Matrix Theory(EE5609) Assignment 7

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Abstract—**This document deals with QR decomposition and Singular Value Decomposition.**

Download latex-tikz codes from

https://github.com/anshum0302/EE5609/blob/master/assignment7/assign7.tex

1 Problem Statement

1. Find the QR decomposition of $\mathbf{V} = \begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix}$
2. Find the vertex $\mathbf{c}$ of the parabola

$$9x^2 + 24xy + 16y^2 - 4y - x + 7 = 0 \quad (1.0.1)$$

using SVD and verify the result using least squares.

2 Solution

2.1 QR decomposition of V

Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be the column vectors of matrix $\mathbf{V}$.Then

$$\mathbf{v_1} = \begin{pmatrix} 9 \\ 12 \end{pmatrix} \quad (2.1.1)$$

$$\mathbf{v_2} = \begin{pmatrix} 12 \\ 16 \end{pmatrix} \quad (2.1.2)$$

We can express column vectors of $\mathbf{V}$ as

$$\mathbf{v_1} = k_1 \mathbf{u_1} \quad (2.1.3)$$

$$\mathbf{v_2} = r_1 \mathbf{u_1} + k_2 \mathbf{u_2} \quad (2.1.4)$$

where

$$k_1 = \|\mathbf{v_1}\| = \sqrt{9^2 + 12^2} = 15 \quad (2.1.5)$$

$$\mathbf{u_1} = \frac{\mathbf{v_1}}{k_1} = \frac{1}{15}\begin{pmatrix} 9 \\ 12 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad (2.1.6)$$

$$r_1 = \frac{\mathbf{u_1}^T \mathbf{v_2}}{\|\mathbf{u_1}\|^2} = \frac{1}{5}\begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} 12 \\ 16 \end{pmatrix} = 20 \quad (2.1.7)$$

$$\mathbf{u_2} = \frac{\mathbf{v_2} - r_1\mathbf{u_1}}{\|\mathbf{v_2} - r_1\mathbf{u_1}\|} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.1.8)$$

$$k_2 = \mathbf{u_2}^T \mathbf{v_2} = \begin{pmatrix} 0 & 0 \end{pmatrix}\begin{pmatrix} 12 \\ 16 \end{pmatrix} = 0 \quad (2.1.9)$$

The equation (2.1.3) and (2.1.4) can be written as

$$\begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} \end{pmatrix} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix}\begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (2.1.10)$$

$$\begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} \end{pmatrix} = \mathbf{QR} \quad (2.1.11)$$

where,

$$\mathbf{Q} = \begin{pmatrix} \mathbf{u_1} & \mathbf{u_2} \end{pmatrix} \quad (2.1.12)$$

$$\mathbf{R} = \begin{pmatrix} k_1 & r_1 \\ 0 & k_2 \end{pmatrix} \quad (2.1.13)$$

Now $\mathbf{Q}$ should be an orthogonal matrix such that

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \quad (2.1.14)$$

Here, we see that the second column vector of $\mathbf{Q}$ is zero since the column vectors of $\mathbf{V}$ are dependent. Therefore we can write $\mathbf{Q}$ as

$$\mathbf{Q} = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} \quad (2.1.15)$$

Therefore, for the given matrix $\mathbf{V}$, we can write $\mathbf{QR}$ decomposition as the product of respective row and column vectors as

$$\mathbf{V} = \mathbf{QR} \quad (2.1.16)$$

$$\implies \begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}\begin{pmatrix} 15 & 20 \end{pmatrix} \quad (2.1.17)$$

2.2 Finding vertex c using SVD

$$\mathbf{V} = \begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix} \quad (2.2.1)$$

$$\mathbf{u} = \begin{pmatrix} -\frac{1}{2} \\ -2 \end{pmatrix} \quad (2.2.2)$$

$$f = 7 \quad (2.2.3)$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -4 & 3 \\ 3 & 4 \end{pmatrix} \quad (2.2.4)$$

$$\eta = 2\mathbf{p_1}^T\mathbf{u} = -\frac{8}{5} \quad (2.2.5)$$

And the expression for vertex $\mathbf{c}$ is given by

$$\begin{pmatrix} \mathbf{u}^T + \eta \mathbf{p_1}^T \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \eta \mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.2.6}$$

Substituting values from (2.2.1) to (2.2.5) in equation (2.2.6) we get

$$\begin{pmatrix} \frac{39}{50} & -\frac{74}{25} \\ 9 & 12 \\ 12 & 16 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -7 \\ \frac{89}{50} \\ \frac{26}{25} \end{pmatrix} \tag{2.2.7}$$

