Introduction to Vector & Matrix Algebra

Data models in data science, vectors and their norms, normalized vectors, vector operations, inner product, cosine similarity, normal vectors, half planes and half hyperplanes.

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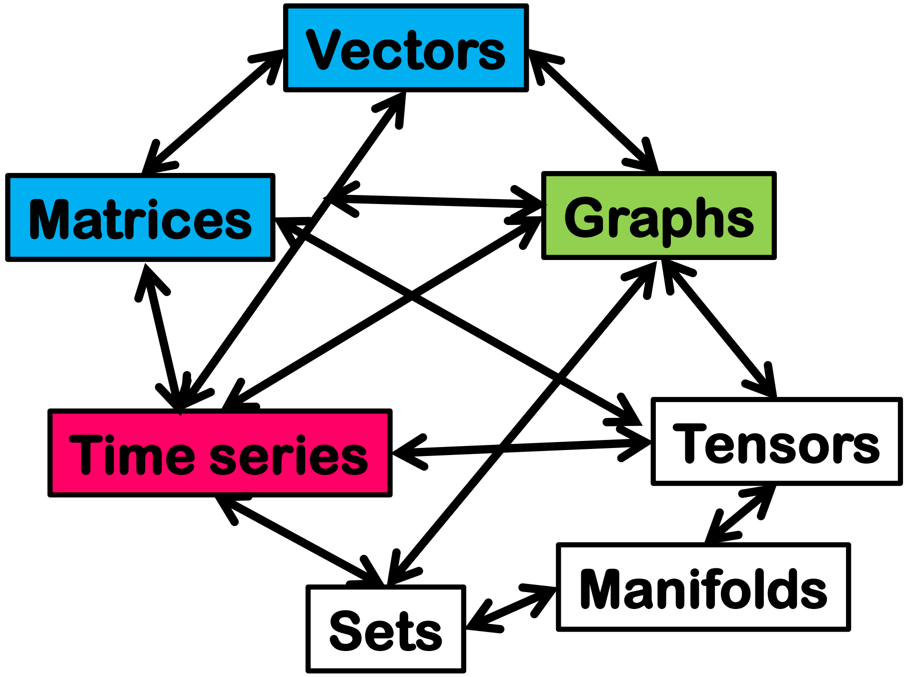
Vector and Matrix Algebra

DATA MODELS IN DATA SCIENCE

Data Model in the Data Science (DS) Process

Application → Data → Questions → Data Objects + Features → Mathematical Data Representation (**Data Model**)



