Applications of Matrix Algebra

Vector Space Model for text data, Latent Semantic Indexing (LSI) for text searches, matrix representation for graph/network data, Recommender Systems as matrix factorization

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Text Analytics

VECTOR SPACE MODEL

Text Analytics Example

Titles
C1: *Human* machine *interface* for LAB ABC *computer* applications
C2: A *survey* of *user* opinion of *computer system response time*
C3: The *EPS user interface* management system
C4: *System* and *human system* engineering testing of *EPS*
C5: Relation of *user*-percieved *response time* to error measurement
M1: The generation of random, binary, unordered *trees*
M2: the intersection *graph* of paths in *trees*
M3: *Graph minors* IV: Widths of *trees* and well-quasi-ordering
M4: *Graph minors*: A *survey*

Italicized words occur and multiple docs and are indexed

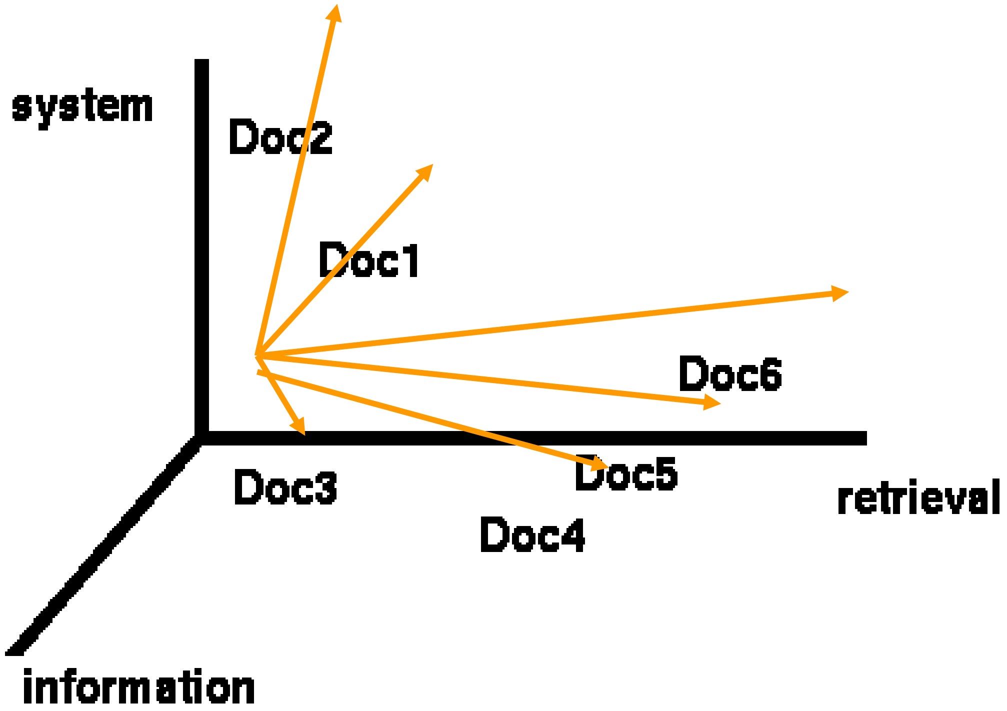
Term-Document Matrix

Terms	Documents								
	c1	c2	c3	c4	c5	m1	m2	m3	m4
Human	1	0	0	1	0	0	0	0	0
Interface	1	0	1	0	0	0	0	0	0
Computer	1	1	0	0	0	0	0	0	0
User	0	1	1	0	1	0	0	0	0
System	0	1	1	2	0	0	0	0	0
Response	0	1	0	0	1	0	0	0	0
Time	0	1	0	0	1	0	0	0	0
EPS	0	0	1	1	0	0	0	0	0
Survey	0	1	0	0	0	0	0	0	0
Trees	0	0	0	0	0	1	1	1	0
Graph	0	0	0	0	0	0	1	1	1
Minors	0	0	0	0	0	0	0	1	1

Vector Space Model

- **Documents are represented as vectors in the "term space"**
  - **Terms are usually *stems***
  - **Documents represented by binary or weighted vectors of terms**
- **Queries are represented the same as documents**
- **Query and Document weights are based on length and direction of their vector**
- **A vector distance measure between the query and documents is used to rank retrieved documents**

Documents in 3D Space



Primary assumption of the Vector Space Model:
Documents that are "close together" in space are similar in meaning.

SVD: Text Analytics

LSI: LATENT SEMANTIC INDEXING

Document Space has High Dimensionality

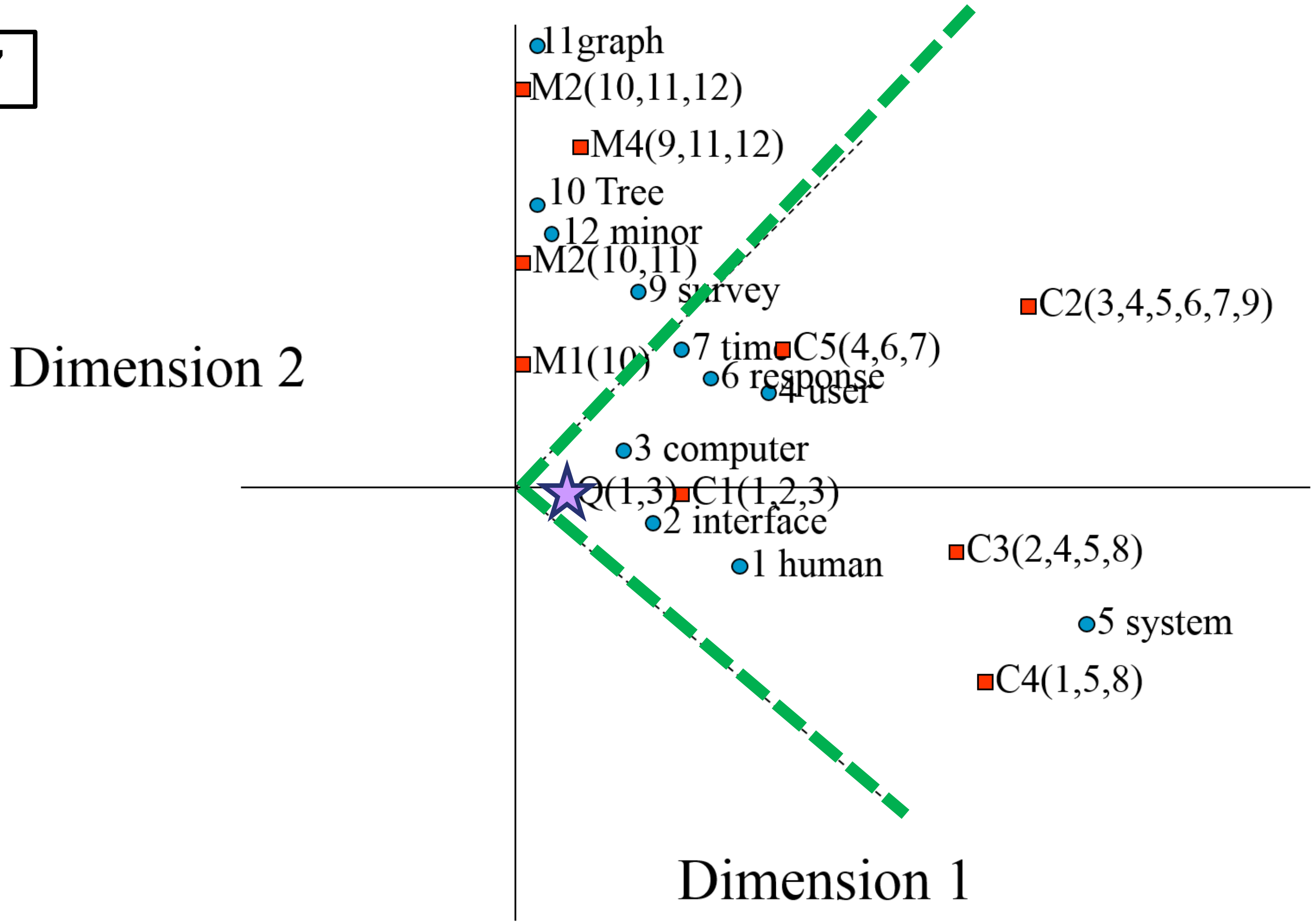
- What happens beyond 2 or 3 dimensions?
- Similarity still has to do with how many tokens are shared in common.
- More terms -> harder to understand which subsets of words are shared among similar documents.
- Approaches to handling (reducing) high dimensionality: Clustering and Latent Semantic Indexing (LSI) (i.e., dimension reduction using SVD)

