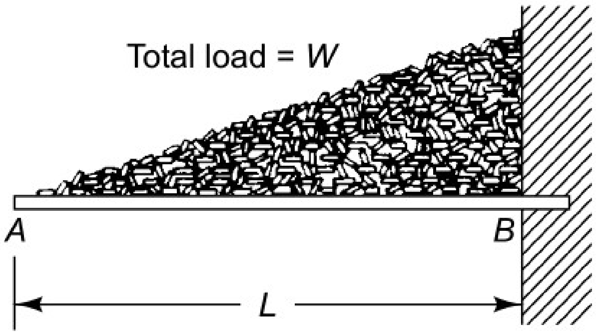
Forces and Moments Transmitted by Slender Members

Example 2

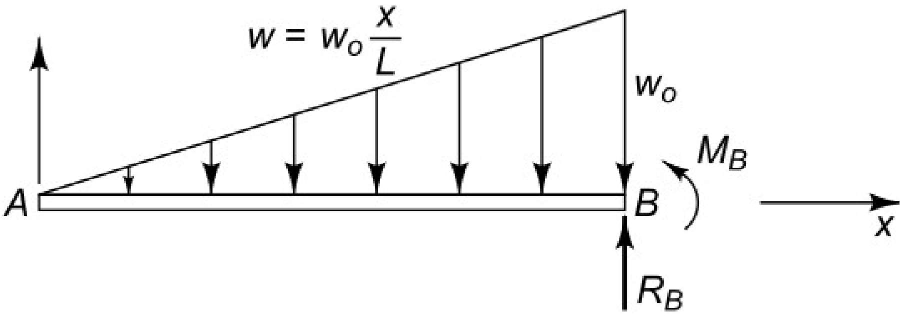
Consider a cantilever beam built in at the right end. Bricks having a total weight W have been piled up in triangular fashion. It is desired to obtain shear-force and bending-moment diagrams.

Total load W can be idealized as a linearly distributed load on the beam.
i.e., $w(x) = A + Bx$. Constants A and B can be determined as,
(i) considering that at $x=0$, $q=0$, thus $A=0$.
(ii) $\int_0^L w(x)dx = \int_0^L Bxdx = W \Rightarrow B = 2W/L^2$.

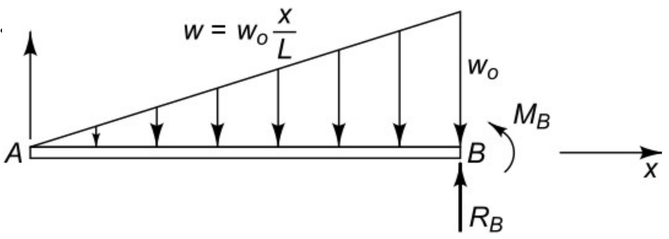
Thus, $w(x) = 2Wx/L^2 = w_0 x/L$, where w_0 is the load at $x=L$, i.e., $w(L) = w_0 = 2W/L$.



Idealization

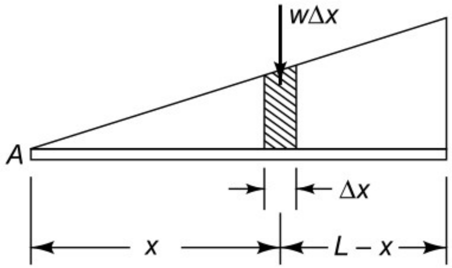


Find the support reactions by applying the equilibrium equations.



$$\sum F_y = R_B - \int_0^L w(x)dx = 0$$

$$\Rightarrow R_B = \int_0^L w_0 x/L dx = w_0 L/2. \qquad \cdots\cdots\cdots\cdots\cdots(2.a)$$



$$\sum M_b = M_B + \int_0^L w(x)(L-x)dx = 0$$

$$\Rightarrow M_B = WL/3. \qquad \cdots\cdots\cdots\cdots\cdots(2.b)$$

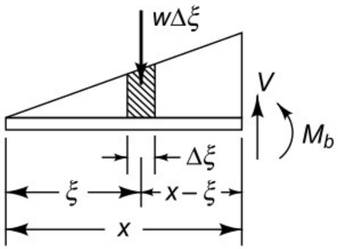
Now, to get the distribution of shear force and bending moment along the length of the beam, we take a section at a distance x and consider the equilibrium of sub-system as,

