# ME232: Dynamics

# Vibrations

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# *Harmonic excitation:*

As a very first step we start with the harmonic force which occurs frequently in engineering.

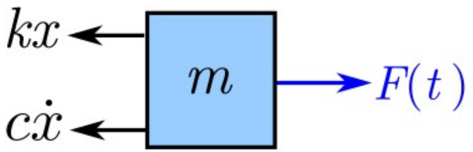
The understanding of the analysis associated with harmonic forces is also a necessary step in the study of more complex forms.

Consider a system where the body is subjected to the external harmonic force $F = F_0 \sin \omega t$, in which $F_0$ is the force amplitude and $\omega$ is the driving frequency (in radians/second).

Note that $\omega_n$, which is a **property of the system**, is different from $\omega$, which is a **property of the force** applied to the system.

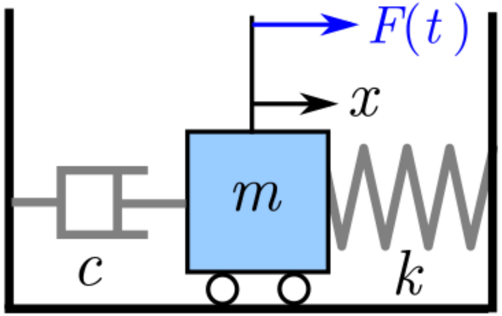
From the free-body diagram, we apply Newton's second law to obtain

$$-kx - c\dot{x} + F_0 \sin \omega t = m\ddot{x} \quad \text{or} \quad m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t.$$




Alternate form of the above equation is,

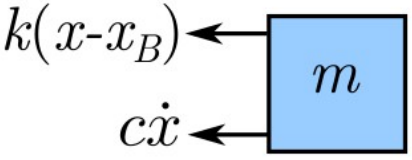
$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m} \sin \omega t. \qquad \cdots\cdots\cdots\cdots(20)$$



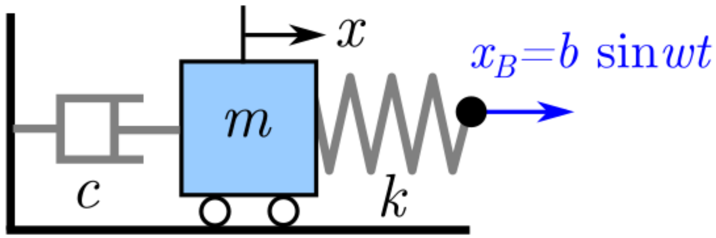
# *Base excitation:*

In many cases, the excitation of the mass is due to **the movement of the base or foundation** to which the mass is connected by springs or other compliant mountings; e.g., seismographs, vehicle suspensions, and structures shaken by earthquakes.

Harmonic movement of the base is **equivalent to the direct application of a harmonic force**.




Consider the system where the spring is attached to a movable base. $x$ is the displacement of mass from the equilibrium position when the base were in its neutral position.




The base, is assumed to have a harmonic movement $x_B = b\ \sin\omega t$. The spring deflection is the difference between the inertial displacements of the mass and the base.

From the free-body diagram, Newton's second law gives

$$-k(x - x_b) - c\dot{x} = m\ddot{x} \quad \text{or} \quad \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{kb}{m}\sin\omega t. \qquad \cdots\cdots\cdots\cdots(21)$$

It can be observed (21) is similar to (20).

## *Undamped force vibration:*

We start with the case where damping is negligible ($c = 0$). The basic equation of motion becomes

$$\ddot{x} + \omega_n^2 x = \frac{F_0}{m}\sin\omega t. \qquad \text{............(22)}$$

The complete solution to (22) is the sum of the **complementary solution** $x_c$, which is the general solution of (22) with the right side set to zero, and **the particular solution** $x_p$, which is any solution to the complete equation. Thus, $x = x_c + x_p$ . The complementary solution in already available (i.e. Equation (5) or (7)). A particular solution is investigated by assuming that **the form of the response to the force should resemble that of the force term.** Hence, we assume

$$x_p = X\sin\omega t \qquad \text{............(23)}$$

where $X$ is the amplitude (in units of length) of the particular solution. Substituting this expression into (22) and solving for $X$ yield

$$X = \frac{F_0/k}{1-(\omega/\omega_n)^2}.$$

Thus the particular solution is,

$$x_p = \frac{F_0/k}{1-(\omega/\omega_n)^2}\sin\omega t. \qquad \cdots\cdots\cdots\cdots(24)$$

The complementary solution, known as **the transient solution**, is of no special interest here since, **with time, it dies out with the small amount of damping** which is always unavoidably present.

