ME232: Dynamics

3D dynamics of rigid bodies

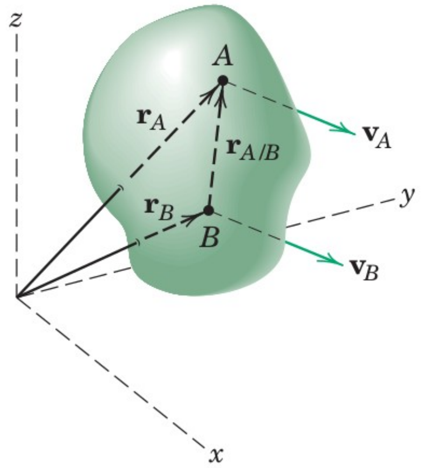
Anshul Faye
afaye@iitbhilai.ac.in
Room # 106

Introduction

- Although a large percentage of dynamics problems in engineering can be solved by the principles of plane motion, modern developments have focused increasing attention on problems which call for the analysis of motion in three dimensions.
- Inclusion of the third dimension **adds considerable complexity** to the kinematic and kinetic relationships.
- Introduction of third dimension not only add a third component to vectors such as force, linear velocity, linear acceleration, and linear momentum, but it also adds the **possibility of two additional components for vectors representing angular quantities** including moments of forces, angular velocity, angular acceleration, and angular momentum. Thus in three-dimensional motion application of vector analysis helps a lot.

Kinematics

Translation

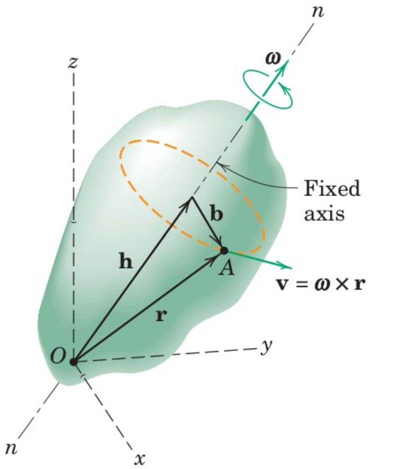


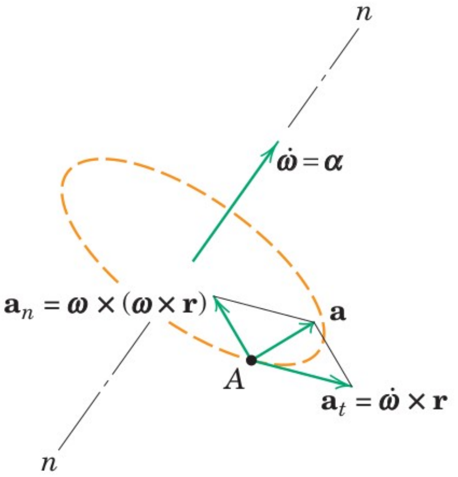
$$\boldsymbol{r}_A = \boldsymbol{r}_B + \boldsymbol{r}_{A/B}$$
$$\boldsymbol{v}_A = \boldsymbol{v}_B$$
$$\boldsymbol{a}_A = \boldsymbol{a}_B$$

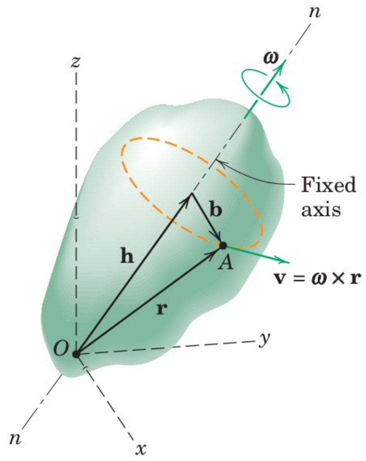
Fixed-axis rotation

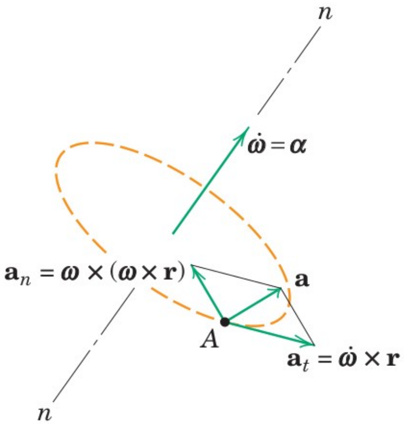
Consider the rotation of a rigid body about a fixed axis n-n in space with an angular velocity ω. $\boldsymbol{\omega}$ is a vector in the direction of the rotation axis with a sense established by the right-hand rule. For fixed-axis rotation, $\boldsymbol{\omega}$ does not change its direction since it lies along the axis. We choose the origin O of the fixed coordinate system on the rotation axis for convenience. Any point such as A which is not on the axis moves in a circular arc in a plane normal to the axis and has a velocity

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} = \boldsymbol{\omega} \times (\boldsymbol{b} + \boldsymbol{h}) = \boldsymbol{\omega} \times \boldsymbol{b} \quad \cdots\cdots\cdots\cdots\cdots (1)$$









