ME232: Dynamics

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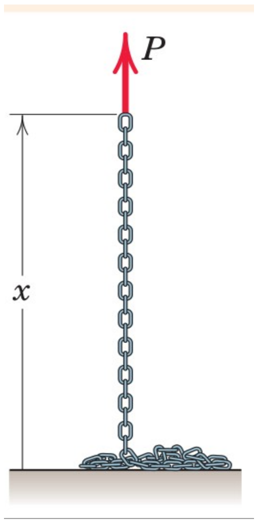
Room # 106

Example 5

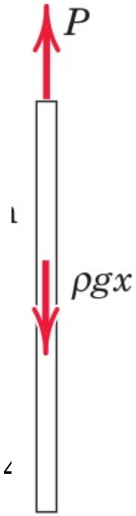


The end of a chain of length L and mass per unit length which is piled on a platform is lifted vertically with a constant velocity v by a variable force P. Find P as a function of the height x of the end above the platform.

Variable-Mass Approach:

We apply the equation of motion to the moving part of the chain of length x which is gaining mass. The force summation ΣF includes all forces acting on the moving part except the force exerted by the particles which are being attached. From the diagram we have

$$\Sigma F_x = P - \rho g x$$



The velocity is constant so that $\dot{v} = 0$. The rate of increase of mass is $\dot{m} = \rho v$, and the relative velocity with which the attaching particles approach the moving part is $u = v - 0 = v$. Thus,

$$[\Sigma F = m\dot{v} + \dot{m}u], \qquad P - \rho g x = 0 + \rho v(v), \qquad P = \rho(gx + \mathrm{v}^2)$$

The force P consists of the two parts, (i) $\rho g x$, the weight of the moving part of the chain, and (ii) ρv^2, the force required to change the momentum of the links on the platform from a condition at rest to a velocity v.

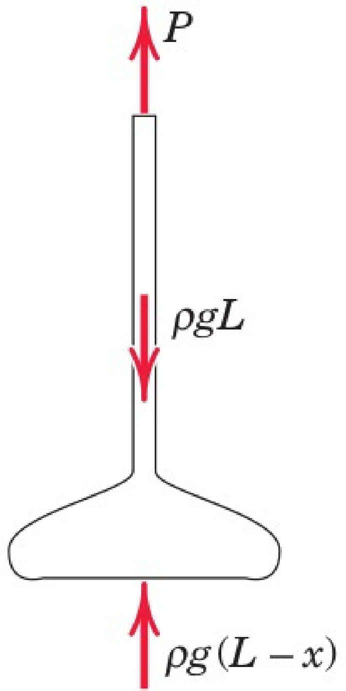
Constant-Mass Approach:

The principle of impulse and momentum for a system of particles can be applied to the entire chain considered as the system of constant mass. The free-body diagram of the system shows the unknown force P, the total weight of all links $\rho g L$, and the force $\rho g(L - x)$ exerted by the platform on those links which are at rest on it. The momentum of the system at any position is $G_x = \rho x v$ and the momentum equation gives

$$\Sigma F_x = dG_x / dt, \qquad P + \rho g(L - x) - \rho g L = d(\rho x v)/dt,$$

$$P = \rho(gx + v^2)$$

Again the force P is seen to be equal to the weight of the portion of the chain which is off the platform plus the added term which accounts for the time rate of increase of momentum of the chain.



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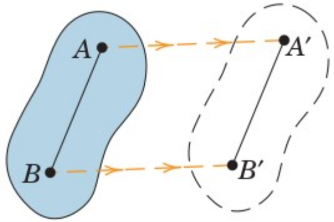
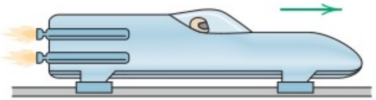
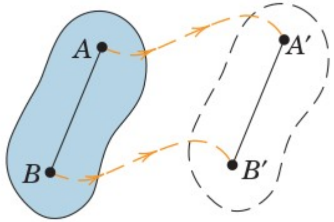
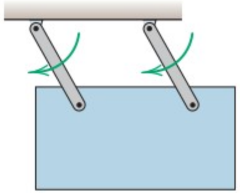
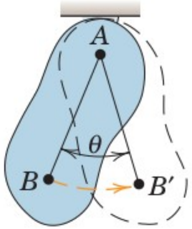
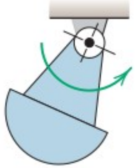
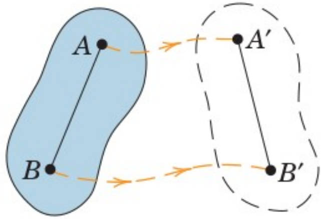
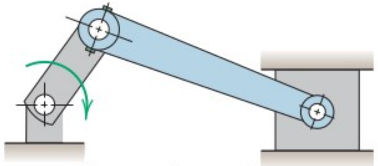
Introduction

- In rigid-body kinematics we use all the relationships derived for kinematics of particles and system of particles, in addition to accounting for the **rotational motion of the body**. Thus rigid-body kinematics **involves both linear and angular displacements, velocities, and accelerations**.
- The motion of rigid bodies need to be described for two important reasons.
  - Need to generate, transmit, or control certain motions by the use of cams, gears, and linkages of various types. Here we must analyze the displacement, velocity, and acceleration of the motion to determine the design geometry of the mechanical parts. Furthermore, as a result of the motion generated, forces may be developed which must be accounted for in the design of the parts.
  - Determine the motion of a rigid body caused by the forces applied to it. Calculation of the motion of a rocket under the influence of its thrust and gravitational attraction is an example of such a problem.

