Inverse Ackermann without pain

(Updated June 2009)

The inverse Ackermann function is an **extremely slow-growing function** which occasionally turns up in computer science and mathematics. The function is denoted **α(*n*)** (alpha of *n*).

This function is most well-known in connection with the **Union-Find problem**: The optimal algorithm for the Union-Find problem runs in time $O(m\alpha(n) + n)$, where n is the number of elements and m is the total number of Union and Find operations performed. (See Cormen et al., Introduction to Algorithms, Second Edition, Chapter 21, MIT Press, 2001.) (A more precise bound is $O(m\alpha(m, n) + n)$, with a two-parameter version of the inverse Ackermann function, which we will explain below.)

The inverse Ackermann function also arises in **computational geometry**. For example, the maximum complexity of the **lower envelope of *n* segments** in the plane is $\Theta(n\alpha(n))$. (See J. Matoušek, Lectures on Discrete Geometry, Chapter 7, Springer-Verlag, New York, 2002.)

For some reason the inverse Ackermann function gets much less attention than it deserves. This is probably due to the perception that just **defining** $\alpha(n)$ is **complicated**, never mind working with it.

It may come as a surprise, then, that there is a very **simple and elegant way** to define the inverse Ackermann function and derive its asymptotic properties. Moreover, there is no need to make any mention of A, the very quickly-growing Ackermann function.

In other words, **dealing with α(*n*) does not have to be painful!**

There are several different versions of the inverse Ackermann function in the literature. In fact, usually one needs to define a specific version of the function for each application. However, at the end of the day, all definitions yield equivalent asymptotic behavior; namely, we have $|\alpha(n) - \alpha'(n)| = O(1)$ for any two versions α and α'. Thus, it is convenient to have a **canonical definition** of $\alpha(n)$, which we would like to be as simple and elegant as possible.

The inverse Ackermann hierarchy

The **inverse Ackermann hierarchy** is a sequence of functions $\alpha_k(n)$, for $k = 1, 2, 3, \ldots$, where each function in the hierarchy grows much more slowly than the previous one.

Let [] denote the ceiling function (rounding *up* to the nearest integer). Then the inverse Ackermann hierarchy is defined as follows. We first let

$\alpha_1(n) = \lceil n / 2 \rceil$.

Then, for each $k \geq 2$, we let $\alpha_k(n)$ be the number of times we have to apply the function α_{k-1}, starting from n, until we reach 1. Formally, for $k \geq 2$, we let

$$\alpha_k(1) = 0; \qquad \alpha_k(n) = 1 + \alpha_k(\alpha_{k-1}(n)), \quad n \geq 2.$$

The following table shows the first values of $\alpha_k(n)$:

n	=	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	2
$\alpha_1(n)$	=	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11	11	1
$\alpha_2(n)$	=	0	1	2	2	3	3	3	3	4	4	4	4	4	4	4	4	5	5	5	5	5	5	[illegible]
$\alpha_3(n)$	=	0	1	2	2	3	3	3	3	3	3	3	3	3	3	3	3	4	4	4	4	4	4	[illegible]
$\alpha_4(n)$	=	0	1	2	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	[illegible]
...																								

We have $\alpha_2(n) = \lceil \log_2 n \rceil$, and $\alpha_3(n)$ is the iterated logarithm function, denoted $\log^* n$.

Claim 1: If $n \geq 4$ then $\alpha_k(n) \leq n - 2$.

Proof: By induction on k. The case $k = 1$ is clear. So assume $k \geq 2$.

If $n = 4$, then $\alpha_k(n) = 2$; and if $n = 5$ or 6, then $\alpha_k(n) = 3$. So let $n \geq 7$. Then, by induction on k and n,

$$\alpha_k(n) = 1 + \alpha_k(\alpha_{k-1}(n)) \leq 1 + \alpha_k(n - 2) \leq 1 + n - 4 < n - 2.$$
QED

Claim 2: We have $\alpha_{k+1}(n) \leq \alpha_k(n)$ for all k and n. Moreover, for $k \geq 2$ the inequality is strict if and only if $\alpha_k(n) \geq 4$.