Different DS methods require different *abstractions for data representation*

Not one hat fits all

More than one models is needed

Models are related but often in a complementary way

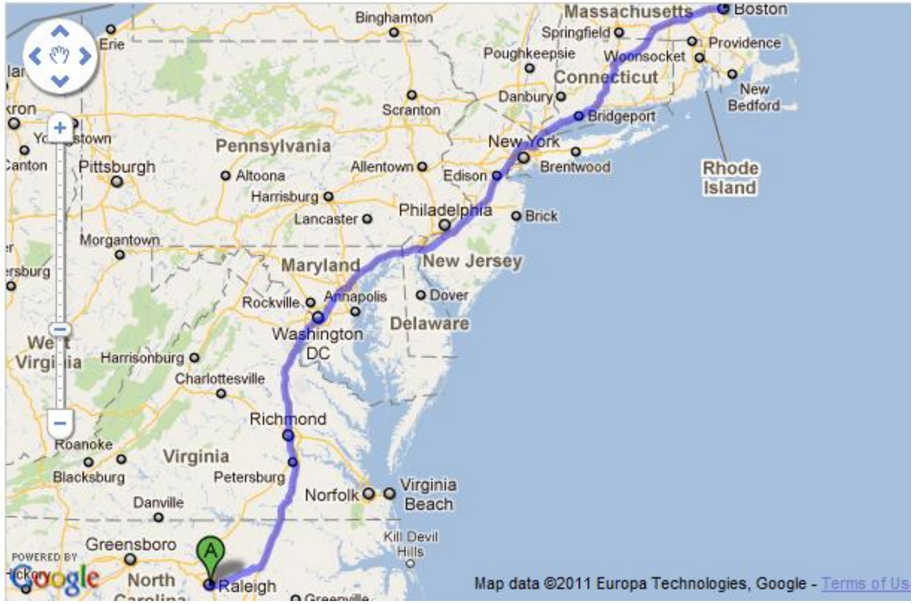
Data object as Vector with Components…

Vector components:
- **Features**, or
- **Attributes**, or
- **Dimensions**

City=(Latitude, Longitude) is a *2-dimensional object*

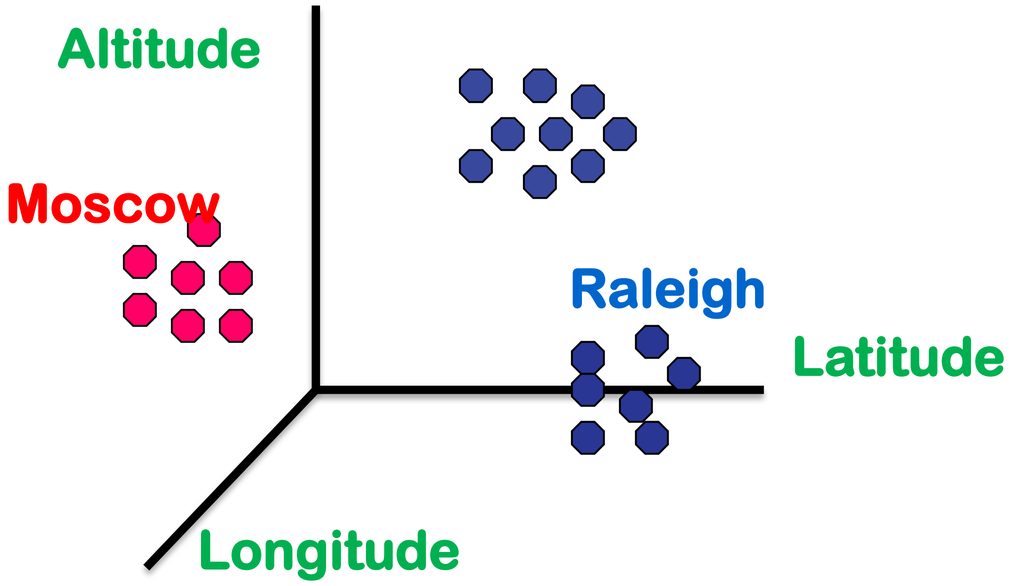
Raleigh=(35.46, 78.39) **Boston=(42.21, 71.5)**

<u>Proximity (Raleigh, Boston) = ?</u>
- Geodesic distance
- Euclidean distance
- Length of the interstate route



A Set of Data Objects as Vector Spaces

3-dimensional vector space



Mining such data ~ studying vector spaces

as Matrices…

Example: A collection of text documents on the Web

Original Documents

D1:	Child Safety at Home
D2:	Infant & Toddler First Aid
D3:	Your Baby's Health and Safety: From Infant to Toddler

Parsed Documents

D1:	Child Safety Home
D2:	Infant Toddler
D3:	Bab Health Safety Infant Toddler

Terms=Features=Dimensions

T1:	Bab
T2:	Child
T3:	Health
T4:	Home
T5:	Infant
T6:	Safety
T7:	Toddler

t-d term-document matrix

	D1:	D2:	D3:
T1:	0	0	1
T2:	1	0	0
T3:	0	0	1
T4:	1	0	0
T5:	0	1	1
T6:	1	0	1
T7:	0	1	1

Mining such data ~ studying matrices

or as trees…

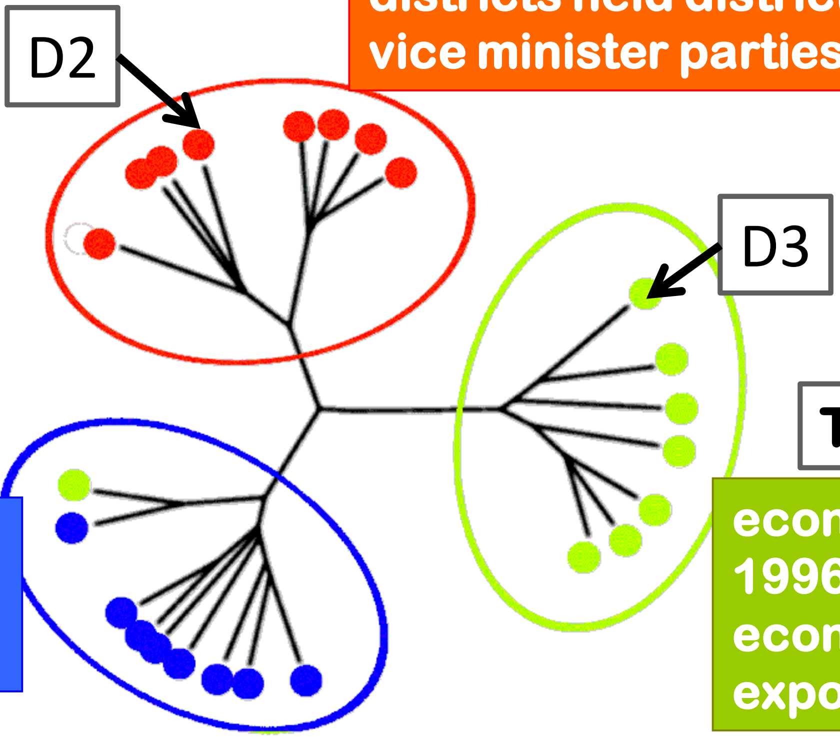
t-d term-document matrix

document

	D1:	D2:	D3:
T1:	0	0	1
T2:	1	0	0
T3:	0	0	1
T4:	1	0	0
T5:	0	1	1
T6:	1	0	1
T7:	0	1	1

Is D2 similar to D3?

president government party election political elected national districts held district independence vice minister parties



Terms

economy million products 1996 growth copra economic 1997 food scale exports rice fish