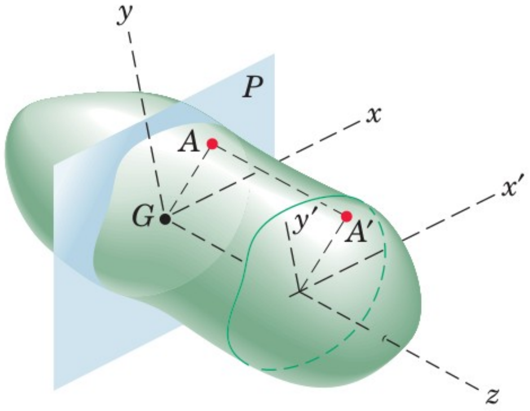
Acceleration of A is given by the time derivative of (1) as,

$$\boldsymbol{a} = \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \quad \cdots\cdots\cdots\cdots\cdots(2)$$

Normal and tangential component of acceleration are shown in the figure.

Parallel-plane motion

When all points in a rigid body moves in a plane P, we have a general plane motion. The reference plane is customarily taken through the mass center G and is call the plane of motion. Because each point in the body has a motion identical with the motion of corresponding point in the plane P, it follows that the kinematics of plane motion completely describes the motion of body.



Rotation about a fixed point

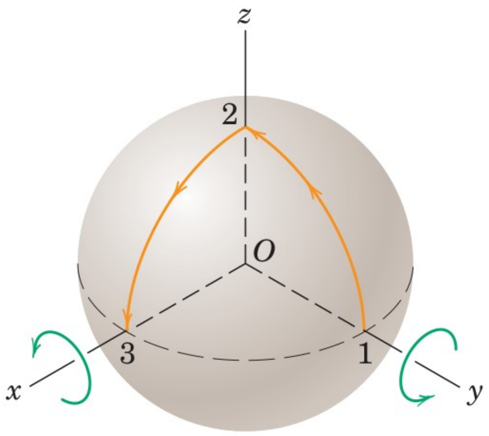
When a body rotates about a fixed point, the angular-velocity vector no longer remains fixed in direction, and this change calls for a more general concept of rotation.

Rotation and Proper vectors:

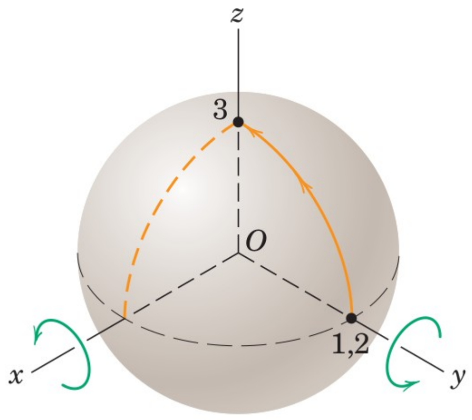
Consider a solid sphere which is cut from a rigid body confined to rotate about a fixed point O. The x-y-z axes here are taken as fixed in space and do not rotate with the body.

Now, consider two successive 90° rotations of the sphere, first about the x-axis and, second about the y-axis, which result in the motion of point 1 to point 2 and then to point 3, successively.

Now reverse the order of rotation, i.e. first rotate about y-axis and then about x-axis. We see that for the two cases do not produce the same final position.



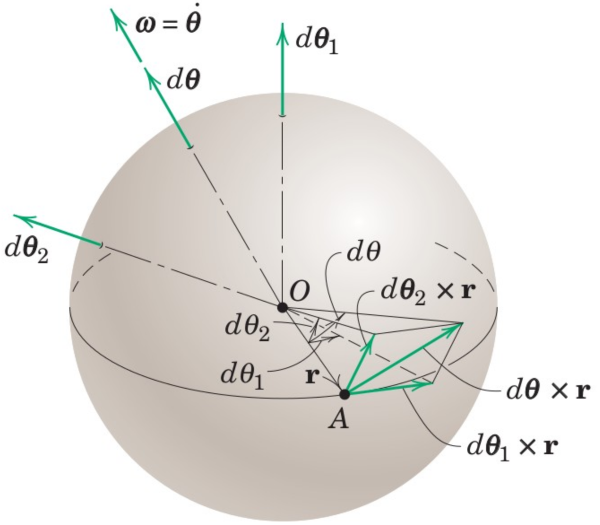
θ_x then θ_y



θ_y then θ_x

The previous example shows that **finite rotations do not generally obey the parallelogram law of vector addition and are not commutative.** Thus, finite rotations may not be treated as vectors.

However, **infinitesimal rotations do obey the parallelogram law of vector addition.** Figure shows the combined effect of two infinitesimal rotations d_1 and d_2 of a rigid body about the respective axes through the fixed point O. As a result either order of infinitesimal rotations clearly produce the same resultant displacement, which is $d_1 \times \boldsymbol{r} + d_2 \times \boldsymbol{r}$. Thus, the two rotations are equivalent to the single rotation $d = d_1 + d_2$. It follows that the angular velocities $\boldsymbol{\omega}_1 = \dot{\boldsymbol{\theta}}_1$ and $\boldsymbol{\omega}_2 = \dot{\boldsymbol{\theta}}_2$ may be added vectorially to give $\boldsymbol{\omega} = \dot{\boldsymbol{\theta}} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$.

