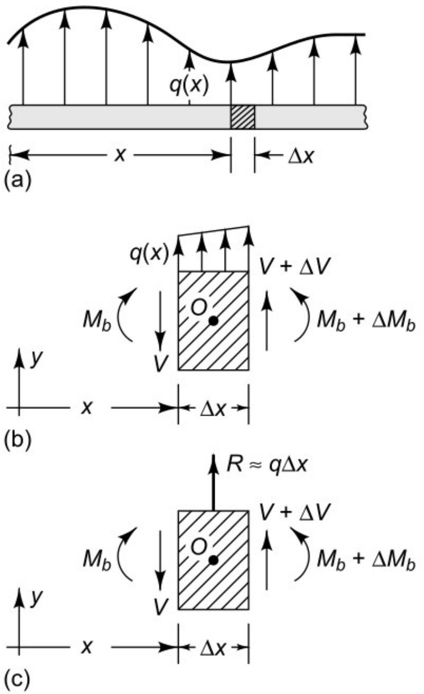
ME231: Solid Mechanics-I

Forces and Moments Transmitted by Slender Members

Differential equilibrium relationship

Another procedure for obtaining internal forces and moments along a slender element, is to consider a very small element of the beam as a free body and applying equilibrium conditions to it. This results in differential equations connecting the load, the shear force, and the bending moment. By integrating these relationships for particular cases we can evaluate shear forces and bending moments.



Consider an element of size Δx at a distance x. Distributed force for the element, shear forces and moments at two sections of the element are shown. As Δx is very small we replace distributed force with a concentrated force $R = q\Delta x$.

Now considering the equilibrium of the element as,

$$\sum F_y = V + \Delta V - V + R = 0,$$

$$\Rightarrow \frac{\Delta V}{\Delta x} + q(x) = 0. \qquad \cdots\cdots\cdots\cdots(1)$$

$$\sum M_O = (V + \Delta V)(\Delta x/2) + V(\Delta x/2) + M_b + \Delta M_b - M_b = 0,$$

$$\Rightarrow V\Delta x + \Delta V \Delta x/2 + \Delta M_b = 0 \quad \text{or} \quad \frac{\Delta M_b}{\Delta x} + V = -\frac{\Delta V}{2}. \qquad \cdots\cdots\cdots\cdots(2)$$

As $\Delta x \to 0$, (1) and (2) becomes,

$$\frac{dV}{dx} + q(x) = 0, \qquad \cdots\cdots\cdots\cdots(3)$$

$$\frac{dM_b}{dx} + V = 0. \qquad \cdots\cdots\cdots\cdots(4)$$

(3) and (4) are the basic differential equations relating the load intensity $q(x)$ with the shear force $V(x)$ and bending moment $M_b(x)$ in a beam. These equations can be integrated at any section from $x = x_1$ to $x = x_2$, which yields

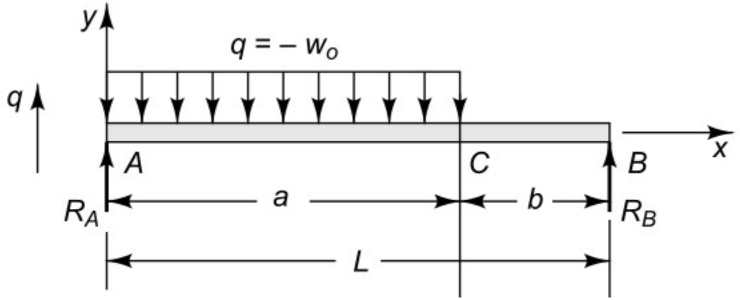
$$V(x_2) - V(x_1) + \int_{x_1}^{x_2} q(x)dx = 0, \qquad \cdots\cdots\cdots\cdots(5)$$

$$M_b(x_2) - M_b(x_1) + \int_{x_1}^{x_2} V(x)dx = 0, \qquad \cdots\cdots\cdots\cdots(6)$$

Let us look at the application of (5) and (6) through an example.

