ME231: Solid Mechanics-I

Stresses due to bending

Strain energy in bending

Pure bending

$$U = \frac{1}{2}\int_V \sigma_x \varepsilon_x dV = \int_V \frac{\sigma_x^2}{2E} dV,$$

$$\Rightarrow U = \int_V \frac{1}{2E}\left(\frac{M_b y}{I_{zz}}\right)^2 dV$$

$$\Rightarrow U = \int_L \frac{1}{2E}\left(\frac{M_b}{I_{zz}}\right)^2 dx \int_A y^2 dA,$$

$$\Rightarrow U = \int_L \frac{M_b^2}{2EI_{zz}} dx. \qquad \text{.............}(18)$$

Bending with transverse loads

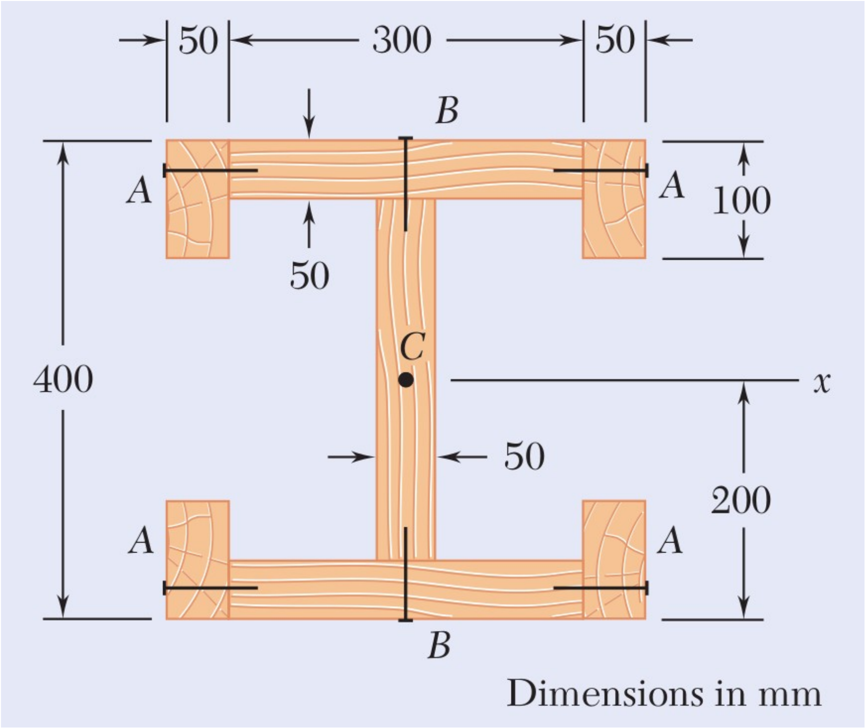
$$U = \frac{1}{2}\int_V \left(\sigma_x \varepsilon_x + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz}\right) dV$$

$$\Rightarrow U = \int_V \frac{\sigma_x^2}{2E} + \frac{1}{2G}\left(\tau_{xy}^2 + \tau_{xz}^2\right) dV.$$

