ME232: Dynamics

3D dynamics of rigid bodies

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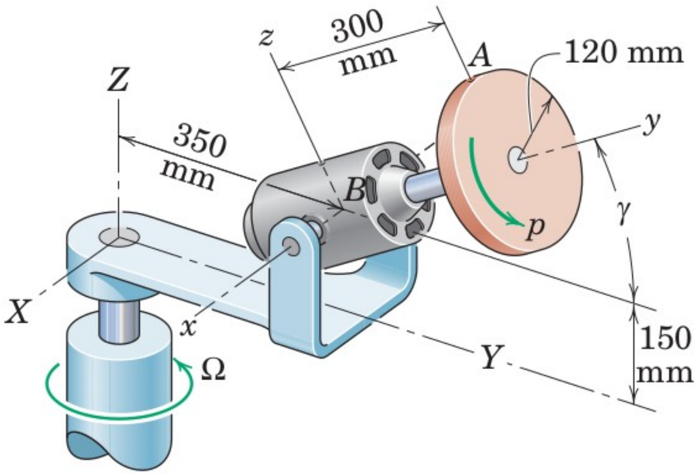
Example 3

The motor housing and its bracket rotate about the Z-axis at the constant rate $\Omega = 3$ rad/s. The motor shaft and disk have a constant angular velocity of spin $p = 8$ rad/s with respect to the motor housing in the direction shown. If γ is constant at 30°, determine the velocity and acceleration of point A at the top of the disk and the angular acceleration α of the disk.



The rotating reference axes x-y-z are attached to the motor housing, and the rotating base for the motor has the momentary orientation shown with respect to the fixed axes X-Y-Z. We will use both X-Y-Z components with unit vectors $\boldsymbol{I}$, $\boldsymbol{J}$, $\boldsymbol{K}$ and x-y-z components with unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$. The angular velocity of the x-y-z axes becomes $\boldsymbol{\Omega} = \Omega\boldsymbol{K} = 3\boldsymbol{K}$ rad/s.

The velocity of A is given by

$$\boldsymbol{v}_A = \boldsymbol{v}_B + \boldsymbol{\Omega} \times \boldsymbol{r}_{A/B} + \boldsymbol{v}_{\mathrm{rel}}$$

$$\boldsymbol{v}_B = \boldsymbol{\Omega} \times \boldsymbol{r}_B = 3\boldsymbol{K} \times 0.350\boldsymbol{J} = -1.05\boldsymbol{I} = -1.05\boldsymbol{i} \text{ m/s}$$

$$\boldsymbol{\Omega} \times \boldsymbol{r}_{A/B} = 3\boldsymbol{K} \times (0.300\boldsymbol{j} + 0.120\boldsymbol{k}) = -0.599\boldsymbol{i} \text{ m/s}$$

$$\boldsymbol{v}_{\mathrm{rel}} = \boldsymbol{p} \times \boldsymbol{r}_{A/B} = 8\boldsymbol{j} \times (0.300\boldsymbol{j} + 0.120\boldsymbol{k}) = 0.960\boldsymbol{i} \text{ m/s}$$

Thus, $\boldsymbol{v}_A = -1.05\boldsymbol{i} - 0.599\boldsymbol{i} + 0.960\boldsymbol{i} = -0.689\boldsymbol{i}$ m/s.

The acceleration of A is given by

$$\boldsymbol{a}_A = \boldsymbol{a}_B + \dot{\boldsymbol{\Omega}} \times \boldsymbol{r}_{A/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_{A/B}) + 2\boldsymbol{\Omega} \times \boldsymbol{v}_{\text{rel}} + \boldsymbol{a}_{\text{rel}}$$

where,

$$\boldsymbol{a}_B = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_b) = -3.15\boldsymbol{J} = -2.73\boldsymbol{j} + 1.575\boldsymbol{k} \text{ m/s}^2$$

$$\dot{\boldsymbol{\Omega}} = \boldsymbol{0}$$

$$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_{A/B}) = -1.557\boldsymbol{j} + 0.899\boldsymbol{k} \text{ m/s}^2$$

$$2\boldsymbol{\Omega} \times \boldsymbol{r}_{\text{rel}} = -4.99\boldsymbol{j} - 2.88\boldsymbol{k} \text{ m/s}^2$$

$$\boldsymbol{a}_{\text{rel}} = \boldsymbol{p} \times \left(\boldsymbol{p} \times \boldsymbol{r}_{A/B}\right) = 8\boldsymbol{j} \times [8\boldsymbol{j} \times (0.3\boldsymbol{j} + 0.120\boldsymbol{k})] = -7.68\boldsymbol{k}$$

