ME632: Fracture Mechanics

Timings	
Monday	10:00 to 11:20
Thursday	08:30 to 09:50

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Westergaard's approach

There is another method to solve the stress and displacement field at the crack-tip. Westergaard (1939) gave a solution using complex variable approach. Let us first look into some of the basics of complex variable theory. An advantage complex variable offer is that it reduces the number of variable from two to one.

A complex variable is given as

$$z = x_1 + ix_2. \qquad \cdots\cdots\cdots\cdots\cdots(72)$$

In polar coordinate complex variable is $z = re^{i\theta}$.

The complex conjugate $\overline{z}$ of the variable z is $\overline{z} = x_1 - ix_2 = re^{i\theta}$. $\cdots\cdots\cdots\cdots\cdots(73)$

Using the definition of complex variable following differential operators can be defined.

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial z}\frac{\partial z}{\partial x_1} + \frac{\partial}{\partial \overline{z}}\frac{\partial \overline{z}}{\partial x_1} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}, \qquad \frac{\partial}{\partial z} = \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial z} + \frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right),$$

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial z}\frac{\partial z}{\partial x_2} + \frac{\partial}{\partial \overline{z}}\frac{\partial \overline{z}}{\partial x_2} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\right). \qquad \frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial \overline{z}} + \frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right).$$

$\cdots\cdots\cdots\cdots\cdots(74)$

A function of complex function variables may be written as

$$f(z) = f(x_1 + ix_2) = u(x_1, x_2) + iv(x_1, x_2), \qquad \cdots\cdots\cdots\cdots\cdots(75)$$

where $u(x_1, x_2) = \mathrm{Re}(f)$ and $v(x_1, x_2) = \mathrm{Im}(f)$.

A complex conjugate function is defined as,

$$\overline{f(z)} = \bar{f}(\bar{z}) = u(x_1, x_2) - iv(x_1, x_2), \qquad \cdots\cdots\cdots\cdots\cdots(76)$$

For e.g.,

$$f(z) = az + bz^2$$
$$\Rightarrow a(x_1 + ix_2) + b(x_1 + ix_2)^2$$
$$\Rightarrow (ax_1 + bx_1^2 - bx_2^2) + i(ax_2 + 2bx_1x_2).$$

Here,

$$u(x_1, x_2) = (ax_1 + bx_1^2 - bx_2^2),$$
$$v(x_1, x_2) = (ax_2 + 2bx_1x_2).$$

Thus, $f(z) = u(x_1, x_2) + iv(x_1, x_2)$.

$$\overline{f(z)} = \bar{f}(\bar{z}) = a\bar{z} + b\bar{z}^2$$
$$\Rightarrow a(x_1 - ix_2) + b(x_1 - ix_2)^2$$
$$\Rightarrow (ax_1 + bx_1^2 - bx_2^2) - i(ax_2 + 2bx_1x_2).$$

Thus,

$$\overline{f(z)} = u(x_1, x_2) - iv(x_1, x_2).$$

Differentiation of complex function are

$$f'(z) = \frac{\partial}{\partial z}(u + iv) = \frac{1}{2}\left(\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial x_2}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x_1} - \frac{\partial u}{\partial x_2}\right). \qquad \cdots\cdots\cdots\cdots\cdots(77)$$

Following the basic definition of differentiation, it can be shown that Cauchy-Riemann equations for analyticity of function f is

$$\frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_2}, \quad \frac{\partial u}{\partial x_2} = -\frac{\partial v}{\partial x_1}, \quad \frac{\partial}{\partial x_1}\mathrm{Re}\, f = \frac{\partial}{\partial x_2}\,\mathrm{Im}\, f, \quad \frac{\partial}{\partial x_2}\,\mathrm{Re}\, f = -\frac{\partial}{\partial x_1}\,\mathrm{Im}\, f, \qquad \cdots\cdots\cdots\cdots\cdots(78)$$

By simple differentiation of (78) we can show that,

$$\nabla^2 u = 0, \nabla^2 v = 0, \qquad \cdots\cdots\cdots\cdots\cdots(79)$$

Thus real and imaginary part of an analytic complex function must be a solution of Laplace's equation and hence they are harmonic functions.

We have already seen that solution of boundary value problems in elasticity can be obtained in the form of Airy stress function Φ and it must satisfy the biharmonic equation.

One way to solve the biharmonic equation is to represent Φ in terms of another complex functions. Westergaard suggested function Φ for mode-I and mode-II problems.