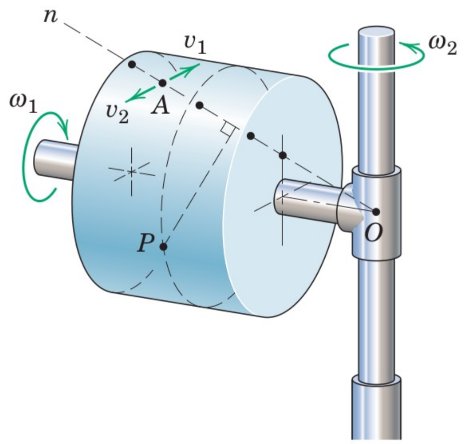


Therefore it concludes that **at any instant of time a body with one fixed point is rotating instantaneously about a particular axis passing through the fixed point.**

Instantaneous axis of rotation

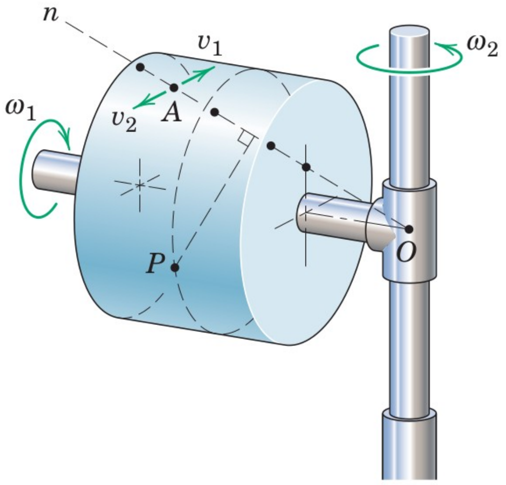
Consider a solid cylindrical rotor made of clear plastic containing many black particles embedded in the plastic. The rotor is spinning about its shaft axis at constant angular velocity ω_1, and its shaft, in turn, is rotating about the fixed vertical axis at with constant angular velocity ω_2.

If the rotor is photographed at a certain instant during its motion, the resulting picture would show **one line of black dots sharply defined**, indicating that, momentarily, **their velocity was zero**. This line of points with no velocity defines the instantaneous position of the axis of rotation O-n. Any dot on this line, such as A, would have **equal and opposite velocity components, v_1 due to ω_1 and v_2 due to ω_2**.



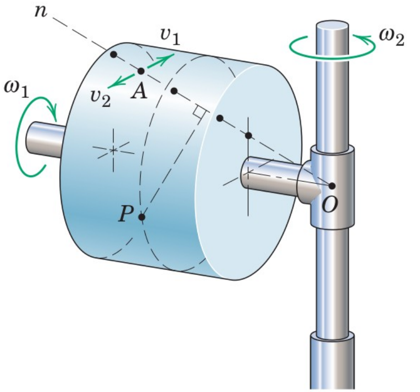
All other dots, such as the one at P, would appear blurred, and their movements would show as short streaks in the form of small circular arcs in planes normal to the axis O-n. Thus, all such particles of the body, are **momentarily rotating in circular arcs about the instantaneous axis of rotation**.

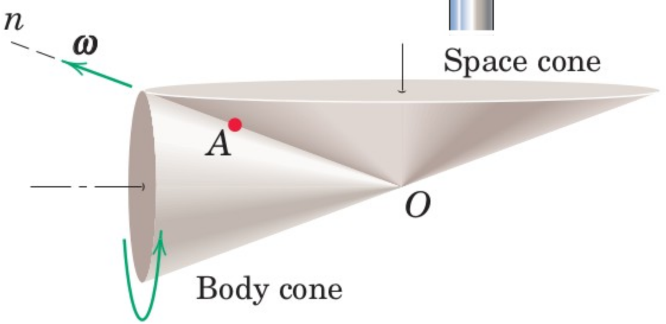
If a succession of photographs were taken, we would observe in each photograph that the rotation axis would be defined by a new series of sharply-defined dots and that the axis would change position both in space and relative to the body. For rotation of a rigid body about a fixed point, then, it is seen that the rotation axis is, in general, not a line fixed in the body.



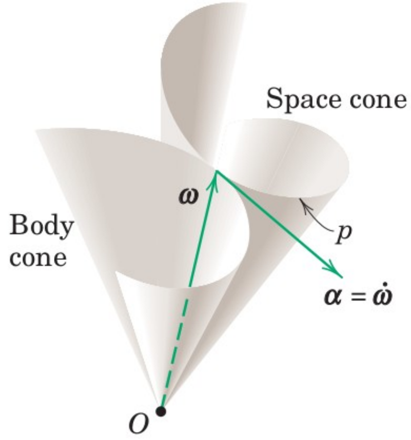
Body and space cones:

Relative to the plastic cylinder, the instantaneous axis of rotation O-A-n generates a right-circular cone about the cylinder axis called the **body cone**. As the two rotations continue and the cylinder swings around the vertical axis, the instantaneous axis of rotation also generates a right-circular cone about the vertical axis called the **space cone**. These cones are shown for this particular example.





