ME232: Dynamics

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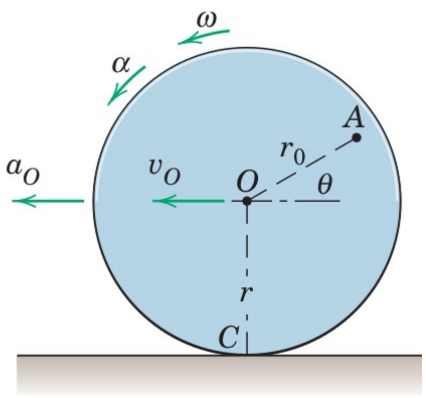
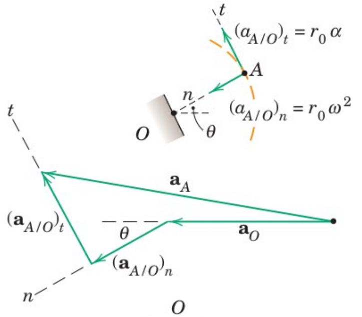
Room # 106

Example 6



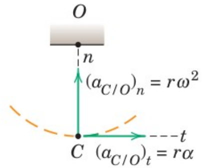
The wheel of radius r rolls to the left without slipping and, at the instant considered, the center O has a velocity $\boldsymbol{v}_O$ and an acceleration $\boldsymbol{a}_O$ to the left. Determine the acceleration of points A and C on the wheel for the instant considered.



Angular velocity $\omega = v_O/r$, angular acceleration $\alpha = a_O/r$.

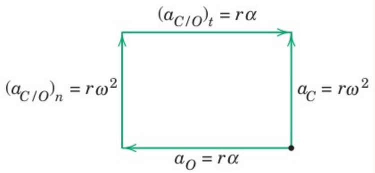
The acceleration of A, $\boldsymbol{a}_A = \boldsymbol{a}_O + \boldsymbol{a}_{A/O} = \boldsymbol{a}_O + (\boldsymbol{a}_{A/O})_n + (\boldsymbol{a}_{A/O})_t$.

where, magnitudes $(a_{A/O})_n = r_0\omega^2 = r_0(v_O/r)^2$, $(a_{A/O})_t = r_0\alpha = r_0(a_O/r)$.



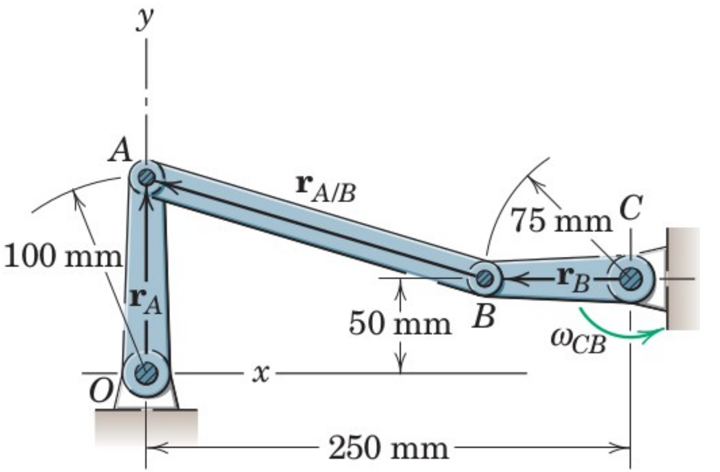
The acceleration of C, $\boldsymbol{a}_C = \boldsymbol{a}_O + \boldsymbol{a}_{C/O} = \boldsymbol{a}_O + (\boldsymbol{a}_{C/O})_n + (\boldsymbol{a}_{C/O})_t$.

where, magnitudes $(a_{C/O})_n = r\omega^2$, $(a_{C/O})_t = r\alpha$.



Example 7

Crank CB has a constant counterclockwise angular velocity of 2 rad/s in the position shown during a short interval of its motion. Determine the angular acceleration of links AB and OA for this position. Solve by using vector algebra.



We already solved for ω_{AB} = -6/7 rad/s, ω_{OA} = -3/7 rad/s.

