Q1. The probability mass function for a multinomial Distribution for class C_i is given as

$$P(x = (x_1 \dots x_n) | C_i) = P(x = (x_1, \dots x_n) | P_{i1}, \dots P_{in})$$

$$= \frac{m!}{x_1! \dots x_n!} * (P_{i1}^{x_1} \dots P_{in}^{x_n})$$

where $\sum_{j=1}^{n} P_{ij} = 1$ and $\sum_{j=1}^{n} x_j = m$ for all K categories.

The mixture density of K multinomial distributions

$$P(x) = \sum_{i=1}^{K} P(x|C_i) P(C_i) = \sum_{i=1}^{K} (\pi_i) \left(\frac{m!}{x_1! \dots x_n!} \right) \left(P_{i1}^{x_1} \dots P_{in}^{x_n} \right)$$

{where z_i^t is the binary indicator of sample t in cluster i}

$$E(z_i^t) = \frac{\sum_{z_i^t} z_i^t \left(z_i^t \frac{m!}{x! \dots x_n!} P_{i1}^{x_1} \dots P_{in}^{x_n} \right)}{\sum_{k=1}^{K} z_k^t \left(z_k \frac{m!}{x_1! \dots x_n!} P_{k1}^{x_1} \dots P_{kn}^{x_2} \right)}$$

$$= \frac{\pi_i \left(\frac{m!}{x_1! \dots x_n!} \right) \left(P_{i1}^{x_1} \dots P_{in}^{x_n} \right)}{\sum_{k=1}^{K} \pi_k \left(\frac{m!}{x_1! \dots x_n!} \right) \left(P_{k1}^{x_1} \dots P_{kn}^{x_n} \right)}$$

$$E(z_i^t) = \frac{\pi_i \left(P_{i1}^{x_1} \dots P_{in}^{x_n} \right)}{\sum_{k=1}^{K} \pi_k \left(P_{k1}^{x_1} \dots P_{kn}^{x_n} \right)} = \cancel{\gamma}\, \gamma(z_i^t)$$

we now have a ~~functio~~ formula to decide the responsibility $(\gamma(z_i^t))$

~~we know will derive the~~ $_\wedge$ ~~log likelihood estimation is~~

we can now derive the complete log likelihood function and formulas to increment our parameters.

$$E_c[\ln P(X, c \mid p_1, p_2 \ldots p_n, \pi)]$$

$$\mathcal{L} = \sum_{t=1}^{N} \sum_{i=1}^{K} \gamma(z_i^t) \left(\ln(\pi_i) + \ln\left(\frac{m!}{x_1^t \ldots x_n^t} p_{i1}^{x_1^t} \ldots p_{in}^{x_n^t} \right) \right.$$

Partially deriving the equation w.r.t P_{ij}

$$\frac{\partial \mathcal{L}}{\partial P_{ij}} = \frac{\partial}{\partial P_{ij}} \left(\left\{ \sum_{t=1}^{N} \sum_{i=1}^{K} \gamma(z_i^t) \left[\ln \pi_i + \ln\left(\frac{m!}{x_1^t \ldots x_n^t} p_{11}^{x_1^t} \ldots p_{in}^{x_n^t} \right) \right] \right. \right.$$

$$+ \lambda(\Sigma P_{ij} - 1) \Big)$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial P_{ij}} = \sum_{t=1}^{N} \gamma(z_i^t) \frac{x_j^t}{P_{ij}} + \lambda$$

$$0 = \sum_{t=1}^{N} \gamma(z_i)^t \frac{x_j}{P_{ij}} + \lambda.$$

$$\lambda = \sum_{t=1}^{N} \gamma(z_i^t) \left(\frac{(x_j)^t}{P_{ij}} \right)$$

$$p_{ij} = -\left(\frac{\sum_{t=1}^{N} y(z_i^t) + (x_j)^t}{\lambda} \right)$$

We have that $\sum_{j=1}^{n} p_{ij} = 1$

$$\sum_{j=1}^{n} p_{ij} = 1 = -\sum_{j=1}^{n} \left(\frac{\sum_{t=1}^{N} y(z_i^t)\, x_j^t}{\lambda} \right)$$