Rectangular Matrix Decomposition with SVD

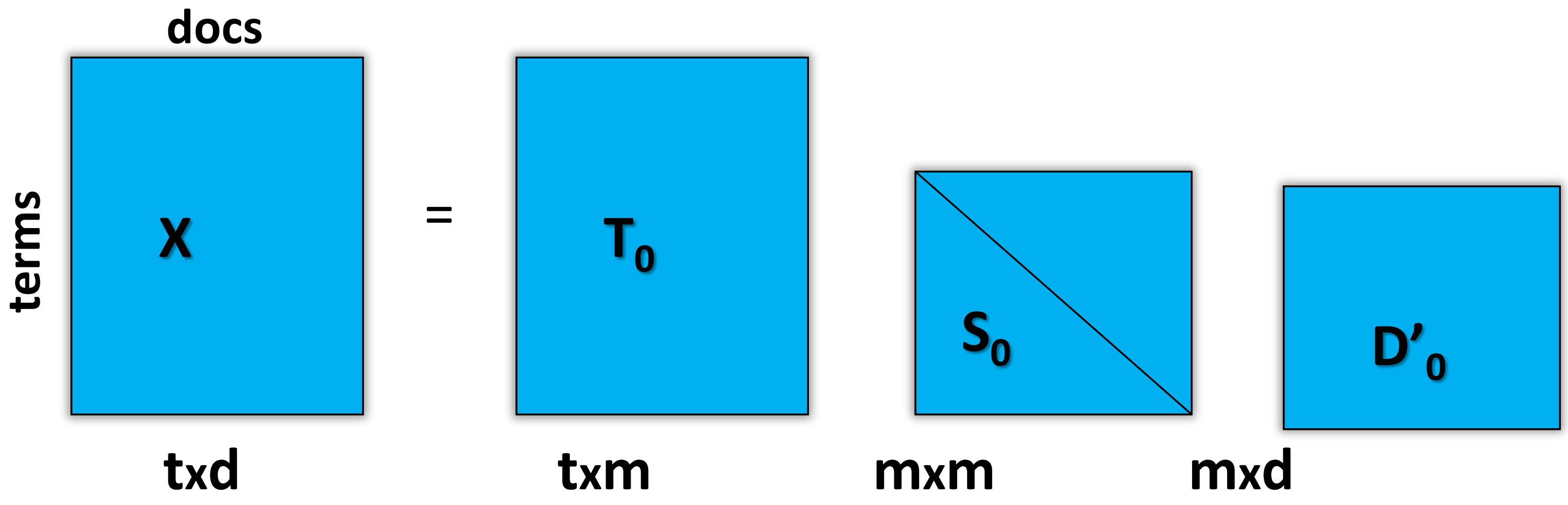
<u>Query</u>: "Human Computer Interaction"

SVD to 2 dimensions:

Blue dots are terms
Documents are red squares
Purple star is a query
Dotted cone is cosine .9 from Query
-- even docs with no terms in common (c3 and c5) lie within cone.



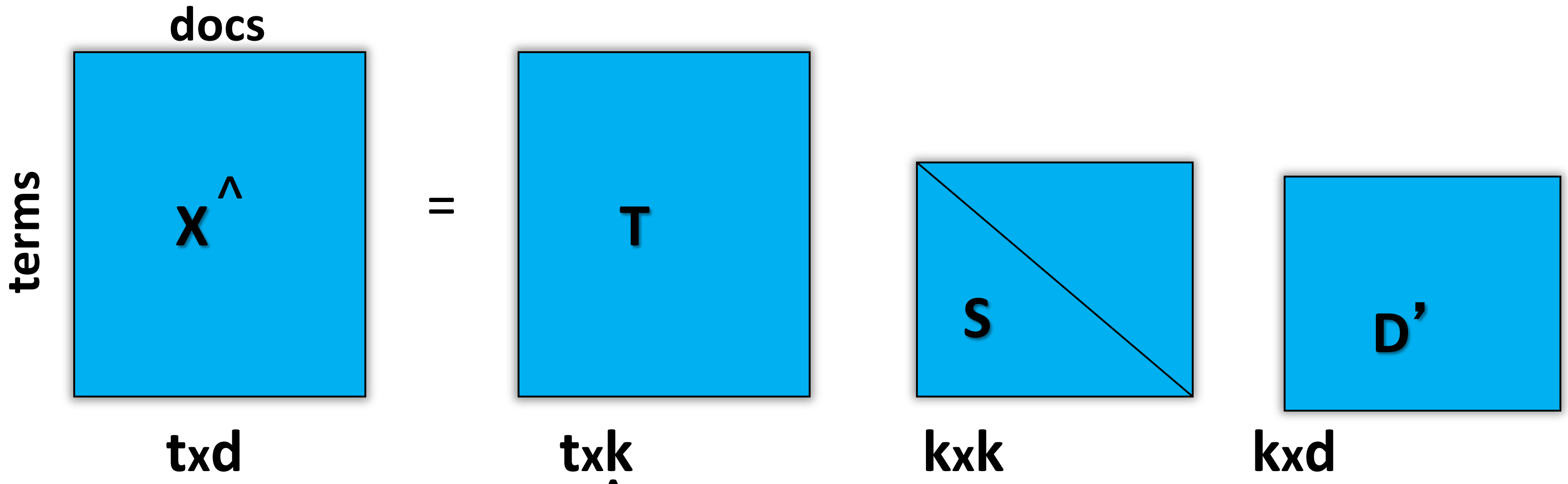
SVD Decomposition of Term-Document Matrix



$$X = T_0 S_0 D'_0$$

T_0 has orthogonal, unit-length columns ($T'_0 T_0 = 1$)
D_0 has orthogonal, unit-length columns ($D'_0 D_0 = 1$)
S_0 is the diagonal matrix of singular values
t is the number of rows in X
d is the number of columns in X
m is the rank of X (<= min(t,d))

Low-Dimensional Approximation of Term-Document Matrix



$$\hat{X} = TSD'$$

T has orthogonal, unit-length columns ($T'T = 1$)
D has orthogonal, unit-length columns ($D'D = 1$)
S_0 is the diagonal matrix of singular values
t is the number of rows in X
d is the number of columns in X
m is the rank of X (<= min(t,d)
k is the chosen number of dimensions in the reduced model (k <= m)

Comparing Two Documents

- The dot product or cosine angle between two column vectors of the matrix X(hat) tells the extent to which two documents have a similar profile of terms/words

Matrices in Graphs / Networks

GRAPHS: MOTIVATION

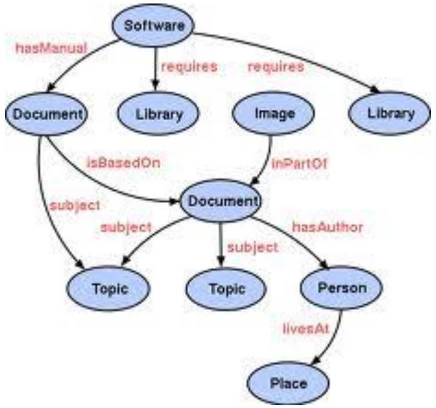
What apps naturally deal w/ graphs?

Social Networks

twitter

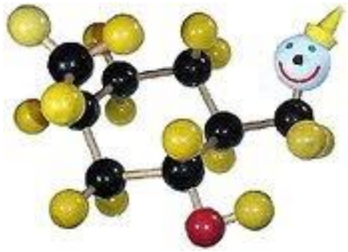
facebook

Semantic Web



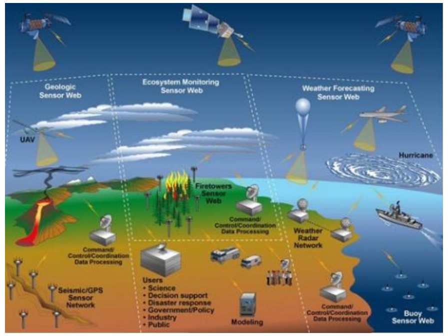
World Wide Web

Drug Design,
Chemical compounds



Computer networks

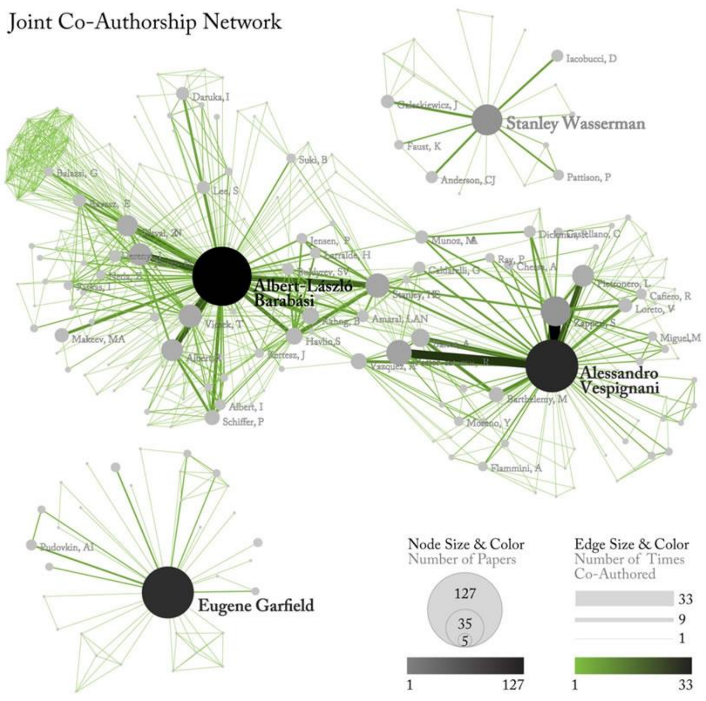
Sensor networks



Credit: Images are from Google images via search of keywords

Real-world Graphs, or Networks

- **Graphs:** consist of objects (vertices) & their relationships (edges/links)
  - **Social:** Facebook, Twitter, Linkedin
  - **Physical:** Internet, power-grid
  - **Biological:** Protein-protein interactions
  - **Academic Collaboration:** Citation, co-author



