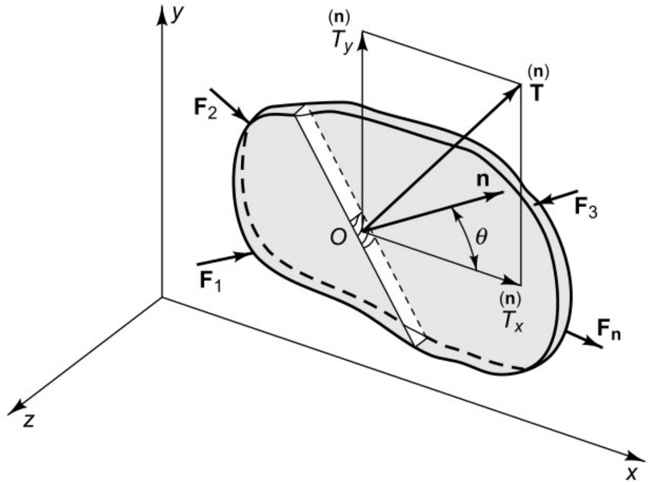
ME231: Solid Mechanics-I

Stress and Strain

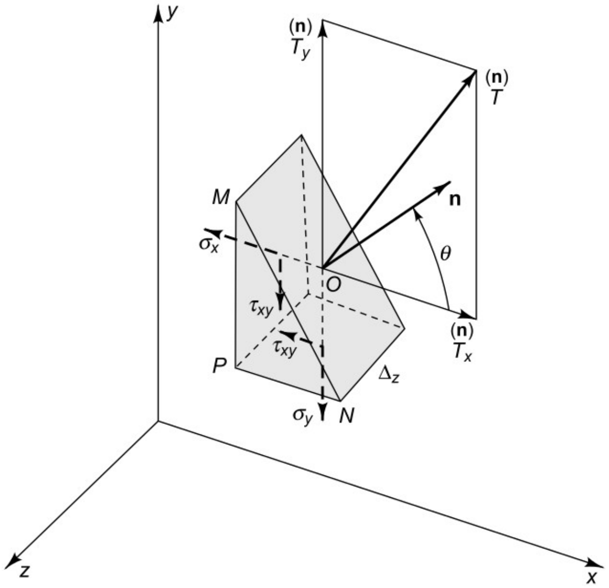
Stress components associated with arbitrarily oriented faces in plane stress

While deriving the stress components at a point corresponding to x, y and z faces, we discussed that stress components at arbitrary plane passing through O can be related to these components. Let us find the relations for plane stress case.



Let us assume that stress component at point O associated with x and y planes are σ_x, σ_y, and τ_{xy}. Consider an arbitrary plane having a normal $\boldsymbol{n}$ and which passes through O.

Consider the equilibrium of a wedge element which obtained when plane $\boldsymbol{n}$ slices the elemental cuboid around point O.



$$\sum F_x = t_x^{\boldsymbol{n}} \Delta z \overline{MN} - \sigma_{xx} \overline{MP} \Delta z - \tau_{xy} \Delta z \overline{PN} = 0,$$

$$\Rightarrow t_x^{\boldsymbol{n}} \Delta z \overline{MN} - \sigma_{xx} \overline{MN} \cos\theta - \tau_{xy} \Delta z \overline{MN} \sin\theta = 0,$$

$$\Rightarrow t_x^{\boldsymbol{n}} = \sigma_{xx} \cos\theta + \tau_{xy} \sin\theta. \qquad \cdots\cdots\cdots(18)$$

$$\sum F_y = t_y^{\boldsymbol{n}} \Delta z \overline{MN} - \sigma_{yy} \overline{PN} \Delta z - \tau_{xy} \Delta z \overline{MP} = 0,$$

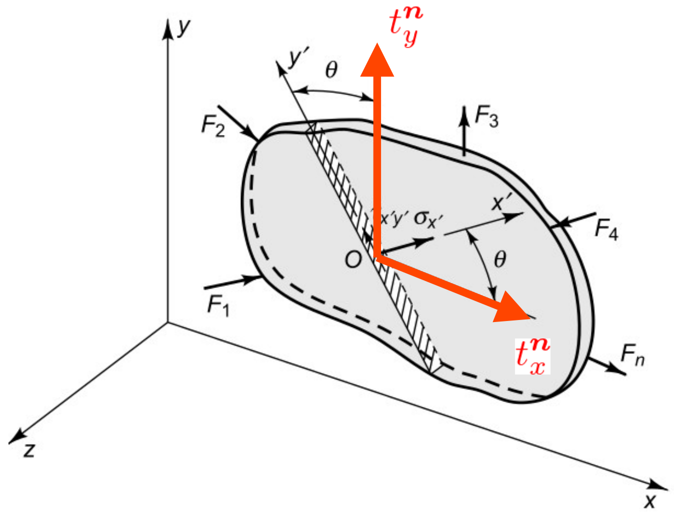
$$\Rightarrow t_y^{\boldsymbol{n}} \Delta z \overline{MN} - \sigma_{yy} \overline{MN} \sin\theta - \tau_{xy} \Delta z \overline{PN} \cos\theta = 0,$$