The acceleration of A, $\boldsymbol{a}_A = \boldsymbol{a}_B + (\boldsymbol{a}_{A/B})_n + (\boldsymbol{a}_{A/B})_t$.

$$\boldsymbol{a}_A = \boldsymbol{\alpha}_{OA} \times \boldsymbol{r}_A + \boldsymbol{\omega}_{OA} \times (\boldsymbol{\omega}_{OA} \times \boldsymbol{r}_A)$$

$$= \alpha_{OA}\, \boldsymbol{k} \times 100\, \boldsymbol{j} + (-3/7\, \boldsymbol{k}) \times (-3/7\, \boldsymbol{k} \times 100\, \boldsymbol{j})$$

$$= -100\, \alpha_{OA} \boldsymbol{i}\ \ -100(3/7)^2\, \boldsymbol{j}\ \text{mm/s}^2$$

$$\boldsymbol{a}_B = \boldsymbol{\alpha}_{CB} \times \boldsymbol{r}_B + \boldsymbol{\omega}_{CB} \times (\boldsymbol{\omega}_{CB} \times \boldsymbol{r}_B)$$

$$= \mathbf{0} + (2\boldsymbol{k}) \times (2\boldsymbol{k} \times -75\ \boldsymbol{i}) = 300\ \boldsymbol{i}\ \text{mm/s}^2$$

The acceleration of A, $\boldsymbol{a}_A = \boldsymbol{a}_B + (\boldsymbol{a}_{A/B})_n + (\boldsymbol{a}_{A/B})_t$.

$(\boldsymbol{a}_{A/B})_n = \boldsymbol{\omega}_{AB}\times(\boldsymbol{\omega}_{AB}\times\boldsymbol{r}_{AB}) = [-6/7\ \boldsymbol{k}] \times [-6/7\ \boldsymbol{k} \times (-175\ \boldsymbol{i} + 50\ \boldsymbol{j})]$

$= (6/7)^2\ (175\ \boldsymbol{i} - 50\ \boldsymbol{j})$ mm/s^2

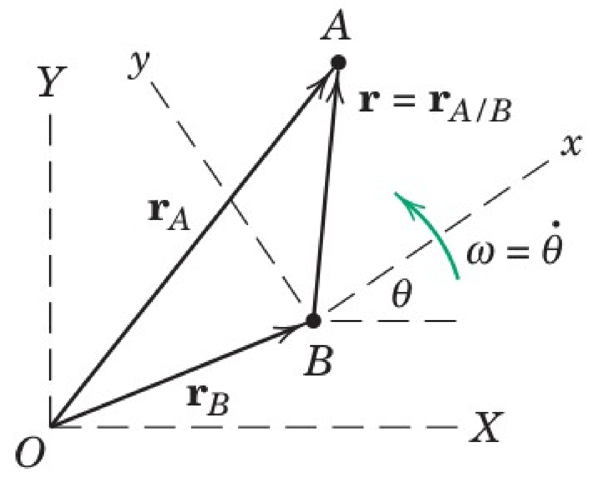
$(\boldsymbol{a}_{A/B})_t = \boldsymbol{\alpha}_{AB}\times\ \boldsymbol{r}_{AB} = \alpha_{AB}\,\boldsymbol{k} \times (-175\ \boldsymbol{i} + 50\ \boldsymbol{j}) = -\ 50\ \alpha_{AB}\,\boldsymbol{k}\ -175\ \alpha_{AB}\,\boldsymbol{j}$ mm/s^2 .

$\alpha_{AB} = -0.1050$ rad/s^2, $\alpha_{OA} = -4.34$ rad/s^2.

