ME632: Fracture Mechanics

Timings	
Monday	10:00 to 11:20
Thursday	08:30 to 09:50

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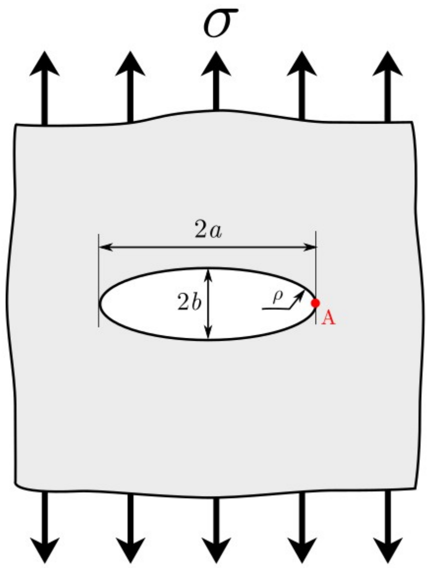
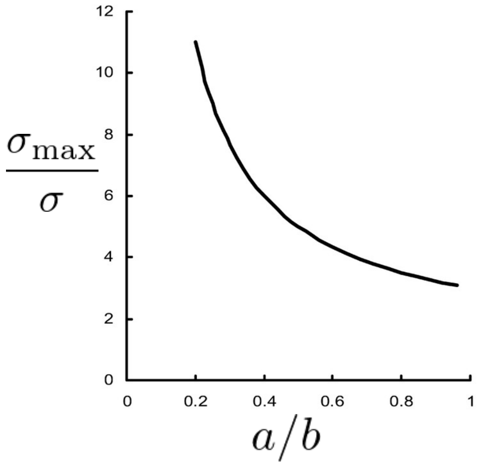
Room No. # 106

Stress concentration due to a flaw





Inglis (1913) gave analytical solution for stress near a two dimensional elliptical hole in an infinite plate. He showed that the stress at the tip of major axis (i.e., point A) is given as

$$\sigma_A = \sigma\left(1 + \frac{2a}{b}\right). \qquad \cdots\cdots\cdots\cdots\cdots(11)$$

As major axis, a, increases relative to b the hole begin to take the shape of a sharp crack. In this case it is more convenient to express (11) in terms of radius of curvature at the tip as

$$\sigma_A = \sigma\left(1 + 2\sqrt{\frac{a}{\rho}}\right). \quad (\text{where, } \rho = b^2/a) \qquad \cdots\cdots\cdots\cdots\cdots(12)$$

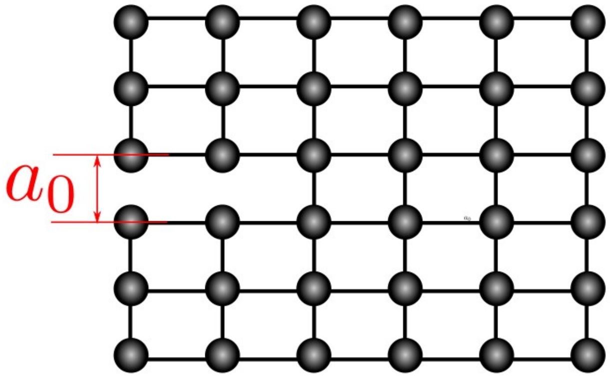
When $a \gg b$ then,

$$\sigma_A = 2\sigma\sqrt{\frac{a}{\rho}} \qquad \cdots\cdots\cdots\cdots\cdots(13)$$

Equation (13) gives an infinite stress at the tip of an infinitely sharp crack (i.e., ρ=0).

As no material is capable of withstanding infinite stress this result was a caused a concern at first. A material that contains a sharp crack should theoretically fail upon the application of an infinitesimal load, which is not observed in practice.

Real materials are made of atoms. For an elastic material the minimum radius a crack tip can have is on the order of the equilibrium distance between molecules.