Substituting in expression of $\boldsymbol{a}_A$ we get, $\boldsymbol{a}_A = 0.703\ \boldsymbol{j}$ -8.09 $\boldsymbol{k}$ m/s^2.

Angular acceleration $\boldsymbol{\alpha}$ is calculated as,

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \boldsymbol{\Omega} \times \boldsymbol{\omega} = -20.8\boldsymbol{i} \text{ rad/s}^2$$

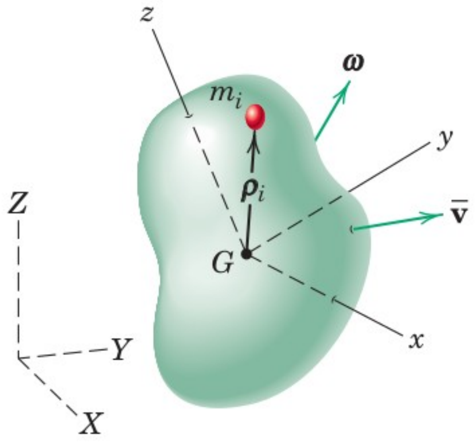
Kinetics

The force equation for a rigid or nonrigid in three-dimensional motion is the simple extension of Newton's second law.

The moment equation for three-dimensional motion is not as simple as for plane motion since the change of angular momentum has a number of additional components which are absent in plane motion.

Angular momentum:

Consider a rigid body moving with any general motion in space. Axes x-y-z are attached to the body with origin at the mass center G. Thus, the angular velocity $\boldsymbol{\omega}$ of the body becomes the angular velocity of the x-y-z axes as observed from the fixed reference axes X-Y-Z. The absolute angular momentum $\boldsymbol{H}_G$ of the body about its mass center G is the sum of the moments about G of the linear momenta of all elements of the body as,



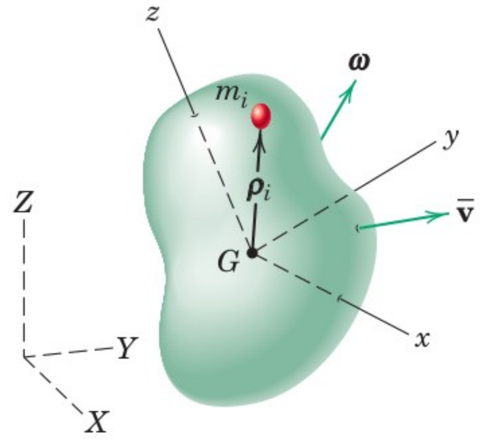
$$\boldsymbol{H}_G = \sum(\boldsymbol{\rho}_i \times m_i \boldsymbol{v}_i).$$

For a rigid body,

$$\boldsymbol{v}_i = \overline{\boldsymbol{v}} + \boldsymbol{\omega} \times \boldsymbol{\rho}_i$$

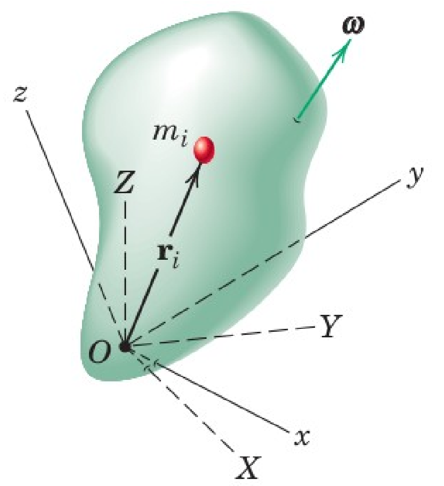
and thus,

$$\boldsymbol{H}_G = -\overline{\boldsymbol{v}} \times \sum m_i \boldsymbol{\rho}_i + \sum \left[\boldsymbol{\rho}_i \times m_i(\boldsymbol{\omega} \times \boldsymbol{\rho}_i)\right].$$