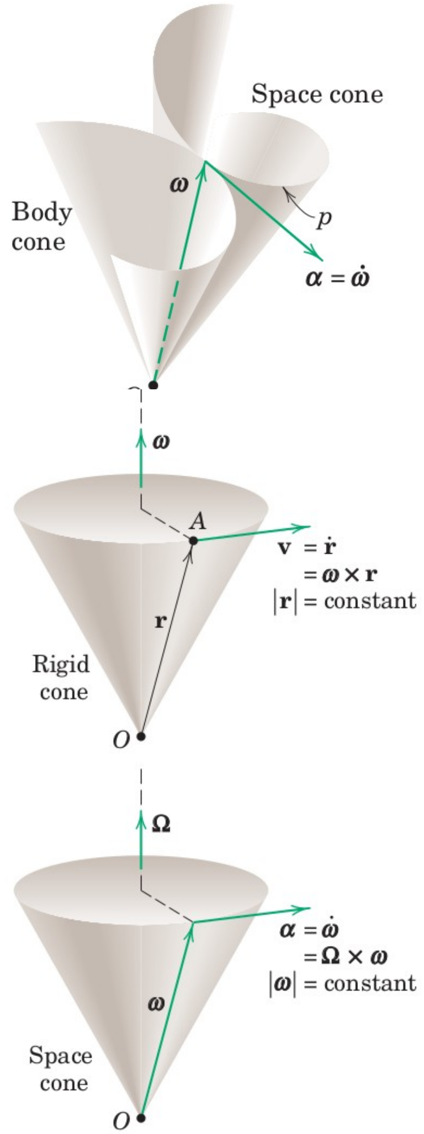
We see that the body cone rolls on the space cone and that the **angular velocity ω of the body is a vector which lies along the common element of the two cones.** For a more general case where the rotations are not steady, the space and body cones are not right-circular cones but the body cone still rolls on the space cone.

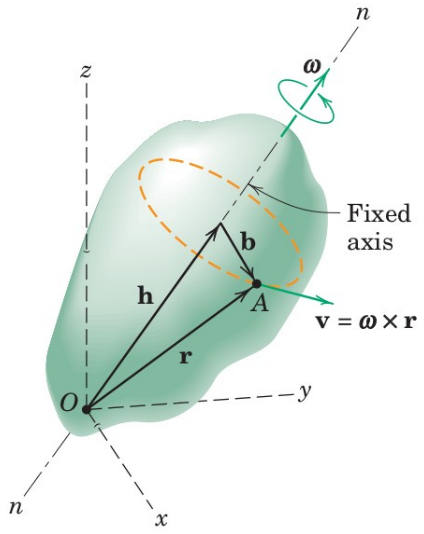


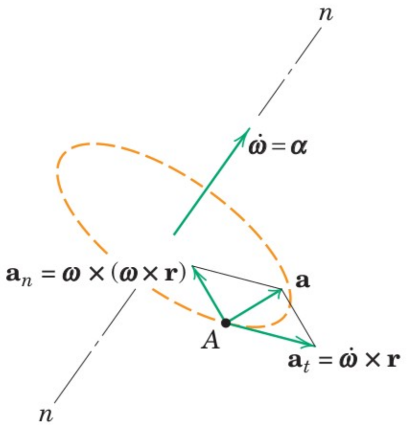
Angular accelerations:

The angular acceleration $\boldsymbol{\alpha}$ of a rigid body in three-dimensional motion is the time derivative of its angular velocity, $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$. In contrast to the case of rotation in a single plane where the scalar α measures only the change in magnitude of $\boldsymbol{\omega}$, in three-dimensional motion **$\boldsymbol{\alpha}$ reflects the change in direction of $\boldsymbol{\omega}$ as well as its change in magnitude**. Thus as the angular velocity vector $\boldsymbol{\omega}$ follows the space curve p and changes in both magnitude and direction, the angular acceleration $\boldsymbol{\alpha}$ becomes a vector tangent to this curve in the direction of the change in $\boldsymbol{\omega}$.

When the magnitude of $\boldsymbol{\omega}$ remains constant, the angular acceleration $\boldsymbol{\alpha}$ is normal to $\boldsymbol{\omega}$. For this case, if we let $\boldsymbol{\Omega}$ stand for the angular velocity with which the vector $\boldsymbol{\omega}$ itself rotates (*precesses*) as it forms the space cone, the angular acceleration may be written $\boldsymbol{\alpha} = \boldsymbol{\Omega} \times \boldsymbol{\omega}$.







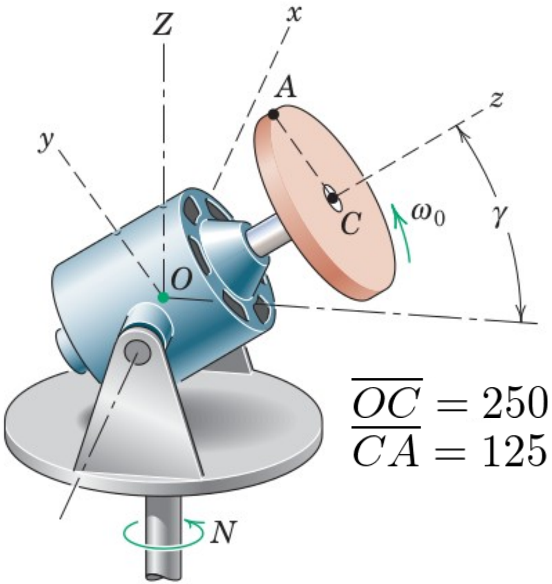
If the figure represent a rigid body rotating about a fixed point O with the instantaneous axis of rotation n-n, we see that the velocity $\boldsymbol{v}$ and acceleration $\boldsymbol{a} = \dot{\boldsymbol{v}}$ of any point A in the body are given by the same expressions as apply to the case in which the axis is fixed.