Can you think of real-world graphs/networks?
- **What are their nodes/vertices?**
- **What are their relationships/edges/links?**

Examples of Real-World Graphs

Graph / Network	Node / Vertex	Edge / Arc / Relationship
Internet	Computer/router	Cable/wireless data connection
WWW	Web page	Hyperlink
Facebook	Person	Friendship
Power Grid	Generating station	Transmission line
Airport Network	Airport in a city	Flight connection
US Roads	City	Roads
Neural Network	Neuron	Synapse

How Big Are These Graphs?

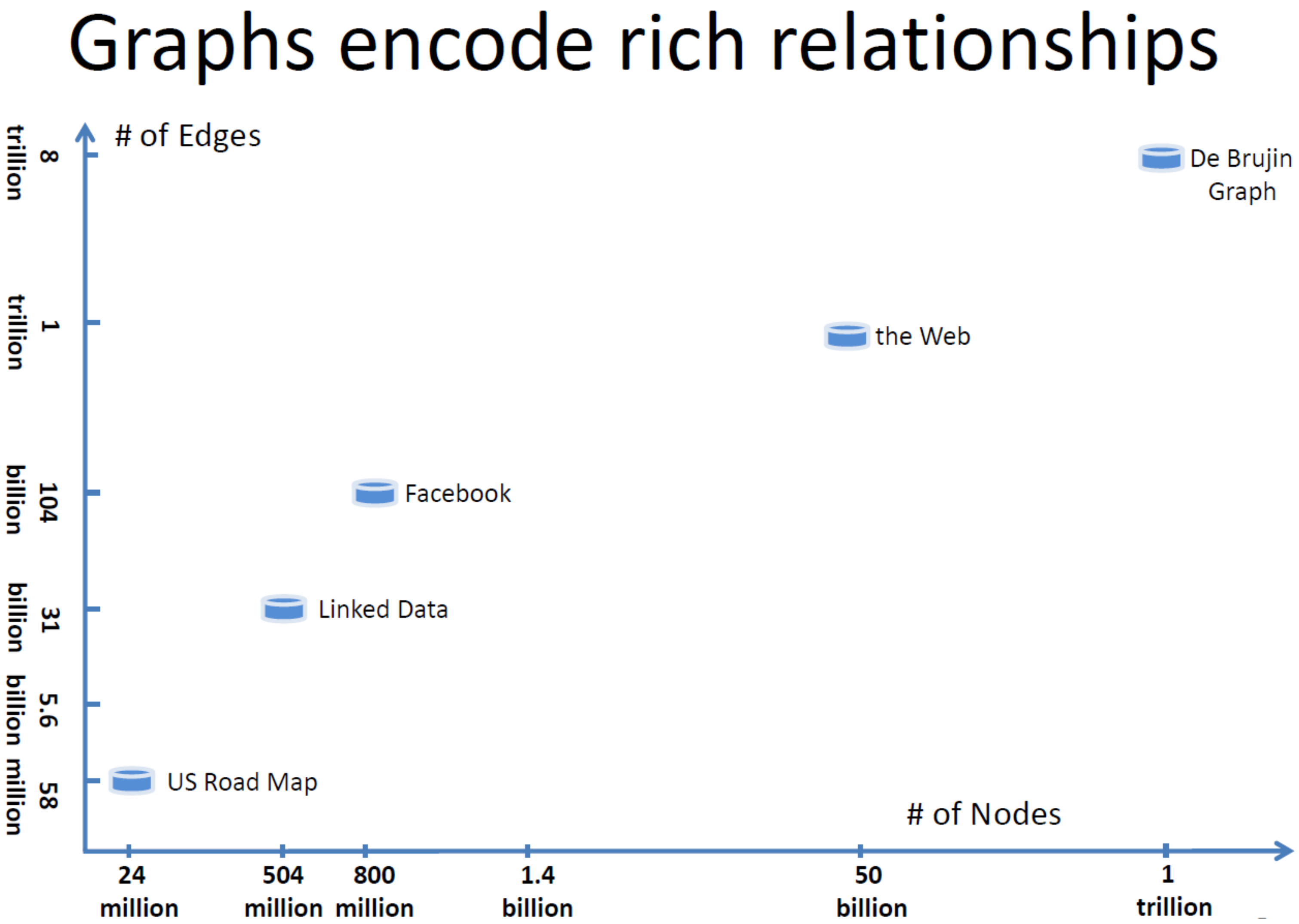


Image source: Haixun Wang, KDD 2012

Graphs Challenge Many Algorithms

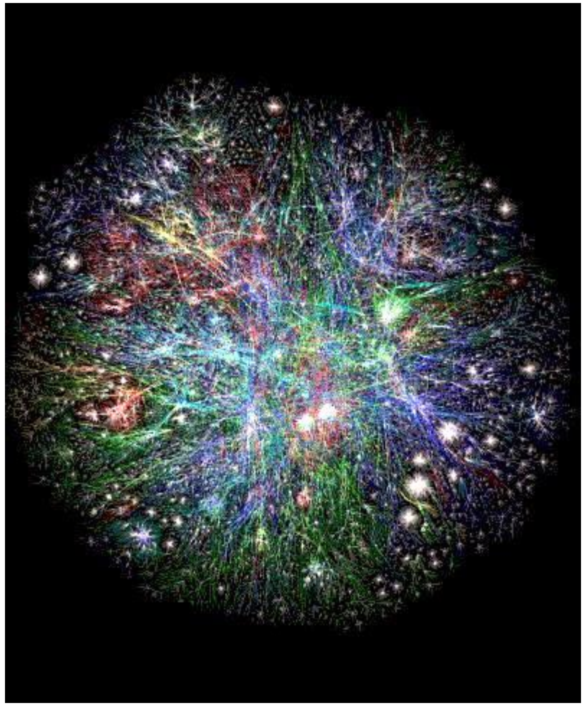
Massive scale

– Can't use conventional algorithms on large graphs

– The graph itself may not even fit in memory

Facebook: |V| = 721M, |E| = 137B
Common crawl: |V| = 3.5B, |E| = 128B

$O(n^2)$ ☹



Matrices in Graphs / Networks

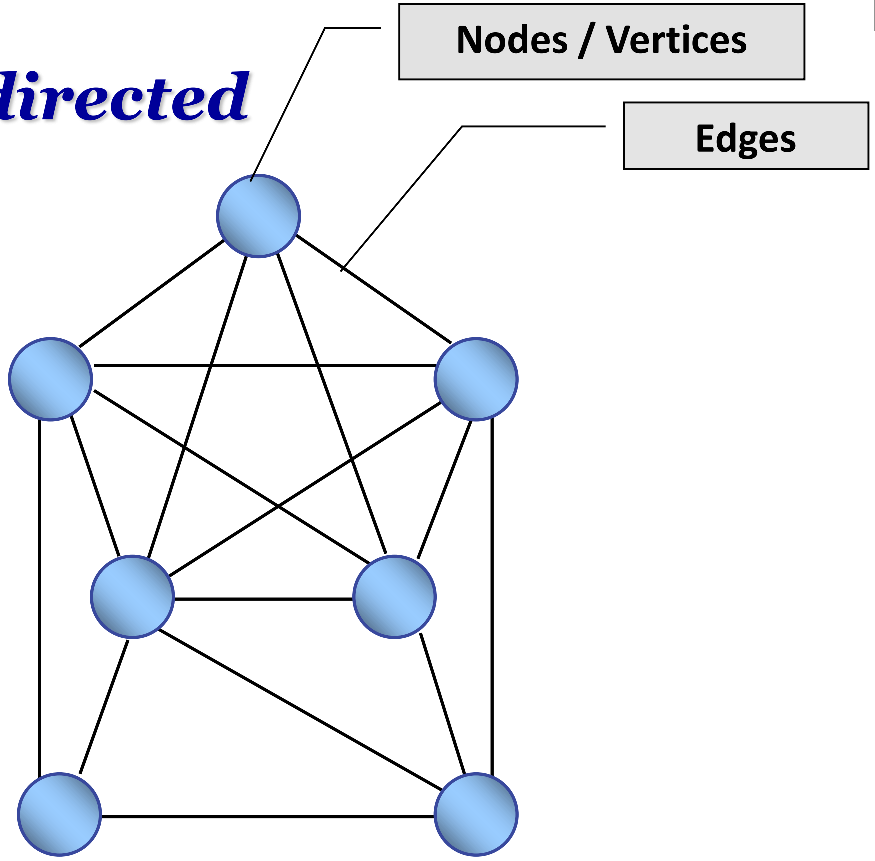
GRAPH THEORY PRIMER

Graphs

Graph with 7 nodes and 16 edges

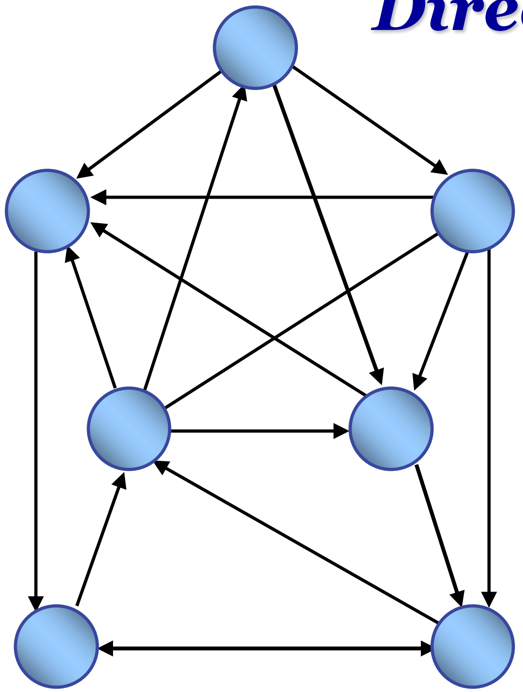