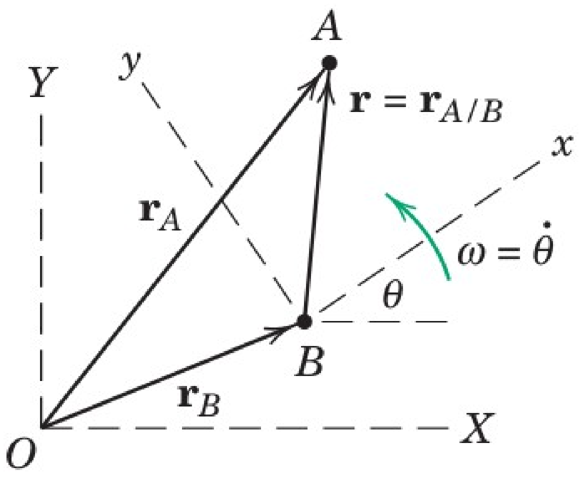
Motion relative to rotating axes

- Till now in our discussion we have used non-rotating reference axes to describe relative velocity and relative acceleration.
- Use of rotating reference axes greatly facilitates the solution of many problems in kinematics where motion is generated **within a system** or **observed from a system which itself is rotating**.
- An example of such a motion is the movement of a fluid particle along the curved vane of a centrifugal pump, where the **path relative to the vanes** of the impeller becomes an important design consideration.

Consider the plane motion of two particles A and B in the fixed X-Y plane. For the time being, we will consider A and B to be moving independently of one another for the sake of generality. We observe the motion of A from a moving reference frame x-y which has its origin attached to B and which rotates with an angular velocity $\omega = \dot{\theta}$.



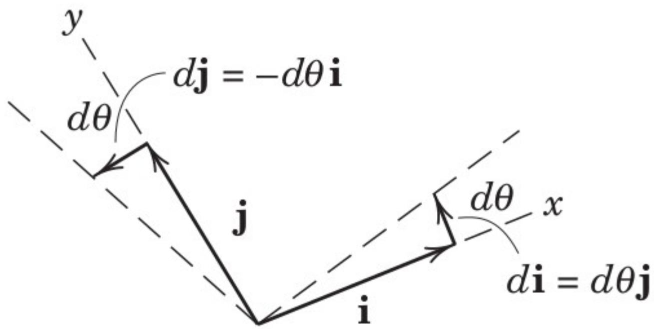
The angular velocity vector is $\boldsymbol{\omega} = \omega \boldsymbol{k}$. The absolute position vector of A is given,



$$\boldsymbol{r}_A = \boldsymbol{r}_B + \boldsymbol{r}_{A/B} = \boldsymbol{r}_B + (x\,\boldsymbol{i} + y\,\boldsymbol{j}), \qquad \cdots\cdots\cdots\cdots\cdots\cdots(9)$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors attached to the x-y frame.

To obtain the velocity and acceleration equations we successively differentiate (9) w.r.t. time. The unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ are now rotating with the x-y axes and, therefore, have time derivatives. These derivatives may be seen from the figure, which shows the infinitesimal change in each unit vector during time dt as the reference axes rotate through an angle $d\theta = \omega\, dt$.



The differential change in $\boldsymbol{i}$ is $d\boldsymbol{i} = d\theta\ \boldsymbol{j}$, and the differential change in $\boldsymbol{j}$ is $d\boldsymbol{j} = -d\theta\ \boldsymbol{i}$. Thus,

$$\dot{\boldsymbol{i}} = \omega \boldsymbol{j} = \boldsymbol{\omega} \times \boldsymbol{i}, \text{ and } \dot{\boldsymbol{j}} = -\omega \boldsymbol{i} = \boldsymbol{\omega} \times \boldsymbol{j}. \qquad \cdots\cdots\cdots\cdots\cdots\cdots(10)$$

Relative velocity:

Expression for relative velocity can be obtained by differentiating (10) w.r.t. time as,

$$\dot{\boldsymbol{r}}_A = \dot{\boldsymbol{r}}_B + \dot{\boldsymbol{r}}_{A/B} = \dot{\boldsymbol{r}}_B + \left[(\dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j}) + (x\dot{\boldsymbol{i}} + y\dot{\boldsymbol{j}})\right]$$

$$\boldsymbol{v}_A = \boldsymbol{v}_B + [\boldsymbol{v}_{\text{rel}} + (x\boldsymbol{\omega} \times \boldsymbol{i} + y\boldsymbol{\omega} \times \boldsymbol{j})]$$

$$\boldsymbol{v}_A = \boldsymbol{v}_B + \boldsymbol{v}_{\text{rel}} + \boldsymbol{\omega} \times (x\boldsymbol{i} + y\boldsymbol{j})$$

$$\boldsymbol{v}_A = \boldsymbol{v}_B + \boldsymbol{v}_{\text{rel}} + \boldsymbol{\omega} \times \boldsymbol{r} \qquad \text{..................}(11)$$