Rigid body assumption and Plane motion

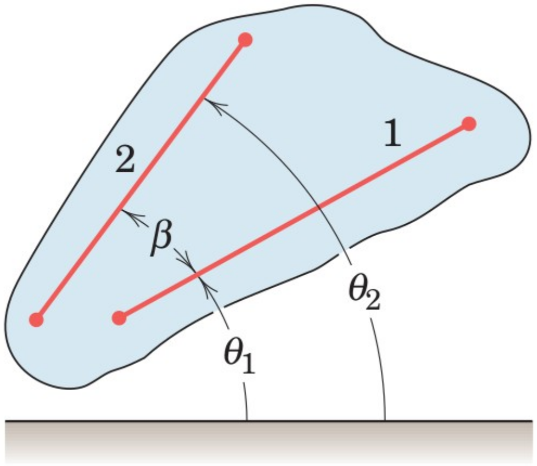
- A rigid body is a system of particles for which the **distances between the particles remain unchanged**. Thus, if each particle of such a body is located by a position vector from reference axes attached to and rotating with the body, there will be no change in any position vector as measured from these axes.
- A rigid body executes plane motion when all parts of the body move in parallel planes. For convenience, we generally consider the plane of motion to be the **plane which contains the mass center**, and the body is treated as a thin slab whose motion is confined to the plane of the slab. This idealization adequately describes a very large category of rigid-body motions encountered in engineering.

Plane motion

	Type of Rigid-Body Plane Motion	Example
(*a*) Rectilinear translation		 Rocket test sled
(*b*) Curvilinear translation		 Parallel-link swinging plate
(*c*) Fixed-axis rotation		 Compound pendulum
(*d*) General plane motion		 Connecting rod in a reciprocating engine

Rotation

The rotation of a rigid body is described by its angular motion. A rigid body is shown during the rotation in a plane. The angular positions of any two lines 1 and 2 attached to the body are specified by θ_1 and θ_2 measured from any convenient fixed reference direction. Because the angle β is invariant, the relation $\theta_2 = \theta_1 + \beta$ upon differentiation with respect to time gives,



$\dot{\theta}_1 = \dot{\theta}_2$, and $\ddot{\theta}_1 = \ddot{\theta}_2$, or during a finite interval $\Delta\theta_2 = \Delta\theta_1$.

Thus, all lines on a rigid body in its plane of motion have the same angular displacement, the same angular velocity, and the same angular acceleration.

Angular motion relations

The angular velocity ω and angular acceleration α of a rigid body in plane rotation are, respectively, the first and second time derivatives of the angular position coordinate θ of any line in the plane of motion of the body. These definitions give

$$\omega = \frac{d\theta}{dt} = \dot{\theta},$$

$$\alpha = \frac{d\omega}{dt} = \dot{\omega}, \quad \text{or} \quad \alpha = \frac{d^2\theta}{dt^2} = \ddot{\theta}, \qquad \text{.................}(1)$$

$$\omega d\omega = \alpha d\theta, \quad \text{or} \quad \dot{\theta} d\dot{\theta} = \ddot{\theta} d\theta.$$

For rotation with constant angular acceleration, the integrals of (1) becomes

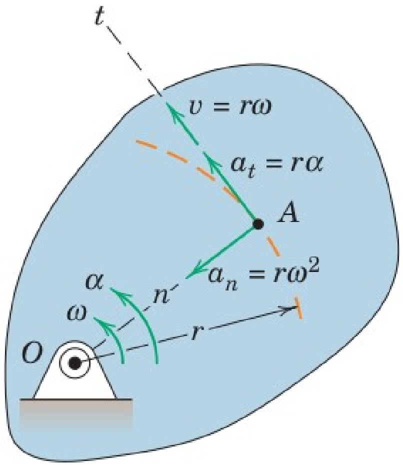
$$\omega = \omega_0 + \alpha t,$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2, \qquad \text{.................}(2)$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0).$$

Rotation about a fixed axis

When a rigid body rotates about a fixed axis, all points other than those on the axis move in concentric circles about the fixed axis. Thus, for the rigid body rotating about a fixed axis normal to the plane of the figure through O, any point such as A moves in a circle of radius r. Thus, the relations for the angular velocity and angular acceleration, respectively, of the body are



$$v = r\omega,$$

$$a_n = r\dot{\theta}^2 = r\omega^2 = v^2/r = v\omega, \qquad \cdots\cdots\cdots\cdots(3)$$

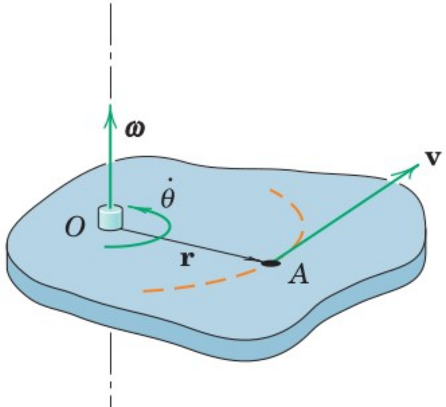
$$a_t = r\ddot{\theta} = r\alpha.$$

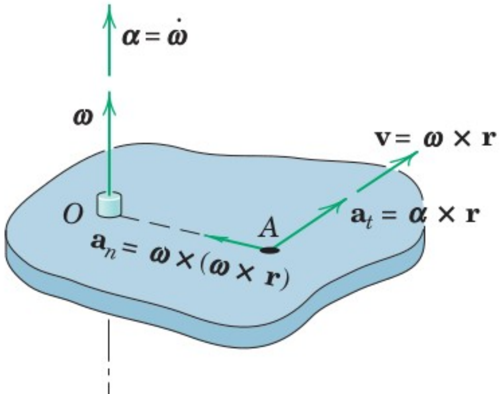
Vector form of (3) can be written as,

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \boldsymbol{\omega} \times \boldsymbol{r},$$

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times \dot{\boldsymbol{r}} = \boldsymbol{\alpha} \times \boldsymbol{r} + \boldsymbol{\omega} \times \dot{\boldsymbol{v}}$$