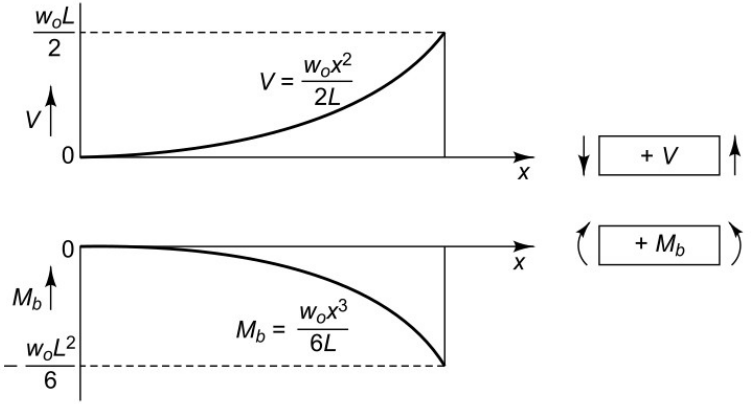
$$\sum F_y = V - \int_0^x w(\xi)d\xi = 0$$

$$\Rightarrow V = \frac{w_0 x^2}{2L}. \qquad \cdots\cdots\cdots\cdots\cdots(2.c)$$

$$\sum M = M_b + \int_0^x w(\xi)(L-\xi)d\xi = 0$$

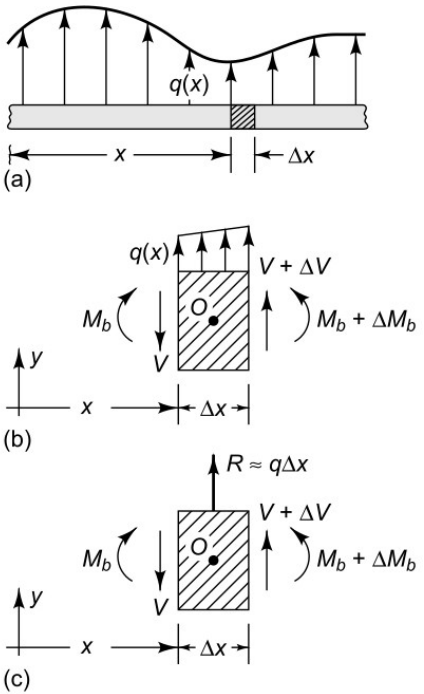
$$\Rightarrow M_b = -\frac{w_0 x^3}{6L}. \qquad \cdots\cdots\cdots\cdots\cdots(2.d)$$





Differential equilibrium relationship

Another procedure for obtaining internal forces and moments along a slender element, is to consider a very small element of the beam as a free body and applying equilibrium conditions to it. This results in differential equations connecting the load, the shear force, and the bending moment. By integrating these relationships for particular cases we can evaluate shear forces and bending moments.



Consider an element of size Δx at a distance x. Distributed force for the element, shear forces and moments at two sections of the element are shown. As Δx is very small we replace distributed force with a concentrated force $R = q\Delta x$.

Now considering the equilibrium of the element as,

$$\sum F_y = V + \Delta V - V + R = 0,$$

$$\Rightarrow \frac{\Delta V}{\Delta x} + q(x) = 0. \qquad \cdots\cdots\cdots\cdots(1)$$

$$\sum M_O = (V + \Delta V)(\Delta x/2) + V(\Delta x/2) + M_b + \Delta M_b - M_b = 0,$$

$$\Rightarrow V\Delta x + \Delta V \Delta x/2 + \Delta M_b = 0 \quad \text{or} \quad \frac{\Delta M_b}{\Delta x} + V = -\frac{\Delta V}{2}. \qquad \cdots\cdots\cdots\cdots(2)$$