The one difference between the case of rotation about a fixed axis and rotation about a fixed point lies in the fact that for rotation about a fixed point, the angular accelera$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$ will have **a component normal to ω due to the change in direction of ω**, as well as **a component in the direction of ω to reflect any change in the magnitude of ω**. Although any point on the rotation axis n-n momentarily will have zero velocity, it will not have zero acceleration as long as $\boldsymbol{\omega}$ is changing its direction.

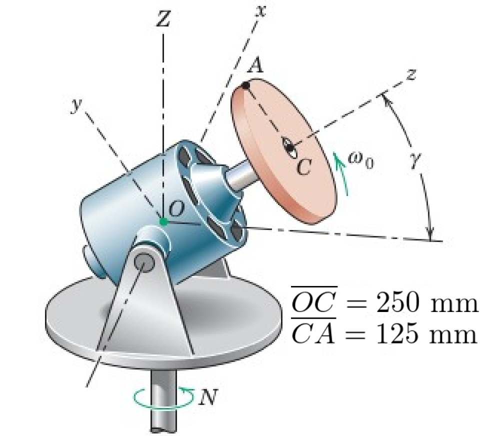
On the other hand, for rotation about a fixed axis, $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$ has **only the one component along the fixed axis to reflect the change in the magnitude of ω.** Furthermore, points which lie on the fixed rotation axis clearly have no velocity or acceleration.

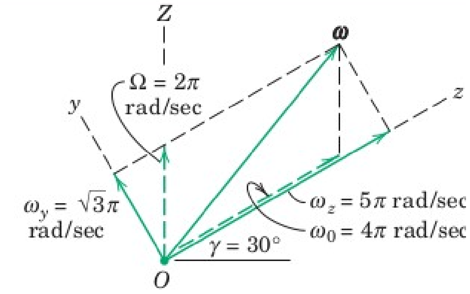
Example 1

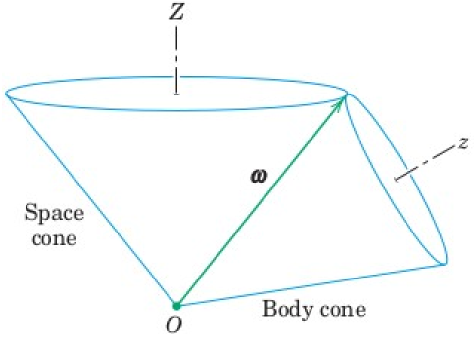
The electric motor with an attached disk is running at a constant low speed of 120 rpm in the direction shown. Its housing and mounting base are initially at rest. The entire assembly is next set in rotation about the vertical Z-axis at the constant rate $N = 60$ rpm with a fixed angle γ of 30°. Determine (a) the angular velocity and angular acceleration of the disk, (b) the space and body cones, and (c) the velocity and acceleration of point A at the top of the disk for the instant shown.



$\overline{OC} = 250$ mm
$\overline{CA} = 125$ mm







The axes x-y-z with unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are attached to the motor frame, with the z-axis coinciding with the rotor axis and the x-axis coinciding with the horizontal axis through O about which the motor tilts. The Z-axis is vertical and carries the unit vector $\boldsymbol{K} = \boldsymbol{j} \cos \gamma + \boldsymbol{k} \sin \gamma$.

(a) The rotor and disk have two components of angular velocity: $\omega_0 = 120(2\pi)/60 = 4\pi$ rad/sec about the z-axis and $\Omega = 60(2\pi)/60 = 2\pi$ rad/sec about the Z-axis. Thus, the angular velocity becomes

$$\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \boldsymbol{\Omega} = \omega_0 \boldsymbol{k} + \Omega \boldsymbol{K}$$

$$\boldsymbol{\omega} = \omega_0 \boldsymbol{k} + \Omega(\boldsymbol{j} \cos \gamma + \boldsymbol{k} \sin \gamma) = \pi(\sqrt{3}\boldsymbol{j} + 5\boldsymbol{k}) \text{ rad/s}$$

The angular acceleration of the disk

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \boldsymbol{\Omega} \times \boldsymbol{\omega} = 68.4\boldsymbol{i} \text{ rad/s}^2$$

(c) The position vector of point A for the instant considered is

$$\boldsymbol{r} = 0.125\boldsymbol{j} + 0.250\boldsymbol{k} \text{ m}$$

Velocity of point A,

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} = -0.1920\pi\boldsymbol{i} \text{ m/s}$$