$$G = (V, E)$$
$$V = \{v_1, v_2, \ldots, v_n\}$$
$$E = \{e_k = (v_i, v_j) \mid v_i, v_j \in V, k = 1, \ldots, m\}$$

Undirected



$$(v_i, v_j) = (v_j, v_i)$$

Directed



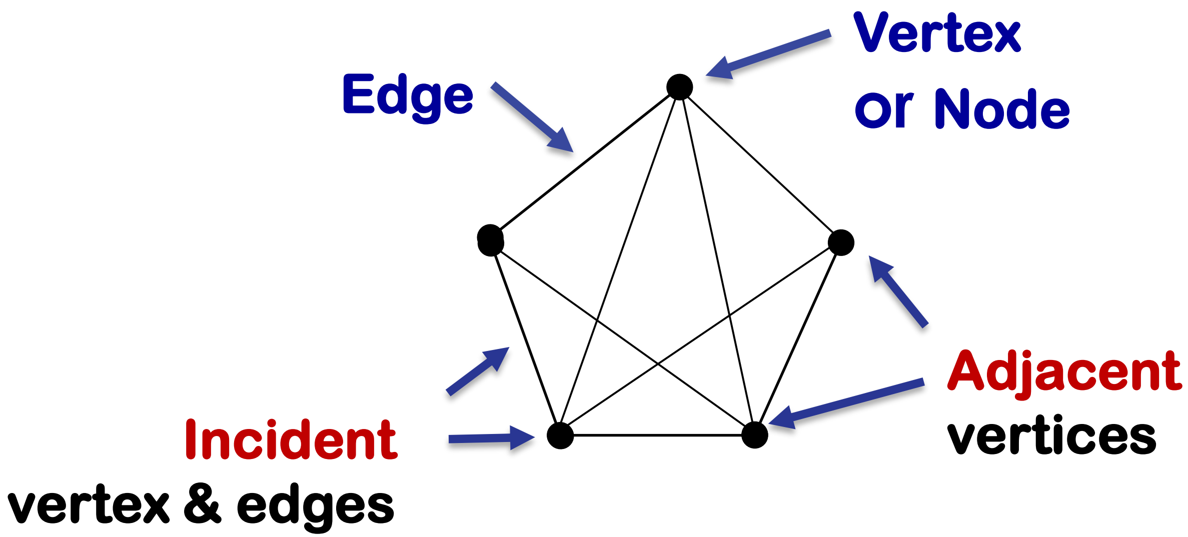
$$(v_i, v_j) \neq (v_j, v_i)$$

Graph Theory

A graph is a collection of *vertices* (nodes) and *edges*.

A vertex represents some object.

An edge connects two vertices and represents some relationship between them.

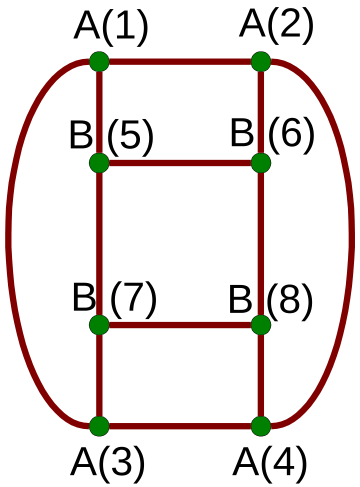


Graph Representation as a Matrix

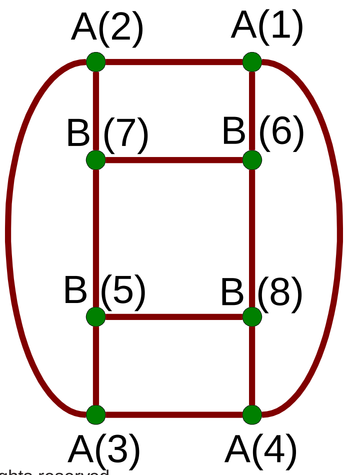
- **Adjacency Matrix** (vertex vs. vertex)
- Incidence Matrix (vertex vs. edge)
- **Sparse** (lots of zero's) vs. **Dense** Matrices

Adjacency Matrix Representation

The representation is *NOT unique*.
Some algorithms are *order-sensitive*.



	A(1)	A(2)	A(3)	A(4)	B(5)	B(6)	B(7)	B(8)
A(1)	1	1	1	0	1	0	0	0
A(2)	1	1	0	1	0	1	0	0
A(3)	1	0	1	1	0	0	1	0
A(4)	0	1	1	1	0	0	0	1
B(5)	1	0	0	0	1	1	1	0
B(6)	0	1	0	0	1	1	0	1
B(7)	0	0	1	0	1	0	1	1
B(8)	0	0	0	1	0	1	1	1

