ME531: Advanced Mechanics of Solids

Motion, Strain and Stress

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Material and Spatial derivatives

Material field: Field which is a function of material coordinates $\boldsymbol{X}$ and time t.

Spatial field: Field which is a function of spatial coordinates $\boldsymbol{x}$ and t.

***Material time derivative* of a material field:**

Consider a smooth material field $\mathcal{F}(\boldsymbol{X}, t)$. Material time derivative or *total time derivative* of the field is defined as,

$$\dot{\mathcal{F}}(\boldsymbol{X}, t) = \frac{D\mathcal{F}(\boldsymbol{X}, t)}{Dt} = \left(\frac{\partial \mathcal{F}(\boldsymbol{X}, t)}{\partial t}\right)_{\boldsymbol{X}} \text{ or } \dot{\mathcal{F}}(X_A, t) = \left(\frac{\partial \mathcal{F}(X_A, t)}{\partial t}\right)_{X_A}.$$

***Material gradient* of a material field:**

Material gradient of the material field $\mathcal{F}(\boldsymbol{X}, t)$ is given as,

$$\operatorname{Grad} \mathcal{F}(\boldsymbol{X}, t) = \left(\frac{\partial \mathcal{F}(\boldsymbol{X}, t)}{\partial \boldsymbol{X}}\right)_t$$

***Spatial time derivative* of a spatial field:**

Consider a smooth spatial field $f(\boldsymbol{x}, t)$. The spatial time derivative of the field is the derivative of $\boldsymbol{x}$ w.r.t time t holding the current position $\boldsymbol{x}$. fixed. It is also referred as *local time derivative*. It is simply denoted as,

$$\left(\frac{\partial f}{\partial t}\right)_{\boldsymbol{x}}.$$

It represents the change in f with time t as seen by an observer currently situated at $\boldsymbol{x}$.

***Spatial gradient* of a spatial field:** $f(\boldsymbol{x}, t)$

Spatial gradient of the spatial field , grad$(\boldsymbol{x})$ is the derivative w.r.t. the current position $\boldsymbol{x}$ holding time t fixed.

$$\operatorname{grad} f(\boldsymbol{x}, t) = \left(\frac{\partial f(\boldsymbol{x}, t)}{\partial \boldsymbol{x}}\right)_t$$

Material time derivative of a spatial field:

The material time derivative of the spatial field $f(\boldsymbol{x}, t)$ is the time derivative of f holding $\boldsymbol{X}$ fixed.

$$\dot{f}(\boldsymbol{x}, t) = \frac{Df(\boldsymbol{x}, t)}{Dt} = \left(\frac{\partial f[\boldsymbol{\chi}(X, t), t]}{\partial t}\right)_{\boldsymbol{X}=\chi^{-1}(\boldsymbol{x}, t)}.$$

Now, consider a smooth scaler valued spatial field $\phi(\boldsymbol{x}, t)$, by chain rule

$$\dot{\phi}(\boldsymbol{x}, t) = \left(\frac{\partial \phi(\boldsymbol{x}, t)}{\partial t}\right)_{\boldsymbol{x}} + \left(\frac{\partial \phi(\boldsymbol{x}, t)}{\partial \boldsymbol{x}}\right)_{\boldsymbol{t}} \cdot \left(\frac{\partial \boldsymbol{\chi}(X, t)}{\partial t}\right)_{\boldsymbol{X}=\chi^{-1}(\boldsymbol{x}, t)}.$$

$$\dot{\phi}(\boldsymbol{x}, t) = \frac{D\phi(\boldsymbol{x}, t)}{Dt} = \boxed{\frac{\partial \phi(\boldsymbol{x}, t)}{\partial t}} + \boxed{\operatorname{grad} \phi(\boldsymbol{x}, t) \cdot \boldsymbol{v}(\boldsymbol{x}, t)} \quad \text{or} \quad \dot{\phi}(\boldsymbol{x}, t) = \frac{D\phi}{Dt} = \boxed{\frac{\partial \phi}{\partial t}} + \boxed{\frac{\partial \phi}{\partial x_i} v_i}.$$

Here, the first term on the right hand side is the *spatial or local time derivative* of spatial scaler field ϕ, while the second term is called the ***convective rate of change*** of ϕ, which describes the changing position of a partial.

