Continuous-Time Dynamics

- The most general/generic for a smooth system:

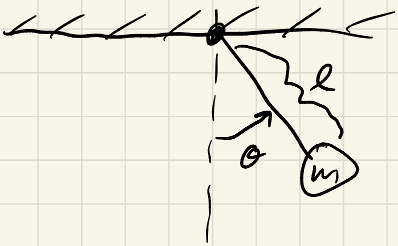
$$\underbrace{\dot{x}}_{\text{state time derivative}} = \underbrace{f}_{\text{"dynamics"}}(\underbrace{x}_{\text{"state"} \in \mathbb{R}^n}, \underbrace{u}_{\text{"input"} \in \mathbb{R}^m})$$

- For a mechanical system: $x = \begin{bmatrix} q \\ v \end{bmatrix}$

"configuration" / "pose" (not always a vector)

"velocity"

· Example (Pendulum):



$$\boxed{m\ell^2 \ddot{\theta} + mg\ell \sin(\theta) = \tau}$$

$$q = \theta, \quad v = \dot{\theta}, \quad u = \tau$$

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \Rightarrow \dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} \dot{\theta} \\ -\frac{g}{\ell}\sin(\theta) + \frac{1}{m\ell^2} u \end{bmatrix}}_{f(x,u)}$$

$q \in S^1$ (circle) , $x \in S^1 \times \mathbb{R}$ (cylinder)

* Control-Affine Systems

$$\dot{x} = \underbrace{f_0(x)}_{\text{"drift"}} + \underbrace{B(x)}_{\text{"input Jacobian"}} u$$

- most systems can be put in this form
- Pendulum:

$$f_0(x) = \begin{bmatrix} \dot{\theta} \\ -\frac{g}{\ell}\sin(\theta) \end{bmatrix}, \quad B(x) = \begin{bmatrix} 0 \\ 1/m\ell^2 \end{bmatrix}$$

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* Manipulator Dynamics:

$$\underbrace{M(q)}_{\text{"Mass Matrix"}} \dot{v} + \underbrace{C(q,v)}_{\substack{\text{"Dynamics Bias"} \\ \text{(Coriolis + Gravity)}}} = \underbrace{B(q)}_{\text{"Input Jacobian"}} u + \overset{\text{"external forces"}}{F}$$

$$\underbrace{\dot{q} = G(q)\, v}_{\text{"Velocity Kinematics"}} \qquad \dot{x} = f(x,u) = \begin{bmatrix} G(q)\, v \\ M(q)^{-1}\left(B(q)\, u + F - C\right) \end{bmatrix}$$

- Pendulum:

$$M(q) = m\ell^2, \quad C(q,v) = mg\ell \sin(\theta), \quad B = I, \quad G = I$$

- All mechanical systems can be written like this
- This is just a way of re-writing the Euler-Lagrange equation for:

$$L = \underbrace{\tfrac{1}{2} v^T M(q)\, v}_{\text{Kinetic Energy}} - \underbrace{U(q)}_{\text{Potential Energy}}$$

* Linear Systems:

$$\dot{x} = A(t)x + B(t)u$$

- Called "time invariant" if $A(t) = A$, $B(t) = B$
- Called "time varying" otherwise
- Super important in control
- We often approximate nonlinear systems with linear ones:

$$\dot{x} = f(x, u) \Rightarrow A = \frac{\partial f}{\partial x}, \quad B = \frac{\partial f}{\partial u}$$

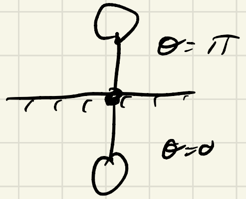
Equilibria:

- A point where a system will "remain at rest"

$$\Rightarrow \dot{x} = f(x,u) = 0$$

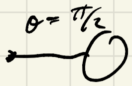
- Algebraically, roots of the dynamics
- Pendulum:

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ -\frac{g}{l}\sin(\theta) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} \dot{\theta} = 0 \\ \theta = 0, \pi \end{matrix}$$



* First Control Problem

- Can I move the equilibria



$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ -\frac{g}{\ell}\sin(\pi/2) + \frac{1}{m\ell^2}u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{1}{m\ell^2}u = \frac{g}{\ell}\sin(\pi/2) \Rightarrow u = mg\ell$$

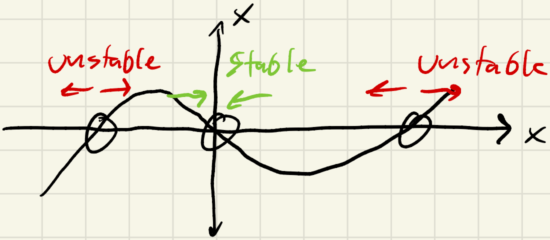
- In general, we get a root-finding problem in u:

$$f(x^*, u) = 0$$

Stability of Equilibria:

- When will we stay "near" an equilibrium point under perturbations?

- Look at a 1D system $(x \in \mathbb{R})$



$$\frac{\partial f}{\partial x} < 0 \Rightarrow \text{stable} \qquad \frac{\partial f}{\partial x} > 0 \Rightarrow \text{unstable}$$

- In higher dimensions:

$$\frac{\partial f}{\partial x} \text{ is a Jacobian Matrix}$$

- Take an eigendecomposition $\Rightarrow$ decouple into n 1D systems

$$Re\left[\text{eigvals}\left(\frac{\partial f}{\partial x}\right)\right] < 0 \Rightarrow \text{stable}$$

$$\text{otherwise} \Rightarrow \text{unstable}$$

- Pendulum:

$$f(x) = \begin{bmatrix} \dot{\theta} \\ -\frac{g}{\ell}\sin(\theta) \end{bmatrix} \Rightarrow \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -g/\ell \cos(\theta) & 0 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{\theta=\pi} = \begin{bmatrix} 0 & 1 \\ g/\ell & 0 \end{bmatrix} \Rightarrow \text{eigvals}\left(\frac{\partial f}{\partial x}\right) = \pm\sqrt{\frac{g}{\ell}}$$

$$\Rightarrow \text{unstable}$$

$$\left.\frac{\partial f}{\partial x}\right|_{\theta=0} = \begin{bmatrix} 0 & 1 \\ -g/\ell & 0 \end{bmatrix} \Rightarrow \text{eigvals}\left(\frac{\partial f}{\partial x}\right) = \pm i\sqrt{\frac{g}{\ell}}$$

$$\Rightarrow \text{undamped oscillations}$$

- Add damping (e.g. $u = -k_d\dot{\theta}$) results in strictly negative real part.