Example 6

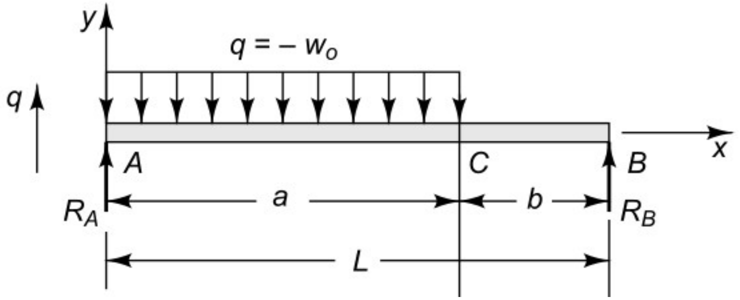
A beam with simple transverse supports at A and B and loaded with a uniformly distributed load $q = -w_0$ over a portion of the length. It is desired to obtain the shear-force and bending-moment diagrams.



Find reaction forces from equilibrium conditions,

$$\sum F_y = R_A + R_B - w_0 a = 0, \qquad \Rightarrow R_A = w_0 a \left(1 - \frac{a}{2L}\right), \quad R_B = \frac{w_0 a^2}{2L}.$$
$$\sum M_A = R_B L - w_0 a^2/2 = 0.$$

As the load is acting on only a part of the beam, we must consider the two parts of the beam (i) the part in which the load acts and (ii) the part in which no load acts. Then we write differential equations accordingly and integrate accordingly using valid boundary conditions.



For part AC (using 3),

$$\frac{dV_1}{dx} - w_0 = 0,$$

$$V_1 = w_0 x + c_1,$$

For part CB (using 3),

$$\frac{dV_2}{dx} = 0,$$

$$V_2 = c_2, \qquad \cdots\cdots\cdots\cdots(6.a)$$

Now, using (4) for part AC and CB, we get

$$\frac{dM_{b1}}{dx} + V_1 = \frac{dM_{b1}}{dx} + w_0 x + c_1 = 0, \qquad \text{and} \qquad \frac{dM_{b2}}{dx} + V_2 = \frac{dM_{b2}}{dx} + c_2 = 0.$$

After integrating

$$M_{b1} + \frac{w_0 x^2}{2} + c_1 x + c_3 = 0, \qquad \text{and} \qquad M_{b2} + c_2 x + c_4 = 0. \qquad \cdots\cdots\cdots\cdots(6.b)$$

Constants c_1 to c_4 can be determined using following conditions,

(i) Boundary conditions: $M_{b1} = 0$ at $x = 0$ and $M_{b2} = 0$ at $x = L$, ············(6.c)

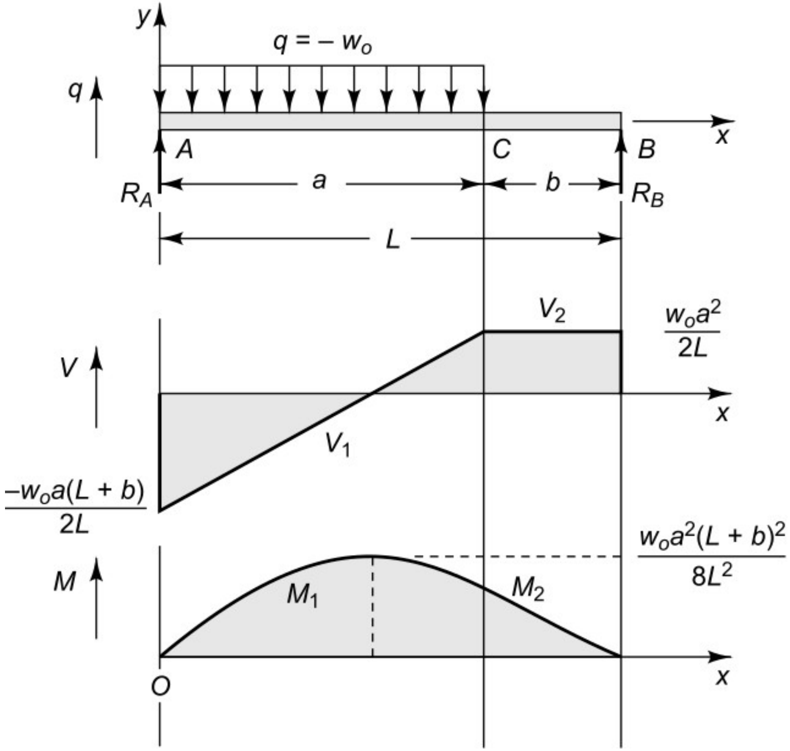
(ii) Conditions from equilibrium: $V_1 = V_2$ at $x = a$, and $M_{b1} = M_{b2}$ at $x = a$. ············(6.d)

