Balance Principles

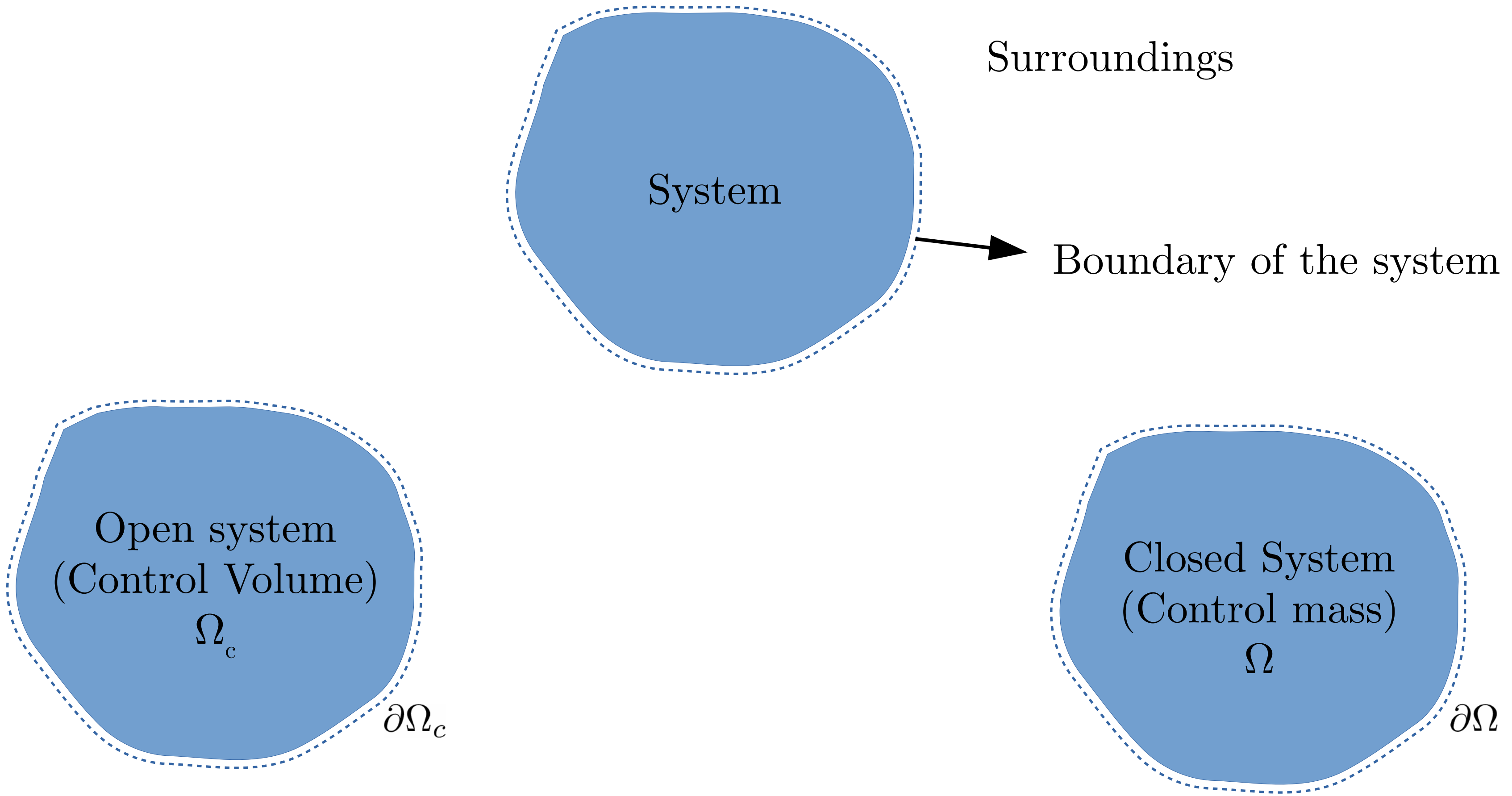
In this chapter we discuss some important fundamental balance principles such as,

- conservation of mass,
- the momentum balance principle (linear and angular momentum), and
- balance of energy.

These principles are valid in all branches of continuum mechanics. They must be satisfied for all times.

We will also discuss some fundamentals laws which are inequalities such as second law of thermodynamics.