To obtain a rough estimate of failure stress for continuum materials, we apply (13) at the atomic level (though it is valid for a continuum) and use $\rho = a_0$.

$$\sigma_A = 2\sigma\sqrt{\frac{a}{a_0}}. \qquad \text{...............(14)}$$

Now, if we assume that the failure happens when $\sigma_A = \sigma_c$, then

$$\sigma_A = \sigma_c = 2\sigma_f\sqrt{\frac{a}{a_0}}, \qquad \text{...............(15)}$$

where σ_f is the applied remote stress at failure. Using (10),

$$\sigma_c = \sqrt{\frac{\gamma_s E}{a_0}} = 2\sigma_f\sqrt{\frac{a}{a_0}} \quad \Rightarrow \sigma_f = \sqrt{\frac{E\gamma_s}{4a}}. \qquad \text{...............(16)}$$

Similar results have been obtained from numerical simulation of a crack in a two-dimensional lattice (Gehlen and Kanninen, 1970) as,

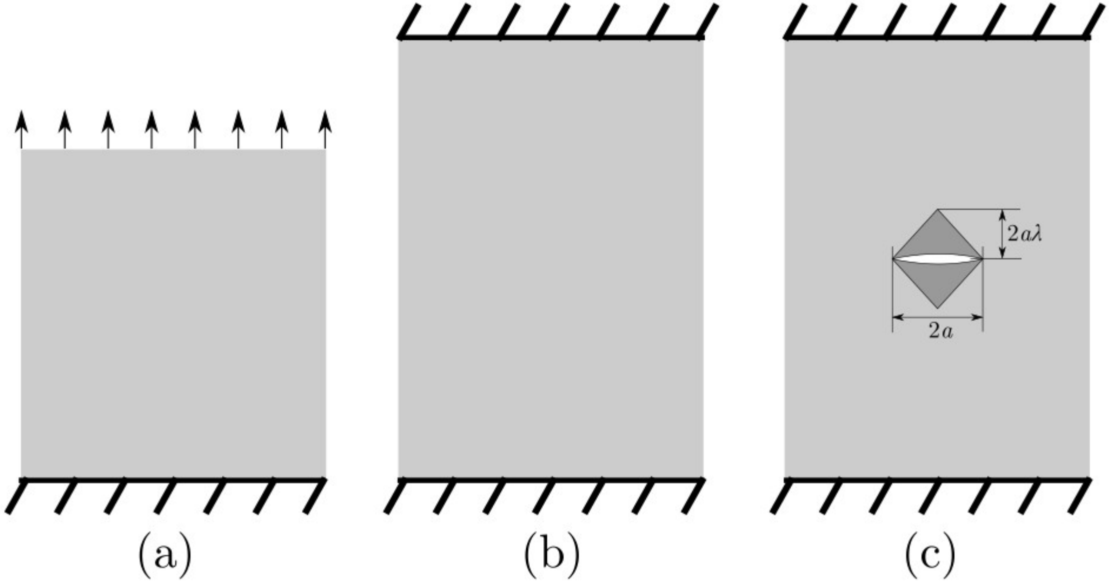
$$\sigma_f = \alpha\sqrt{\frac{E\gamma_s}{a}}, \qquad \text{...............(17)}$$

where a is a constant, on the order of unity, which depends slightly on the assumed atomic force-displacement law.

Griffith's theory

As we have already seen that the result obtained by Inglis suggest that the sharp crack must fail even at infinitesimal remote loading, i.e., the strength of the plate is independent of the flaw size. This paradox motivated Griffith to develop a fracture theory based on energy rather than local stress.

Think of the following experiment. Consider an elastic plate with no prior crack (Fig.a). The plate is pulled and then maintained in tension between supports (Fig.b). Then a crack normal to the direction of tension is cut with a knife at the center of the plate. The crack length is then gradually increased with the help of a knife. A critical stage is reached when the crack starts growing on its own; i.e., without any need of the knife.



Questions which we need to answer are following:

(a) What is the critical length after which crack grows on its own?

(b) How do we predict it?

To look for these answer note the following.

(i) Stiffness of the plate decreases with increasing crack length.

