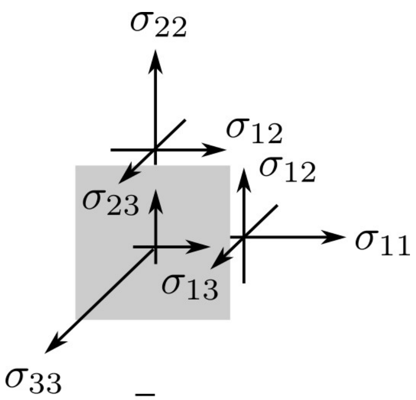
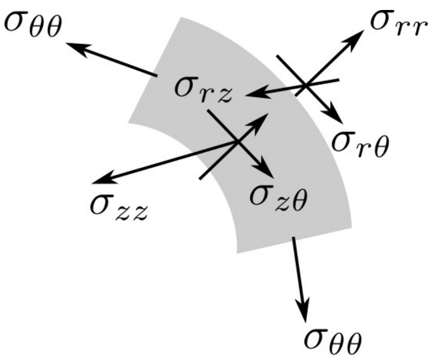
ME632: Fracture Mechanics

Timings	
Monday	10:00 to 11:20
Thursday	08:30 to 09:50

Anshul Faye
afaye@iitbhilai.ac.in
Room No. # 106

Williams' asymptotic method





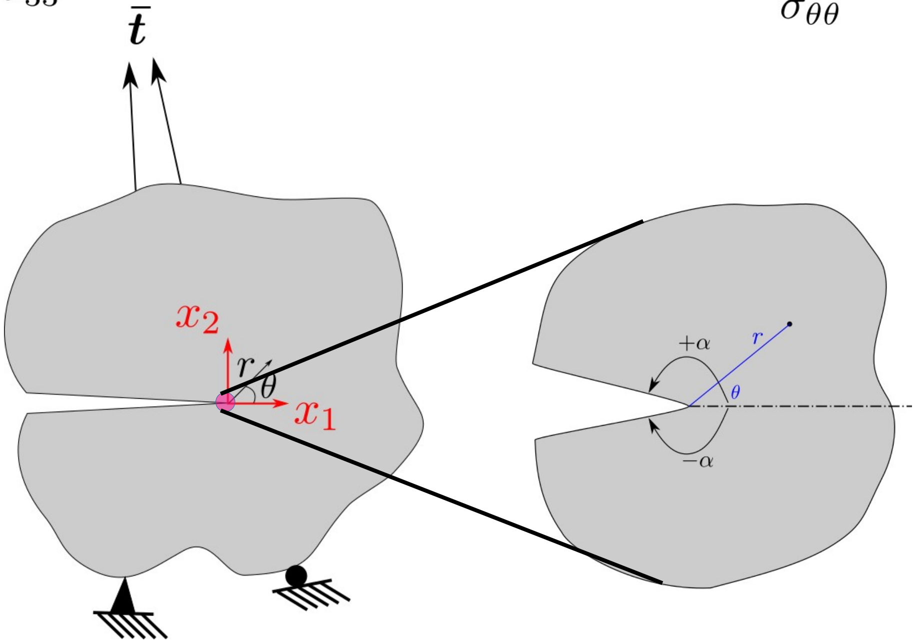


Figure shows a body with a notch. It is loaded by tractions on the remote boundaries. Williams developed a method of exploring the nature of the stress field near the crack-tip by defining a set of polar coordinates centered at the tip and expanding the stress field as an asymptotic series in powers of r.

We are concerned only with the stress components in the notch at very small values of r and hence we imagine looking at the corner through a strong microscope, so that we see the wedge. We magnify upto an extent so that the other surfaces of the body, including the loaded boundaries, appear far enough away for us and we can treat the wedge as infinite which is 'loading at infinity'.

