Linear elasticity

Small strain linear elasticity

- Engineering structures are generally designed within the elastic limit of metals. In this limit, strains and deformations are very small.
- Since deformations are small, difference between the reference and current coordinates is negligible; hence it is advisable to refer to only one of the configuration, i.e., the current configurations, for analysis.
- Higher order derivatives of displacements can also be neglected for defining strains. Thus, the small strain tensor become

$$E_{ij} \approx e_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right] \approx \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] = \varepsilon_{ij}$$

$$\text{or} \quad \boldsymbol{\varepsilon} = \frac{1}{2}\left[\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T\right] \qquad \cdots\cdots\cdots\cdots\cdots\cdots(1)$$

Strain compatibility for small strains

- Strain-displacement relations are *six* equations in terms of three displacements.
- If we specify 3 single valued displacement components u, v, and w then through differentiation resulting strain field will be well behaved.
- However converse is not true always; i.e. if strain fields are given then integration of these equations getting does not necessarily produce a single-valued continuous displacements fields. It is because we are trying to solve six equations for only three unknown displacement components.
- Thus, to ensure continuous, single-valued displacements, the strains must satisfy additional relations; they are called **compatibility equations**.
- Also remember that, we can never completely recover the displacement field that gives rise to a particular strain field. Any rigid motion produces no strain, so the displacements can only be completely determined if there is some additional information (besides the strain) that will tell us how much the solid has rotated and translated. However, integrating the strain field can tell us the displacement field to within an arbitrary rigid motion.

To develop the additional conditions so that integration of strain yields continuous single-valued displacement fields, we start with the definition of small strains (1),

The idea is to eliminate displacements from these equations. For that we differentiate the equation twice w.r.t. x, as

$$\varepsilon_{ij,kl} = \frac{1}{2}\left(u_{i,jkl} + u_{j,ikl}\right). \qquad \cdots\cdots\cdots\cdots\cdots\cdots(2)$$

Simple exchange of subscript in the previous equation results in following additional relations,

$$\varepsilon_{kl,ij} = \frac{1}{2}\left(u_{k,lij} + u_{l,kij}\right), \ \varepsilon_{jl,ik} = \frac{1}{2}\left(u_{j,lik} + u_{l,jik}\right), \ \varepsilon_{ik,jl} = \frac{1}{2}\left(u_{i,kjl} + u_{k,ijl}\right).$$

$$\cdots\cdots\cdots\cdots\cdots\cdots(3)$$

With the assumption of continuous displacement field $\boldsymbol{u}$, order of differentiation on $\boldsymbol{u}$ can be interchanged and displacements can be eliminated from the previous equations, which results in the following equations, called **Saint Vinent Compatibility equations**.

