ME232: Dynamics

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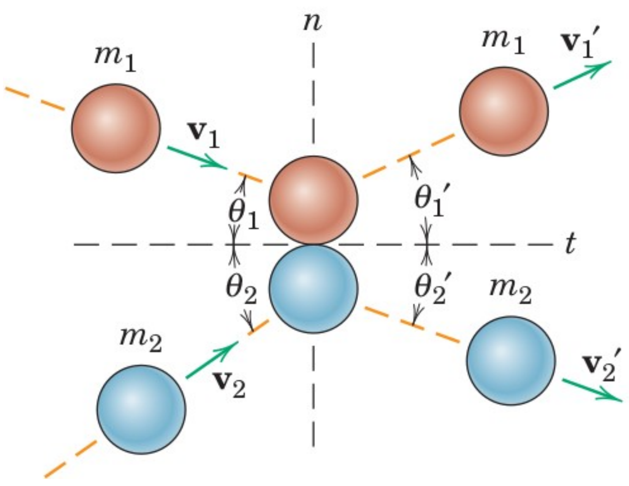
Room # 106

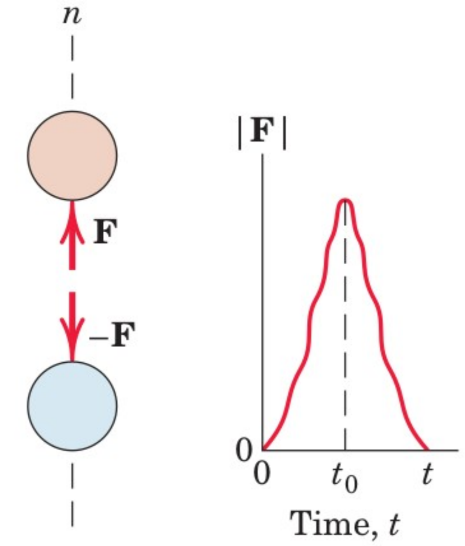
Oblique Central Impact:

Now we discuss the case where the initial and final velocities are not parallel.



Spherical particles of mass m_1 and m_2 have initial velocities v_1 and v_2 in the same plane and approach each other on a collision course. The directions of the velocity vectors are measured from the direction tangent to the contacting surfaces.

Thus, the initial velocity components along the t- and n-axes are $(v_1)_n = -v_1 \sin\theta_1$, $(v_1)_t = v_1 \cos\theta_1$, $(v_2)_n = v_2 \sin\theta_2$, and $(v_2)_t = v_2 \cos\theta_2$.



The final rebound conditions are also shown. The impact forces are $\boldsymbol{F}$ and -$\boldsymbol{F}$. They vary from zero to their peak value during the deformation portion of the impact and back again to zero during the restoration period.

For given initial conditions of m_1, m_2, $(v_1)_n$, $(v_1)_t$, $(v_2)_n$, and $(v_2)_t$, there will be four unknowns, namely, $(v'_1)_n$, $(v'_1)_t$, $(v'_2)_n$, and $(v'_2)_t$. The four equations are obtained as follows:

(1) Momentum of the system is conserved in the n-direction. This gives

$$m_1(v_1)_n + m_2(v_2)_n = m_1(v'_1)_n + m_2(v'_2)_n.$$

(2) and (3) The momentum for each particle is conserved in the t-direction since there is no impulse on either particle in the t-direction. Thus,

$$m_1(v_1)_t = m_1(v'_1)_t, \quad m_2(v_2)_t = m_2(v'_2)_t.$$

(4) The coefficient of restitution, as in the case of direct central impact, is the positive ratio of the recovery impulse to the deformation impulse. Equation (53) applies, then, to the velocity components in the n-direction. Thus,

$$e = \frac{|\text{relative velocity of seperation}|}{|\text{relative velocity of approach}|} = \frac{(v'_2)_n - (v'_1)_n}{(v_1)_n - (v_2)_n}. \qquad \cdots\cdots\cdots(54)$$

Once the four final velocity components are found, the angles θ'_1 and θ'_2 may be easily determined.

Application: Relative motion

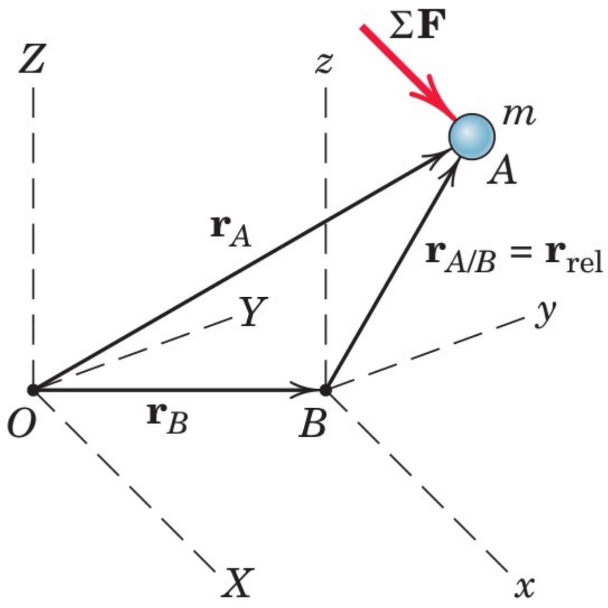
We now consider a particle A of mass m, whose motion is observed from a set of axes x-y-z which translate with respect to a fixed reference frame X-Y-Z.

