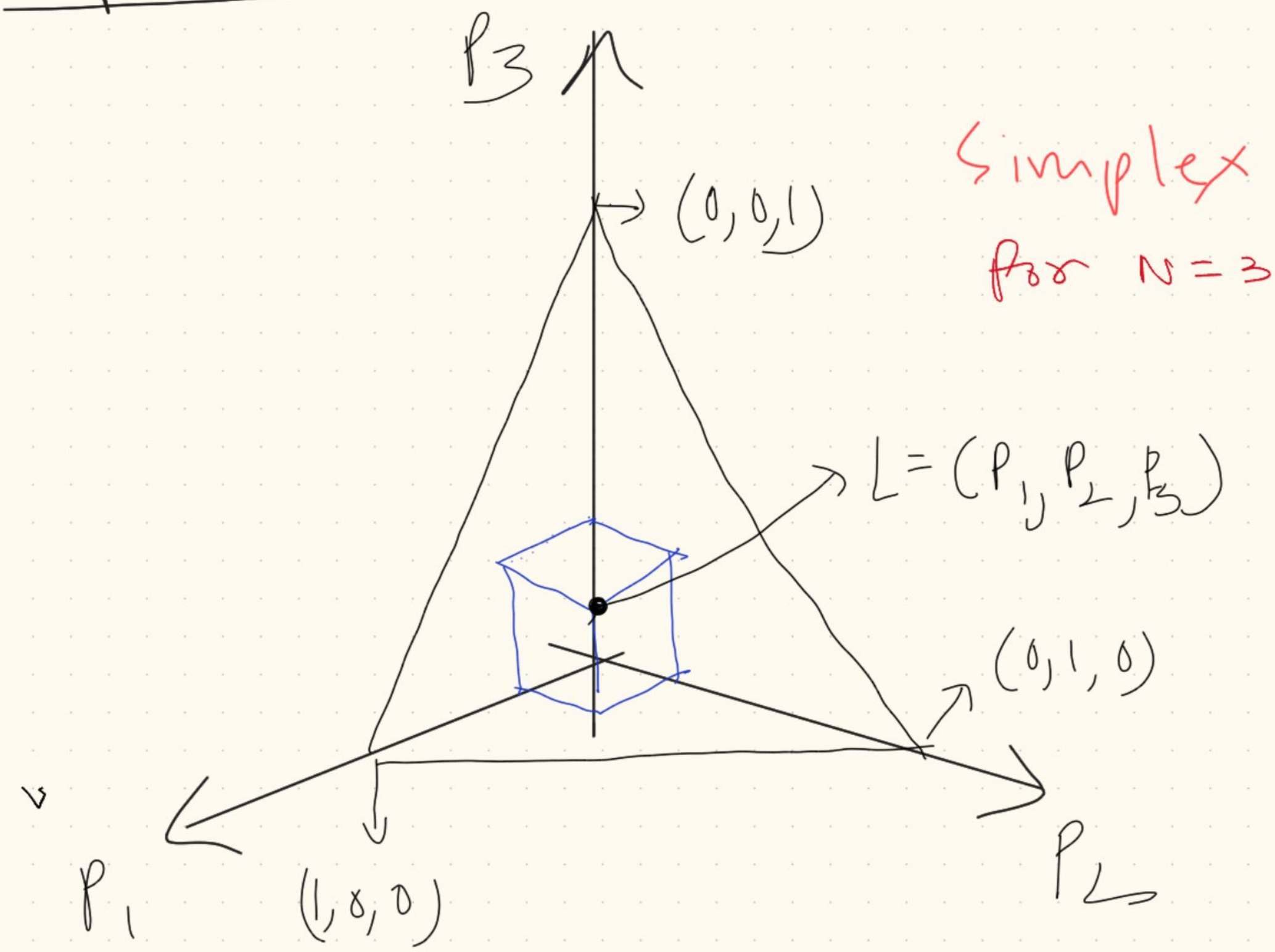
Chp-6 {Lottery}

6B Expected utility theory

Def 6 B 1 – A simple lottery L is a list of all probabilities of outcomes N in a vector form

ie $L = (p_1, p_2 \ldots p_N)$ with $p_n \geq 0$

Simplex notation of a Lottery



A simplex is a geometrical representation of a Lottery with N outcomes having the axes listed as the probability of the outcome n, occuring with sureity.

Notation

$$\Delta = \{ p \in R^N_+ : p_1 + p_2 \cdots p_N = 1 \}$$

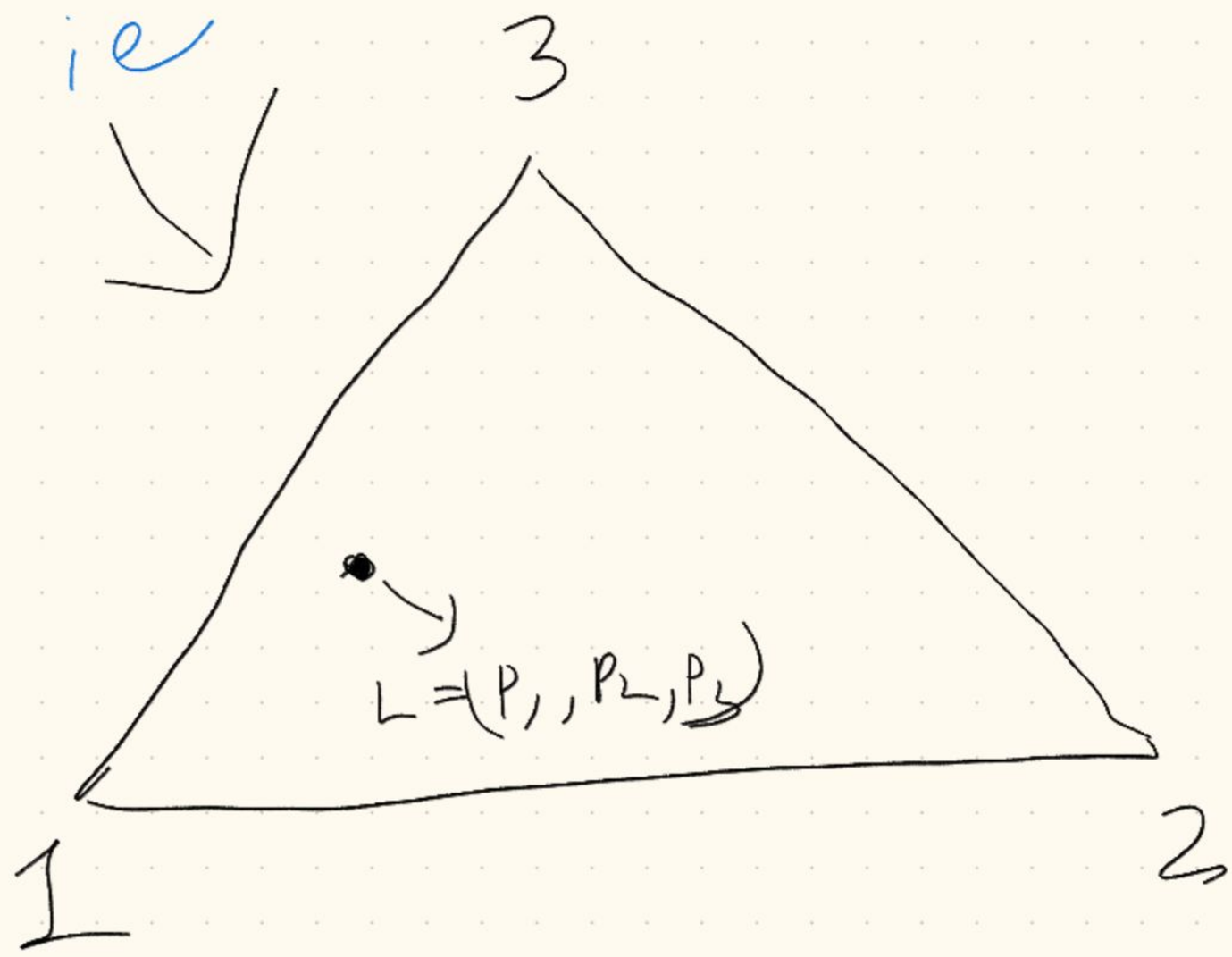
$\downarrow$

simplex notation of Lottery L

where $L = (p_1, p_2, p_3 \cdots p_N)$ st

$$\sum_{n=1}^{\infty} p_n = 1$$

NOTE: if we have a Lottery of $N = 3$ outcomes, a simplex drawn will be of $(N-1) = 2$ dimesional figure ie



Def - 6.B.2 (compound lottery

where taking into account a single lottery $L = (p_1, p_2 \dots p_N)$ with N outcomes, $\sum_{n=1}^{N} p_n = 1$. However if there are K lotteries, then each lottery L_k, $\forall k = 1 \dots K$, will have its own individual probability of occurance as well ie if there L_k lotteries

$$\begin{array}{ccccc} L_1 & L_2 & L_3 & \dots & L_K \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ \alpha_1 & \alpha_2 & \alpha_3 & & \alpha_K \end{array}$$

probabilities

we say such lotteries are combined together to be a compound lottery ie

$$(L_1, L_2 \ldots L_k \mid \alpha_1, \alpha_2, \alpha_3 \ldots \alpha_k)$$

<u>Transforming a compound Lottery → Reduced Lottery</u>

Given a compound Lottery like

$$\vec{L} = (L_1, L_2, L_3 \ldots L_k \mid \alpha_1, \alpha_2, \alpha_3 \ldots \alpha_k)$$

can be written into a simple lottery

like $L = (P_1, P_2, P_3 \ldots P_N)$ (How?)

$$L^1 = (P^1_1, P^1_2, P^1_3, P^1_4 \ldots P^1_n)$$

Lottery 1

outcome 1

probability of outcome 1 in lottery 1 & so on for the rest

$$\vdots$$

$$L^k = (P^k_1, P^k_2, P^k_3 \ldots P^k_n)$$

now, if we want to analyse what P_1 in the reduced lottery L means — It means the sum

weighted probability of all outcome 1s' in different Lotteries $L_k \; \forall k = \{1 \dots k\}$. Then we can write this probability P_1 as

$$P_1 = \alpha_1 P_1^1 + \alpha_2 P_1^2 + \alpha_3 P_1^3 \dots \alpha_k P_1^k$$

P_n^k → Lottery no.