System



Fixed amount of volume.
Mass and energy can cross the boundaries

Fixed amount of mass.
Energy can cross the boundaries

Conservation of Mass

Every continuum body has mass, (denoted by m), which is a fundamental property indicating the amount of matter contained in the body. We will assume that the mass is continuously distributed over an arbitrary region Ω.

It is assumed that during the motion neither the mass is created (no mass sources) nor destroyed (no mass sinks). So mass of a particle during the motion remain unchanged. Thus considering a closed system (i.e. fixed mass system)

$$\boxed{m(\Omega_0) = m(\Omega) > 0, \text{ at all time } t.}$$

This relation is known as the **conservation of mass**. In *differential* (or *local*) form

$$\boxed{\mathrm{d}m(\boldsymbol{X}) = \mathrm{d}m(\boldsymbol{x},\mathrm{t}) > 0,}$$

where $\mathrm{d}m$ is infinitesimal element mass. It should be noted that the mass m is independent of the motion, and hence the material time derivative of the mass m is,

$$\frac{D}{Dt}m(\Omega_0) = \frac{D}{Dt}m(\Omega) = 0.$$

The mass is characterized by continuous (or atleast piecewise continuous) scaler fields, i.e. $\rho_0 = \rho_0(\boldsymbol{X}) > 0$ and $\rho = \rho(\boldsymbol{x}, t) > 0$, respectively.

$\rho_0(\boldsymbol{X})$ is called the reference mass density (independent of time) and $\rho(\boldsymbol{x}, t)$ is called the spatial mass density during the motion.

The mass densities are defined as,

$$\rho_0(\boldsymbol{X}) = \lim_{\Delta V \to 0} \frac{\Delta m(\Omega_0)}{\Delta V(\Omega_0)} = \frac{dm}{dV}, \quad \rho(\boldsymbol{x}, t) = \lim_{\Delta v \to 0} \frac{\Delta m(\Omega)}{\Delta v(\Omega)} = \frac{dm}{dv},$$

where $\mathrm{d}V$ and $\mathrm{d}v$ are infinitesimal volume element in reference and spatial configurations.

Differential form of conservation of mass in terms of density can now be written as,

$$\boxed{\rho_0(\boldsymbol{X})dV = \rho(\boldsymbol{x}, t)dv > 0,}$$

which suggests that the volume increases when density decreases.

Total mass of the system will be,

$$\boxed{m = \int_{\Omega_0} \rho_0(\boldsymbol{X}) dV = \int_{\Omega} \rho(\boldsymbol{x}, t) dv = \text{constant} > 0, \ \text{at all times } t.}$$

Hence, the rate form is,

$$\boxed{\dot{m} = \frac{Dm}{Dt} = \frac{D}{Dt} \int_{\Omega} \rho(\boldsymbol{x}, t) dv = 0.}$$

which means that the material time derivative of m is zero for region Ω at all time t during the motion. Above equations are known as *integral* (or *global*) form of conservation of mass.

We want to determine the relationship between the referential mass density $\rho_0(\boldsymbol{X})$ and the spacial mass density $\rho(\boldsymbol{x}, t)$. We start with the integral form of conservation of mass.

Using the relation between the volume elements in the reference and current configurations, we can rewrite the integral form as,

$$m = \int_{\Omega_0} \rho_0(\boldsymbol{X}) dV = \int_{\Omega_0} \rho(\boldsymbol{\chi}(\boldsymbol{X}, t), t) J(\boldsymbol{X}, t) dV = \text{constant} > 0, \text{ at all times } t$$
$$\Rightarrow \int_{\Omega_0} \left(\rho_0(\boldsymbol{X}) - J(\boldsymbol{X}, t)\rho(\boldsymbol{\chi}(\boldsymbol{X}, t), t)\right) dV = 0.$$

Considering that V is an *arbitrary* volume of region Ω_0 implies that, the integrand must vanish everywhere (following the *localization theorem*), hence,

$$\rho_0(\boldsymbol{X}) - J(\boldsymbol{X}, t)\rho(\boldsymbol{\chi}(\boldsymbol{X}, t), t) = 0 \quad \text{or} \quad \boxed{\rho_0(\boldsymbol{X}) = J(\boldsymbol{X}, t)\rho(\boldsymbol{\chi}(\boldsymbol{X}, t), t),}$$

which is known as the **continuity mass equation** in *Lagrangian description*, which relates the referential density to the spatial density. The rate form in the material description becomes,

$$\frac{\partial \rho_0(\boldsymbol{X})}{\partial t} = \dot{\rho}_0(\boldsymbol{X}) = 0.$$