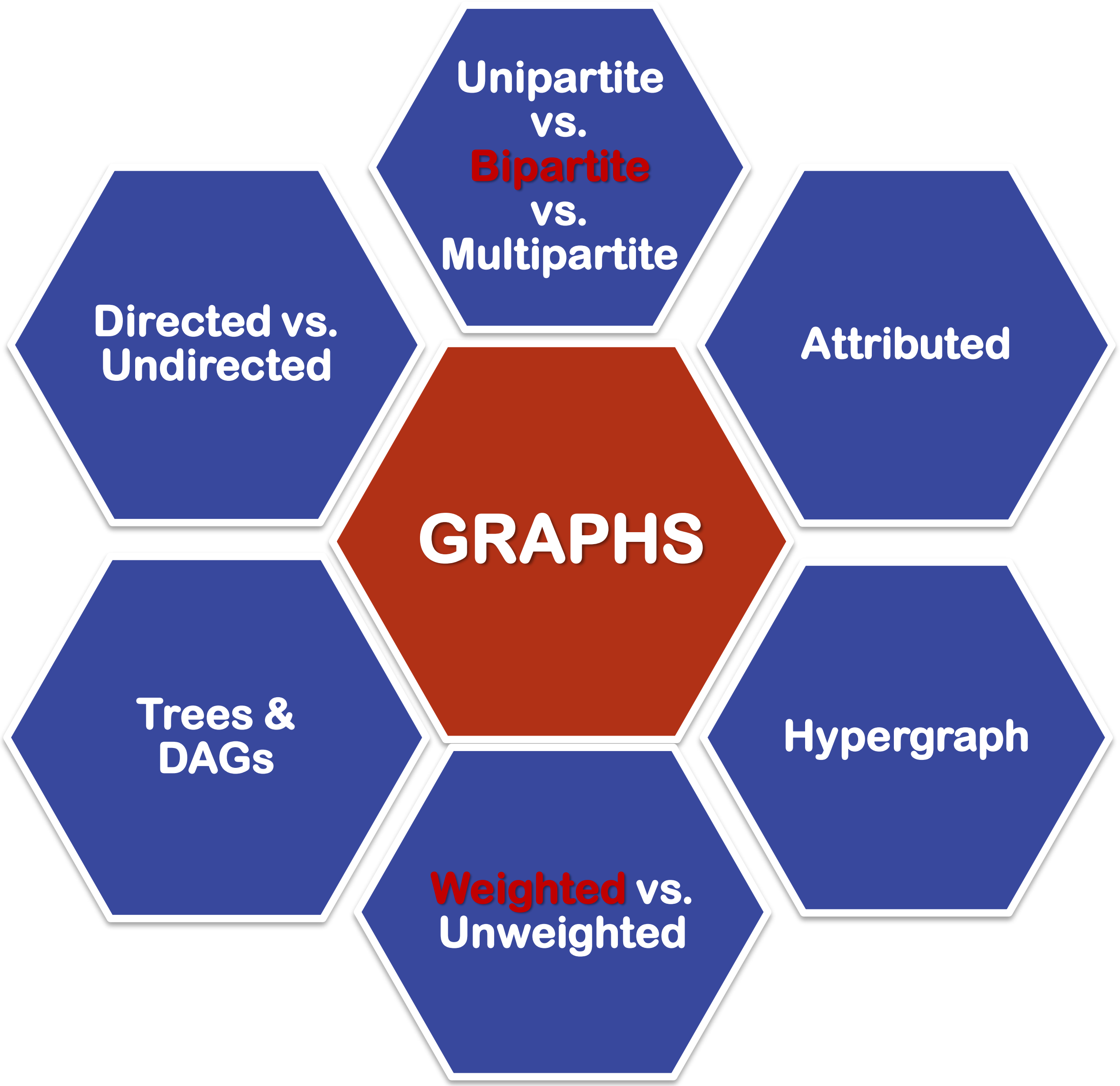


	A(1)	A(2)	A(3)	A(4)	B(5)	B(6)	B(7)	B(8)
A(1)	1	1	0	1	0	1	0	0
A(2)	1	1	1	0	0	0	1	0
A(3)	0	1	1	1	1	0	0	0
A(4)	1	0	1	1	0	0	0	1
B(5)	0	0	1	0	1	0	1	1
B(6)	1	0	0	0	0	1	1	1
B(7)	0	1	0	0	1	1	1	0
B(8)	0	0	0	1	1	1	0	1

Types of Graphs

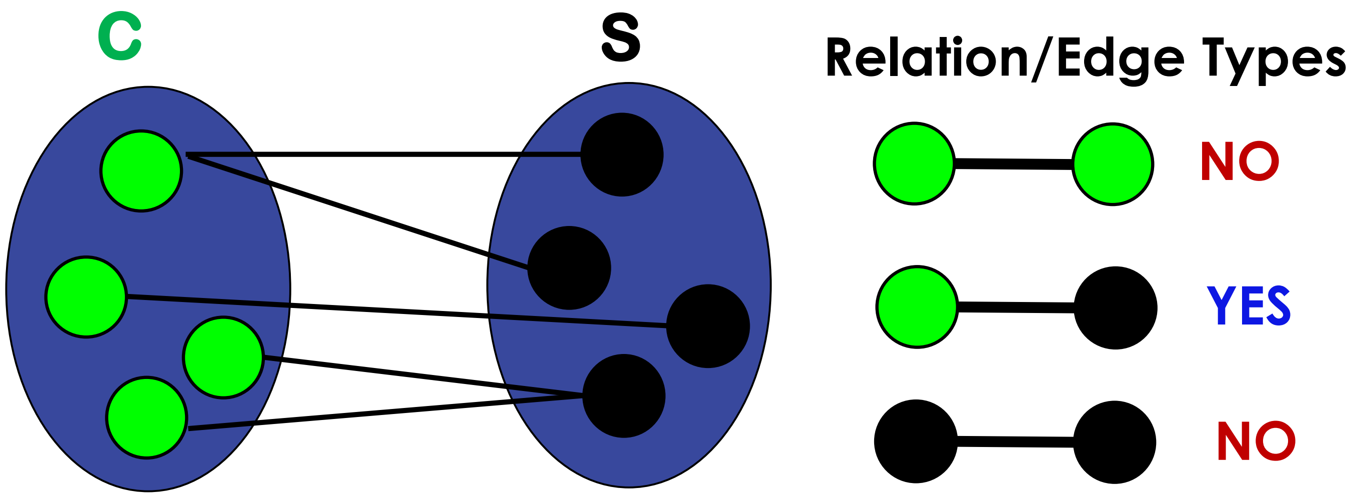
BIPARTITE GRAPHS

Types of Graphs

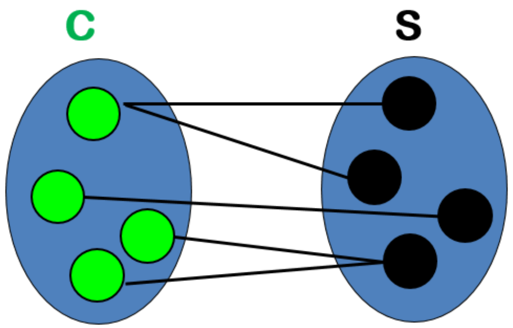


Bipartite Graphs

A graph B is bipartite if its vertices V can be partitioned into two non-intersecting sets C and S such that all relationships/edges go between C and S (no edges go from C to C or from S to S.

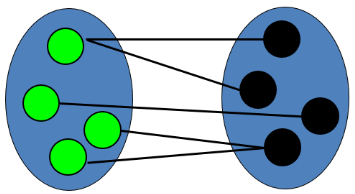


Bipartite Graphs in Business Intelligence (BI)



Business, B	Customers, C	Services, S	Relationships, R
Amazon	Customers	Products	Purchasing
Netflix	Subscribers	Movies	Watching
Pandora	Subscribers	Songs	Listening to
CISCO	Manufacturers	Parts	Supplying to CISCO

Observations about Bipartite BI Graphs



- Adjacency matrix is very **sparse**:($r_{cs} = 0$ or $r_{cs} =???$)
- Relationships are not always binary, but **weighted**:
  - $r_{cs} =*****$: How customer rated a product
  - $r_{cs} = 25$: How many times subscriber listened to the song
  - $r_{cs} = High$: How reliable manufacturer was in supplying the part
- **Missing** relationships ($r_{cs} = 0$ or $r_{cs} =???$) are **business opportunities**:
  - Recommend new products to customers
  - Find reliable manufacturers
  - Ensure continued customer subscriptions to services

Business, B	Customers, C	Services, S	Relationships, R
Amazon	Customers	Products	Purchasing /Rating
Netflix	Subscribers	Movies	Watching
Pandora	Subscribers	Songs	Listening to
CISCO	Manufacturers	Parts	Supplying to CISCO