Above equations can further be written (by considering mass $\boldsymbol{m}_i$ to be elemental mass dm) as,

$$\boldsymbol{H}_G = \int \boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})dm. \qquad \cdots\cdots\cdots\cdots\cdots(6)$$



Also consider the case of a rigid body rotating about a fixed point O, the x-y-z axes are attached to the body, and both body and axes have an angular velocity $\boldsymbol{\omega}$. The angular momentum about O can be written as,

$$\boldsymbol{H}_O = \int \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})dm. \qquad \cdots\cdots\cdots\cdots\cdots(7)$$

Moments and Product of inertia:

Observe that in (6) and (7), the position vectors $\boldsymbol{\rho}$ and $\boldsymbol{r}$ are given by the same expression $x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. Thus, they are identical in form.

Let us expand the integrand in the two expressions for angular momentum, recognizing that the components of $\boldsymbol{\omega}$ are invariant with respect to the integrals over the body and thus become constant multipliers of the integrals, and we get,

$$
\begin{aligned}
d\boldsymbol{H} =& \boldsymbol{i}\left[(y^2+z^2)\omega_x - xy\omega_y - xz\omega_z\right]dm \\
& \boldsymbol{j}\left[-xy\omega_x + (z^2+x^2)\omega_y - yz\omega_z\right]dm \\
& \boldsymbol{k}\left[-zx\omega_x - yz\omega_y + (x^2+y^2)\omega_z\right]dm
\end{aligned}
\quad
\begin{aligned}
\Rightarrow \boldsymbol{H} =& (I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\,\boldsymbol{i} \\
& + (-I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z)\,\boldsymbol{j} \\
& + (-I_{zx}\omega_x - I_{yz}\omega_y + I_{zz}\omega_z)\,\boldsymbol{k}
\end{aligned}
\qquad \text{...............(8)}
$$

where,
$$I_{xx} = \int (y^2+z^2)dm, \quad I_{xy} = \int xy\, dm, \quad I_{yy} = \int (z^2+x^2)dm,$$

$$I_{xz} = \int xz\, dm, \quad I_{zz} = \int (x^2+y^2)dm, \quad I_{yz} = \int yz\, dm. \qquad \text{...............(9)}$$

The quantities I_{xx}, I_{yy}, I_{zz} are called the moments of inertia of the body about the respective axes, and I_{xy}, I_{xz}, I_{yz} are the products of inertia with respect to the coordinate axes.

Observe that $I_{xy} = I_{yx}$, $I_{xz} = I_{zx}$, and $I_{yz} = I_{zy}$ and the components of $\boldsymbol{H}$ are

$$\begin{aligned} H_x &= I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\ H_y &= -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z \\ H_z &= -I_{zx}\omega_x - I_{yz}\omega_y + I_{zz}\omega_z . \end{aligned} \qquad \cdots\cdots\cdots\cdots\cdots(10)$$

(8) represents the general expression for the angular momentum either about the mass center G or about a fixed point O for a rigid body rotating with an instantaneous angular velocity $\boldsymbol{\omega}$.

In each of the two cases represented, the reference axes x-y-z are attached to the rigid body, which makes **the moment-of-inertia integrals and the product-of-inertia integrals invariant with time.**

If the x-y-z axes were to rotate with respect to an irregular body, then these inertia integrals would be functions of the time, which would introduce an undesirable complexity into the angular-momentum relations.