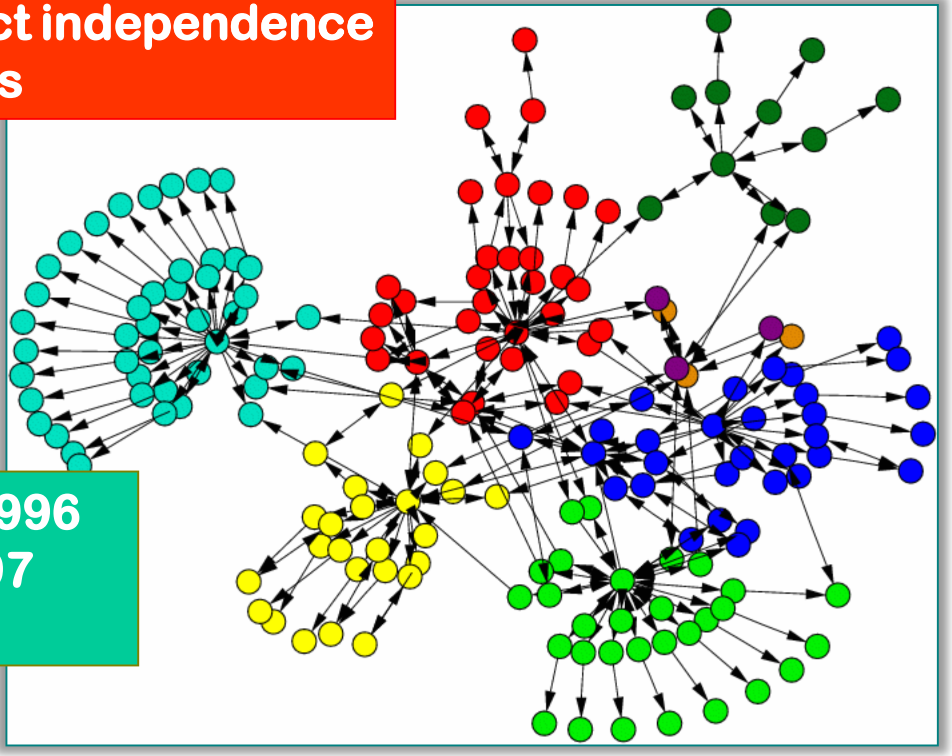
population area climate city miles province land topography total season 1999 square rate

Mining such data ~ studying trees

Or as networks, or graphs w/ nodes & links

Nodes = Documents
Links = Document similarity (e.g., if document references another document)

president government party election political elected national districts held district independence vice minister parties



population area climate city miles province land topography total season 1999 square rate

economy million products 1996 growth copra economic 1997 food scale exports rice fish

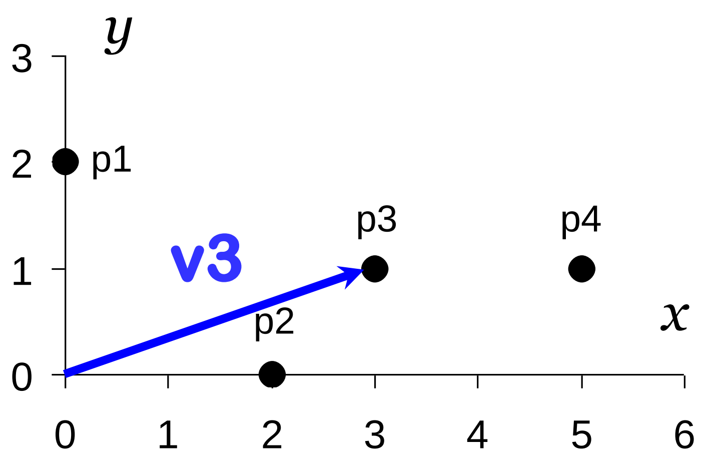
Mining such data ~ studying graphs, or graph mining

Vector Algebra

VECTORS AND VECTOR NORM

Vectors in Low-dimensional Spaces

Points in 2-dimensional space



Data Points in 2-d

point	x	y
p1	0	2
p2	2	0
p3	3	1
p4	5	1

Point ⟷ Vector

$$\boxed{p3 = (x, y) \leftrightarrow v3 = (x, y)}$$

Vector Length or **Norm** or **L_2-norm**
(Pythagorean theorem)

$$\| v3 \|_2 = \sqrt{x^2 + y^2} = \sqrt{3^2 + 1^1} = \sqrt{10}$$

Vector has:

- The **origin** (0,0)
- The **direction**
- The **length/norm**: ||v||

Vectors in Higher-dimensional Spaces ($d = 3, 4, \dots$)

Point ⟷ Vector

$$\boldsymbol{p} = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d \longleftrightarrow \boldsymbol{v} = (v_1, v_2, \cdots, v_d) \in \mathbb{R}^d$$

Point in *d*-dimensional space

Corresponding vector in *d*-dimensional space

$$\boxed{\begin{array}{c} v_1 = x_1 \\ v_2 = x_2 \\ \vdots \\ v_d = x_d \end{array}}$$

Vector has:

- The **origin** (0,0)
- The **direction**
- The **length/norm**: ||v||

Examples of vectors in 3-d and 4-d

$$(\mathbf{1}, \mathbf{3}, -\mathbf{4}) \in \mathbb{R}^{\mathbf{3}}$$
$$(\mathbf{0.5}, -\mathbf{1.2}, \mathbf{3.7}, \mathbf{9.6}) \in \mathbb{R}^{\mathbf{4}}$$