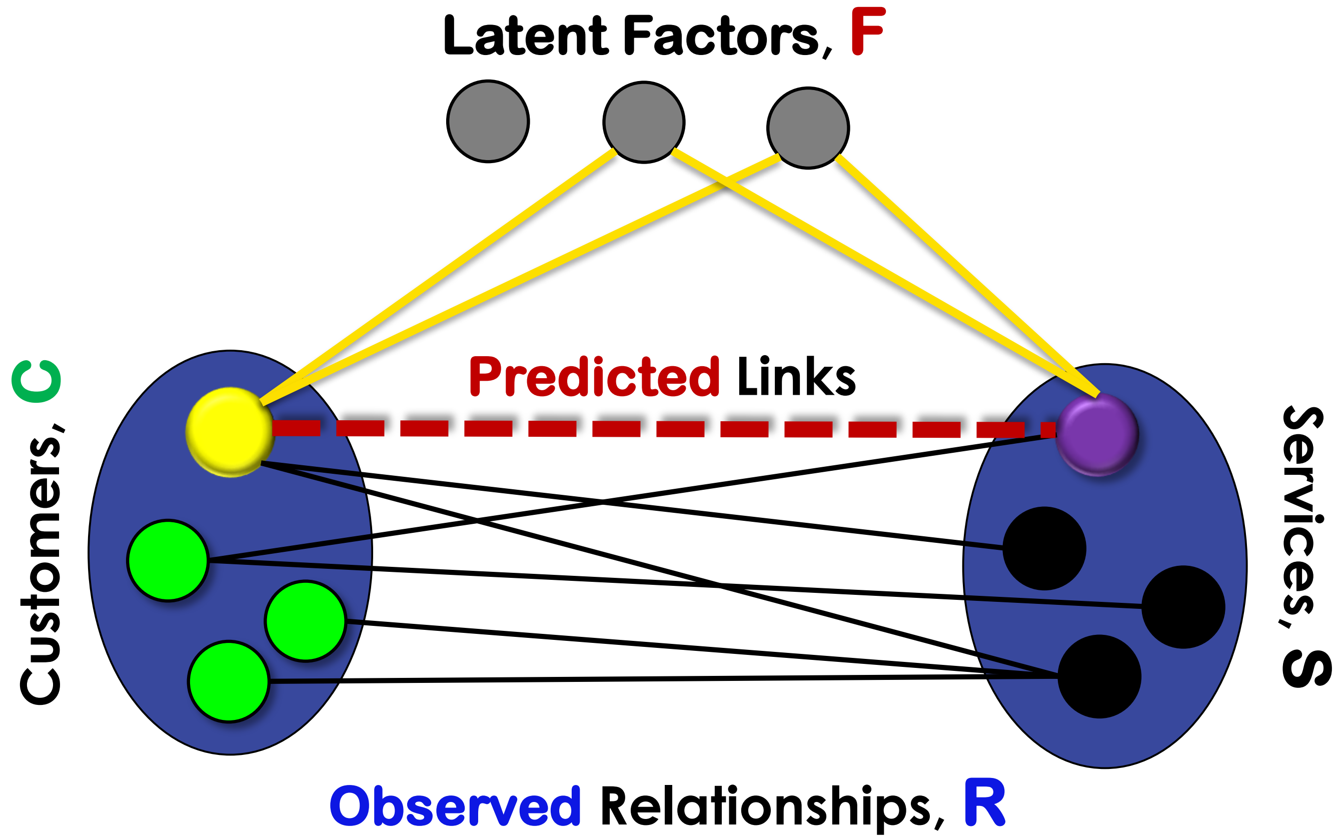
Bipartite Graphs

MISSING LINKS

Assumptions: Hidden Factors & Missing Links

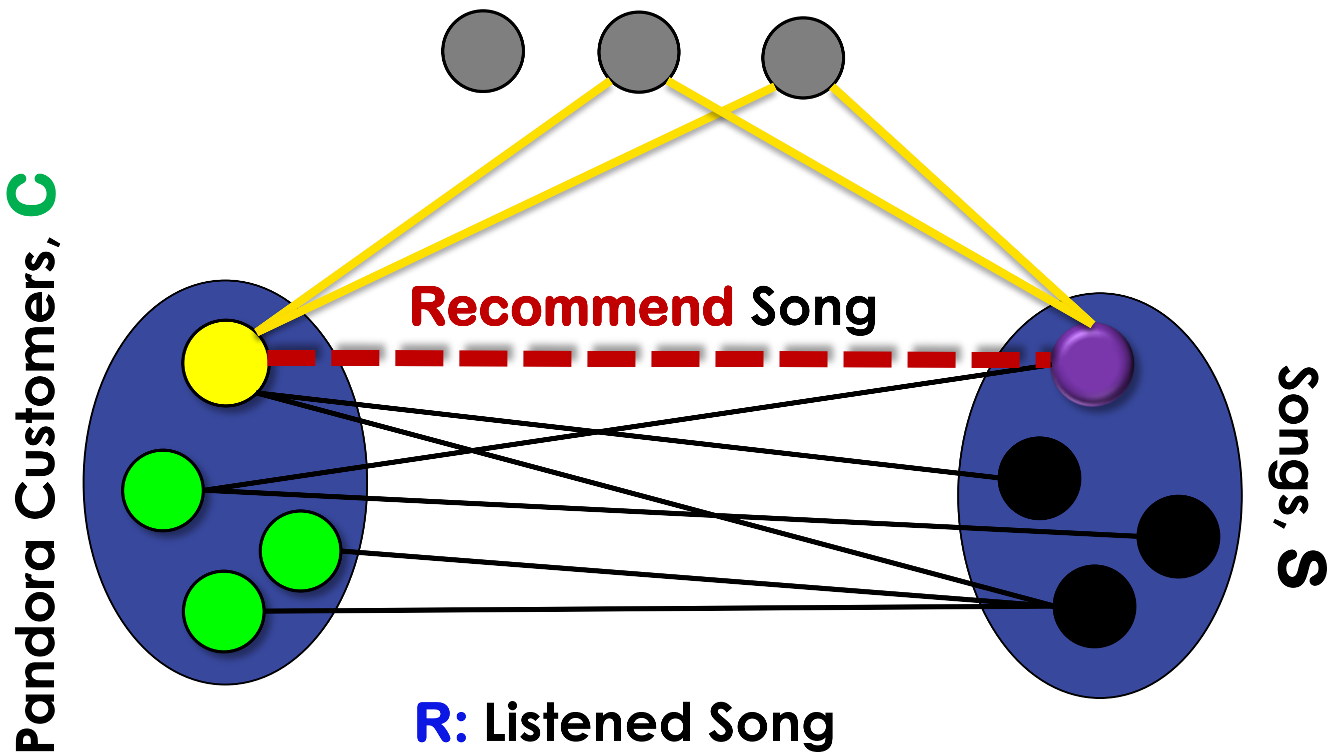
There are a few **hidden** or **latent factors** $F = \{f_1, f_2, \dots, f_h\}$ that define/drive the **relationships** between C and S.

These **latent / hidden factors** can be used **to complete missing relationships** between C and S.