Similarly, material time derivative of a smooth vector-valued spatial field $\boldsymbol{v}(\boldsymbol{x},t)$ will be,

$$\dot{\boldsymbol{v}}(\boldsymbol{x},t) = \frac{D\boldsymbol{v}(\boldsymbol{x},t)}{Dt} = \frac{\partial \boldsymbol{v}(\boldsymbol{x},t)}{\partial t} + \operatorname{grad}\boldsymbol{v}\cdot\boldsymbol{v}(\boldsymbol{x},t) \quad \text{or} \quad \dot{v}_i(\boldsymbol{x},t) = \frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j}v_j.$$

If $\boldsymbol{v}(\boldsymbol{x},t)$ is considered as spatial description of velocity then $\dot{\boldsymbol{v}}(\boldsymbol{x},t)$ describes the spatial acceleration field ($\boldsymbol{a}$). Hence,

$$\boldsymbol{a} = \frac{D\boldsymbol{v}}{Dt} = \frac{\partial \boldsymbol{v}}{\partial t} + \operatorname{grad}\boldsymbol{v}\cdot\boldsymbol{v} \quad \text{or} \quad a_i = \frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j}v_j.$$

Again, the first term on the right hand side $\partial v_i/\partial t$ describes the local acceleration (local rate of change of the velocity field). The second term describes the *convective acceleration* (convective rate of change of the velocity field).

Spatial velocity field $\boldsymbol{v}(\boldsymbol{x},t)$ is defined as,

$$\boldsymbol{v}(\boldsymbol{x},t) = \frac{Dx(\boldsymbol{X},t)}{Dt} = \left(\frac{\partial \boldsymbol{x}(\boldsymbol{X},t)}{\partial t}\right)_{\boldsymbol{X}} = \frac{\partial \boldsymbol{x}}{\partial t}$$

Rate of deformation tensor

Consider the spatial derivative of spatial velocity field $\boldsymbol{v}(\boldsymbol{x},t)$ defined by

$$\boldsymbol{l}(\boldsymbol{x},t) = \frac{\partial \boldsymbol{v}(\boldsymbol{x},t)}{\partial \boldsymbol{x}} = \operatorname{grad} \boldsymbol{v}(\boldsymbol{x},t) \quad \text{or} \quad l_{ij} = \frac{\partial v_i}{\partial x_j},$$

where spatial field $\boldsymbol{l}(\boldsymbol{x},t)$ is known as *spatial velocity gradient*, which is in general a non-symmetric second order tensor.

Material velocity gradient of a material velocity field $\boldsymbol{V}(\boldsymbol{X},t)$ is defined as

$$\frac{\partial \boldsymbol{V}(\boldsymbol{X},t)}{\partial \boldsymbol{X}} = \frac{\partial}{\partial \boldsymbol{X}}\left(\frac{\partial \boldsymbol{\chi}(\boldsymbol{X},t)}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{\chi}(\boldsymbol{X},t)}{\partial \boldsymbol{X}}\right) = \dot{\boldsymbol{F}}(\boldsymbol{X},t)$$

$$\dot{\boldsymbol{F}}(\boldsymbol{X},t) = \frac{\partial \boldsymbol{V}(\boldsymbol{X},t)}{\partial \boldsymbol{X}} = \operatorname{Grad} \boldsymbol{V}$$

Using the chain rule of differentiation, spatial velocity gradient can be written as,

$$\boldsymbol{l}(\boldsymbol{x},t) = \frac{\partial \boldsymbol{v}(\boldsymbol{x},t)}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{v}(\boldsymbol{x},t)}{\partial \boldsymbol{X}} \frac{\partial \boldsymbol{X}(\boldsymbol{x},t)}{\partial \boldsymbol{x}} = \frac{\partial \dot{\boldsymbol{\chi}}(\boldsymbol{X},t)}{\partial \boldsymbol{X}} \frac{\partial \boldsymbol{X}(\boldsymbol{x},t)}{\partial \boldsymbol{x}}$$
$$= \frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{\chi}(\boldsymbol{X},t)}{\partial \boldsymbol{X}}\right) \frac{\partial \boldsymbol{X}(\boldsymbol{x},t)}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{F}}{\partial t}\boldsymbol{F}^{-1}$$

Thus, $\boldsymbol{l} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}$, hence $\dot{\boldsymbol{F}} = \boldsymbol{l}\boldsymbol{F}$, or $l_{ij} = \dot{F}_{ik}F^{-1}_{kj}$, hence $\dot{F}_{ij} = l_{ik}F_{kj}$.