As $\Delta x \to 0$, (1) and (2) becomes,

$$\frac{dV}{dx} + q(x) = 0, \qquad \cdots\cdots\cdots\cdots(3)$$

$$\frac{dM_b}{dx} + V = 0. \qquad \cdots\cdots\cdots\cdots(4)$$

(3) and (4) are the basic differential equations relating the load intensity $q(x)$ with the shear force $V(x)$ and bending moment $M_b(x)$ in a beam. These equations can be integrated at any section from $x = x_1$ to $x = x_2$, which yields

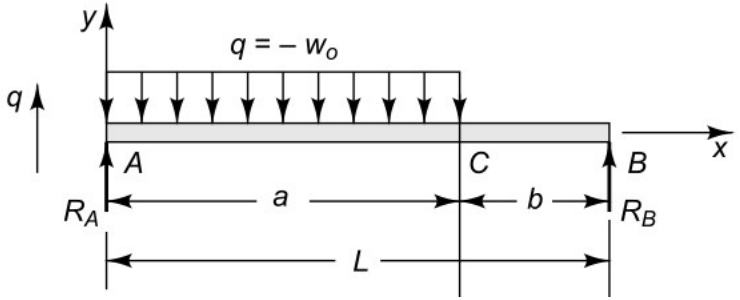
$$V(x_2) - V(x_1) + \int_{x_1}^{x_2} q(x)dx = 0, \qquad \cdots\cdots\cdots\cdots(5)$$

$$M_b(x_2) - M_b(x_1) + \int_{x_1}^{x_2} V(x)dx = 0, \qquad \cdots\cdots\cdots\cdots(6)$$

Let us look at the application of (5) and (6) through an example.

Example 3

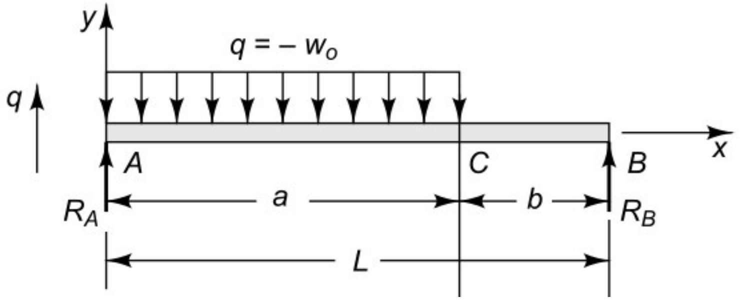
A beam with simple transverse supports at A and B and loaded with a uniformly distributed load $q = -w_0$ over a portion of the length. It is desired to obtain the shear-force and bending-moment diagrams.



Find reaction forces from equilibrium conditions,

$$\sum F_y = R_A + R_B - w_0 a = 0, \qquad \sum M_A = R_B L - w_0 a^2/2 = 0. \qquad \Rightarrow R_A = w_0 a\left(1 - \frac{a}{2L}\right), \quad R_B = \frac{w_0 a^2}{2L}.$$

As the load is acting on only a part of the beam, we must consider the two parts of the beam (i) the part in which the load acts and (ii) the part in which no load acts. Then we write differential equations accordingly and integrate accordingly using valid boundary conditions.



For part AC (using 3),

$$\frac{dV_1}{dx} - w_0 = 0,$$

$$V_1 = w_0 x + c_1,$$

For part CB (using 3),

$$\frac{dV_2}{dx} = 0,$$

$$V_2 = c_2, \qquad \cdots\cdots\cdots\cdots(a.3)$$

Now, using (4) for part AC and CB, we get

$$\frac{dM_{b1}}{dx} + V_1 = \frac{dM_{b1}}{dx} + w_0 x + c_1 = 0, \qquad \text{and} \qquad \frac{dM_{b2}}{dx} + V_2 = \frac{dM_{b2}}{dx} + c_2 = 0.$$

After integrating

$$M_{b1} + \frac{w_0 x^2}{2} + c_1 x + c_3 = 0, \qquad \text{and} \qquad M_{b2} + c_2 x + c_4 = 0. \qquad \cdots\cdots\cdots\cdots(b.3)$$

