ME232: Dynamics

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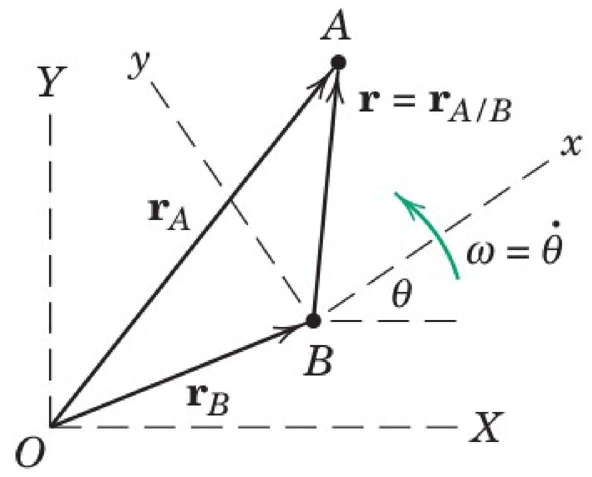
Room # 106

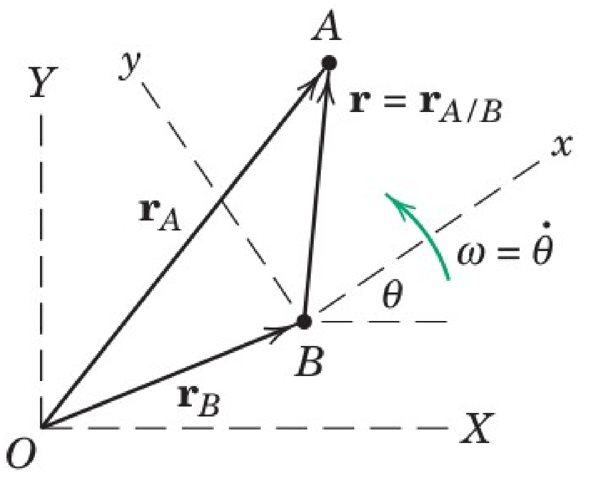
Motion relative to rotating axes

- Till now in our discussion we have used non-rotating reference axes to describe relative velocity and relative acceleration.
- Use of rotating reference axes greatly facilitates the solution of many problems in kinematics where motion is generated **within a system** or **observed from a system which itself is rotating**.
- An example of such a motion is the movement of a fluid particle along the curved vane of a centrifugal pump, where the **path relative to the vanes** of the impeller becomes an important design consideration.

Consider the plane motion of two particles A and B in the fixed X-Y plane. For the time being, we will consider A and B to be moving independently of one another for the sake of generality. We observe the motion of A from a moving reference frame x-y which has its origin attached to B and which rotates with an angular velocity $\omega = \dot{\theta}$.



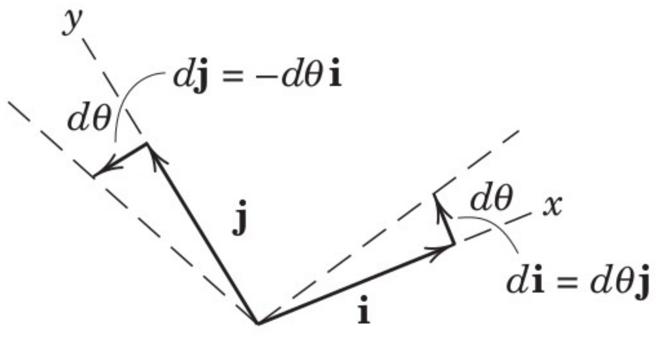
We may write this angular velocity as the vector $\boldsymbol{\omega} = \omega \boldsymbol{k}$ (direction of the vector is established by the right-hand rule). The absolute position vector of A is given,



$$\boldsymbol{r}_A = \boldsymbol{r}_B + \boldsymbol{r}_{A/B} = \boldsymbol{r}_B + (x\,\boldsymbol{i} + y\,\boldsymbol{j}), \qquad \text{.................}(9)$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors attached to the x-y frame.

To obtain the velocity and acceleration equations we successively differentiate (9) w.r.t. time. The unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ are now rotating with the x-y axes and, therefore, have time derivatives. These derivatives may be seen from the figure, which shows the infinitesimal change in each unit vector during time dt as the reference axes rotate through an angle $d\theta = \omega\, dt$.