The acceleration of the origin B of x-y-z is $\boldsymbol{a}_B$. The acceleration of A as observed from or relative to x-y-z is $\boldsymbol{a}_{\text{rel}} = \boldsymbol{a}_{A/B} = \ddot{\boldsymbol{r}}_{A/B}$, and by the relative-motion principle, the absolute acceleration of A is

$$\boldsymbol{a}_A = \boldsymbol{a}_B + \boldsymbol{a}_{A/B} = \boldsymbol{a}_B + \boldsymbol{a}_{\text{rel}}.$$

Thus, Newton's second law $\Sigma \boldsymbol{F} = m\boldsymbol{a}_A$ becomes

$$\sum \boldsymbol{F} = m\left(\boldsymbol{a}_B + \boldsymbol{a}_{\text{rel}}\right). \qquad \cdots\cdots\cdots(66)$$



$\Sigma \boldsymbol{F}$ can be identified by a complete free-body diagram. This diagram will appear the same to an observer in x-y-z or to one in X-Y-Z as long as only the real forces acting on the particle are represented. Then we can conclude that Newton's second law does not hold with respect to an accelerating system, as $\Sigma \boldsymbol{F} \neq m\boldsymbol{a}_{\text{rel}}$.

Reg. D'Alembert's Principle:

Opinion differs concerning the original interpretation of D'Alembert's principle, but the principle in the form in which it is generally known is regarded in this book as being mainly of historical interest. It evolved when understanding and experience with dynamics were extremely limited and was a means of explaining dynamics in terms of the principles of statics, which were more fully understood. This excuse for using an artificial situation to describe a real one is no longer justified, as today a wealth of knowledge and experience with dynamics strongly supports the direct approach of thinking in terms of dynamics rather than statics. It is somewhat difficult to understand the long persistence in the acceptance of statics as a way of understanding dynamics, particularly in view of the continued search for the understanding and description of physical phenomena in their most direct form.

Engineering Mechanics:Dynamics
By J. L. Meriam

[†]In the 1700s, Jean-Baptiste le Rond d'Alembert expressed Newton's second law as $\Sigma \mathbf{F} - m\mathbf{a} = 0$ so he could solve dynamics problems using the principles of statics. The $-m\mathbf{a}$ term has been called a fictitious *inertial force*, but it is important for you to realize that there is no such thing as inertial forces (or centrifugal forces that "push" you outward when going around a curve). D'Alembert's principle (also called dynamic equilibrium) is seldom used in modern engineering.

Vector Mechanics for Engineers:
Statics and Dynamics
By Beer and Johnston

Constant-Velocity, Nonrotating Systems:

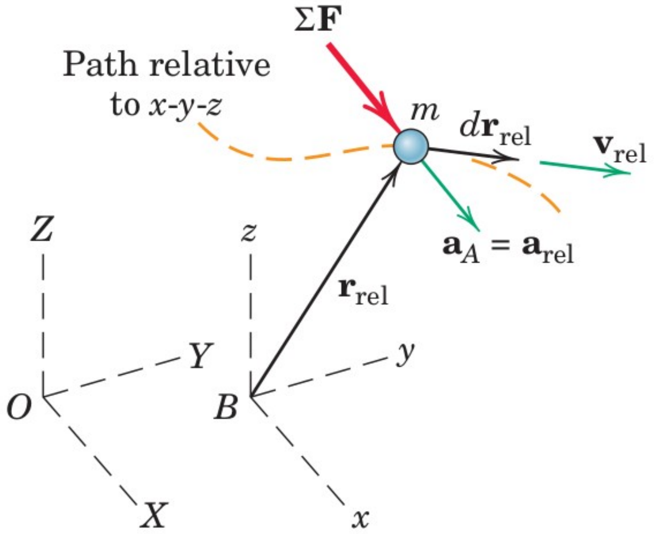
In discussing particle motion relative to moving reference systems, we should note the special case where the reference system has a constant velocity and no rotation. If the x-y-z axes have a constant velocity, then $\boldsymbol{a}_B = 0$ and the acceleration of the particle is $\boldsymbol{a}_{\mathrm{A}} = \boldsymbol{a}_{\mathrm{rel}}$. Therefore, we write (66) as

$$\sum \boldsymbol{F} = m\boldsymbol{a}_{\mathrm{rel}}, \qquad \cdots\cdots\cdots(67)$$

which tells us that Newton's second law holds for measurements made in a system moving with a constant velocity. Such a system is known as **an inertial system** or as **a Newtonian frame of reference**. Observers in the moving system and in the fixed system will also agree on the designation of the resultant force acting on the particle from their identical free-body diagrams, provided they avoid the use of any so-called "inertia forces."