(iii) $V_1 = -R_A$ at $x = 0$ and $V_2 = R_B$ at $x = L$. ············(6.e)

Using (6.c)-(6.e) with (6.a) and (6.b) we can determine all constants as,

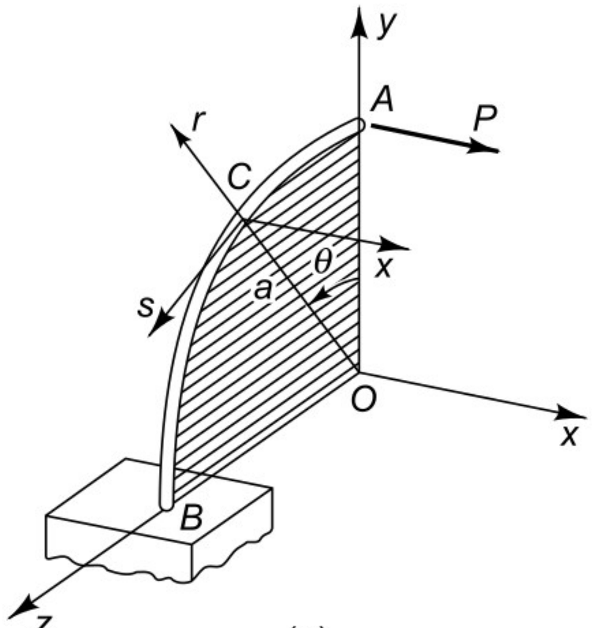
$$c_3 = 0, \quad c_4 = \tfrac{1}{2} w_0 a^2, \quad c_2 = \frac{w_0 a^2}{2L}, \quad c_1 = w_0 a \left(\frac{a}{2L} - 1 \right). \qquad \text{...........(6.f)}$$

Substituting all constants in (6.a) and (6.b) we get the expression for shear-force and bending moment distribution along the beam, which can be drawn as,



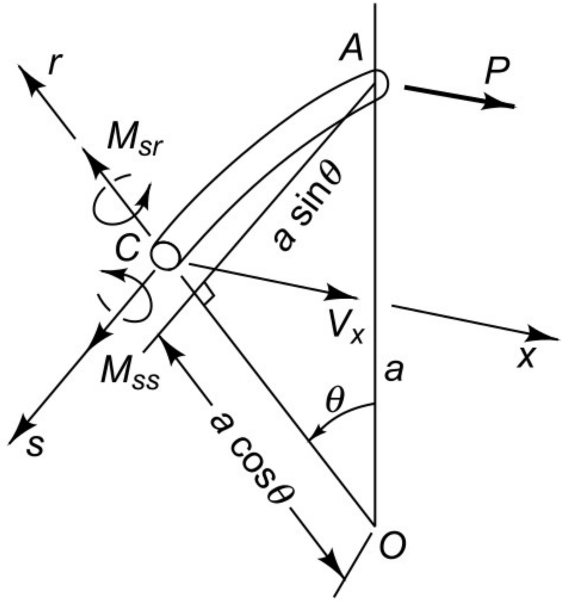
Example 7

A curved slender member AB is anchored at B and bent into a quadrant of a circle of center O and radius a. It is desired to obtain force and moment diagrams for this segment of pipe when a transverse load P is acting as shown.



(a)

It is a three dimensional problem. Basic procedure remain same. We cut a section at any point C. Define the forces and moments at that section and then consider the equilibrium of subsection to determine the forces and moments. As it a curved member, we use a local coordinate system x-r-s along the length of the member. For three-dimensional problems, it will be advisable to use the vector form of equilibrium conditions



FBD of the sub-system along with forces and moments at the section is shown. Applying equilibrium conditions as

$$\sum \boldsymbol{F} = P\boldsymbol{i} + \boldsymbol{V} = 0, \text{ which gives } V_x = -P, V_y = V_z = 0. \quad \cdots\cdots\cdots\cdots(a.5)$$

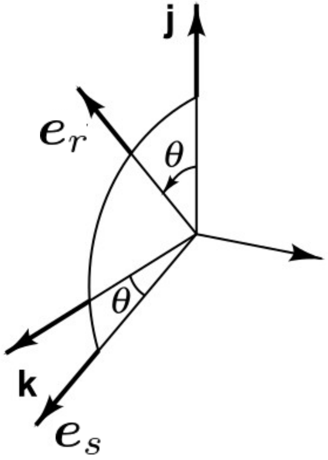
$$\sum \boldsymbol{M}_c = \boldsymbol{r}_{CA} \times P\boldsymbol{i} + M_{sr}\boldsymbol{e}_r + M_{ss}\boldsymbol{e}_s = 0, \quad \cdots\cdots\cdots\cdots(b.5)$$

where $\boldsymbol{e}_r$ and $\boldsymbol{e}_s$ are unit vectors along r- and s- axis.