The differential change in $\boldsymbol{i}$ is $d\boldsymbol{i} = d\theta\ \boldsymbol{j}$, and the differential change in $\boldsymbol{j}$ is $d\boldsymbol{j} = -d\theta\ \boldsymbol{i}$. Thus,

$$\dot{\boldsymbol{i}} = \omega \boldsymbol{j} = \boldsymbol{\omega} \times \boldsymbol{i}, \text{ and } \dot{\boldsymbol{j}} = -\omega \boldsymbol{i} = \boldsymbol{\omega} \times \boldsymbol{j}. \ \text{.................}(10)$$

Relative velocity:

Expression for relative velocity can be obtained by differentiating (10) w.r.t. time as,

$$\dot{\boldsymbol{r}}_A = \dot{\boldsymbol{r}}_B + \dot{\boldsymbol{r}}_{A/B} = \dot{\boldsymbol{r}}_B + \left[(\dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j}) + (x\dot{\boldsymbol{i}} + y\dot{\boldsymbol{j}})\right]$$

$$\boldsymbol{v}_A = \boldsymbol{v}_B + [\boldsymbol{v}_{\text{rel}} + (x\boldsymbol{\omega} \times \boldsymbol{i} + y\boldsymbol{\omega} \times \boldsymbol{j})]$$

$$\boldsymbol{v}_A = \boldsymbol{v}_B + \boldsymbol{v}_{\text{rel}} + \boldsymbol{\omega} \times (x\boldsymbol{i} + y\boldsymbol{j})$$

$$\boldsymbol{v}_A = \boldsymbol{v}_B + \boldsymbol{v}_{\text{rel}} + \boldsymbol{\omega} \times \boldsymbol{r} \qquad \text{.................}(11)$$

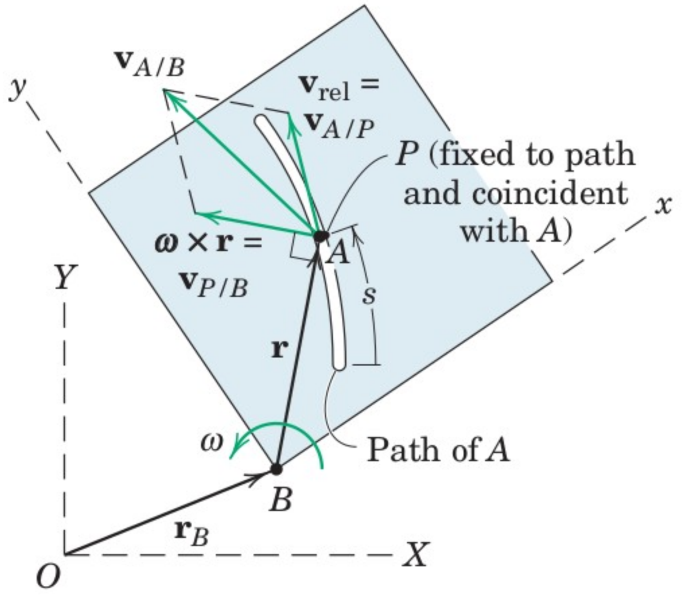
Comparing (11) with the expression for non-rotating reference axes it is observed that $\boldsymbol{v}_{A/B} = \boldsymbol{\omega} \times \boldsymbol{r} + \boldsymbol{v}_{\text{rel}}$, from which it can be concluded that **the term $\boldsymbol{\omega} \times \boldsymbol{r}$ is the difference between the relative velocities as measured from non-rotating and rotating axes.**

To visualize the meaning of the last two terms of (11), we observe the motion of particle A in a curved slot in a plate which represents the rotating x-y reference system.

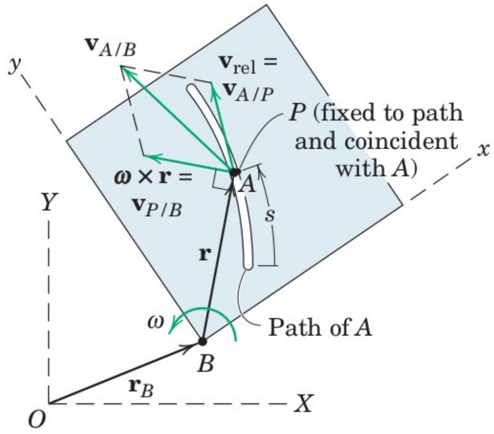
The velocity of A as measured relative to the plate, $\boldsymbol{v}_{\text{rel}}$, would be tangent to the path fixed in the x-y plate and would have a magnitude $\dot{s}$, where s is measured along the path.