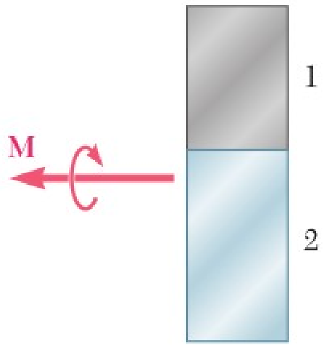
$$\text{.............}(19)$$

Example 4

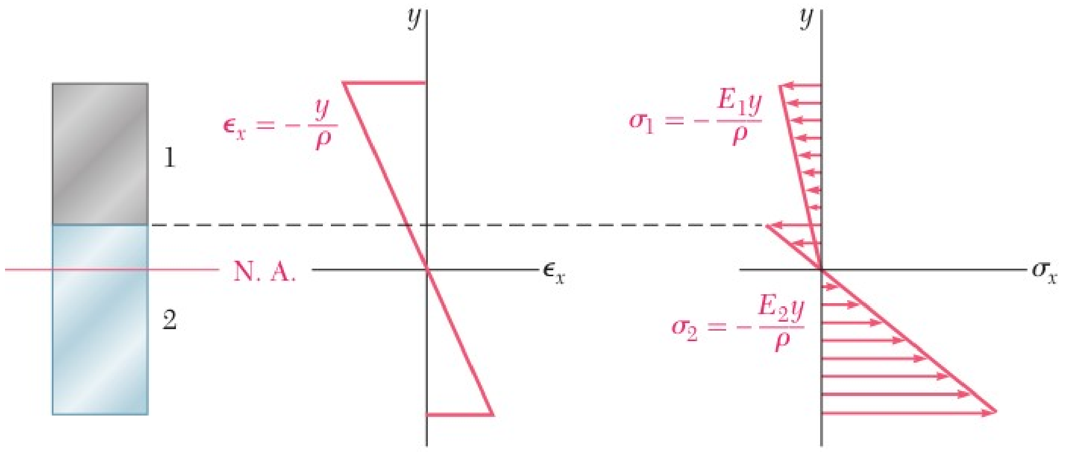
The built-up wooden beam shown is subjected to a vertical shear of 8 kN. Knowing that the nails are spaced longitudinally every 60 mm at A and every 25 mm at B, determine the shearing force in the nails (a) at A, (b) at B. (Given: $I_x = 1.504 \times 10^9$ mm^4.)



Members made of composite materials

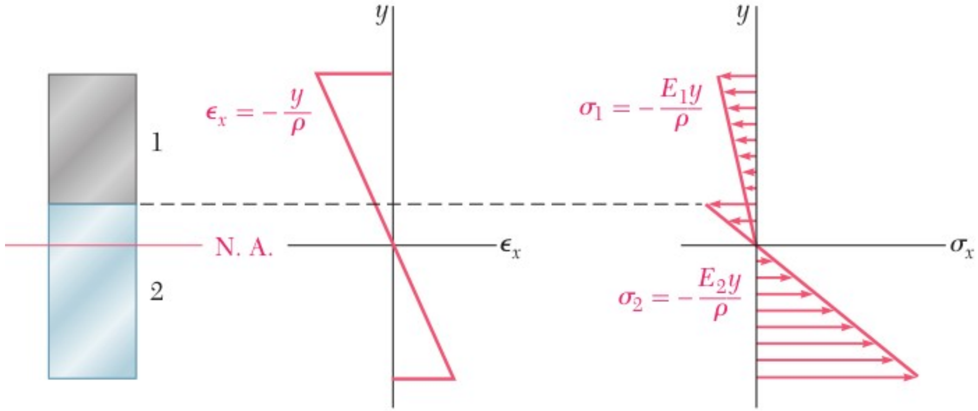


Consider a bar consisting of two different materials bonded together. Deformation of this composite bar will deform will follow the bending theory we discussed, since its cross section remains the same throughout its entire length, and since no assumption was made regarding the stress-strain relationship of the material or materials involved. Thus, the normal strain still varies linearly with the distance y from the neutral axis of the section as



$$\varepsilon_x = -y/\rho \qquad \cdots\cdots\cdots\cdots(20)$$

However, it cannot be assumed that the neutral axis passes through the centroid of the composite section, and one of the goals of this analysis is to determine the location of this axis.



Since the moduli of elasticity E_1 and E_2 of the two materials are different, the equations for the normal stress in each material are

$$\sigma_1 = E_1 \varepsilon_x = -\frac{E_1 y}{\rho}, \qquad \cdots\cdots\cdots\cdots(21)$$

$$\sigma_2 = E_2 \varepsilon_x = -\frac{E_2 y}{\rho}, \qquad \cdots\cdots\cdots\cdots(22)$$

Stress distribution is shown. Now the the force dF_1 exerted on an element of area dA of the upper portion of the cross section is

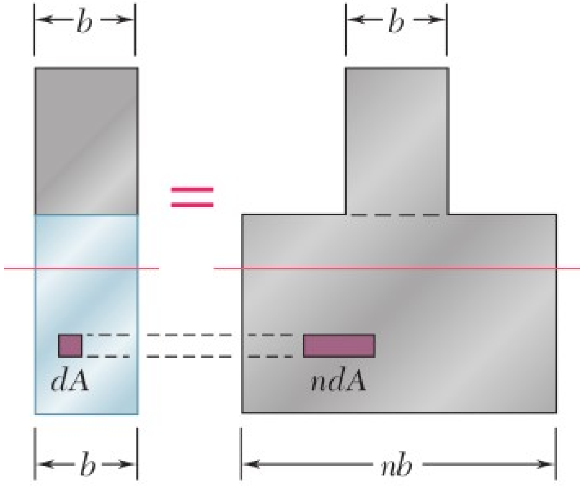
$$dF_1 = \sigma_1 dA = -\frac{E_1 y}{\rho} dA, \quad \text{and} \qquad \cdots\cdots\cdots\cdots(23)$$

$$dF_2 = \sigma_2 dA = -\frac{E_2 y}{\rho} dA. \qquad \cdots\cdots\cdots\cdots(24)$$