Thus, $\boldsymbol{r}_{CA} = a\boldsymbol{j} + a\boldsymbol{e}_r.$ $\cdots\cdots\cdots\cdots$(c.5)

Vector $\boldsymbol{j}$ can be related to $\boldsymbol{e}_r$ and $\boldsymbol{e}_s$ as,

$$\boldsymbol{j} = \boldsymbol{e}_r \cos\theta - \boldsymbol{e}_s \sin\theta. \quad \cdots\cdots\cdots\cdots(d.5)$$



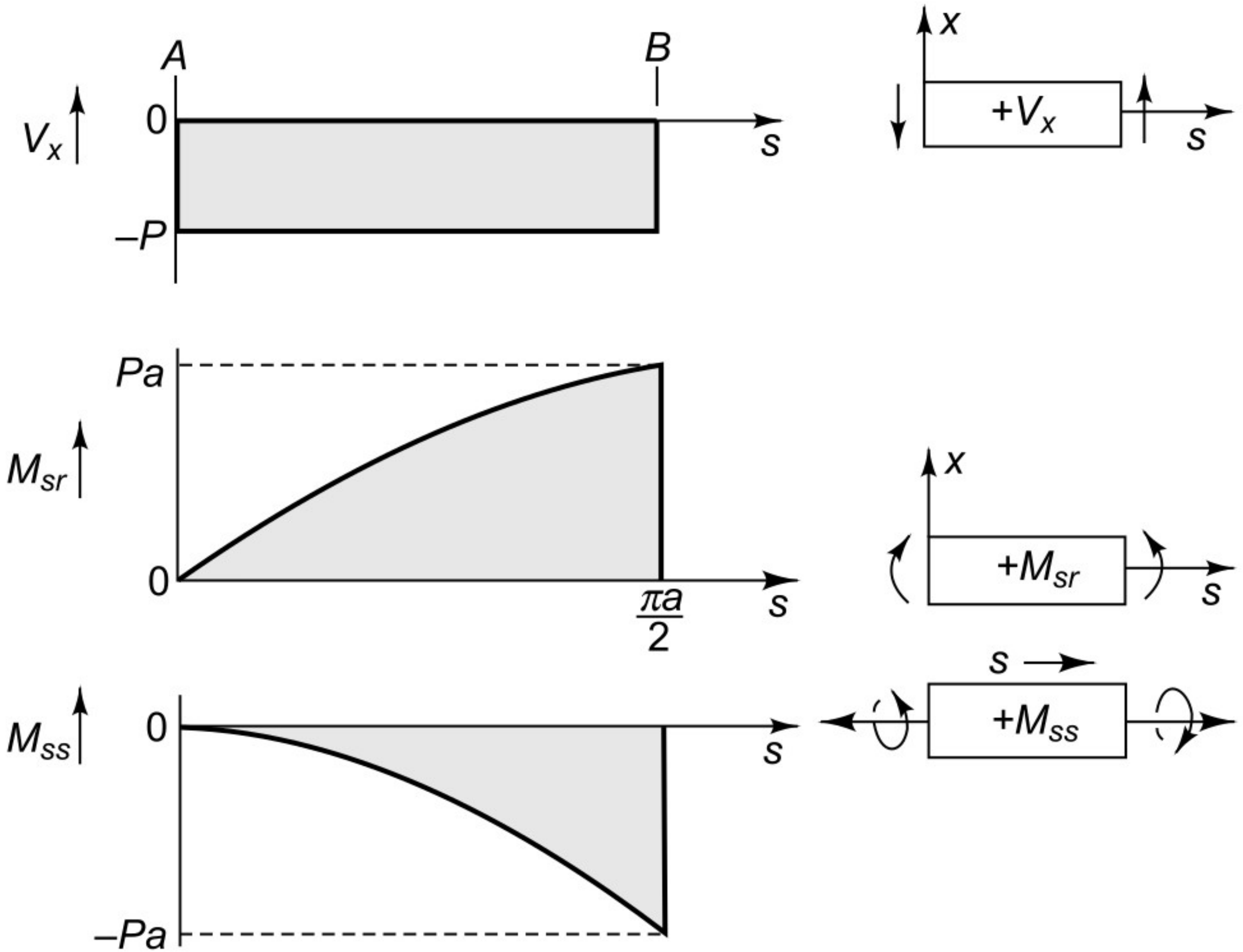
Using (d.5),(c.5) and substituting in (b.5),