P_n → outcome no.

where α_1 = probability that L_1^k is selected and P_1^1 is probability that outcome occurs when L_1^k is selected. So the Total probability of outcome 1 ie P_1 is the sum of all probabilities of all outcome 1's in different lotteries with their own probability of occurance ie $\alpha_k \; \forall k = 1 \dots k$.

$$P_1 = \alpha_1 P_1^1 + \alpha_2 P_1^2 + \dots \alpha_k P_1^k$$

$$P_2 = \alpha_1 P_2^1 + \alpha_2 P_2^2 + \dots \alpha_k P_2^k$$

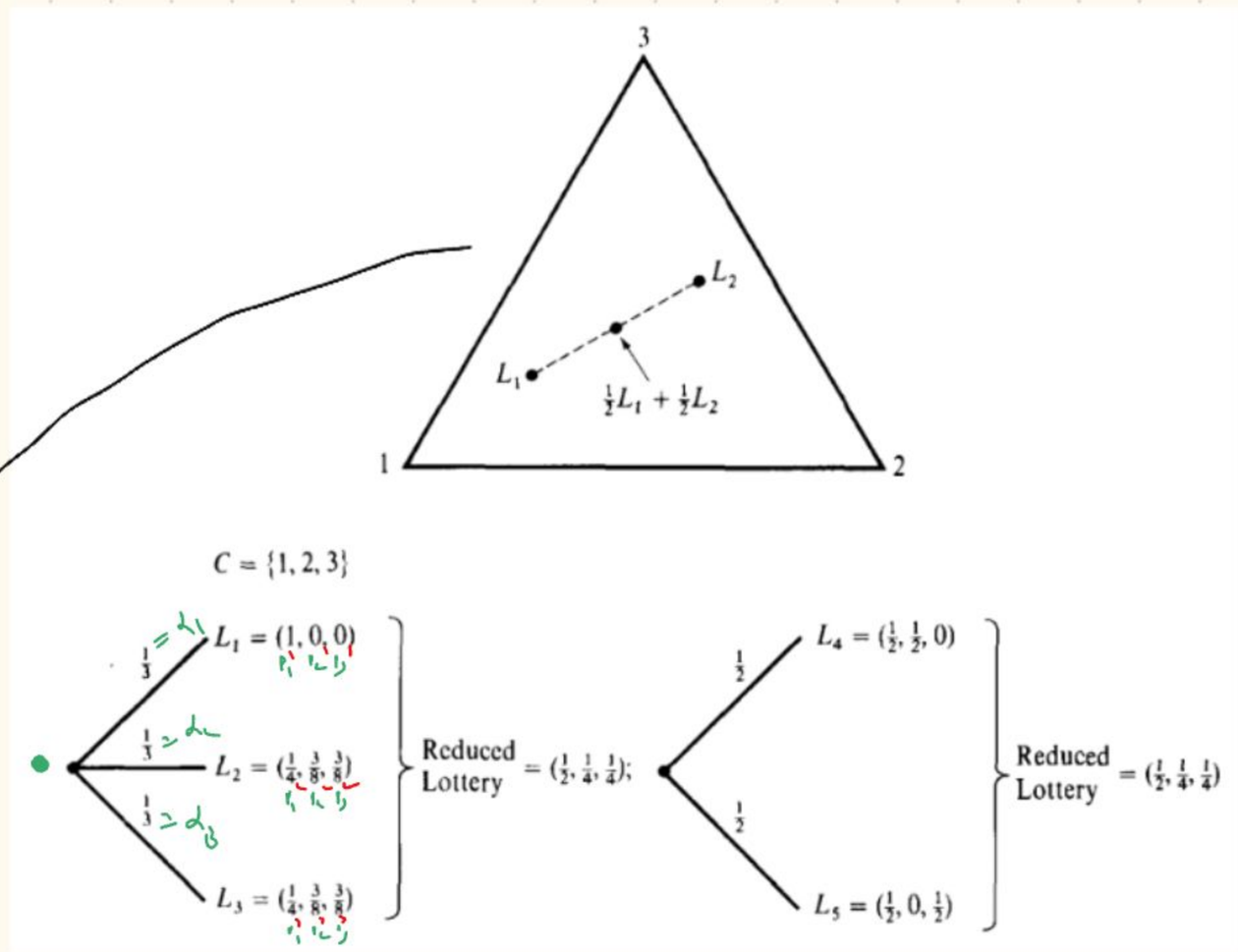
$$\vdots$$

$$P_k = \alpha_1 P_k^1 + \dots \dots \alpha_k P_n^k$$

Reduced Lottery $\tilde{L} = (P_1, P_2, P_3 \dots P_n)$

where $P_n = \sum \alpha_k P_n^k =$

eg Compound to reduced



↓

figure represents a simple Lotteries L_1 & L_2 with the reduced form of a compound Lottery.

Suppose a compound Lottery is given ie you flip a coin, then if heads turns up, you roll a die, and the no that shows up that amount is paid to the person who plays the lottery

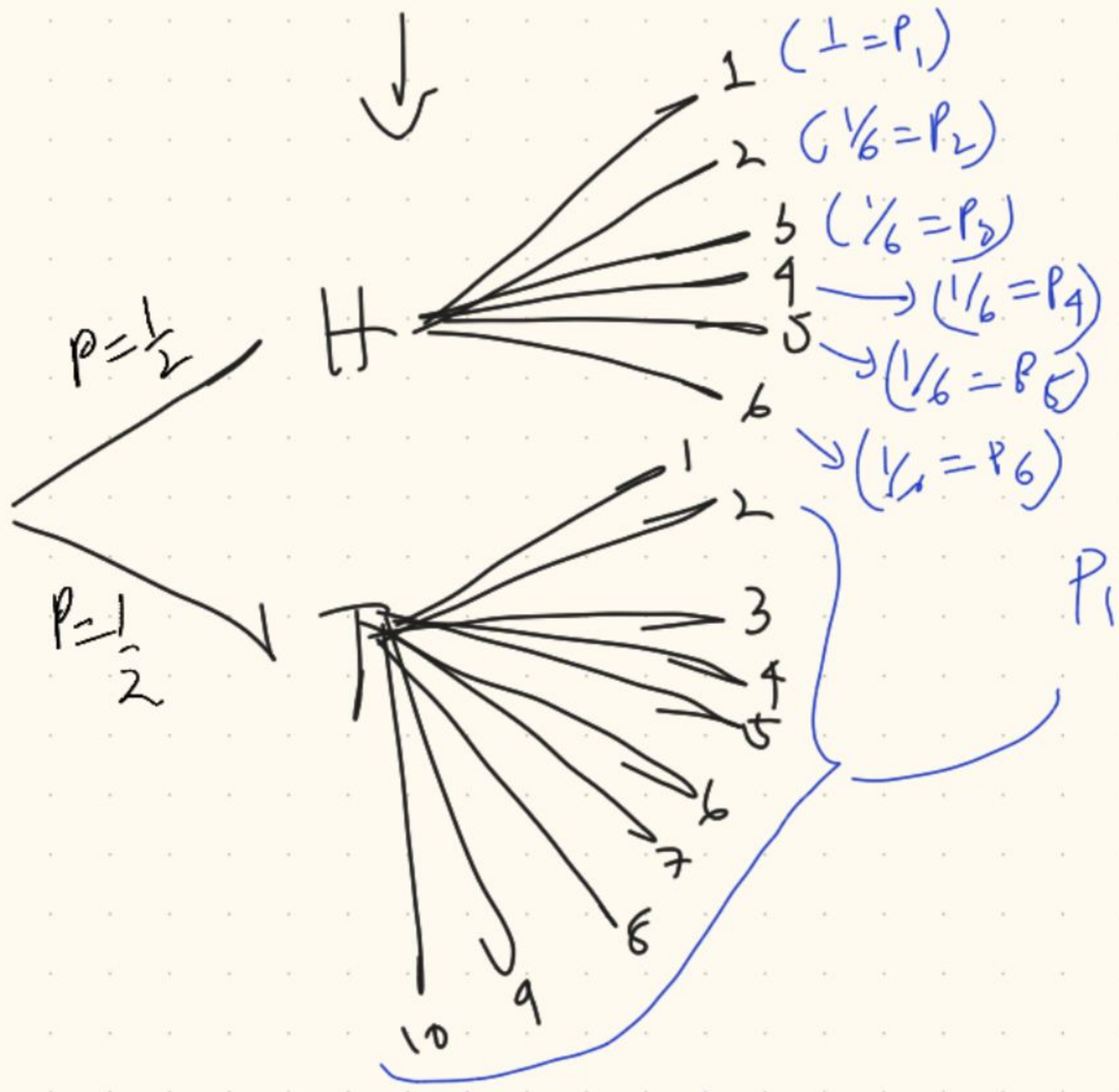
If tail, turns up, then the person who plays the lottery has to select between the two no. (1, 10). (The lottery here is a compound lottery.)