Solution of Mode-I crack

For mode-I crack Westergaard suggested Φ in the form of a complex function Z_I as,

$$\Phi = \mathrm{Re}\,\overline{\overline{Z}}_I + x_2\,\mathrm{Im}\,\overline{Z}_I, \qquad \cdots\cdots\cdots\cdots\cdots(80)$$

where $Z_I = \dfrac{\overline{Z}_I}{dz}$, and $\overline{Z}_I = \dfrac{\overline{\overline{Z}}_I}{dz}$.

Check that function Φ satisfies the Biharmonic equation.

Using (9) we can show that,

$$\begin{aligned} \sigma_{11} &= \mathrm{Re}\,Z_I - x_2\,\mathrm{Im}\,Z_I', \\ \sigma_{22} &= \mathrm{Re}\,Z_I + x_2\,\mathrm{Im}\,Z_I', \\ \sigma_{12} &= -x_2\,\mathrm{Re}\,Z_I'. \end{aligned} \qquad \cdots\cdots\cdots\cdots\cdots(81)$$

To solve a given problem, the proper form of the Westergaard function $Z_I(z)$ is chosen such that the stress components, determined through (81), satisfy all the boundary conditions. Once such a function is obtained, the stress field in the vicinity of the crack tip can be easily obtained. The Westergaard function does not solve a problem completely; it only aids in solving a problem. We still have to guess the form of the complex function Z_I in a specific problem.

To determine the displacement field u_1 and u_2 we first write strains in the form of complex functions using stress-strain relations (5) and (6). For plane stress case,

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = \frac{1}{E}\left[(\operatorname{Re} Z_I - x_2 \operatorname{Im} Z_I') - \nu\,(\operatorname{Re} Z_I + x_2 \operatorname{Im} Z_I')\right] = \frac{1}{2\mu}\left[\left(\frac{1-\nu}{1+\nu}\right)\operatorname{Re} Z_I - x_2 \operatorname{Im} Z_I'\right],$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = \frac{1}{E}\left[(\operatorname{Re} Z_I + x_2 \operatorname{Im} Z_I') - \nu\,(\operatorname{Re} Z_I - x_2 \operatorname{Im} Z_I')\right] = \frac{1}{2\mu}\left[\left(\frac{1-\nu}{1+\nu}\right)\operatorname{Re} Z_I + x_2 \operatorname{Im} Z_I'\right],$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = -\frac{x_2}{2\mu}\operatorname{Re} Z_I'. \qquad \cdots\cdots\cdots\cdots\cdots(82)$$

Now first and second equations of (82) can be integrated to determine u_1 and u_2 as

$$u_1 = \frac{1}{2\mu}\left[\left(\frac{1-\nu}{1+\nu}\right)\operatorname{Re} \overline{Z}_I - x_2 \operatorname{Im} Z_I'\right] + f(x_2),$$

$$u_2 = \frac{1}{2\mu}\left[\left(\frac{2}{1+\nu}\right)\operatorname{Im} \overline{Z}_I - x_2 \operatorname{Re} Z_I'\right] + g(x_1),$$

It can be shown that in (83) functions $f(x_2)$ and $g(x_1)$ contribute to rigid body displacement and rotation. Since they do not contribute to any strain or stress, they can be discarded safely.

Thus, for plane stress,

$$
\begin{aligned}
u_1 &= \frac{1}{2\mu}\left[\left(\frac{1-\nu}{1+\nu}\right)\mathrm{Re}\,\overline{Z}_I - x_2\,\mathrm{Im}\,Z_I'\right], \\
u_2 &= \frac{1}{2\mu}\left[\left(\frac{2}{1+\nu}\right)\mathrm{Im}\,\overline{Z}_I - x_2\,\mathrm{Re}\,Z_I'\right].
\end{aligned}
\qquad \cdots\cdots\cdots\cdots\cdots(83)
$$