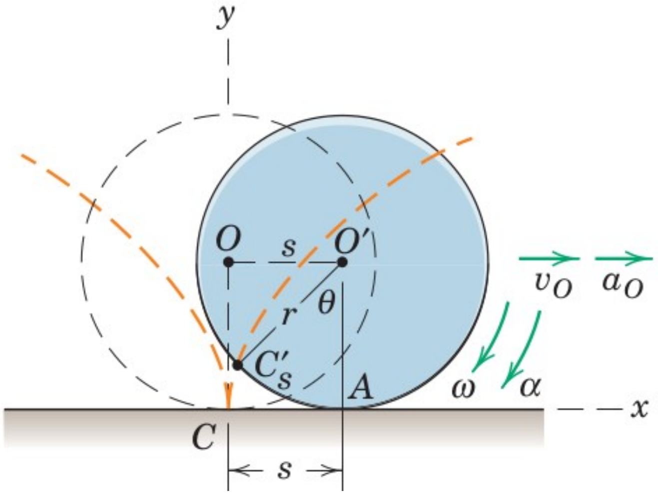
$$\cdots\cdots\cdots\cdots(4)$$





Example 1

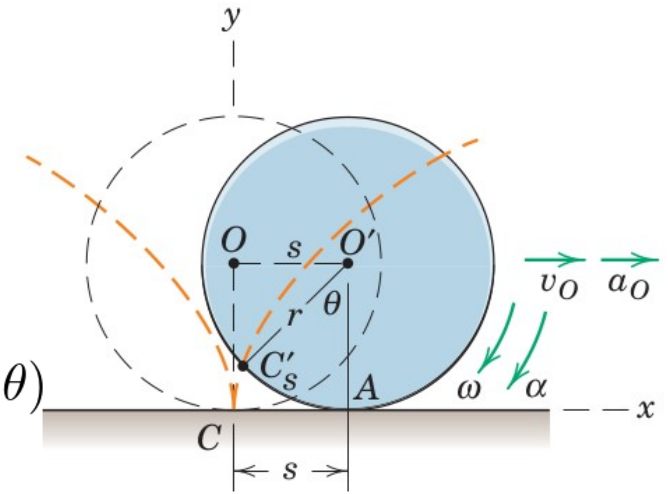
A wheel of radius r rolls on a flat surface without slipping. Determine the angular motion of the wheel in terms of the linear motion of its center O. Also determine the acceleration of a point on the rim of the wheel as the point comes into contact with the surface on which the wheel rolls.



The figure shows the wheel rolling to the right from the dashed to the full position without slipping. The linear displacement of the center O is s, which is also the arc length $C'A$ along the rim on which the wheel rolls. The radial line CO rotates to the new position $C'O'$ through the angle θ which is measured from the vertical direction. If the wheel does not slip, the arc $C'A$ must equal the distance s. Thus, the displacement relationship and its two time derivatives give

$$s = r\theta, \quad \dot{s} = v_O = r\dot{\theta} = r\omega, \quad \dot{v}_O = a_O = r\dot{\omega} = r\alpha.$$

The origin of fixed coordinates is taken arbitrarily but conveniently at the point of contact between on the rim of the wheel and the ground (point C). When point C has moved along its cycloidal path to C', its new coordinates and their time derivatives become



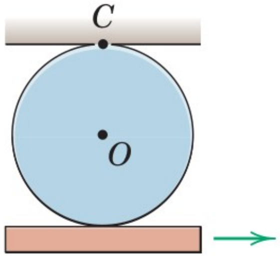
$$x = s - r\cos\theta = r(\theta - \sin\theta), \quad y = r - r\cos\theta = r(1 - \cos\theta)$$

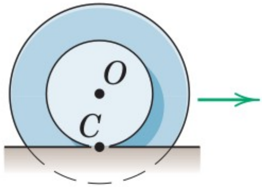
$$\dot{x} = r\dot{\theta}(1 - \cos\theta) = v_O(1 - \cos\theta), \quad \dot{y} = r\dot{\theta}\sin\theta = v_O\sin\theta$$

$$\ddot{x} = \dot{v}_O(1 - \cos\theta) + v_O\dot{\theta}\sin\theta = a_O(1 - \cos\theta) + r\omega^2\sin\theta$$

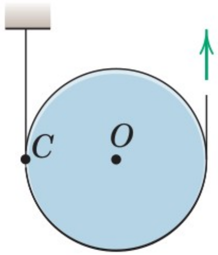
$$\ddot{y} = \dot{v}_O\sin\theta + v_O\dot{\theta}\cos\theta = a_O\sin\theta + r\omega^2\cos\theta.$$

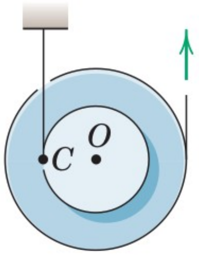
At the given instant, $\theta = 0$, hence, $\ddot{x} = 0$ and $\ddot{y} = r\omega^2$.





Thus, the acceleration of the point C on the rim at the instant of contact with the ground depends only on r and ω and is directed toward the center of the wheel. If desired, the velocity and acceleration of C at any position θ may be obtained by writing the expressions $\boldsymbol{v} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j}$ and $\boldsymbol{a} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j}$.





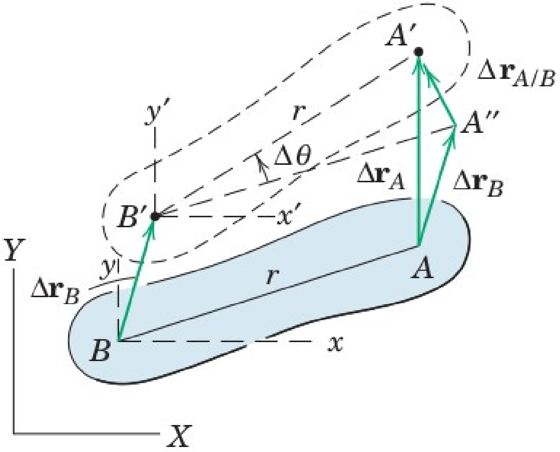
Relative velocity

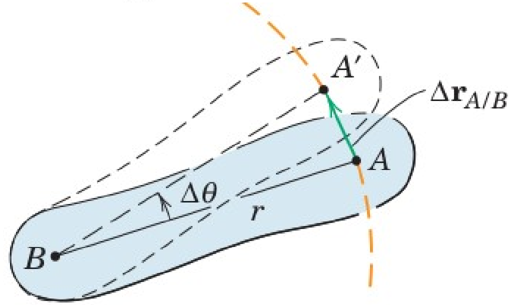
Figure shows a rigid body moving in the plane of the figure from position AB to $A'B'$ during time Δt. This movement may be visualized as occurring in two parts.