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(4)$$

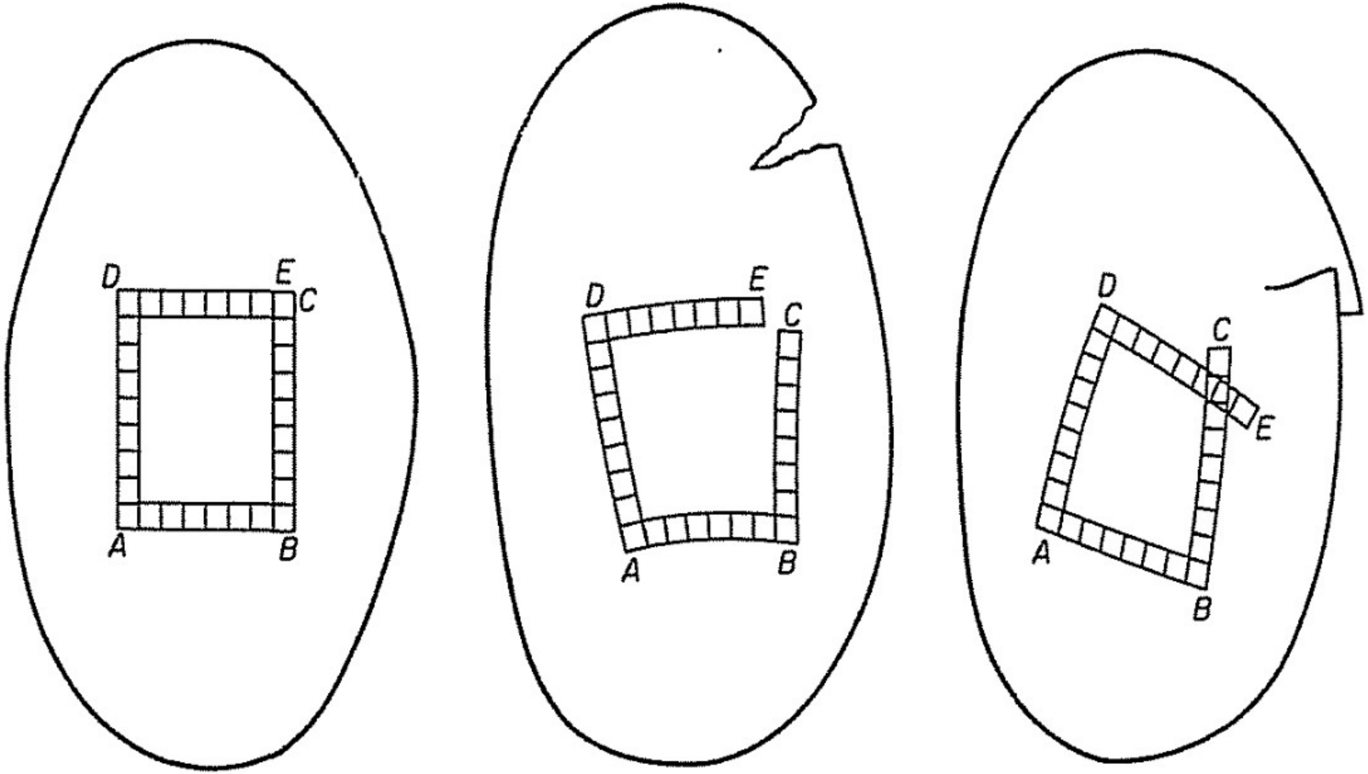
Saint Vinent compatibility equations derived in the previous slide are total of 81 equations, out of which only 6 are meaningful (others are simple identities or repetition). These equations are following.

$$
\begin{aligned}
&\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 x_2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = 0, \qquad &&\frac{\partial^2 \varepsilon_{11}}{\partial x_2 x_3} - \frac{\partial}{\partial x_1}\left(-\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3}\right) = 0,\\
&\frac{\partial^2 \varepsilon_{22}}{\partial x_3^2} - 2\frac{\partial^2 \varepsilon_{23}}{\partial x_1 x_3} + \frac{\partial^2 \varepsilon_{33}}{\partial x_2^2} = 0, \qquad &&\frac{\partial^2 \varepsilon_{22}}{\partial x_3 x_1} - \frac{\partial}{\partial x_2}\left(-\frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3} + \frac{\partial \varepsilon_{23}}{\partial x_1}\right) = 0,\\
&\frac{\partial^2 \varepsilon_{11}}{\partial x_3^2} - 2\frac{\partial^2 \varepsilon_{13}}{\partial x_1 x_3} + \frac{\partial^2 \varepsilon_{33}}{\partial x_1^2} = 0, \qquad &&\frac{\partial^2 \varepsilon_{33}}{\partial x_1 x_2} - \frac{\partial}{\partial x_3}\left(-\frac{\partial \varepsilon_{12}}{\partial x_3} + \frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2}\right) = 0.
\end{aligned}
$$

$$\cdots\cdots\cdots\cdots\cdots\cdots(5)$$

It can be shown that these six equations are equivalent to three independent fourth-order relations.

Physical interpretation of strain compatibility



- For isotropic materials the elasticity tensor is

$$\mathbb{C}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}. \quad \cdots\cdots\cdots\cdots(6)$$

Thus the stress and strain are related as,

$$\sigma_{ij} = \mathbb{C}_{ijkl}\varepsilon_{kl} = \left(\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}\right)\varepsilon_{kl}$$

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} \text{ or } \boldsymbol{\sigma} = \lambda\,\mathrm{tr}\boldsymbol{\varepsilon}\,\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}. \quad \cdots\cdots\cdots\cdots(7)$$

Here, λ and μ are called Lame's constant. They are related to other elastic constants Young's modulus (E), Poisson's ratio (ν), bulk modulus (K), and shear modulus (μ).