Vector Length, Norm or L_2-norm (used interchangeably)

$\boldsymbol{v} = (v_1, v_2, \cdots, v_d) \in \mathbb{R}^d$

Vector in ***d***-dimensional space

Vector has:
- **The origin (0,0)**
- **The direction**
- **The length/norm: ||v||**

Vector length/norm (e.g. L_2-norm):

scalar

$$\boxed{\| v \|_2 = \sqrt{\sum_{k=1}^{d} (v_k)^2} \in R}$$

$\sum$ **: Summation symbol**

Do not mix with *length(v)* in R:
- **number of vector components, *d***

Examples: Vector norms in 3-d and 4-d:

$\boldsymbol{v} = (1, 3, -4) \in \mathbb{R}^3 \rightarrow \|\boldsymbol{v}\| = \sqrt{1^2 + 3^2 + (-4)^2} = \sqrt{26}$ ← **scalar, number**

$\boldsymbol{u} = (0.5, -1.2, 3.7, 9.6) \in \mathbb{R}^4 \rightarrow \|\boldsymbol{u}\| = \sqrt{0.5^2 + (-1.2)^2 + 3.7^2 + 9.6^2}$

Normalized Vectors ≡ Vectors of Unit Length

$$\boldsymbol{v} = (v_1, v_2, \cdots, v_d) \in \mathbb{R}^d$$

Vector in *d*-dimensional space

Vector length/norm (e.g. L_2-norm):

scalar

$$\boxed{\| v \|_2 = \sqrt{\sum_{k=1}^{d} (v_k)^2} \in R}$$

Normalized Vector (L_2-norm=1):

$$\boxed{u = \frac{v}{\| v \|}}$$

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Ex #1: Vector Norm

Consider the vector in a two-dimensional space:

$$\boldsymbol{v} = (\boldsymbol{1}, -\boldsymbol{2}) \in \mathbb{R}^{\boldsymbol{2}}$$

a. What is the length, i.e., the L_2 −norm of this vector? Show calculations by hand.
Validate the result by showing the R or Python code that does the same.

$$||\boldsymbol{v}|| =$$

b. Normalize this vector to the unit length? Show calculations by hand.
Validate the result by showing the R or Python code that does the same.

$$||\boldsymbol{v_n}|| =$$

Vector Algebra

VECTOR OPERATIONS

Vector Operations: Scaling

$\boldsymbol{u} = (u_1, u_2, \dots, u_d) \in \mathbb{R}^d$ and a constant $\alpha \in \mathbb{R}$, a real number (e.g., 5.0, -3,8)

Scaling:

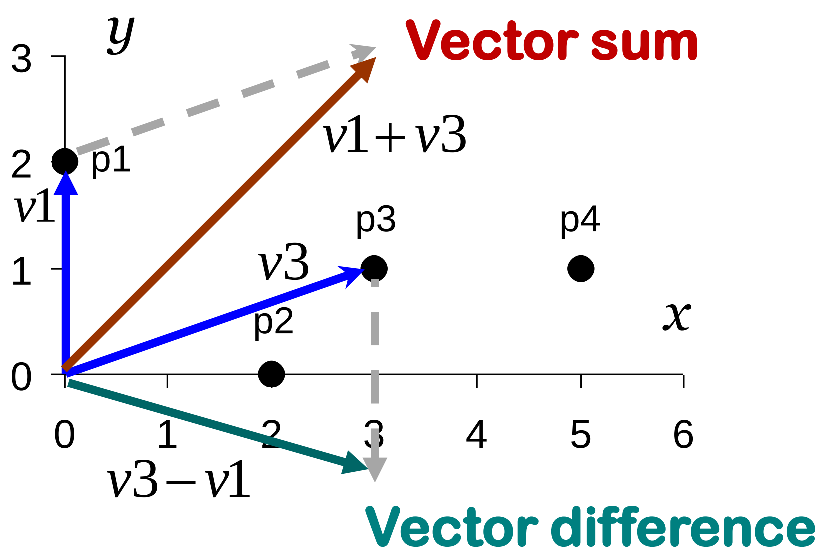
vector → $\boxed{\alpha \cdot \boldsymbol{u} = (\alpha \cdot u_1, \alpha \cdot u_2, \dots, \alpha \cdot u_d) \in \mathbb{R}^d}$

Examples: Scaling a vector by a constant

$\boldsymbol{u} = (\mathbf{1}, \mathbf{3})$ **and** $\boldsymbol{\alpha} = -\mathbf{2}$ **then** $\boldsymbol{\alpha} \cdot \boldsymbol{u} = (-\mathbf{2}, -\mathbf{6})$ **– a vector with opposite direction and larger length**
$\boldsymbol{u} = (\mathbf{1}, \mathbf{3})$ **and** $\boldsymbol{\alpha} = \mathbf{0.5}$ **then** $\boldsymbol{\alpha} \cdot \boldsymbol{u} = (\mathbf{0.5}, \mathbf{1.5})$ **– a vector with the same direction but shorter length**