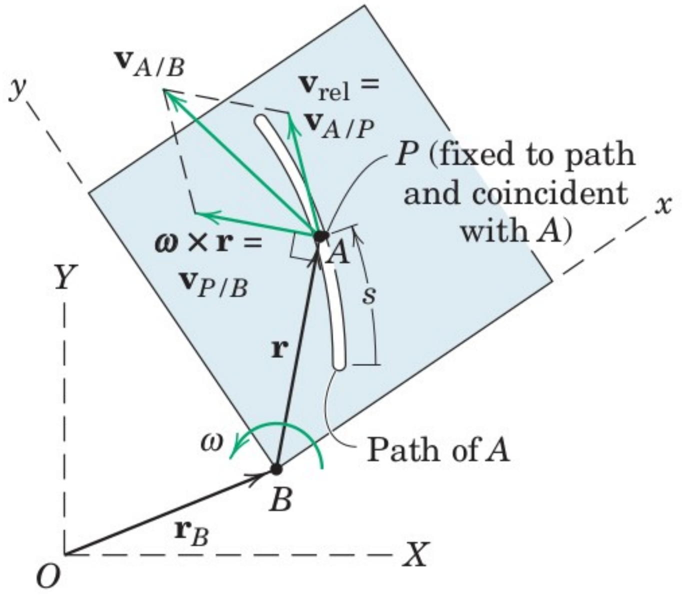
Comparing (11) with the expression for non-rotating reference axes it is observed that $\boldsymbol{v}_{A/B} = \boldsymbol{\omega} \times \boldsymbol{r} + \boldsymbol{v}_{\text{rel}}$, from which it can be concluded that **the term $\boldsymbol{\omega} \times \boldsymbol{r}$ is the difference between the relative velocities as measured from non-rotating and rotating axes.**

To visualize the meaning of the last two terms of (11), we observe the motion of particle A in a curved slot in a plate which represents the rotating x-y reference system.

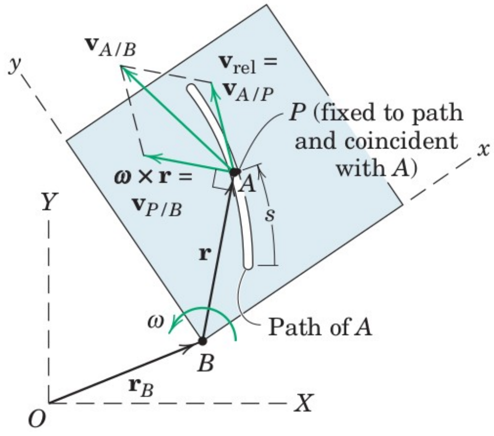
The velocity of A as measured relative to the plate, $\boldsymbol{v}_{\text{rel}}$, would be tangent to the path fixed in the x-y plate and would have a magnitude $\dot{s}$, where s is measured along the path.

This relative velocity may also be viewed as the velocity $\boldsymbol{v}_{A/P}$ **relative to a point P attached to the plate and coincident with A at the instant under consideration.**

The term $\boldsymbol{\omega} \times \boldsymbol{r}$ has a magnitude $r\dot{\theta}$ and a direction normal to $\boldsymbol{r}$ and is the **velocity of point P relative to B as seen from nonrotating axes attached to B.**



The following comparison will help establish the equivalence of, and clarify the differences between, the relative-velocity equations written for rotating and nonrotating reference axes:



$$
\begin{aligned}
\boldsymbol{v}_A &= \boldsymbol{v}_B + \boldsymbol{\omega} \times \boldsymbol{r} + \boldsymbol{v}_{\text{rel}} \\
\boldsymbol{v}_A &= \boldsymbol{v}_B + \boldsymbol{v}_{P/B} + \boldsymbol{v}_{A/P} \\
\boldsymbol{v}_A &= \boldsymbol{v}_P + \boldsymbol{v}_{A/P} \\
\boldsymbol{v}_A &= \boldsymbol{v}_B + \boldsymbol{v}_{A/B}
\end{aligned}
\qquad \text{..................(11a)}
$$