First, the body translates to the parallel position $A''B'$ with the displacement $\Delta \boldsymbol{r}_B$.

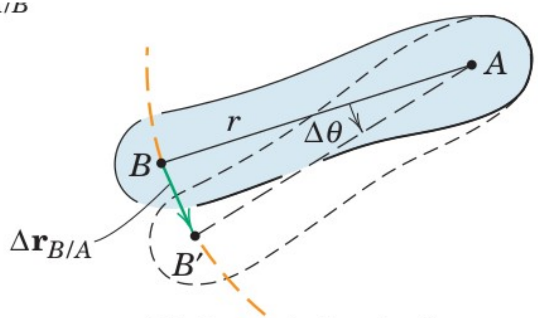
Second, the body rotates about B' through the angle $\Delta\theta$. From the nonrotating reference axes x'-y' attached to the reference point B', it is observed that the remaining motion of the body is one of simple rotation about B', giving rise to the displacement $\Delta \boldsymbol{r}_{A/B}$ of A with respect to B.

To the nonrotating observer attached to B, the body appears to undergo fixed-axis rotation about B with A executing circular motion. Therefore, the relationships developed for circular describe the relative portion of the motion of point A.





Motion relative to B



Motion relative to A

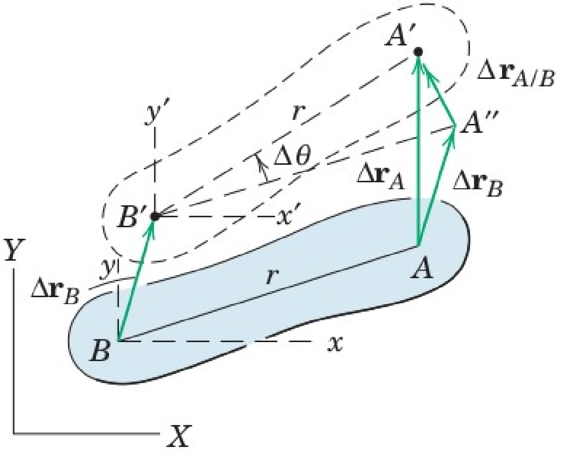
Point B was arbitrarily chosen as the reference point for attachment of our nonrotating reference axes x-y. Point A could have been used just as well, in which case we would observe B to have circular motion about A considered fixed. We see that the sense of the rotation is the same whether we choose A or B as the reference, and we see that

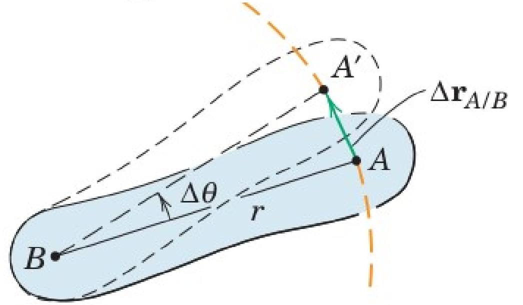
$$\Delta \boldsymbol{r}_{B/A} = -\Delta \boldsymbol{r}_{A/B} \,.$$

With B as the reference point, we see that the total displacement of A is $\Delta \boldsymbol{r}_A = \Delta \boldsymbol{r}_B + \Delta \boldsymbol{r}_{A/B}$,

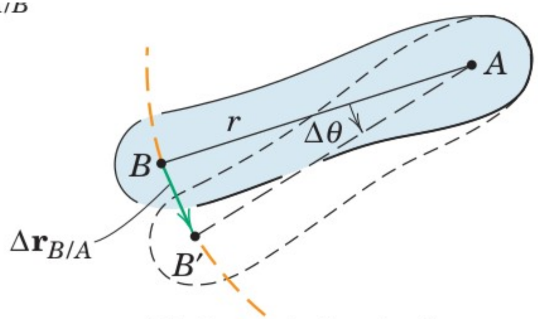
where $\Delta \boldsymbol{r}_{A/B}$ has the magnitude $r\Delta\theta$ as $\Delta\theta$ approaches zero. We note that the relative linear motion $\Delta \boldsymbol{r}_{A/B}$ is accompanied by the absolute angular motion $\Delta\theta$, as seen from the translating axes x'-y'. Dividing the expression for $\Delta \boldsymbol{r}_A$ by the corresponding time interval, we get,

$$\Delta \boldsymbol{v}_A = \Delta \boldsymbol{v}_B + \Delta \boldsymbol{v}_{A/B} \,. \qquad \dots\dots\dots\dots\dots\dots(4)$$