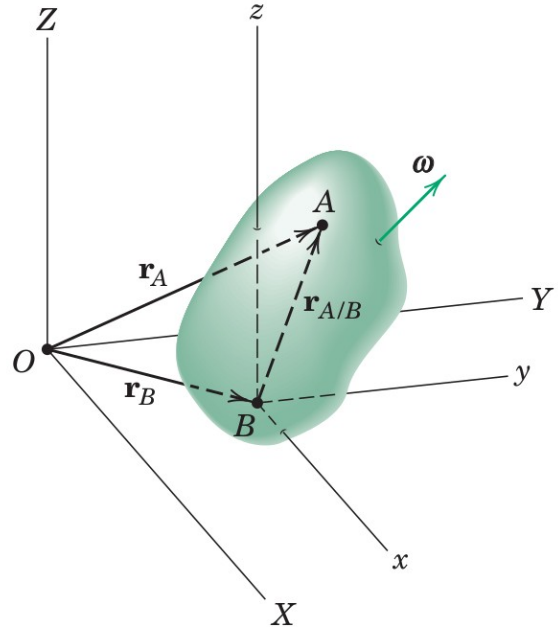
Acceleration of point A,

$$\boldsymbol{a} = \boldsymbol{\alpha} \times \boldsymbol{r} + \boldsymbol{\omega} \times \boldsymbol{v} = -26.6\boldsymbol{j} + 11.83\boldsymbol{k} \text{ m/s}^2$$

General motion

Translating reference axis:

Figure shows a rigid body which has an angular velocity $\boldsymbol{\omega}$. We may choose any convenient point B as the origin of a translating reference system x-y-z. The velocity $\boldsymbol{v}$ and acceleration $\boldsymbol{a}$ of any other point A in the body are given by the relative-velocity and relative-acceleration expressions, which were developed for plane motion of rigid bodies.



$$\boldsymbol{v}_A = \boldsymbol{v}_B + \boldsymbol{v}_{A/B}$$
$$\boldsymbol{a}_A = \boldsymbol{a}_B + \boldsymbol{a}_{A/B}$$

As already discussed earlier that for rigid-body motion in space the distance AB remains constant. Thus, from an observer's position on x-y-z, the body appears to rotate about the point B and point A appears to lie on a spherical surface with B as the center. Consequently, we may view the general motion as a translation of the body with the motion of B plus a rotation of the body about B.

The relative-motion terms represent the effect of the rotation about B and are identical to the velocity and acceleration expressions discussed for rotation of a rigid body about a fixed point. Therefore, the relative-velocity and relative-acceleration equations may be written

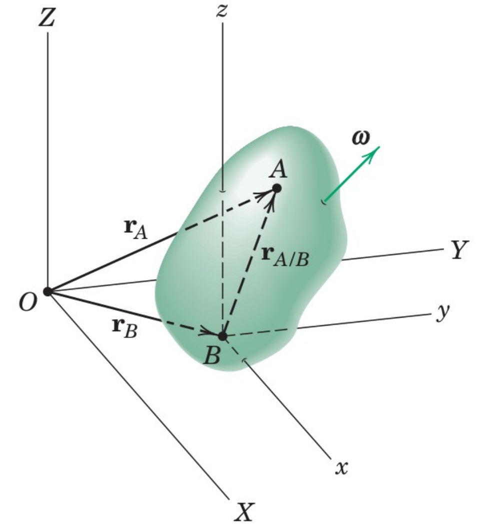
$$\begin{aligned} \boldsymbol{v}_A &= \boldsymbol{v}_B + \boldsymbol{\omega} \times \boldsymbol{r}_{A/B} \\ \boldsymbol{a}_A &= \boldsymbol{a}_B + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_{A/B} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{A/B}) \end{aligned} \qquad \text{...............(3)}$$

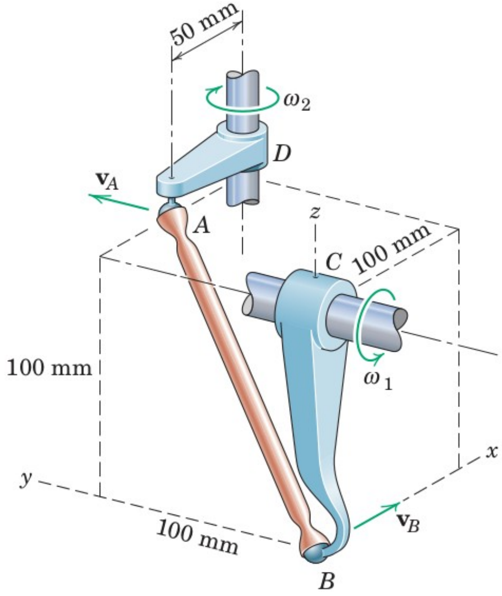
where $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\omega}}$ are the instantaneous angular velocity and angular acceleration of the body, respectively.

The selection of the reference point B is quite arbitrary in theory. In practice, point B is chosen for convenience as some point in the body whose motion is known in whole or in part. If point A is chosen as the reference point, the relative-motion equations will be written accordingly.