This is of the form

$$\mathbf{Ac} = \mathbf{b} \tag{2.2.8}$$

where

$$\mathbf{A} = \begin{pmatrix} \frac{39}{50} & -\frac{74}{25} \\ 9 & 12 \\ 12 & 16 \end{pmatrix} \tag{2.2.9}$$

$$\mathbf{b} = \begin{pmatrix} -7 \\ \frac{89}{50} \\ \frac{26}{25} \end{pmatrix} \tag{2.2.10}$$

To solve (2.2.8) we perform Singular value decomposition of $\mathbf{A}$ as follows:

$$\mathbf{A} = \mathbf{USV}^T \tag{2.2.11}$$

where columns of $\mathbf{V}$ are eigenvectors of $\mathbf{A}^T\mathbf{A}$, columns of $\mathbf{U}$ are eigenvectors of $\mathbf{AA}^T$ and $\mathbf{S}$ is diagonal matrix of the square roots of eigen values of $\mathbf{A}^T\mathbf{A}$. Substituting (2.2.11) in (2.2.8) we get

$$\mathbf{USV}^T\mathbf{c} = \mathbf{b} \tag{2.2.12}$$

$$\implies \mathbf{c} = \mathbf{VS_+U}^T\mathbf{b} \tag{2.2.13}$$

Where $\mathbf{S}_+$ is the Moore-Penrose pseudo-Inverse of $\mathbf{S}$.Now using (2.2.9) we get

$$\mathbf{AA}^T = \begin{pmatrix} \frac{937}{100} & -\frac{57}{2} & -38 \\ -\frac{57}{2} & 225 & 300 \\ -38 & 300 & 400 \end{pmatrix} \tag{2.2.14}$$

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} \frac{564021}{2500} & \frac{186057}{625} \\ \frac{186057}{625} & \frac{255476}{625} \end{pmatrix} \tag{2.2.15}$$

Eigen values of $\mathbf{AA}^T$ can be found as

$$\left|\mathbf{AA}^T - \lambda\mathbf{I}\right| = 0 \tag{2.2.16}$$

$$\implies \begin{vmatrix} \frac{937}{100} - \lambda & -\frac{57}{2} & -38 \\ -\frac{57}{2} & 225 - \lambda & 300 \\ -38 & 300 & 400 - \lambda \end{vmatrix} = 0 \tag{2.2.17}$$

$$\implies -\lambda^3 + \frac{63437}{100}\lambda^2 - 3600\lambda = 0 \tag{2.2.18}$$

Solving (2.2.18) we get

$$\lambda_1 = \frac{63437 + \sqrt{3880252969}}{200} \tag{2.2.19}$$

$$\implies \lambda_1 = 628.6434 \tag{2.2.20}$$

$$\lambda_2 = \frac{63437 - \sqrt{3880252969}}{200} \tag{2.2.21}$$

$$\implies \lambda_2 = 5.7266 \tag{2.2.22}$$

$$\lambda_3 = 0 \tag{2.2.23}$$

The normalized eigenvectors corresponding to these eigenvalues are

$$\mathbf{u}_1 = \begin{pmatrix} -0.07648 \\ 0.59823 \\ 0.79764 \end{pmatrix} \tag{2.2.24}$$

$$\mathbf{u}_2 = \begin{pmatrix} 0.99707 \\ 0.04589 \\ 0.06118 \end{pmatrix} \tag{2.2.25}$$

$$\mathbf{u}_3 = \begin{pmatrix} 0 \\ -0.8 \\ 0.6 \end{pmatrix} \tag{2.2.26}$$

Hence we obtain $\mathbf{U}$ of (2.2.11) as

$$\mathbf{U} = \begin{pmatrix} -0.07648 & 0.99707 & 0 \\ 0.59823 & 0.04589 & -0.8 \\ 0.79764 & 0.06118 & 0.6 \end{pmatrix} \tag{2.2.27}$$