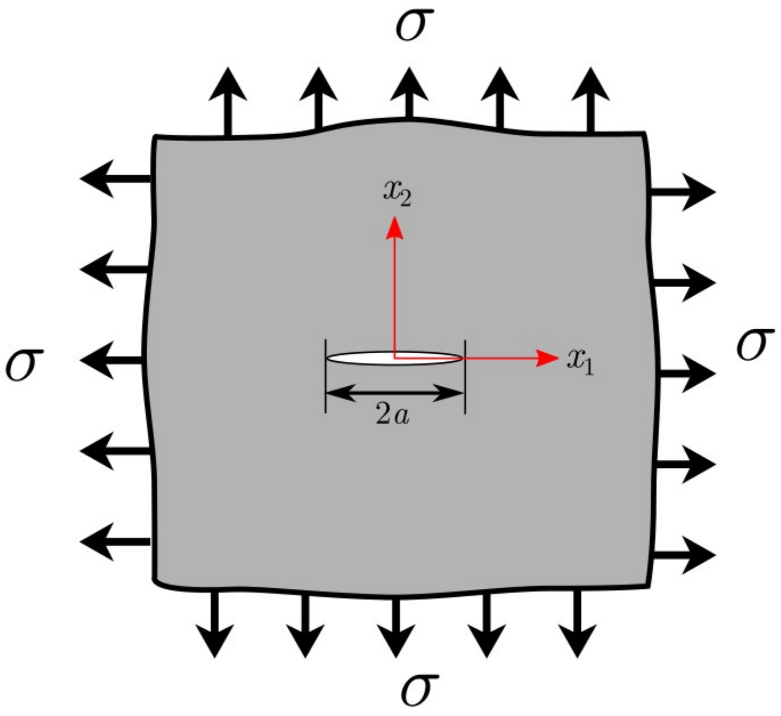
For plane strain,

$$
\begin{aligned}
u_1 &= \frac{1}{2\mu}\left[(1-2\nu)\,\mathrm{Re}\,\overline{Z}_I - x_2\,\mathrm{Im}\,Z_I'\right], \\
u_2 &= \frac{1}{2\mu}\left[2\,(1-\nu)\,\mathrm{Im}\,\overline{Z}_I - x_2\,\mathrm{Re}\,Z_I'\right].
\end{aligned}
\qquad \cdots\cdots\cdots\cdots\cdots(83a)
$$

So we have stress and displacement in terms of Westergaard Function Z_I, which is yet to be determined.

Now let us consider the case of infinite plate with through-thickness crack of length $2a$ under a biaxial field of stress σ. By infinite plate we mean that the exterior dimensions of the plate are much larger than the crack length.

The boundary conditions that should be satisfied by the Westergaard function are following:



(i) At the crack tip, (i.e., $x_1 = \pm a$)

$$\sigma_{22} = \infty.$$

(ii) On the crack surface, (i.e., $x_2 = 0, -a < x_1 < a$)

$$\sigma_{22} = \sigma_{12} = 0.$$

(iii) From away from the crack tip, (i.e., $|z| \to \infty$)

$$\sigma_{11} = \sigma_{22} = \sigma \text{ and } \sigma_{12} = 0.$$

Now while guessing the function Z_I above boundary conditions should be taken care by keeping in mind equations (81). Thus the following function is suitable,

$$Z_I(z) = \frac{\sigma z}{\sqrt{(z-a)(z+a)}} = \frac{\sigma z}{\sqrt{(z^2 - a^2)}}. \quad \cdots\cdots\cdots\cdots(84)$$

Now, let us determine the stress field near the crack tip. To do that it is convenient to transform the origin from the center of the crack-tip by substituting $z = a + z_0$, where z_0 is measured from the crack-tip. Thus Z_I becomes,

$$Z_I(z_0) = \frac{\sigma(z_0 + a)}{\sqrt{z_0(z_0 + 2a)}} = \frac{\sigma a(1 + z_0/a)}{\sqrt{2az_0}\sqrt{1 + z_0/2a}} = \frac{\sigma\sqrt{a}(1 + z_0/a)}{\sqrt{2z_0}\sqrt{1 + z_0/2a}}. \quad \cdots\cdots\cdots\cdots\cdots(85)$$

In the vicinity of the crack-tip $|z_0| \ll a$, then

$$Z_I(z_0) \simeq \frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi z_0}}. \quad \cdots\cdots\cdots\cdots\cdots(86)$$

Similarly from (81), we can also determine Z_I' in the vicinity of the crack-tip,

$$Z_I'(z_0) \simeq -\frac{\sigma a^2}{(2az_0)^{3/2}}. \quad \cdots\cdots\cdots\cdots\cdots(87)$$

Writing z_0 in polar coordinate as $z_0 = r\left(\cos\theta + i\sin\theta\right)$, we get,