This relative velocity may also be viewed as the velocity $\boldsymbol{v}_{A/P}$ **relative to a point P attached to the plate and coincident with A at the instant under consideration.**

The term $\boldsymbol{\omega} \times \boldsymbol{r}$ has a magnitude $r\dot{\theta}$ and a direction normal to $\boldsymbol{r}$ and is the **velocity of point P relative to B as seen from nonrotating axes attached to B.**



The following comparison will help establish the equivalence of, and clarify the differences between, the relative-velocity equations written for rotating and nonrotating reference axes:



$$
\begin{aligned}
\boldsymbol{v}_A &= \boldsymbol{v}_B + \boldsymbol{\omega} \times \boldsymbol{r} + \boldsymbol{v}_{\text{rel}} \\
\boldsymbol{v}_A &= \boldsymbol{v}_B + \boldsymbol{v}_{P/B} + \boldsymbol{v}_{A/P} \\
\boldsymbol{v}_A &= \boldsymbol{v}_P + \boldsymbol{v}_{A/P} \\
\boldsymbol{v}_A &= \boldsymbol{v}_B + \boldsymbol{v}_{A/B}
\end{aligned}
\qquad \text{..................(11a)}
$$

In the second equation, the term $\boldsymbol{v}_{P/B}$ is measured from a nonrotating position. The term $\boldsymbol{v}_{A/P}$ is the same as $\boldsymbol{v}_{\text{rel}}$ and is the velocity of A as measured in the x-y frame. In the third equation, $\boldsymbol{v}_P$ is the absolute velocity of P and represents the effect of the moving coordinate system, both transnational and rotational. The fourth equation is the same as that developed for nonrotating axes, and it is seen that

$$\boldsymbol{v}_{A/B} = \boldsymbol{v}_{P/B} + \boldsymbol{v}_{A/P} = \boldsymbol{\omega} \times \boldsymbol{r} + \boldsymbol{v}_{\text{rel}}.$$

Transformation of a time derivative

(11) represents a transformation of the time derivative of the position vector between rotating and nonrotating axes. This result can be generalized to apply to the time derivative of any vector quantity $\boldsymbol{V} = V_x\ \boldsymbol{i} + V_y\ \boldsymbol{j}$. Accordingly, the total time derivative with respect to the X-Y system is

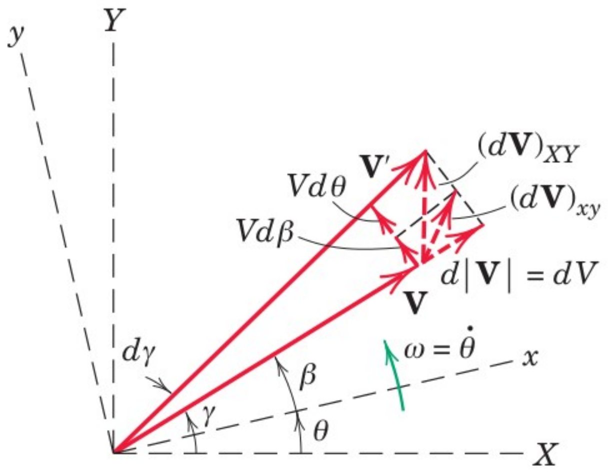
$$\left(\frac{d\boldsymbol{V}}{dt}\right)_{XY} = \left(\dot{V}_x\boldsymbol{i} + \dot{V}_y\boldsymbol{j}\right) + \left(V_x\dot{\boldsymbol{i}} + V_y\dot{\boldsymbol{j}}\right)$$

$$\left(\frac{d\boldsymbol{V}}{dt}\right)_{XY} = \left(\frac{d\boldsymbol{V}}{dt}\right)_{xy} + \boldsymbol{\omega} \times (V_x\boldsymbol{i} + V_y\boldsymbol{j})$$

$$\left(\frac{d\boldsymbol{V}}{dt}\right)_{XY} = \left(\frac{d\boldsymbol{V}}{dt}\right)_{xy} + \boldsymbol{\omega} \times \boldsymbol{V} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(12)$$

The first term represents **the part of the total derivative of *V* which is measured relative to the *x*-*y* reference system**, and the second term represents **the part of the derivative due to the rotation of the reference system.**

To understand the physical significance of (12) consider a vector $\boldsymbol{V}$ at time t, observed both in the fixed axes X-Y and in the rotating axes x-y. Because we are dealing with the effects of rotation only, the vector may be drawn through the coordinate origin without loss of generality. During time dt, the vector swings to position $\boldsymbol{V}'$, and the observer in x-y measures the two components



(a) dV due to its change in magnitude, and

(b) $Vd\beta$ due to its rotation $d\beta$ relative to x-y.