<u>Exercise</u>: Plot this vector and its scaled versions in 2-d

Vector Operations: Addition and Subtraction



$$u = (u_1, u_2, \ldots., u_d) \in R^d \text{ and } v = (v_1, v_2, \ldots., v_d) \in R^d$$

Vector Sum:

vector → $u + v = (u_1 + v_1, .., u_d + v_d) \in R^d$

Vector Difference:

vector → $u - v = (u_1 - v_1, .., u_d - v_d) \in R^d$

Data Points in 2-d

point	x	y
p1	0	2
p2	2	0
p3	3	1
p4	5	1

Exercise: Plot these two vectors, their sum and difference in 2-d. Verify the vector coordinates using the Python and R code below:

Example in Python:

```
v = np.array([3,1])
u = np.array([2,3])
u + v
u - v
```

Example in R:

```
v = c (3,1)
u = c(2,3)
u + v
u - v
```

Vector Operations: Scalar or Inner Product

$$u = (u_1, u_2, \dots., u_d) \in R^d \text{ and } v = (v_1, v_2, \dots., v_d) \in R^d$$

Scalar or Inner Product of Two Vectors:

scalar

$$(u, v) = u_1 \cdot v_1 + \ldots + u_d \cdot v_d \in R$$

Exercise: Compute by hand the scalar product of the two vectors below. Verify your answer using the Python and R code below:

Example in Python:

```
v = np.array([5,1,3])
u = np.array([2,5,5])
np.matmul(v,u)
```

Example in R:

```
v = c (5,1,3)
u = c(2,5,5)
sum (u * v)
```

Vector Operation: Cosine between Two Vectors, Orthogonal Vectors

$$u = (u_1, u_2, ..., u_d) \in R^d \text{ and } v = (v_1, v_2, ..., v_d) \in R^d$$

Scalar Product of Two Vectors:

scalar

$$(u, v) = u_1 \cdot v_1 + ... + u_d \cdot v_d \in R$$

Vector length/norm (e.g. L_2-norm):

scalar

$$\| v \|_2 = \sqrt{\sum_{k=1}^{d} (v_k)^2} \in R$$

Cosine between Two Vectors:

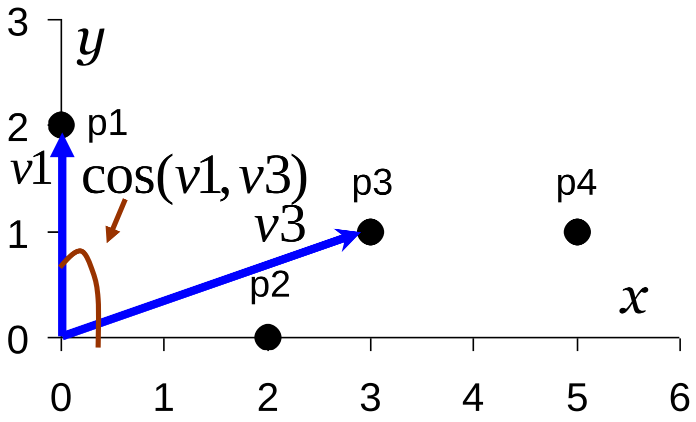
scalar

$$\cos(u, v) = \frac{(u, v)}{\| u \| \cdot \| v \|} \in R$$

Orthogonal Vectors:

$$u \perp v \Rightarrow \cos(u, v) = 0 \Rightarrow (u, v) = 0$$

$$u = (1,1), v = (1,-1)$$



Data Points in 2-d

point	x	y
p1	0	2
p2	2	0
p3	3	1
p4	5	1

$$\| v1 \| = \sqrt{4} = 2$$

$$\| v3 \| = \sqrt{10}$$

$$(v1, v3) = 3 \cdot 0 + 1 \cdot 2 = 2$$

$$\cos(v1, v3) = \frac{2}{2 \cdot \sqrt{10}} = \frac{1}{\sqrt{10}}$$

Ex #2: Scalar/Inner Product & Cosine

Consider two vectors in a two-dimensional space:

$$\boldsymbol{v} = (\mathbf{1}, -\mathbf{2}), \boldsymbol{u} = (\mathbf{2}, \mathbf{1}) \in \mathbb{R}^{\mathbf{2}}$$

a. What is the scalar product (aka inner product) of these two vectors?
Show calculations by hand. Validate the result by showing the Python or R code that does the same. Is scalar product symmetric, i.e. $(\boldsymbol{v}, \boldsymbol{u}) = (\boldsymbol{u}, \boldsymbol{v})$**?**

$$(\boldsymbol{v}, \boldsymbol{u}) =$$
$$(\boldsymbol{u}, \boldsymbol{v}) =$$

b. What is the value of $cos(u, v)$**? Show calculations by hand.**
Validate the result by showing the Python or R code that does the same.
Are these two vectors perpendicular, i.e., angle is 90°**?**

$$\boldsymbol{cos}(\boldsymbol{u}, \boldsymbol{v}) =$$

Ex #3: Scalar/Inner Product & Cosine

Consider two vectors in a four-dimensional space:

$$\boldsymbol{v} = (\mathbf{1}, -\mathbf{2}, \mathbf{1}, -\mathbf{2}), \boldsymbol{u} = (\mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}) \in \mathbb{R}^{\mathbf{4}}$$