Principal axis:

The array of moments and products of inertia $\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{zy} & I_{zz} \end{bmatrix}$

is called the **inertia matrix or inertia tensor.**

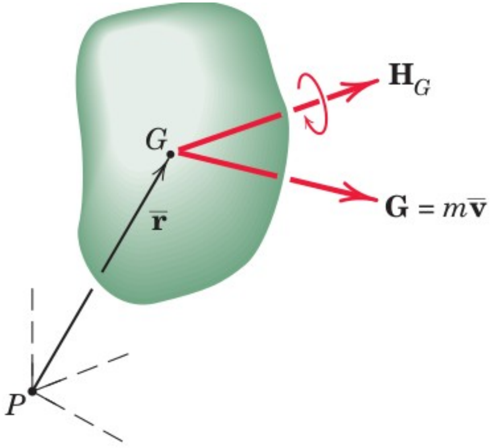
With the change of the orientation of the axes relative to the body, the moments and products of inertia will also change in value. It can be shown that there is one unique orientation of axes *x*-*y*-*z* for a given origin for which **the products of inertia vanish** and the moments of inertia I_{xx} , I_{yy} , I_{zz} take on stationary values. For this orientation, the inertia matrix takes the form

$$\begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

and is said to be diagonalized. The axes *x*-*y*-*z* for which the products of inertia vanish are called **the principal axes of inertia**, and I_{xx} , I_{yy} , and I_{zz} are called the **principal moments of inertia**. The principal moments of inertia for a given origin represent the maximum, the minimum, and an intermediate value of the moments of inertia.

Transfer principle for angular momentum:

The momentum properties of a rigid body may be represented by the resultant linear-momentum vector $\boldsymbol{G} = m\overline{\boldsymbol{v}}$ through the mass center and the resultant angular-momentum vector $\boldsymbol{H}_G$ about the mass center.



These vectors have properties analogous to those of a force and a couple. Thus, the angular momentum about any point P equals the free vector $\boldsymbol{H}_G$ plus the moment of the linear-momentum vector G about P. Therefore, we may write

$$\boldsymbol{H}_P = \boldsymbol{H}_G + \overline{\boldsymbol{r}} \times \boldsymbol{G}.$$

Kinetic energy

We already developed the expression for the kinetic energy T of any general system of mass, rigid or nonrigid, and expressed as,

$$T = \frac{1}{2}m\overline{v}^2 + \sum \frac{1}{2}m_i|\dot{\rho}_i|^2$$

where $\overline{v}$ is the velocity of the mass center and $\boldsymbol{\rho}_i$ is the position vector of a representative element of mass m_i with respect to the mass center. The first term is the kinetic energy due to the translation of the system and the second term is the kinetic energy associated with the motion relative to the mass center. The translational term may be written as

$$\frac{1}{2}m\overline{v}^2 = \frac{1}{2}m\dot{\overline{\boldsymbol{r}}} \cdot \dot{\overline{\boldsymbol{r}}} = \frac{1}{2}\overline{\boldsymbol{v}} \cdot \boldsymbol{G}$$

where $\dot{\overline{\boldsymbol{r}}}$ is the velocity $\overline{\boldsymbol{v}}$ of the mass center and $\boldsymbol{G}$ is the linear momentum of the body.

For a rigid body, the relative term becomes the kinetic energy due to rotation about the mass center. Because $\dot{\boldsymbol{\rho}}_i$ is the velocity of the representative particle with respect to the mass center, then for the rigid body we may write it as $\dot{\boldsymbol{\rho}}_i = \boldsymbol{\omega} \times \boldsymbol{\rho}_i$, where $\boldsymbol{\omega}$ is the angular velocity of the body. With this substitution, the relative term in the kinetic energy expression becomes

$$\sum \tfrac{1}{2} m_i |\dot{\boldsymbol{\rho}}_{\boldsymbol{i}}|^2 = \sum \tfrac{1}{2} m_i (\boldsymbol{\omega} \times \boldsymbol{\rho}_i) \cdot (\boldsymbol{\omega} \times \boldsymbol{\rho}_i) = \sum \tfrac{1}{2} m_i \boldsymbol{\omega} \cdot \boldsymbol{\rho}_i \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_i) = \tfrac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{H}_G$$

$$= \tfrac{1}{2} \boldsymbol{\omega} \cdot \sum \boldsymbol{\rho}_i \times m_i (\boldsymbol{\omega} \times \boldsymbol{\rho}_i) = \tfrac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{H}_G$$

