A/B Testing

Treatment and control, randomization, test statistic, Null and Alternative Hypothesis, One-tailed and Two-Tailed Test, p-value, alpha, Bootstrap sampling, Statistical Simulation, independent and paired samples, comparing means (t-test), and comparing proportions (Z-test, chi-squared test, McNamir test)

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Examples: A/B Tests

- Testing two prices to determine which yields more net profit
- Testing two web headlines to determine which produces more clicks
- Testing two web ads to determine which generates more conversions
- …

A/B Tests: The Two Sample Comparison

Is treatment A different from treatment/control B?

- **Applications: Treatment, intervention**
  - a drug
  - a medical device
  - an advertising campaign
  - a price
  - a manufacturing procedure
- **Is the observed difference between two samples due to chance or due to a real difference in the treatments?**

A/B Test

- **A/B Test**
  - An experiment with two groups to establish which of two treatments, products, procedures, or the like is superior
  - Often one of the treatments is the standard existing treatment, or no treatment
- **Control vs. treatment**
  - If a standard (or no treatment) is used, it is called the control
- **Typical Hypothesis**
  - Treatment is better than control
  - Example: *The product price A is more profitable than the price B*
- **Applications**
  - Marketing
  - Web design

A/B Testing: Key Ideas

- **Subjects are assigned (ideally, randomly) to two (A and B) groups**
  - that are treated exactly alike
  - except that the treatment under study differs from one another
- **A metric (test statistic) is used to compare group A to group B**

Test Statistic

- **Test statistic**
  - A metric (e.g., difference in means, difference in proportions, difference in risks) used to compare group A to group B
- **Examples**
  - Binary variables
    - click or no-click
    - buy or don't buy
    - fraud or no fraud
  - Categorical count variables:
    - contracts signed
    - pages visited
  - Continuous variables
    - purchase amount
    - profit
    - revenue per page-view (rather than conversion)

2 x 2 table for eCommerce Experiment

Outcome	Price A	Price B
Conversion	200	182
No conversion	2,353	2,240
Proportion	0.085	0.081

Mean Revenue per Page Conversion

Outcome	Price A	Price B
Revenue	$3.87	$4.11

A/B Testing: Comparing Two Means

INDEPENDENT SAMPLES OF CONTINUOUS VARIABLE

A/B Hypothesis Testing Procedures

Hypothesis to Test

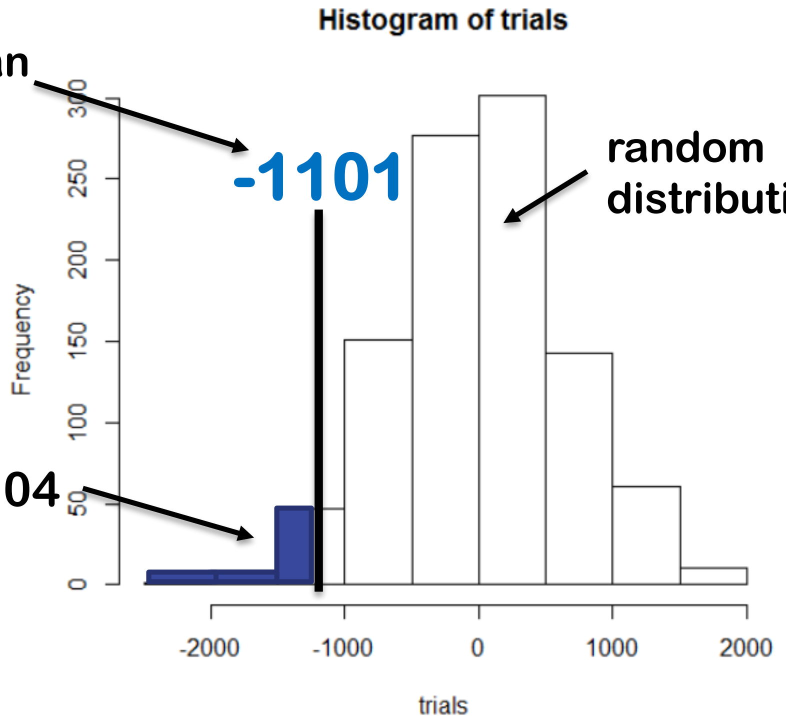
Function	Statistical simulation	Mathematical Formula
t.test() stats.ttest_ind()	see histogram below	$$t = \frac{\overline{x}_B - \overline{x}_A}{\sqrt{\frac{s_B^2}{n_B} + \frac{s_A^2}{n_A}}}$$ t-statistic