a. What is the scalar product (aka inner product) of these two vectors?
Show calculations by hand. Validate the result by showing the Python or R code that does the same. Is scalar product symmetric, i.e. $(\boldsymbol{v}, \boldsymbol{u}) = (\boldsymbol{u}, \boldsymbol{v})$**?**

$$(\boldsymbol{v}, \boldsymbol{u}) =$$

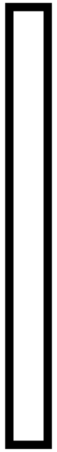
$$(\boldsymbol{u}, \boldsymbol{v}) =$$

b. What is the value of $\boldsymbol{cos}(\boldsymbol{u}, \boldsymbol{v})$**? Show calculations by hand.**
Validate the result by showing the Python or R code that does the same.
Are these two vectors perpendicular, i.e., angle is $\mathbf{90}^{\circ}$**?**

$$\boldsymbol{cos}(\boldsymbol{u}, \boldsymbol{v}) =$$

Vector Transpose (v^T)

v



Column vector

v^T

Row vector = Transposed column vector

Example in Python:

```
v = np.array([5,1,3])
np.transpose(v)
```

Example in R:

```
v = c (5,1,3)
vt = t (v)
help (t)
```

Vector Algebra

NORMAL VECTORS FOR LINES, PLANES, …

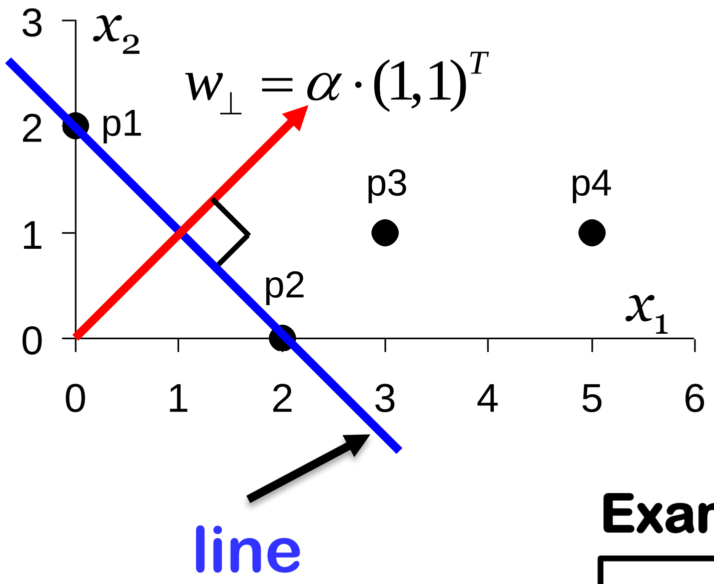
Lines Defined by Normal Vectors

Line in 2-dimensions:

$$y = ax + b, \text{ or equivalently}$$
$$l : a_1x_1 + a_2x_2 + b = 0$$

Line, l ⟷ **Normal Vector** (orthogonal to line l)

$$l \leftrightarrow w_\perp = (a_1, a_2)^T$$



$$l : x_1 + x_2 - 2 = 0$$
$$w_\perp = (1,1)^T \text{ and } b = -2$$

$$x = (x_1, x_2)$$
$$w_\perp = (a_1, a_2)^T$$
$$l : x \cdot w_\perp + b = 0$$

Example: Normal vector for a line in 2-d

$\mathbf{2x_1 - 3x_2 + 5 = 0}$ → **line in 2-dimensional space**

$\boldsymbol{w_\perp} = (2, -3)^T \in \mathbb{R}^2$ → **normal (orthogonal) vector**

$$\mathbf{0} = (\boldsymbol{x}, \boldsymbol{w_\perp}) + \boldsymbol{b} = (\boldsymbol{x_1}, \boldsymbol{x_2}) \circ (\mathbf{2}, -\mathbf{3})^{\boldsymbol{T}} + \mathbf{5} =$$
$$= \mathbf{2}\boldsymbol{x_1} + (-\mathbf{3})\boldsymbol{x_2} + \mathbf{5} = \mathbf{0}$$

Planes and Hyperplanes Defined by Normal Vectors

Hyper-plane in *d*-dimensions: ⟷ **Normal Vector (orthogonal to hyperplane)**

$$\boxed{a_1x_1 + a_2x_2 + \ldots + a_d x_d + b = 0}$$

$$\boxed{l \leftrightarrow w_\perp = (a_1, a_2, \ldots, a_d)^T}$$

Plane in 3-dimensions (d=3):

