Introduction to Tensors

Algebra of second order tensors

A second order tensor $\boldsymbol{A}$ is a linear transformation mapping of a vector to another vector, i.e.

$$\boldsymbol{u} = \boldsymbol{A}\boldsymbol{v}.$$

As $\boldsymbol{A}$ is a linear transformation, it implies

$$\boldsymbol{A}(\alpha\boldsymbol{u} + \beta\boldsymbol{v}) = \alpha\boldsymbol{A}\boldsymbol{u} + \beta\boldsymbol{A}\boldsymbol{v}.$$

The *tensor product* or *the dyad* of two vectors is a second order tensor defined as,

$$(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w} = \boldsymbol{u}(\boldsymbol{v} \cdot \boldsymbol{w}) = (\boldsymbol{v} \cdot \boldsymbol{w})\boldsymbol{u}$$

Note that the dot product is between the two immidiate adjacent vectors which are not connected by $\otimes$ symbol.

Sometimes dyad is simply written as $\boldsymbol{uv}$.

It also follows, $(\alpha\boldsymbol{u} + \beta\boldsymbol{v}) \otimes \boldsymbol{w} = \alpha\boldsymbol{u}\boldsymbol{w} + \beta\boldsymbol{v}\boldsymbol{w}$

Another relation,

$$(\boldsymbol{u} \otimes \boldsymbol{v}) \cdot (\boldsymbol{x} \otimes \boldsymbol{y}) = (\boldsymbol{v} \cdot \boldsymbol{x})\boldsymbol{u} \otimes \boldsymbol{y} = \boldsymbol{u} \otimes \boldsymbol{y}(\boldsymbol{v} \cdot \boldsymbol{x})$$

Notice the vectors for which dot prodcut is taken.

A second order tensor can also be represented as a dyadic or tensor product of cartesian basis vectors $\boldsymbol{e}_i$ $(i \in [1, 2, 3])$ as,

$$\boldsymbol{A} = A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \text{ or } \boldsymbol{A} = A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j$$

where $\boldsymbol{e}_i\boldsymbol{e}_j$ may be though as a 'base tensor' in terms of which tensor $\boldsymbol{A}$ may be expanded in Cartesian frame. It is analogus to a vector (first order tensor) being expanded in terms of 'base vectors' $\boldsymbol{e}_i$.

The components of second order tensor $\boldsymbol{A}$ in a particular cooridnate system can be represented as a 3×3 matrix as:

$$[\boldsymbol{A}] = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

The components of a *unit* or *identity* tensor $\boldsymbol{I}$ in represented as:

$$\boldsymbol{I} = \delta_{ij} \boldsymbol{e}_i \boldsymbol{e}_j = \boldsymbol{e}_i \boldsymbol{e}_i$$

$$[\boldsymbol{I}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Examples

Example 5: For a given second order tensor $\boldsymbol{A}$ find the $(m,n)^{\text{th}}$ component of the tensor.

The $(m,n)^{\text{th}}$ component of tensor $\boldsymbol{A}$ can be extracted by post-multiplying with $\boldsymbol{e}_n$,and then pre-multiplying with $\boldsymbol{e}_m$ as,

$$
\begin{aligned}
\boldsymbol{e}_m \cdot \boldsymbol{A}\boldsymbol{e}_n &= \boldsymbol{e}_m \cdot \left(A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j\right)\boldsymbol{e}_n \\
&\Rightarrow \boldsymbol{e}_m \cdot \left(A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j\right)\boldsymbol{e}_n \\
&\Rightarrow \boldsymbol{e}_m \cdot \left(A_{ij}\boldsymbol{e}_j \cdot \boldsymbol{e}_n\right)\boldsymbol{e}_i \\
&\Rightarrow \boldsymbol{e}_m \cdot \left(A_{ij}\delta_{jn}\right)\boldsymbol{e}_i \\
&\Rightarrow A_{ij}\delta_{jn}\boldsymbol{e}_m \cdot \boldsymbol{e}_i \\
&\Rightarrow A_{in}\boldsymbol{e}_m \cdot \boldsymbol{e}_i \\
&\Rightarrow A_{in}\delta_{mi} \\
&\Rightarrow A_{mn}
\end{aligned}
$$

Examples

Example 6: Show that $\boldsymbol{v} = \boldsymbol{A}\boldsymbol{u}$ in the tensorial form can be written as

$$v_i = A_{ij} u_j \,.$$

We start by writing $\boldsymbol{A}$ as $A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j$ and $\boldsymbol{u}$ as $u_k\boldsymbol{e}_k$, then

$$\begin{aligned}
\boldsymbol{A}\boldsymbol{u} &= (A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j)(u_k\boldsymbol{e}_k) \\
&\Rightarrow A_{ij}u_k(\boldsymbol{e}_j \cdot \boldsymbol{e}_k)\boldsymbol{e}_i \\
&\Rightarrow A_{ij}u_k\delta_{jk}\boldsymbol{e}_i \\
&\Rightarrow A_{ij}u_j\boldsymbol{e}_i = v_i\boldsymbol{e}_i
\end{aligned}$$