sample mean difference

-1101

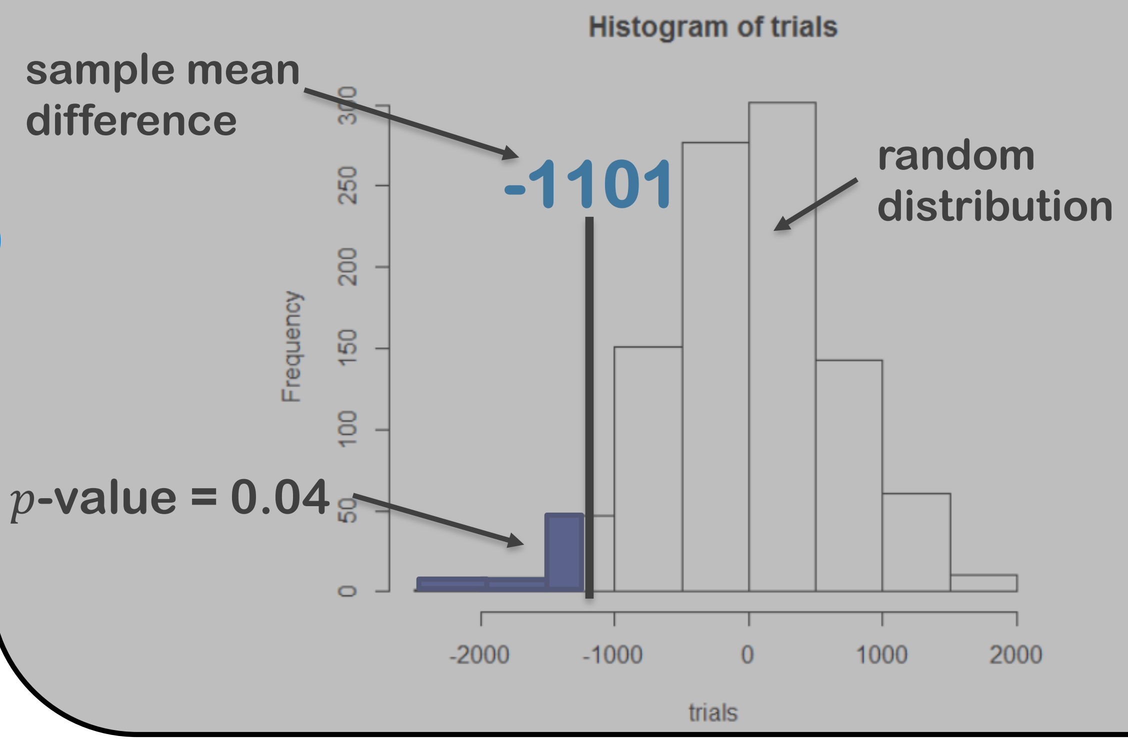
random distribution

p-value = 0.04



A/B Hypothesis Testing Procedures

Hypothesis to Test

R function	Statistical simulation	Mathematical Formula
t.test() stats.ttest_ind()		$$t = \frac{\overline{x}_B - \overline{x}_A}{\sqrt{\frac{s_B^2}{n_B} + \frac{s_A^2}{n_A}}}$$ t-statistic

Example: Comparing Two Means

Is the difference in means between Treatment & Control statistically significant?

- Ten pigs were randomly assigned to a Treatment group with a new blood clotting agent
- Ten pigs were assigned to a Control group that did not receive the clotting agent
- Each pig's liver was injured in a controlled manner and blood loss was measured

- **Sample Statistic**
  - Blood loss
  - Continuous variable
  - Compare means

```r
df <- read.table(file = "../data_raw/pigblood.table.txt",
                 header=TRUE)
control = df[,1]
treatment = df [,2]

cat("Mean Control: ", mean(control), "\n")
cat("Mean Treatment: ", mean(treatment), "\n")
cat("Difference in means: ",
    mean(treatment) - mean(control),"\n")
```

File: AB_test_compare_means.R

Control	Treatment
786	543
375	666
4446	455
2886	823
478	1716
587	797
434	2828
4764	1251
3281	702
3837	1078

Mean Control: 2187

Mean Treatment: 1086

Difference: -1101

Approach #1: Comparing Two Means via t.test()

Alternative Hypothesis: H1: Control Mean is greater than Treatment Mean

- **Ho**: no difference in the mean blood loss between two groups except for what chance might produce

```
# alternative = "greater":
#     x has a larger mean than y
t.test (x=control,
        y=treatment,
        alternative="greater")
```

File: AB_test_compare_means.R

Mean Control: 2187
Mean Treatment: 1086
Difference: -1101

```
        Welch Two Sample t-test

data:  control and treatment
t = 1.777, df = 11.717, p-value = 0.05075
alternative hypothesis: true difference in means is
 greater than 0
95 percent confidence interval:
 -5.494969       Inf
sample estimates:
mean of x mean of y
   2187.4    1085.9
```

p-value is statistically significant (alpha = 0.1)

There is a ~5% chance of observing such a difference by chance

Examples: Null and Alternative Hypotheses

- **Null = Ho = "no difference between the means of group A and group B"**
  - Alternative = H1 = "A is different from B" (could be bigger or smaller)
- **Null = Ho = "A $\leq$ B"**
  - Alternative = H1 = "B > A"
- **Null = Ho = "A $>$ B"**
  - Alternative = H1 = "B $\leq$ A"

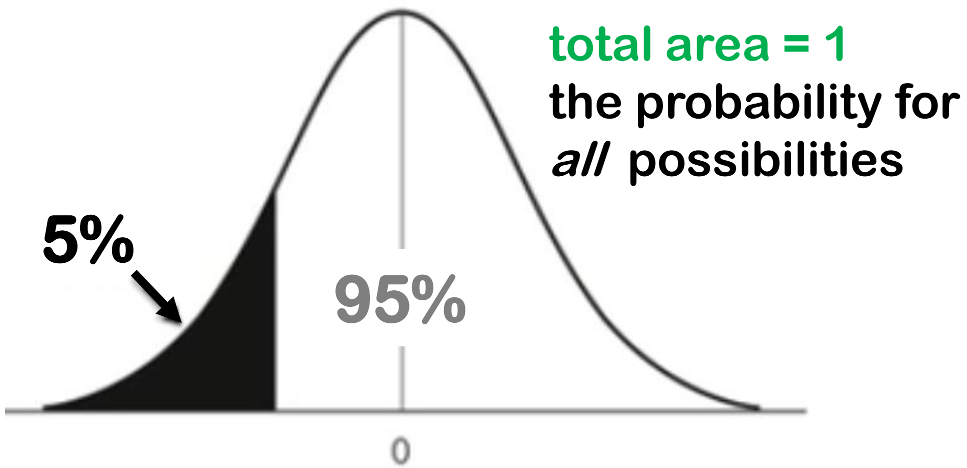
> - The Null and Alternative Hypotheses must account for *all* possibilities.
> - The nature of the null hypothesis determines the structure of the hypothesis test.

One-tailed or Two-tailed Hypothesis Tests

Statistical Significance Level, $\alpha = 0.05$

defined before the experiment starts

One-tailed test

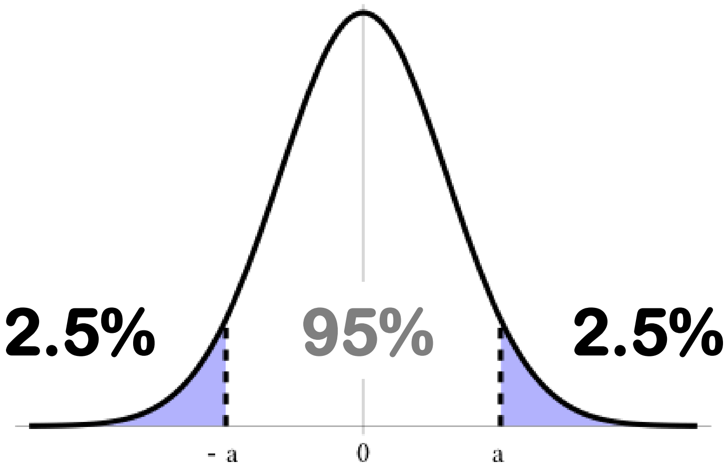


p-value < 0.05 = α

area under the curve < 0.05

alternative = c("less")