If $E_2/E_1 = n$, then we can write,

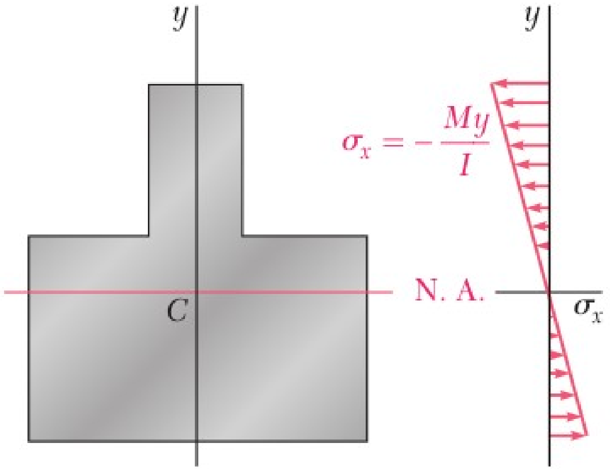
$$dF_2 = -\frac{nE_1 y}{\rho} dA = -\frac{E_1 y}{\rho}(n dA). \quad \cdots\cdots\cdots\cdots(25)$$

Compare (23) and (25), which suggest that the same force dF_2 would be exerted on an element of area ndA of the first material. Thus, the resistance to bending of the bar would remain the same if both portions were made of the first material, provided that the width of each element of the lower portion were multiplied by the factor n. Note that this widening (if $n>1$) or narrowing (if $n<1$) must be in a direction parallel to the neutral axis of the section, since it is essential that the distance y of each element from the neutral axis remain the same. This new cross section is called the transformed section of the member.



Now the transformed section is made of same material, NA and stresses can be determined as