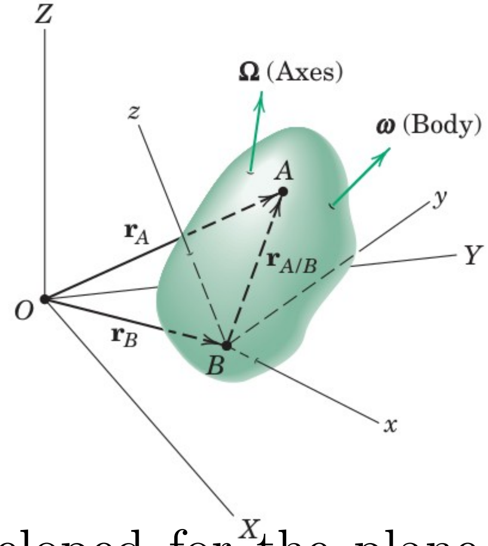
If points A and B represent the ends of a rigid control link in a spatial mechanism where the end connections act as ball-and-socket joints, it is necessary to impose certain kinematic requirements. Clearly, any rotation of the link about its own axis AB does not affect the action of the link. Thus, the angular velocity $\boldsymbol{\omega}_n$ whose vector is normal to the link describes its action. It is necessary, therefore, that $\boldsymbol{\omega}_n$ and $\boldsymbol{r}_{A/B}$ be at right angles, and this condition is satisfied if $\boldsymbol{\omega}_n \cdot \boldsymbol{r}_{A/B} = 0.$
Similarly, it is only the component $\boldsymbol{\alpha}_n$ of the angular acceleration of the link normal to AB which affects its action, so that must also hold.

$$\boldsymbol{\alpha}_n \cdot \boldsymbol{r}_{A/B} = 0$$





Rotating reference axis:



A more general formulation of the motion of a rigid body in space calls for the use of reference axes which rotate as well as translate. The reference axes whose origin is attached to the reference point B rotate with an absolute angular velocity $\boldsymbol{\Omega}$ which may be different from the absolute angular velocity $\boldsymbol{\omega}$ of the body.

Motion of the body can be described using the expressions developed for the plane motion of a rigid body with the use of rotating axes. The extension of these relations from two to three dimensions is easily accomplished by merely including the z-component of the vectors. The expressions for the velocity and acceleration of point A become

$$\boldsymbol{v}_A = \boldsymbol{v}_B + \boldsymbol{\Omega} \times \boldsymbol{r}_{A/B} + \boldsymbol{v}_{\text{rel}}$$

$$\boldsymbol{a}_A = \boldsymbol{a}_B + \dot{\boldsymbol{\Omega}} \times \boldsymbol{r}_{A/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}_{A/B}) + 2\boldsymbol{\Omega} \times \boldsymbol{v}_{\text{rel}} + \boldsymbol{a}_{\text{rel}} \qquad \text{...............(4)}$$

where $\boldsymbol{v}_{\text{rel}} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k}$ and $\boldsymbol{a}_{\text{rel}} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{z}\boldsymbol{k}$ are, respectively, the velocity and acceleration of point A measured relative to x-y-z by an observer attached to x-y-z.

Note that $\boldsymbol{r}_{A/B}$ remains constant in magnitude for points A and B fixed to a rigid body, but it will change direction with respect to x-y-z when the angular velocity $\boldsymbol{\Omega}$ of the axes is different from the angular velocity $\boldsymbol{\omega}$ of the body. Further, if x-y-z are rigidly attached to the body, $\boldsymbol{\Omega} = \boldsymbol{\omega}$ and $\boldsymbol{v}_{\text{rel}}$ and $\boldsymbol{a}_{\text{rel}}$ are both zero, which makes the equations (4) identical to (3).

The relationship developed earlier between the time derivative of a vector $\boldsymbol{V}$ as measured in the fixed X-Y system and the time derivative of $\boldsymbol{V}$ as measured relative to the rotating x-y system. For our three-dimensional case, this relation becomes

$$\left(\frac{d\boldsymbol{V}}{dt}\right)_{XYZ} = \left(\frac{d\boldsymbol{V}}{dt}\right)_{xyz} + \boldsymbol{\Omega} \times \boldsymbol{V}. \qquad \cdots\cdots\cdots\cdots\cdots(5)$$

Using the above transformation to the expression of relative position vector, and velocity vector expressions (4) can be derived.

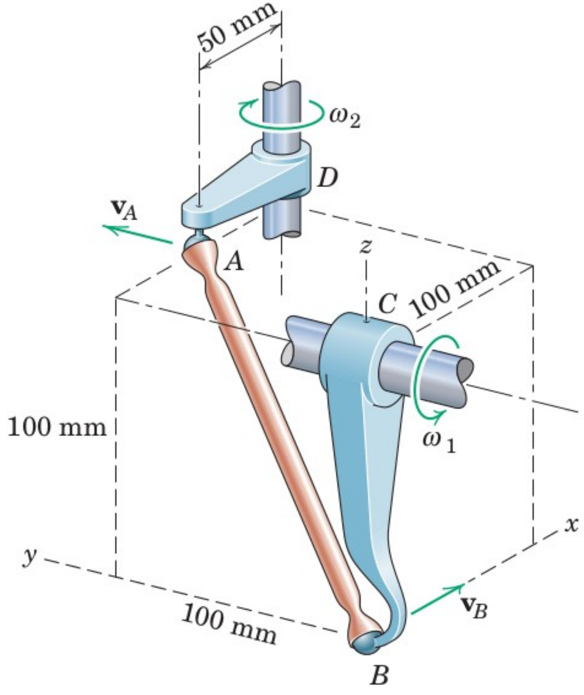
Example 2

Crank CB rotates about the horizontal axis with an angular velocity $\omega_1 = 6$ rad/s which is constant for a short interval of motion which includes the position shown. The link AB has a ball-and-socket fitting on each end and connects crank DA with CB. For the instant shown,

(a) determine the angular velocity ω_2 of crank DA and the angular velocity $\boldsymbol{\omega}_n$ of link AB.

