ME531: Advanced Mechanics of Solids

Motion, Strain and Stress

Anshul Faye
afaye@iitbhilai.ac.in
Room No. # 106

Other stress measures

There are other definitions of stress which of practical applications in analysis. Most of them do not have a direct physical interpretation.

Kirchhoff stress tensor: It is defined the as Cauchy stress tensor time the volume ratio J.

$$\boldsymbol{\tau} = J\boldsymbol{\sigma} \text{ or } \tau_{ij} = J\sigma_{ij}.$$

Second Piola-Kirchhoff stress tensor: It is denoted as $\boldsymbol{S}$ and defined as,

$$\boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{\tau}\boldsymbol{F}^{-T} \text{ or } S_{ij} = F_{im}^{-1}F_{jn}^{-1}\tau_{mn}.$$

With its inverse,

$$\boldsymbol{\sigma} = J^{-1}\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^{T} \text{ or } \sigma_{ij} = J^{-1}F_{im}F_{jn}^{-1}S_{mn}.$$

Second Piola-Kirchhoff stress tensor in Lagrangian description and is a symmetric tensor. It is related with the First Piola-Kirchhoff stress tensor as

$$\boldsymbol{P} = \boldsymbol{S}\boldsymbol{F}^{T} \text{ or } P_{ij} = S_{ik}F_{jk}.$$

2^{nd} PK stress derivation

Considering that, the force is a vector quantity, we use following transformation rule for mapping of a force in the deformed configuration to the undeformed configuration,

$$d\boldsymbol{F} = \boldsymbol{F}^{-1} d\boldsymbol{f} \quad \text{or} \quad dF_i = F_{ij}^{-1} df_j .$$

Force in the deformation configuration is $df_j = t_j da = \sigma_{kj} n_k da$. Thus,

$$dF_i = F_{ij}^{-1} \sigma_{kj} n_k da,$$

$$dF_i = F_{ij}^{-1} \sigma_{kj} J F_{pk}^{-1} N_p dA.$$

(Using Nanson's relationship, i.e., $n_k da = J F_{pk}^{-1} N_p dA$)

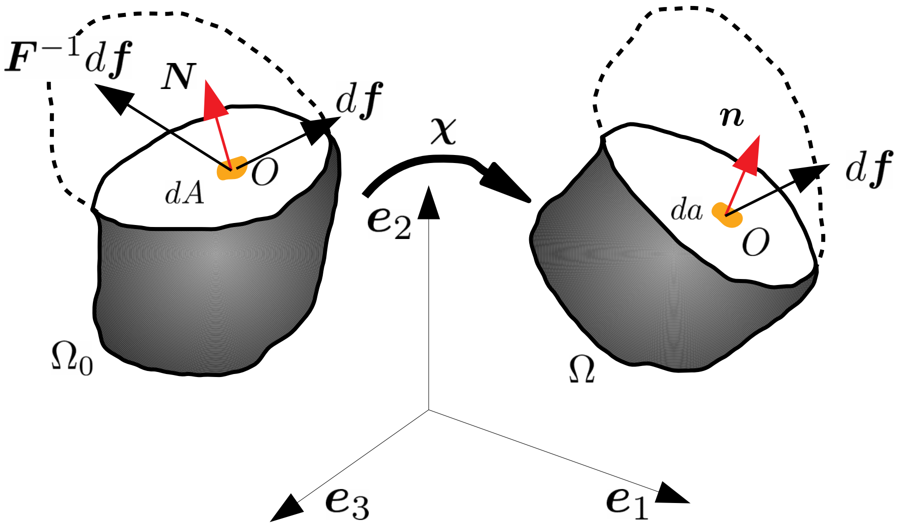
$dF_i = T_i dA = F_{ij}^{-1} \sigma_{kj} J F_{pk}^{-1} N_p dA$, or

$T_i = (J F_{ij}^{-1} \sigma_{kj} F_{pk}^{-1}) N_p, \quad \text{or} \quad T_i = S_{pi} N_p.$

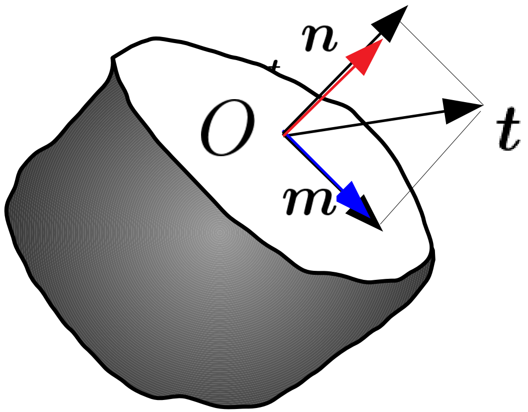
where, $S_{pi} = J F_{ij}^{-1} \sigma_{kj} F_{pk}^{-1}$ or $\boldsymbol{S} = J \boldsymbol{F}^{-1} \boldsymbol{\sigma} \boldsymbol{F}^{-T}$, is Second P-K Stress Tensor

Summary

- The *Cauchy stress tensor* ($\boldsymbol{\sigma}$) is defined to be the *current force* ($d\boldsymbol{f}$) per unit *deformed area* (da).
- The *first Piola—Kirchhoff stress tensor* ($\boldsymbol{P}$) gives the *current force* ($d\boldsymbol{f}$) per unit *undeformed area* (dA).
- The *second Piola—Kirchhoff stress tensor* ($\boldsymbol{S}$) gives the *transformed current force* ($d\boldsymbol{F} = \boldsymbol{F}^{-1} d\boldsymbol{f}$) per unit *undeformed area* (dA).



Normal and shear stresses



Consider a traction vector $\boldsymbol{t}$ on at a surface having unit normal vector $\boldsymbol{n}$ and a tangential vector $\mathbf{m}$ which is perpendicular to $\boldsymbol{n}$.

Traction vector $\boldsymbol{t}$ can be decomposed into two vector along $\boldsymbol{n}$ and $\boldsymbol{m}$ directions namely *normal* and *shear traction*.

Normal traction is $\boldsymbol{t}_n = (\boldsymbol{n} \cdot \boldsymbol{t})\boldsymbol{n} = \sigma \boldsymbol{n}$, and

Shear traction is $\boldsymbol{t}_m = (\boldsymbol{m} \cdot \boldsymbol{t})\boldsymbol{m} = \tau \boldsymbol{m} = \boldsymbol{t} - \boldsymbol{t}_n$,

where σ and τ are magnitudes of normal and shear traction called normal and shear stress. If σ >0 normal stresses are said to be tensile, while negative normal stresses (i.e. σ <0) are call compressive stresses.

Now, $\boldsymbol{t} = \boldsymbol{t}_n + \boldsymbol{t}_m = \sigma \boldsymbol{n} + \tau \boldsymbol{m}$ and $|\boldsymbol{t}|^2 = \sigma^2 + \tau^2$.