Constants c_1 to c_4 can be determined using following conditions,

(i) Boundary conditions: $M_{b1} = 0$ at $x = 0$ and $M_{b2} = 0$ at $x = L$, $\cdots\cdots\cdots\cdots$(c.3)

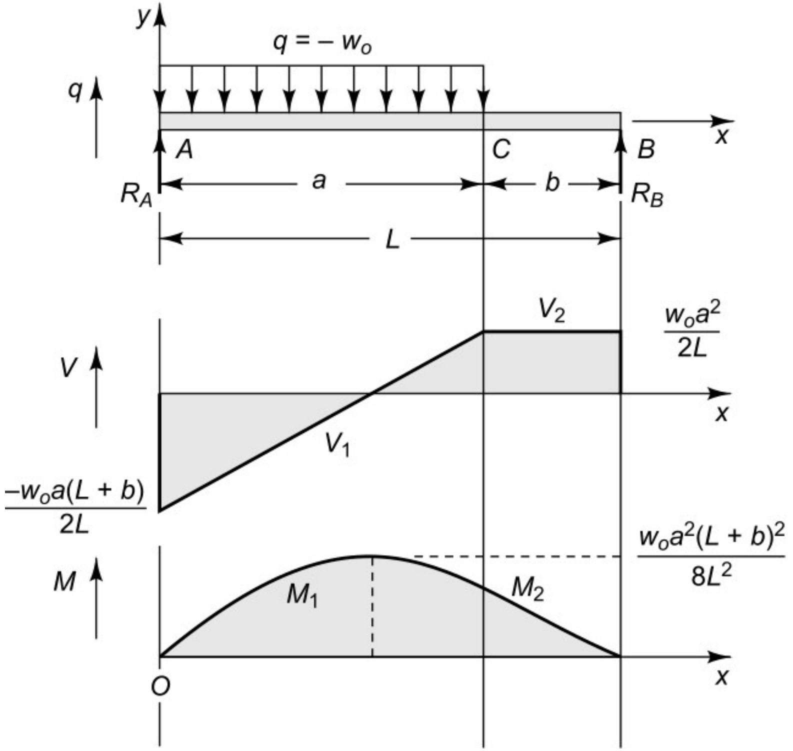
(ii) Conditions from equilibrium: $V_1 = V_2$ at $x = a$, and $M_{b1} = M_{b2}$ at $x = a$. $\cdots\cdots\cdots\cdots$(d.3)

(iii) $V_1 = -R_A$ at $x = 0$ and $V_2 = R_B$ at $x = L$. $\cdots\cdots\cdots\cdots$(e.3)

Using (c.3)-(e.3) with (a.3) and (b.3) we can determine all constants as,

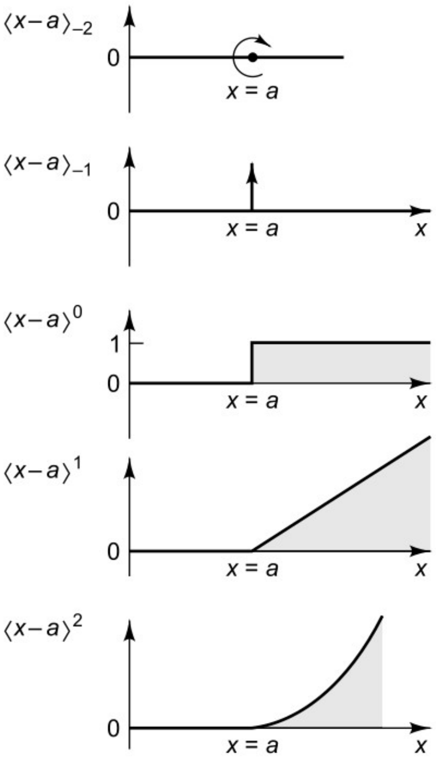
$$c_3 = 0, \quad c_4 = \frac{1}{2} w_0 a^2, \quad c_2 = \frac{w_0 a^2}{2L}, \quad c_1 = w_0 a \left(\frac{a}{2L} - 1 \right). \qquad \text{...........(f.3)}$$