Hence, the derivative $(d\boldsymbol{V}/dt)_{xy}$ which the rotating observer measures has the components dV/dt and $Vd\beta/dt = V\dot{\beta}$.

The remaining part of the total time derivative not measured by the rotating observer has the magnitude $Vd\theta/dt$ and, expressed as a vector, is $\boldsymbol{\omega} \times \boldsymbol{V}$. Thus, we see from the diagram that

$$(\dot{\boldsymbol{V}})_{XY} = (\dot{\boldsymbol{V}})_{xy} + \boldsymbol{\omega} \times \boldsymbol{V}.$$

Relative acceleration:

Expression for relative acceleration can be obtained by differentiating (11) w.r.t. time as,

$$\dot{\boldsymbol{v}}_A = \dot{\boldsymbol{v}}_B + \dot{\boldsymbol{v}}_{\text{rel}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times \dot{\boldsymbol{r}}$$

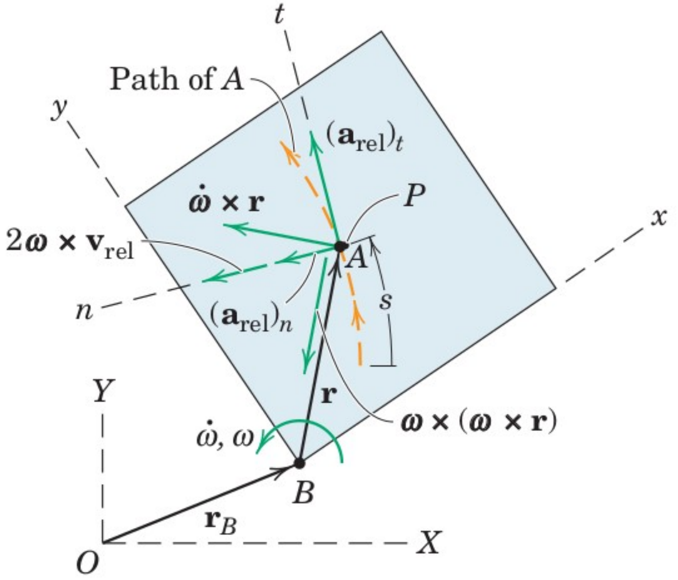
$$\boldsymbol{a}_A = \boldsymbol{a}_B + (\boldsymbol{a}_{\text{rel}} + \boldsymbol{\omega} \times \boldsymbol{v}_{\text{rel}}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times (\boldsymbol{v}_{\text{rel}} + \boldsymbol{\omega} \times \boldsymbol{r}) \qquad \text{(from 12)}$$

$$\boldsymbol{a}_A = \boldsymbol{a}_B + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + 2\boldsymbol{\omega} \times \boldsymbol{v}_{\text{rel}} + \boldsymbol{a}_{\text{rel}} \qquad \text{..................(13)}$$

Equation (13) is the general vector expression for the absolute acceleration of a particle A in terms of its acceleration $\boldsymbol{a}_{\text{rel}}$ measured relative to a moving coordinate system which rotates with an angular velocity $\boldsymbol{\omega}$ and an angular acceleration $\dot{\boldsymbol{\omega}}$.

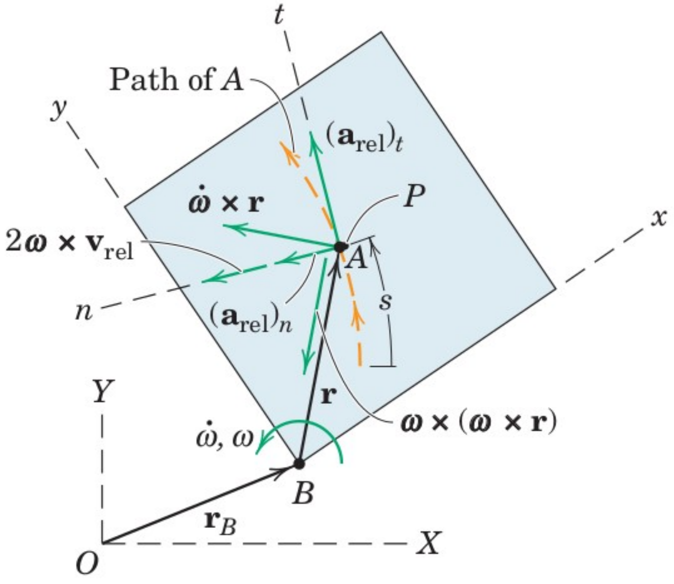
The terms $\dot{\boldsymbol{\omega}} \times \boldsymbol{r}$ and $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})$ are shown.