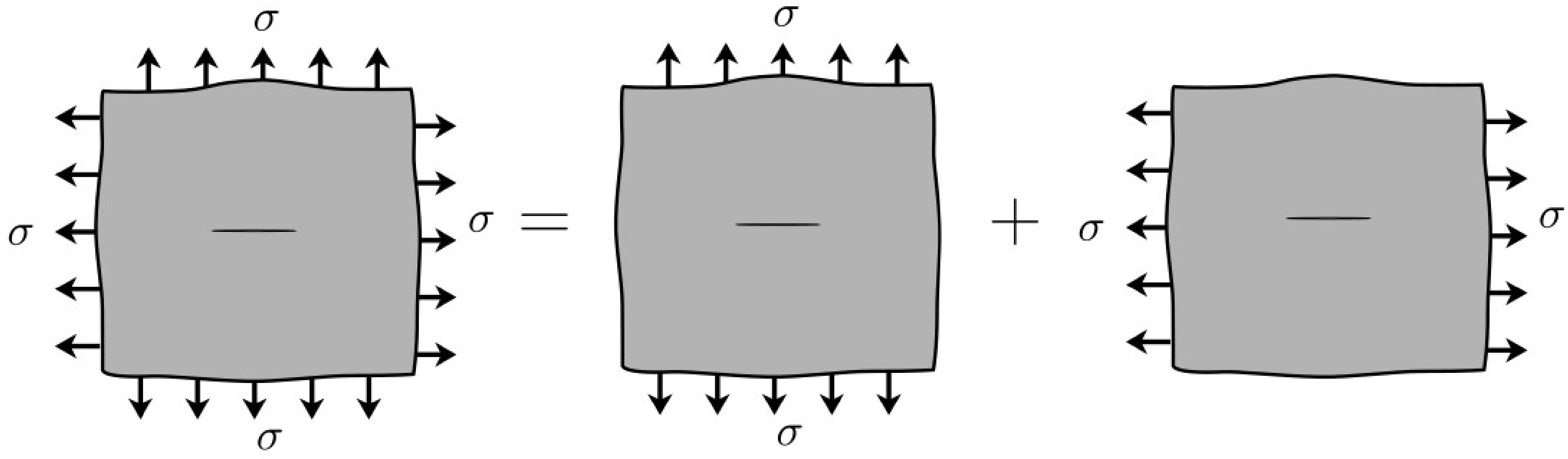
$$Z_I = \frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi r}}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right), \qquad Z_I' = \frac{\sigma\sqrt{\pi a}}{2\sqrt{2\pi}r^{3/2}}\left(\cos\frac{3\theta}{2} - i\sin\frac{3\theta}{2}\right). \quad \cdots\cdots\cdots\cdots\cdots(88)$$

Using (88) in (81), we can now determine the stress components,

$$\sigma_{11} = \frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right),$$
$$\sigma_{11} = \frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right), \qquad \cdots\cdots\cdots\cdots\cdots(89)$$
$$\sigma_{12} = \frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi r}} \sin\theta\sin\frac{3\theta}{2}\cos\frac{3\theta}{2}.$$

Comparing (89) will (49) it can be observed that, for an infinite body having through-thickness crack with far-field stress σ stress intensity factor is $K_I = \sigma\sqrt{\pi a}$.

It may be argued that comparing (89) and (49) is not valid, as both are not the same configuration. However, we will justify the comparison by showing that the solution of biaxial loading is not different from the mode-I loading.



Since we are working with small deformations and linear elasticity, principal of superposition is applicable. The biaxial loading can be assumed as the superposition of two configurations,

(i) loading perpendicular to the crack,

(ii) loading parallel to the crack.

Observe that the configuration (i) is nothing but mode-I which tries to open the crack, whereas configuration (ii) does not try to open the crack. Though it modifies the stress-field near the crack-tip to a certain extent. However, the solution of configuration (ii) is not simple and we usually neglect its effect in engineering solutions to fracture mechanics.

Therefore, for most of the practical purposes, the solution of a biaxially loaded plate is employed for both uniaxially and biaxially loaded problems.

Solution for the displacement can be obtained by calculating $\overline{Z_I}$ from (86) and substituting (83) as,

(for plane stress)

$$u_1 = \frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}\left(\frac{1-\nu}{1+\nu}+\sin^2\frac{\theta}{2}\right),$$
$$u_2 = \frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left(\frac{2}{1+\nu}-\cos^2\frac{\theta}{2}\right). \qquad \cdots\cdots\cdots\cdots(90)$$

(for plane strain)

