Introduction to Tensors

Tensor functions

Tensor functions have *one* or *more tensor variables* as argument and their values are *scalars*, *vectors* or *tensors*.

For example: $\mathit{\Phi}(\boldsymbol{A})$, $\boldsymbol{u}(\boldsymbol{A})$, $\boldsymbol{F}(\boldsymbol{A})$ are scaler-valued, vector-valued and tensor-valued *tensor functions* of one tensor variable $\boldsymbol{A}$, respectively.

Similarly, $\mathit{\Phi}(\boldsymbol{v})$, $\boldsymbol{u}(\boldsymbol{v})$, $\boldsymbol{F}(\boldsymbol{v})$ are scaler-valued, vector-valued and tensor-valued *vector functions* of one vector variable, respectively.

In general, we will call all those functions, whose arguments are tensors, vectors or scalers as *tensor functions*.

Tensor functions

Usual rules of differentiation apply to tensor function of one scaler variable
For e.g. to find the derivative of $\boldsymbol{A}^{-1}$, where $\boldsymbol{A}$ is a function of scaler variable t, we use the identity,

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$$

$$\frac{D}{Dt}\boldsymbol{A}\boldsymbol{A}^{-1} = \frac{D}{Dt}\boldsymbol{I}$$

$$\Rightarrow \dot{\boldsymbol{A}}\boldsymbol{A}^{-1} + \boldsymbol{A}\dot{\boldsymbol{A}}^{-1} = 0$$

$$\Rightarrow \boldsymbol{A}\dot{\boldsymbol{A}}^{-1} = -\dot{\boldsymbol{A}}\boldsymbol{A}^{-1}$$

$$\Rightarrow \dot{\boldsymbol{A}}^{-1} = -\boldsymbol{A}^{-1}\dot{\boldsymbol{A}}\boldsymbol{A}^{-1}$$

Notice that usual chain rule of differentiation is applied in the above derivation.

Gradient of a scaler function of tensor varialbe can be obtained by realizing that $\phi\ (\boldsymbol{A}) = \phi\ (A_{11}, A_{12}, A_{13} ...)$, so that the total derivation of ϕ is given as,

$$
\begin{aligned}
d\phi &= \frac{\partial \phi}{\partial A_{11}} dA_{11} + \frac{\partial \phi}{\partial A_{12}} dA_{12} + \frac{\partial \phi}{\partial A_{13}} dA_{13} \dots \\
&\Rightarrow \frac{\partial \phi}{\partial A_{ij}} dA_{ij} \\
&\Rightarrow \frac{\partial \phi}{\partial \boldsymbol{A}} : d\boldsymbol{A} \\
&\Rightarrow \dot{\phi} = \frac{\partial \phi}{\partial \boldsymbol{A}} : \dot{\boldsymbol{A}}
\end{aligned}
$$

where, $\dfrac{\partial \phi}{\partial \boldsymbol{A}} = \dfrac{\partial \phi}{\partial A_{ij}} \boldsymbol{e}_i \boldsymbol{e}_j$ is a second order tensor called the gradient of ϕ.

Similarly for a tensor function of tensor variable, we can write,

$$
\begin{aligned}
dF_{ij} &= \frac{\partial F_{ij}}{\partial A_{mn}} dA_{mn} \\
&\Rightarrow \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{A}} : d\boldsymbol{A} \\
&\Rightarrow \dot{\boldsymbol{F}} = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{A}} : \dot{\boldsymbol{A}}
\end{aligned}
$$

where, the gradient of $\boldsymbol{F}$, $\dfrac{\partial \boldsymbol{F}}{\partial \boldsymbol{A}} = \dfrac{\partial F_{ij}}{\partial A_{mn}} \boldsymbol{e}_i \boldsymbol{e}_j \boldsymbol{e}_m \boldsymbol{e}_m$ is a fourth order tensor.