The particular solution $x_p$ describes the continuing motion and is called the **steady-state solution**. Its period is $\tau = 2\pi/\omega$, the same as that of the forcing function.

We are primarily interested in amplitude $X$ of the motion. If $\delta_{st}$ be the magnitude of the static deflection of the mass under a static load $F_0$, (i.e., $\delta_{st} = F_0/k$), then we can write,

$$M = \frac{x_p}{\delta_{st}} = \frac{1}{1-(\omega/\omega_n)^2}. \qquad \cdots\cdots\cdots\cdots(25)$$

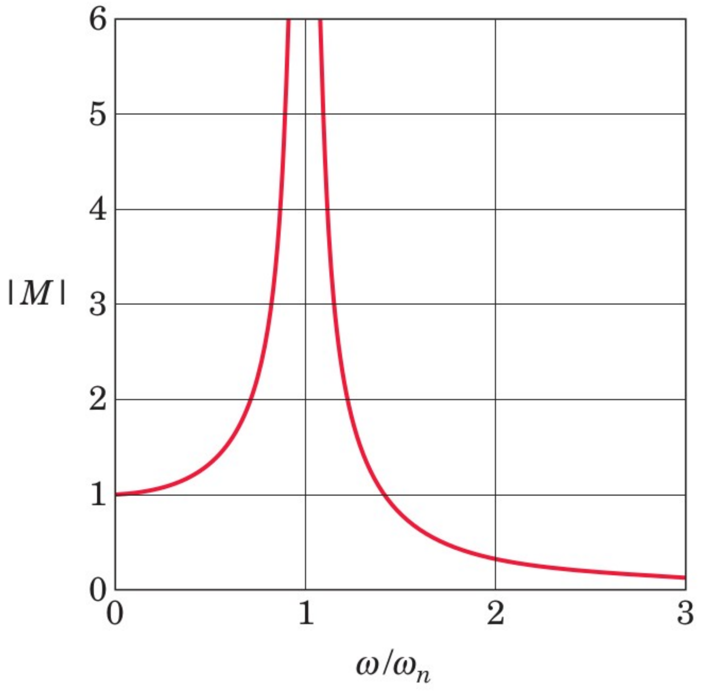
The ratio $M$ is called **the amplitude ratio or magnification factor and is a measure of the severity of the vibration**.

$M$ approaches infinity as $\omega$ approaches $\omega_n$. Consequently, if the system possesses no damping and is excited by a harmonic force whose frequency $\omega$ approaches the natural frequency $\omega_n$ of the system, then $M$, and thus $X$, increase without limit. Physically, this means that the motion amplitude would reach the limits of the attached spring, **which is a condition to be avoided.**

The value $\omega_n$ is called the **resonant or critical frequency of the system**, and the condition of $\omega$ being close in value to $\omega_n$ with the resulting large displacement amplitude $X$ is called **resonance**.

For $\omega<\omega_n$, the magnification factor $M$ is positive, and the vibration is in phase with the force $F$. For $\omega>\omega_n$, the magnification factor is negative, and the vibration is 180° out of phase with $F$.

Figure shows a plot of the absolute value of $M$ as a function of the driving-frequency ratio $\omega/\omega_n$.



## *Damped force vibration:*

We now reintroduce damping in our expressions for forced vibration. Our basic differential equation of motion is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}\sin\omega t. \qquad \cdots\cdots\cdots\cdots(26)$$

Again, the complete solution is the sum of the complementary solution $x_c$, which is the general solution of (26) with the right side equal to zero, and the particular solution $x_p$, which is any solution to the complete equation. We have already developed the complementary solution $x_c$ (i.e., Eq.(16)).

