Clausius-Duhem inequality

The rate of entropy input is closely related to the rate of thermal work (the total heat fluxes and sources). Entropy fluxes $\boldsymbol{h}$, $\boldsymbol{H}$ and entropy sources $\tilde{r}, \tilde{R}$, and heat fluxes $\boldsymbol{q}$, $\boldsymbol{Q}$ and heat sources r, R are assumed to be related as,

$$\boldsymbol{h} = \frac{\boldsymbol{q}}{\Theta}, \quad \tilde{r} = \frac{r}{\Theta}, \quad \boldsymbol{H} = \frac{\boldsymbol{Q}}{\Theta}, \quad \tilde{R} = \frac{R}{\Theta},$$

where, $\Theta{=}\Theta(\boldsymbol{X},t){>}0$ is a time-dependent scaler field known as absolute temperature, measured in Kelvin.

From the above relations second law of thermodynamics can be rewritten as,

$$\Gamma(t) = \frac{D}{Dt}\int_{\Omega} \eta_c(\boldsymbol{x},t)dv + \int_{\partial\Omega} \frac{\boldsymbol{q}}{\Theta}\cdot \boldsymbol{n}ds - \int_{\Omega} \frac{r}{\Theta}dv \geq 0,$$

$$\boxed{\Gamma(t) = \frac{D}{Dt}\int_{\Omega_0} \eta(\boldsymbol{X},t)dV + \int_{\partial\Omega_0} \frac{\boldsymbol{Q}}{\Theta}\cdot \boldsymbol{N}dS - \int_{\Omega_0} \frac{R}{\Theta}dV \geq 0,}$$

which is known as the **Clausius-Duhem inequality**.

To derive the local form of the inequality in the material description we use following relationship,

$$\int_{\partial\Omega_0} \frac{\boldsymbol{Q}}{\Theta} \cdot \boldsymbol{N} dS = \int_{\Omega_0} \mathrm{Div}\left(\frac{\boldsymbol{Q}}{\Theta}\right) dV = \int_{\Omega_0} \left(\frac{1}{\Theta}\mathrm{Div}\boldsymbol{Q} - \frac{1}{\Theta^2}\boldsymbol{Q} \cdot \mathrm{Grad}\Theta\right) dV,$$

and substitute in the global form to obtain,

$$\boxed{\dot{\eta} - \frac{R}{\Theta} + \frac{1}{\Theta}\mathrm{Div}\boldsymbol{Q} - \frac{1}{\Theta^2}\boldsymbol{Q} \cdot \mathrm{Grad}\Theta \geq 0, \quad \text{or} \quad \dot{\eta} - \frac{R}{\Theta} + \frac{1}{\Theta}\frac{\partial Q_i}{\partial X_i} - \frac{1}{\Theta^2}Q_i\frac{\partial \Theta}{\partial X_i} \geq 0.}$$

Alternate form of the above equation can be obtained by using the local form of first law of thermodynamics as,

$$\boxed{\boldsymbol{P} : \dot{\boldsymbol{F}} - \dot{e} + \Theta\dot{\eta} - \frac{1}{\Theta}\boldsymbol{Q} \cdot \mathrm{Grad}\Theta \geq 0, \quad \text{or} \quad P_{ij} : \dot{F}_{ij} - \dot{e} + \Theta\dot{\eta} - \frac{1}{\Theta}Q_i\frac{\partial \Theta}{\partial X_i} \geq 0.}$$

The last terms determine the **entropy production by conduction of heat.**

Clausius-Planck inequality and heat conduction

Based on the physical observations, heat flows from the warmer region to the colder region of a body (free from source of heat), not vice versa. Hence entropy production by conduction of heat must be non-negative, i.e.

$$-\frac{1}{\Theta}\boldsymbol{Q}\cdot\mathrm{Grad}\Theta \geq 0, \quad \text{or} \quad -\frac{1}{\Theta}\boldsymbol{q}\cdot\mathrm{grad}\Theta \geq 0.$$

Accordingly an even stronger from the second law of thermodynamics, which is also referred as **Clausius-Planck inequality** is,

$$\boxed{\mathcal{D}_{\text{int}} = \boldsymbol{P} : \dot{\boldsymbol{F}} - \dot{e} + \Theta\dot{\eta} \geq 0, \quad \text{or} \quad \mathcal{D}_{\text{int}} = P_{ij} : \dot{F}_{ij} - \dot{e} + \Theta\dot{\eta} \geq 0.}$$

with internal dissipation or local production of entropy $\mathcal{D}_{\text{int}} \geq 0$, which is required to be non-negative at any particle of the body all the times.