We will also check the validity of the work-energy equation and the impulse-momentum equation relative to a constant-velocity, nonrotating system.

Again, we take the x-y-z axes to be moving with a constant velocity $\boldsymbol{v}_B = \dot{\boldsymbol{r}}_B$ relative to the fixed axes X-Y-Z. The path of the particle A relative to x-y-z is governed by $\boldsymbol{r}_{\text{rel}}$.



The work done by $\Sigma\boldsymbol{F}$ relative to x-y-z is $dU_{\text{rel}} = \Sigma\boldsymbol{F}\cdot d\boldsymbol{r}_{\text{rel}}$. But $\Sigma\boldsymbol{F} = m\boldsymbol{a}_A = m\boldsymbol{a}_{\text{rel}}$ since $\boldsymbol{a}_B = 0$.

Also $\boldsymbol{a}_{\text{rel}}\cdot d\boldsymbol{r}_{\text{rel}} = \boldsymbol{v}_{\text{rel}}\cdot d\boldsymbol{v}_{\text{rel}}$ (similar to $\boldsymbol{a}_t$ ds $= vdv$ for curvilinear motion). Thus, we have

$$dU_{\text{rel}} = m\boldsymbol{a}_{\text{rel}}\cdot d\boldsymbol{r}_{\text{rel}} = mv_{\text{rel}}dv_{\text{rel}} = d\left(\tfrac{1}{2}mv_{\text{rel}}^2\right).$$

We define the kinetic energy relative to x-y-z as $T_{\text{rel}} = \frac{1}{2}mv_{\text{rel}}^2$, so now we have,

$$dU_{\text{rel}} = dT_{\text{rel}} \quad \text{or} \quad U_{\text{rel}} = \Delta T_{\text{rel}}. \qquad \cdots\cdots\cdots(68)$$

which shows that the work-energy equation holds for measurements made relative to a constant-velocity, nonrotating system.

Relative to x-y-z, the impulse on the particle during time dt is $\Sigma \boldsymbol{F}\, dt = m\boldsymbol{a}_A\, dt = m\boldsymbol{a}_{\text{rel}}\, dt$. But $m\boldsymbol{a}_{\text{rel}}\, dt = m\, d\boldsymbol{v}_{\text{rel}} = d(m\boldsymbol{v}_{\text{rel}})$ so $\Sigma \boldsymbol{F}\, dt = d(m\boldsymbol{v}_{\text{rel}})$.

We define the linear momentum of the particle relative to x-y-z as $\boldsymbol{G}_{\text{rel}} = m\boldsymbol{v}_{\text{rel}}$, which gives us $\Sigma \boldsymbol{F}\, dt = d\boldsymbol{G}_{\text{rel}}$. Dividing by dt and integrating give,

$$\sum \boldsymbol{F} = \dot{\boldsymbol{G}}_{\text{rel}} \quad \text{and} \quad \int \sum \boldsymbol{F} dt = \Delta \boldsymbol{G}_{\text{rel}}. \qquad \cdots\cdots\cdots(69)$$

Thus, the impulse-momentum equations for a fixed reference system also hold for measurements made relative to a constant-velocity, nonrotating system.

Finally, we define the relative angular momentum of the particle about a point in x-y-z, such as the origin B, as the moment of the relative linear momentum.

Thus, $(\boldsymbol{H}_B)_{\text{rel}} = \boldsymbol{r}_{\text{rel}} \times \boldsymbol{G}_{\text{rel}}$. The time derivative gives $(\dot{\boldsymbol{H}}_B)_{\text{rel}} = \dot{\boldsymbol{r}}_{\text{rel}} \times \boldsymbol{G}_{\text{rel}} + \boldsymbol{r}_{\text{rel}} \times \dot{\boldsymbol{G}}_{\text{rel}}$. The first term become 0, and the second term becomes $\boldsymbol{r}_{\text{rel}} \times \Sigma \boldsymbol{F} = \Sigma \boldsymbol{M}_B$, the sum of the moments about B of all forces on m. Thus, we have

$$\sum \boldsymbol{M}_B = (\dot{\boldsymbol{H}}_B)_{\text{rel}}. \qquad \cdots\cdots\cdots(70)$$

Equation (70) shows that the moment-angular momentum relation holds with respect to a constant-velocity, nonrotating system.