Williams (1952) proposed the following form of Φ for crack-field solution,

$$\Phi = r^{\lambda+1}F(\theta). \qquad \cdots\cdots\cdots\cdots(14)$$

Φ in (14) must satisfy (11). So substituting (14) in (11), we get

$$\nabla^2\left(\nabla^2\Phi\right) = \nabla^2\left(\nabla^2 r^{\lambda+1}F(\theta)\right) = 0. \qquad \cdots\cdots\cdots\cdots(15)$$

Now using (12)

$$\begin{aligned}
&\nabla^2\left(r^{\lambda+1}F(\theta)\right)\\
\Rightarrow&\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(r^{\lambda+1}F(\theta)\right)\\
\Rightarrow&\lambda(\lambda+1)r^{\lambda-1}F(\theta) + (\lambda+1)r^{\lambda-1}F(\theta) + r^{\lambda-1}F''(\theta)\\
\Rightarrow&\left[\frac{\partial^2}{\partial\theta^2} + (\lambda+1)^2\right]F(\theta)r^{\lambda-1}. \qquad \cdots\cdots\cdots\cdots(17)
\end{aligned}$$

Similarly using (17) it can be shown that

$$\nabla^4\Phi = \left[\frac{\partial^2}{\partial\theta^2} + (\lambda+1)^2\right]\left[\frac{\partial^2}{\partial\theta^2} + (\lambda-1)^2\right]F(\theta)r^{\lambda-3} = 0 \qquad \cdots\cdots\cdots\cdots(18)$$

Equation (18) can be satisfied if,

$$\left[\frac{\partial^2}{\partial\theta^2}+(\lambda-1)^2\right]F(\theta)r^{\lambda-3}=0, \Rightarrow F(\theta)=c_1\cos\left[(\lambda-1)\theta\right]+c_2\sin\left[(\lambda-1)\theta\right]. \qquad \cdots\cdots\cdots\cdots(19)$$

or,

$$\left[\frac{\partial^2}{\partial\theta^2}+(\lambda+1)^2\right]F(\theta)r^{\lambda-3}=0, \Rightarrow F(\theta)=c_3\cos\left[(\lambda+1)\theta\right]+c_4\sin\left[(\lambda+1)\theta\right]. \qquad \cdots\cdots\cdots\cdots(20)$$

Thus, the general solution can now be written as,

$$F(\theta)=c_1\cos\left[(\lambda-1)\theta\right]+c_2\sin\left[(\lambda-1)\theta\right]+c_3\cos\left[(\lambda+1)\theta\right]+c_4\sin\left[(\lambda+1)\theta\right]. \qquad \cdots\cdots\cdots\cdots(21)$$

Here c_1, c_2, c_3 and c_4 are constant which can be determined by applying boundary conditions. Stress components are

Also,

$$\begin{aligned}F'(\theta)=&-c_1(\lambda-1)\sin\left[(\lambda-1)\theta\right]+c_2(\lambda-1)\cos\left[(\lambda-1)\theta\right]\\&-c_3(\lambda+1)\sin\left[(\lambda+1)\theta\right]+c_4(\lambda+1)\cos\left[(\lambda+1)\theta\right].\end{aligned} \qquad \cdots\cdots\cdots\cdots(22)$$

$$\begin{aligned}F''(\theta)=&-c_1(\lambda-1)^2\cos\left[(\lambda-1)\theta\right]-c_2(\lambda-1)^2\sin\left[(\lambda-1)\theta\right]\\&-c_3(\lambda+1)^2\cos\left[(\lambda+1)\theta\right]-c_4(\lambda+1)^2\sin\left[(\lambda+1)\theta\right].\end{aligned} \qquad \cdots\cdots\cdots\cdots(23)$$

Stress components are

$$\sigma_{rr} = r^{\lambda-1}\left[(\lambda+1)F(\theta) + F''(\theta)\right],$$

$$\sigma_{\theta\theta} = \lambda(\lambda+1)r^{\lambda-1}F(\theta), \qquad \cdots\cdots\cdots\cdots(24)$$

$$\sigma_{r\theta} = -\lambda r^{\lambda-1}F'(\theta).$$

We apply boundary conditions that crack faces are traction free, i.e.,

$$\sigma_{\theta\theta} = \sigma r\theta = 0 \text{ at } \theta = \pm\alpha. \qquad \cdots\cdots\cdots\cdots(25)$$