$$\sum \boldsymbol{M}_c = \left[a\left(\boldsymbol{e}_r \cos\theta - \boldsymbol{e}_s \sin\theta\right) + a\boldsymbol{e}_r\right] \times P\boldsymbol{i} + M_{sr}\boldsymbol{e}_r + M_{ss}\boldsymbol{e}_s = 0,$$

Simplifying and collecting components,

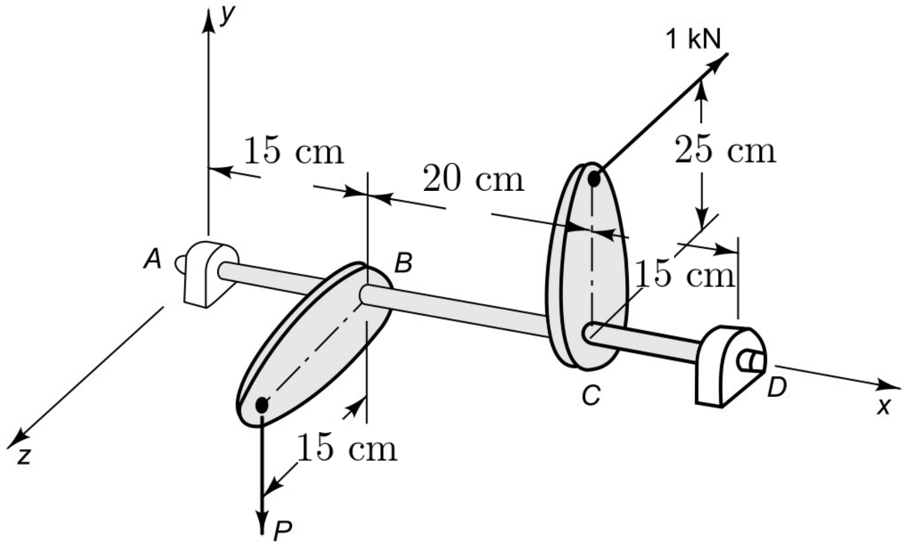
$$-Pa(1+\cos\theta) + M_{ss} = 0, \quad -Pa\sin\theta + M_{sr} = 0.$$

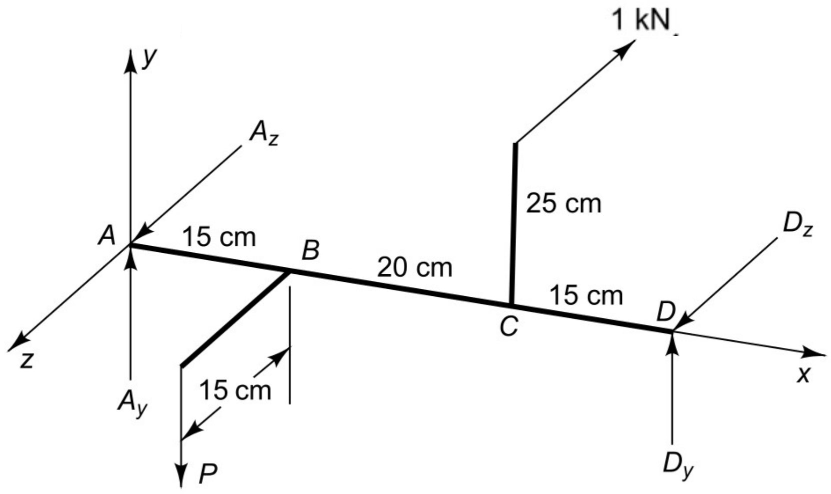
Thus, $M_{ss} = Pa(1+\cos\theta), \quad M_{sr} = Pa\sin\theta.$