Relations among elastic constants

	E	v	k	μ	λ
E, v	E	v	$\dfrac{E}{3(1-2v)}$	$\dfrac{E}{2(1+v)}$	$\dfrac{Ev}{(1+v)(1-2v)}$
E, k	E	$\dfrac{3k-E}{6k}$	k	$\dfrac{3kE}{9k-E}$	$\dfrac{3k(3k-E)}{9k-E}$
E, μ	E	$\dfrac{E-2\mu}{2\mu}$	$\dfrac{\mu E}{3(3\mu-E)}$	μ	$\dfrac{\mu(E-2\mu)}{3\mu-E}$
E, λ	E	$\dfrac{2\lambda}{E+\lambda+R}$	$\dfrac{E+3\lambda+R}{6}$	$\dfrac{E-3\lambda+R}{4}$	λ
v, k	$3k(1-2v)$	v	k	$\dfrac{3k(1-2v)}{2(1+v)}$	$\dfrac{3kv}{1+v}$
v, μ	$2\mu(1+v)$	v	$\dfrac{2\mu(1+v)}{3(1-2v)}$	μ	$\dfrac{2\mu v}{1-2v}$
v, λ	$\dfrac{\lambda(1+v)(1-2v)}{v}$	v	$\dfrac{\lambda(1+v)}{3v}$	$\dfrac{\lambda(1-2v)}{2v}$	λ
k, μ	$\dfrac{9k\mu}{6k+\mu}$	$\dfrac{3k-2\mu}{6k+2\mu}$	k	μ	$k-\dfrac{2}{3}\mu$
k, λ	$\dfrac{9k(k-\lambda)}{3k-\lambda}$	$\dfrac{\lambda}{3k-\lambda}$	k	$\dfrac{3}{2}(k-\lambda)$	λ
μ, λ	$\dfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\dfrac{\lambda}{2(\lambda+\mu)}$	$\dfrac{3\lambda+2\mu}{3}$	μ	λ

Problems in Elasticity

Total 15 unknown scalar variables (3 Displacements, 6 strain components and 6 stress components.

Available field Equations:

6 Strain Displacement relations

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left[\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T\right] \quad \text{or} \quad \varepsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right], \qquad \text{....................(8)}$$

3 Equilibrium equations:

$$\text{div}\boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0}, \quad \text{or} \quad \sigma_{ji,j} + b_i = 0, \qquad \text{....................(9)}$$

6 Constitutive equations:

$$\boldsymbol{\sigma} = \lambda\, \text{tr}\boldsymbol{\varepsilon}\, \boldsymbol{I} + 2\mu\boldsymbol{\varepsilon} \quad \text{or} \quad \sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}. \qquad \text{....................(10)}$$

Additionally 6 scalar compatibility equations ensure a single-valued displacement solution.

Lame-Navier Equation

Substituting strain-displacement relations (8) into Linear-elastic constitutive relation (10), we get

$$\sigma_{ij} = \lambda\delta_{ij}u_{k,k} + \mu\left(u_{i,j} + u_{j,i}\right). \qquad \cdots\cdots\cdots\cdots\cdots\cdots(11)$$

Now, substituting this expression into equilibrium equations (9),

$$\begin{aligned}
&\sigma_{ji,j} + b_i = 0,\\
\Rightarrow &\lambda\delta_{ij}u_{k,kj} + \mu\left(u_{i,jj} + u_{j,ij}\right) + b_i = 0,\\
\Rightarrow &\lambda u_{k,ki} + \mu\left(u_{i,jj} + u_{j,ij}\right) + b_i = 0,\\
\Rightarrow &\lambda u_{k,ik} + \mu\left(u_{i,jj} + u_{k,ik}\right) + b_i = 0,\\
\Rightarrow &(\lambda+\mu)u_{k,ik} + \mu u_{i,jj} + b_i = 0, \quad \text{or} \quad (\lambda+\mu)\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{u}\right) + \mu\nabla^2\boldsymbol{u} + \boldsymbol{b} = \mathbf{0}. \quad \cdots\cdots\cdots\cdots(12)
\end{aligned}$$