$$\boxed{a_1x_1 + a_2x_2 + a_3x_3 + b = 0}$$

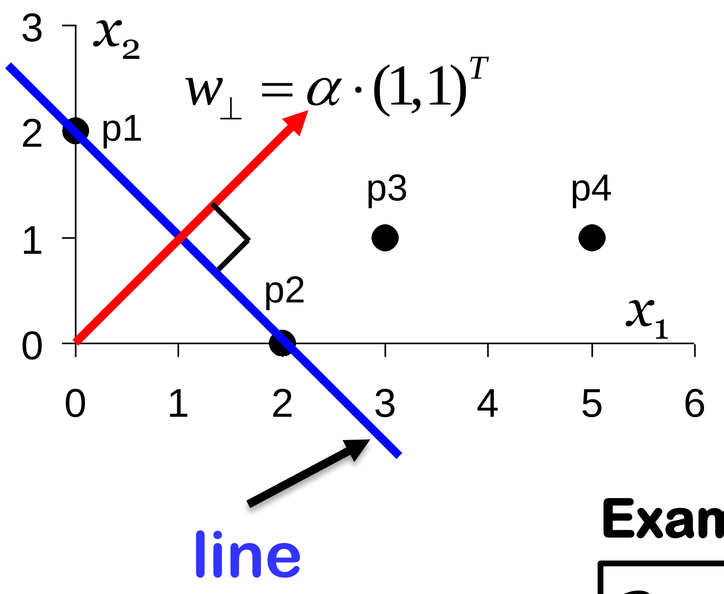
Lines Defined by Normal Vectors

Line in 2-dimensions:

$$y = ax + b, \text{ or equivalently}$$
$$l : a_1 x_1 + a_2 x_2 + b = 0$$

Line, l $\longleftrightarrow$ **Normal Vector** (orthogonal to line l)

$$l \leftrightarrow w_\perp = (a_1, a_2)^T$$



$$l : x_1 + x_2 - 2 = 0$$
$$w_\perp = (1,1)^T \text{ and } b = -2$$

$$x = (x_1, x_2)$$
$$w_\perp = (a_1, a_2)^T$$
$$l : x \cdot w_\perp + b = 0$$

Example: Normal vector for a line in 2-d

$\mathbf{2x_1 - 3x_2 + 5 = 0}$ $\rightarrow$ **line in 2-dimensional space**

$\boldsymbol{w_\perp = (2, -3)^T \in \mathbb{R}^2}$ $\rightarrow$ **normal (orthogonal) vector**

$$\mathbf{0 = (x, w_\perp) + b = (x_1, x_2) \circ (2, -3)^T + 5 =}$$
$$\mathbf{= 2x_1 + (-3)x_2 + 5 = 0}$$

Planes and Hyperplanes Defined by Normal Vectors

Hyper-plane in *d*-dimensions: ⟷ **Normal Vector** (orthogonal to hyperplane)

$$\boxed{a_1x_1 + a_2x_2 + \ldots + a_dx_d + b = 0}$$

$$\boxed{l \leftrightarrow w_\perp = (a_1, a_2, \ldots, a_d)^T}$$

Plane in 3-dimensions (d=3):

$$\boxed{a_1x_1 + a_2x_2 + a_3x_3 + b = 0}$$

Example: Normal vector for a plane in 3-d

$\boldsymbol{x_1 - 2x_2 + 4x_3 - 5 = 0} \rightarrow$ **plane in 3-dimensional space**

$w_\perp = (1, -2, 4)^T \in \mathbb{R}^3 \rightarrow$ **normal (orthogonal) vector**

$$\boldsymbol{0 = (x, w_\perp) + b = (x_1, x_2, x_3) \circ (1, -2, 4)^T - 5 =}$$
$$\boldsymbol{= x_1 + (-2)x_2 + 4x_3 - 5 = 0}$$

Ex #4: Normal Vector to a Line

Consider the line in a 2-dimensional space:

$$\boldsymbol{l}\colon\ \boldsymbol{x_1} - \boldsymbol{x_2} + \mathbf{2} = \mathbf{0} \in \mathbb{R}^{\mathbf{2}}$$

a. What is the normal vector $w_\perp$ for the line, i.e. the perpendicular vector to this line?

$$\boldsymbol{w}_\perp(\boldsymbol{l}) = ?$$

b. What is the value of the intercept b for this line?

$$\boldsymbol{b} = ?$$

c. Choose any point p that lies on this line and give its coordinates:

$$\boldsymbol{p} = (\boldsymbol{x_1} = \quad , \boldsymbol{x_2} = \quad) \in \boldsymbol{l}$$

d. Show (by manual calculations) that the following is true:

$$(\boldsymbol{p}, \boldsymbol{w}_\perp) + \boldsymbol{b} = \mathbf{0}$$

Ex #5: Normal Vector to a Plane

Consider the plane in a 3-dim. space:

$$\boldsymbol{\alpha}\text{: } \boldsymbol{x_1} + \boldsymbol{x_2} - \boldsymbol{x_3} - \mathbf{1} = \mathbf{0} \in \mathbb{R}^{\mathbf{3}}$$

a. What is the normal vector $w_\perp$ for the plane, i.e. perpendicular vector to this plane?

$$\boldsymbol{w}_\perp(\boldsymbol{\alpha}) = ?$$

b. What is the value of the intercept b for this plane?

$$\boldsymbol{b} = ?$$

c. Choose any point p that lies on this plane and give its coordinates:

$$\boldsymbol{p} = (\boldsymbol{x_1} = \quad, \boldsymbol{x_2} = \quad, \boldsymbol{x_3} = \quad) \in \boldsymbol{\alpha}$$