It can be shown that in the present of damping a single sine or cosine term, such as we were able to use for the undamped case, is not sufficiently general for the particular solution. So we try

$$x_p = X_1\cos\omega t + X_2\sin\omega t \quad \text{or} \quad x_p = X\sin(\omega t - \phi).$$

Substituting in (26), match coefficients of sin$\omega t$ and cos$\omega t$, and solve the resulting two equations to obtain

$$X = \frac{F_0/k}{\sqrt{[1-(\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}} \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2}\right) \quad \cdots\cdots\cdots\cdots(27)$$

The complete solution is now known, and for underdamped systems it can be written as

$$x = Ce^{-\zeta\omega_n t}\sin(\omega_d t + \psi) + X\sin(\omega t - \phi). \qquad \cdots\cdots\cdots\cdots(28)$$

Because the first term on the right side **diminishes with time**, it is known as the **transient solution**.

The particular solution $x_p$ is the steady-state solution and is the part of the solution in which we are primarily interested.

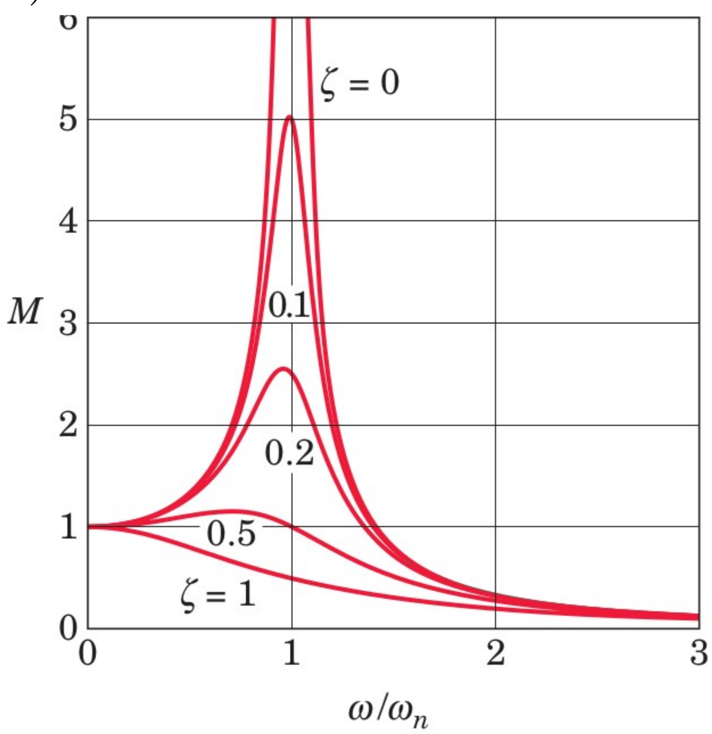
All quantities on the right side of (28) are properties of the system and the applied force, except for $C$ and $\psi$ (which are determinable from initial conditions) and the running time variable $t$.

## *Magnification factor and phase angle:*

Near resonance the magnitude $X$ of the steady-state solution is a strong function of the damping ratio $\zeta$ and the nondimensional frequency ratio $\omega/\omega_n$. It is again convenient to form the nondimensional ratio $M = X/(F_0/k)$, which is called the **amplitude ratio** or **magnification factor**

$$M = \frac{1}{\sqrt{[1-(\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2}}. \qquad \cdots\cdots\cdots\cdots(29)$$

Figure shows a plot of $M$ vs. $\omega/\omega_n$ for various values of the damping ratio $\zeta$ .




We can conclude that if a motion amplitude is excessive, two possible solutions would be to

(a) increase the damping (to obtain a larger value of $\zeta$) or

(b) alter the driving frequency so that $\omega$ is farther from the resonant frequency $\omega_n$.

(c)The addition of damping is most effective near resonance.

The phase angle $\phi$, given by (27), can vary from 0 to $\pi$ and represents the part of a cycle (and thus the time) by which the response $x_p$ lags the forcing function $F$.

The phase angle $\phi$ varies with the frequency ratio is plotted for various values of the damping ratio $\zeta$. Note that the value of $\phi$, when $\omega/\omega_n = 1$, is 90° for all values of $\zeta$.

Two examples of the variation of $F$ and $x_p$ with $\omega t$ are shown. They show the phase difference between the response and the forcing function.