$$\sigma_x = -\frac{My}{I_z},$$

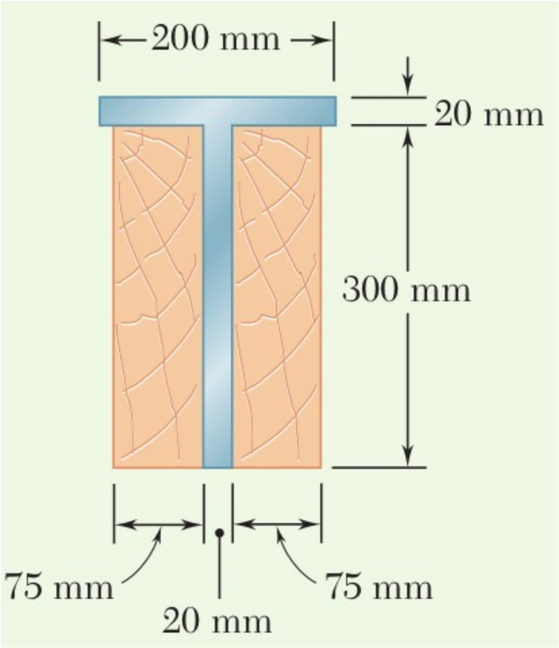


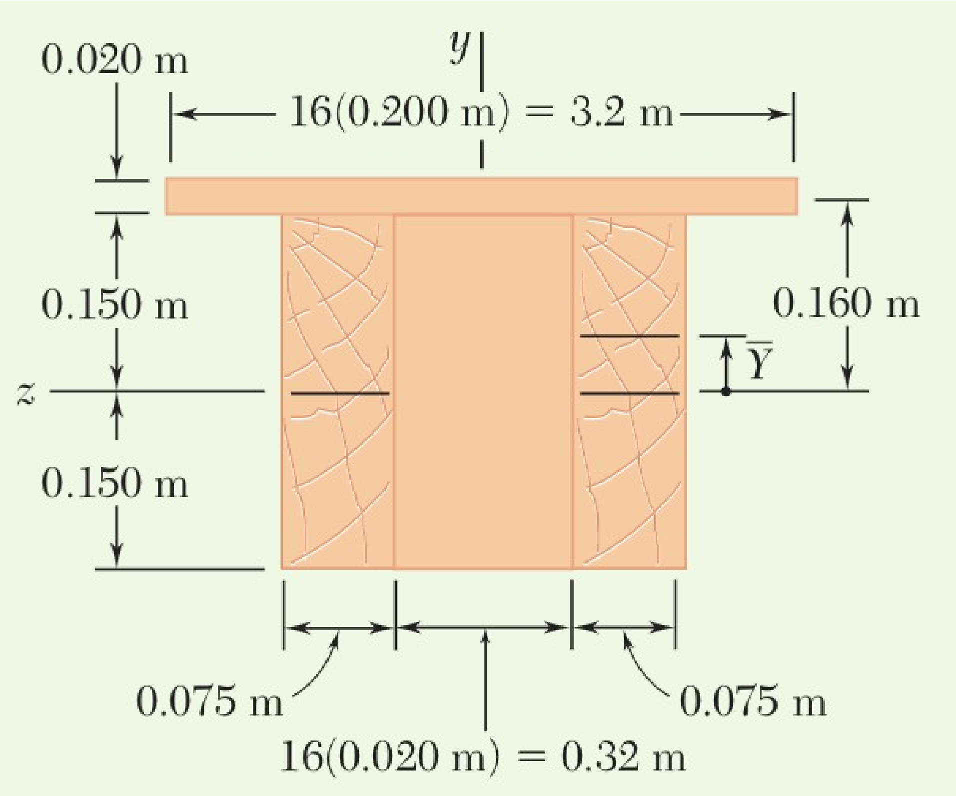
where y is the distance from the neutral surface and I is the moment of inertia of the transformed section with respect to its centroidal axis.

Now the stress σ_1 at a point located in the upper portion of the cross section of the original composite bar can be obtained by computing the stress σ_x at the corresponding point of the transformed section. However, to obtain the stress σ_2 at a point in the lower portion of the cross section, n must be multiplied to the stress σ_x computed at the corresponding point of the transformed section.

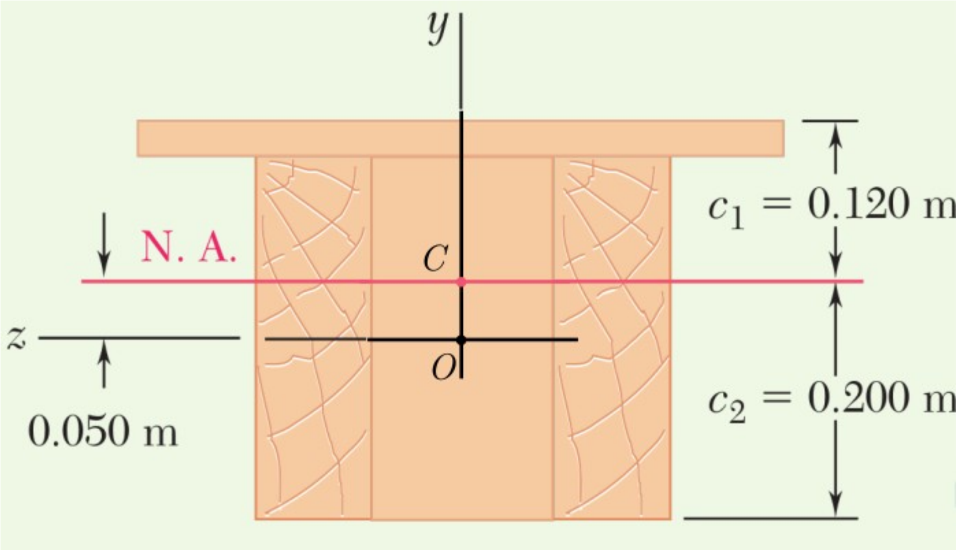
Example 5

Two steel plates have been welded together to form a beam in the shape of a T that has been strengthened by securely bolting to it the two oak timbers shown in the figure. The modulus of elasticity is 12.5 GPa for the wood and 200 GPa for the steel. Knowing that a bending moment M=50 kN·m is applied to the composite beam, determine (a) the maximum stress in the wood and (b) the stress in the steel along the top edge.