Proof: The claim is easily established for $\alpha_k(n) \leq 3$, so suppose $\alpha_k(n) \geq 4$. By Claim 1,

$$\alpha_{k+1}(n) = 1 + \alpha_{k+1}(\alpha_k(n)) \leq 1 + \alpha_k(n) - 2 < \alpha_k(n). \quad \text{QED}$$

Corollary 3: We have $\alpha_k(n) = o(n)$ for all $k \geq 2$.

Proof: By Claim 2, since $\alpha_2(n) = \Theta(\log n) = o(n)$. QED

Claim 4: We have $\alpha_{k+1}(n) = o(\alpha_k(n))$ for all $k \geq 1$.

Proof: By Corollary 3 we have

$$\alpha_{k+1}(n) = 1 + \alpha_{k+1}(\alpha_k(n)) = 1 + o(\alpha_k(n)). \quad \text{QED}$$

In fact, Claim 4 can be strengthened. Given an integer $r \geq 1$, let $f^{(r)}$ denote the r-th-fold composition of the function f. Then,

Claim 5: $\alpha_{k+1}(n) = o(\alpha_k^{(r)}(n))$ for all fixed k and r.

Proof: Iterating r times the definition of $\alpha_{k+1}(n)$, and applying Corollary 3,

$$\alpha_{k+1}(n) = r + \alpha_{k+1}(\alpha_k^{(r)}(n)) = r + o(\alpha_k^{(r)}(n)). \quad \text{QED}$$

Thus, we have $\log^* n = o(\log \log \log n)$, $\alpha_4(n) = o(\log^* \log^* \log^* \log^* n)$, etc.

The inverse Ackermann function

By Claim 2, for every fixed $n \geq 5$, the sequence

$$\alpha_1(n), \alpha_2(n), \alpha_3(n), \ldots$$

decreases strictly until it settles at 3. For example, for $n = 9876!$ we obtain the sequence

$$3.06\times10^{35163}, 116812, 6, 4, 3, 3, 3, \ldots$$

The **inverse Ackermann function** $\alpha(n)$ assigns to each integer n the smallest k for which $\alpha_k(n) \leq 3$:

$$\alpha(n) = \min \{ k : \alpha_k(n) \leq 3 \}.$$

Thus, $\alpha(9876!) = 5$.

Claim 6: We have $\alpha(n) = o(\alpha_k(n))$ for every fixed k.

Proof: Let $m = \alpha_{k+1}(n)$. Then the $(m - 2)$-nd term of the sequence

$$\alpha_{k+1}(n), \alpha_{k+2}(n), \alpha_{k+3}(n), \ldots,$$

namely $\alpha_{k+m-2}(n)$, already equals 3. Thus,

$$\alpha(n) \leq k + m - 2 = k - 2 + \alpha_{k+1}(n) = o(\alpha_k(n)). \quad \text{QED}$$

The two-parameter version of the inverse Ackermann function

There is also a **two-parameter version** of the inverse Ackermann function that sometimes comes up (for example, in the running time of the Union-Find algorithm mentioned above). This two-parameter function can be defined as:

$$\alpha(m, n) = \min \{ k : \alpha_k(n) \leq 3 + m / n \}.$$

This definition differs by at most a small additive constant from the "usual" definition of $\alpha(m, n)$ found in the literature. And as before, we defined it directly, without making mention of the rapidly-growing Ackermann function.

The function $\alpha(m, n)$ satisfies the following properties:

- $\alpha(m, n) \leq \alpha(n)$ for every m and n.
- $\alpha(m, n)$ is nonincreasing in m.
- If $m = n\alpha_k(n)$ then $\alpha(m, n) \leq k$.

See also

R. Seidel, Understanding the inverse Ackermann function (PDF presentation).