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Examples

Example 10: If $\boldsymbol{A}$ is a second order invertible tensor then show that,

$$\frac{\partial \boldsymbol{A}^{-1}}{\partial \boldsymbol{A}} : \boldsymbol{B} = -\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{A}^{-1}.$$

We start from the fact that,

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$$

$$\Rightarrow \frac{\partial\left(\boldsymbol{A}\boldsymbol{A}^{-1}\right)}{\partial \boldsymbol{A}} = \frac{\partial\left(A_{im}A^{-1}_{mj}\right)}{\partial A_{kl}} = 0$$

$$\Rightarrow \frac{\partial A_{im}}{\partial A_{kl}} A^{-1}_{mj} + A_{im}\frac{\partial A^{-1}_{mj}}{\partial A_{kl}} = 0$$

$$\Rightarrow A_{im}\frac{\partial A^{-1}_{mj}}{\partial A_{kl}} = -\delta_{ik}\delta_{ml}A^{-1}_{mj}$$

Multiplying bothside by A^{-1}_{ni}

$$\Rightarrow A^{-1}_{ni}A_{im}\frac{\partial A^{-1}_{mj}}{\partial A_{kl}} = -A^{-1}_{ni}\delta_{ik}\delta_{ml}A^{-1}_{mj}$$

$$\Rightarrow \delta_{nm}\frac{\partial A^{-1}_{mj}}{\partial A_{kl}} = -A^{-1}_{nk}A^{-1}_{lj}$$

$$\Rightarrow \frac{\partial A^{-1}_{nj}}{\partial A_{kl}} = -A^{-1}_{nk}A^{-1}_{lj}$$

$$\Rightarrow \frac{\partial A^{-1}_{nj}}{\partial A_{kl}}B_{kl} = -A^{-1}_{nk}B_{kl}A^{-1}_{lj}$$

$$\Rightarrow \frac{\partial \boldsymbol{A}^{-1}}{\partial \boldsymbol{A}} : \boldsymbol{B} = -\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{A}^{-1}$$

Examples

Example 11: Show that,

$$\frac{\partial \det \boldsymbol{A}}{\partial \boldsymbol{A}} = \left(\boldsymbol{A}^T\right)^2 - I_1 \boldsymbol{A}^T + I_2 \boldsymbol{I}.$$

We first find,

$$I_1 = \mathrm{tr}\boldsymbol{A} = A_{ii}$$

$$\frac{\partial I_1}{\partial A_{mn}} = \frac{\partial A_{ii}}{\partial A_{mn}} = \delta_{im}\delta_{in} = \delta_{mn} = \boldsymbol{I}$$

also,

$$I_2 = \frac{1}{2}\left[(\mathrm{tr}\boldsymbol{A})^2 - \mathrm{tr}\boldsymbol{A}^2\right] = \frac{1}{2}(A_{ii}A_{jj} - A_{ij}A_{ji})$$

$$\frac{\partial I_2}{\partial A_{mn}} = \frac{1}{2}\left(\frac{\partial A_{ii}}{\partial A_{mn}}A_{jj} + A_{ii}\frac{\partial A_{jj}}{\partial A_{mn}} - \frac{\partial A_{ij}}{\partial A_{mn}}A_{ji} - \frac{\partial A_{ji}}{\partial A_{mn}}A_{ij}\right)$$

$$\Rightarrow \frac{1}{2}\left(\delta_{im}\delta_{in}A_{jj} + A_{ii}\delta_{jm}\delta_{jn} - \delta_{im}\delta_{jn}A_{ji} - \delta_{jm}\delta_{in}A_{ij}\right)$$

$$\Rightarrow \frac{1}{2}\left(\delta_{mn}A_{jj} + A_{ii}\delta_{mn} - A_{nm} - A_{nm}\right) = A_{ii}\delta_{mn} - A_{nm} = I_1\boldsymbol{I} - \boldsymbol{A}^T$$

Examples

Now, We start with the Cayley- Hamilton equation,

$$\boldsymbol{A}^3 - I_1\boldsymbol{A}^2 + I_2\boldsymbol{A} - I_3\boldsymbol{I} = \boldsymbol{O}$$

$$A_{ip}A_{pq}A_{qj} - I_1A_{ip}A_{pj} + I_2A_{ij} - I_3\delta_{ij} = 0$$

$$\Rightarrow \left(A_{ip}A_{pq}A_{qj} - I_1A_{ip}A_{pj} + I_2A_{ij} - I_3\delta_{ij}\right)\delta_{ij} = 0$$

$$\Rightarrow A_{ip}A_{pq}A_{qi} - I_1A_{ip}A_{pi} + I_2A_{ii} - I_3\delta_{ii} = 0$$

$$\Rightarrow 3I_3 = A_{ip}A_{pq}A_{qi} - I_1A_{ip}A_{pi} + I_2I_1$$

$$\Rightarrow 3\frac{\partial I_3}{\partial A_{mn}} = \delta_{im}\delta_{pn}A_{pq}A_{qi} + \delta_{pm}\delta_{qn}A_{ip}A_{qi} + \delta_{qm}\delta_{in}A_{ip}A_{pq} - \frac{\partial I_1}{\partial A_{mn}}A_{ip}A_{pi} - I_1\delta_{im}\delta_{pn}A_{pi} - I_1\delta_{pm}\delta_{in}A_{ip} + \frac{I_2}{\partial A_{mn}}I_1 + I_2\frac{\partial I_1}{\partial A_{mn}}$$