Thus, $v_i = A_{ij}u_j$

Transpose of a tensor

The *transpose* of a tensor $\boldsymbol{A}$ is denoted by $\boldsymbol{A}^T$ and is defined as,

$$\boldsymbol{A}^T = A_{ji}\boldsymbol{e}_i\boldsymbol{e}_j \text{ or } \left(\boldsymbol{A}^T\right)_{ij} = A_{ji}.$$

Definition of tranpose is governed by the following identity. For any two vector $\boldsymbol{u}$ and $\boldsymbol{v}$,

$$\boldsymbol{v}\cdot\boldsymbol{A}^T\boldsymbol{u} = \boldsymbol{u}\cdot\boldsymbol{A}\boldsymbol{v} = \boldsymbol{A}\boldsymbol{v}\cdot\boldsymbol{u}.$$

Proof:

$$\begin{aligned}\boldsymbol{v}\cdot\boldsymbol{A}^T\boldsymbol{u} &= (v_m\boldsymbol{e}_m)\cdot(A_{ji}\boldsymbol{e}_i\boldsymbol{e}_j)u_k\boldsymbol{e}_k \\ &\Rightarrow (v_m\boldsymbol{e}_m)\cdot A_{ji}u_k\delta_{jk}\boldsymbol{e}_i \\ &\Rightarrow A_{ji}u_kv_m\delta_{jk}\delta_{mi} \\ &\Rightarrow A_{ji}u_jv_i\end{aligned}$$

$$u_jA_{ji}v_i = \boldsymbol{u}\cdot\boldsymbol{A}\boldsymbol{v}$$

or

$$A_{ji}v_iu_j = \boldsymbol{A}\boldsymbol{v}\cdot\boldsymbol{u}$$

From the definition following identities immidialtely follow,

$$\left(\boldsymbol{A}^T\right)^T = \boldsymbol{A}, \quad (\boldsymbol{A}\boldsymbol{B})^T = \boldsymbol{B}^T\boldsymbol{A}^T, \quad (\boldsymbol{u}\otimes\boldsymbol{v})^T = \boldsymbol{v}\otimes\boldsymbol{u}$$

Contraction

- *Contraction* is an operation in which we identity two indices and sum over them. Contraction is characterized as a dot.
- *Double contraction* or scaler product of two tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ is characterized as two dots and yield a scaler,

$$\boldsymbol{A} : \boldsymbol{B} = A_{ij} B_{ij}$$

Proof:

$$\boldsymbol{A} : \boldsymbol{B} = (A_{ij} \boldsymbol{e}_i \boldsymbol{e}_j) : (B_{kl} \boldsymbol{e}_k \boldsymbol{e}_l)$$

$$\Rightarrow A_{ij} B_{kl} \delta_{ik} \delta_{jl} = A_{ij} B_{ij}$$

(Notice the order in which dot product of basis vectors are taken)

- *Double contraction* of any tensors $\boldsymbol{A}$ with identity tensor yields the *trace* of tensor $\boldsymbol{A}$.

$$\boldsymbol{A} : \boldsymbol{I} = A_{ij} \delta_{ij} = A_{ii} = \mathrm{tr} \boldsymbol{A} = A_{11} + A_{22} + A_{33}$$

Examples

Example 7: Show that $\boldsymbol{A} \colon (\boldsymbol{B} \cdot \boldsymbol{C}) = (\boldsymbol{B}^T \cdot \boldsymbol{A}) \colon \boldsymbol{C} = (\boldsymbol{A} \cdot \boldsymbol{C}^T) \colon \boldsymbol{B}$.