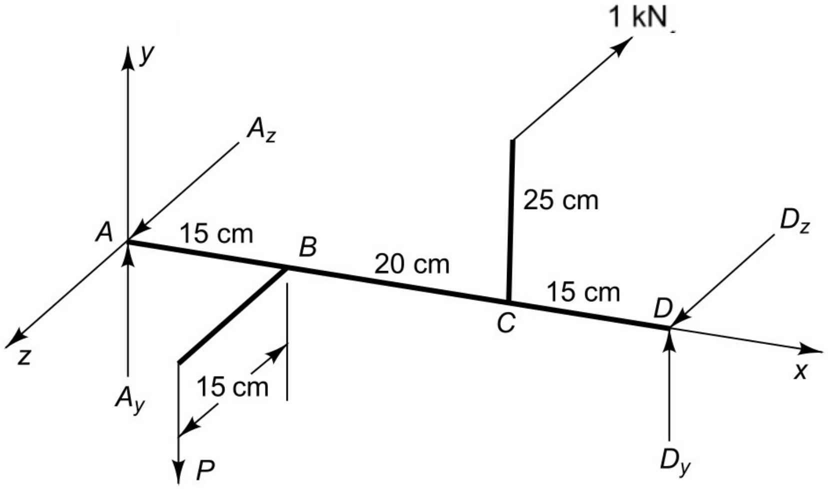

A
B
V_x
0
$-P$
s
x
$+V_x$
s
Pa
M_{sr}
0
$\frac{\pi a}{2}$
s
x
$+M_{sr}$
s
s
M_{ss}
0
s
$+M_{ss}$
$-Pa$

Example 8

Consider the offset bell-crank mechanism. A shaft supported in journal bearings at A and D is loaded as shown and has offset links attached at B and C. The problem is to obtain diagrams showing the variation of shear force, bending moment, and twisting moment in the shaft AD.







Applying equilibrium conditions,

$$\sum F_y = A_y + D_y - P = 0, \qquad \cdots\cdots\cdots\cdots(a.6)$$

$$\sum F_z = A_z + D_z - 1 = 0, \qquad \cdots\cdots\cdots\cdots(b.6)$$

$$\sum M_A = -0.15\,(\boldsymbol{k} + \boldsymbol{i})\times(-P\boldsymbol{j}) + (-0.25\boldsymbol{j} - 0.35\boldsymbol{i})\times(-1\boldsymbol{k}) - 0.5\boldsymbol{i}\times(D_z\boldsymbol{k} + D_y\boldsymbol{j}) = 0, \qquad \cdots\cdots\cdots\cdots(c.6)$$

Now, by simplifying and collecting components, we get,

$D_z = 0.7$ kN (from c.6)

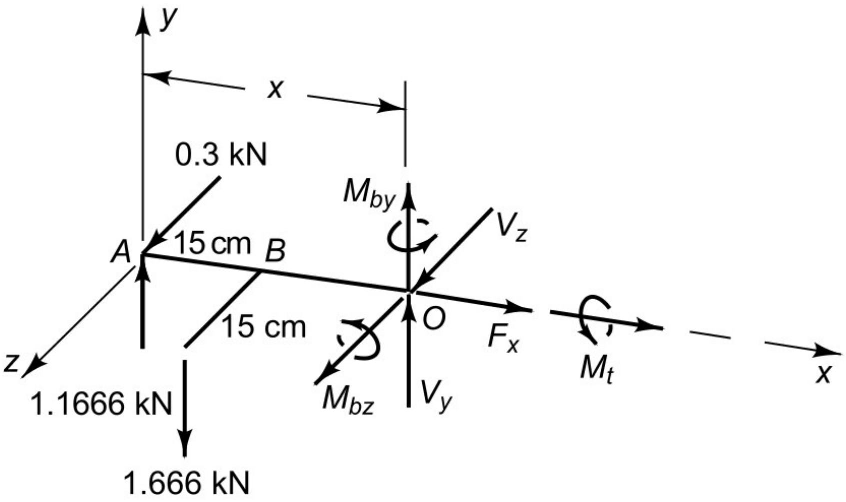
$A_z = 0.3$ kN (from b.6)

$P = 1.666$ kN (from c.6)

$D_y =$ -0.5 kN (from c.6)

$A_y = 1.666$ kN (from a.6)

After determining all support reactions and load P required for equilibrium, now let us consider the equilibrium of sub-sections.