Thus, **the internal dissipation** $=$ **the work-conjugate pair, i.e. the stress-power per unit reference volume** $+$ **rate of internal energy per unit volume** $+$ **absolute temperature multiplied by the rate of entropy**

For reversible process internal dissipation is equal to zero and the inequality holds for irreversible process.
Recall the balance of energy derived for a thermodynamic system,

$$\dot{e} = \boldsymbol{P} : \dot{\boldsymbol{F}} - \mathrm{Div}\boldsymbol{Q} + R \quad \text{or} \quad \dot{e} = P_{ij} : \dot{F}_{ij} - \frac{\partial Q_i}{\partial X_i} + R.$$

From the Clausius-Planck inequality, we can replace rate of internal energy in above equations as,

$$\dot{e} = \boldsymbol{P} : \dot{\boldsymbol{F}} + \Theta\dot{\eta} - \mathcal{D}_{\mathrm{int}} \quad \text{or} \quad \dot{e} = P_{ij} : \dot{F}_{ij} + \Theta\dot{\eta} - \mathcal{D}_{\mathrm{int}}.$$

Equating above expressions we get the local form of balance of energy in the **entropy form**,

$$\boxed{\Theta\dot{\eta} = \mathcal{D}_{\mathrm{int}} - \mathrm{Div}\boldsymbol{Q} + R \quad \text{or} \quad \Theta\dot{\eta} = \mathcal{D}_{\mathrm{int}} - \frac{\partial Q_i}{\partial X_i} + R,}$$

in which **local evolution of entropy in explicitly appear**.

For elastic materials it is convenient to work with strain energy functions Ψ. However, in the thermodynamic regime incorporating thermal variables such as Θ and η, Ψ is referred to as Helmholtz free energy function (or free energy). Helmholtz free energy function can be expressed in terms of internal energy as,

$$\Psi = e - \Theta\eta.$$

By using the material time derivative of the free energy Ψ, Clausius-Planck inequality can be written as,

$$\boxed{\mathcal{D}_{\text{int}} = \boldsymbol{P} : \dot{\boldsymbol{F}} - \dot{\Psi} + \dot{\Theta}\eta \geq 0, \quad \text{or} \quad \mathcal{D}_{\text{int}} = P_{ij} : \dot{F}_{ij} - \dot{\Psi} + \dot{\Theta}\eta \geq 0.}$$

Note that for the case of purely mechanical theory, i.e. if thermal effects are ignored, then the inequality becomes,

$$\boxed{\mathcal{D}_{\text{int}} = \boldsymbol{P} : \dot{\boldsymbol{F}} - \dot{\Psi} \geq 0, \quad \text{or} \quad \mathcal{D}_{\text{int}} = \boldsymbol{P} : \dot{\boldsymbol{F}} - \dot{e} \geq 0.}$$

For a reversible process, internal dissipation is zero, then the rate of internal mechanical work per unit reference volume equals $\dot{\Psi}$ (or $\dot{e}$).

Master Balance Principle

Global form:

The present status of a set of particles occupying a region Ω of a continuum body with boundary surface $\partial\Omega$ at time t may be characterized by $I = \int_\Omega f dv$, where $f = f(\boldsymbol{x}, t)$ is a smooth tensor field per unit volume of order n.

f may characterize some physical quantity such as **density, internal energy, linear or angular momentum** and so forth.

A change of these quantities may now be expressed as the following **master balance principle**, as presented here in the global form,

$$\boxed{\frac{D}{Dt}\int_\Omega f(\boldsymbol{x}, t)dv = \int_{\partial\Omega} \phi(\boldsymbol{x}, \boldsymbol{n}, t)ds + \int_\Omega \Sigma(\boldsymbol{x}, t)dv.}$$

$\phi(\boldsymbol{x},\boldsymbol{n},t)$ is the surface density, which is a tensor of order n. The surface density is defined per unit current area and distributed over the boundary $\partial\Omega$. Note that ϕ depends not only on the position x and time t, but also on the normal $\boldsymbol{n}$ to an infinitesimal area element ds at point $\boldsymbol{x}$. Thus, ϕ is not a tensor field.

$\Sigma(\boldsymbol{x},t)$ is called volume density, which consists of internal and external sources. It is a spatial tensor field of order n defined per unit volume and distributed over region Ω. This expression represents the most general structure of tensor valued balance principles in the global form.