$$\Rightarrow t_y^{\boldsymbol{n}} = \sigma_{yy} \sin\theta + \tau_{xy} \cos\theta. \qquad \cdots\cdots\cdots(19)$$

Thus, (18) and (19) relates the stress components at $\boldsymbol{n}$–plane to stress components at x – and y – planes.

$$\begin{aligned} t_x^{\boldsymbol{n}} &= \sigma_{xx} \cos\theta + \tau_{xy} \sin\theta. \\ t_y^{\boldsymbol{n}} &= \sigma_{yy} \sin\theta + \tau_{xy} \cos\theta. \end{aligned} \qquad \cdots\cdots\cdots(20)$$

Components $t_x^{\boldsymbol{n}}$ and $t_y^{\boldsymbol{n}}$ directs along x- and y- coordinate axes. However, when talking about any plane, knowledge of stress components perpendicular and parallel to the plane (along x' and y') is important.



With the knowledge of $t_x^{\boldsymbol{n}}$ and $t_y^{\boldsymbol{n}}$, we can simply transform forces in x and y direction along x' and y' direction to find the stress components σ'_{xx} and τ'_{xy}.

Thus,

$$\sigma'_{xx}\Delta A = t_x^{\boldsymbol{n}}\Delta A\cos\theta + t_y^{\boldsymbol{n}}\Delta A\sin\theta,$$

$$\text{and } \tau'_{xy}\Delta A = -t_x^{\boldsymbol{n}}\Delta A\sin\theta + t_y^{\boldsymbol{n}}\Delta A\cos\theta. \qquad \text{..........(21)}$$

Substituting (20) in to (21) and simplifying, we get,

$$\sigma'_{xx} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta.$$

$$\tau'_{xy} = (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta). \qquad \text{..........(22)}$$

Another component σ'_{yy} can be determined by considering another plane having normal perpendicular to $\boldsymbol{n}$ and considering the equilibrium of wedge element near point O.

The same can be obtained by replacing θ with $(\theta+90^\circ)$ in (22). By doing that we get,

$$\sigma'_{yy} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta. \qquad \cdots\cdots\cdots\cdot(23)$$

To summarize, if stress components at a point in x-y coordinate system is known, then it is possible to know the stress components for all possible orientations of faces through the point using following equations. We say that we know the **state of stress** at the point.

$$\begin{aligned}
\sigma'_{xx} &= \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta. \\
\sigma'_{yy} &= \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta. \qquad \cdots\cdots\cdots\cdot(24) \\
\tau'_{xy} &= (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta).
\end{aligned}$$

