Matrix Theory(EE5609) Assignment 9

Anshum Agrawal
Roll No- AI20MTECH11006

***Abstract*—Given AB = -BA, this documents finds value of trace(A) and trace(B).**

Download latex-tikz codes from

https://github.com/anshum0302/EE5609/blob/master/assignment9/assign9.tex

1 Problem Statement

Let $\mathbf{A}$ and $\mathbf{B}$ be real invertible matrices such that

$$\mathbf{AB} = -\mathbf{BA}. \tag{1.0.1}$$

Then
1. trace($\mathbf{A}$) = trace($\mathbf{B}$) = 0
2. trace($\mathbf{A}$) = trace($\mathbf{B}$) = 1
3. trace($\mathbf{A}$) = 0, trace($\mathbf{B}$) = 1
4. trace($\mathbf{A}$) = 1, trace($\mathbf{B}$) = 0

2 Solution

Proof of trace($\mathbf{XY}$) = trace($\mathbf{YX}$) for some matrices $\mathbf{X}$ and $\mathbf{Y}$ of size $N \times N$.

$$trace(\mathbf{XY}) = \sum_{i=1}^{N} (\mathbf{XY})_{ii} \tag{2.0.1}$$
$$= \sum_{i=1}^{N} \sum_{k=1}^{N} \mathbf{X}_{ik}\mathbf{Y}_{ki} \tag{2.0.2}$$
$$= \sum_{i=1}^{N} \sum_{k=1}^{N} \mathbf{Y}_{ki}\mathbf{X}_{ik} \tag{2.0.3}$$
$$= \sum_{k=1}^{N} \sum_{i=1}^{N} \mathbf{Y}_{ki}\mathbf{X}_{ik} \tag{2.0.4}$$
$$= \sum_{k=1}^{N} (\mathbf{YX})_{kk} \tag{2.0.5}$$
$$\implies trace(\mathbf{XY}) = trace(\mathbf{YX}) \tag{2.0.6}$$

Now from equation (1.0.1)

$$\mathbf{AB} = -\mathbf{BA} \tag{2.0.7}$$
$$\implies \mathbf{A} = -\mathbf{BAB}^{-1} \tag{2.0.8}$$

Taking trace of $\mathbf{A}$ in (2.0.8)

$$trace(\mathbf{A}) = trace(-\mathbf{BAB}^{-1}) \tag{2.0.9}$$
$$= -trace((\mathbf{BA})\mathbf{B}^{-1}) \tag{2.0.10}$$

using (2.0.6) in (2.0.10) with $\mathbf{X} = \mathbf{BA}$ and $\mathbf{Y} = \mathbf{B^{-1}}$ we get

$$trace(\mathbf{A}) = -trace(\mathbf{B}^{-1}(\mathbf{BA})) \tag{2.0.11}$$
$$= -trace((\mathbf{B}^{-1}\mathbf{B})\mathbf{A}) \tag{2.0.12}$$
$$= -trace(\mathbf{A}) \tag{2.0.13}$$
$$\implies 2trace(\mathbf{A}) = 0 \tag{2.0.14}$$
$$trace(\mathbf{A}) = 0 \tag{2.0.15}$$

Similarly from equation (1.0.1)

$$\mathbf{AB} = -\mathbf{BA} \tag{2.0.16}$$
$$\implies \mathbf{B} = -\mathbf{A}^{-1}\mathbf{BA} \tag{2.0.17}$$

Taking trace of $\mathbf{B}$ in (2.0.17)

$$trace(\mathbf{B}) = trace(-\mathbf{A}^{-1}\mathbf{BA}) \tag{2.0.18}$$
$$= -trace(\mathbf{A}^{-1}(\mathbf{BA})) \tag{2.0.19}$$

using (2.0.6) in (2.0.19) with $\mathbf{X} = \mathbf{A}^{-1}$ and $\mathbf{Y} = \mathbf{BA}$ we get

$$trace(\mathbf{B}) = -trace((\mathbf{BA})\mathbf{A}^{-1}) \tag{2.0.20}$$
$$= -trace(\mathbf{B}(\mathbf{AA}^{-1})) \tag{2.0.21}$$
$$= -trace(\mathbf{B}) \tag{2.0.22}$$
$$\implies 2trace(\mathbf{B}) = 0 \tag{2.0.23}$$
$$trace(\mathbf{B}) = 0 \tag{2.0.24}$$

From (2.0.15) and (2.0.24) we conclude that equation trace($\mathbf{A}$) = trace($\mathbf{B}$) = 0 is correct.