$$\Rightarrow 3\frac{\partial I_3}{\partial A_{mn}} = A_{nq}A_{qm} + A_{im}A_{ni} + A_{np}A_{pm} - \delta_{mn}A_{ip}A_{pi} - I_1A_{nm} - I_1A_{nm} + \frac{I_2}{\partial A_{mn}}I_1 + I_2\delta_{mn}$$

$$\Rightarrow 3\frac{\partial I_3}{\partial A_{mn}} = A_{qm}A_{nq} + A_{im}A_{ni} + A_{pm}A_{np} - \delta_{mn}A_{ip}A_{pi} - 2I_1A_{nm} + \frac{\partial I_2}{\partial A_{mn}}I_1 + I_2\delta_{mn}$$

$$\Rightarrow 3\frac{\partial I_3}{\partial A_{mn}} = 3A_{qm}A_{nq} - \delta_{mn}\mathrm{tr}\boldsymbol{A}^2 - 2I_1A_{nm} + (I_1\delta_{mn} - A_{nm})I_1 + I_2\delta_{mn}$$

$$\Rightarrow 3\frac{\partial I_3}{\partial A_{mn}} = 3A_{qm}A_{nq} + \delta_{mn}(I_1^2 - \mathrm{tr}\boldsymbol{A}^2) - 2I_1A_{nm} - A_{nm}I_1 + I_2\delta_{mn}$$

$$\Rightarrow 3\frac{\partial I_3}{\partial A_{mn}} = 3A_{qm}A_{nq} + 2I_2\delta_{mn} - 3I_1A_{nm} + I_2\delta_{mn} = 3(A_{qm}A_{nq} + I_2\delta_{mn} - I_1A_{nm})$$

$$\Rightarrow \frac{\partial I_3}{\partial \boldsymbol{A}} = \left(\boldsymbol{A}^T\right)^2 + 2I_2\boldsymbol{I} - I_1\boldsymbol{A}^T$$

Gradient, Curl and Divergence of a vector field

Different operation of $\boldsymbol{\nabla}$ operator are governed by following rules:

$$\boldsymbol{\nabla} \cdot (\bullet) = \frac{\partial(\bullet)}{\partial x_i} \cdot \boldsymbol{e}_i, \quad \boldsymbol{\nabla} \times (\bullet) = \boldsymbol{e}_i \times \frac{\partial(\bullet)}{\partial x_i}, \quad \boldsymbol{\nabla} \otimes (\bullet) = \frac{\partial(\bullet)}{\partial x_i} \otimes \boldsymbol{e}_i.$$

Following above rules, following are defined.

Divergence of a vector field $\boldsymbol{u}$,

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial x_i} \cdot \boldsymbol{e}_i = \frac{\partial u_m}{\partial x_i} \boldsymbol{e}_m \cdot \boldsymbol{e}_i = \frac{\partial u_m}{\partial x_i} \delta_{mi} = \frac{\partial u_i}{\partial x_i}.$$

Curl of a vector field $\boldsymbol{u}$,

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{e}_i \times \frac{\partial \boldsymbol{u}}{\partial x_i} = \boldsymbol{e}_i \times \frac{\partial u_m}{\partial x_i} \boldsymbol{e}_m = \frac{\partial u_m}{\partial x_i} \boldsymbol{e}_i \times \boldsymbol{e}_m.$$

Gradient, Curl and Divergence of a vector field

Gradient of a vector field $\boldsymbol{u}$,

$$\boldsymbol{\nabla} \otimes \boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial x_i} \otimes \boldsymbol{e}_i = \frac{\partial u_m}{\partial x_i} \boldsymbol{e}_m \otimes \boldsymbol{e}_i .$$

In matrix notations,

$$[\boldsymbol{\nabla} \otimes \boldsymbol{u}] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

Transposed gradient of a vector field $\boldsymbol{u}$,

$$\boldsymbol{u} \otimes \boldsymbol{\nabla} = \boldsymbol{e}_i \otimes \frac{\partial \boldsymbol{u}}{\partial x_i} = \frac{\partial u_m}{\partial x_i} \boldsymbol{e}_i \otimes \boldsymbol{e}_m .$$