Thus, the general expression for the kinetic energy of a rigid body moving with mass-center velocity $\overline{v}$ and angular velocity $\boldsymbol{\omega}$ is

$$T = \tfrac{1}{2} \overline{\boldsymbol{v}} \cdot \boldsymbol{G} + \tfrac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{H}_G. \qquad \text{...............(11)}$$

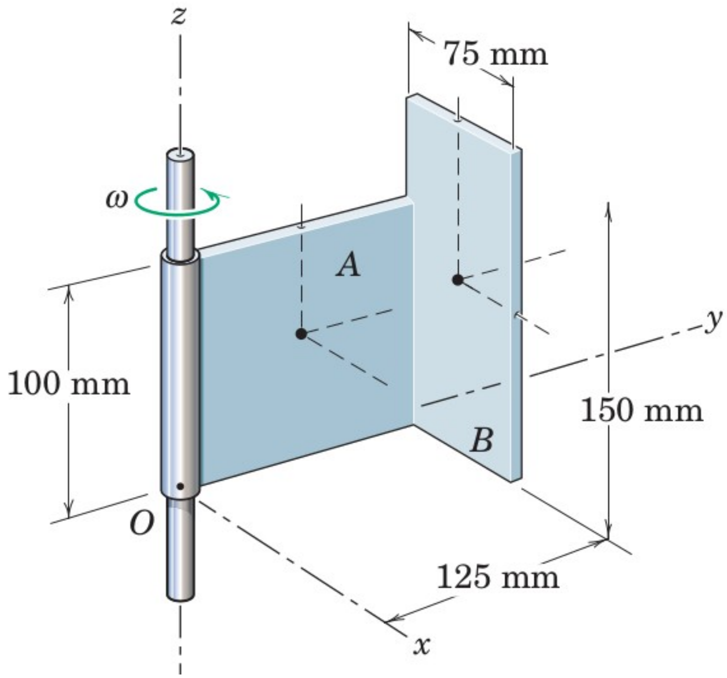
When a rigid body is pivoted about a fixed point O or when there is a point O in the body which momentarily has zero velocity, the kinetic energy is

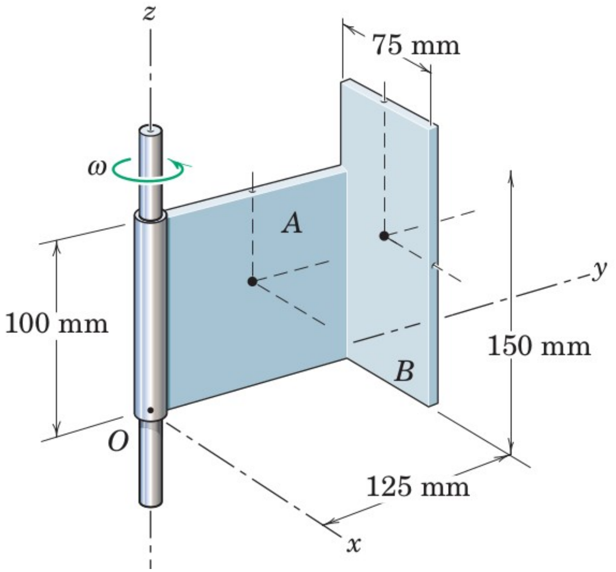
$$T = \tfrac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{H}_O, \qquad \text{...............(11a)}$$

where $\boldsymbol{H}_O$ is the angular momentum about O.

Example 4

The bent plate has a mass of 70 kg/m^2 of surface area and revolves about the z-axis at the rate $\omega = 30$ rad/s. Determine (a) the angular momentum $\boldsymbol{H}$ of the plate about point O and (b) the kinetic energy T of the plate. Neglect the mass of the hub and the thickness of the plate compared with its surface dimensions.





The masses of the parts are

$m_A = (0.100)(0.125)(70) = 0.875$ kg, and

$m_B = (0.075)(0.150)(70) = 0.788$ kg.