Out comes :- All outcomes

H :- 1, 2, 3, 4, 5, 6

T - 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

Forming a baysian tree



Thus probability condition Head shows up

$$P(1|H) = \frac{P(1 \cap H)}{P(H)} \quad \text{or} \quad P(1|H) \cdot P(H) = P(1 \cap H)$$

$$\Rightarrow \frac{1}{2} \times \frac{1}{6} = P(1 \cap H)$$

$$\Rightarrow \frac{1}{12}$$

Similarly $P(2 \cap H) = P(3 \cap H) \cdots P(6 \cap H) = \frac{1}{12}$

Thus probability condition tail shows up

$$P(1|T) = \frac{P(1|T)}{P(T)} \quad \text{or} \quad P(1|T) \cdot P(T) = P(1 \cap H)$$

$$\Rightarrow \frac{1}{2} \times \frac{1}{10} = P(1 \cap H)$$

$$\Rightarrow \frac{1}{20}$$

Thus Total probability of the compound Lottery is

$$L = \frac{1}{2}\left(\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}\right) + \frac{1}{2}\left(\frac{1}{10} \text{ -- 10 times}\right)$$

$$= \left(\frac{2}{15}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}\right)$$

Preferences over Lotteries

Definition 6.B.3: The preference relation $\succsim$ on the space of simple lotteries $\mathscr{L}$ is *continuous* if for any $L, L', L'' \in \mathscr{L}$, the sets

$$\{\alpha \in [0,1]: \alpha L + (1-\alpha)L' \succsim L''\} \subset [0,1]$$

and

$$\{\alpha \in [0,1]: L'' \succsim \alpha L + (1-\alpha)L'\} \subset [0,1]$$

are closed.

Before explaining the above definition, we will want to be to express the utility achieved through the lotteries. And for utility to be achieved, preferences have to be continuous over which lottery is chosen.

Like consumer theory, X is the consumption set of the consumer, while x is the bundle

st: $x \in X \subset R^L_+$

In case of lotteries

$\tilde{L}$ is such set of lotteries st

$L \in \tilde{L} \subset R^N_+$

We assume that consumer is a consequentialist

↳ some who cares about the probability distribution of the final outcomes, not the type of lottery ie

simple compound

↳ through which the outcome is achieved.

So it's best practice to reduce the compound lotteries into simple lottery { like the coin toss example)

given the assumption of long equentialist & reduction is satisfied, we move to

Continuity

For utility function to be representing the preferences which are continous, should also be continous (chp-3)

→ continuity implies you cannot pass from lower contour set to upper countour set without passing through the Indifference set.

*expressed below

Here, in case of lotteries, we talk about continuity of probabilities,

<u>Suppose</u>

$L > L'$

Here $L, L' \in \tilde{L}$ and are probability vectors.

So by continuity we mean continuity to be expressed for L & L' which are probability vectors.

<u>Now</u>

adding a small probability vector L'' to $L > L'$, should not change the preference ordering {no reversal of preference}

then

if $L \succsim L'$, then $L \succsim (1-\alpha)L' + \alpha L''$ $\forall \alpha > 0$

then $\succsim$ is continous on the $\tilde{L}$

*expressed below,

$\{\alpha \in (0,1) : L \succsim (1-\alpha)L' + \alpha L''\}$

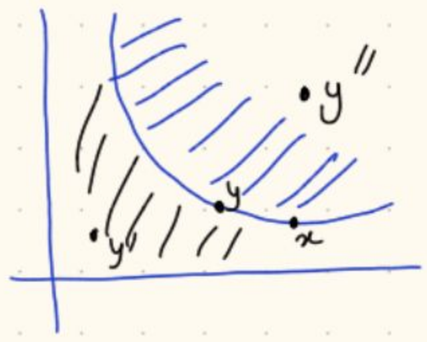
$x \succsim y$

and $\{a \in (0,1) : (1-\alpha)L' + \alpha L'' \succsim L\}$

$y \succsim x$

Side note

contour set



uppercontour set

$y'' \succsim x$

Lower cont. set

$x \succsim y'$

IC

$y \sim x$

Continuity is mostly a technical assumption needed to get the existence of a utility function. But, notice that its validity is less compelling than it was in the certainty case. For example, suppose L is a trip to Bermuda and L' is \$3000, and that you prefer L' to L. Now suppose we introduce L'', which is violent, painful, death. If preferences are continuous, then the trip to Bermuda should also be less preferred than \$3000 with probability $1-a$ and violent painful death with probability a, provided that a is sufficiently small. For many people, there is no probability $a > 0$ such that this would be the case, even though when $a = 0$, L' is preferred to L.

If the consumer has rational, continuous preferences over $\tilde{L}$, we know that there is a utility function $U()$ such that $U()$ represents those preferences. In order to get a utility function of the expected utility form that represents those preferences, the consumer's preferences must also satisfy the **independence axiom**.

L = Bermuda

L' = 3000 & $L' > L$

L'' = violent death.

according to continuity:

$$L' > L$$

then

$$\alpha L' + (1-\alpha) L'' > L$$

ie possibility of violent death given \$3000 in lottery should also be preferred over trip to Bermuda,

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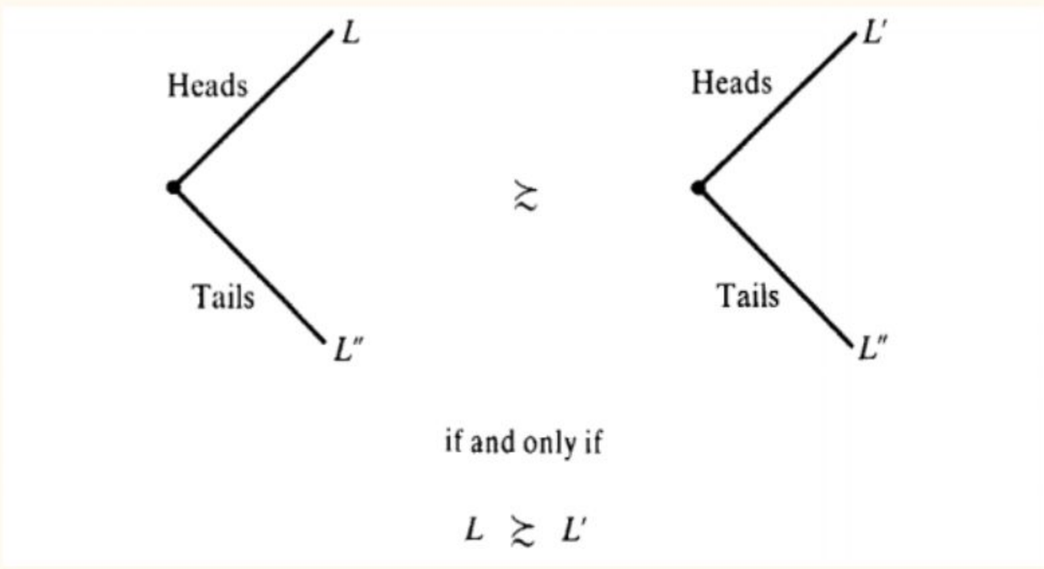
Independence

Definition 6.B.4: The preference relation $\succsim$ on the space of simple lotteries $\mathscr{L}$ satisfies the *independence axiom* if for all $L, L', L'' \in \mathscr{L}$ and $\alpha \in (0, 1)$ we have

$$L \succsim L' \quad \text{if and only if} \quad \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''.$$

The essence of Independence axiom is that If there exist 2 lotteries, where preference is set, then inclusion of a 3rd lottery should not hamper the preferences.



In the above figure, suppose you have 2 lotteries $L \succsim L'$ and you include another lottery of coin toss such that

Suppose

$$L \;:\; \$5 \text{ with probability } \frac{1}{5},\ 0 \text{ with probability } \frac{4}{5}$$
$$L' \;:\; \$12 \text{ with probability } \frac{1}{10},\ 0 \text{ with probability } \frac{9}{10}.$$

Then i flip a coin, if head comes up, i offer you choice between L & L', if tail comes up, you get nothing. What the independence axiom says that if i preferred $L \succsim L'$ before coin flip then after coin flip

Exercise 6.B.1

Exercise 6.B.1: Show that if the preferences $\succsim$ over $\mathscr{L}$ satisfy the independence axiom, then for all $\alpha \in (0,1)$ and $L, L', L'' \in \mathscr{L}$ we have

$$L \succ L' \quad \text{if and only if} \quad \alpha L + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L''$$

and

$$L \sim L' \quad \text{if and only if} \quad \alpha L + (1-\alpha)L'' \sim \alpha L' + (1-\alpha)L''.$$

Show also that if $L \succ L'$ and $L'' \succ L'''$, then $\alpha L + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L'''$.