Two-tailed test

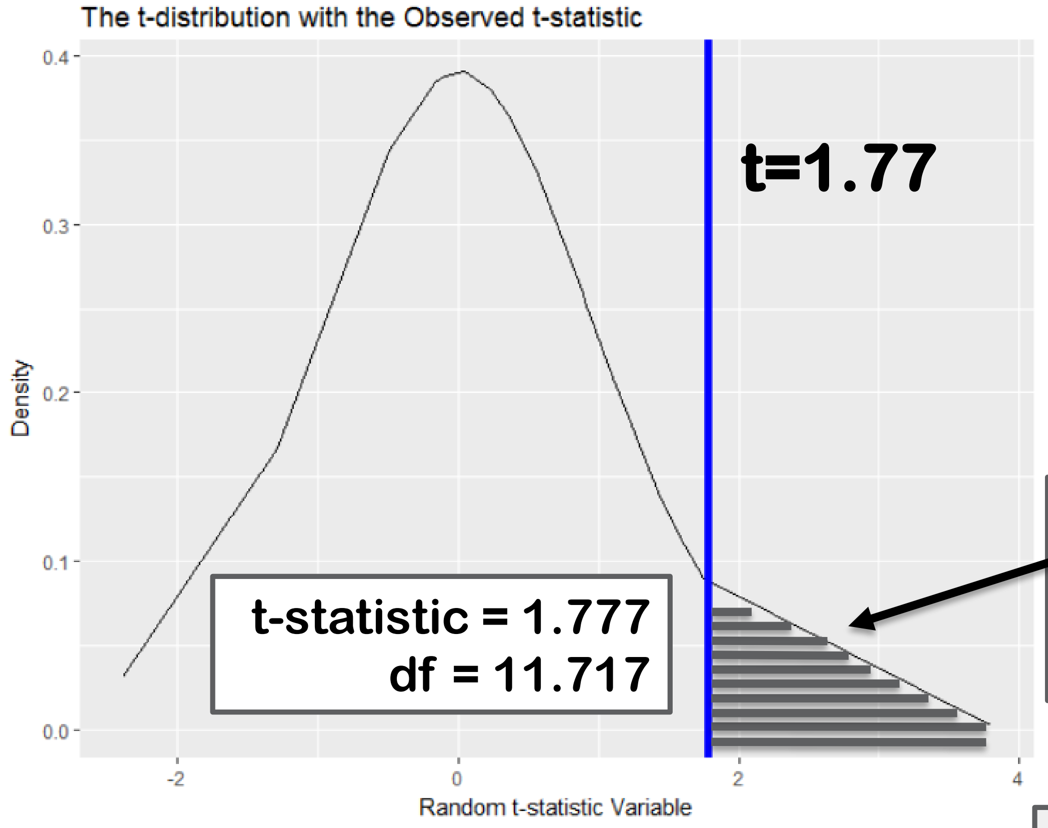


p-value < 0.025 = $\frac{\alpha}{2}$

alternative = c("two.sided")

p-value is the probability of observing such an effect by random chance

Approach #1: t-statistic and df reported by t.test()




```
# access individual Values
res <- t.test (x=control,
               y=treatment,
               alternative="greater")
res$p.value
res$statistic
res$parameter
```

t-statistic
degrees of freedom

File: AB_test_compare_means.R

```
# P[X > t.statistic]
pt (t.statistic,
    df=degrees.of.freedom,
    lower.tail = FALSE)
```

File: AB_test_t_distribution.R

```
# Generate n random numbers from a t-distribution
# with the specified degrees of freedom

tRandom <- rt (n, df=degrees.of.freedom)
```

Statistical Distributions & Functions in R

Distribution	Random Number Generator	Density	Distribution	Quantile
Normal	rnorm	dnorm	pnorm	qnorm
t	rt	dt	pt	qt
F	rf	df	pf	qf
χ^2	rchisq	dchisq	pchisq	qchisq

{**dpqr**}*distribution_abbreviation*()

- **d** = density
- **p** = distribution function
- **q** = quantile function
- **r** = random generation

- **pnorm(a)** $\equiv P(X \leq a)$: probability that a or smaller number occurs
- **pnorm(b) – pnorm(a)** $\equiv P(a \leq X \leq b)$: probability that the variable falls between two points
- qnorm(): given the cumulative probability distribution, it returns the quantile

Statistical Distributions: Mean and Variance

Distribution	Degrees of freedom	Mean	Variance
Normal		μ	σ^2
t	n	0	$n/(n-2)$
F	n_1 and n_2	$n_2/(n_2-2)$	a/b
χ^2	r	r	$2r$

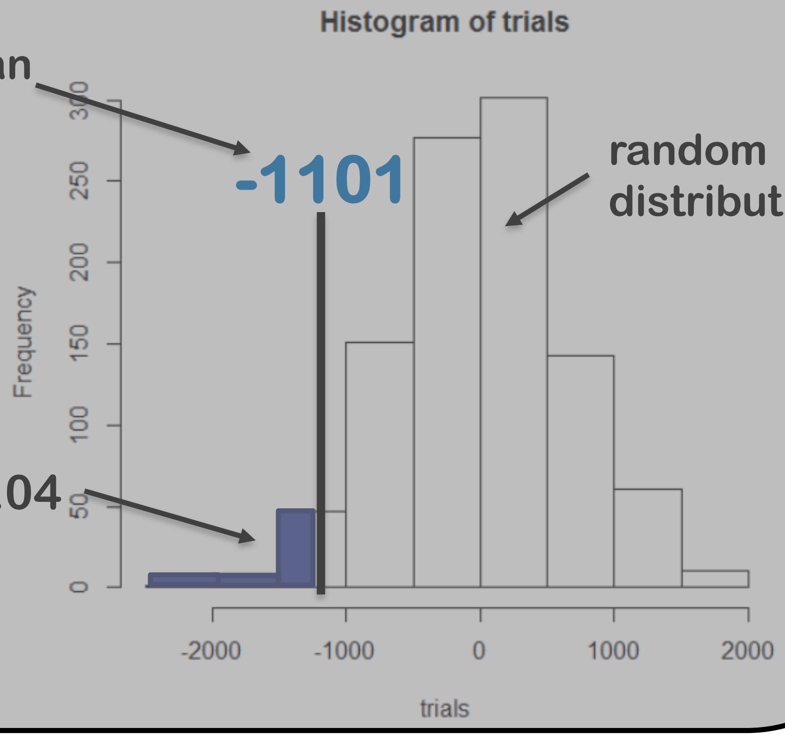
$$a = 2n_2^2(n_1 + n_2 - 2)$$

$$b = n_1(n_2 - 2)^2\,(n_2 - 4)$$

A/B Hypothesis Testing Procedures

Hypothesis to Test

R function	Statistical simulation	Mathematical Formula
t.test()	sample mean difference -1101 random distribution p-value = 0.04	$$t = \frac{\bar{x}_B - \bar{x}_A}{\sqrt{\frac{s_B^2}{n_B} + \frac{s_A^2}{n_A}}}$$ t-statistic