Laplacian and Hessian

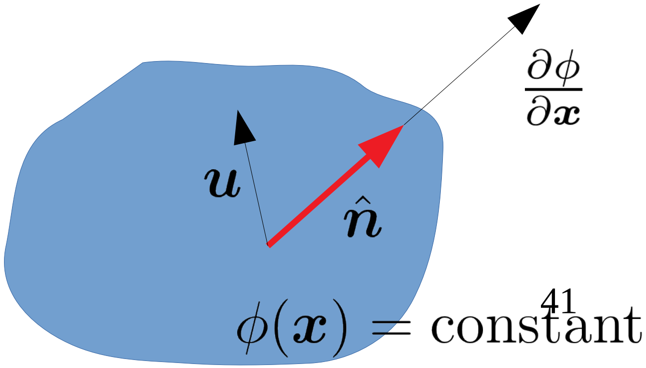
$\boldsymbol{\nabla}$ operator dotted with itself gives *Laplacian* as,

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}(\bullet) = \frac{\partial}{\partial x_i}\boldsymbol{e}_i \cdot \frac{\partial(\bullet)}{\partial x_j}\boldsymbol{e}_j = \frac{\partial}{\partial x_i}\frac{\partial(\bullet)}{\partial x_j}\delta_{ij} = \frac{\partial^2(\bullet)}{\partial x_i^2} = \nabla^2(\bullet).$$

Similarly, $\boldsymbol{\nabla}\otimes\boldsymbol{\nabla}$ gives *Hessian* as,

$$\boldsymbol{\nabla}\otimes\boldsymbol{\nabla}(\bullet) = \frac{\partial}{\partial x_i}\boldsymbol{e}_i \otimes \frac{\partial(\bullet)}{\partial x_j}\boldsymbol{e}_j = \frac{\partial}{\partial x_i}\frac{\partial(\bullet)}{\partial x_j}\boldsymbol{e}_i\otimes\boldsymbol{e}_j = \frac{\partial^2(\bullet)}{\partial x_i \partial x_j}\boldsymbol{e}_i\otimes\boldsymbol{e}_j.$$

An important concept of *directinal derivative* can be introduced here. $\boldsymbol{\nabla}\phi\cdot\boldsymbol{u}$ is the directinal derivative of ϕ with respect to $\boldsymbol{x}$ in the direction of vector $\boldsymbol{u}$.



Examples

Example 12:

If $\boldsymbol{u}(\boldsymbol{x}) = x_1x_2x_3\boldsymbol{e}_1 + x_1x_2\boldsymbol{e}_2 + x_1\boldsymbol{e}_3$, then determine $\boldsymbol{\nabla}\boldsymbol{u}$, $\boldsymbol{\nabla}\cdot\boldsymbol{u}$, and $\nabla^2\boldsymbol{u}$.

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial x_i}\cdot\boldsymbol{e}_i$$

$$\Rightarrow (x_2x_3\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + \boldsymbol{e}_3)\cdot\boldsymbol{e}_1 + (x_1x_3\boldsymbol{e}_1 + x_1\boldsymbol{e}_2)\cdot\boldsymbol{e}_2 + (x_1x_2\boldsymbol{e}_1)\cdot\boldsymbol{e}_3$$

$$\Rightarrow x_2x_3 + x_1$$

$$\boldsymbol{\nabla}\boldsymbol{u} = \frac{\partial u_i}{\partial x_j}\boldsymbol{e}_i\boldsymbol{e}_j$$

$$\Rightarrow x_2x_3\boldsymbol{e}_1\boldsymbol{e}_1 + x_1x_3\boldsymbol{e}_1\boldsymbol{e}_2 + x_1x_2\boldsymbol{e}_1\boldsymbol{e}_3$$

$$+ x_2\boldsymbol{e}_2\boldsymbol{e}_1 + x_1\boldsymbol{e}_2\boldsymbol{e}_2 + \boldsymbol{e}_3\boldsymbol{e}_1$$

In matrix notations,

$$[\boldsymbol{\nabla}\boldsymbol{u}] = \begin{bmatrix} x_2x_3 & x_1x_3 & x_1x_2 \\ x_2 & x_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\nabla^2 \boldsymbol{u} = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \boldsymbol{u} = \frac{\partial \boldsymbol{\nabla} \boldsymbol{u}}{\partial x_i} \cdot \boldsymbol{e}_i$$

$$\Rightarrow (x_3 \boldsymbol{e}_1 \boldsymbol{e}_2 + x_2 \boldsymbol{e}_1 \boldsymbol{e}_3 + \boldsymbol{e}_2 \boldsymbol{e}_2) \cdot \boldsymbol{e}_1 + (x_3 \boldsymbol{e}_1 \boldsymbol{e}_1 + x_1 \boldsymbol{e}_1 \boldsymbol{e}_3 + \boldsymbol{e}_2 \boldsymbol{e}_1) \cdot \boldsymbol{e}_2$$