Show also that if $L \succ L'$ and $L'' \succ L'''$, then $\alpha L + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L'''$.

$L \succ L'$

$L'' \succ L'''$

$\alpha L + (1-\alpha)L'' > \alpha L' + (1-\alpha)L'''$

$\Rightarrow \alpha L'' + (1-\alpha)L > \alpha L'' + (1-\alpha)L'$ —— ①

$\alpha L''' + (1-\alpha)L > \alpha L''' + (1-\alpha)L'$ ——

Proof Ex-6.B.1

$L > L' \iff \alpha L + (1-\alpha) L'' > \alpha L' + (1-\alpha) L''$ — (I)

&

$L \sim L' \iff \alpha L + (1-\alpha) L'' \sim \alpha L' + (1-\alpha) L''$ — (II)

eq^n (I) proof

$L > L' \implies L \succsim L'$ & $L' \not\succsim L$.

then

$$\alpha L + (1-\alpha) L'' \succsim \alpha L' + (1-\alpha) L''$$

ie adding a small L'' probability won't effect the probabilities in L & L', and preferences remain the same (independence axiom)

Similarly,

$L' \not\succsim L \Rightarrow \alpha L' + (1-\alpha) L'' \not\succsim \alpha L + (1-\alpha) L''$

(through independence)

hence now we have, $L \succsim L' \Rightarrow \alpha L + (1-\alpha) L'' \succsim \alpha L' + (1-\alpha) L''$

&

$L' \not\succsim L \Rightarrow \alpha L' + (1-\alpha) L'' \not\succsim \alpha L + (1-\alpha) L''$

hence we have

$$\boxed{L > L' \Rightarrow \alpha L + (1-\alpha) L'' > \alpha L' + (1-\alpha) L''}$$

($\Rightarrow$) proved

<u>Now</u> $\Leftarrow$

$$\alpha L + (1-\alpha) L'' > \alpha L' + (1-\alpha) L''.$$

$\alpha L + (1-\alpha) L'' \succsim \alpha L' + (1-\alpha) L''$.

$\alpha L' + (1-\alpha) L'' \not\succsim \alpha L + (1-\alpha) L''$

then removing L'' from both sides

$L \succsim L'$ & $L' \not\succsim L$

$\Rightarrow$ $\boxed{L \succ L'}$

$$\boxed{\alpha L + (1-\alpha) L'' > \alpha L' + (1-\alpha) L'' = L \succ L'}$$

$$\boxed{L \succ L' \Longleftrightarrow \alpha L + (1-\alpha) L'' > \alpha L' + (1-\alpha) L''}$$

$\hookrightarrow$ eq^n (I) proved

Similarly we can prove eq^n (II)

now,

Show also that if $L \succ L'$ and $L'' \succ L'''$, then $\alpha L + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L'''$.

given all the information of eq^n (I) & eq^n (II) are satisfied, we prove the last part of the question.

we have to show that

$L > L'$ & $L'' > L''' \Rightarrow \alpha L + (1-\alpha) L'' > \alpha L' + (1-\alpha) L'''$

Through independence axiom

if $L > L'$

then a small probability L'', st $L'' > L'''$

is added to L & L' st:

$$\alpha L + (1-\alpha) L'' > \alpha L' + (1-\alpha) L'' \quad (1)$$

now since the $L'' > L'''$, it also satisfies

that

$$\alpha L + (1-\alpha) L'' > \alpha L' + (1-\alpha) L'''$$

<u>Def - 6.B.5</u> (vNM utility function)

A utility function has an expected utility form when all the outcomes of the Lottery L ie N outcome associated with it's probabilities ie $(p_1, p_2 .. p_n)$ it's assigned a number $U_1, U_2, U_3 ... U_n$ for each outcome:

$$\boxed{U(L) = U_1 P_1 + U_2 P_2 \cdots U_n P_n.}$$

↳ VNM utility function.

↳ it is expressed as a Linear function in the probabilities π

$$U(L) = \sum_{n=1}^{N} U_n P_n$$

Proposition - 6.B.1

Proposition 6.B.1: A utility function $U: \mathscr{L} \to \mathbb{R}$ has an expected utility form if and only if it is *linear*, that is, if and only if it satisfies the property that

$$U\left(\sum_{k=1}^{K} \alpha_K L_k\right) = \sum_{k=1}^{K} \alpha_k U(L_k) \qquad (6.B.1)$$

for any K lotteries $L_k \in \mathscr{L}$, $k = 1, \ldots, K$, and probabilities $(\alpha_1, \ldots, \alpha_K) \geq 0$, $\sum_k \alpha_k = 1$.

(very tricky important)

NOTE Utility function is of Expected Utility form iff it's Linear

proof:

To prove

$$U\left(\sum_{k=1}^{K} \alpha_k L_k\right) = \sum_{k=1}^{K} \alpha_k U(L_k)$$

α_k is the probability of L_k occuring.

LHS (Expected utility → Linear)

$$U\left(\sum_{k=1}^{K} \alpha_k L_k\right)$$

$\Rightarrow$ since $L_k = (p_1^k, p_2^k \dots p_n^k)$

$$\Rightarrow U\left(\sum_{k=1}^{K} \alpha_k p_n^k\right)$$

$$\Rightarrow \sum_n u_n \sum_{k=1}^{K} \alpha_k p_n^k$$

$$\Rightarrow \sum_n \underbrace{u_n p_n^k}_{U(L)} \sum_{k=1}^{K} \alpha_k$$

$$\Rightarrow \sum_{k=1}^{K} \alpha_k U(L) \text{ (Linear)}$$

RHS (Linear → Exp utility form)

$L = (p_1, p_2 \ldots p_n)$

Let L^n be convex combination of n Lotteries, st

$U\left(\sum p_n L^n\right)$

↳ probability of L^n occuring

$\Rightarrow \sum p_n U(L^n)$

$\Rightarrow \sum p_n U_n$ (expected utility form)

Proposition 6.B.2: Suppose that $U: \mathscr{L} \to \mathbb{R}$ is a v.N–M expected utility function for the preference relation $\succsim$ on $\mathscr{L}$. Then $\tilde{U}: \mathscr{L} \to \mathbb{R}$ is another v.N–M utility function for $\succsim$ if and only if there are scalars $\beta > 0$ and γ such that $\tilde{U}(L) = \beta U(L) + \gamma$ for every $L \in \mathscr{L}$.

positive affine transformation property of Expected utility function.

It says:- if $U(L)$ is an VNM utility function, &
$\tilde{U}(L)$ is another VNM utility function, then

if $\beta > 0$ & γ, $\tilde{U}(L) = \beta U(L) + \gamma$.

↳ implying that VNM utility form of utility, retains cardinality of a utility function.

<u>Proof</u> (1st Direction → to prove expected utility form)

$\underline{L}, \bar{L} \in \tilde{L}$ st $\bar{L} \succsim L \succsim \underline{L}$

and $\bar{L} > \underline{L}$

To prove

$\tilde{U}(L) = \beta U(L) + \gamma$ is a VNM utility form, where $U(L)$ is a VNM utility form

1st job for proving any utility function to expected utility is to prove Linearity.

So Let $U(L)$ is VNM (given)

and $\tilde{U}(L) = \beta U(L) + \gamma$ (assume)

then

since $U(L_k) = U(\sum \alpha_k L_k)$

$\Rightarrow \tilde{U}(\sum \alpha_k L_k) = \beta U(\sum \alpha_k L_k) + \gamma$

Through Linearity assumption

$\Rightarrow \tilde{U}(\sum \alpha_k L_k) = \beta \{\alpha_k U(L_k) + \gamma$

$\Rightarrow \tilde{U}(\{\alpha_k, L_k) = \{\alpha_k \underbrace{\beta U(L_k) + \gamma}$

$\Rightarrow \tilde{U}(\{\alpha_k, L_k) = \{\alpha_k \tilde{U}(L_k)$

$\therefore \tilde{U}(\sum \alpha_k L_k)$ is of Expected utility form

(2nd Direction → to prove, given Expected utility → $\tilde{U}(L) = \beta U(L) + \gamma$

→ Let $U(L) = \lambda U(\bar{L}) + (1-\lambda) U(\underline{L})$

$\Rightarrow U(L) = \lambda U(\bar{L}) + U(\underline{L}) - \lambda(U\underline{L})$

$$\Rightarrow \frac{U(L) - U(\underline{L})}{U(\bar{L}) - U(\underline{L})} = \lambda \quad \text{— (1)}$$

<u>Through Linearity</u>

$$U\left(\sum \alpha^k L_k\right) = \sum \alpha_k U(L_k)$$

then

$$U(\lambda \bar{L} + (1-\lambda)\underline{L}) = \lambda U(\bar{L}) + (1-\lambda) U(\underline{L})$$

<u>now</u>

$$\bar{L} \succsim L \succsim \underline{L}$$

$$L = \lambda \bar{L} + (1-\lambda)\underline{L}$$

(as L lies in between $\underline{L}$ & $\bar{L}$, so it must be a convex combination of both.)

then $\tilde{U}(L) =$

$$\tilde{U}(L) = \tilde{U}(\lambda \bar{L} + (1-\lambda)\underline{L})$$

and through linearity

$$\tilde{U}(\lambda \bar{L} + (1-\lambda)\underline{L}) = \lambda \tilde{U}(\bar{L}) + (1-\lambda)\tilde{U}(\underline{L})$$

$$\Rightarrow \lambda \tilde{U}(\bar{L}) + \tilde{U}(\underline{L}) - \lambda \tilde{U}(\underline{L})$$

$$\Rightarrow \lambda(\tilde{U}(\bar{L}) - \tilde{U}(\underline{L})) + \tilde{U}(\underline{L}) = \tilde{U}(L)$$

$\hookrightarrow$ 11

Substituting eq^n (I) into (II)

Let

$$\beta = \frac{\tilde{U}(\bar{L}) - \tilde{U}(\underline{L})}{U(\tilde{L}) - U(\underline{L})}$$

&

$$\gamma = \tilde{U}(L) - U(\underline{L}) \frac{\tilde{U}(\bar{L}) - \tilde{U}(\underline{L})}{U(\tilde{L}) - U(\underline{L})}$$

$$\boxed{\tilde{U}(L) = \beta\, U(L)) + \gamma}$$

proved.