Although the work-energy and impulse-momentum equations hold relative to a system translating with a constant velocity, the individual expressions for work, kinetic energy, and momentum differ between the fixed and the moving systems. Thus,

$$\left(dU = \sum \boldsymbol{F} \cdot d\boldsymbol{r}_A\right) \neq \left(dU_{\text{rel}} = \sum \boldsymbol{F} \cdot d\boldsymbol{r}_{\text{rel}}\right)$$
$$\left(T = \frac{1}{2}mv_A^2\right) \neq \left(T_{\text{rel}} = \frac{1}{2}mv_{\text{rel}}^2\right)$$
$$(\boldsymbol{G} = m\boldsymbol{v}_A) \neq (\boldsymbol{G}_{\text{rel}} = mv_{\text{rel}})$$

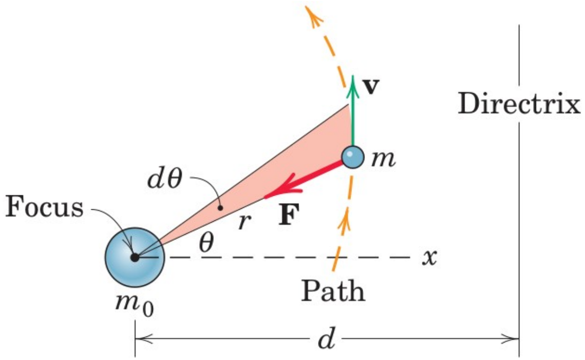
Application: Central force motion

Motion of a particle under the influence of a force directed toward a fixed center of attraction is called **central-force motion**. The most common example of central-force motion is the **orbital movement of planets and satellites**. The laws which govern this motion were deduced from observation of the motions of the planets by J. Kepler (1571–1630). An understanding of such motion is required to design high-altitude rockets, earth satellites, and space vehicles.

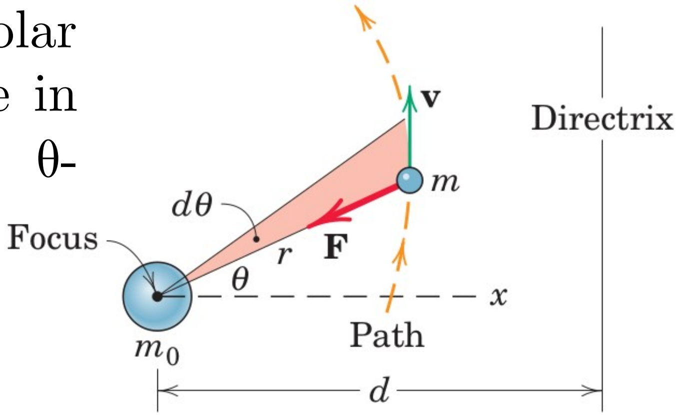
Consider a particle of mass m moving under the action of the central gravitational attraction

$$F = G\frac{mm_0}{r^2},$$

where m_0 is the mass of the attracting body, which is assumed to be fixed, G is the universal gravitational constant, and r is the distance between the centers of the masses. The particle of mass m could represent the earth moving about the sun, the moon moving about the earth, or a satellite in its orbital motion about the earth above the atmosphere.



The most convenient coordinate system to use is polar coordinates in the plane of motion since $\boldsymbol{F}$ will always be in the negative r-direction and there is no force in the θ-direction. Thus, we can write,



$$-G\frac{mm_0}{r^2} = m\boldsymbol{a}_n = m(\ddot{r} - r\dot{\theta}^2),$$
$$0 = m\boldsymbol{a}_\theta = m(r\ddot{\theta} - 2\dot{r}\dot{\theta}). \qquad \cdots\cdots\cdots(55)$$

By multiplying second equation of (55) with r/m, it can be rewritten as

$$\frac{d}{dt}(r^2\dot{\theta}) = 0 \quad \text{or} \quad r^2\dot{\theta} = h \quad \text{(a constant).} \qquad \cdots\cdots\cdots(56)$$

The physical significance of (56) become clear when we note that the angular momentum $\boldsymbol{r} \times m\boldsymbol{v}$ of m about m_0 has the magnitude $mr^2\dot{\theta}$. Thus, (56) merely states that **the angular momentum of m about m_0 remains constant (is conserved)**. This statement is easily deduced from (48), which shows that the angular momentum $\boldsymbol{H}_O$ is conserved if there is no moment acting on the particle about a fixed point O.

We observe that during time dt, the radius vector sweeps out an area, shaded in figure, equal to

$dA = (\frac{1}{2} r)(r d\theta)$. Therefore, the rate at which area is swept by the radius vector is $\dot{A} = 1/2 r^2 \dot{\theta}$,

which is constant according (56). This conclusion is expressed in **Kepler's second law** of planetary motion, which states that **the areas swept through in equal times are equal.**