Test Statistic: Typical General Form

$$\text{Test Statistic} = \frac{\text{Observed Differences}}{\text{Standard Error}}$$

- Observed difference
  - Difference between the observed mean of group A and observed mean of group B
  - Difference between the observed proportions for group A and observed proportions for group B (e.g., proportions: ad clicks vs. total web page visits)
- Standard error:
  - The variability (standard deviation) of a sample statistic over many samples

- Standard deviation:
  - refers to variability of individual data values within a single sample

Approach #2: Formula for t-statistic to compare two means

The difference between the mean of the treatment and the mean of the control comes from the t-distribution. The statistical significance of the observed difference is the probability P [X > t.statistic] (one-tailed).

t-statistic formula

$$\boxed{t = \frac{\bar{x}_B - \bar{x}_A}{\sqrt{\frac{s_B^2}{n_B} + \frac{s_A^2}{n_A}}}}$$

$\bar{x}_B$ = The sample average for treatment B

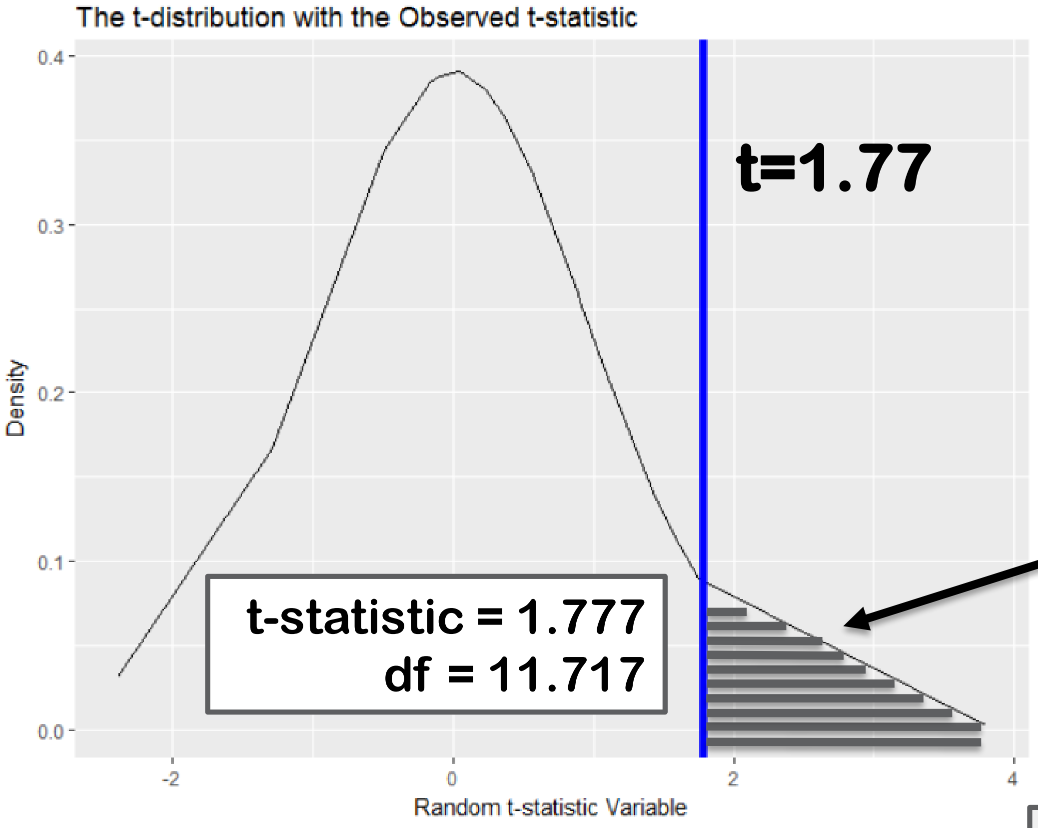
s_B^2 = The variance for treatment B

n_B = The sample size for treatment B

```
df <- read.table(file = "../data_raw/pigblood.table.txt",
                 header=TRUE)
control <- df[,1]
treatment <- df [,2]
n.control <- length(control)
n.treatment <- length(treatment)

observed.difference <- mean (control) -
                       mean (treatment)

standard.error <- sqrt (var(treatment)/n.treatment +
                        var(control)/n.control)

t.statistic <- observed.difference / standard.error
t.statistic
```

File: AB_test_formula.R

Approach #2: t-statistic and df from math formulas



$$t = \frac{\bar{x}_B - \bar{x}_A}{\sqrt{b + a}}$$

$$df = \frac{b + a}{\frac{b^2}{n_B - 1} + \frac{a^2}{n_A - 1}}$$

$$a = \frac{s_A^2}{n_A}$$

$$b = \frac{s_B^2}{n_B}$$

P [X > t.statistic] (one-tailed)

```
pt (t.statistic,
    df=degrees.of.freedom,
    lower.tail = FALSE)
```

File: AB_test_t_distribution.R

```
# Generate n random numbers from a t-distribution
# with the specified degrees of freedom

tRandom <- rt (n, df=degrees.of.freedom)
```

A/B Hypothesis Testing Procedures

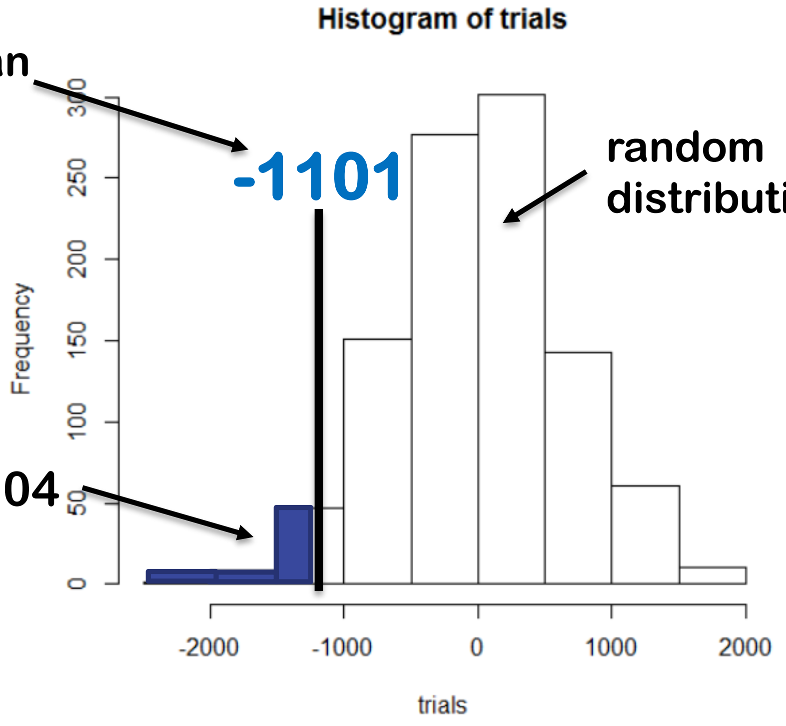
Hypothesis to Test

R function

t.test()