They represent, respectively, the tangential and normal components of the acceleration $\boldsymbol{a}_{P/B}$ of the coincident point P in its circular motion with respect to B.



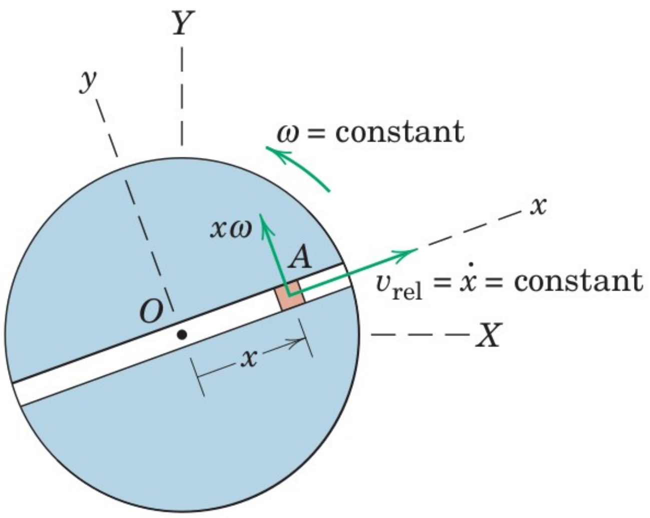
This motion would be observed from a set of nonrotating axes moving with B. The magnitude of $\dot{\boldsymbol{\omega}} \times \boldsymbol{r}$ is $r\ddot{\theta}$ and its direction is tangent to the circle. The magnitude of $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})$ is $r\omega^2$ and its direction is from P to B along the normal to the circle.

The acceleration of A relative to the plate along the path, $\boldsymbol{a}_{\text{rel}}$, may be expressed in rectangular, normal and tangential, or polar coordinates in the rotating system. Frequently, n- and t-components are used, and these components are shown in figure. The tangential component has the magnitude $(a_{\text{rel}})_t = \ddot{s}$, where s is the distance measured along the path to A. The normal component has the magnitude $(a_{\text{rel}})_n = v^2_{\text{rel}}/\rho$, where ρ is the radius of curvature of the path as measured in x-y. The sense of this vector is always toward the center of curvature.

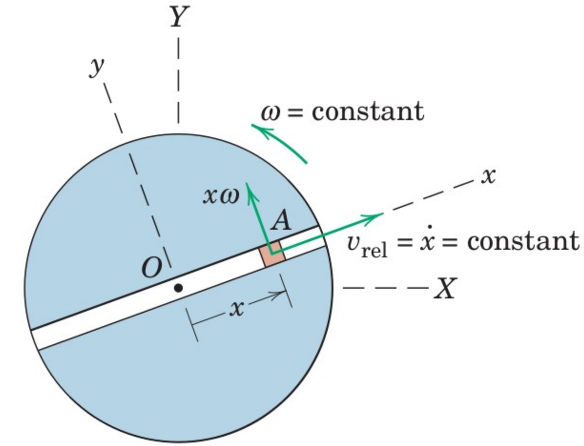


The term $2\boldsymbol{\omega}\times\boldsymbol{v}_{\text{rel}}$, is called the **Coriolis acceleration**. It represents the **difference between the acceleration of A relative to P as measured from nonrotating axes and from rotating axes.** The direction is always normal to the vector $\boldsymbol{v}_{\text{rel}}$, and the sense is established by the right-hand rule for the cross product.

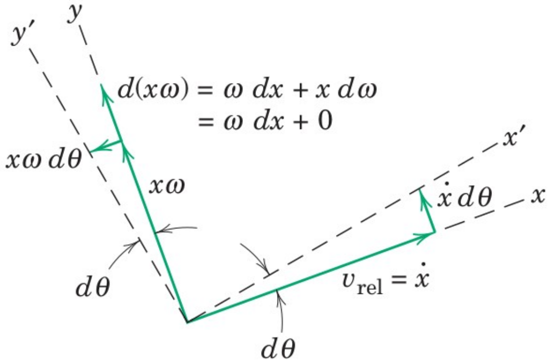
The Coriolis acceleration $\boldsymbol{a}_{\text{Cor}} = 2\boldsymbol{\omega}\times\boldsymbol{v}_{\text{rel}}$ is difficult to visualize because it is composed of two separate physical effects. To help with this visualization, we will consider the simplest possible motion in which this term appears. Consider a rotating disk with a radial slot in which a small particle A is confined to slide. Let the disk turn with a constant angular velocity $\boldsymbol{\omega}$ and let the particle move along the slot with a constant speed $\boldsymbol{v}_{\text{rel}} = \dot{\boldsymbol{x}}$ relative to the slot. The velocity of A has the two components (a) $\dot{\boldsymbol{x}}$ due to motion along the slot, and (b) $\boldsymbol{x\omega}$ due to the rotation of the slot.



