Matrix Theory(EE5609) Assignment 6

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***Abstract*—This Assignment is about tracing a curve**

Download all python codes from

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https://github.com/anshum0302/EE5609/blob/
    master/assignment6/figure.py
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Download latex-tikz codes from

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https://github.com/anshum0302/EE5609/blob/
    master/assignment6/assign6.tex
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1 Problem Statement

Trace the parabola

$$9x^2 + 24xy + 16y^2 - 4y - x + 7 = 0 \tag{1.0.1}$$

2 Solution

The general second degree equation can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.1}$$

Comparing (1.0.1) and (2.0.1) we get

$$\mathbf{V} = \begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{u} = \begin{pmatrix} \frac{-1}{2} \\ -2 \end{pmatrix} \tag{2.0.3}$$

$$f = 7 \tag{2.0.4}$$

The characteristic equation of $\mathbf{V}$ is given as

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = 0 \tag{2.0.5}$$

$$\implies \begin{vmatrix} 9-\lambda & 12 \\ 12 & 16-\lambda \end{vmatrix} = 0 \tag{2.0.6}$$

$$\implies \lambda^2 - 25\lambda = 0 \tag{2.0.7}$$

The roots of (2.0.7) are eigenvalue of $\mathbf{V}$ and are given by

$$\lambda_1 = 0, \lambda_2 = 25$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.0.8}$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{2.0.9}$$

For $\lambda_1 = 0$

$$(\mathbf{V} - \lambda\mathbf{I}) = \begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix} \xleftrightarrow{R_2 = R_2 - \frac{4}{3}R_1} \begin{pmatrix} 9 & 12 \\ 0 & 0 \end{pmatrix} \tag{2.0.10}$$

Substituting equation (2.0.10) in equation (2.0.9) and upon normalization we get

$$\mathbf{p_1} = \frac{1}{5}\begin{pmatrix} -4 \\ 3 \end{pmatrix} \tag{2.0.11}$$

For $\lambda_2 = 25$

$$(\mathbf{V} - \lambda\mathbf{I}) = \begin{pmatrix} -16 & 12 \\ 12 & -9 \end{pmatrix} \xleftrightarrow{R_2 = R_2 + \frac{3}{4}R_1} \begin{pmatrix} -16 & 12 \\ 0 & 0 \end{pmatrix} \tag{2.0.12}$$

Substituting equation (2.0.12) in equation (2.0.9) and upon normalization we get

$$\mathbf{p_2} = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{2.0.13}$$

By eigenvalue decomposition, $\mathbf{P}^T\mathbf{V}\mathbf{P} = \mathbf{D}$ where

$$\mathbf{P} = \begin{pmatrix} \mathbf{p1} & \mathbf{p2} \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -4 & 3 \\ 3 & 4 \end{pmatrix} \tag{2.0.14}$$

and

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix} \tag{2.0.15}$$

Then for the parabola

$$\eta = 2\mathbf{p_1}^T\mathbf{u} = -\frac{8}{5} \tag{2.0.16}$$

$$focal\ length = \left|\frac{\eta}{\lambda_2}\right| = \frac{8}{125} \tag{2.0.17}$$

When $\left|\mathbf{V}\right| = 0$,equation (2.0.1) can be written as

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \tag{2.0.18}$$

And the vertex $\mathbf{c}$ is given by

$$\begin{pmatrix}\mathbf{u}^T + \frac{\eta}{2}\mathbf{p_1}^T \\ \mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-f \\ \frac{\eta}{2}\mathbf{p_1} - \mathbf{u}\end{pmatrix} \tag{2.0.19}$$

Substituting values from (2.0.2), (2.0.3), (2.0.4), (2.0.11), (2.0.16) in (2.0.19)

$$\begin{pmatrix}\frac{7}{50} & -\frac{124}{50} \\ 9 & 12 \\ 12 & 16\end{pmatrix}\mathbf{c} = \begin{pmatrix}-7 \\ \frac{57}{50} \\ \frac{76}{50}\end{pmatrix} \tag{2.0.20}$$

To find $\mathbf{c}$,performing row reduction in augmented matrix as follows

$$\begin{pmatrix}\frac{7}{50} & -\frac{124}{50} & -7 \\ 9 & 12 & \frac{57}{50} \\ 12 & 16 & \frac{76}{50}\end{pmatrix} \xleftrightarrow[R_1 \leftarrow \frac{50}{7}R_1]{R_3 \leftarrow R_3 - \frac{4}{3}R_2} \begin{pmatrix}1 & -\frac{124}{7} & -50 \\ 9 & 12 & \frac{57}{50} \\ 0 & 0 & 0\end{pmatrix}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - 9R_1} \begin{pmatrix}1 & -\frac{124}{7} & -50 \\ 0 & \frac{1200}{7} & \frac{22557}{50} \\ 0 & 0 & 0\end{pmatrix}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{7}{1200}R_2} \begin{pmatrix}1 & -\frac{124}{7} & -50 \\ 0 & 1 & 2.63165 \\ 0 & 0 & 0\end{pmatrix}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{124}{7}R_2} \begin{pmatrix}1 & 0 & -3.3822 \\ 0 & 1 & 2.63165 \\ 0 & 0 & 0\end{pmatrix}$$

Thus

$$\mathbf{c} = \begin{pmatrix}-3.3822 \\ 2.63165\end{pmatrix} \tag{2.0.21}$$

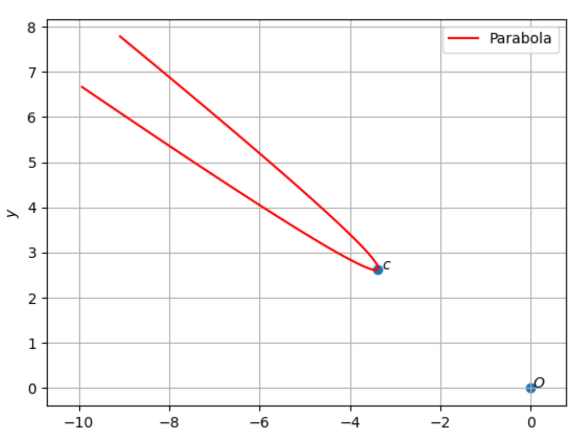

Fig. 1: Graph of $9x^2 + 24xy + 16y^2 - 4y - x + 7 = 0$