The spatial quantities ϕ and Σ describes the action of the surroundings (in the form of a *resultant force and moment*, *external mechanical power* and *thermal power*)on the set of particles with in region Ω.

All balance principles (mass, balance, linear and angular momentum balance and first law of thermodynamics) can be expressed by replacing f, ϕ, and Σ with appropriate variables. For different balance principles f, ϕ, and Σ are identified as

Balance Principles	$\boldsymbol{f}$	$\boldsymbol{\phi}$	Σ
Mass balance	ρ	0	0
Linear momentum	$\rho\boldsymbol{v}$	$\boldsymbol{t}$	$\boldsymbol{b}$
Angular momentum	$\boldsymbol{r}\times\rho\boldsymbol{v}$	$\boldsymbol{r}\times\boldsymbol{t}$	$\boldsymbol{r}\times\boldsymbol{b}$
First law of thermodynamics	$\rho\boldsymbol{v}^2/2+e_c$	$\boldsymbol{t}\cdot\boldsymbol{n}+q_n$	$\boldsymbol{b}+\tilde{r}$

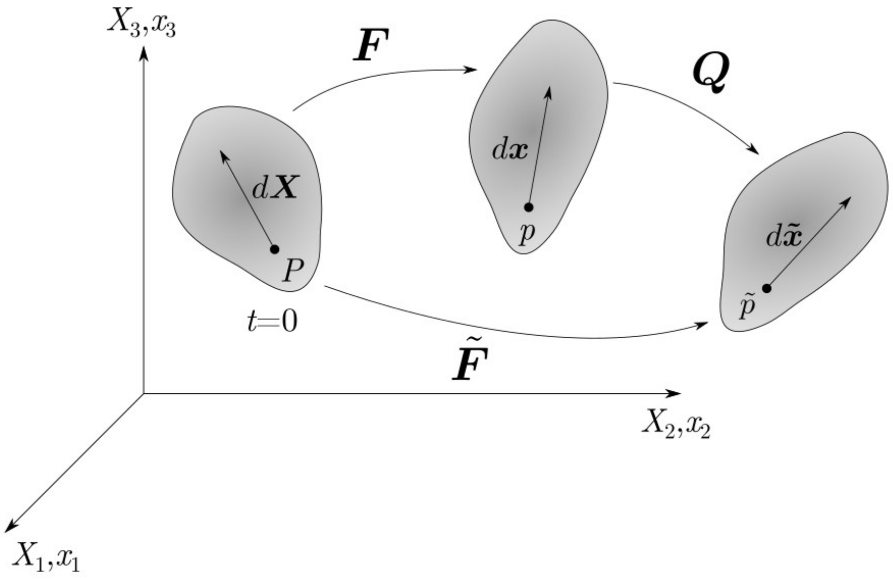
Material Objectivity

The natural/physical processes do not depend on the change of observer. Hence, The mathematical representation of physical phenomena must reflect this invariance.

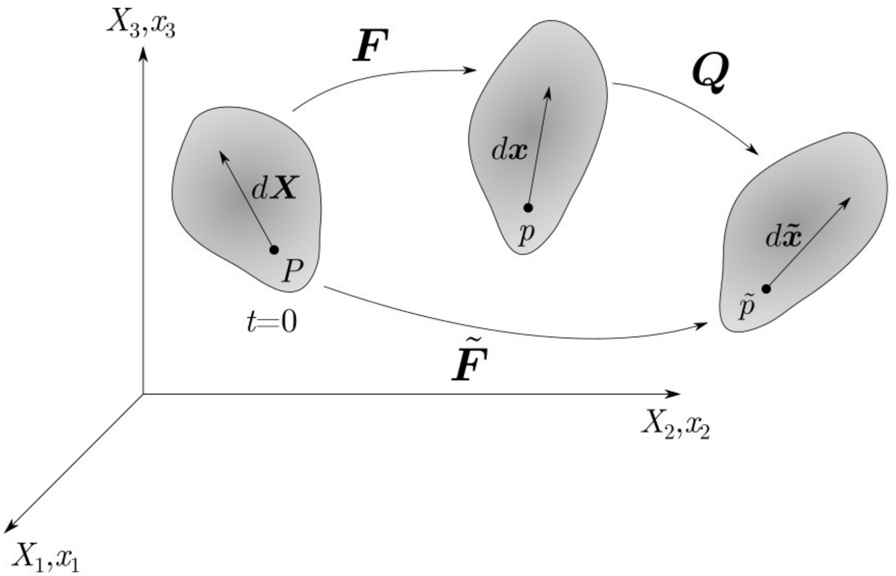
That is, two observers even if in the realtive motion with respect to each other observe the same process.