The changes in these two velocity components due to the rotation of the disk are shown for the interval dt, during which the x-y axes rotate with the disk through the angle $d\theta$ to x'-y'.



The velocity increment due to the change in direction of $\boldsymbol{v}_{\mathrm{rel}}$ is $\dot{x}\ d\theta$ and that due to the change in magnitude of $x\omega$ is ωdx, both being in the y-direction normal to the slot. Dividing each increment by dt and adding give the sum $\omega\dot{x} + \dot{x}\omega = 2\dot{x}\omega$, which is the magnitude of the Coriolis acceleration $2\boldsymbol{\omega} \times \boldsymbol{v}_{\mathrm{rel}}$.



Dividing the remaining velocity increment $x\omega d\theta$ due to the change in direction of $x\omega$ by dt gives $x\omega\dot{\theta}$ or $x\omega^2$, which is the acceleration of a point P fixed to the slot and momentarily coincident with the particle A.

Now applying (13) for the same motion, we see that

$$\boldsymbol{a}_A = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + 2\boldsymbol{\omega} \times \boldsymbol{v}_{\mathrm{rel}}$$

Replacing $\boldsymbol{r}$ by $x\boldsymbol{i}$, $\boldsymbol{\omega}$ by $\omega\boldsymbol{k}$, and $\boldsymbol{v}_{\mathrm{rel}}$ by $\dot{x}\boldsymbol{i}$ gives

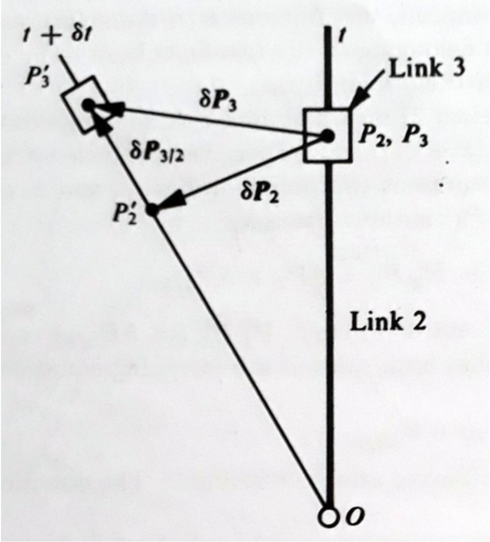
$$\boldsymbol{a}_A = \omega^2 x\boldsymbol{i} + 2\omega\dot{x}\boldsymbol{j}$$

Motion difference between two instantaneously coincident points:

$$\boldsymbol{OP}_3 = \boldsymbol{OP}_2 + \boldsymbol{P}_2\boldsymbol{P}_3$$

$$\boldsymbol{v}_{P_3} = \boldsymbol{v}_{P_2} + \boldsymbol{\omega} \times \boldsymbol{P}_3\boldsymbol{P}_2 + \boldsymbol{v}_{P_3/P_2}$$

$$\boldsymbol{a}_{P_3} = \boldsymbol{a}_{P_2} + \dot{\boldsymbol{\omega}} \times \boldsymbol{P}_3\boldsymbol{P}_2 + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{P}_3\boldsymbol{P}_2) + 2\boldsymbol{\omega} \times \boldsymbol{v}_{P_3/P_2} + \boldsymbol{a}_{P_3/P_2}$$