To obtain the rate form of the continuity equation in spatial description, we start with,

$$\dot{\rho}_0(\boldsymbol{X}) = \frac{D\rho_0(\boldsymbol{X})}{Dt} = \frac{D(J\rho)}{Dt} = \dot{\overline{J\rho}} = 0.$$

Using the product rule, $\dot{\overline{J\rho}} = J\dot{\rho} + \dot{J}\rho = 0.$

Using the material time derivative of spatial density field $\rho(\boldsymbol{x}, t)$ and J, we obtain the spatial form of continuity equation as,

$$\boxed{\dot{\rho}(\boldsymbol{x}, t) + \rho(\boldsymbol{x}, t)\mathrm{div}\boldsymbol{v}(\boldsymbol{x}, t) = 0, \quad \text{or} \quad \dot{\rho} + \rho\frac{\partial v_i}{\partial x_i} = 0.}$$

Following expressions will be used to derive the above result.

$$\dot{\rho} = \frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathrm{grad}\rho \cdot \boldsymbol{v}.$$

Exercise

$$\dot{J}(\boldsymbol{F}, t) = \frac{\partial J}{\partial \boldsymbol{F}} : \dot{\boldsymbol{F}}, \quad \boxed{\frac{\partial J}{\partial \boldsymbol{F}} = J\boldsymbol{F}^{-T}, \quad \det \boldsymbol{A} = \frac{1}{6}e_{ijk}e_{rst}A_{ir}A_{js}A_{kt},} \quad \dot{\boldsymbol{F}} = \boldsymbol{l}\boldsymbol{F}.$$

Reynolds' transport theorem

Let us consider a spatial scaler field $\phi(\boldsymbol{x},\mathrm{t})$, which may indicate any physical quantity (such as mass, entropy, internal energy or heat etc.) of a particle in space per unit *volume* at any time t. ϕ is smooth and continuously differentiable. Thus the following integral gives the integration of the field over the spatial volume v of the body Ω,

$$I(t) = \int_{\Omega} \phi(\boldsymbol{x}, t) dv.$$

Now we compute the material time derivative of the volume integral I.

$$\dot{I}(t) = \frac{D}{Dt} \int_{\Omega} \phi(\boldsymbol{x}, t) dv = \frac{D}{Dt} \int_{\Omega_0} \phi(\boldsymbol{\chi}(\boldsymbol{X}, t), t) J(\boldsymbol{X}, t) dV.$$

Not as the region of integration is reference configuration, which is independent of time, hence the integration and differentiation commute. Thus,

$$\frac{D}{Dt} \int_{\Omega} \phi(\boldsymbol{x}, t) dv = \int_{\Omega_0} \left[\dot{\phi}(\boldsymbol{\chi}(\boldsymbol{X}, t), t) J(\boldsymbol{X}, t) + \phi(\boldsymbol{\chi}(\boldsymbol{X}, t), t) \dot{J}(\boldsymbol{X}, t) \right] dV.$$

$$\frac{D}{Dt}\int_{\Omega}\phi(\boldsymbol{x},t)dv = \int_{\Omega_0}\left[\dot{\phi}(\boldsymbol{\chi}(\boldsymbol{X},t),t)J(\boldsymbol{X},t) + \phi(\boldsymbol{\chi}(\boldsymbol{X},t),t)J(\boldsymbol{X},t)\text{div}\boldsymbol{v}\right]dV$$

$$\Rightarrow \int_{\Omega_0}\left[\dot{\phi}(\boldsymbol{\chi}(\boldsymbol{X},t),t) + \phi(\boldsymbol{\chi}(\boldsymbol{X},t),t)\text{div}\boldsymbol{v}\right]J(\boldsymbol{X},t)dV$$

$$\Rightarrow \int_{\Omega}\left[\dot{\phi}(\boldsymbol{x},t) + \phi(\boldsymbol{x},t)\text{div}\boldsymbol{v}(\boldsymbol{x},t)\right]dv$$

Thus,

$$\boxed{\frac{D}{Dt}\int_{\Omega}\phi dv = \int_{\Omega}\left[\dot{\phi} + \phi\text{div}\boldsymbol{v}\right]dv.}$$