An important concept in solid mechanics is the notion of **objectivity**. The concept of objectivity can be understood by studying the effect of a rigid body motion superimposed on the deformed configuration.

From the point of view of **an observer attached to and rotating with the body** many quantities describing the behavior of the solid will remain unchanged. Such quantities, for example the distance between any two particles and, among others, the state of stresses in the body, are **said to be objective.**



Although the intrinsic nature of these quantities remains unchanged, their spatial description may change.



To express these concepts in a mathematical framework, consider an elemental vector $d\boldsymbol{X}$ in the initial configuration that deforms to $d\boldsymbol{x}$ and is subsequently rotated to $d\tilde{\boldsymbol{x}}$.

Following relationships can be written,

$$d\tilde{\boldsymbol{x}} = \boldsymbol{Q}d\boldsymbol{x} = \boldsymbol{Q}\boldsymbol{F}d\boldsymbol{X}.$$

Although the vector $d\tilde{\boldsymbol{x}}$ is different from $d\boldsymbol{x}$, their magnitudes are equal. In this sense it can be said the $d\boldsymbol{x}$ is objective under rigid body motions.

We can extend this definition to any vector $\boldsymbol{a}$ that transforms according to $\tilde{\boldsymbol{a}} = \boldsymbol{Q}\boldsymbol{a}$.

Following relation is also valid for position vector of any points p and $\tilde{p}$,

$$\tilde{\boldsymbol{x}} = \boldsymbol{Q}\boldsymbol{x}.$$

Now, let us check the objectivity of velocity vector by differentiating the previous relation w.r.t. time,

$$\tilde{\boldsymbol{v}} = \frac{d\tilde{\boldsymbol{x}}}{dt} = \frac{d}{dt}\left(\boldsymbol{Q}\boldsymbol{x}\right) = \boldsymbol{Q}\boldsymbol{v} + \dot{\boldsymbol{Q}}\boldsymbol{x}.$$

Observe that the magnitudes of $\boldsymbol{v}$ and $\tilde{\boldsymbol{v}}$ are not equal because of the presence of the term $\dot{\boldsymbol{Q}}\boldsymbol{x}$, which violates the objectivity criteria. Hence, **velocity is an example of a non-objective vector**. Similarly it can be shown that the acceleration is also a non-objective quantitiy.

Now, we extend the definition of objectivity to second-order tensors. Note that for the deformation gradients we can write,

$$\tilde{\boldsymbol{F}} = \boldsymbol{Q}\boldsymbol{F}.$$

Now, consider a tensor $\boldsymbol{A}$, which maps one elemental vector $d\boldsymbol{x}$ to another elemental vector d$\boldsymbol{y}$ in current configuration.

Hence, $$d\boldsymbol{y} = \boldsymbol{A}\, d\boldsymbol{x}.$$

Similarly, a tensor $\tilde{\boldsymbol{A}}$ maps one elemental vector $d\tilde{\boldsymbol{x}}$ to another elemental vector $d\tilde{\boldsymbol{y}}$ in the rotated configuration as,

$$d\tilde{\boldsymbol{y}} = \tilde{\boldsymbol{A}} d\tilde{\boldsymbol{x}}.$$

A relation between $\boldsymbol{A}$ and $\tilde{\boldsymbol{A}}$ can be eshtablished as $\tilde{\boldsymbol{A}} = \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T$, which is the transformation rule for the objective second-order tensor; i.e., all sencond-order tensors following the above transformation rule will be an objective tensor.

However, not all second-order tensors are objective. Let us examine objectivity of the velocity gradient tensor $\boldsymbol{l}$. We know that,

$$\boldsymbol{l} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}.$$

Thus,

$$\tilde{\boldsymbol{l}} = \dot{\tilde{\boldsymbol{F}}}\tilde{\boldsymbol{F}}^{-1} = \left(\overset{\cdot}{\overline{\boldsymbol{Q}\boldsymbol{F}}}\right)(\boldsymbol{Q}\boldsymbol{F})^{-1},$$

$$\tilde{\boldsymbol{l}} = \dot{\boldsymbol{Q}}\boldsymbol{Q}^{-1} + \boldsymbol{Q}\dot{\boldsymbol{F}}\boldsymbol{F}^{-1}\boldsymbol{Q}^{-1} = \boldsymbol{\Omega} + \boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^T,$$

Hence, this tensor is affected by the superimposed rigid body motion and it is not objective.