Statistical simulation

sample mean difference



random distribution

p-value = 0.04

Mathematical Formula

$$t = \frac{\bar{x}_B - \bar{x}_A}{\sqrt{\frac{s_B^2}{n_B} + \frac{s_A^2}{n_A}}}$$

t-statistic

Approach #3: Statistical Simulation to Compare two Means

Null Hypothesis: **Ho: Differences between Control &Treatment Means are the same**

- **Ho**: no difference in the mean blood loss between two groups except for what chance might produce

Original Sample

Control	Treatment
786	543
375	666
4446	455
2886	823
478	1716
587	797
434	2828
4764	1251
3281	702
3837	1078

Difference: -1101

1. Shuffle Combined Control and Treatment (A+B) data

```
> shuf = sample (vector.data, replace = FALSE)
> shuf
 [1]  543  823 4446 3281  786 3837 1716  587  666 1251
[11]  455 2828  375 1078  797 2886  702  478 4764  434
```

2. Split into New Control and New Treatment data

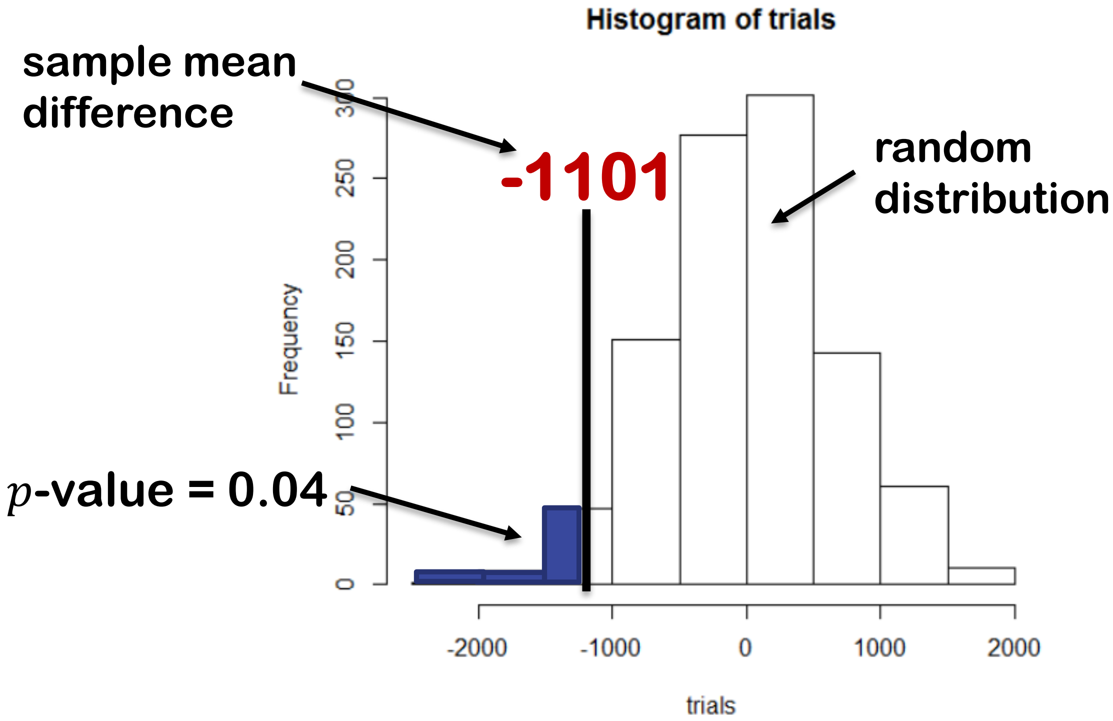
```
> control = shuf[1:10]
> treatment = shuf[11:20]
> control
 [1]  543  823 4446 3281  786 3837 1716  587  666 1251
> treatment
 [1]  455 2828  375 1078  797 2886  702  478 4764  434
```

3. Compare means for New Control and New Treatment data

```
> trial = mean (treatment) - mean (control)
> trial
[1] -313.9
```

4. Repeat steps 1-3 for 1,000 trials, plot histogram, compute p-value

A/B Test: Pig Blood



p-value = 0.044 < 0.05

A difference in blood loss ≤ -1101 ml occurred only 44 times out of 1,000 under the chance model → chance is not responsible and treatment is effective.

Result is statistically significant → Reject the Null Hypothesis

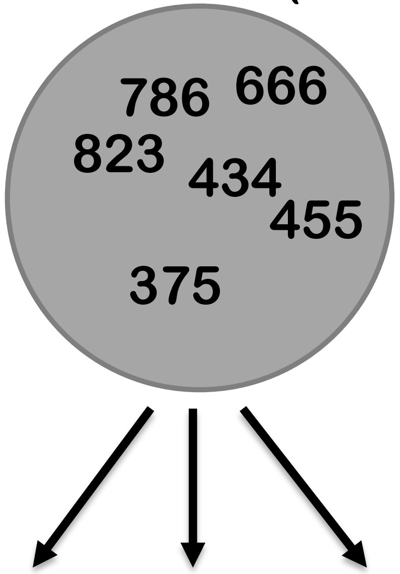
```r
n.repeats <- 1000
with.replacement <- TRUE
trials <- replicate(n.repeats,
                    simulation.trial(vector.data,
                                     with.replacement,
                                     mean))
hist(trials)
# The mean.diff is negative, so use <=
pval <- ifelse(trials <= mean.diff, 1, 0)
cat("p-value: ",sum(pval)/n.repeats,"\n")