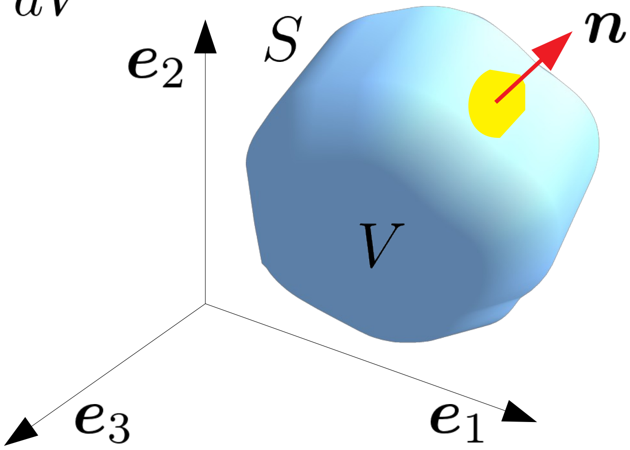
$$+ (x_2 \boldsymbol{e}_1 \boldsymbol{e}_1 + x_1 \boldsymbol{e}_1 \boldsymbol{e}_2) \cdot \boldsymbol{e}_3$$

$$\Rightarrow 0$$

Integral theorems

We introduce two important integral theorems. First one is known as *Gauss' divergence theorem* which transforms a surface integral into volume integral and states that,

$$\int_S \boldsymbol{u}\cdot\boldsymbol{n}\,dS = \int_V \boldsymbol{\nabla}\cdot\boldsymbol{u}\,dV \quad \text{or} \quad \int_S u_i n_i\,dS = \int_V \frac{\partial u_i}{\partial x_i}\,dV$$



where $\boldsymbol{u}(\boldsymbol{x})$ is a smooth vector field defined in space.

Similarly for a smooth tensor field $\boldsymbol{A}(\boldsymbol{x})$ in space,

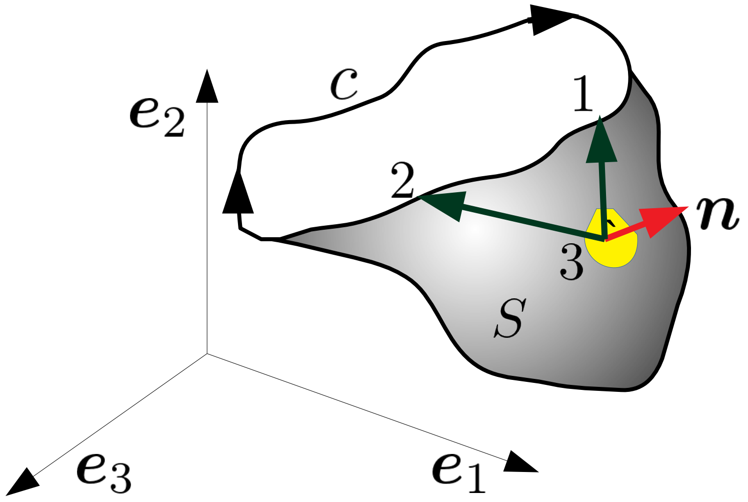
$$\int_S \boldsymbol{A}\cdot\boldsymbol{n}\,dS = \int_V \boldsymbol{\nabla}\cdot\boldsymbol{A}\,dV \quad \text{or} \quad \int_S A_{ij} n_j\,dS = \int_V \frac{\partial A_{ij}}{\partial x_j}\,dV$$

Another theorem is known as *Stoke's theorem* which is related to *open surfaces*. It relates the *surface integral over the open surface* to the *line integral around the bounding closed curve* in space.

$$\oint_c \boldsymbol{u} \cdot d\boldsymbol{x} = \int_S (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot \boldsymbol{n} dS$$

or

$$\oint_c u_k dx_k = \int_S e_{ijk} \frac{\partial u_k}{\partial x_j} n_i dS$$



Note that the sence of curve $\boldsymbol{c}$ and the direction of normal $\boldsymbol{n}$ will be such that the vectors connecting points 1,2, and 3 form a right handed set of vectors.