(ii) Stress near the crack surfaces (shaded region) decreases and hence stored strain energy also decreases. This strain energy is said to be released.

For answering the above questions and understand Griffith's analysis, we make following assumptions. The plate width is infinitely large compared to the largest crack size, which ensures that the stress far from the crack can be assumed to be constant.

As the crack advances, most of the energy release comes from region near the crack surfaces (shaded region), because they are traction free. For the sake of convenience of calculations, we consider this area as a triangle on each side of the crack plane. In fact, other shapes such as a parabola will serve the purpose. With the increase in crack length the base ($2a$) and the height, which is proportional to the base ($2a\lambda$), of triangles increase and, therefore, the area from which the strain energy is released is proportional to the square of the crack length.

So the total release of strain energy

$$E_R = \frac{\sigma^2}{2E} \times 2(\text{Area of triangle}) \times (\text{thickness}),$$

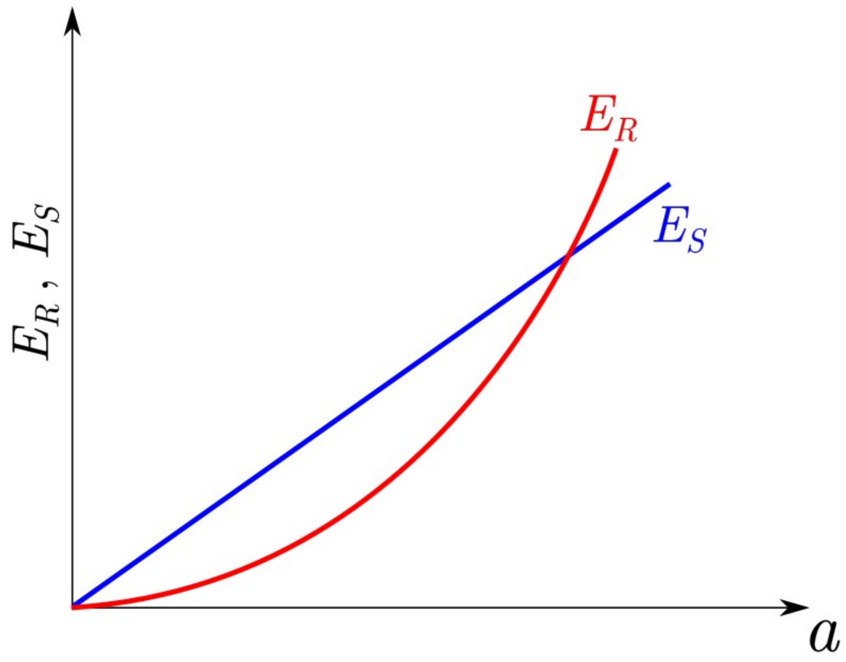
$$E_R = \frac{\sigma^2}{2E} \times 2\left[\frac{1}{2}(2a)(2a\lambda)\right] \times B = \frac{2\lambda a^2 B\sigma^2}{E}.$$

We will show it later that $\lambda = \pi/2$ for a thin plate (i.e., plane stress case), hence,

$$E_R = \frac{\pi a^2 B\sigma^2}{E}. \qquad \cdots\cdots\cdots\cdots\cdots(18)$$

Now to create new surfaces energy is required. If γ is the surfaces energy per unit area of the surface, then the surface energy required to to create a crack of length is,