```

```r
simulation.trial <- function(data.vector,
                             replace=TRUE,
                             fun.name) {
  shuffle <- sample(data.vector,
                    length(data.vector),
                    replace)
  shuffle.matrix <- matrix(shuffle,ncol=2,
                           nrow=length(data.vector),
                           byrow=TRUE)
  statistic <- fun.name(shuffle.matrix[,2]) -
               fun.name(shuffle.matrix[,1])
  return(statistic)
}
```

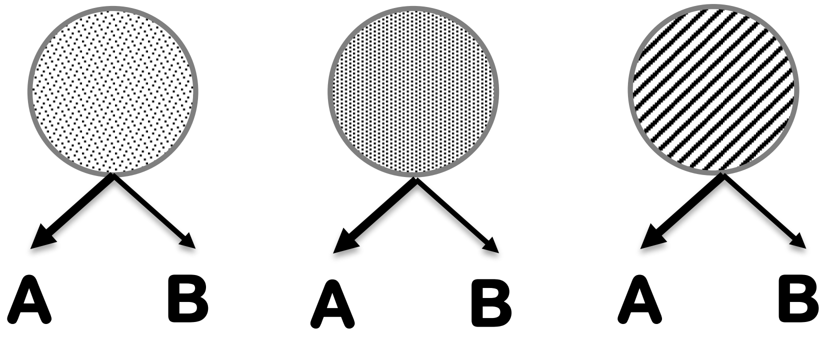
File: AB_test_compare_means_simulation.R

Bootstrap Sampling: With or Without Replacement

Original Combined (A+B) Sample



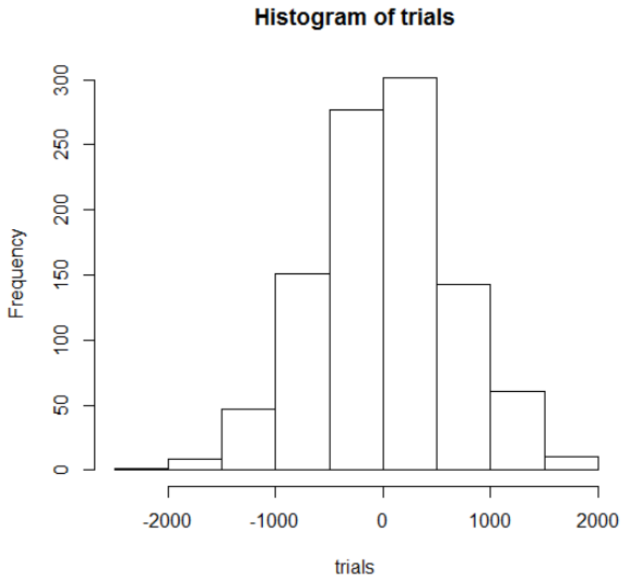
Draw 1,000 **resamples**, **with** or **without replacement**: the **same size** as A+B




```r
simulation.trial <- function(data.vector,
                             replace=TRUE,
                             fun.name) {
  shuffle <- sample(data.vector,
                    length(data.vector),
                    replace)
  shuffle.matrix <- matrix(shuffle,ncol=2,
                           nrow=length(data.vector),
                           byrow=TRUE)
  statistic <- fun.name(shuffle.matrix[,2]) -
               fun.name(shuffle.matrix[,1])
  return(statistic)
}
```

File: AB_test_compare_means_simulation.R

histogram of:
mean(B) – mean(A)



Summary: What we covered so far

- **A/B Test**
  - Control (A) and Treatment (B) groups or
  - Treatment (A) and Treatment (B)
- **Null Hypothesis and Alternative Hypothesis**
  - One-tailed vs. Two-tailed Test
  - p-value (p-value < alpha): to have a statistically significant conclusion
  - statistical significance, alpha: probability of having the observed effect due to chance
- **Hypothesis Testing Procedures:**
  - Function call
  - Mathematical formula
  - Statistical simulation
    - Bootstrap sampling (with or without replacement)
- **Test Statistics Metric**
  - t-statistic: comparing the differences in group means (continuous variable)
- **t-distribution**
  - rt() and pt()

A/B Testing: Comparing Two Means

PAIRED SAMPLES OF CONTINUOUS VARIABLE

Control (A) and Treatment (B) Groups

Two (2) Samples: A and B

Independent	Dependent, or Paired

• Customers randomly assigned to one of the two groups	• Follow-up studies:
	  • Group A: At the beginning of the study
	  • Group B: At the end of the study
	
	• The same customers exposed to both:
	  • Marketing strategy A
	  • Marketing strategy B

Paired Comparison: Dependent Samples

- **Example: Contribution of music to cognitive learning**
  - Reading comprehension scores for 11 subjects:
    - Sample-A: initially after reading a passage without background music
    - Sample-B: a week later after reading a similar passage with background music

- **Sample-A and Sample-B are NOT independent:**
  - the same subjects participated in the intervention
  - the observed mean difference is 1.45

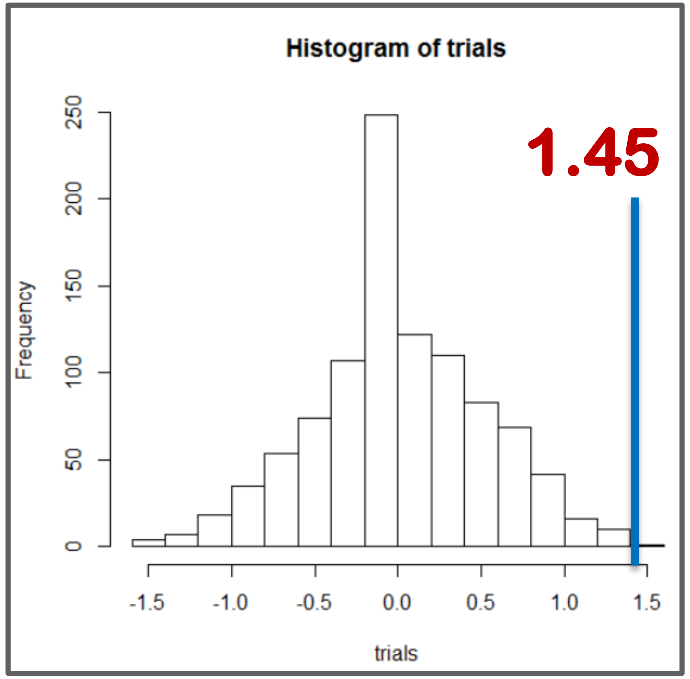
Without Music	With Music	Difference
24	27	+3
79	80	+1
17	18	+1
50	50	0
98	99	+1
45	47	+2
97	97	0
67	70	+3
78	79	+1
85	87	+2
76	78	+2

Question:
Is this observed difference statistically significant?