Moments and products of inertia can be calculated w.r.t. x-y-z axis.

For part A:

$$I_{xx} = \overline{I}_{xx} + m_A d^2 = 0.00747 \text{ kg} \cdot \text{m}^2$$

$$I_{yy} = \tfrac{1}{3} m_A l^2 = 0.00292 \text{ kg} \cdot \text{m}^2$$

$$I_{zz} = \tfrac{1}{3} m_A l^2 = 0.00456 \text{ kg} \cdot \text{m}^2$$

$$I_{xy} = I_{xz} = 0$$

$$I_{yz} = \overline{I}_{yz} + m_A d_y d_z = 0.00273 \text{ kg} \cdot \text{m}^2$$

For part B:

$$I_{xx} = \overline{I}_{xx} + m_B d^2 = 0.01821 \text{ kg} \cdot \text{m}^2$$

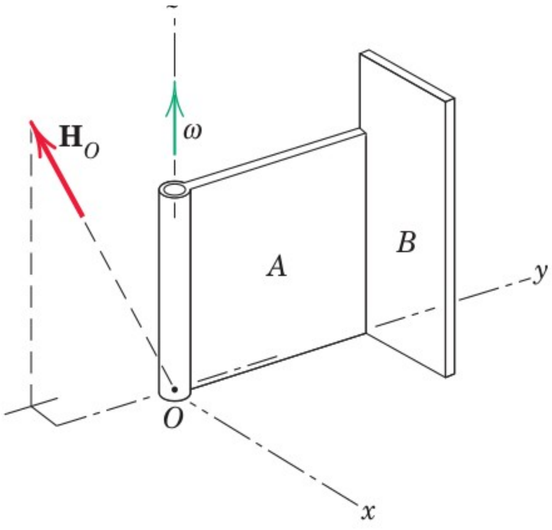
$$I_{yy} = \overline{I}_{yy} + m_B d^2 = 0.00738 \text{ kg} \cdot \text{m}^2$$

$$I_{zz} = \overline{I}_{zz} + m_B d^2 = 0.01378 \text{ kg} \cdot \text{m}^2$$

$$I_{xy} = \overline{I}_{xy} + m_A d_x d_y = 0.00369 \text{ kg} \cdot \text{m}^2$$

$$I_{xz} = \overline{I}_{xz} + m_A d_x d_z = 0.00221 \text{ kg} \cdot \text{m}^2$$

$$I_{yz} = \overline{I}_{yz} + m_A d_y d_z = 0.00738 \text{ kg} \cdot \text{m}^2$$



Angular momentum of the body is

$$\boldsymbol{H}_O = (I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\,\boldsymbol{i} + (-I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z)\,\boldsymbol{j} + (-I_{zx}\omega_x - I_{yz}\omega_y + I_{zz}\omega_z)\,\boldsymbol{k}$$

$\omega_x = \omega_y = 0$ and $\omega_z = \omega$.

Thus, H_O can be calculated.

The kinetic energy can be calculated as $\mathrm{T} = \frac{1}{2}\,\boldsymbol{\omega}\cdot\boldsymbol{H}_O = 8.25$ J.

Momentum and energy equations of motion

Momentum equations:

The general linear- and angular-momentum equations for a system of constant mass are

$$\sum \boldsymbol{F} = \dot{\boldsymbol{G}} \quad \text{and} \quad \sum \boldsymbol{M} = \dot{\boldsymbol{H}}.$$

In the derivation of the moment principle, the derivative of $\boldsymbol{H}$ was taken with respect to an absolute coordinate system. When $\boldsymbol{H}$ is expressed in terms of components measured relative to a moving coordinate system x-y-z which has an angular velocity $\boldsymbol{\Omega}$, then we can write,

$$\begin{aligned}\sum M &= \left(\frac{d\boldsymbol{H}}{dt}\right)_{xyz} + \boldsymbol{\Omega} \times \boldsymbol{H} \\ &= \left(\dot{H}_x\boldsymbol{i} + \dot{H}_y\boldsymbol{j} + \dot{H}_z\boldsymbol{k}\right) + \boldsymbol{\Omega} \times \boldsymbol{H}\end{aligned}$$

$$\begin{aligned}\Rightarrow \sum \boldsymbol{M} =& (\dot{H}_x - H_y\Omega_z + H_z\Omega_y)\boldsymbol{i} \\ &+ (\dot{H}_y - H_z\Omega_x + H_x\Omega_z)\boldsymbol{j} \\ &+ (\dot{H}_z - H_x\Omega_y + H_y\Omega_x)\boldsymbol{k}\end{aligned}$$