Motion relative to B



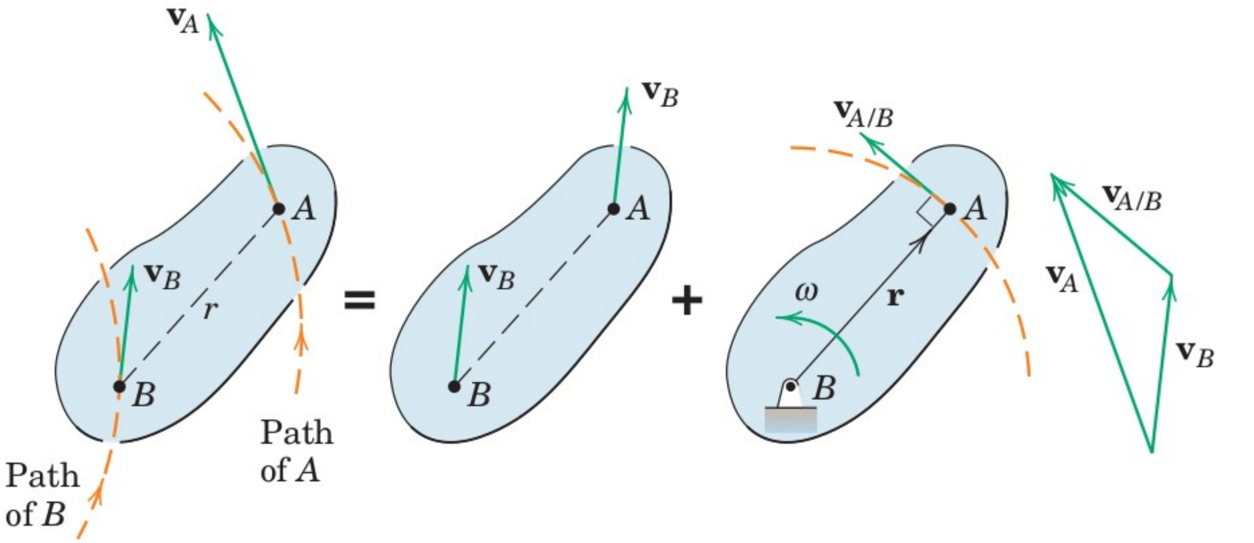
Motion relative to A

Expression (4) is the same as the one we derived earlier for the motion of particle with the one restriction that the distance r between A and B remains constant. The magnitude of the relative velocity is thus seen to be

$$v_{A/B} = \lim_{\Delta t \to 0} (|\Delta \boldsymbol{r}_{A/B}|)/\Delta t = \lim_{\Delta t \to 0} r\Delta\theta/\Delta t = r\omega, \quad \text{where} \quad \omega = \dot{\theta}.$$

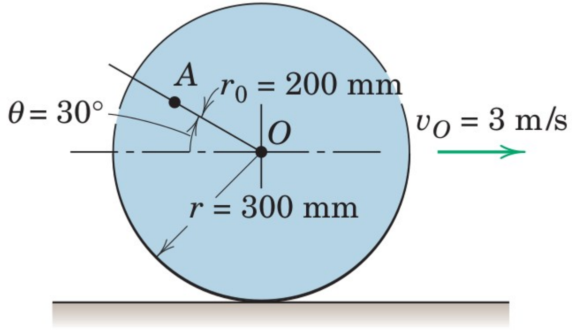
..................(5)

In the vector form, $\boldsymbol{v}_{A/B} = \boldsymbol{\omega} \times \boldsymbol{r}_{A/B}$. ....(6)



Example 2

The wheel of radius $r = 300$ mm rolls to the right without slipping and has a velocity $v_O = 3$ m/s of its center O. Calculate the velocity of point A on the wheel for the instant represented.



The center O is chosen as the reference point for the relative-velocity equation since its motion is given. Thus,

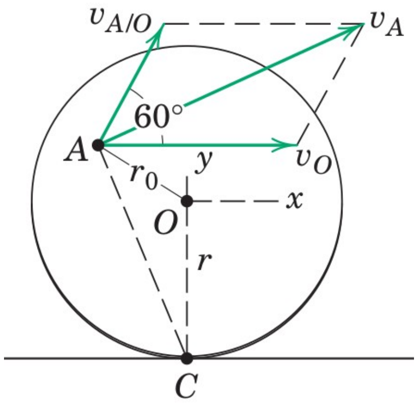
$$\boldsymbol{v}_A = \boldsymbol{v}_O + \boldsymbol{v}_{A/O},$$

$$\boldsymbol{v}_A = \boldsymbol{v}_O + \boldsymbol{\omega} \times \boldsymbol{r}_0,$$

where $\boldsymbol{\omega} = -10\boldsymbol{k}$ rad/s

$$\boldsymbol{r}_0 = 0.2(-\boldsymbol{i}\cos 30° + \boldsymbol{j}\sin 30°) = -0.1732\boldsymbol{i} + 0.1\boldsymbol{j} \text{ m}$$

$$\boldsymbol{v}_O = 3\boldsymbol{i} \text{ m/s}$$

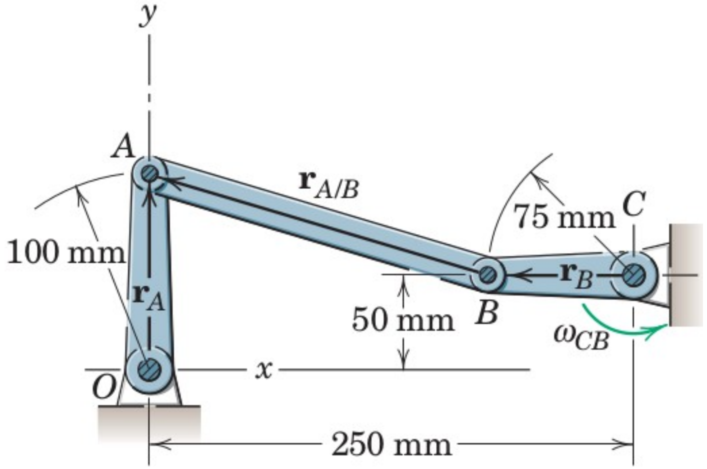


Solution gives,

$$\boldsymbol{v}_A = 4\boldsymbol{i} + 1.732\boldsymbol{j} \text{ m/s}$$

Example 3

Crank CB oscillates about C through a limited arc, causing crank OA to oscillate about O. When the linkage passes the position shown with CB horizontal and OA vertical, the angular velocity of CB is 2 rad/s counterclockwise. For this instant, determine the angular velocities of OA and AB.