Using (22)-(24) we get,

$$\sigma_{\theta\theta}|_{\theta=\alpha} = \lambda(\lambda+1)r^{\lambda-1}\left[c_1\cos(\lambda-1)\alpha + c_2\sin(\lambda-1)\alpha + c_3\cos(\lambda+1)\alpha + c_4\sin(\lambda+1)\alpha\right] = 0$$

$$\sigma_{\theta\theta}|_{\theta=-\alpha} = \lambda(\lambda+1)r^{\lambda-1}\left[c_1\cos(\lambda-1)\alpha - c_2\sin(\lambda-1)\alpha + c_3\cos(\lambda+1)\alpha - c_4\sin(\lambda+1)\alpha\right] = 0$$

$$\begin{aligned}\sigma_{r\theta}|_{\theta=\alpha} = -\lambda r^{\lambda-1}[&-c_1(\lambda-1)\sin(\lambda-1)\alpha + c_2(\lambda-1)\cos(\lambda-1)\alpha \\ &-c_3(\lambda+1)\sin(\lambda+1)\alpha + c_4(\lambda+1)\cos(\lambda+1)\alpha] = 0\end{aligned}$$

$$\begin{aligned}\sigma_{r\theta}|_{\theta=-\alpha} = -\lambda r^{\lambda-1}[&c_1(\lambda-1)\sin(\lambda-1)\alpha + c_2(\lambda-1)\cos(\lambda-1)\alpha \\ &+c_3(\lambda+1)\sin(\lambda+1)\alpha + c_4(\lambda+1)\cos(\lambda+1)\alpha] = 0\end{aligned} \qquad \cdots\cdots\cdots\cdots(26)$$

(26) is a set of 4 homogeneous equations, which will have non-trivial solution only for few values of λ. λ=0 is one solution as it is common in all four equation. To determine other values of λ we first simplify these equations by adding and subtraction first two equations and last two equations and get

$$\begin{aligned}(\lambda+1)\left[c_1\cos(\lambda-1)\alpha+c_3\cos(\lambda+1)\alpha\right]&=0\\(\lambda+1)\left[c_2\sin(\lambda-1)\alpha+c_4\sin(\lambda+1)\alpha\right]&=0\\c_2(\lambda-1)\cos(\lambda-1)\alpha+c_4(\lambda+1)\cos(\lambda+1)\alpha&=0\\c_1(\lambda-1)\sin(\lambda-1)\alpha+c_3(\lambda+1)\sin(\lambda+1)\alpha&=0\end{aligned} \qquad \cdots\cdots\cdots\cdots(27)$$

This procedure gave us two independent matrix equations as follows,

$$\begin{bmatrix}(\lambda+1)\cos(\lambda-1)\alpha & (\lambda+1)\cos(\lambda+1)\alpha\\(\lambda-1)\sin(\lambda-1)\alpha & (\lambda+1)\sin(\lambda+1)\alpha\end{bmatrix}\begin{Bmatrix}c_1\\c_3\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix} \qquad \cdots\cdots\cdots\cdots(28)$$

$$\begin{bmatrix}(\lambda+1)\sin(\lambda-1)\alpha & (\lambda+1)\sin(\lambda+1)\alpha\\(\lambda-1)\cos(\lambda-1)\alpha & (\lambda+1)\cos(\lambda+1)\alpha\end{bmatrix}\begin{Bmatrix}c_2\\c_4\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix} \qquad \cdots\cdots\cdots\cdots(29)$$

For non-trivial solutions of c_1 and c_3, determinant of coefficient matrix of (28) must be zero, which gives characteristic equation as,

$$\lambda \sin 2\alpha + \sin 2\lambda\alpha = 0. \qquad \cdots\cdots\cdots\cdots(30)$$

Similar for non-trivial solutions of c_2 and c_4, determinant of coefficient matrix of (29) must be zero and the characteristic equation is,

$$\lambda \sin 2\alpha - \sin 2\lambda\alpha = 0. \qquad \cdots\cdots\cdots\cdots(31)$$

From (30) and (31) we conclude that,

$$\sin 2\lambda\alpha = 0, \quad \text{and} \quad \lambda \sin 2\alpha = 0. \qquad \cdots\cdots\cdots\cdots(32)$$