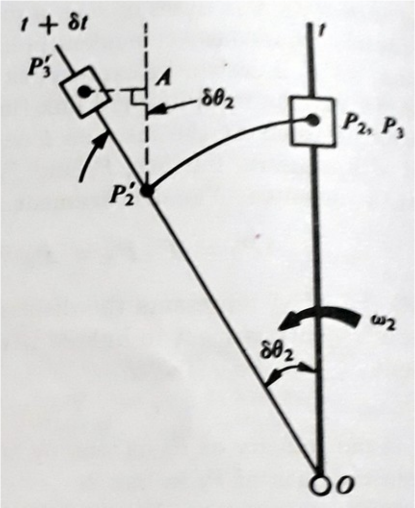
Substitute $\boldsymbol{P}_2\boldsymbol{P}_3$=0 now,

$$\boldsymbol{v}_{P_3} = \boldsymbol{v}_{P_2} + \boldsymbol{v}_{P_3/P_2}$$

$$\boldsymbol{a}_{P_3} = \boldsymbol{a}_{P_2} + 2\boldsymbol{\omega} \times \boldsymbol{v}_{P_3/P_2} + \boldsymbol{a}_{P_3/P_2}$$

The Coriolis component of acceleration can be understood as follows:

$$AP_3' = P_2'P_3' \cdot \delta\theta_2 = V_{P_3/P_2}\delta t \cdot \omega_2 \delta t = V_{P_3/P_2} \cdot \omega_2 \delta t^2$$

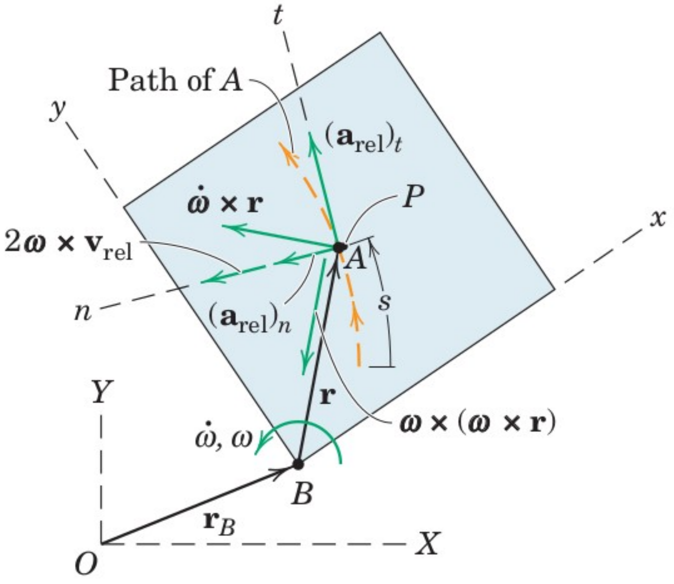


This displacement term is proportional to the square of time elapsed. Therefore this displacement must be due to an additional acceleration of P_3 in transverse direction. If the magnitude of this additional acceleration is a_c, then

$$\frac{1}{2}a_c\delta t^2 = V_{P_3/P_2} \cdot \omega_2 \cdot \delta t^2 \Rightarrow a_c = 2V_{P_3/P_2}\omega_2.$$

Rotating vs. Nonrotating system

The following comparison will help to establish the equivalence of, and clarify the differences between, the relative-acceleration equations written for rotating and nonrotating reference axes:



$$\boldsymbol{a}_A = \boldsymbol{a}_B + \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + 2\boldsymbol{\omega} \times \boldsymbol{v}_{\mathrm{rel}} + \boldsymbol{a}_{\mathrm{rel}}$$

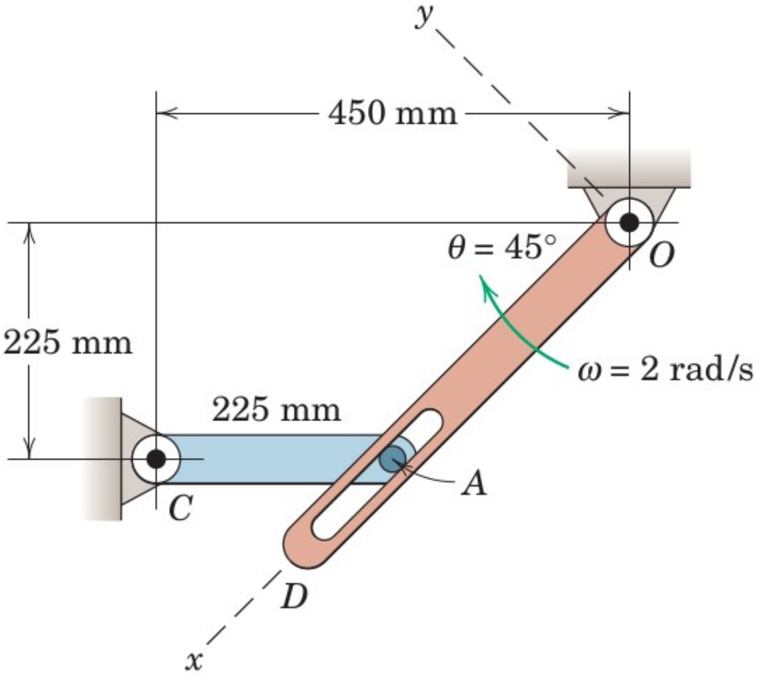
$$\boldsymbol{a}_A = \boldsymbol{a}_B + \boldsymbol{a}_{P/B} + \boldsymbol{a}_{A/P}$$