Ex-6·B·2 ??

if $\succsim$ on $\mathcal{L}$ is representable by $U(\cdot)$ which is of exp utility form, then $\succsim$ satisfies independence axiom.

ie If $\tilde{U}(L) = \sum u_n p_n$

and $\tilde{U}(L) \geq \tilde{U}(L')$

then $\alpha \tilde{U}(L) + (1-\alpha)\tilde{U}(L'') \geq \alpha \tilde{U}(L') + (1-\alpha) U(L'')$

This can only hold iff

$\alpha L + (1-\alpha) L'' \succsim \alpha L' + (1-\alpha) L''$ holds tr

Expected utility theorem

Prop-6·B·3

Proposition 6.B.3: (*Expected Utility Theorem*) Suppose that the rational preference relation $\succsim$ on the space of lotteries $\mathscr{L}$ satisfies the continuity and independence axioms. Then $\succsim$ admits a utility representation of the expected utility form. That is, we can assign a number u_n to each outcome $n = 1, \ldots, N$ in such a manner that for any two lotteries $L = (p_1, \ldots, p_N)$ and $L' = (p'_1, \ldots, p'_N)$, we have

$$L \succsim L' \quad \text{if and only if} \quad \sum_{n=1}^{N} u_n p_n \geq \sum_{n=1}^{N} u_n p'_n. \tag{6.B.4}$$

ie one lottery L is better than L', if

$U(L) \geq U(L')$

$\downarrow$

$\sum u_n p_n \geq \sum u_n p'_n$

Proof

Allais paradox

Suppose you have lotteries

~~Lot A~~ L_A : $(0.11)\$1M + (0.89)\$0$

L_B : $(0.1)\$5M + (0.9)\$0$

suppose by probability i choose L_B as $\{$as $0.1 \times 5\}$ $= 0.5$ Expected value of L_B

while $L_A = (0.11 \times 1)$ $= 0.11$ Expected value of L_C

choose one! $\rightarrow$ $\boxed{L_B \succsim L_A}$ (Based on their expected value)

Now again you have 2 another lotteries

L_C : $(1)\$1M$.

L_D : $(0.1)\$5M + (0.89)\$1M + (0.01)\$0$.

$L_C \succsim L_D$

choose one! $\rightarrow$ $\boxed{L_C \succsim L_D}$ (Based on their expected value)

Suppose we write all lotteries together

L_A : $(0.11)\$1M + (0.89)\$0$

L_B : $(0.1)\$5M + (0.01)\$0 + (0.89)\$0$

or $(0.11)\,[(10/11)\$5 + (1/11)\$0] + 0.89\$0 \rightarrow$ Nested Format.

L_C : $(1)\$1M$.

L_D : $(0.1)\$5M + (0.89)\$1M + (0.01)\$0$.

we can re write lottery B as a nested lottery like we wrote for in the independence anxiom's coin flip.

comparing L_A & L_B. as both probabilities are equal here.

$\rightarrow$ ignore $0.89\$0$ as there is no pay, we focus on 0.11.

~~where~~ Now From here as probability 0.11 is same, compare the nested lottery value into L_B to $L_A = 0.11 \times 1 + 0.89 \times 0$ ie

$\frac{10}{11} \times 5 + \frac{1}{11} \times 0$

$= 4.54$. $\therefore L_B \succsim L_A$

~~Sam~~ do the same story for L_C & L_D

L_C : \$ 1M

L_D : $(0.1)\$5 + (0.01)\$0 + (0.89)\$1M$.

we can ~~wr~~ rewrite L_C and L_D as

L_C : $(0.11)\$1M + (0.89)\$1M$

L_D : $(0.11)\$\left[\left(\frac{10}{11}\right)\$5 + \left(\frac{1}{11}\right)\$0\right] + (0.89)\$1M$.

Now we can compare Lottery C & D. we ~~compare~~ wont compare the parts ~~which~~ ~~has~~ ~~where~~ o where both Lotteries offer 89% \$1M ie 0.89 \$1M; we compare $\underbrace{0.11\$1M}_{L_C}$ & $\underbrace{0.11\left[\frac{10}{11}\$5 + \frac{1}{11}\$0\right]}_{L_B}$

$\therefore$ L_C : $0.11 \times 1 = 0.11$

L_D : $0.11 \times [4.54] = 0.49$

here is where problem arises

originally; $L_C \succsim L_D$

but now $L_D \succsim L_C$ ✓ this is violates independence.

ie if you chose A & C or B & D, you are good to go.

However if you chose B & C or A & D, your preferences violate independence axiom.

6.C Money Lotteries & Risk Aversion

Like the discrete case which has been discussed till now, if there are 1 ... N outcomes, then $U_1 \ldots U_N$ are the utility numbers associated with those N outcomes, with the Lottery L being the probability vector of (1 ... N) outcomes denoted as $L = (P_1 \ldots P_n)$

For continuous case {continous variable}

Like N, let x represent the monetary outcomes of a Lottery and a Lottery L, be denoted as F(x), where F(x) is a distribution function (cdf) representing the probabilities (like the discrete case where, Lottery was a probability vector). The probability

as basically probability density

which are represented through

$$\int_a^b f(x)\,dx = 1 \quad , \quad f(x) = \text{probability density of } x$$

outcome.

Then $F(X) = \int_{-\infty}^{b} f(x)\,dx = L(x)$

Then, our utility representation can be :-

$$U(L) = \int_{-\infty}^{\infty} U(x)\, f(x)\, dx = \int_{-\infty}^{\infty} U(x)\, dF(x)$$

Discrete case

$$U(L) = \sum_{n=1}^{K} U_n P_n$$

, where $U(L)$ is utility in Expected utility form called VNM utility.

Continous case

$$U(F) = \int_{-\infty}^{\infty} U(x) \cdot f(x)\, dx$$

, where $U(x)$ represents utility from monetary outcome x, just like discrete case.

$$= \int_{-\infty}^{\infty} U(x) \cdot dF(x)$$

$$\boxed{F(x) = \int_{-\infty}^{b} f(x)\,dx \quad \therefore dF(x) = f(x)\,dx}$$

$U(x)$ is called the Bernoulli utility function, while $U(F)$ still represents our VNM Expected Utility Form.

The Bernoulli utility function provides a convinient way to think about a person's attitude towards risk.

Definition 6.C.1: A decision maker is a *risk averter* (or exhibits *risk aversion*) if for any lottery $F(\cdot)$, the degenerate lottery that yields the amount $\int x\, dF(x)$ with certainty is at least as good as the lottery $F(\cdot)$ itself. If the decision maker is always [i.e., for any $F(\cdot)$] indifferent between these two lotteries, we say that he is *risk neutral*. Finally, we say that he is *strictly risk averse* if indifference holds only when the two lotteries are the same [i.e., when $F(\cdot)$ is degenerate].

↳ explaination of key terms

↳ degenerate Lottery → that you are able to decide which lottery to choose from, for the outcomes to maximise utility under uncertainity.

↳ $\int x\, dF(x)$ is expected value of Lottery.

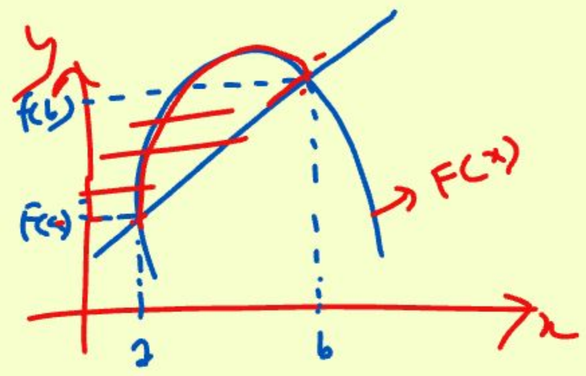
Definition 6.C.1

↳ tells us that, given $\succsim$ is continous, independent & reduced on $\tilde{L}$, then the expected utility representation is $\int U(x) \cdot dF(x) = U(F)$, then A person is risk averse; iff

$$\int U(x) \cdot dF(x) \leq U\left(\int x\, dF(x)\right) \text{ For all } F(\cdot)$$

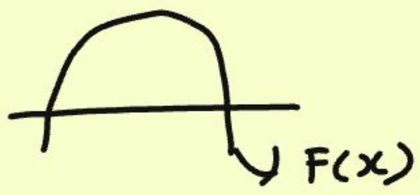
mean $U\left(\int x\, dF(x)\right)$ represents 'Utility' from Expected

value of Lottery, while $\int u(x)\,dF(x)$ is the Expected utility from playing the Lottery. It subtly means, If i find that the Utility I get From the Lottery For sure ($p=1$) (expected value) is greater than Utility from playing the Lottery with uncertainity ($p \in (0,1)$) then i would avoid such Lottery. ie I won't gamble



$F(\lambda a + (1-\lambda) b) \geq \lambda F(a) + (1-\lambda) F(b)$

Concavity :-



- $F(\lambda a + (1-\lambda) b) \geq \lambda F(a) + (1-\lambda) F(b)$
  ↳ then the function is concave.