Now we consider the case of sharp crack, which means $\alpha = \pi$. In this case second equation of (32) is trivially satisfied and the first equation gives,

$$\sin 2\lambda\pi = 0 = \sin n\pi$$

$$\Rightarrow \lambda = n/2, \quad \text{where} \quad n = 0, \pm 1, \pm 2, \pm 3 \ldots \qquad \cdots\cdots\cdots\cdots(33)$$

One should realize that the displacements near the crack-tip will be proportional to r^{λ}. Then all negative values of λ lead to infinite displacements at the crack-tip, which is not physical. Hence negative values of λ are not allowed.

λ=0 gives displacement field which is independent of distance from the crack-tip, hence not allowed (because this is also not physical).

So now for other possible values of λ, we determine the values of constants c_1, c_2, c_3 and c_4 using (27).

$$c_{1n}\cos\left(\frac{n}{2}-1\right)\pi + c_{3n}\cos\left(\frac{n}{2}+1\right)\pi = 0$$

$$c_{2n}\sin\left(\frac{n}{2}-1\right)\pi + c_{4n}\sin\left(\frac{n}{2}+1\right)\pi = 0$$

$$\qquad\cdots\cdots\cdots\cdots(34)$$

$$c_{2n}\left(\frac{n}{2}-1\right)\cos\left(\frac{n}{2}-1\right)\pi + c_{4n}\left(\frac{n}{2}+1\right)\cos\left(\frac{n}{2}+1\right)\pi = 0$$

$$c_{1n}\left(\frac{n}{2}-1\right)\sin\left(\frac{n}{2}-1\right)\pi + c_{3n}\left(\frac{n}{2}+1\right)\sin\left(\frac{n}{2}+1\right)\pi = 0$$

For $n = 1, 3, 5,\ldots$ second and fourth equation of (34) gives,

$$c_{4n} = -c_{2n},$$

and

$$c_{3n} = -c_{1n}\frac{n-2}{n+2}. \qquad \cdots\cdots\cdots\cdots(35)$$

For $n = 2, 4, 6, \ldots$ first and third equation of (34) gives,

$$c_{3n} = -c_{1n},$$

and ....(36)

$$c_{4n} = -c_{2n}\frac{n-2}{n+2}.$$

So finally stress components are,

$$\sigma_{rr} = \sum_n r^{n/2-1}\left[\left\{\left(\frac{n}{2}+1\right) - \left(\frac{n}{2}-1\right)^2\right\}\left\{c_{1n}\cos\left(\frac{n}{2}-1\right)\theta + c_{2n}\sin\left(\frac{n}{2}-1\right)\theta\right\} + \left\{\left(\frac{n}{2}+1\right) - \left(\frac{n}{2}+1\right)^2\right\}\left\{c_{3n}\cos\left(\frac{n}{2}+1\right)\theta + c_{4n}\sin\left(\frac{n}{2}+1\right)\theta\right\}\right]$$

............(37)

$$\sigma_{r\theta} = \sum_{n=1,3,5,\ldots} \frac{n}{2}\left(\frac{n}{2}-1\right) r^{n/2-1}\left[c_{1n}\left(\frac{n}{2}-1\right)\left\{\sin\left(\frac{n}{2}-1\right)\theta - \frac{n-2}{n+2}\left(\frac{n}{2}+1\right)\sin\left(\frac{n}{2}+1\right)\theta\right\}+\right.$$

$$\left. c_{2n}\left\{-\left(\frac{n}{2}-1\right)\cos\left(\frac{n}{2}-1\right)\theta - \left(\frac{n}{2}+1\right)\cos\left(\frac{n}{2}+1\right)\theta\right\}\right]+$$

$$\sum_{n=2,4,6,\ldots} \frac{n}{2} r^{n/2-1}\left[c_{1n}\left(\frac{n}{2}-1\right)\left\{\sin\left(\frac{n}{2}-1\right)\theta - \left(\frac{n}{2}+1\right)\sin\left(\frac{n}{2}+1\right)\theta\right\}+\right.$$