Example: Hidden Factors & Missing Links

Latent Factors, F:
- Tastes
- Genres

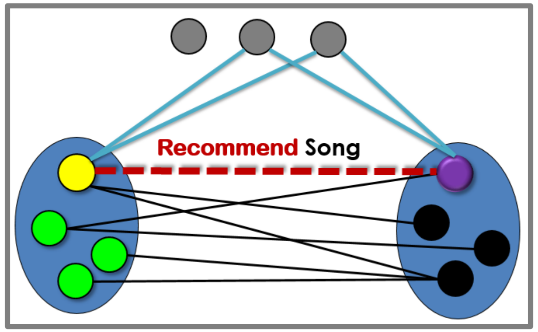


Bipartite Graph Representation

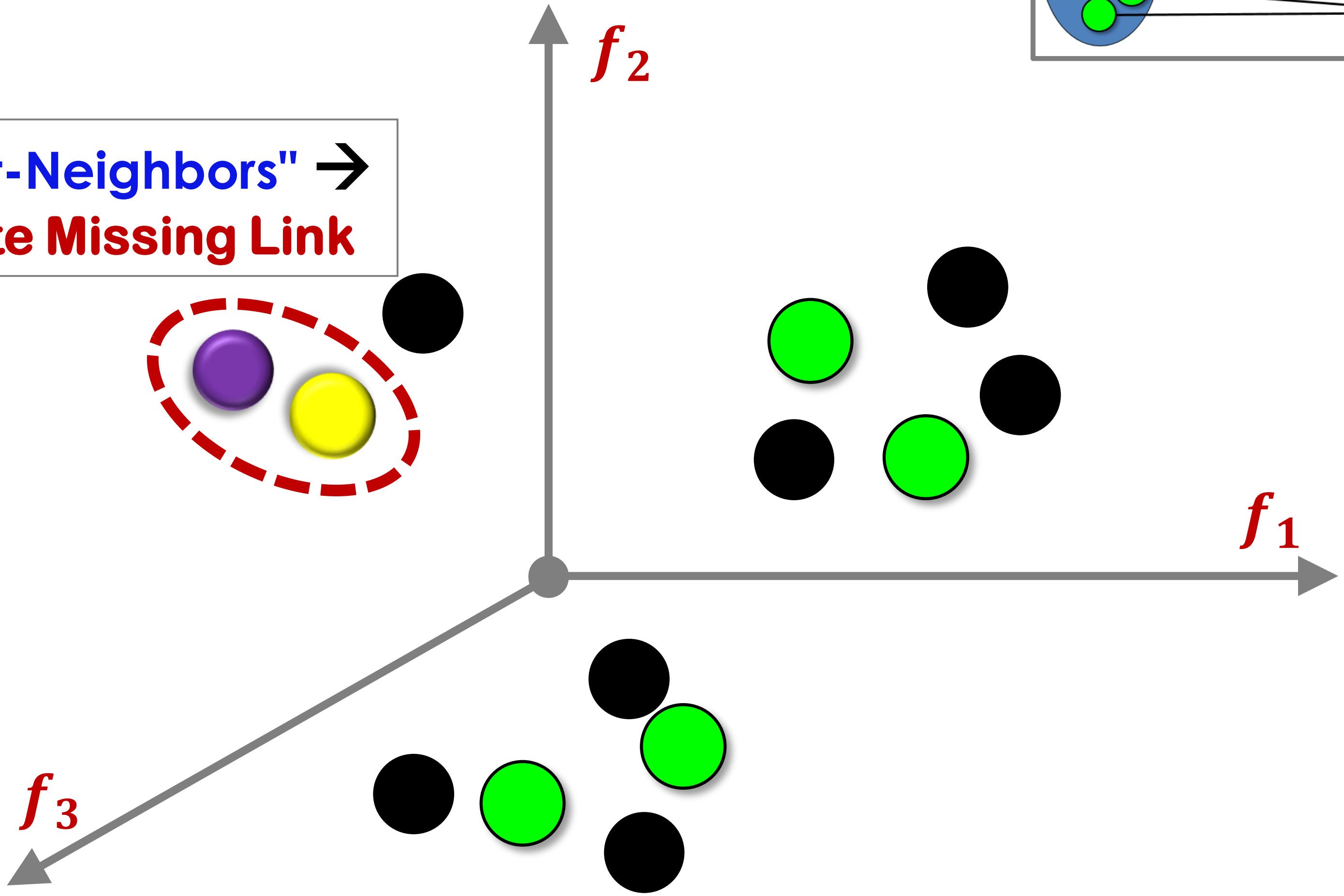
LATENT FACTOR SPACE

Latent Factor Space View

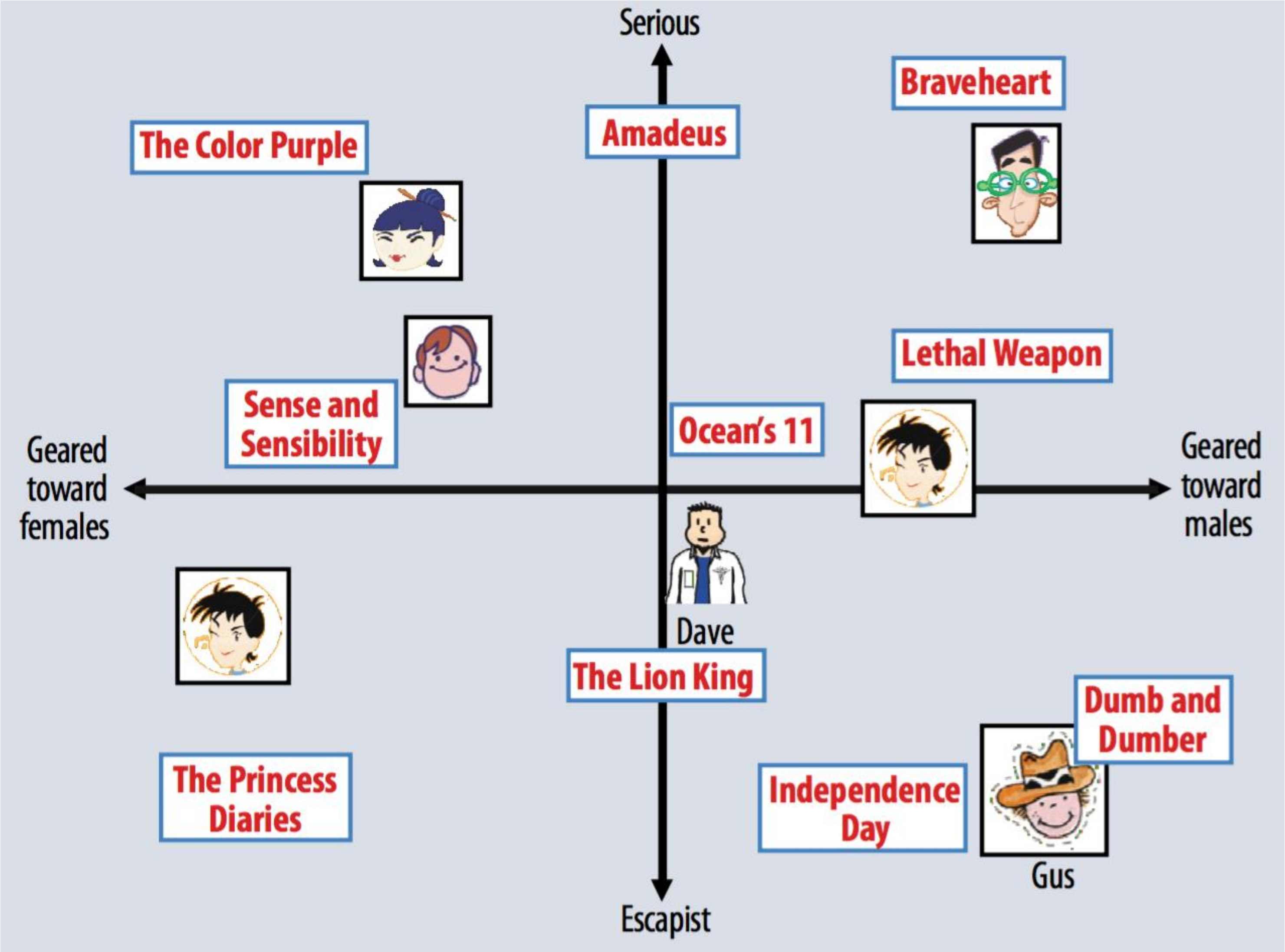
Customer-Service Relations in the Latent Factor Space:
$F = \{f_1, f_2, \ldots, f_h\}$



"Nearest-Neighbors" → Complete Missing Link



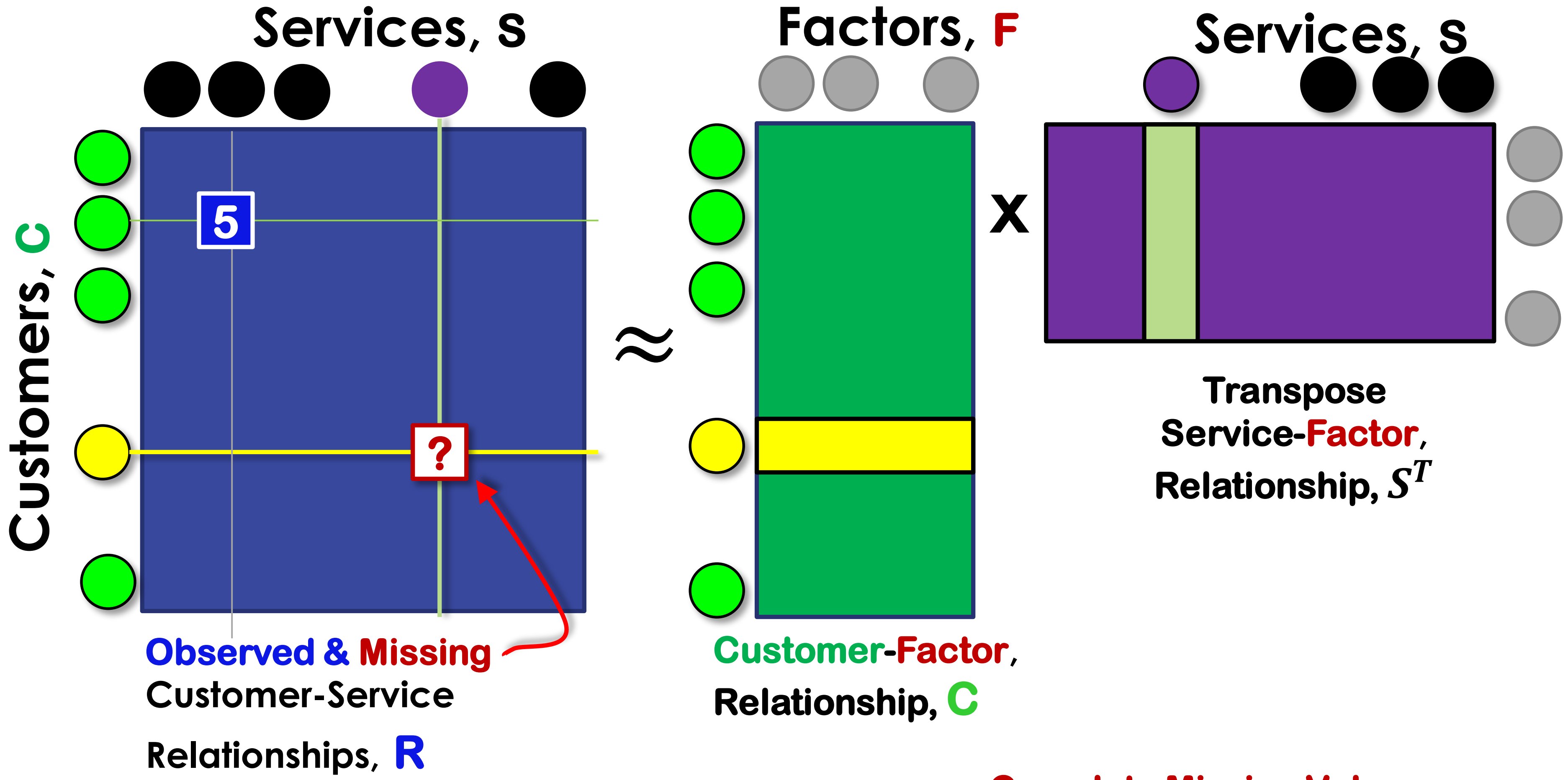
Ex: User-Movie Relations in a Latent Factor Space