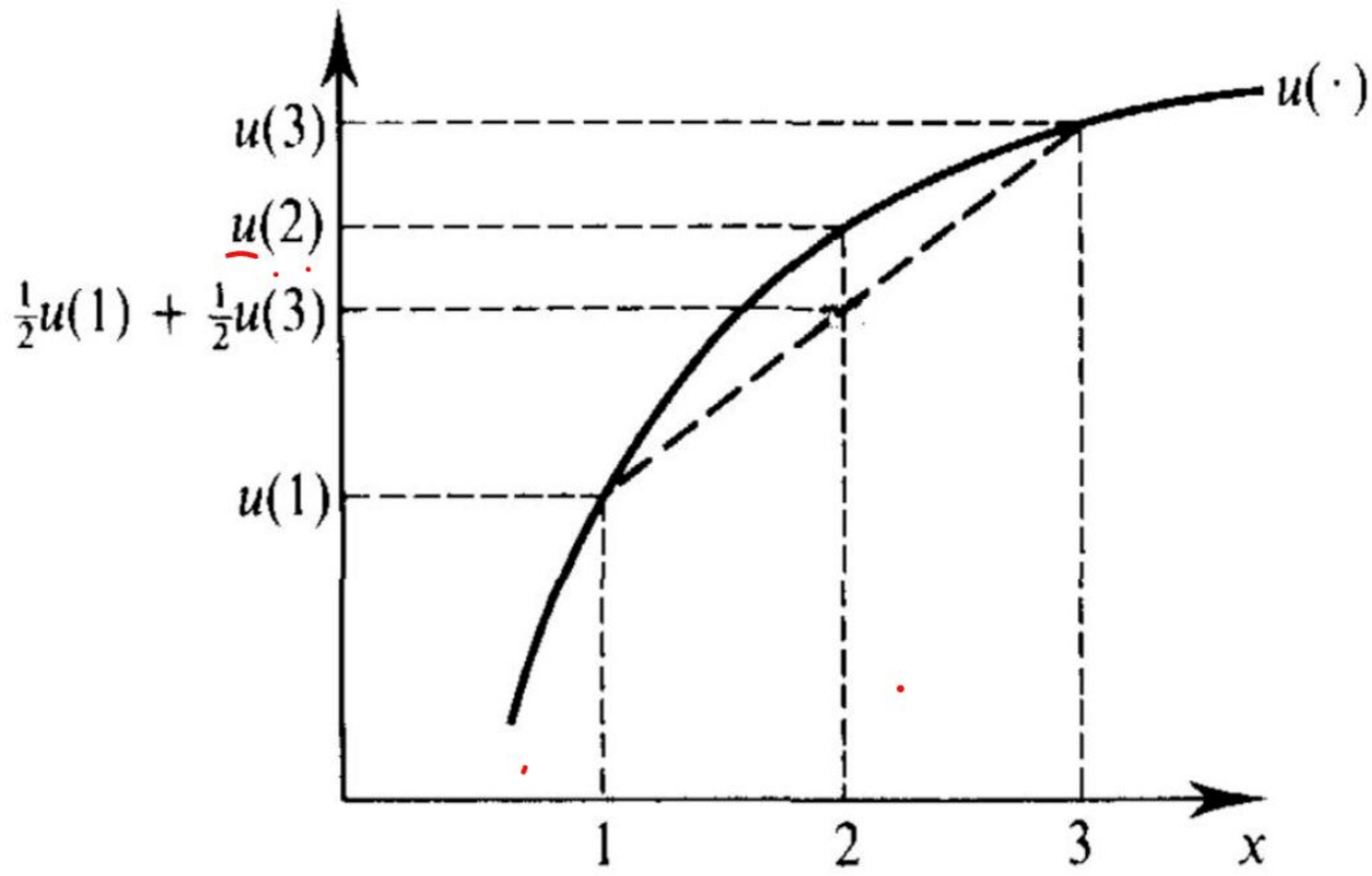
if we look at the risk averse, jensen's inequality

$$\int u(x) \cdot dF(x) \leq u\left(\int x\, dF(x)\right) \quad \text{For all } F(\cdot)$$

or

$$u\left(\int x\, dF(x)\right) \geq \int u(x)\, dF(x)$$

Concave Utility function



u(2)

Figure 6.C.2 Risk aversion (a)

risk aversion is equivalent to the concavity of $U(x)$, and strict risk aversion is equivalent to strict concavity.

strict concavity means, there is a diminishing need of good y as good x increases. For risk terms; The marginal utility of money is decreasing as wealth (in x axis) is increased. The utility gain from an extra dollar is lesser than utility loss of having a dollar in Absolute Form. It means risk of gaining or losing a dollar is not worth taking given equal prob.

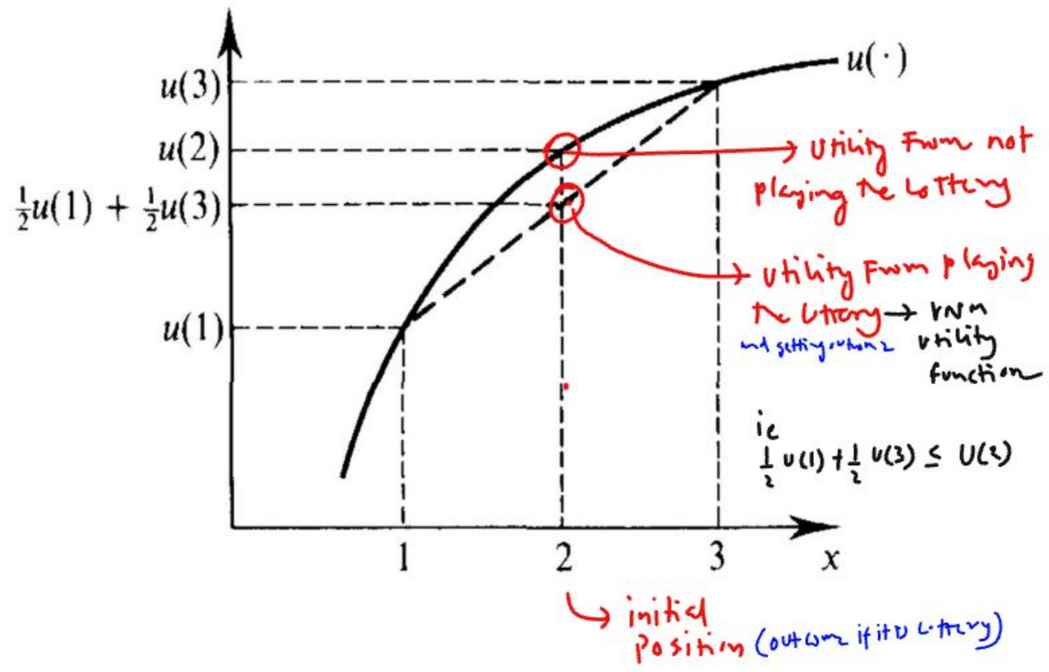
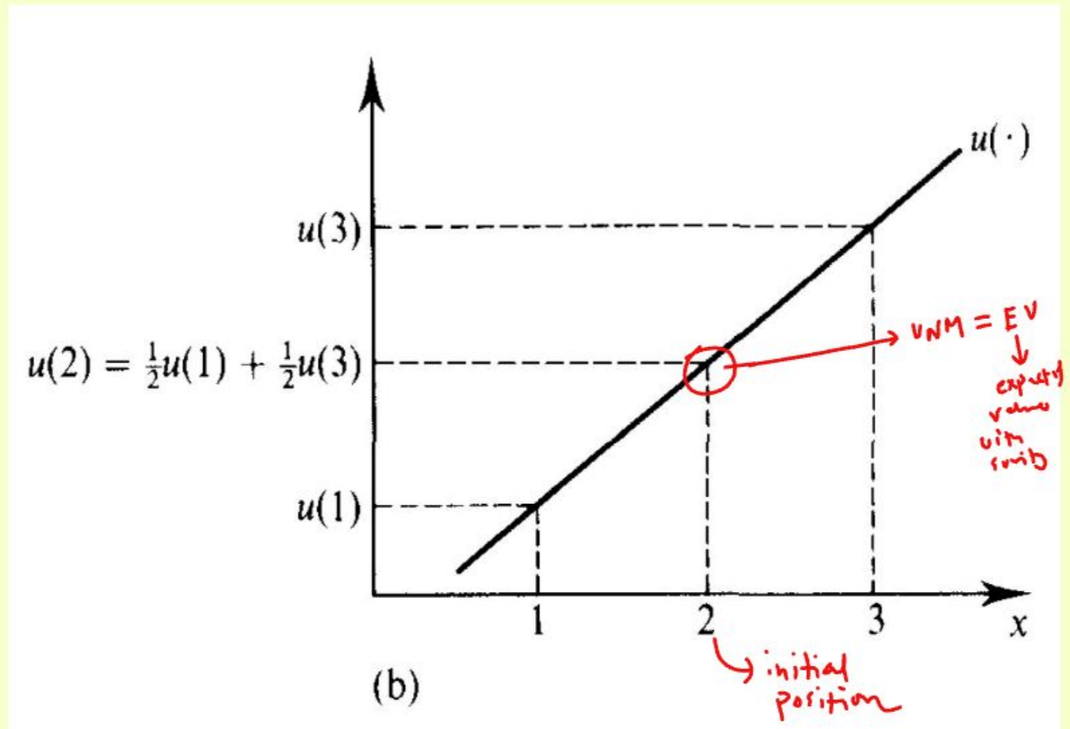


Figure 6.C.2 Risk aversion

Risk neutral

For risk neutral utility maximisers,

$$\int u(x)\,dF(x) = u\left(\int x\,dF(x)\right)$$



risk neutrality (b).