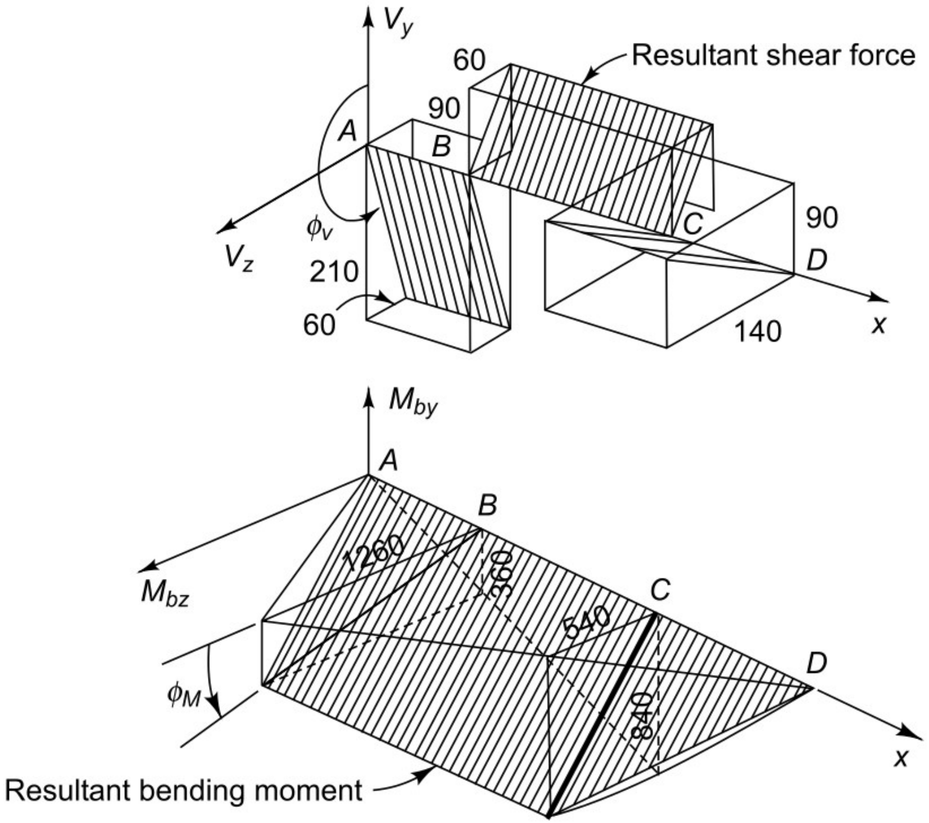


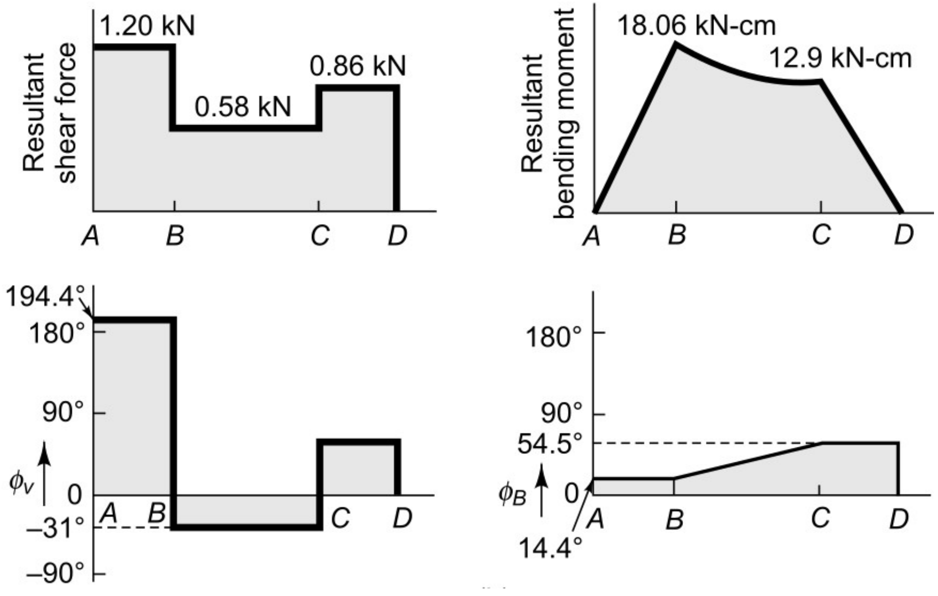
Take a section at a distance $0.35 < x < 0.5$ m. Determine the expressions defining the distribution of all section forces (F_x, V_y, V_z) and moments (M_t, M_{by}, M_{bz}) by applying equilibrium equations. These expressions will be valid for $0.35 < x < 0.5$ m.