These are Lame-Navier equations, which represent the equilibrium equation in terms of the displacement field. They provide three scalar field equations for three unknown scalar displacement fields. For any displacement field satisfying, the corresponding strain field is given by the strain-displacement equations. These equations can be used when one wish to determine only the displacement field.

Lame-Navier Equation

Cartesian coordinates:

$$\mu\nabla^2 u_1 + (\lambda+\mu)\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)+b_1=0$$
$$\mu\nabla^2 u_2 + (\lambda+\mu)\frac{\partial}{\partial x_2}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)+b_2=0$$
$$\mu\nabla^2 u_3 + (\lambda+\mu)\frac{\partial}{\partial x_3}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)+b_3=0. \qquad \cdots\cdots\cdots\cdots(13)$$

Cylindrical coordinates:

$$\mu\left(\nabla^2 u_r - \frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right)+(\lambda+\mu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial x_z}\right)+b_r=0$$
$$\mu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2}-\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right)+(\lambda+\mu)\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial x_z}\right)+b_\theta=0$$
$$\mu\nabla^2 u_z+(\lambda+\mu)\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial x_z}\right)+b_z=0. \qquad \cdots\cdots\cdots\cdots(14)$$

Spherical coordinates:

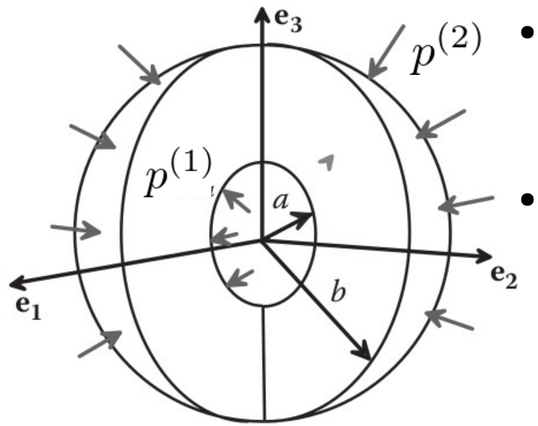
$$\mu\left(\nabla^2 u_R - \frac{2u_R}{R^2} - \frac{2}{R^2}\frac{\partial u_\phi}{\partial \phi} - \frac{2u_\phi \cot\phi}{R^2} - \frac{2}{R^2\sin\phi}\frac{\partial u_\theta}{\partial \theta}\right) +$$
$$(\lambda+\mu)\frac{\partial}{\partial R}\left(\frac{1}{R^2}\frac{\partial}{\partial R}(R^2 u_R) + \frac{1}{R\sin\phi}\frac{\partial}{\partial \phi}(u_\phi \sin\phi) + \frac{1}{R\sin\phi}\frac{\partial u_\theta}{\partial \theta}\right) + b_R = 0$$

$$\mu\left(\nabla^2 u_\phi + \frac{2}{R^2}\frac{\partial u_R}{\partial \phi} - \frac{u_\phi}{R^2\sin^2\phi} - \frac{2\cos\phi}{R^2\sin^2\phi}\frac{\partial u_\theta}{\partial \theta}\right) +$$
$$(\lambda+\mu)\frac{1}{R}\frac{\partial}{\partial \phi}\left(\frac{1}{R^2}\frac{\partial}{\partial R}(R^2 u_R) + \frac{1}{R\sin\phi}\frac{\partial}{\partial \phi}(u_\phi \sin\phi) + \frac{1}{R\sin\phi}\frac{\partial u_\theta}{\partial \theta}\right) + b_\phi = 0$$

$$\mu\left(\nabla^2 u_\theta - \frac{u_\theta}{R^2\sin^2\phi} + \frac{2}{R^2\sin^2\phi}\frac{\partial u_R}{\partial \theta} + \frac{2\cos\phi}{R^2\sin^2\phi}\frac{\partial u_\theta}{\partial \theta}\right) +$$
$$(\lambda+\mu)\frac{1}{R\sin\phi}\frac{\partial}{\partial \theta}\left(\frac{1}{R^2}\frac{\partial}{\partial R}(R^2 u_R) + \frac{1}{R\sin\phi}\frac{\partial}{\partial \phi}(u_\phi \sin\phi) + \frac{1}{R\sin\phi}\frac{\partial u_\theta}{\partial \theta}\right) + b_\theta = 0$$