...............(12)

(12) is the most general form of the moment equation about a fixed point O or about the mass center G.

For a rigid body where the coordinate axes are attached to the body, *the moments and products of inertia are invariant with time*, (when expressed in x-y-z coordinates) and $\boldsymbol{\Omega} = \boldsymbol{\omega}$. Thus, for axes attached to the body, (12) becomes

$$\begin{aligned}\sum \boldsymbol{M} =&(\dot{H}_x - H_y\omega_z + H_z\omega_y)\boldsymbol{i}\\&+ (\dot{H}_y - H_z\omega_x + H_x\omega_z)\boldsymbol{j}\\&+ (\dot{H}_z - H_x\omega_y + H_y\omega_x)\boldsymbol{k}\end{aligned} \qquad \cdots\cdots\cdots\cdots\cdots(13)$$

(13) is the most general moment equations for rigid-body motion with axes attached to the body. They hold w.r.t. axes through a fixed point O or through the mass center G.

Energy equations:

The resultant of all external forces acting on a rigid body may be replaced by the resultant force $\Sigma \boldsymbol{F}$ acting through the mass center and a resultant couple $\Sigma \boldsymbol{M}_G$ acting about the mass center. Work is done by the resultant force and the resultant couple at the respective rates $\sum \boldsymbol{F} \cdot \overline{\boldsymbol{v}}$ and $\sum \boldsymbol{M}_G \cdot \boldsymbol{\omega}$, where $\overline{\boldsymbol{v}}$ is the linear velocity of the mass center and $\boldsymbol{\omega}$ is the angular velocity of the body. Integration over the time from condition 1 to condition 2 gives the total work done during the time interval. Equating the work done to the respective changes in the kinetic energy as (11) gives

$$\int_{t_1}^{t_2} \sum \boldsymbol{F} \cdot \overline{\boldsymbol{v}}\, dt = \frac{1}{2}\overline{\boldsymbol{v}} \cdot \boldsymbol{G}\Big|_1^2, \quad \int_{t_1}^{t_2} \sum \boldsymbol{M}_G \cdot \overline{\boldsymbol{\omega}}\, dt = \frac{1}{2}\overline{\boldsymbol{\omega}} \cdot \boldsymbol{H}_G\Big|_1^2, \qquad \cdots\cdots\cdots\cdots\cdots(14)$$

These equations express the change in translational and rotational kinetic energy, respectively, for the interval during which $\Sigma \boldsymbol{F}$ or $\Sigma \boldsymbol{M}_G$ acts, and the sum of two expressions equals ΔT.

Parallel-plane motion:

When all particles of a rigid body moves in planes which are parallel to a fixed plane, the body has a general form of plane motion. Every line is such a body which is normal to the fixed plane remains parallel to itself at all times. Considering G as the origin of coordinates x-y-z which are attached to the body, with the x-y plane coinciding with the plane of motion P. The components of the angular velocity of both the body and he attached axes become $\omega_x=\omega_y=0$, $\omega_z\neq 0$. For this case, the angular-momentum components are

$$H_x = -I_{xz}\omega_z \qquad H_y = -I_{yz}\omega_z \qquad H_z = I_{zz}\omega_z .$$

and the moment relations reduce to

$$\sum M_x = -\, I_{xz}\dot{\omega}_z + I_{yz}\omega_z^2, \quad \sum M_y = \quad -I_{yz}\dot{\omega}_z + I_{xz}\omega_z^2, \quad \sum M_z = -\, I_{zz}\dot{\omega}_z .$$