$$\boldsymbol{a}_A = \boldsymbol{a}_P + \boldsymbol{a}_{A/P}$$

$$\boldsymbol{a}_A = \boldsymbol{a}_B + \boldsymbol{a}_{A/B}$$

Example 9

The pin A of the hinged link AC is confined to move in the rotating slot of link OD. The angular velocity of OD is $\omega = 2$ rad /s clockwise and is constant for the interval of motion concerned. For the position where $\theta = 45°$ with AC horizontal, determine (a) the velocity of pin A and the velocity of A relative to the rotating slot in OD,
(b) determine the angular acceleration of AC and the acceleration of A relative to the rotating slot in arm OD.



Pin A always moves along the slot, hence we use a rotating coordinate system x-y attached to arm OD. The expression of velocity of A, with the origin at the fixed point O as,

$$\boldsymbol{v}_A = \boldsymbol{\omega}_{OD} \times \boldsymbol{r} + \boldsymbol{v}_{\text{rel}}.$$

Point A also moves in a circular path about C, hence

$$\boldsymbol{v}_A = \boldsymbol{\omega}_{CA} \times \boldsymbol{r}_{CA} = \omega_{CA}\,\boldsymbol{k} \times (225/\sqrt{2})(-\boldsymbol{i} - \boldsymbol{j}) = (225/\sqrt{2})\omega_{CA}\,(\boldsymbol{i} - \boldsymbol{j})$$

where ω_{CA} is assumed to be clockwise arbitrarily.

The vector from the origin to the point P on OD coincident with A is

$$\boldsymbol{r} = OP\,\boldsymbol{i} = 225\sqrt{2}\,\boldsymbol{i}\text{ mm}.$$

Thus, $\boldsymbol{\omega}_{OD} \times \boldsymbol{r} = 450\sqrt{2}\,\boldsymbol{j}$ mm/s.

Finally, the relative-velocity term $\boldsymbol{v}_{\text{rel}}$ is the velocity measured by an observer attached to the rotating reference frame and is $\boldsymbol{v}_{\text{rel}} = \dot{x}\,\boldsymbol{i}$.

Substituting into the relative-velocity equation and solving it gives, $\boldsymbol{\omega}_{CA} = -4$ rad/s and $\dot{\boldsymbol{x}} = \boldsymbol{v}_{\text{rel}} = -450\sqrt{2}$ mm/s. Now $\boldsymbol{v}_A$ can also be calculated.

The expression for acceleration of point A with the origin at the fixed point O,

$$\boldsymbol{a}_A = \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + 2\boldsymbol{\omega} \times \boldsymbol{v}_{\text{rel}} + \boldsymbol{a}_{\text{rel}}$$

Acceleration of point A can also be written as,

$$\boldsymbol{a}_A = \dot{\boldsymbol{\omega}}_{CA} \times \boldsymbol{r}_{CA} + \boldsymbol{\omega}_{CA} \times (\boldsymbol{\omega}_{CA} \times \boldsymbol{r}_{CA})$$

Substituting these values from solution of (a),

$$\boldsymbol{a}_A = \dot{\omega}_{CA}\boldsymbol{k} \times \frac{225}{\sqrt{2}}(-\boldsymbol{i} - \boldsymbol{j}) - 4\boldsymbol{k} \times \left(-4\boldsymbol{k} \times \frac{225}{\sqrt{2}}[-\boldsymbol{i} - \boldsymbol{j}]\right)$$

As $\boldsymbol{\omega}$ is constant, $\dot{\boldsymbol{\omega}} \times \boldsymbol{r} = 0$.

Other terms can also be calculated except, $\boldsymbol{a}_{\text{rel}} = \ddot{x}\boldsymbol{i}$.

Substituting all values in the first expression $\dot{\boldsymbol{\omega}}_{CA}$ and $\ddot{\boldsymbol{x}}$ and then $\boldsymbol{a}_A$ can be calculated.