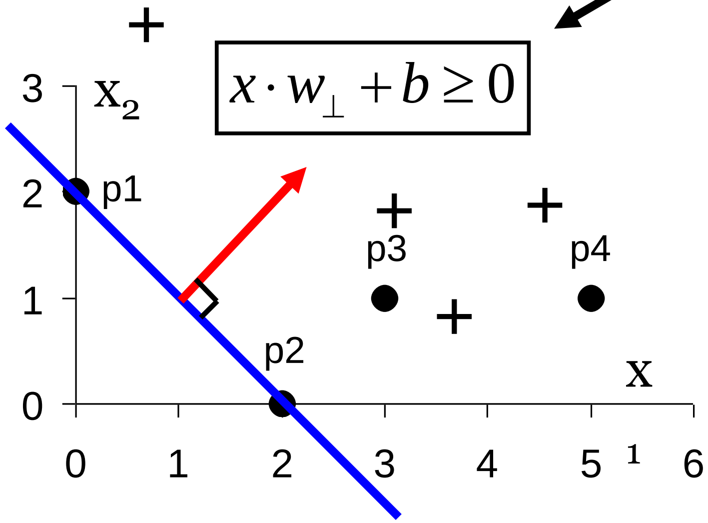
d. Show (by manual calculations) that the following is true:

$$(\boldsymbol{p}, \boldsymbol{w}_\perp) + \boldsymbol{b} = \mathbf{0}$$

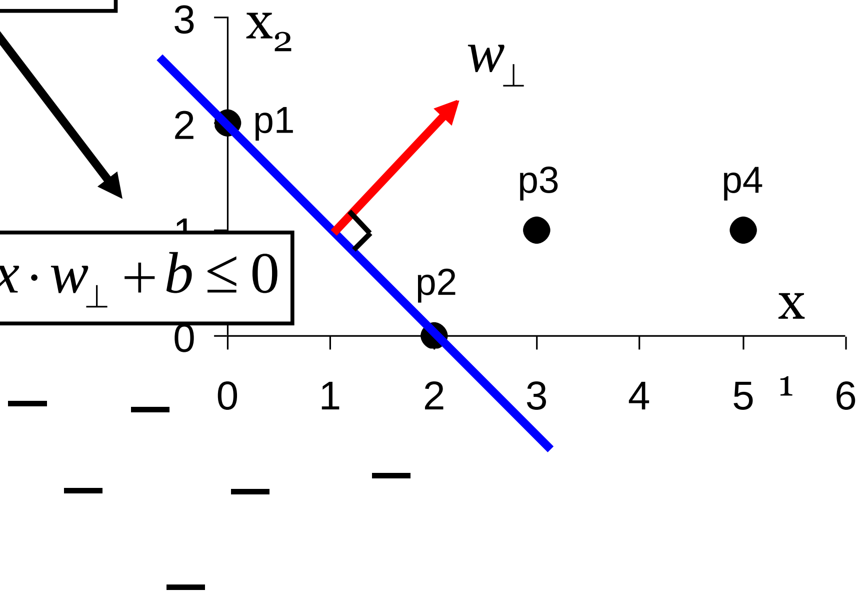
Half-Planes, Half-Spaces, Half-Hyperspaces

$$x = (x_1, x_2, \ldots, x_d)$$
$$w_\perp = (a_1, a_2, \ldots, a_d)^T$$
$$x \cdot w_\perp + b = 0$$

$$x \cdot w_\perp + b \geq 0$$



$$x \cdot w_\perp + b \leq 0$$



$$x = p3 = (3,1)$$
$$w_\perp = (1,1)^T \text{ and } b = -2$$
$$p3 \cdot w_\perp + b = 3 \cdot 1 + 1 \cdot 1 - 2 = 2 \geq 0$$

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Ex #6: Half-Planes

Consider the plane in a 3-dim. space:

$$\boldsymbol{\alpha}\text{: } \boldsymbol{x_1} + \boldsymbol{x_2} - \boldsymbol{x_3} - \mathbf{1} = \mathbf{0} \ \in \mathbb{R}^3$$

a. Give coordinates of any point p that lies in the positive half-plane of this plane:

$$\boldsymbol{p} = (?\,,?\,,?\,) \in \boldsymbol{\alpha}$$

b. Show (by manual calculations) that the following is true:

$$(\boldsymbol{p}, \boldsymbol{w}_\perp) + \boldsymbol{b} > \mathbf{0}$$

d. Give coordinates of any point q that lies in the negative half-plane of this plane:

$$\boldsymbol{q} = (?\,,?\,,?\,) \in \boldsymbol{\alpha}$$

e. Show (by manual calculations) that the following is true:

$$(\boldsymbol{q}, \boldsymbol{w}_\perp) + \boldsymbol{b} < \mathbf{0}$$

Summary: Vector Algebra

- **Vector** is one of the core **data models** in data science
- Vector is defined by its **origin**, **direction** and length (or **norm**)
- **Norm** is a **Euclidean distance** from the origin to the end point of the vector
- **Normalized vector** is a vector of unit length
- Vector operations:
  - scaling, addition, subtraction, scalar or inner product, cosine angle
- **Normal vector** is a vector orthogonal (or perpendicular) to:
  - line in 2-d, plane in 3-d, hyperplane in d-dimensional space
- **Positive** and **negative half planes**, **half spaces**, or **half hyperspaces**:
  - The normal vector points toward the positive sub-spaces