Check the objectivity of symmetric part of $\boldsymbol{l}$, i.e., $\boldsymbol{d}$ and show that it is an objective tensor.

Note that in general, **the rate of an objective second-order tensor is not objective.**

$$\dot{\tilde{\boldsymbol{A}}} = \overset{\cdot}{\overline{\boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T}} = \dot{\boldsymbol{Q}}\boldsymbol{A}\boldsymbol{Q}^T + \boldsymbol{Q}\dot{\boldsymbol{A}}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{A}\dot{\boldsymbol{Q}}^T.$$

Considering the fact that the rate of second order tensor is not objecitve, several definitions for an objective rate of an objective tensor are proposed. For example,

Jaumann-Zaremba rate – $\overset{\triangledown}{\boldsymbol{A}} = \dot{\boldsymbol{A}} - \boldsymbol{W}\boldsymbol{A} - \boldsymbol{A}\boldsymbol{W}^T$

Truesdell rate – $\overset{\circ}{\boldsymbol{A}} = \dot{\boldsymbol{A}} - \boldsymbol{L}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{L}^T - \mathrm{tr}(\boldsymbol{L})\boldsymbol{A}$

Oldroyd rates – $\overset{\diamond}{\boldsymbol{A}} = \dot{\boldsymbol{A}} - \boldsymbol{L}\boldsymbol{A} - \boldsymbol{A}\boldsymbol{L}^T$

It is left for your exercise to verify that these rate tensors are objective.

Push-forward and pull-back operation

The transformations between material and spatial quantities are typically called a push-forward operation and a pull-back operation.

Push-forward operation of a second order tensor is written as

$$\phi_*(\bullet) = \boldsymbol{F}^{-T}(\bullet)\boldsymbol{F}^{-1}.$$

Pull-back operation of a second order tensor is written as

$$\phi_*^{-1}(\bullet) = \boldsymbol{F}^{T}(\bullet)\boldsymbol{F}.$$

For example,

$\boldsymbol{E}$	Push forward $\longrightarrow$	$\boldsymbol{e}$
$\boldsymbol{S}$	$\longleftarrow$ Pull back	$\boldsymbol{\tau}$

Problem Set

Problem 1: Show that Cauchy's first equation of motion may also be written in the following equivalent form,

$$\frac{\partial(\rho \boldsymbol{v})}{\partial t} = \operatorname{div}(\boldsymbol{\sigma} - \rho \boldsymbol{v} \otimes \boldsymbol{v}) + \boldsymbol{b}.$$

Problem 2: Show that $\dot{J} = J \operatorname{div} v$.

Problem 3: Given the following stress field in a thick-wall elastic cylinder:

$$\sigma_{rr} = A + \frac{B}{r^2}, \quad \sigma_{\theta\theta} = A - \frac{B}{r^2}, \quad \sigma_{r\theta} = \sigma_{rz} = \sigma_{\theta z} = \sigma_{zz} = 0,$$

where A and B are constants.

(a) Verify that the given state of stress satisfies the equations of equilibrium in the absence of body forces.

(b) Find the stress vector on a cylindrical surface $r=a$, and

(c) if the surface traction on the inner surface $r=r_i$ is a uniform pressure p_i and the outer surface $r=r_o$ is free of surface traction, find the constant A and B.

Problem 4: For a beam of circular cross-section, analysis from elementary strength of materials theory yields the following stresses:

$$\sigma_x = -\frac{My}{I}, \quad \tau_{xy} = \frac{V\left(R^2 - y^2\right)}{3I}, \quad \sigma_y = \sigma_z = \tau_{xz} = \tau_{yz} = 0,$$

where R is the section radius, $I = \pi R^4/4$, M is the bending moment, V is the shear force, and $\mathrm{d}M/\mathrm{d}x = V$. Assuming zero body show that these stresses do not satisfy the equilibrium equations. This result in one of many that indicates the approximate nature of strength of materials theory.

Problem 5: Consider the Cauchy stress distribution for a continuum in equilibrium given with reference to a rectangular x_1,x_2,x_3-coordinate system. The components of the Cauchy stress tensor are given the form

$$[\boldsymbol{\sigma}] = \begin{bmatrix} x_1 x_2 & x_1^2 & -x_2 \\ x_1^2 & 0 & 0 \\ -x_2 & 0 & x_1^2 + x_2^2 \end{bmatrix}.$$

Find the body force $\boldsymbol{b}$ that acts on this continuum.