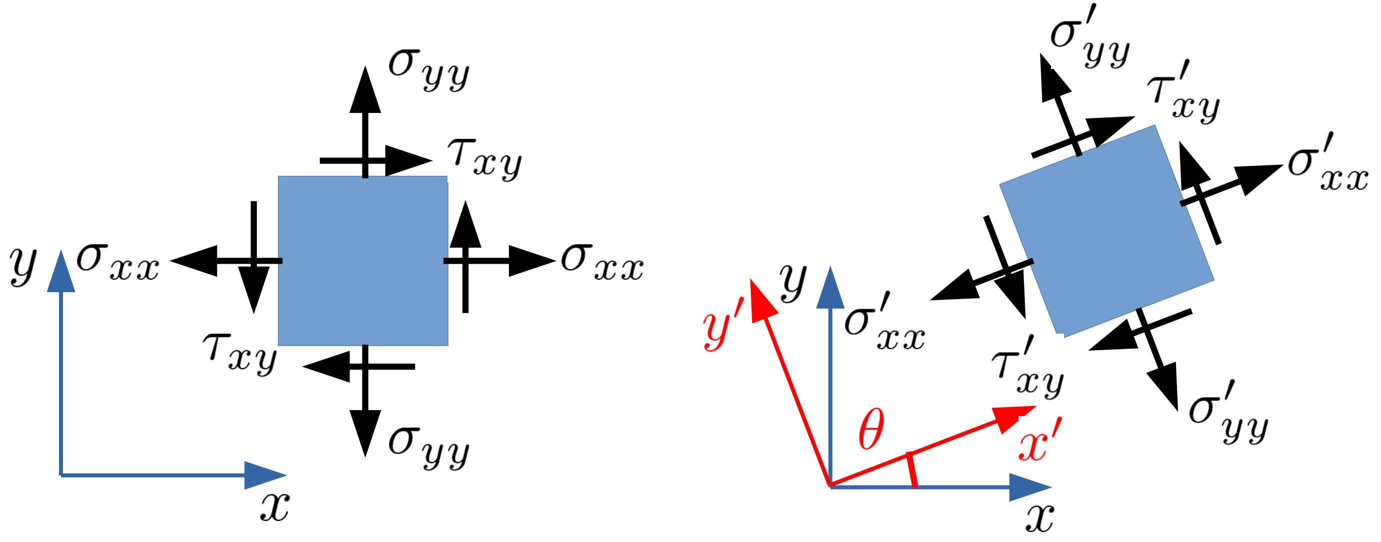
Be careful not to confuse a single stress component with the state of stress at the point.

Determining stress components in x'–y' coordinates system using known stress components in x–y direction is called **stress transformation.**

Using trigonometric identities (24) can be re-written as,

$$
\begin{aligned}
\sigma'_{xx} &= \frac{(\sigma_{xx}+\sigma_{yy})}{2} + \frac{(\sigma_{xx}-\sigma_{yy})}{2}\cos 2\theta + \tau_{xy}\sin 2\theta,\\
\sigma'_{yy} &= \frac{(\sigma_{xx}+\sigma_{yy})}{2} - \frac{(\sigma_{xx}-\sigma_{yy})}{2}\cos 2\theta - \tau_{xy}\sin 2\theta,\\
\tau'_{xy} &= \tau_{xy}\cos 2\theta - \frac{(\sigma_{xx}-\sigma_{yy})}{2}\sin 2\theta.
\end{aligned}
\qquad \cdots\cdots\cdots(25)
$$

Stress transformation can also be define using the transformation matrix as follow.



The transformation matrix is defined as,

$$[\boldsymbol{Q}] = \begin{bmatrix} \cos(x,x') & \cos(x,y') \\ \cos(y,x') & \cos(y,y') \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{..........(26)}$$

Now, the stress components in x'-y' coordinates is given as,

$$\begin{bmatrix} \sigma'_{xx} & \tau'_{xy} \\ \tau'_{yx} & \sigma'_{yy} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \quad \text{..........(27)}$$

Consider the expressions of σ'_{xx} and τ'_{xy} from (25). Eliminating θ from both the equations as,

$$(\sigma'_{xx} - \sigma_m)^2 + (\tau'_{xy})^2 = \left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2, \qquad \text{..........(28)}$$