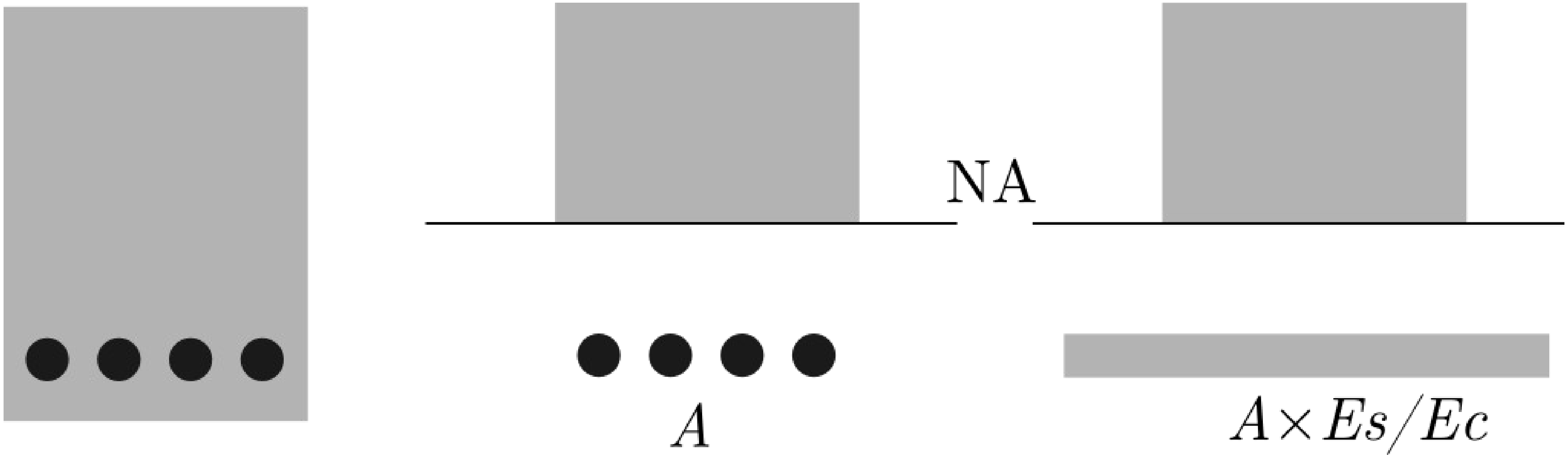
0.020 m
y
16(0.200 m) = 3.2 m
0.150 m
0.160 m
z
Ȳ
0.150 m
0.075 m
0.075 m
16(0.020 m) = 0.32 m



y
c_1 = 0.120 m
N. A.
C
z
O
0.050 m
c_2 = 0.200 m

Example 6: Reinforced concrete beams

Concrete is a brittle material which has good strength in compression but very little strength in tension. Despite its low tensile strength, economic use can be made of concrete in reinforced-concrete construction in which steel bars are embedded in the concrete to provide tensile action. For a reinforced-concrete beam it can be safely assumed that no tensile stresses are carried by the concrete and that the tensile stress in the steel is uniform over the bars.



ME231: Solid Mechanics-I

Deflections due to bending

The moment-curvature relationship

For a symmetrical, linearly elastic beam element subjected to pure bending, we have derived the following moment-curvature relationship.

$$\frac{1}{\rho} = \frac{d\phi}{ds} = \frac{M_b}{EI_{zz}} \qquad \cdots\cdots\cdots\cdots(1)$$

The curvature of the neutral axis completely defines the deformation of an element in pure bending.