Substituting all constants in (a.3) and (b.3) we get the expression for shear-force and bending moment distribution along the beam, which can be drawn as,



Singularity functions

In previous examples we have seen that when there are concentrated-force and concentrated-moment or when the distributed load suddenly changes its magnitude calculations becomes complicated. A family of singularity functions discussed here are specifically designed to handle such cases.



$$f_n(x) = \langle x - a\rangle^n = \begin{cases} (x-a)^n & \text{when } x \geq a \\ 0 & \text{when } x < a \end{cases}$$

- $<x - a>^0$ - the unit step starting at $x = a$
- $<x - a>^1$ - the unit ramp starting at $x = a$
- $<x - a>_{-1}$ - the unit concentrated load or the unit impulse function
- $<x - a>_{-2}$ - the unit concentrated moment or the unit doublet function

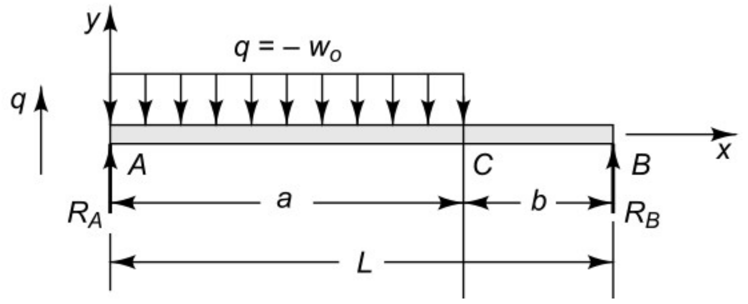
The integration and differentiation law for these functions is

$$\int \langle x - a\rangle^n dx = \frac{1}{n+1}\langle x - a\rangle^{n+1} \quad \text{for } n \geq 0,$$

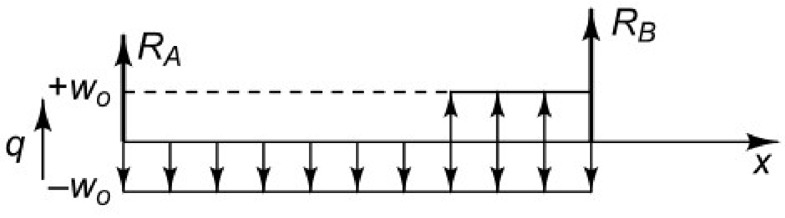
$$\frac{d}{dx}\langle x - a\rangle^n dx = n\langle x - a\rangle^{n-1} \quad \text{for } n \geq 1.$$

Example 4

Let us solve example 3 again using singular function approach.



To implement the singular functions we modify the loading to an equivalent loading on the beam as follows. The load intensity form is now presented as,



$$q(x) = -w_0 + w_0 \langle x - a \rangle^0 \qquad \cdots\cdots\cdots\cdots(\text{a.4})$$

Now (a.4) is valid from $0 < x < L$.

Using (3) we write the differential equation for shear force,

$$\frac{dV}{dx} = -q(x) = w_0 - w_0 \langle x - a \rangle^0 .$$

Integrating above equation gives,

$$V(x) = w_0 x - w_0 \langle x - a \rangle^1 + c_1. \qquad \cdots\cdots\cdots\cdots(\text{b.4})$$

Using boundary condition that at $x = 0, V(0) = c_1 = -R_A$,

where $R_A = w_0 a \left(1 - \dfrac{a}{2L}\right)$.