Definition 6.C.2: Given a Bernoulli utility function $u(\cdot)$ we define the following concepts:

(i) The *certainty equivalent of* $F(\cdot)$, denoted $c(F, u)$, is the amount of money for which the individual is indifferent between the gamble $F(\cdot)$ and the certain amount $c(F, u)$; that is,

$$u(c(F, u)) = \int u(x)\,dF(x). \qquad (6.C.3)$$

(ii) For any fixed amount of money x and positive number ε, the *probability premium* denoted by $\pi(x, \varepsilon, u)$, is the excess in winning probability over fair odds that makes the individual indifferent between the certain outcome x and a gamble between the two outcomes $x + \varepsilon$ and $x - \varepsilon$. That is

$$u(x) = (\tfrac{1}{2} + \pi(x, \varepsilon, u))\,u(x + \varepsilon) + (\tfrac{1}{2} - \pi(x, \varepsilon, u))\,u(x - \varepsilon). \qquad (6.C.4)$$

① Certainity Equivalence

Suppose you are offered a choice between a gamble with probability

$L = \left(\frac{1}{2}, \frac{1}{2}\right)$ st $EV = \underbrace{\frac{1}{2} \times 100}_{\$100\text{'s ko jitne ki prob} = 1/2} + \underbrace{\frac{1}{2} \times 0}_{\text{no win prob} = 1/2} = \$50$ & \$50 for sure. most will choose the

prob of winning \$100

prob of getting no money

$50 For sure. Let C be the amount for which i will be willing to play the Lottery instead of sure thing. This is called as certainity Equivalence of Lottery, which means it is that certain amount of money that offers the same utility as the Lottery (expected utility)

Suppose my certainity equivalence is 40
& certainity valu outcome is 50
50 − 40 = 10 which is called the risk premium.

Risk premium is that money which you pay to avoid getting into a Lottery type situation. ie, take the risk away & replace the Lottery with it's expected value

Certainity Equivalence C hence can be written as

$$\int U(x) \cdot dF(x) \leq U\left(\int x \, dF(x)\right) \quad \text{For all } F(\cdot)$$

then, if C is the amount that equals the utility of the Lottery & Expected value

$$\int U(x)\, dF(x) = U(C(F, U))$$

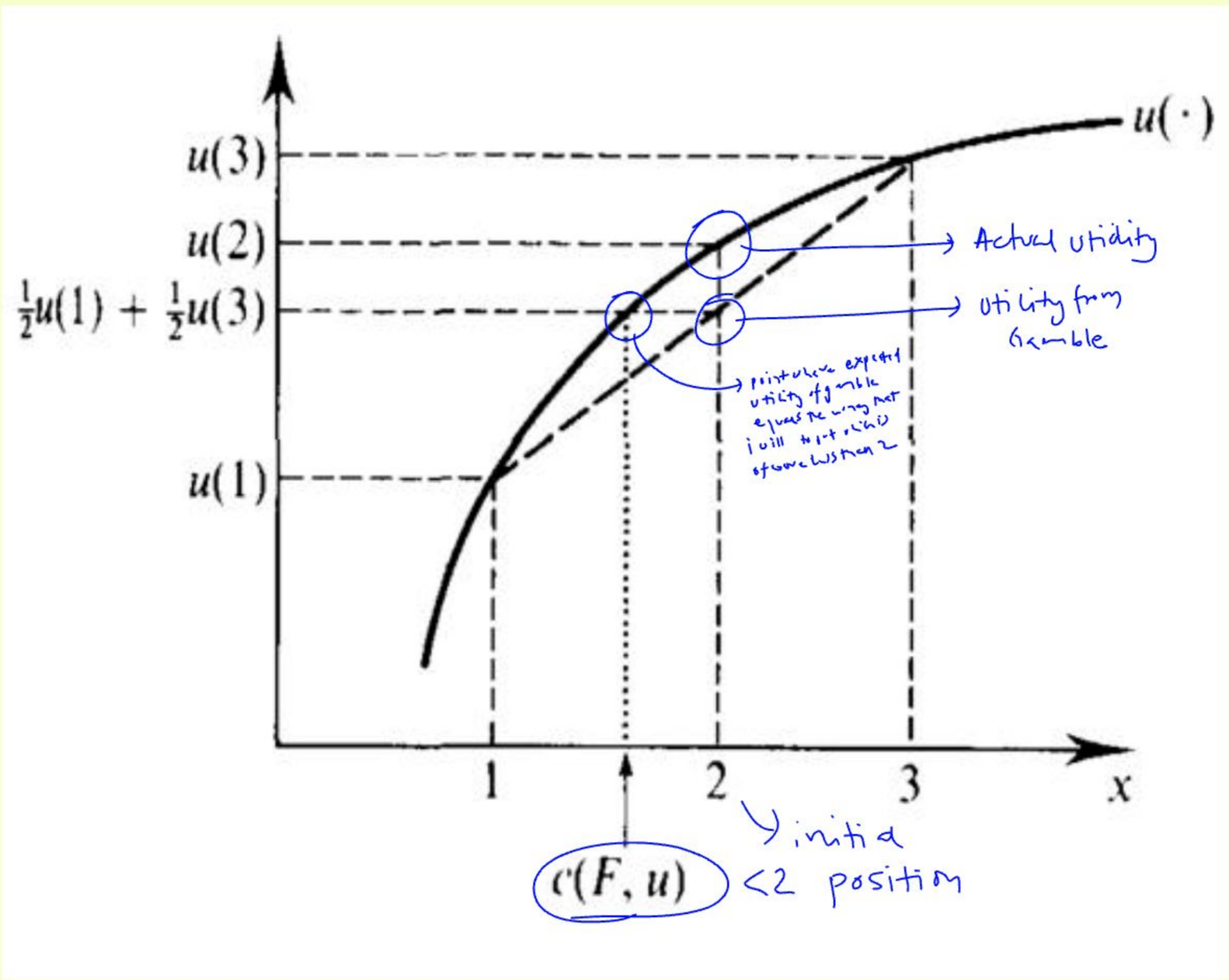
Four classes of utility division

1) risk Averse = $\int U(x) \cdot dF(x) \leq U\left(\int x \, dF(x)\right)$ For all $F(\cdot)$

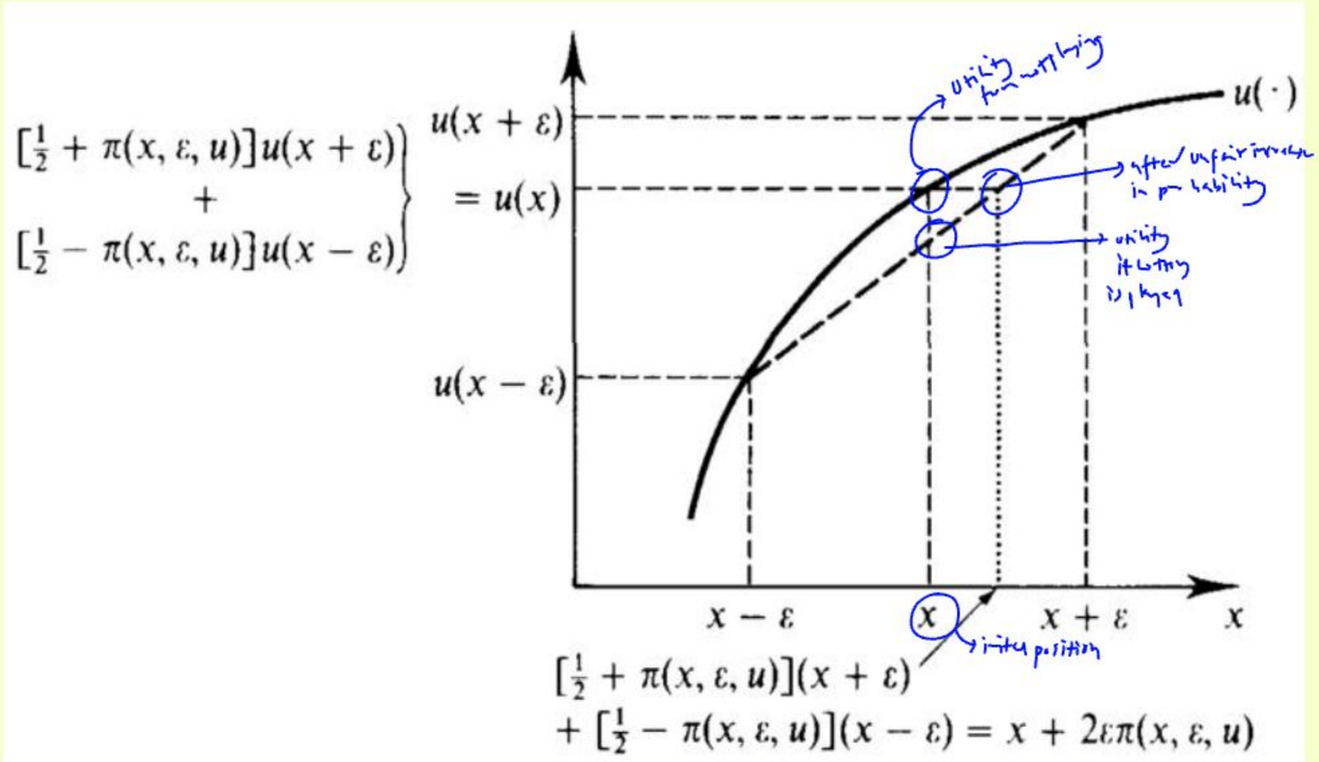
2) risk Lover = $\int U(x) \cdot dF(x) \geq U\left(\int x \, dF(x)\right)$ For all $F(\cdot)$