Another form of Reynolds' transport theorem can be written by using the definition of material time derivative as,

$$\frac{D}{Dt}\int_{\Omega}\phi dv = \int_{\Omega}\left[\frac{\partial\phi}{\partial t} + \text{grad}\phi\cdot\boldsymbol{v} + \phi\text{div}\boldsymbol{v}\right]dv = \int_{\Omega}\left[\frac{\partial\phi}{\partial t} + \text{div}\phi\boldsymbol{v}\right]dv$$

After applying the Gauss-Divergence theorem,

$$\boxed{\frac{D}{Dt}\int_{\Omega}\phi dv = \int_{\Omega}\frac{\partial\phi}{\partial t}dv + \int_{\partial\Omega}\phi\boldsymbol{v}\cdot\boldsymbol{n}ds.}$$

The first term on the right denotes the **local time rate of change of $\boldsymbol{\phi}$ in region Ω,** whereas the second term characterizes the **rate of transport (or the outward normal flux)** of $\phi\boldsymbol{v}$ across the surface $\partial\Omega$ out of region Ω. This contribution arises from the moving region.

For a spatial field ψ, describing a physical quantity of a particle per unit mass at time t integration over the volume of body Ω will be,

$$\bar{I}(t) = \int_{\Omega} \rho\psi(\boldsymbol{x}, t) dv.$$

Using the Reynolds' transport theorem we can write,

$$\frac{D}{Dt}\int_{\Omega} \rho\psi dv = \int_{\Omega} \left[\dot{\overline{\rho\psi}} + \rho\psi \mathrm{div}\boldsymbol{v}\right] dv, \Rightarrow \int_{\Omega} \left[\dot{\rho}\psi + \rho\dot{\psi} + \rho\psi \mathrm{div}\boldsymbol{v}\right] dv,$$

Now, from the mass continuity equation, we have,

$$\dot{\rho} + \rho \mathrm{div}\boldsymbol{v} = 0, \quad \Rightarrow \dot{\rho} = -\rho \mathrm{div}\boldsymbol{v}.$$

This implies,

$$\frac{D}{Dt}\int_{\Omega} \rho\psi dv = \int_{\Omega} \left[\rho\dot{\psi} - \rho\psi \mathrm{div}\boldsymbol{v} + \rho\psi \mathrm{div}\boldsymbol{v}\right] dv = \int_{\Omega} \rho\dot{\psi} dv,$$

$$\boxed{\frac{D}{Dt}\int_{\Omega} \rho(\boldsymbol{x}, t)\psi(\boldsymbol{x}, t) dv = \int_{\Omega} \rho(\boldsymbol{x}, t)\dot{\psi}(\boldsymbol{x}, t) dv.}$$

Linear and Angular Momentum

The **total linear momentum** of a continuum body (closed system) occupying a region Ω in the space is defined as,

$$\boxed{\boldsymbol{L}(t) = \int_{\Omega} \rho(\boldsymbol{x}, t)\boldsymbol{v}(\boldsymbol{x}, t)dv = \int_{\Omega_0} \rho_0(\boldsymbol{X})\boldsymbol{V}(\boldsymbol{X}, t)dV,}$$

where ρ and ρ_0 are spatial and material density, $\boldsymbol{v}$ and $\boldsymbol{V}$ are spatial and material velocity field.

The *total angular momentum* relative to a fixed point (whose position vector is $\boldsymbol{x}_0$) is defined as,

$$\boxed{\boldsymbol{J}(t) = \int_{\Omega} \boldsymbol{r} \times \rho(\boldsymbol{x}, t)\boldsymbol{v}(\boldsymbol{x}, t)dv = \int_{\Omega_0} \boldsymbol{r} \times \rho_0(\boldsymbol{X})\boldsymbol{V}(\boldsymbol{X}, t)dV,}$$

where $\boldsymbol{r}(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{x}_0$.

Angular momentum is also referred as **moment of momentum** or **the rotational momentum**.

Momentum balance principle

Total time derivative of linear and angular momentum of a continuum body results in following momentum balance principles.

The balance of linear momentum balance is,

$$\boxed{\dot{\boldsymbol{L}}(t) = \frac{D}{Dt}\int_{\Omega} \rho(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)dv = \frac{D}{Dt}\int_{\Omega_0} \rho_0(\boldsymbol{x})\boldsymbol{V}(\boldsymbol{X},t)dV = \boldsymbol{F}(t),}$$

where $\boldsymbol{F}(\mathrm{t})$ is the resultant force on the body.