A/B Hypothesis Testing Procedures

Hypothesis to Test

R function	Statistical simulation	Mathematical Formula
t.test() paired = TRUE		$$t = \frac{\bar{d}}{s_d/\sqrt{n}}$$ t-statistic



Approach #1: Paired t.test()

Alternative Hypothesis: H1: Treatment Mean is greater than Control Mean
Ho: no difference in the mean reading scores between two groups except for what chance might produce

Paired t.test()

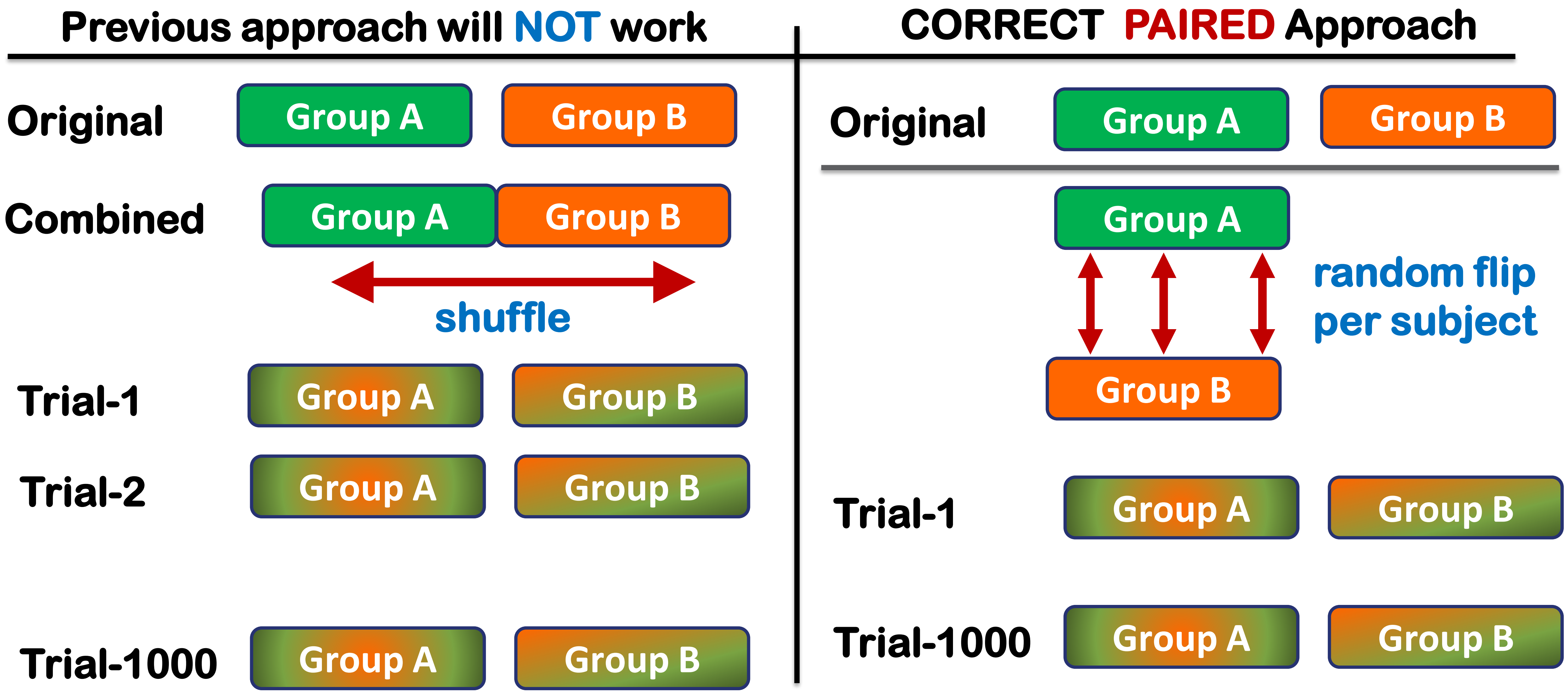
```r
# alternative = "greater":
#     x has a larger mean than y
# paired = TRUE
t.test (x=treatment,
        y=control,
        alternative="greater",
        paired = TRUE)
```

AB_paired_test_music_affect_on_reading.R

Output

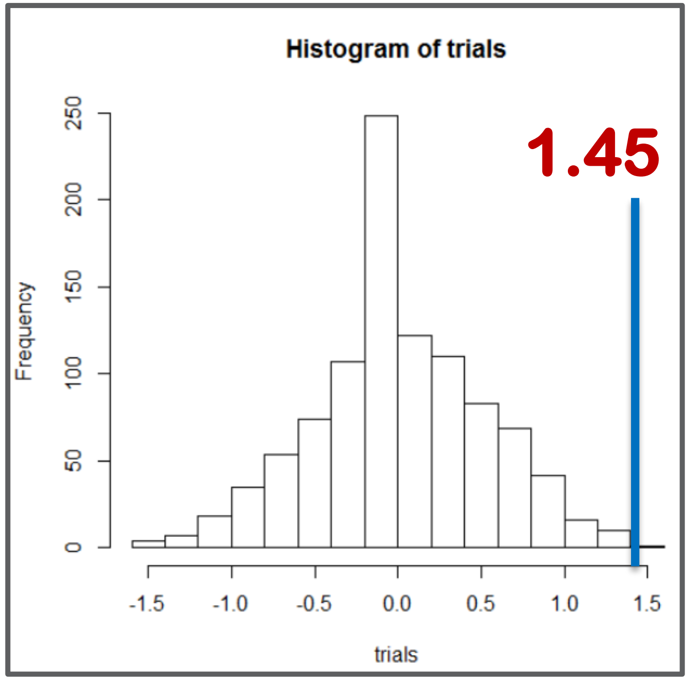
```
        Paired t-test

data:  treatment and control
t = 4.6578, df = 10, p-value = 0.0004487
alternative hypothesis: true difference in means is
 greater than 0
95 percent confidence interval:
 0.8885447       Inf
sample estimates:
mean of the differences
               1.454545
```