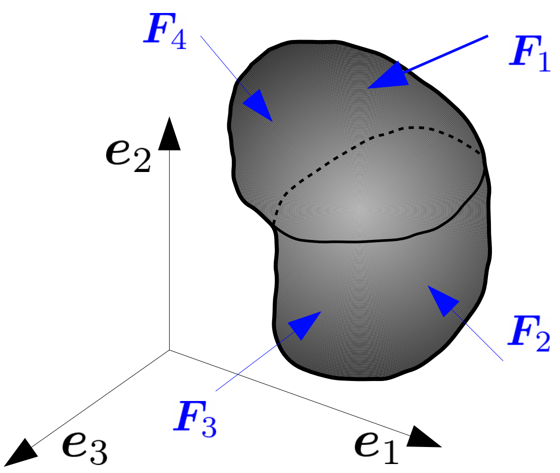
Spatial velocity gradient can be additively decomposed into a symmetric and anti-symmetric part as,

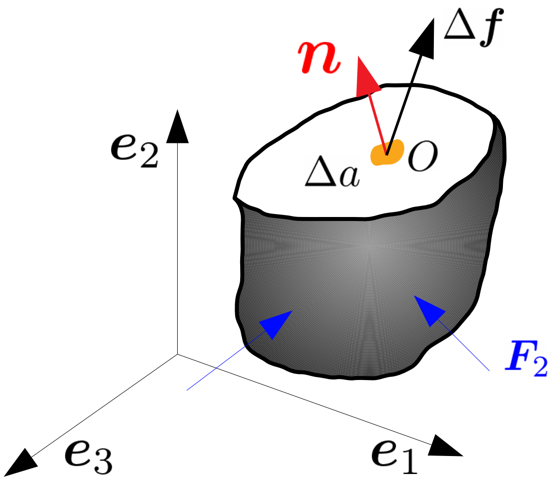
$$\boldsymbol{l}(\boldsymbol{x},\boldsymbol{t}) = \boldsymbol{d}(\boldsymbol{x},t) + \boldsymbol{w}(\boldsymbol{x},t),$$

where $\boldsymbol{d}(\boldsymbol{x},t)$ is called the rate of deformation tensor and $\boldsymbol{w}(\boldsymbol{x},t)$ is called the spin tensor, and they are defined as

$$\boldsymbol{d} = \frac{1}{2}\left(\boldsymbol{l} + \boldsymbol{l}^T\right), \text{ and } \boldsymbol{w} = \frac{1}{2}\left(\boldsymbol{l} - \boldsymbol{l}^T\right).$$

Stress





Consider a body in the deformed configuration acted upon by several external forces. Let the body be cut by an imaginary plane passing though point O.

Now, consider an elemental area ΔA in the neighborhood of point O. The reaction force on the area is $\Delta \boldsymbol{F}$, and normal to the area is $\boldsymbol{n}$.

We define a traction vector $\boldsymbol{t}$ corresponding to normal $\boldsymbol{n}$ and point O is defined as,

$$\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}) = \lim_{\Delta a \to 0} \frac{\Delta \boldsymbol{f}}{\Delta a} = \frac{d\boldsymbol{f}}{da}$$

where $\boldsymbol{t}(\boldsymbol{x},\boldsymbol{n})$ is called *Cauchy-traction vector* (force per unit surface area in the current configuration), exerted on the area da having normal $\boldsymbol{n}$. Traction vectors are also referred to as *surface traction*, *contact forces*, *stress vectors* or *loads*.

Cauchy's stress theorem

There exist a unique second-order tensor fields so that,

$$\boldsymbol{t}(\boldsymbol{x},\boldsymbol{n}) = \boldsymbol{n}\cdot\boldsymbol{\sigma}(\boldsymbol{x}), \quad \text{or} \quad \boldsymbol{t}(\boldsymbol{x},\boldsymbol{n}) = \boldsymbol{\sigma}^T(\boldsymbol{x})\cdot\boldsymbol{n}, \quad \text{or} \quad t_i = \sigma_{ji}n_j.$$

where $\boldsymbol{\sigma}$ is known as *Cauchy stress tensor.* This relation between the stress tensor and the traction vector is known as *Cauchy's stress theorem.* It states that if traction vector $\boldsymbol{t}$ depends upon the outward unit normal $\boldsymbol{n}$ then it must be linear in $\boldsymbol{n}$. It immediately follows that

$$\boldsymbol{t}(\boldsymbol{x},\boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{x},-\boldsymbol{n}),$$

for all unit normal vectors n. This is known as *Newton's (third) law of action and reaction.*