The relative-velocity equation

$$\boldsymbol{v}_A = \boldsymbol{v}_B + \boldsymbol{v}_{A/B}$$

$$\boldsymbol{\omega}_{OA} \times \boldsymbol{r}_A = \boldsymbol{\omega}_{OB} \times \boldsymbol{r}_B + \boldsymbol{\omega}_{AB} \times \boldsymbol{r}_{AB}$$

Everything except $\boldsymbol{\omega}_{AB}$ and $\boldsymbol{\omega}_{OA}$ is unknown, which can be found by substituting all known quantities.

$\boldsymbol{\omega}_{AB} =$ -6/7 rad/s and $\boldsymbol{\omega}_{OA} =$ -3/7 rad/s

Another approach of solution is geometric.

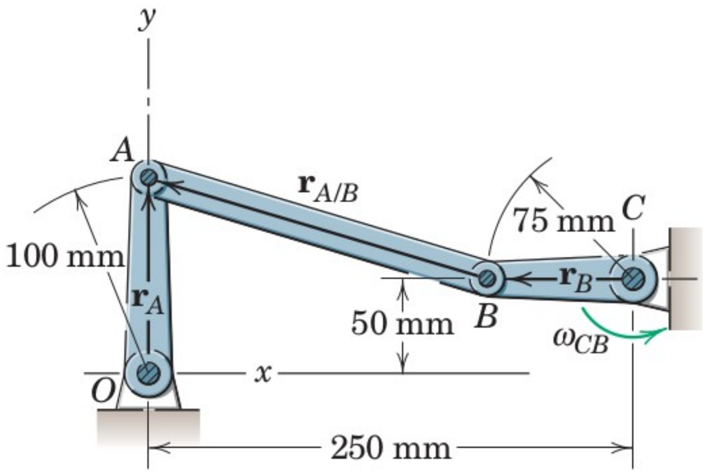
Velocity of point B is $v_B = \omega_{CB} \cdot r_B = 150$ mm/s, direction is perpendicular to CB in downward direction.

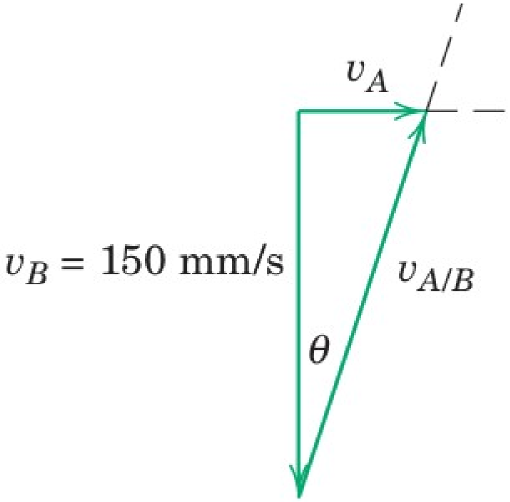
For velocity of point A, magnitude is $\omega_{OA} \cdot r_A$, which is unknown. However, the direction is known, which is perpendicular to OA.

The vector $\boldsymbol{v}_{A/B}$ must be perpendicular to AB.

the angle θ between $\boldsymbol{v}_{A/B}$ and $\boldsymbol{v}_B$ is also the angle made by AB with the horizontal direction.

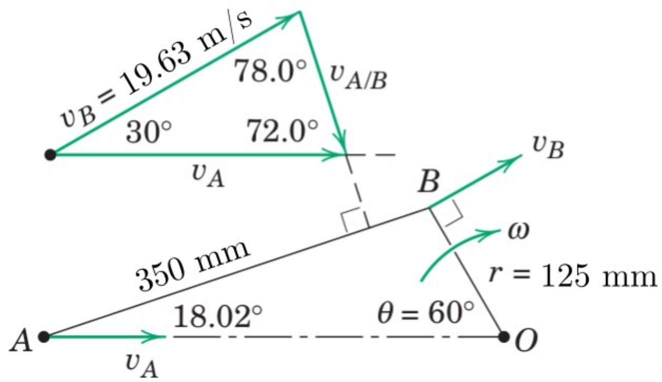
Now all unknown quantities can be calculated simply by applying geometric relations.

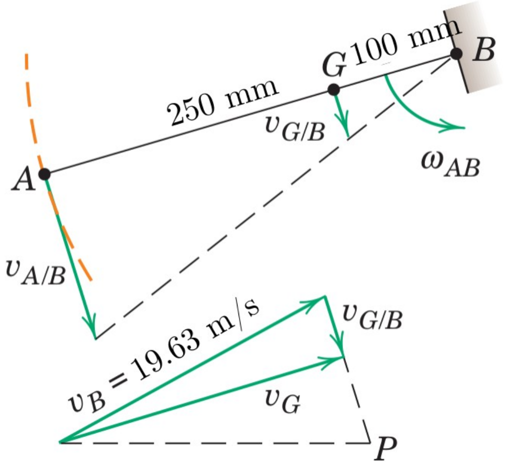


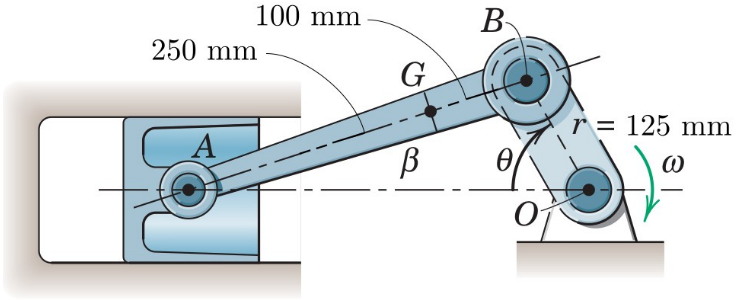


Example 4

The common configuration of a reciprocating engine is that of the slider-crank mechanism shown. If the crank OB has a clockwise rotational speed of 1500 rev/min, determine for the position where $\theta = 60°$ the velocity of the piston A, the velocity of point G on the connecting rod, and the angular velocity of the connecting rod.