Approach #2: Paired Comparison Test: Simulation



Approach #2: Paired Comparison Test: Simulation

1. Randomly shuffle the two scores for the first subject into columns 1 and 2
2. Repeat the shuffle for the remaining 10 subjects
3. Calculate the mean score for columns 1 and 2 and record the difference: col 2 – col 1
4. Repeat steps one through three 1,000 times
5. Draw a histogram of the resampled differences and find out how often the resampled difference exceeds the observed value of 1.45




```r
reading.scores <- scan(file="../data_raw/reading.txt")
score.matrix <- matrix(reading.scores,ncol=2,nrow=11,byrow=TRUE)
# verify the matrix is correct
write.table(score.matrix, sep=" ", row.names=FALSE,
            col.names=c("Without Music", "With Music"))

set.seed (2018)

res <- function() {
  # apply applies the sample function to each *row* in mat
  shuffle.within.rows <- apply(score.matrix, 1,
                               function(mat) sample(mat,replace=F))
  # must transpose the shuffled matrix to get 2 column format
  mat2 <- t(shuffle.within.rows)
  diffmeans <- mean(mat2[,2])-mean(mat2[,1])
  return(diffmeans)
}

n <- 1000
trials <- replicate(n, res())
hist(trials)
pval <- sum(ifelse(trials>=1.45,1,0))
cat("Paired comparison p-value: ", pval/n,"\n")
```

AB_paired_test_music_affect_on_reading_simulation.R

p-value = 0.001 < 0.05

A/B Test: Comparing Two Proportions

INDEPENDENT SAMPLES OF BINARY / CATEGORICAL VARIABLE

Test Statistic

- **Test statistic**
  - A metric (e.g., difference in means, difference in proportions, difference in risks) used to compare group A to group B
- **Examples**
  - Binary variables
    - click or no-click
    - buy or don't buy
    - fraud or no fraud
  - Categorical count variables:
    - contracts signed
    - pages visited
  - Continuous variables
    - purchase amount
    - profit
    - revenue per page-view

2 x 2 table for eCommerce Experiment

Outcome	Price A	Price B
Conversion	200	182
No conversion	2,353	2,240
Proportion	0.085	0.081

Mean Revenue per Page Conversion

Outcome	Price A	Price B
Revenue	$3.87	$4.11

Comparing Two Proportions

2 x 2 table for eCommerce Experiment

Outcome	Price A	Price B
Conversion	200	182
No conversion	2,353	2,240
Proportion	0.085	0.081

Test

Parametric: Z-test

$$Z = \frac{\hat{p}_B - \hat{p}_A}{\sqrt{p(1-p)\left(\frac{1}{n_B} + \frac{1}{n_A}\right)}}$$

Z-statistic

<u>Assumptions</u>: the probability of common success is approximate 0.5, and the number of games is very high: i.e., a binomial distribution approximates a gaussian distribution).

Non-parametric: Chi-squared Test: prop.test()

```r
converters <- c(200, 182)
group.sizes <- c(2353, 2240)

prop.test (x=converters, n=group.sizes,
           alternative = "greater",
           correct = FALSE)
```

AB_test_compare_proportions.R

```
        2-sample test for equality of proportions
 without continuity correction

data:  converters out of group.sizes
X-squared = 0.21139, df = 1, p-value = 0.3228
alternative hypothesis: greater
95 percent confidence interval:
 -0.00965319  1.00000000
sample estimates:
    prop 1     prop 2
0.08499788 0.08125000
```

Statistical Distributions & Functions in R

Distribution	Random Number Generator	Density	Distribution	Quantile
Normal	rnorm	dnorm	pnorm	qnorm
t	rt	dt	pt	qt
F	rf	df	pf	qf
χ^2	rchisq	dchisq	pchisq	qchisq

{dpqr}*distribution_abbreviation*()

- **d** = density
- **p** = distribution function
- **q** = quantile function
- **r** = random generation

- **pnorm(a)** $\equiv P(X \leq a)$: probability that a or smaller number occurs
- **pnorm(b) – pnorm(a)** $\equiv P(a \leq X \leq b)$: probability that the variable falls between two points
- qnorm(): given the cumulative probability distribution, it returns the quantile

Example: Z-statistic for two proportions

- **Metric:**
  - a percentage
  - a proportion

$$Z = \frac{\hat{p}_B - \hat{p}_A}{\sqrt{p(1-p)\left(\frac{1}{n_B} + \frac{1}{n_A}\right)}}$$

Outcome	Price A	Price B
Conversions, X	42,480	42,551
Sample size, n	50,332	49,981

n_B = The sample size for treatment B

X_B = The number of conversions for treatment B

$\hat{p}_B = \frac{X_B}{n_B}$ = The point estimate for the proportion of converted for treatment B

$p = \frac{X_A + X_B}{n_A + n_B}$ = The combined conversion rate

Standard error for two proportions:

- obtained by combining all values (irrespective of original group), computing the combined proportion, and then normalizing by the two sample sizes

A/B Testing: Summary

Two Samples	Compare means	Compare proportions
Independent	Student's t-test: t.test()	Non-parametric Chi-squared test: prop.test() or chisq.test() Parametric: Z-test
Paired	t.test(paired=TRUE)	McNemar's test: mcnemar.test()

See more examples: http://yatani.jp/teaching/doku.php?id=hcistats:chisquare

Randomization in A/B Testing

- **Randomization**
  - The process of randomly assigning subjects to treatment groups
- **Why is randomization necessary?**
  - To make sure that any difference between treatment groups is due to one of two things:
    - The effect of the different treatments
    - Random assignment may have resulted in the naturally better-performing subjects being concentrated in A or B:
      - e.g. most African-American athletes from the sample ended up being on the same basketball team
      - e.g., most kids from economically advantageous families ended up being assigned to the same group for new computerized test design A vs. paper-based test design B

Why Have a Control Group?

- **Why not skip the control group and just run an experiment applying the treatment of interest to only group, and compare the outcome to prior experience?**
- **Rationale:**
  - Without a control group, there is no assurance that "other things are equal" and that any differences is really due to the treatment (or to chance)
  - Control groups is subject to the SAME conditions (EXCEPT for the treatment of interest) as the treatment group
  - If comparison is made simply to "baseline" or prior experience, then other factors, besides the treatment, might differ

Requesting Permission

- **In studies involving human or animal subjects, it is typically necessary to get their permissions (special process)**
- **Facebook Experiment with Emotional tone is users' newsfeeds (2014)**
  - Used sentiment analysis to classify newsfeed posts as positive or negative
  - Then altered the pos/neg balance in what it showed to users:
    - some randomly selected users experienced more positive posts
    - while others more negative ones
  - Finding: users exposed to more positive newsfeed were more likely to post positively themselves and vice versa
- **Issue with Facebook Experiment**
  - Users were subjects without their knowledge
  - What if Facebook pushed some extremely depressed users over the edge if they got negative version of their feed?