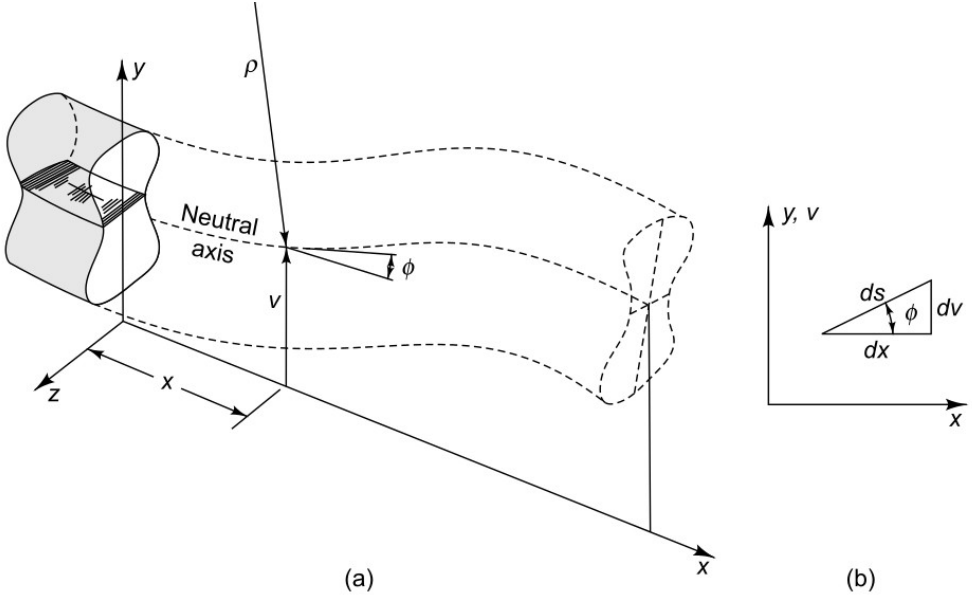
We can extend this to the case of general bending where the bending moment varies along the length of the beam.

We **assume that the shear forces do not contribute** significantly to the overall deformation.

Thus we assume that the **deformation is still defined by the curvature** and that the curvature is still given by above relation.

Accordingly, if we know how the bending moment varies along the length of the beam, we will then know how the curvature varies.

To determine the bent shape of the beam, we deduce the deflection of the neutral axis from the knowledge of its curvature. To facilitate this, we first derive a differential equation relating the curvature $d\phi/ds$ to the deflection $v(x)$.



We start with the definition of the slope of the neutral axis in

$$\frac{dv}{dx} = \tan\phi. \qquad \cdots\cdots\cdots\cdots(2)$$

Differentiating (2) w.r.t. s,

$$\frac{d^2v}{dxds} = \frac{d^2v}{dx^2}\frac{dx}{ds} = \sec^2\phi\frac{d\phi}{ds},$$

or the curvature,

$$\frac{d\phi}{ds} = \cos^2\phi\frac{d^2v}{dx^2}\frac{dx}{ds} = \frac{d^2v/dx^2}{\left[1+(dv/dx)^2\right]^{3/2}}, \qquad \left(\because \cos\phi = \frac{dx}{ds} = \frac{1}{\left[1+(dv/dx)^2\right]^{1/2}}\right)$$

$$\cdots\cdots\cdots\cdots(3)$$

From (1) and (3) we get,

$$\frac{M_b}{EI_{zz}} = \frac{d^2v/dx^2}{[1+(dv/dx)^2]^{3/2}}.$$

When the slope angle ϕ is small, then dv/dx is small compared to unity. If we neglect $(dv/dx)^2$ in the denominator, we obtain a simple approximation as

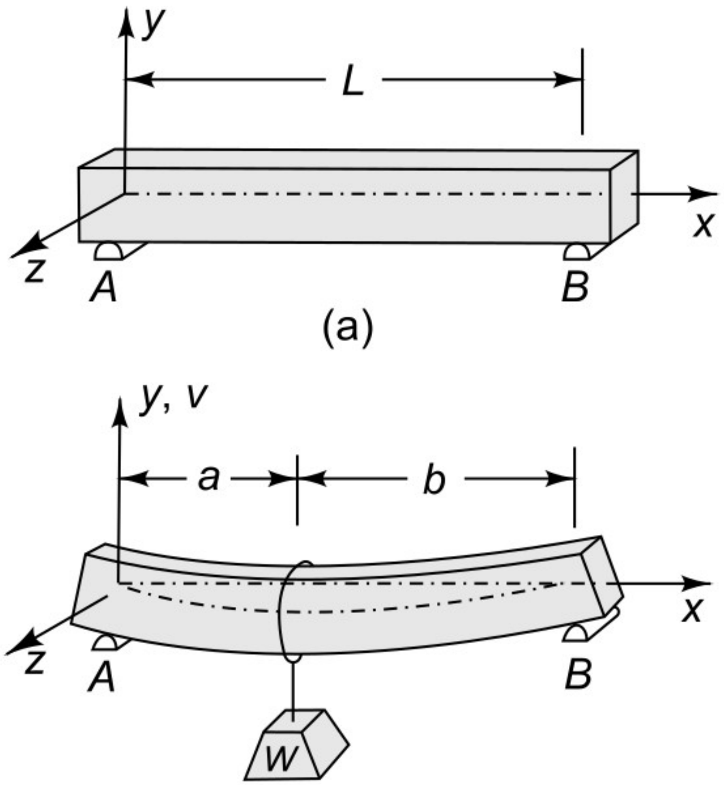
$$\frac{M_b}{EI_{zz}} \approx \frac{d^2v}{dx^2}. \qquad \cdots\cdots\cdots\cdots(4)$$