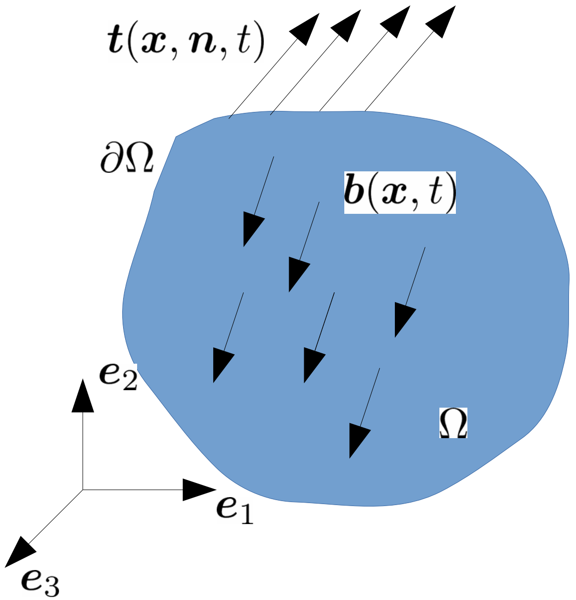
The balance of angular momentum balance is,

$$\boxed{\dot{\boldsymbol{J}}(t) = \frac{D}{Dt}\int_{\Omega} \boldsymbol{r}\times\rho(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)dv = \frac{D}{Dt}\int_{\Omega_0} \boldsymbol{r}\times\rho_0(\boldsymbol{X})\boldsymbol{V}(\boldsymbol{X},t)dV = \boldsymbol{M}(t),}$$

where $\boldsymbol{M}(t)$ is the resultant moment about $\boldsymbol{x}_0$.

Using Reynolds' transport theorem, alternate form of the momentum balance principles can be written as follows,

$$\boxed{\begin{aligned} \dot{\boldsymbol{L}}(t) &= \int_{\Omega} \rho \dot{\boldsymbol{v}} dv = \int_{\Omega_0} \rho_0 \dot{\boldsymbol{V}} dV = \boldsymbol{F}(t), \\ \dot{\boldsymbol{J}}(t) &= \int_{\Omega} \boldsymbol{r} \times \rho \dot{\boldsymbol{v}} dv = \frac{D}{Dt} \int_{\Omega_0} \boldsymbol{r} \times \rho_0 \dot{\boldsymbol{V}} dV = \boldsymbol{M}(t). \end{aligned}}$$



Consider a body in the current configuration with a volume Ω and the surface area as $\partial\Omega$. $\boldsymbol{t}(\boldsymbol{x},\boldsymbol{n},t)$ is the Cauchy traction vector and $\boldsymbol{b}(\boldsymbol{x},t)$ is a spatial vector field called body force.

Now, resultant force $\boldsymbol{F}(t)$ and the moment $\boldsymbol{M}(t)$ on the body will be given as,

$$\boldsymbol{F}(t) = \int_{\partial\Omega} \boldsymbol{t}ds + \int_{\Omega} \boldsymbol{b}dv, \text{ and}$$

$$\boldsymbol{M}(t) = \int_{\partial\Omega} \boldsymbol{r} \times \boldsymbol{t}ds + \int_{\Omega} \boldsymbol{r} \times \boldsymbol{b}dv.$$

Now, the global form of linear and angular momentum balance in spatial description can be written as,

$$\boxed{\begin{aligned} \dot{\boldsymbol{L}} = \frac{D}{Dt}\int_{\Omega} \rho \boldsymbol{v} dv = \int_{\Omega} \rho \dot{\boldsymbol{v}} dv = \int_{\partial\Omega} \boldsymbol{t} ds + \int_{\Omega} \boldsymbol{b} dv, \\ \dot{\boldsymbol{J}} = \frac{D}{Dt}\int_{\Omega} \boldsymbol{r} \times \rho \boldsymbol{v} dv = \int_{\Omega} \boldsymbol{r} \times \rho \dot{\boldsymbol{v}} dv = \int_{\partial\Omega} \boldsymbol{r} \times \boldsymbol{t} ds + \int_{\Omega} \boldsymbol{r} \times \boldsymbol{b} dv. \end{aligned}}$$

These are fundamental equations in the continuum mechanics.

Note that for balance of angular momentum we have assumed that the distributed resultant couples are neglected.