Let's start from LHS:

$$
\begin{aligned}
\boldsymbol{A} \colon (\boldsymbol{B} \cdot \boldsymbol{C}) &= A_{mn} \boldsymbol{e}_m \boldsymbol{e}_n \colon (B_{ij} \boldsymbol{e}_i \boldsymbol{e}_j \cdot C_{kl} \boldsymbol{e}_k \boldsymbol{e}_l) \\
&\Rightarrow A_{mn} \boldsymbol{e}_m \boldsymbol{e}_n \colon B_{ij} C_{kl} \boldsymbol{e}_i \boldsymbol{e}_l \delta_{jk} \\
&\Rightarrow A_{mn} \boldsymbol{e}_m \boldsymbol{e}_n \colon B_{ik} C_{kl} \boldsymbol{e}_i \boldsymbol{e}_l \\
&\Rightarrow A_{mn} B_{ik} C_{kl} \delta_{mi} \delta_{nl} \\
&\Rightarrow A_{mn} B_{mk} C_{kn}
\end{aligned}
$$

Above can also be written as following,

$$A_{mn} B_{mk} C_{kn} = B_{mk} A_{mn} C_{kn} = \left(\boldsymbol{B}^T \cdot \boldsymbol{A}\right) : \boldsymbol{C}$$

$$A_{mn} B_{mk} C_{kn} = A_{mn} C_{kn} B_{mk} = \left(\boldsymbol{A} \cdot \boldsymbol{C}^T\right) : \boldsymbol{B}$$

Examples

Example 8: Show that $(\boldsymbol{A} \otimes \boldsymbol{B})\colon \boldsymbol{C} = \boldsymbol{A}(\boldsymbol{B}\colon \boldsymbol{C})$.

Let's start from LHS:

$$
\begin{aligned}
(\boldsymbol{A} \otimes \boldsymbol{B})\colon \boldsymbol{C} &= (A_{mn}\boldsymbol{e}_m\boldsymbol{e}_n \otimes B_{ij}\boldsymbol{e}_i\boldsymbol{e}_j)\colon C_{kl}\boldsymbol{e}_k\boldsymbol{e}_l \\
&\Rightarrow A_{mn}\boldsymbol{e}_m\boldsymbol{e}_n B_{ij} C_{kl}\delta_{ik}\delta_{jl} \\
&\Rightarrow A_{mn}\boldsymbol{e}_m\boldsymbol{e}_n B_{kl} C_{kl} = \boldsymbol{A}(\boldsymbol{B}\colon \boldsymbol{C})
\end{aligned}
$$

Determinant and Inverse of a tensor

The *determinant* of a second order tensor is a scaler and is given as

$$\det(\boldsymbol{A}) = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix},$$

with properties,

$$\det(\boldsymbol{AB}) = \det \boldsymbol{A} \det \boldsymbol{B} \text{ and } \det(\boldsymbol{A}^T) = \det(\boldsymbol{A}).$$

A tensor $\boldsymbol{A}$ is said to be singular *if and only if* $\det(\boldsymbol{A}){=}0$

For a non-singular tensor $\boldsymbol{A}$, their exists a unique inverse tensor $\boldsymbol{A}^{-1}$ such that,

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$$

Inverse of a tensor

Invertable tensors have the following imortant properties:

$$
\begin{aligned}
(\boldsymbol{AB})^{-1} &= \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}, \\
(\boldsymbol{A}^{-1})^{-1} &= \boldsymbol{A}, \\
(\boldsymbol{A}^{-1})^{T} &= (\boldsymbol{A}^{T})^{-1} = \boldsymbol{A}^{-T}, \\
\det(\boldsymbol{A}^{-1}) &= (\det \boldsymbol{A})^{-1}.
\end{aligned}
$$

An *orthogonal tensor* is a special tensor whose inverse is same as its transpose, i.e.

$$\boldsymbol{Q}^T = \boldsymbol{Q}^{-1},$$

which follows, $\boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}$.

Also, $\det(\boldsymbol{Q}^T\boldsymbol{Q}) = (\det \boldsymbol{Q})^2 = 1$.

If $\det \boldsymbol{Q} = +1$ tensor is called *proper* orthogonal tensor, and if $\det \boldsymbol{Q} = -1$ it is called *improper* orthogonal tensor.

Symmetric and skew tensor

A second order tensor is symmetric if $\boldsymbol{S} = \boldsymbol{S}^T$ or $S_{ij} = S_{ji}$.

A tensor is called skew or antisymmetric if $\boldsymbol{W} = -\boldsymbol{W}^T$ or $W_{ij} = -W_{ji}$.

Any tensor $\boldsymbol{A}$ can be decomposed into a symmetric and skew tensor as,

$$\boldsymbol{A} = \boldsymbol{S} + \boldsymbol{W}$$

where,

$$\boldsymbol{S} = \frac{\boldsymbol{A} + \boldsymbol{A}^T}{2}, \text{and } \boldsymbol{W} = \frac{\boldsymbol{A} - \boldsymbol{A}^T}{2},$$

which have following forms,

$$[\boldsymbol{S}] = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{pmatrix} \quad \text{and} \quad [\boldsymbol{W}] = \begin{pmatrix} 0 & W_{12} & W_{13} \\ -W_{12} & 0 & W_{23} \\ -W_{13} & -W_{23} & 0 \end{pmatrix}$$