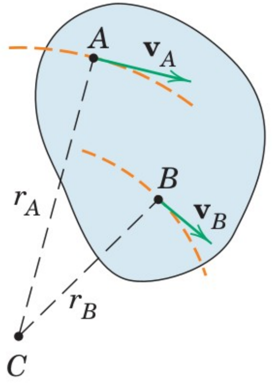


Instantaneous center of zero velocity

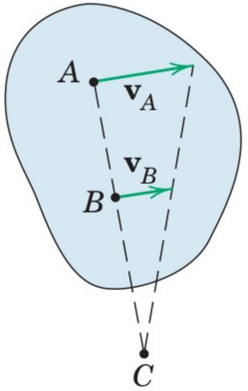
- We determined the velocity of a point on a rigid body in plane motion by adding the relative velocity due to rotation about a convenient reference point to the velocity of the reference point.
- We now solve the problem by choosing a unique reference point which **momentarily has zero velocity.**
- As far as velocities are concerned, the body may be considered to be in pure rotation about an axis, normal to the plane of motion, passing through this point. This axis is called the **instantaneous axis of zero velocity**, and the intersection of this axis with the plane of motion is known as the **instantaneous center of zero velocity**.
- This approach provides a valuable means for visualizing and analyzing velocities in plane motion.

Assume that the directions of the absolute velocities of any two points A and B on the body are known and are not parallel.

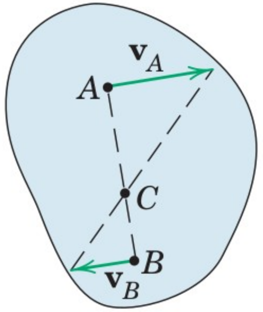
The point about which A has absolute circular motion at the instant considered must lie on the normal to $\boldsymbol{v}_A$ through A. Similarly for B too. The intersection of the two perpendiculars fulfills the requirement for an absolute center of rotation **at the instant considered.**



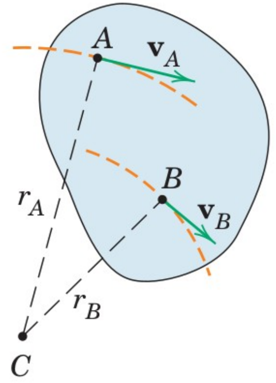
Point C is the instantaneous center of zero velocity and may lie on or off the body. If it lies off the body, it may be visualized as lying on an imaginary extension of the body. **The instantaneous center need not be a fixed point in the body or a fixed point in the plane.**



If we also know the magnitude of the velocity of one of the points, say, $\boldsymbol{v}_A$, we may easily obtain the angular velocity ω of the body and the linear velocity of every point in the body. Thus, the angular velocity of the body is $\omega = v_A/r_A$, which is also the **angular velocity of every line in the body.**

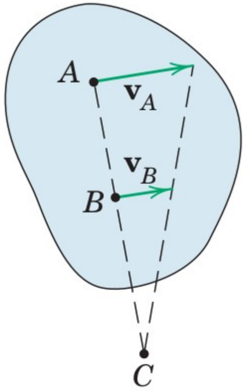


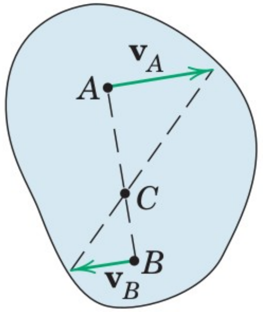
Therefore, the velocity of B is $v_B = r_B\,\omega = (r_B/r_A)v_A$.



Once the instantaneous center is located, the direction of the instantaneous velocity of every point in the body can be easily found.

If the velocities of two points in a body having plane motion are parallel, and the line joining the points is perpendicular to the direction of the velocities, the instantaneous center is located by direct proportion as shown.

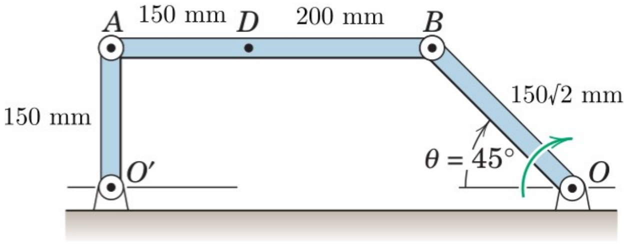


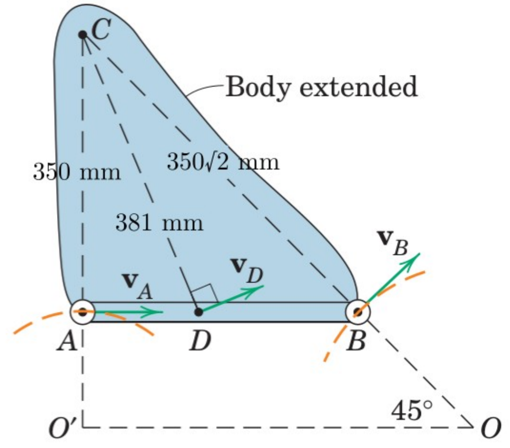


It can be seen that as the parallel velocities become equal in magnitude, the instantaneous center moves farther away from the body and approaches infinity in the limit as the body stops rotating and translates only.

Example 5

Arm OB of the linkage has a clockwise angular velocity of 10 rad/sec in the position shown where $\theta = 45°$. Determine the velocity of A, the velocity of D, and the angular velocity of link AB for the position shown.





The directions of the velocities of A and B are tangent to their circular paths about the fixed centers O' and O.

The intersection of the two perpendiculars to the velocities from A and B locates the instantaneous center C for the link AB. The distances AC, BC, and DC can be computed or scaled from the drawing.