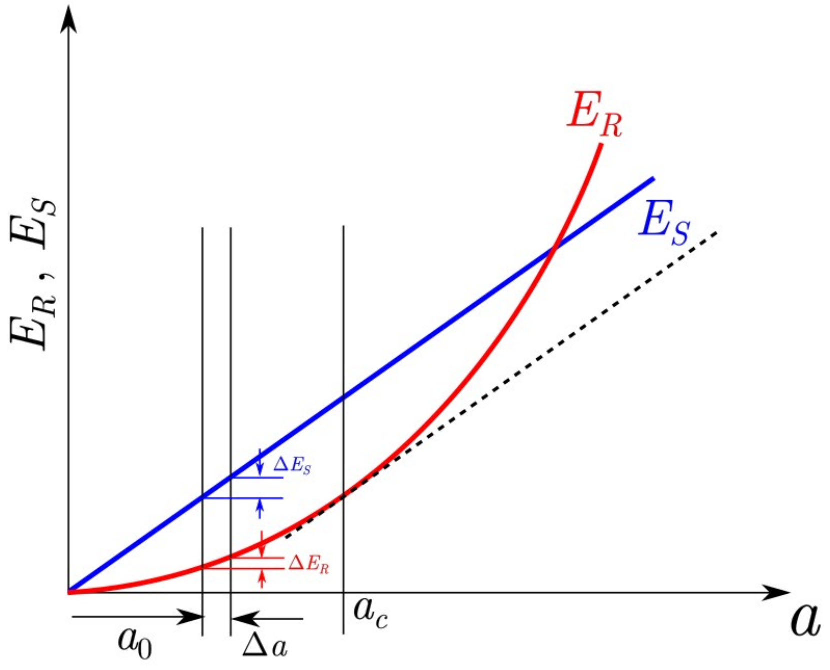
$$E_s = 2(2aB)\gamma = 4aB\gamma. \qquad \cdots\cdots\cdots\cdots\cdots(19)$$



Now consider a initial crack of length $2a_0$. If the crack length increases by an increment Δa, then the energy released during the incremental growth Δa is ΔE_R, whereas energy required for the increment is ΔE_S.

The crack will advance only when requirement of energy ΔE_S is fulfilled by the released strain energy ΔE_R.

If $\Delta E_S > \Delta E_R$ during the increment Δa, then the crack would not grow or would remain sub-critical unless additional energy is supplied through external sources (for e.g., through knife).



Now if the crack is slowly advanced by cutting through knife, and at certain crack length $\Delta E_S = \Delta E_R$ for advancing the crack by Δa length, then the crack becomes critical. Thus the condition for the crack to become critical is,

$$\frac{dE_R}{da} \geq \frac{dE_S}{da}. \qquad \text{...............}(20)$$

For the plate under tension, critical length of the crack can be determined using (18)-(20) as,

$$\frac{2a_c\pi B\sigma^2}{E} \geq 4B\gamma \quad \Rightarrow \quad a_c \geq \frac{2E\gamma}{\pi\sigma^2}. \qquad \text{...............(21)}$$

Crack will be safe when $a \leq \dfrac{2E\gamma}{\pi\sigma^2}.$ ....(22)

Similarly, critical load for a given crack length a to be critical (in plane stress) can be determined from (21) as,

$$\sigma_c \geq \sqrt{\frac{2E\gamma}{\pi a}}. \qquad \text{...............(23)}$$

For the plane strain case, it can expression (21) and (22) can be obtained as,

$$a_c \geq \frac{2E\gamma}{(1-\nu^2)\pi\sigma^2}, \qquad \sigma_c \geq \sqrt{\frac{2E\gamma}{(1-\nu^2)\pi a}}. \qquad \text{...............(24)}$$

So from Griffith's analysis we conclude followings:

- The critical stress depends on modulus E, surface energy γ and crack length a.
- Higher the value of the surface energy of a material higher the critical stress.
- Longer crack reduces the critical stress.
- Larger modulus means that the plate is capable of storing less energy, thereby resulting into smaller energy release which, in turn requires higher stress for making the crack critical.
- Also note from the product $\sigma\sqrt{a}$ depends only on material properties (E, ν and γ). Therefore, it may be treated as a new variable in fracture mechanics.
- Results obtained from Griffith's analysis [Equation (23)] is consistent with (16) and (17).
- The change in the stored energy with crack formation [Equation (18)] is insensitive to the notch radius as long as $a >> b$; thus, the Griffith model implies that the fracture stress is insensitive to ρ. It is contradictory to the solution of Inglis.
- The actual material behavior is somewhere between these extremes; fracture stress does depend on notch root radius, but not to the extent implied by the Inglis stress analysis.