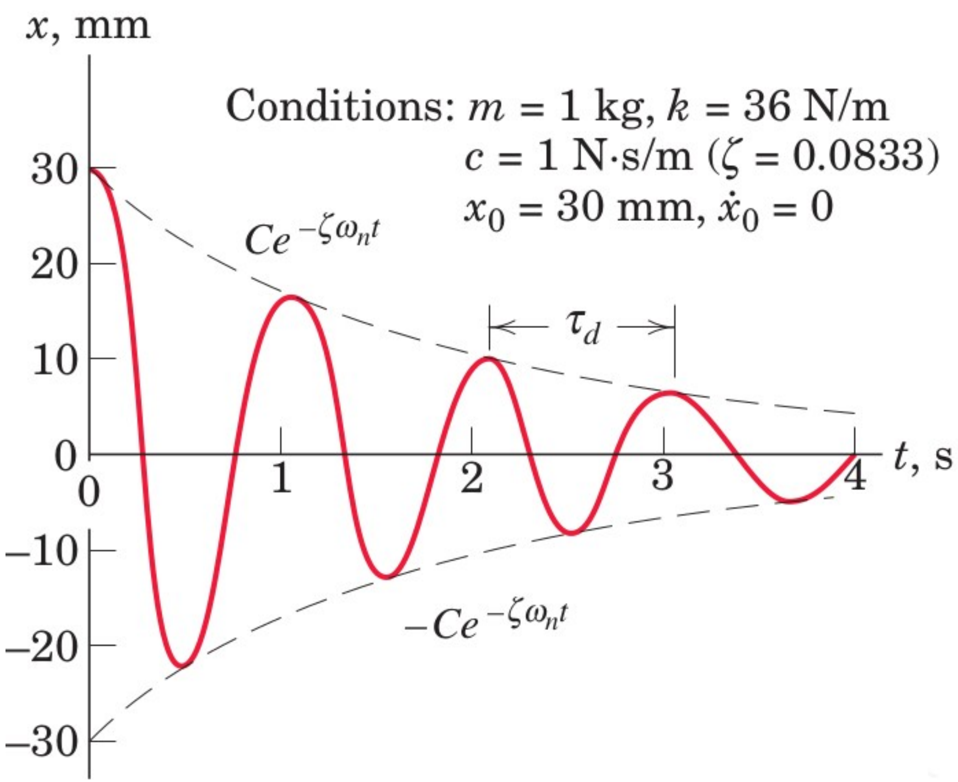
Use of the Euler formula (16) can be rewritten as

$$x = e^{-\zeta\omega_n t}[A_3 \cos\omega_d t + A_4 \sin\omega_d t], \qquad \text{................(16b)}$$

where $A_3 = (A_1 + A_2)$ and $A_4 = i\,(A_1 \text{-} A_2)$. Alternatively we can also write,

$$x = e^{-\zeta\omega_n t}[C\sin(\omega_d t + \psi)],$$

$$\text{or} \qquad x = Ce^{-\zeta\omega_n t}\sin(\omega_d t + \psi). \qquad \text{.............(16c)}$$



(16) represents an exponentially decreasing harmonic function, as shown in figure for specific numerical values. The frequency

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

is called the damped natural frequency. The damped period is given by $\tau_d = 2\pi/\omega_d$.

To find C and ψ if damping is present we use (16) and apply initial conditions, i.e., at $t = 0$, initial displacement is x_0 and initial velocity is $\dot{x}_0$, respectively.