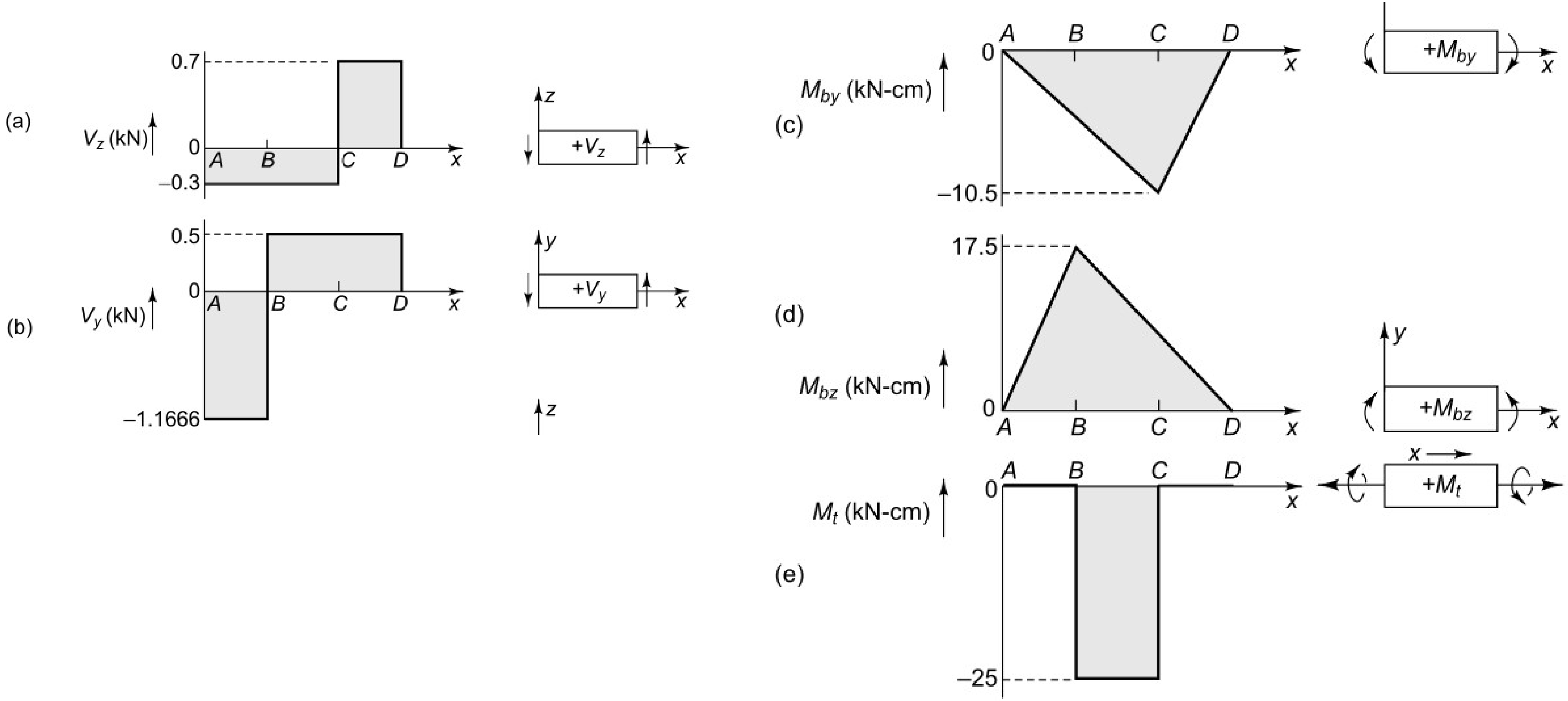
Repeat similar exercise by considering a section between 0 and 0.35 m length.

After completion of above exercise we will have all information to draw the distribution of all forces and moments along the length of the shaft.

SFDs and BMDs drawn in the previous slide can be combined to draw resultant shear force and bending moment and their directions as follows.






(a)
V_z (kN)
0.7
0
–0.3
A
B
C
D
x
z
$+V_z$
(b)
V_y (kN)
0.5
0
–1.1666
y
$+V_y$
z
(c)
M_{by} (kN-cm)
–10.5
$+M_{by}$
(d)
M_{bz} (kN-cm)
17.5
$+M_{bz}$
(e)
M_t (kN-cm)
–25
$+M_t$