where,

$$\sigma_m = \frac{\sigma_{xx} + \sigma_{yy}}{2}$$

This equation represents the equation of a circle, whose center is at,

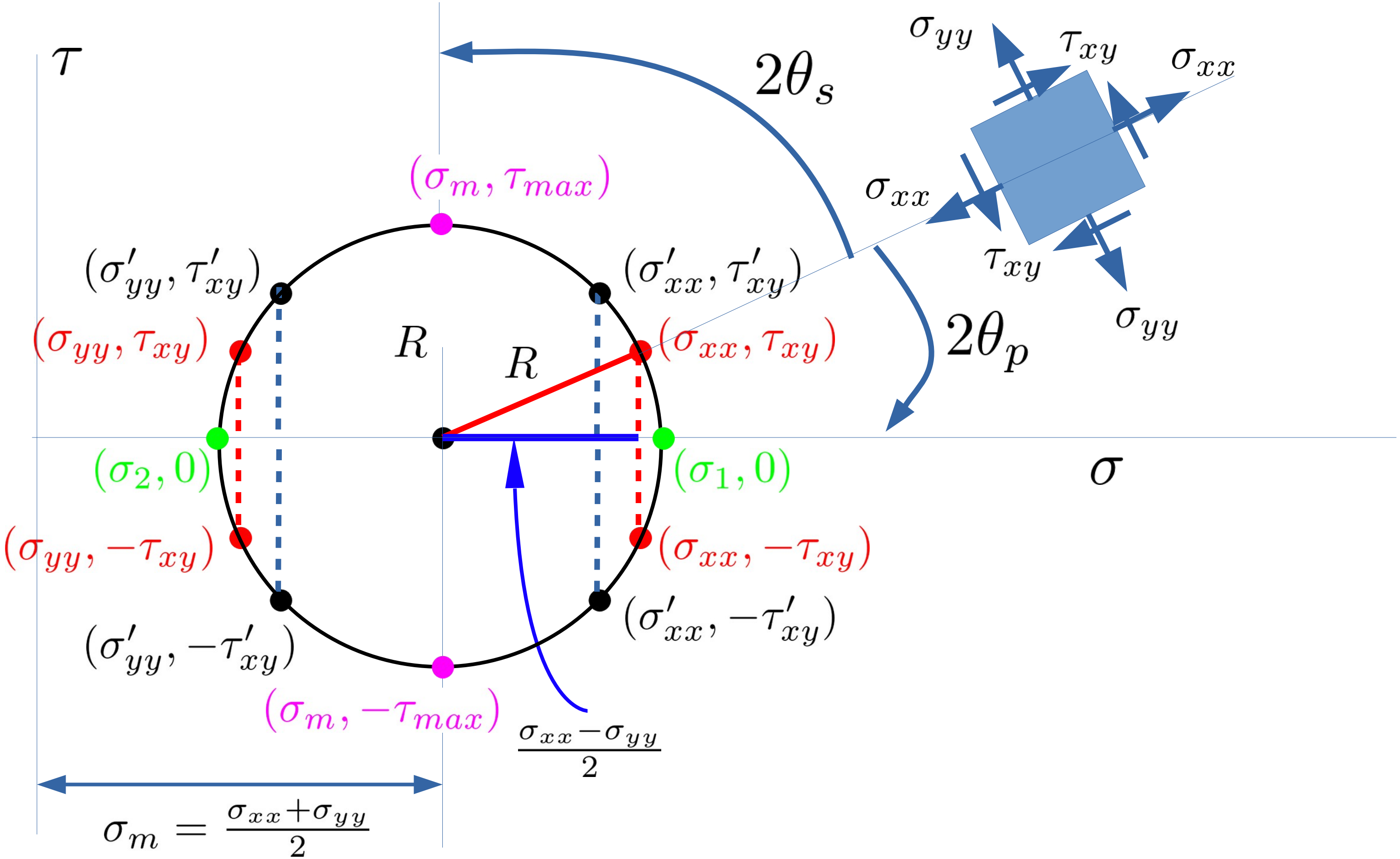
$$(\sigma_m, 0)\,,\ \text{and radius is } R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}. \qquad \text{..........(29)}$$

Points $(\sigma'_{xx}, \tau'_{xy})$ and $(\sigma'_{xx}, -\tau'_{xy})$ lie on this circle.

Similarly, considering equations for σ'_{yy} and τ'_{xy} and eliminating θ results in the equation representing the same circle and we see that points $(\sigma'_{yy}, \tau'_{xy})$ and $(\sigma'_{yy}, -\tau'_{xy})$ also lie on the same circle.

It should be observed that points (σ_{xx}, τ_{xy}), $(\sigma_{xx}, -\tau_{xy})$, (σ_{yy}, τ_{xy}), and $(\sigma_{yy}, -\tau_{xy})$, Also satisfy (27), i.e., all these points also lie on the same circle.

Mohr's circle: Graphical method



Principal stresses and maximum shear stress

- From the equations of transformed stresses, we see that the state of stress at a particular plane depends upon its inclination from x-axis (i.e. θ).
- From the equation of τ'_{xy}, we can find the plane at which shear stress will be zero.

$$\tau'_{xy} = \tau_{xy}\cos 2\theta - \frac{(\sigma_{xx} - \sigma_{yy})}{2}\sin 2\theta = 0,$$

$$\Rightarrow \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}} \qquad \text{..........(30)}$$

- Planes at which **shear stresses are zero** are called **principal planes** and the normal stresses at these planes are called **principal stresses**.
- It can also be seen that maximum value of shear stress is given as,

$$\tau'_{\max} = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}, \text{ at angle } \theta \text{ satisfying } \tan 2\theta_s = -\frac{\sigma_{xx} - \sigma_{yy}}{2\tau_{xy}}. \qquad \text{..........(31)}$$