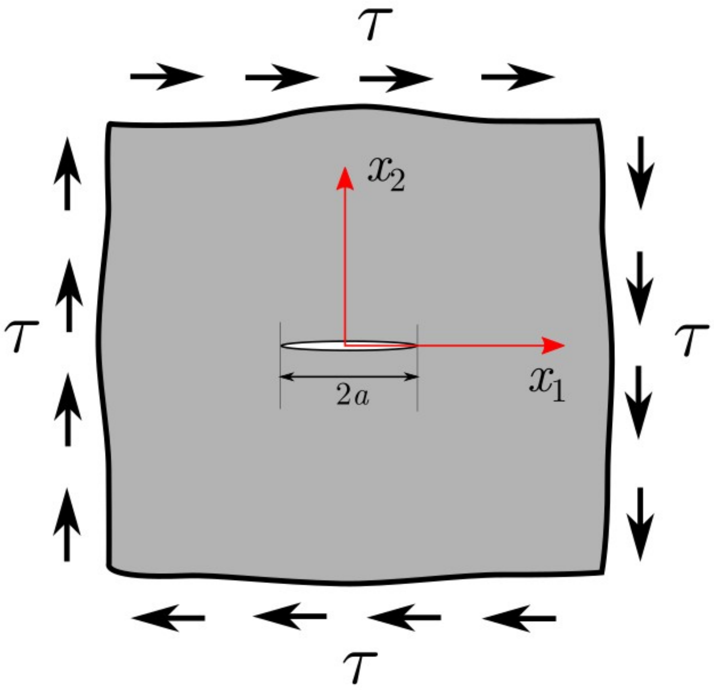
$$u_1 = \frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}\left(1-2\nu+\sin^2\frac{\theta}{2}\right),$$
$$u_2 = \frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left(2-2\nu-\cos^2\frac{\theta}{2}\right). \qquad \cdots\cdots\cdots\cdots(90a)$$

The distance between the two crack faces is as the Crack Opening Displacement (COD) and it is useful from experimental point of view. Thus for plane stress equations (83),

$$\mathrm{COD} = 2\times u_2|_{x_2=0} = \frac{1}{\mu}\frac{2}{1-\nu}(\mathrm{Im}\, Z_I')_{x_2=0}. \qquad \cdots\cdots\cdots\cdots(91)$$

Using (84), we get, $\mathrm{COD} = \dfrac{4\sigma}{E}\sqrt{a^2-x_1^2}$, with $(\mathrm{COD})_{\max} = \dfrac{4\sigma a}{E}$, at $x_1 = 0$. $\cdots\cdots\cdots\cdots(92)$

For Mode-II crack



For an infinite plate with center-crack subjected to Mode II loading the following expression of the Airy Stress Function Φ is suggested:

$$\Phi = -x_2 \operatorname{Re} \overline{Z}_{II}.$$

Stress field is given as,

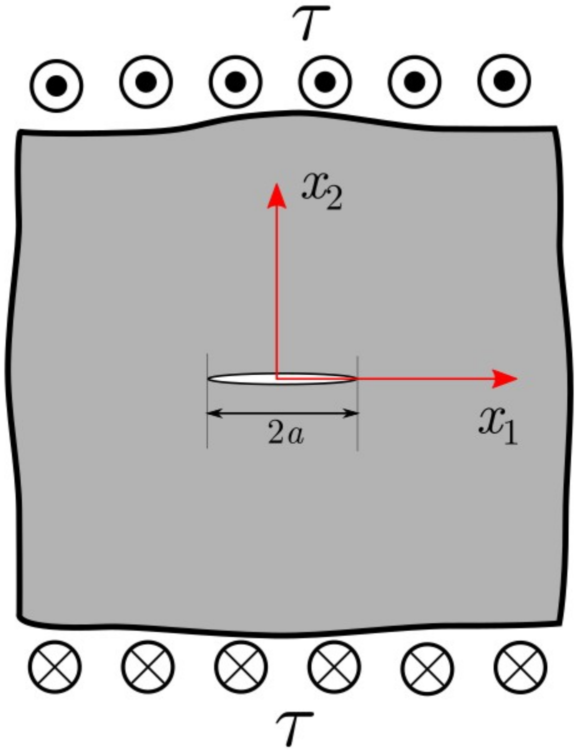
$$\sigma_{11} = 2 \operatorname{Im} Z_{II} + x_2 \operatorname{Re} Z'_{II},$$
$$\sigma_{22} = -x_2 \operatorname{Re} Z'_{II},$$
$$\sigma_{12} = \operatorname{Re} Z_{II} - x_2 \operatorname{Im} Z'_{II}.$$

Displacement field for plane stress and plane strain can be derived.

