Last Time:

- LQR as a QP
- Riccati

Today:

- Infinite-Horizon LQR
- Controllability
- Dynamic Programming

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*Infinite-Horizon LQR

- For time-invariant LQR, K matrices converge to constant values
- For stabilization problems we usually use constant K
- Backward recursion for P:

$$K_n = (R + B^T P_{n+1} B)^{-1} B^T P_{n+1} A$$

$$P_n = Q + A^T P_{n+1} (A - BK_n)$$

- Infinite-horizon limit $\Rightarrow P_{n+1} = P_n = P_{inf}$

$\Rightarrow$ solve as a root-finding / fixed-point problem

- Julia/MATLAB/Python dare function does this for you

Controllability

- How do we know if LQR will work?
- We already know $Q \succeq 0$, $R > 0$
- For the time-invariant case, there is a simple answer
- For any initial state x_0, x_N is given by:

$$\begin{aligned} x_N &= Ax_{N-1} + Bu_{N-1} \\ &= A(Ax_{N-2} + Bu_{N-2}) + Bu_{N-1} \\ &\vdots \\ &= A^N x_0 + A^{N-1}Bu_0 + A^{N-2}Bu_1 + \cdots Bu_{N-1} \\ &= \underbrace{\begin{bmatrix} B & AB & A^2B & \cdots & A^{N-1}B \end{bmatrix}}_{C} \begin{bmatrix} u_{N-1} \\ u_{N-2} \\ \vdots \\ u_0 \end{bmatrix} + A^N x_0 \end{aligned}$$

- This is equivalent to a least-squares problem for $u_{0:N-1}$

$$\begin{bmatrix} u_{N-1} \\ u_{N-2} \\ \vdots \\ u_0 \end{bmatrix} = \underbrace{\left[C^T(CC^T)^{-1}\right]}_{\text{"Pseudo-inverse"}} (x_N - A^N x_0)$$

- For CC^T to be invertable:

$$\Rightarrow \text{rank}(C) = n \quad , \quad n = \dim(x)$$

- I can stop at n time steps in C because the Cayley-Hamilton theorem says that A^N can be written in terms of a linear combination of lower powers of A up to n

$$A^N = \sum_{n=0}^{n-1} \alpha_n A^n \quad \text{(for some } \alpha_n\text{)}$$

- Therefore adding more time steps/columns to C can't increase the rank:

$$\Rightarrow C = \underbrace{[B \; AB \; \cdots \; A^{n-1}B]}$$

"Controllability Matrix"

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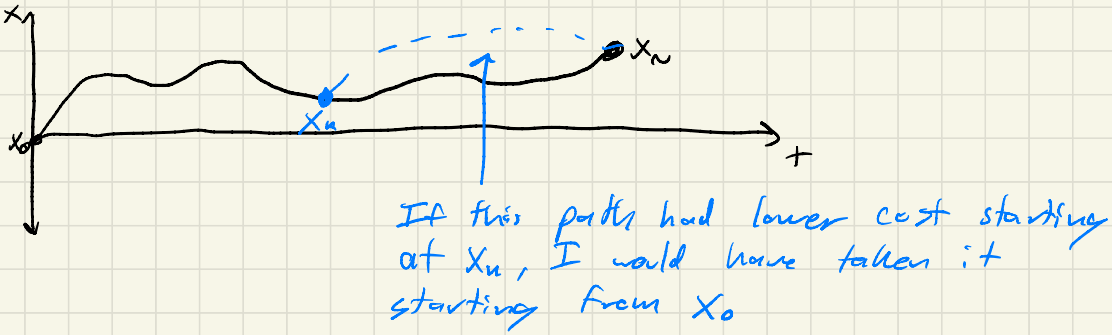
* Bellman's Principle

- Optimal control problems have an inherently sequential structure
- Past control inputs only affect future states, future inputs can't affect past states.
- Bellman's Principle ("Principle of Optimality") states the consequences of this for optimal trajectories.



- Sub-trajectories of optimal trajectories have to be optimal for appropriately defined sub problems

* Dynamic Programming
  - Bellman's Principle suggests starting from the end of the trajectory and working backwards
  - We've already seen hints of this from Riccati
  - Define "optimal cost-to-go" aka "Value function" $V_k(x)$
  - Encodes cost incurred starting from state x at time k if we act optimally

- For LQR

$$V_N(x) = \frac{1}{2} x^T Q_N x = \frac{1}{2} x^T P_N x$$

- back up one step and calculate $V_{N-1}(x)$:

$$V_{N-1} = \min_u \frac{1}{2} x_{N-1}^T Q x_{N-1} + \frac{1}{2} u^T R u + V_N(A_{N-1} x_{N-1} + B u_{N-1}) \quad *$$

$$\Rightarrow \min_u \; u^T R u + (A x_{N-1} + B u_{N-1})^T P_N (A x_{N-1} + B u_{N-1}) \quad \nabla_u = 0$$

$$\Rightarrow R u + B^T P_N (A x_{N-1} + B u) = 0$$

$$\Rightarrow u_{N-1} = -\underbrace{(R + B_{N-1}^T P_N B_{N-1})^{-1} B_{N-1} P_N A_{N-1}}_{K_{N-1}} x_{N-1}$$

- Plug $u = -Kx$ back into $*$

$$\Rightarrow V_{N-1}(x) = \frac{1}{2} x^T \underbrace{[Q + K^T R K + (A - BK)^T P_N (A - BK)]}_{P_{N-1}} x$$

$$\Rightarrow V_{N-1}(x) = \frac{1}{2} x^T P_{N-1} x$$

- Now we have a backward recursion for K and P that we iterate until $k = 0$

* Dynamic Programming Algorithm:

$V_N(x) \leftarrow \ell_N(x)$

$k \leftarrow N$

while $k > 1$

$$V_{k-1}(x) = \min_{u \in \mathcal{U}} \left[\ell(x,u) + V_k(f(x,u)) \right]$$

$k \leftarrow k-1$

end

- If we know $V_k(x)$, the optimal policy is:

$$u_k(x) = \operatorname*{argmin}_{u \in \mathcal{U}} \left[\ell(x,u) + V_{k+1}(f(x,u)) \right]$$

- DP equations written equivalently in terms of "action-value" or "Q" function:

$$S_k(x,u) = \ell(x,u) + V_{k+1}(f(x,u))$$

- Usually denoted $Q(x,u)$ but we'll use $S(x,u)$

- Avoids need for explicit dynamics model

* The Curse

- DP is sufficient for global optimum
- Only tractable for simple problems (LQR, low-dimensional)
- $V(x)$ stays quadratic for LQR but becomes impossible to write analytically even for simple nonlinear problems.
- Even if we could, $\min S(x,u)$ will be non-convex and possibly "hard to solve.
- Cost of DP blows up with state dimension due to cost of representing $V(x)$

* Why do we care?

- Approximate DP with a function approximator for $V(x)$ or $S(x,u)$ is very powerful.
- Forms basis of lots of modern RL
- DP generalizes to stochastic problems (just wrap everything in expectations). Pontryagin does not.

* Finally: What are the Lagrange Multipliers?

- Recall Riccati derivation from QP:

$$\lambda_n = P_n x_n$$

- From DP:

$$V(x) = \frac{1}{2} x^T P x$$

$$\Rightarrow \lambda_n = \nabla_x V_n(x)$$

- Dynamics multipliers are cost-to-go gradients
- Carries over to nonlinear setting (not just LQR)