Examples

Example 9: If $\boldsymbol{S}$ is a symmetric, and $\boldsymbol{W}$ is an antisymmetric tensor, then show that
(i) $\boldsymbol{S}:\boldsymbol{W}=0$,
(ii) $\boldsymbol{S}:\boldsymbol{B}=\boldsymbol{S}:\mathrm{symm}(\boldsymbol{B})$, and
(iii) $\boldsymbol{W}:\boldsymbol{B}=\boldsymbol{W}:\mathrm{asymm}(\boldsymbol{B})$
where $\boldsymbol{B}$ is a second order tensor; $\mathrm{symm}(\boldsymbol{B})$ and $\mathrm{asymm}(\boldsymbol{B})$ are *symmetric* and *antisymmetric* part of $\boldsymbol{B}$, respectively.

Consider the fact that, for any second order tensors $\boldsymbol{S}$ and $\boldsymbol{W}$, we can write,

$$\boldsymbol{S}:\boldsymbol{W}=S_{ij}W_{ij}=S_{ji}W_{ji}.$$

$$\boldsymbol{S}:\boldsymbol{W}=1/2\left(S_{ij}W_{ij}+S_{ij}W_{ij}\right)=1/2\left(S_{ij}W_{ij}+S_{ji}W_{ji}\right)$$

As $\boldsymbol{S}$ is symmetric and $\boldsymbol{W}$ is skew, $S_{ij}=S_{ji}$, and $W_{ij}=-W_{ji}$.

Hence, we can write, $\boldsymbol{S}:\boldsymbol{W}=1/2\left(S_{ij}W_{ij}-S_{ij}W_{ij}\right)=0.$

Now, tensor $\boldsymbol{B}$ can be splitted in to symmetric and antisymmetric part, hence, $\boldsymbol{B} = \text{symm}(\boldsymbol{B}) + \text{asymm}(\boldsymbol{B})$

$\boldsymbol{S} : \boldsymbol{B} = \boldsymbol{S} : (\text{symm}(\boldsymbol{B}) + \text{asymm}(\boldsymbol{B})) = \boldsymbol{S} : \text{symm}(\boldsymbol{B})$,
as $\boldsymbol{S} : \text{asymm}(\boldsymbol{B}) = 0$.

Similarly,
$\boldsymbol{W} : \boldsymbol{B} = \boldsymbol{W} : (\text{symm}(\boldsymbol{B}) + \text{asymm}(\boldsymbol{B})) = \boldsymbol{W} : \text{asymm}(\boldsymbol{B})$,
as $\boldsymbol{W} : \text{symm}(\boldsymbol{B}) = 0$.

Spherical and deviatoric tensor

Any tensors $\boldsymbol{A}$ can be splited into a spherical and a deviatoric part as,

$$\boldsymbol{A} = \alpha\boldsymbol{I} + \mathrm{dev}\boldsymbol{A} \text{ or } A_{ij} = \alpha\delta_{ij} + \mathrm{dev}A_{ij},$$

where, scaler α is given as $\alpha = \dfrac{1}{3}\mathrm{tr}\boldsymbol{A} = \dfrac{1}{3}A_{ii}$.

Deviatoric part is calculated as,

$$\mathrm{dev}A_{ij} = A_{ij} - \alpha\delta_{ij} = A_{ij} - \frac{1}{3}A_{kk}\delta_{ij}.$$

Deviatoric tensor has an important property that,

$$\mathrm{tr}(\mathrm{dev}\boldsymbol{A}) = (\mathrm{dev}\boldsymbol{A})_{mm} = A_{mm} - \tfrac{1}{3}A_{kk}\delta_{mm} = A_{mm} - A_{kk} = 0$$

Thus trace of any deviatoric tensor is always *zero*.

Transformation laws of tensors

- Through a tensor is invarient with respect to coordinate system, its components change if the coordinate system changes.
- Consider two coordinate system denoted by x_i and $\bar{x}_i$ with base vectors $\boldsymbol{e}_i$ and $\bar{e}_i$, respectively.
- As shown in the figure the components of a vector $\boldsymbol{v}$ are $v_i = \boldsymbol{e}_i \cdot \boldsymbol{v}$ in the first system, and $\bar{v}_i = \bar{e}_i \cdot \boldsymbol{v}$ in the second system.
- Similarly, the components of a tensor $\boldsymbol{A}$ is $A_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{A}\boldsymbol{e}_j$ in the first system and $\bar{A}$ is $\bar{A}_{ij} = \bar{e}_i \cdot \boldsymbol{A}\bar{e}_j$ in the second system. We will derive the transformation laws for a tensor, i.e. the relationship between the components A_{ij} and $\bar{A}_{ij}$.