Westergaard function for Mode-II is

$$Z_{II} = \frac{\tau z}{\sqrt{z^2 - a^2}}.$$

For Mode-III crack



For mode-III crack, out-of-plane displacement u_z is defined in terms of complex functions. For an infinite plate with center-crack subjected to Mode III loading the following expression of u_z is suggested:

$$u_z = \frac{1}{\mu} \operatorname{Im} Z_{III}.$$

Non-zero stress components are given as,

$$\sigma_{13} = \operatorname{Im} Z'_{III},$$

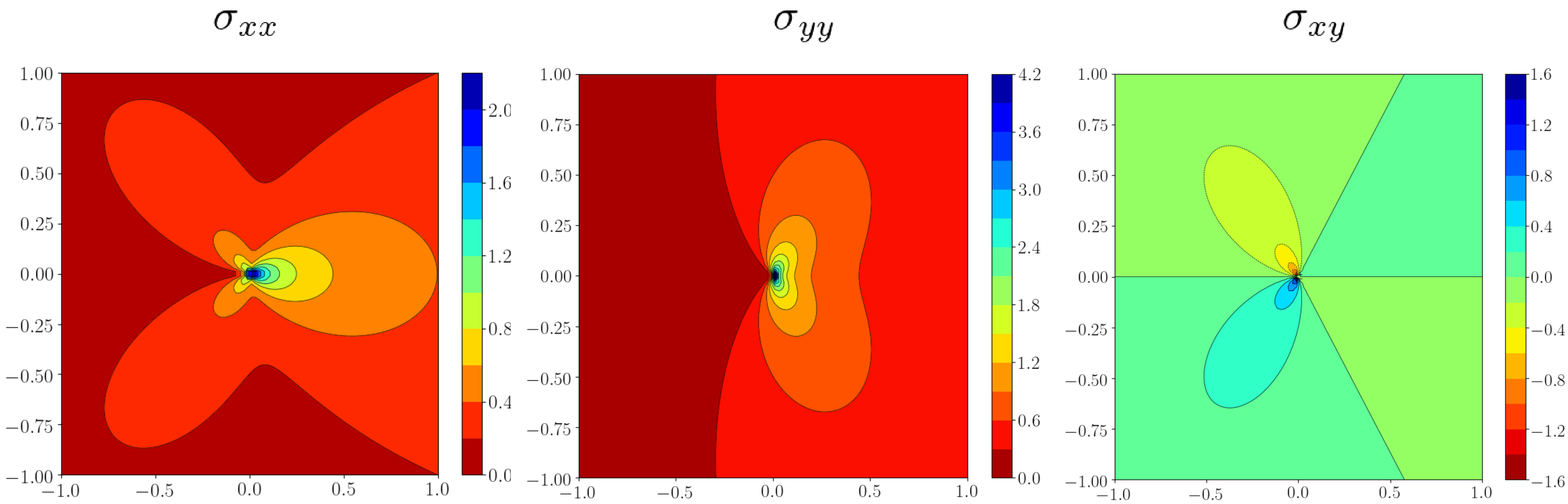
$$\sigma_{23} = \operatorname{Re} Z'_{III}.$$

Subsequently displacement field can be derived.