Latent Factor Space

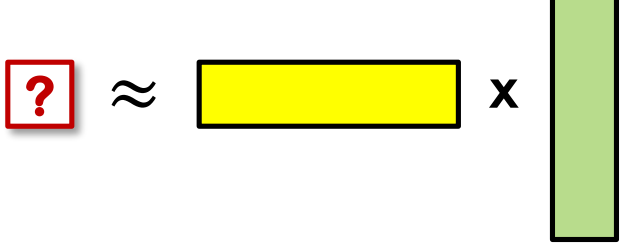
MATRIX FACTORIZATION / DECOMPOSITION

Matrix Factorization / Completion View

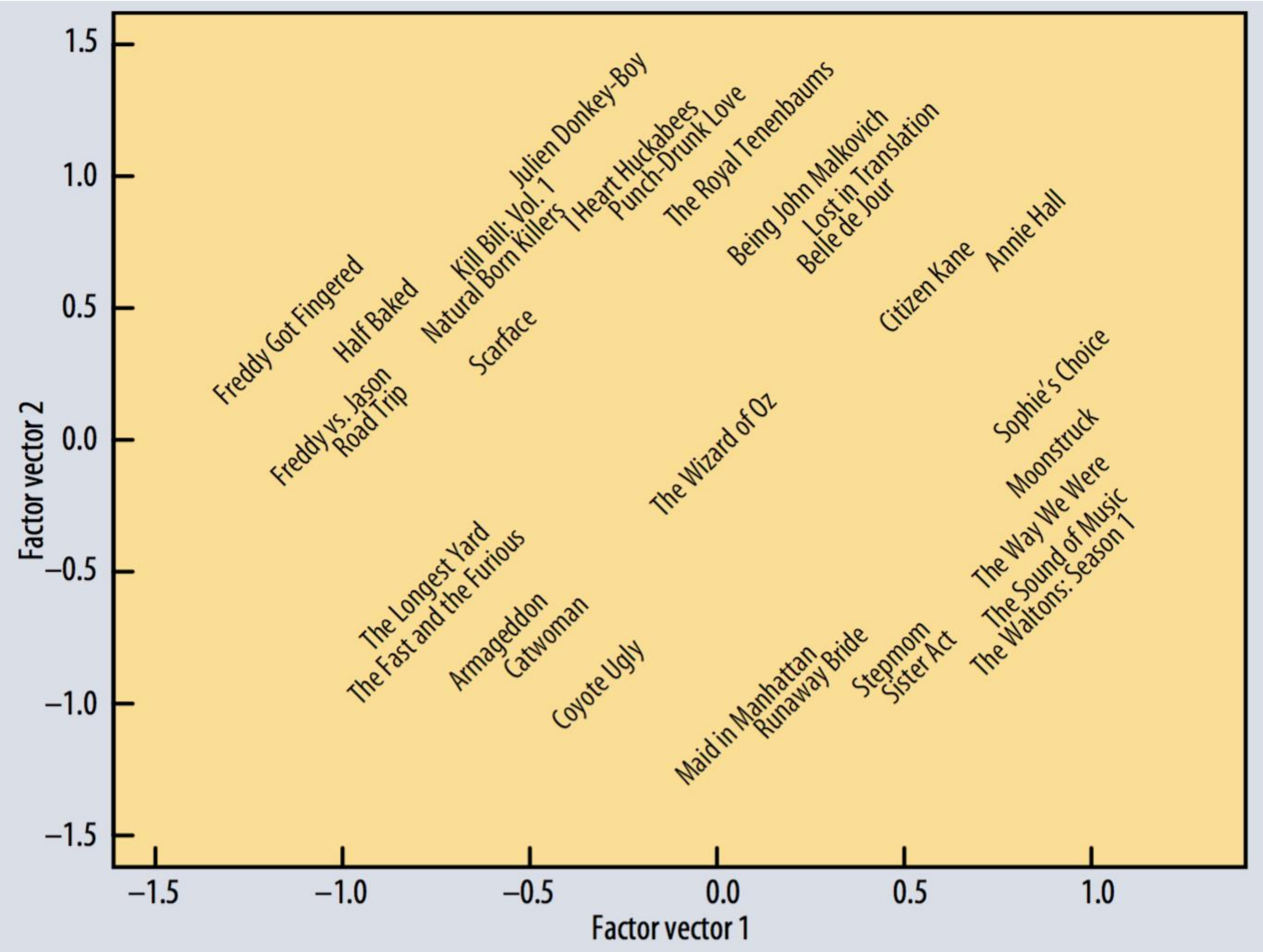


$$\boxed{R \approx C \times S^T}$$

<u>Complete</u> Missing Values



First Two Vectors from Matrix Decomposition

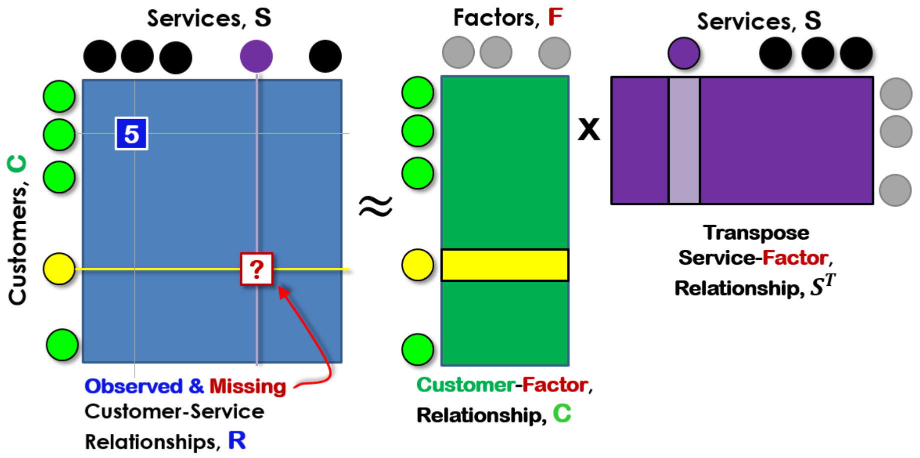


Matrix Factorization / Decomposition

ALTERNATE LEAST SQUARES

Paatero, 1994

Matrix Completion, or Matrix Factorization



Customers: C $= \{c_1, c_2, \dots, c_n\}$

Services: S $= \{s_1, s_2, \dots, s_m\}$

Hidden Factors: $F = \{f_1, f_2, \dots, f_h\}$

$$\boxed{R_{n\times m} \approx \hat{R} = C_{n\times h} \times S^T_{h\times m}}$$

$$\boxed{r_{i,j} \approx \overrightarrow{c_i} \times \overrightarrow{s_j^T}}$$

vector product

R: sparse matrix

$\hat{\text{R}}$: dense matrix

$C \times S^T$: dense matrix

C: dense matrix

S: dense matrix

$h = O(constant)$

$h \ll n$

$h \ll m$

How to Find the "Best" $\hat{R}$?

$$\boxed{R_{n\times m} \approx \hat{R} = C_{n\times h} \times S^T_{h\times m}}$$

known unknown unknown

$$\boxed{r_{i,j} \approx \vec{c_i} \times \vec{s_j^T}}$$

vector product

- **For the sake of simplicity, <u>let's assume</u> that $S^T_{h\times m}$ is known.**
- **How to find $C_{n\times h}$?**
- **Can we multiply on the right by the inverse of $\left(S^T_{h\times m}\right)^{-1}$?**
  - Note: Matrix inverse is only for squared matrices
  - But $S^T_{h\times m}$ is skinny, rectangular ($h \ll m$)
- **How can we create a squared matrix?**

$$\boxed{R_{n\times m} \times S_{m\times h} \approx C_{n\times h} \times (S^T_{h\times m} \times S_{m\times h})}$$

- **$S^T \times S$ is an $h \times h$ squared matrix.**
- **<u>Let's assume</u> that $S^T \times S$ is invertible, then**

$$\boxed{R \times S \times \left(S^T \times S\right)^{-1} \approx C}$$

Assuming S is known and $S^T S$ is invertible

- Goal: Optimize the "best" approximation to the matrix C

$$R \times S \times \left(S^T \times S\right)^{-1} \approx C$$

Minimize Least Squares, or the Frobenius Norm

$$\left\| R \times S \times \left(S^T \times S\right)^{-1} - C \right\|_F \rightarrow min$$

$$E = R \times S \times \left(S^T \times S\right)^{-1} - C$$

Error Matrix, E

$$\|E\|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{h} E_{ij}^2} = \sqrt{trace(E^T E)}$$

Let's assume that $C_{n\times h}$ is known

$$\boxed{R_{n\times m} \approx \hat{R} = C_{n\times h} \times S^T_{h\times m}}$$

known known unknown

$$\boxed{r_{i,j} \approx \overrightarrow{c_i} \times \overrightarrow{s_j^T}}$$

vector product

1. How to find $S^T_{h\times m}$?
2. What other assumptions should be made?
3. What is the optimization function?

$$\boxed{???\approx S}$$

$$\boxed{???\rightarrow \textbf{min}}$$

Alternating Least Square

<u>Optimization Problem:</u> Find $R_{n\times m} \approx \hat{R} = C_{n\times h} \times S^T_{h\times m}$

Minimize Loss Function: $\left\|R - C \times S^T\right\|^2_F \rightarrow min$ or equivalently

$$min_{C,S} \sum_{u,i} (r_{ui} - C_u S_i^T)^2$$ ***

- Optimizing C and S simultaneously is non-convex, hard
- If C <u>or</u> S are fixed, then the system of linear equations: convex & easy
  - Initialize S with random values
  - Solve for C
  - Fix C
  - Solve for S
  - Repeat ("Alternating")
    - Fixed number of iterations or
    - Till the Error stabilizes