$$\lambda = -m N_i$$

$$p_{ij} = \frac{\sum_{t=1}^{N} y(z_i^t)\,(x_j)^t}{m N_i}$$

we now ~~deri~~ get the partial derivative of l wrt to π_i

$$\frac{\partial l}{\partial \pi_i} = \frac{\partial}{\partial \pi_i} \left(\sum_{t=1}^{N} \sum_{i=1}^{k} y(z_i^t) \left\{ \ln \pi_i + \ln \left(\frac{m!}{x_1^t! \cdots x_n^t!} \right) p_{i1}^{x_1^t} \cdots p_{in}^{x_n^t} \right\} + \lambda \left(\sum_{i=1}^{k} \pi_i - 1 \right) \right)$$

Taking the derivative $= 0$.

$$\frac{\partial l}{\partial \pi_i} = \sum_{t=1}^{N} \left(\frac{y(z_i^t)}{\pi_i} \right) + \lambda = 0$$

$$\Rightarrow -\lambda = \frac{\sum_{t=1}^{N} (y(z_i^t))}{\pi_i}$$

$$\Rightarrow \pi_i = -\frac{\sum_{t=1}^{N} (y(z_i^t))}{\lambda} = -\frac{N_i}{\lambda}$$

we know that $\sum_{i=1}^{k} \pi_i = 1$.

by substitution the value of π_i we get.

$$1 = -\sum_{i=1}^{k} \left(\frac{\sum_{t=1}^{N} (y(z_i^t))}{\lambda} \right)$$

$$1 = -\left(\frac{1}{\lambda}\right) N$$

$$\Rightarrow \lambda = -N$$

using this in the equation for π_i we get

$$\pi_i = -\frac{N_i}{\lambda} = \frac{-N_i}{-N} = \left(\frac{N_i}{N}\right)$$

$$\pi_i = \frac{N_i}{N}$$

$$P_{ij} = \frac{\sum_{t=1}^{N} y(z_i^t)\,(x_j^t)}{m N_i}$$

and we have our expectation function

$$y(z_i^t) = \frac{\pi_i \,(P_{i1}^{x_1} \ldots P_{in}^{x_n})}{\sum_{k=1}^{k} \pi_k \,(P_{k1}^{x_1} \ldots P_{kn}^{x_n})}$$

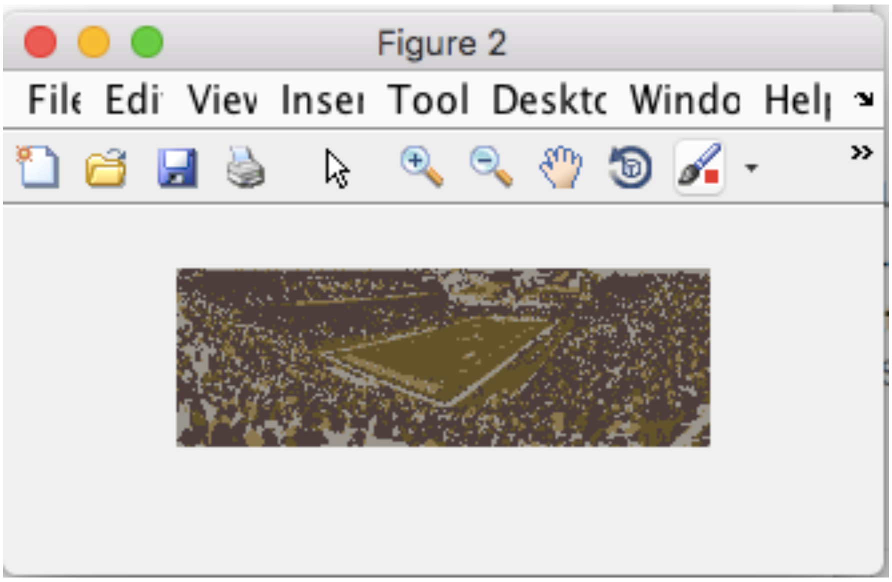
Figure 2
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