$$\left. c_{2n}\left\{\left(\frac{n}{2}-1\right)\cos\left(\frac{n}{2}-1\right)\theta + \frac{n-2}{n+2}\left(\frac{n}{2}+1\right)\sin\left(\frac{n}{2}+1\right)\theta\right\}\right] \qquad \cdots\cdots\cdots\cdots(38)$$

$$\sigma_{\theta\theta} = \sum_{n=1,3,5,\ldots} \frac{n}{2}\left(\frac{n}{2}+1\right) r^{n/2-1}\left[c_{1n}\left\{\cos\left(\frac{n}{2}-1\right)\theta - \frac{n-2}{n+2}\cos\left(\frac{n}{2}+1\right)\theta\right\}+\right.$$

$$\left. c_{2n}\left\{\sin\left(\frac{n}{2}-1\right)\theta - \sin\left(\frac{n}{2}+1\right)\theta\right\}\right]+$$

$$\sum_{n=2,4,6,\ldots} \frac{n}{2}\left(\frac{n}{2}-1\right) r^{n/2-1}\left[c_{1n}\left\{\cos\left(\frac{n}{2}-1\right)\theta - \cos\left(\frac{n}{2}+1\right)\theta\right\}+\right.$$

$$\left. c_{2n}\left\{\sin\left(\frac{n}{2}-1\right)\theta - \frac{n-2}{n+2}\sin\left(\frac{n}{2}+1\right)\theta\right\}\right] \qquad \cdots\cdots\cdots\cdots(39)$$

Note that in all the stress components dominant term corresponds to n=1. Thus,

$$\sigma_{rr} = \frac{1}{\sqrt{r}}\left[c_{11}\cos\frac{\theta}{2}\left(2-\cos^2\frac{\theta}{2}\right)+c_{21}\sin\frac{\theta}{2}\left(2-3\sin^2\frac{\theta}{2}\right)\right] \quad \cdots\cdots\cdots\cdots(40)$$

$$\sigma_{\theta\theta} = \frac{3}{4\sqrt{r}}\left[c_{11}\left(\cos\frac{\theta}{2}+\frac{1}{3}\cos\frac{3\theta}{2}\right)-c_{21}\left(\sin\frac{\theta}{2}+\sin\frac{3\theta}{2}\right)\right] \quad \cdots\cdots\cdots\cdots(41)$$

$$\sigma_{r\theta} = \frac{1}{4\sqrt{r}}\left[c_{11}\left(\sin\frac{\theta}{2}+\sin\frac{3\theta}{2}\right)+c_{21}\left(\cos\frac{\theta}{2}+3\cos\frac{3\theta}{2}\right)\right] \quad \cdots\cdots\cdots\cdots(42)$$

Now we define,

$$(\sigma_{rr})_{\text{symm}} = \frac{K_I}{\sqrt{2\pi r}}\left[\cos\frac{\theta}{2}\left(2-\cos^2\frac{\theta}{2}\right)\right], \qquad (\sigma_{rr})_{\text{asymm}} = \frac{K_{II}}{\sqrt{2\pi r}}\left[\sin\frac{\theta}{2}\left(2-3\sin^2\frac{\theta}{2}\right)\right], \quad \cdots\cdot(43)$$

$$(\sigma_{\theta\theta})_{\text{symm}} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{3}{4}\left(\cos\frac{\theta}{2}+\frac{1}{3}\cos\frac{3\theta}{2}\right)\right], (\sigma_{\theta\theta})_{\text{asymm}} = -\frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{3}{4}\left(\sin\frac{\theta}{2}+\sin\frac{3\theta}{2}\right)\right], \quad \cdots\cdot(44)$$

$$(\sigma_{r\theta})_{\text{symm}} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{1}{4}\left(\sin\frac{\theta}{2}+\sin\frac{3\theta}{2}\right)\right], \ (\sigma_{r\theta})_{asymm} = \frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{1}{4}\left(\cos\frac{\theta}{2}+3\cos\frac{3\theta}{2}\right)\right], \quad \cdots\cdot(45)$$

where, $K_I = c_{11}\sqrt{2\pi}$ and $K_I = c_{21}\sqrt{2\pi}$. $\cdots\cdots\cdots\cdots(46)$

Expressions for strains near the crack-tip can be obtained using (5) (for plane σ) and using (6) (for plane ε) as follows.