(b) determine the angular acceleration of crank AD. Also find the angular acceleration of link A$\dot{\omega}_2$

$\dot{\boldsymbol{\omega}}_n$



The relative-velocity relation will be solved first using translating reference axes attached to B. The equation is

$$\boldsymbol{v}_A = \boldsymbol{v}_B + \boldsymbol{\omega}_n \times \boldsymbol{r}_{A/B} \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\text{(a)}$$

where $\boldsymbol{\omega}_n$ is the angular velocity of link AB taken normal to AB. The velocities of A and B are

$$\boldsymbol{v}_A = 50\omega_2\,\boldsymbol{j}, \qquad \boldsymbol{v}_B = 100(6)\ \boldsymbol{i} = 600\ \boldsymbol{i}\ \text{mm/s}.$$

Also $\boldsymbol{r}_{A/B} = 50\,\boldsymbol{i} + 100\,\boldsymbol{j} + 100\,\boldsymbol{k}$ mm.

After substituting in (a) we get,

$$50\omega_2\,\boldsymbol{j} = 600\ \boldsymbol{i} + (\omega_{nx}\boldsymbol{i} + \omega_{ny}\boldsymbol{j} + \omega_{nz}\boldsymbol{k}) \times (50\,\boldsymbol{i} + 100\,\boldsymbol{j} + 100\,\boldsymbol{k})$$

Comparing both side, we get following equations

$$-6 = \omega_{ny} - \omega_{nz}$$

$$\omega_2 = -2\omega_{nx} + \omega_{nz} \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\text{(b)}$$

$$0 = -2\omega_{nx} - \omega_{ny}$$

Above equations can be solved to get $\omega_2 = 6$ rad/s.

To determine ω_n, we incorporate additional condition, i.e. $\boldsymbol{\omega}_n \cdot \boldsymbol{r}_{A/B} = 0$.

Using this condition with (b), $\boldsymbol{\omega}_n$ can be determined as, $\boldsymbol{\omega}_n = 3\,(-2\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k})$ rad/s.

The accelerations of the links may be found from the following equation

$$\boldsymbol{a}_A = \boldsymbol{a}_B + \dot{\boldsymbol{\omega}}_n \times \boldsymbol{r}_{A/B} + \boldsymbol{\omega}_n \times (\boldsymbol{\omega}_n \times \boldsymbol{r}_{A/B}) \qquad \cdots\cdots\cdots\cdots\cdots\cdots(c)$$

the accelerations of A and B are

$$\begin{aligned} \boldsymbol{a}_A &= 50\omega_2^2\,\boldsymbol{i} + 50\dot{\omega}_2\,\boldsymbol{j} = 1800\,\boldsymbol{i} + \dot{\omega}_2\,\boldsymbol{j} \text{ mm/s}^2 \\ \boldsymbol{a}_B &= 100\omega_1^2\,\boldsymbol{k} = 3600\,\boldsymbol{k} \text{ mm/s}^2 \end{aligned} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(d)$$

Also,

$$\begin{aligned} \boldsymbol{\omega}_n \times (\boldsymbol{\omega}_n \times \boldsymbol{r}_{A/B}) &= 20(50\,\boldsymbol{i} + 100\,\boldsymbol{j} + 100\,\boldsymbol{k}) \text{ mm/s}^2 \\ \dot{\boldsymbol{\omega}}_n \times \boldsymbol{r}_{A/B} &= (100\dot{\omega}_{ny} - 100\dot{\omega}_{nz})\boldsymbol{i} + (50\dot{\omega}_{nz} - 100\dot{\omega}_{nx})\boldsymbol{j} + (100\dot{\omega}_{nx} - 50\dot{\omega}_{ny})\boldsymbol{k} \end{aligned} \qquad \cdots\cdots(e)$$

Substituting (e) and (d) in (c), we get following equations,

$$\begin{aligned} 28 &= \dot{\omega}_{ny} - \dot{\omega}_{nz} \\ \dot{\omega}_2 + 40 &= 2\dot{\omega}_{nx} + \dot{\omega}_{nz} \\ -32 &= 2\dot{\omega}_{nx} - \dot{\omega}_{ny} \end{aligned} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(f)$$

From above equations we can determine, $\dot{\omega}_2 = -36 \text{ rad/s}^2$.

The vector $\dot{\boldsymbol{\omega}}_n$ is normal to $\boldsymbol{r}_{A/B}$ but is not normal to $\boldsymbol{v}_{A/B}$, as was the case with $\boldsymbol{\omega}_n$.
Imposing additional condition, i.e. $\dot{\boldsymbol{\omega}}_n \cdot \boldsymbol{r}_{A/B} = 0$.
Using this condition along with (f), we can determine,

$$\dot{\boldsymbol{\omega}}_n = 4(2\boldsymbol{i} + 4\boldsymbol{j} - 3\boldsymbol{k}) \text{ rad/s}^2.$$