The angular velocity of BC, considered a line on the body extended, is equal to the angular velocity of AC, DC, and AB and is

$$\omega_{BC} = v_B/BC = OB{\cdot}\omega_{OB}/BC = 4.29 \text{ rad/s (CCW)}$$

Thus velocity of A and D are,

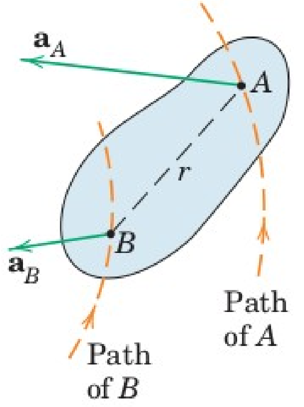
$$v_A = 4.29 \times AC = 1.5 \text{ m/s},\ v_D = 4.29 \times CD = 1.632 \text{ m/s}$$

Relative acceleration

Consider the equation $\boldsymbol{v}_A = \boldsymbol{v}_B + \boldsymbol{v}_{A/B}$, which describes the relative velocities of two points A and B in plane motion in terms of nonrotating reference axes. By differentiating the equation with respect to time, we obtain the relative-acceleration as

$$\boldsymbol{a}_A = \boldsymbol{a}_B + \boldsymbol{a}_{A/B}\,. \qquad \text{..................(7)}$$

If point A and B belong to the same rigid body then during the plane motion, the observer moving with B perceives A to have circular motion about B.

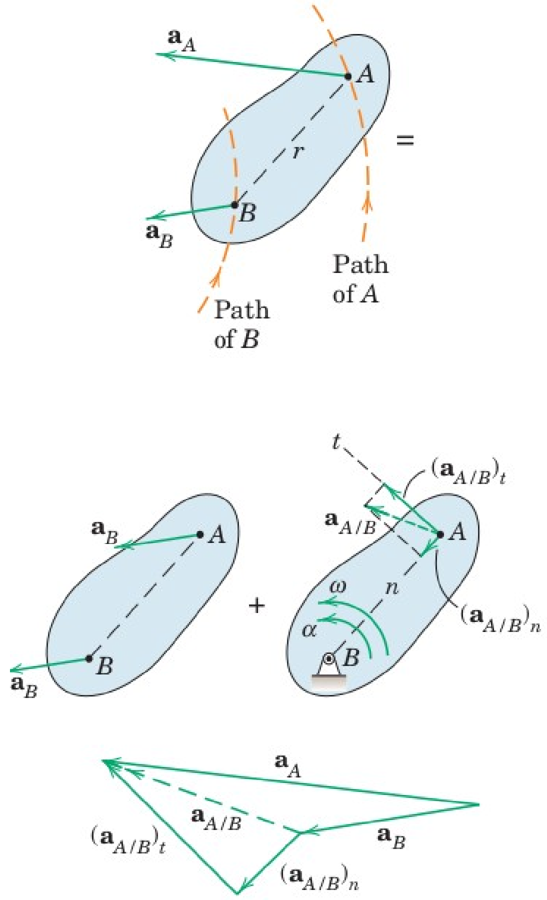


Because the relative motion is circular, it follows that the relative-acceleration term will have both a normal component directed from A toward B due to the change of direction of $\boldsymbol{v}_{A/B}$ and a tangential component perpendicular to AB due to the change in magnitude of $\boldsymbol{v}_{A/B}$.

Thus we may write,

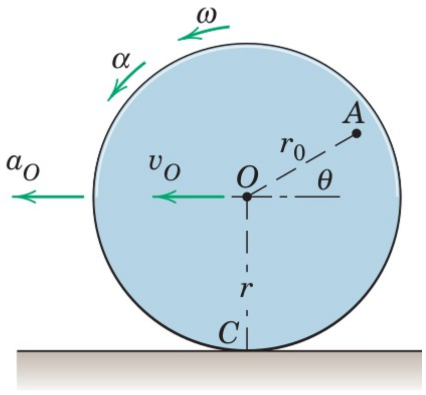
$$\boldsymbol{a}_A = \boldsymbol{a}_B + (\boldsymbol{a}_{A/B})_n + (\boldsymbol{a}_{A/B})_t\,, \qquad \cdots\cdots\cdots\cdots\cdots\cdots(8)$$

where $(\boldsymbol{a}_{A/B})_n = \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})$, and $(\boldsymbol{a}_{A/B})_t = \boldsymbol{\alpha}\times\boldsymbol{r}$.



Example 6

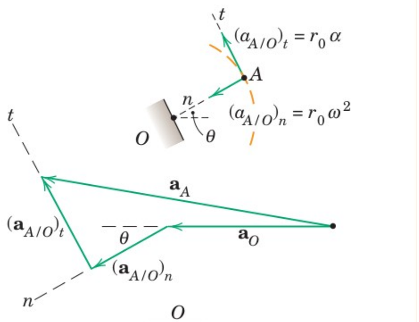
The wheel of radius r rolls to the left without slipping and, at the instant considered, the center O has a velocity $\boldsymbol{v}_O$ and an acceleration $\boldsymbol{a}_O$ to the left. Determine the acceleration of points A and C on the wheel for the instant considered.



Angular velocity $\omega = v_O/r$, angular acceleration $\alpha = a_O/r$.

The acceleration of A, $\boldsymbol{a}_A = \boldsymbol{a}_O + \boldsymbol{a}_{A/O} = \boldsymbol{a}_O + (\boldsymbol{a}_{A/O})_n + (\boldsymbol{a}_{A/O})_t$.

where, magnitudes $(a_{A/O})_n = r_0\omega^2 = r_0(v_O/r)^2$, $(a_{A/O})_t = r_0\alpha = r_0(a_O/r)$.



The acceleration of C, $\boldsymbol{a}_C = \boldsymbol{a}_O + \boldsymbol{a}_{C/O} = \boldsymbol{a}_O + (\boldsymbol{a}_{C/O})_n + (\boldsymbol{a}_{C/O})_t$.

where, magnitudes $(a_{C/O})_n = r\omega^2$, $(a_{C/O})_t = r\alpha$.