(for plane σ)

$$\varepsilon_{rr} = \frac{1}{E}\left[\sigma_{rr} - \nu\sigma_{\theta\theta}\right],$$
$$\varepsilon_{\theta\theta} = \frac{1}{E}\left[\sigma_{\theta\theta} - \nu\sigma_{rr}\right],$$
$$\varepsilon_{r\theta} = \frac{\sigma_{r\theta}}{2\mu} = \frac{1+\nu}{E}\sigma_{r\theta},$$
$$\varepsilon_{zz} = -\frac{\nu}{E}\left(\sigma_{rr} + \sigma_{\theta\theta}\right).$$

(for plane ε)

$$\varepsilon_{rr} = \frac{1}{E'}\left[\sigma_{rr} - \nu'\sigma_{\theta\theta}\right],$$
$$\varepsilon_{\theta\theta} = \frac{1-\nu^2}{E}\left[\sigma_{\theta\theta} - \frac{\nu}{1-\nu}\sigma_{rr}\right], = \frac{1}{E'}\left[\sigma_{\theta\theta} - \nu'\sigma_{rr}\right],$$
$$\varepsilon_{r\theta} = \frac{\sigma_{r\theta}}{2\mu} = \frac{1+\nu}{E}\sigma_{r\theta} = \frac{1+\nu'}{E'}\sigma_{r\theta},$$
$$\varepsilon_{zz} = 0.$$

Crack-tip displacements (u_r, u_θ) follows from strain using strain-displacement relations (2).

$$u_r = \frac{K_I\sqrt{r}}{4\mu\sqrt{2\pi}}\left[(2\kappa-1)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right], \quad u_r = \frac{K_{II}\sqrt{r}}{4\mu\sqrt{2\pi}}\left[-(2\kappa-1)\sin\frac{\theta}{2} + 3\sin\frac{3\theta}{2}\right], \quad \cdots\cdots(47)$$

$$u_\theta = \frac{K_I\sqrt{r}}{4E\sqrt{2\pi}}\left[(2\kappa+1)\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right]. \quad u_\theta = \frac{K_{II}\sqrt{r}}{4E\sqrt{2\pi}}\left[-(2\kappa+1)\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}\right]. \quad \cdots\cdots(48)$$

Here, $\kappa = 3 - 4\nu$ for plane ε and $\kappa = (3-\nu)/(1+\nu)$ for plane σ.

Solution in Cartesian Coordinate system

	(for Mode I)	(for Mode II)
$\sigma_{ij} = \frac{K_{I/II}}{\sqrt{2\pi r}}\hat{\sigma}_{ij}(\theta)$	$\hat{\sigma}_{xx} = \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$ $\hat{\sigma}_{yy} = \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$ $\hat{\sigma}_{xy} = \frac{1}{2}\sin\theta\cos\frac{3\theta}{2}$ ....(49)	$\hat{\sigma}_{xx} = \sin\frac{\theta}{2}\left(-2 - \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$ $\hat{\sigma}_{yy} = \frac{1}{2}\sin\theta\cos\frac{3\theta}{2}$ $\hat{\sigma}_{xy} = \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$ ....(50)
$u_i = \frac{K_{I/II}\sqrt{r}}{2\mu\sqrt{2\pi}}\hat{u}_i(\theta)$	$\hat{u}_x = \left[(\kappa - 1)\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2}\right]$ $\hat{u}_y = \left[(\kappa + 1)\sin\frac{\theta}{2} - \sin\theta\cos\frac{\theta}{2}\right]$ ....(51)	$\hat{u}_x = \sin\frac{\theta}{2}\left[\kappa + 1 + 2\cos^2\frac{\theta}{2}\right]$ $\hat{u}_y = \cos\frac{\theta}{2}\left[\kappa - 1 + 2\sin^2\frac{\theta}{2}\right]$ ....(52)