......................(15)

Pressurized hollow sphere



- No body forces act on the sphere
- The inner surface $R=a$ is subjected to pressure $p^{(1)}$, which implies

$$\sigma_{R\theta} = \sigma_{R\phi} = 0, \quad \sigma_{RR} = -p^{(1)} \quad \text{on} \quad R = a.$$

- The outer surface $R=b$ is subjected to pressure $p^{(2)}$, which implies,

$$\sigma_{R\theta} = \sigma_{R\phi} = 0, \quad \sigma_{RR} = -p^{(2)} \quad \text{on} \quad R = b.$$

Let us solve for displacements first.

As the geometry and loading is symmetric, we also assume that the solution displacement field exhibits the spherical symmetry. The displacement components are then,

$$u_R = U(R), \quad u_\theta = u_\phi = 0.$$

Symmetry also implies that the problem is independent of and ϕ direction.

Then, the Navier's equation become,

$$(\lambda + 2\mu)\frac{d}{dR}\left[\frac{1}{R^2}\frac{d}{dR}(R^2 U)\right] = 0.$$

From the integration of the equation twice, we get,

$$U(R) = AR + B/R^2,$$

where A and B are constants of integration.

Now, strain-displacement relations can be used to find the strain components.

$$\varepsilon_{RR} = \frac{\partial U(R)}{\partial R} = \frac{dU(R)}{\partial R}, \quad \varepsilon_{\theta\theta} = \varepsilon_{\phi\phi} = U(R)/R.$$

Using constitutive relations stress components can be found.

$$\sigma_{RR} = (\lambda + 2\mu)\varepsilon_{RR} + \lambda\varepsilon_{\theta\theta} + \lambda\varepsilon_{\phi\phi} = (\lambda + 2\mu)\frac{dU(R)}{dR} + 2\lambda\frac{U(R)}{R}$$

$$\sigma_{RR} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1+\nu)A - 2(1-2\nu)\frac{B}{R^3}\right].$$

By applying boundary conditions, constant A and B can be found .

$$A = \frac{\left(p^{(2)}b^3 - p^{(1)}a^3\right)(1-2\nu)}{\left(a^3 - b^3\right)E}, B = \frac{\left(p^{(2)} - p^{(1)}\right)a^3b^3(1+\nu)}{2E\left(a^3 - b^3\right)}.$$

Finally, stress components are,

$$
\begin{aligned}
\sigma_{RR} &= \left(\frac{a^3p^{(1)} - b^3p^{(2)}}{b^3 - a^3}\right) - \frac{a^3b^3}{R^3}\frac{p^{(1)} - p^{(2)}}{b^3 - a^3}, \\
\sigma_{\theta\theta} = \sigma_{\phi\phi} &= \left(\frac{a^3p^{(1)} - b^3p^{(2)}}{b^3 - a^3}\right) + \frac{a^3b^3}{R^3}\frac{p^{(1)} - p^{(2)}}{b^3 - a^3}, \\
\sigma_{R\theta} &= \sigma_{\theta\phi} = \sigma_{\phi R} = 0.
\end{aligned}
$$

and displacement components are,

$$
\begin{aligned}
u_R &= \frac{1}{2E\left(b^3 - a^3\right)R^2}\left[2\left(p^{(1)}a^3 - p^{(2)}b^3\right)(1 - 2\nu)R^3 + \left(p^{(1)} - p^{(2)}\right)(1 + \nu)a^3b^3\right], \\
u_\theta &= u_\phi = 0.
\end{aligned}
$$