Equation (4) relates the bending moment to the transverse displacement. Although (4) involves an approximation to the curvature which is valid only for small bending angles, we call it the **moment-curvature relation**. It is essentially a "force-deformation" or "stress-strain" relation in which the bending moment is the "force" or "stress" and the approximate curvature is the resulting "deformation" or "strain."

The relation is a linear one; the constant of proportionality EI is sometimes called the **flexural rigidity or the bending modulus.**

Example 1

The simply supported beam of uniform cross section shown in figure is subjected to a concentrated load W. It is desired to obtain the deflection curve of the deformed neutral axis.



From the bending moment analysis of the beam, it can be shown that the distribution of bending moment is as follows,

For length AC,

$$M_b = \frac{Wb}{L}x. \qquad \cdots\cdots\cdots\cdots(1.a)$$

For length CB

$$M_b = \frac{Wb}{L}x - W(x-a) \qquad \cdots\cdots\cdots\cdots(1.b)$$

For determining the displacement, we use the moment-curvature relationship (4).
First for part AC $(x < a)$as,

$$EI\frac{d^2v}{dx^2} = M_b = \frac{Wb}{L}x \qquad \cdots\cdots\cdots\cdots(1.c)$$

Since EI is constant throughout the beam, integration of (1.c) gives following,

$$EI\frac{dv}{dx} = \frac{Wb}{L}\frac{x^2}{2} + c_1 \qquad \cdots\cdots\cdots\cdots(1.c)$$

$$EIv = \frac{Wb}{L}\frac{x^3}{6} + c_1x + c_2 \qquad \cdots\cdots\cdots\cdots(1.d)$$

For part CD $(x>a)$,

$$EI\frac{d^2v}{dx^2} = M_b = \frac{Wb}{L}x - W(x-a)$$

Integrating twice we get,

$$EI\frac{dv}{dx} = \frac{Wb}{L}\frac{x^2}{2} - W\frac{(x-a)^2}{2} + c_3 \qquad \cdots\cdots\cdots\cdots(1.e)$$

$$EIv = \frac{Wb}{L}\frac{x^3}{6} - W\frac{(x-a)^3}{6} + c_3x + c_4 \qquad \cdots\cdots\cdots\cdots(1.f)$$

Now we apply boundary conditions to determine the constants of integration as follows:
(I) at x=0, v=0, which gives $c_2 = 0$

(II) at x=L, v=0, which gives $$\frac{WbL^2}{6} - W\frac{b^3}{6} + c_3x + c_4 = 0 \qquad \text{..........(1.g)}$$

(III) at x=a, $v_{\mathrm{AC}} = v_{\mathrm{CB}}$ $$\frac{Wb}{L}\frac{a^3}{6} + c_1a = \frac{Wb}{L}\frac{a^3}{6} + c_3x + c_4 \qquad \text{..........(1.h)}$$

(IV) at $x = a$, $$\left.\frac{dv}{dx}\right|_{AC} = \left.\frac{dv}{dx}\right|_{CB}$$

$$\frac{Wb}{L}\frac{a^2}{2} + c_1 = \frac{Wb}{L}\frac{a^2}{2} + c_3 \quad \Rightarrow \quad c_1 = c_3. \qquad \text{..........(1.i)}$$

Finally, we get

$$v = -\frac{W}{6EI}\left[\frac{bx}{L}\left(L^2 - b^2 - x^2\right)\right] \quad (0 \leq x \leq a) \qquad \cdots\cdots\cdots\cdots(1.j)$$

$$v = -\frac{W}{6EI}\left[\frac{bx}{L}\left(L^2 - b^2 - x^2\right) + (x-a)^3\right] \quad (a \leq x \leq L) \qquad \cdots\cdots\cdots\cdots(1.k)$$