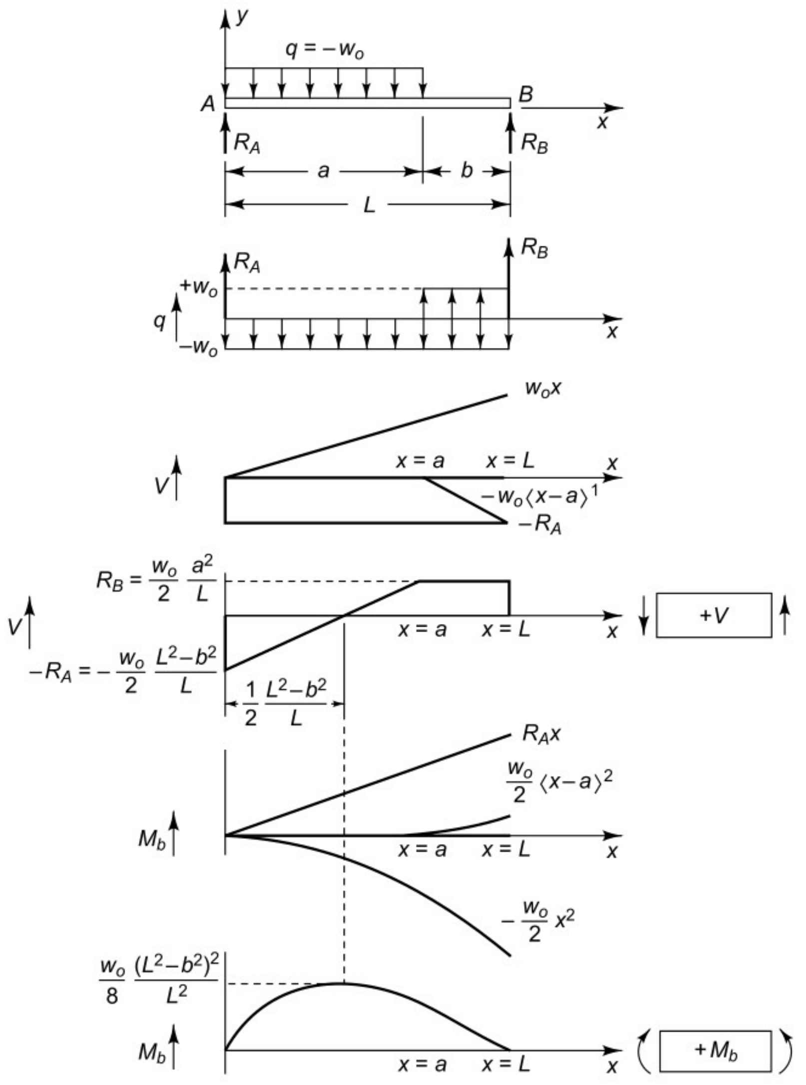
Using relation (4) and integrating (b.4), we get,

$$\frac{dM_b}{dx} = -V(x),$$

$$M_b = -\frac{w_0 x^2}{2} + \frac{w_0}{2} \langle x - a \rangle^2 + R_A x + c_2 \qquad \cdots\cdots\cdots\cdots(\text{c.4})$$

Using condition that $M_b(x{=}0) = 0$, we get $c_2{=}0$.

Equation (b.4) and (c.4) can be plotted as SFD and BMD.



Reaction forces can also be included in the definition of $q(x)$ as

$$q(x) = R_A \langle x \rangle_{-1} - w_0 \langle x \rangle^0 + w_0 \langle x - a \rangle^0 + R_B \langle x - L \rangle_{-1}.$$

Now, using (3) and integrating from -∞ to x, we get,

$$-V(x) = \int_{-\infty}^{x} q(x)dx = R_A \langle x \rangle^0 - w_0 \langle x \rangle^1 + w_0 \langle x - a \rangle^1 + R_B \langle x - L \rangle^0.$$

Note that we have applied the condition that $V(-\infty)=0$.

Again integrating the expression for $V(x)$, we get,

$$M_b(x) = \int_{-\infty}^{x} -V(x)dx = R_A \langle x \rangle^1 - \frac{w_0}{2} \langle x \rangle^2 + \frac{w_0}{2} \langle x - a \rangle^2 + R_B \langle x - L \rangle^1.$$

We have used the condition that $M_b(-\infty)=0$.

R_A and R_B can be obtained by applying equilibrium condition to the beam.