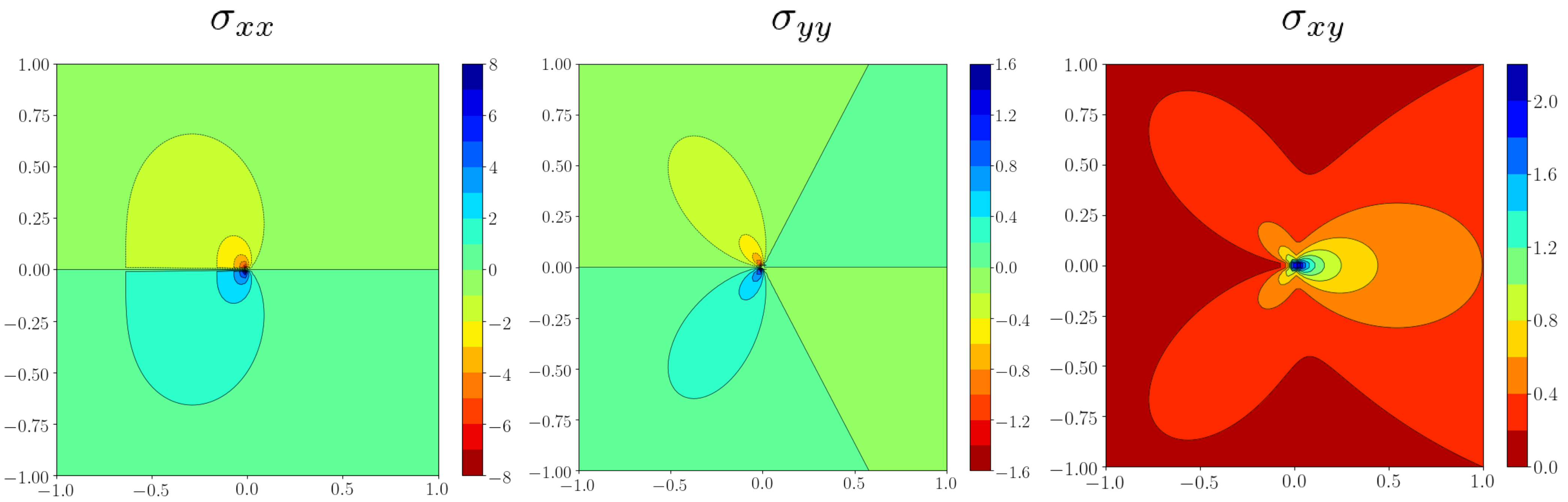
Westergaard function for Mode-III is

$$Z'_{III} = \frac{\tau z}{\sqrt{z^2 - a^2}}.$$

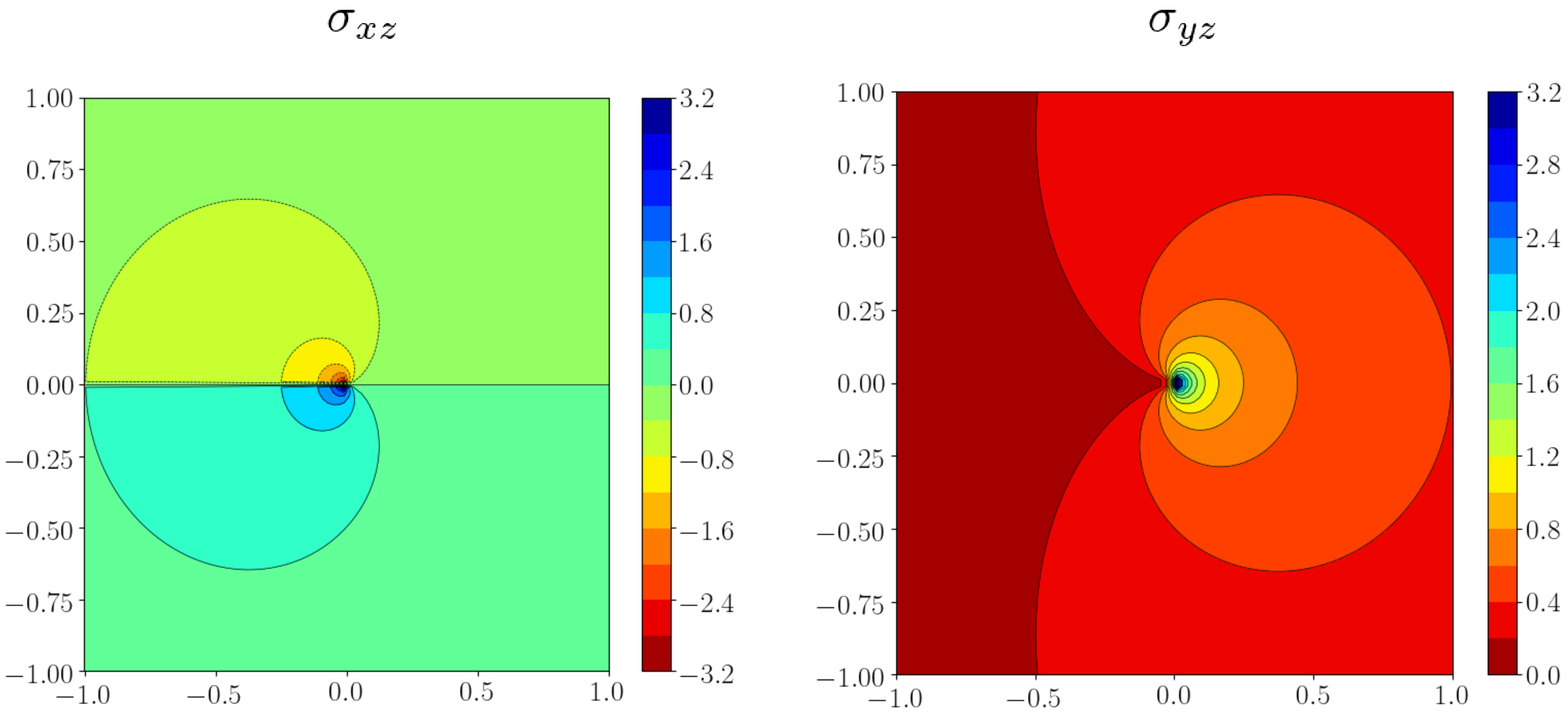
Mode-I stress fields



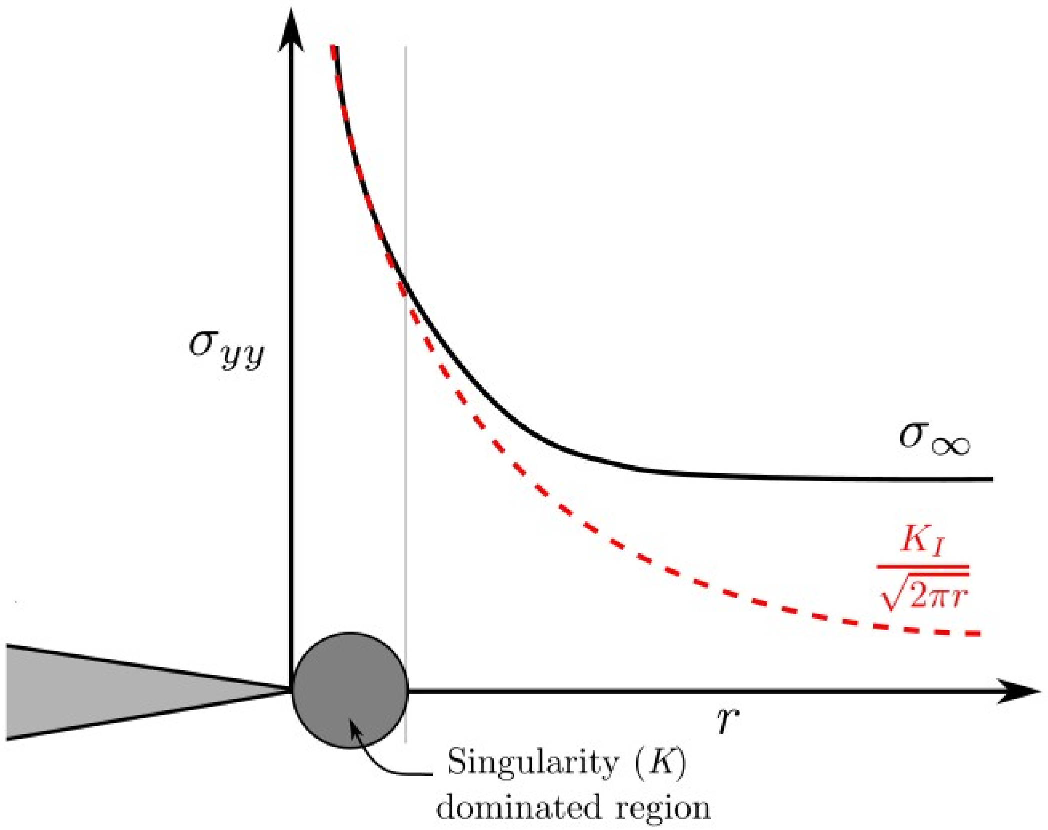
Mode-II stress fields



Mode-III stress fields



Singularity or K-dominant region



Critical stress intensity factor (SIF)

- So we have seen that the stress intensity factor (SIF) defines the amplitude of singularity at the crack-tip.
- We have also seen that SIF is a function of applied stress (σ) and crack length (a).
- For a given crack length (a) crack starts growing at a critical applied stress (σ_c). The SIF corresponding to (a and σ_c) is critical stress intensity factor K_C.
- Similarly for a given applied stress (σ) crack become unstable after a critical crack length (a_c). Again the stress intensity factor corresponding to (σ and a_c) is critical stress intensity factor K_C.
- Critical stress intensity factor K_{IC}, K_{IIC}, K_{IIIC} is material property.
- We can now define a new fracture based design criteria as $K_{I/II/III} \leq K_{IC/IIC/IIIC}$.
- For the case of a mixed crack-tip loading by K_I , K_{II} , and K_{III}, then a generalized fracture criterion $f(K_{IC}, K_{IIC}, K_{IIIC})=0$ must be defined.
- For such a design criteria to follow, we should be able to calculate SIF corresponding to any configuration.