In the second equation, the term $\boldsymbol{v}_{P/B}$ is measured from a nonrotating position. The term $\boldsymbol{v}_{A/P}$ is the same as $\boldsymbol{v}_{\text{rel}}$ and is the velocity of A as measured in the x-y frame. In the third equation, $\boldsymbol{v}_P$ is the absolute velocity of P and represents the effect of the moving coordinate system, both transnational and rotational. The fourth equation is the same as that developed for nonrotating axes, and it is seen that

$$\boldsymbol{v}_{A/B} = \boldsymbol{v}_{P/B} + \boldsymbol{v}_{A/P} = \boldsymbol{\omega} \times \boldsymbol{r} + \boldsymbol{v}_{\text{rel}}.$$

Transformation of a time derivative

(11) represents a transformation of the time derivative of the position vector between rotating and nonrotating axes. This result can be generalized to apply to the time derivative of any vector quantity $\boldsymbol{V} = V_x \boldsymbol{i} + V_y \boldsymbol{j}$. Accordingly, the total time derivative with respect to the X-Y system is

$$\left(\frac{d\boldsymbol{V}}{dt}\right)_{XY} = \left(\dot{V}_x\boldsymbol{i} + \dot{V}_y\boldsymbol{j}\right) + \left(V_x\dot{\boldsymbol{i}} + V_y\dot{\boldsymbol{j}}\right)$$

$$\left(\frac{d\boldsymbol{V}}{dt}\right)_{XY} = \left(\frac{d\boldsymbol{V}}{dt}\right)_{xy} + \boldsymbol{\omega} \times (V_x\boldsymbol{i} + V_y\boldsymbol{j})$$

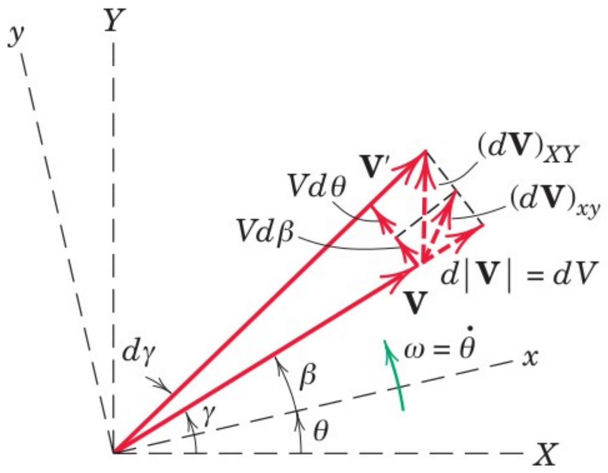
$$\left(\frac{d\boldsymbol{V}}{dt}\right)_{XY} = \left(\frac{d\boldsymbol{V}}{dt}\right)_{xy} + \boldsymbol{\omega} \times \boldsymbol{V} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(12)$$

The first term represents **the part of the total derivative of *V* which is measured relative to the *x-y* reference system**, and the second term represents **the part of the derivative due to the rotation of the reference system.**

To understand the physical significance of (12) consider a vector $\boldsymbol{V}$ at time t, observed both in the fixed axes X-Y and in the rotating axes x-y. Because we are dealing with the effects of rotation only, the vector may be drawn through the coordinate origin without loss of generality. During time dt, the vector swings to position $\boldsymbol{V}'$, and the observer in x-y measures the two components

(a) dV due to its change in magnitude, and

(b) $Vd\beta$ due to its rotation $d\beta$ relative to x-y.



Hence, the derivative $(d\boldsymbol{V}/dt)_{xy}$ which the rotating observer measures has the components dV/dt and $Vd\beta/dt = V\dot{\beta}$.

The remaining part of the total time derivative not measured by the rotating observer has the magnitude $Vd\theta/dt$ and, expressed as a vector, is $\boldsymbol{\omega} \times \boldsymbol{V}$. Thus, we see from the diagram that

$$(\dot{\boldsymbol{V}})_{XY} = (\dot{\boldsymbol{V}})_{xy} + \boldsymbol{\omega} \times \boldsymbol{V}.$$