Similarly eigen values of $\mathbf{A}^T\mathbf{A}$ can be found as

$$\left|\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I}\right| = 0 \tag{2.2.28}$$

$$\implies \begin{vmatrix} \frac{564021}{2500} - \lambda & \frac{186057}{625} \\ \frac{186057}{625} & \frac{255476}{625} - \lambda \end{vmatrix} = 0 \tag{2.2.29}$$

$$\implies \lambda^2 - \frac{63437}{100}\lambda + 3600 = 0 \tag{2.2.30}$$

Solving (2.2.30) we get

$$\lambda_4 = \frac{63437 + \sqrt{3880252969}}{200} \tag{2.2.31}$$

$$\implies \lambda_4 = 628.6434 \tag{2.2.32}$$

$$\lambda_5 = \frac{63437 - \sqrt{3880252969}}{200} \tag{2.2.33}$$

$$\implies \lambda_5 = 5.7266 \tag{2.2.34}$$

The normalized eigenvectors corresponding to these eigenvalues are

$$\mathbf{v}_1 = \begin{pmatrix}0.59412\\0.80437\end{pmatrix} \tag{2.2.35}$$

$$\mathbf{v}_2 = \begin{pmatrix}0.80438\\-0.59412\end{pmatrix} \tag{2.2.36}$$

Hence we obtain $\mathbf{V}$ of (2.2.11) as

$$\mathbf{V} = \begin{pmatrix}0.59412 & 0.80437\\0.80437 & -0.59412\end{pmatrix} \tag{2.2.37}$$

$\mathbf{S}$ of equation (2.2.11) corresponding to λ_4,λ_5 is

$$\mathbf{S} = \begin{pmatrix}\sqrt{628.6434} & 0\\0 & \sqrt{5.7266}\\0 & 0\end{pmatrix} \tag{2.2.38}$$

$$\implies \mathbf{S} = \begin{pmatrix}25.07276 & 0\\0 & 2.39303\\0 & 0\end{pmatrix} \tag{2.2.39}$$

Moore-Penrose Pseudo-Inverse of $\mathbf{S}$ is given by

$$\mathbf{S}_+ = \begin{pmatrix}\frac{1}{25.07276} & 0 & 0\\0 & \frac{1}{2.39303} & 0\end{pmatrix} \tag{2.2.40}$$

$$\mathbf{S}_+ = \begin{pmatrix}0.03988 & 0 & 0\\0 & 0.41788 & 0\end{pmatrix} \tag{2.2.41}$$

Now using values from (2.2.27), (2.2.37), (2.2.41), (2.2.10) in (2.2.13) we get

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix}2.429755\\-6.834179\\-0.8\end{pmatrix} \tag{2.2.42}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix}0.096899\\-2.855867\end{pmatrix} \tag{2.2.43}$$

$$\mathbf{c} = \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix}-2.2396\\1.7747\end{pmatrix} \tag{2.2.44}$$

Verifying this solution using least squares,

$$\mathbf{A}^T\mathbf{A}\mathbf{c} = \mathbf{A}^T\mathbf{b} \tag{2.2.45}$$

Substituting the values from equations (2.2.9) and (2.2.10) here, we get

$$\begin{pmatrix}\frac{564021}{2500} & \frac{186057}{625}\\\frac{186057}{625} & \frac{255476}{625}\end{pmatrix}\mathbf{c} = \begin{pmatrix}\frac{576}{25}\\\frac{1468}{25}\end{pmatrix} \tag{2.2.46}$$

Solving the augmented matrix

$$\begin{pmatrix}\frac{564021}{2500} & \frac{186057}{625} & \frac{576}{25}\\\frac{186057}{625} & \frac{255476}{625} & \frac{1468}{25}\end{pmatrix} \xleftrightarrow{R_1\leftarrow\frac{2500}{564021}R_1} \begin{pmatrix}1 & \frac{744228}{564021} & \frac{57600}{564021}\\\frac{186057}{625} & \frac{255476}{625} & \frac{1468}{25}\end{pmatrix} \tag{2.2.47}$$

$$\xleftrightarrow{R_2\leftarrow R_2-\frac{186057}{625}R_1} \begin{pmatrix}1 & \frac{744228}{564021} & \frac{57600}{564021}\\0 & \frac{9000000}{564021} & \frac{15972300}{564021}\end{pmatrix} \tag{2.2.48}$$

$$\xleftrightarrow{R_2\leftarrow\frac{564021}{9000000}R_2} \begin{pmatrix}1 & \frac{744228}{564021} & \frac{57600}{564021}\\0 & 1 & \frac{17747}{10000}\end{pmatrix} \tag{2.2.49}$$

$$\xleftrightarrow{R_1\leftarrow R_1-\frac{744228}{564021}R_2} \begin{pmatrix}1 & 0 & -\frac{22396}{10000}\\0 & 1 & \frac{17747}{10000}\end{pmatrix} \tag{2.2.50}$$

Therefore

$$\mathbf{c} = \begin{pmatrix}-\frac{22396}{10000}\\\frac{17747}{10000}\end{pmatrix} = \begin{pmatrix}-2.2396\\1.7747\end{pmatrix} \tag{2.2.51}$$

Hence, verified