3) risk neutral = $\int U(x) \cdot dF(x) = U\left(\int x \, dF(x)\right)$ For all $F(\cdot)$

4) none of above.



(II) $\pi(x, \varepsilon, u)$



If there is a probability π, that is unbalanced (not Fair), and weighs such that it makes the individual indifferent between outcome x & gamble of 2 outcomes $(x+\varepsilon, x-\varepsilon)$

ie $U(x) = \left\{\frac{1}{2} + \pi(x, \varepsilon, u)\right\} U(x+\varepsilon) + \left\{\frac{1}{2} - \pi(x, \varepsilon, u)\right\} U(x-\varepsilon)$

Properties of Expected utility maximiser

(I) risk averse
(II) $U(\cdot)$ is concave
(III) $C(F, U) \leq \int x \, dF(x) \; \forall F(\cdot)$
(IV) $\pi(x, \varepsilon, U) \geq 0$

} important

Applications of risk aversion

→ insurance
→ Demand of risky asset
→ General Asset problems

} do them seperately.

Measurement of risk aversion

Definition 6.C.3: Given a (twice-differentiable) Bernoulli utility function $u(\cdot)$ for money, the *Arrow Pratt coefficient of absolute risk aversion* at x is defined as $r_A(x) = -u''(x)/u'(x)$.

It might be understood, from the properties of a risk averse individual, that their utility function is concave, hence, more concavity might imply more risk averse? yes. This is true, but we have a more complicated method of measuring risk.

From the bernoullis utility function, one might be tempted to think that if $U_1'' > U_2''$ {U_1 & U_2 are two Bern utility func} $\forall x$, but there is a problem with this.

ie it is not invariant to +ve linear transformations such as $\boxed{\tilde{U}(L) = \beta U(L) + \gamma}$

Such transformations should let no change happen to consumer's Utility pref ie $U(L) > U(L')$
$\Rightarrow \tilde{U}(L) > \tilde{U}(L')$

However if $U_1'' > U_2''$ is compared

then if $U_2'' = \alpha U_1''$, ie U_2'' is a positive affine transformation of U_1''

then $\alpha U_1'' > U_1''$, then $U_2'' > U_1''$ {ie previously $L > L'$, now $L' > L$: which is a violation}

Thus we normalise the formula by $-\frac{U_2''}{U_2'}$ & $-\frac{U_1''}{U_1'}$

now they are comparable

$r_A(x) = -\frac{U''(x)}{U'(x)}$ is a measure of risk aversion

called Arrow pratt measure of risk aversion.

Example 6.C.4: Consider the utility function $u(x) = -e^{-ax}$ for $a > 0$. Then $u'(x) = ae^{-ax}$ and $u''(x) = -a^2e^{-ax}$. Therefore, $r_A(x, u) = a$ for all x. It follows from the observation just made that the general form of a Bernoulli utility function with an Arrow–Pratt measure of absolute risk aversion equal to the constant $a > 0$ at all x is $u(x) = -\alpha e^{-ax} + \beta$ for some $\alpha > 0$ and β. ■

$U = -e^{-ax}$

$U' = ae^{-ax}$

$U'' = -a^2e^{ax}$

$r_A(x) = \frac{-U''}{U'} \Rightarrow -\left(\frac{-a^2 \cancel{e^{-ax}}}{a \cancel{e^{-ax}}}\right) = a > 0$

Comparision across individuals

Given 2 bernoulli utility function, when can we say that U_2 is more risk averse than U_1.

1) if $r_A(x, U_2) \geq r_A(x, U_1) \; \forall x$

2) $C(F, U_1) \geq C(F, U_2)$

↳ greater CE implying can play a gamble if compensated well

↳ lower CE mean you can't make him/her convinced to play lottery even at a high CE

3) if $U_2 = \psi(U_1(\cdot))$

ie if U_2 is derived from a +ve, strict, concave, transformation

ie if U_1 = concave

ψ_1 is a concave transformation

$\therefore U_2 = \psi_1(U_1)$ ie concave (concave) = concave

4) $\pi(x, \epsilon, U_2) \geq \pi(x, \epsilon, U_1)$

5) If Expected Utility $\int U_2 dF(\cdot) \geq \bar{U}_2(x)$ for a Lottery $F(\cdot)$, then $\int U_1 dF(\cdot) \geq \bar{U}_1(x) \Rightarrow \bar{U}_2 > \bar{U}_1$

use property 3 to prove $r_A(U_2) > r_A(U_1)$ (6.C.2)

3) if $U_2 = \psi(U_1(\cdot))$
ie if U_2 is derived from a +ve, strict, concave, transformation

ie if U_1 = concave
ψ_1 is a concave transformation
$\therefore$ $U_2 = \psi_1(U_1)$ ie concave (concave)
= concave

proof

From $U_2 = \psi(U_1)$

differentiating both sides, and then differentiating twice

1ST diff

$$U_2' = \frac{\partial \psi(U_1)}{\partial U_1} \cdot \frac{\partial U_1}{\partial x_1} = \psi_1' U_1'$$

2nd diff

since $U_2' = \psi_1'(U_1) \cdot U_1'$

$$\Rightarrow U_2'' = \psi''(U_1)(U_1')^2 + \psi_1'(U_1) U_1''$$

$$\Rightarrow \frac{U_2''}{U_2'} = \frac{\psi''(U_1)(U_1')^2 + \psi_1' U_1''}{\psi' U_1'}$$

$$\Rightarrow r_A(U_2) = r_A(U_1) - \frac{\psi_1''}{\underset{<0}{\psi_1'}} \underbrace{U_1}_{>0}$$

$\therefore \gamma_A(x, u_2) \geq \gamma_A(x, u_1)$ iff

$$\psi''_1 \leq 0$$

Proposition 6.C.3

Proposition 6.C.3: The following properties are equivalent:

(i) The Bernoulli utility function $(\cdot)$ exhibits decreasing absolute risk aversion.

(ii) Whenever $x_2 < x_1$, $u_2(z) = u(x_2 + z)$ is a concave transformation of $u_1(z) = u(x_1 + z)$.

(iii) For any risk $F(z)$, the certainty equivalent of the lottery formed by adding risk z to wealth level x, given by the amount c_x at which $u(c_x) = \int u(x + z)\, dF(z)$, is such that $(x - c_x)$ is decreasing in x. That is, the higher x is, the less is the individual willing to pay to get rid of the risk.

(iv) The probability premium $\pi(x, \varepsilon, u)$ is decreasing in x.

(v) For any $F(z)$, if $\int u(x_2 + z)\, dF(z) \geq u(x_2)$ and $x_2 < x_1$, then $\int u(x_1 + z)\, dF(z) \geq u(x_1)$.

What is Left of 6.C

↳ Applications of risk aversion

→ insurance

→ Demand of risky asset

→ General Asset problems

} do them seperately.

most important

↳ eg 6.C.2 continued sums

Exercise 6.C.8

Def 6.C.5 ✓

proposition 6.C.4 ✓

Ex- 6.6.2

Demand for risky Asset (maximisation problem)

what is an Asset?

A asset is a divisible claim to a Financial return in the Future

Suppose there is a customer who has invested his/her wealth in 2 Financial assets. Then suppose let one asset be risky, while other be safe. Let customer's total wealth be w which is allocated into these assets in form α & β

$\alpha \to$ risky asset, $\beta \to$ safe asset.

Return per dollar invested
- $\to$ Risky asset $= Z$ dollars, per dollar invested
- $\to$ safe asst $= 1$ dollar, per dollar invested.

Then individual's portfolio's (α, β) return $= \alpha Z + \beta$, given $\alpha + \beta = w$

Note

Z is a return from a random risky aoset, thus having distributi function $F(Z)$

st

$$\int Z \, dF(Z) > 1$$

(very weird Assump)

$\to$ distribution of Safe asset.

$\downarrow$ meaning return from risky asset exceed safe asset.

Our main intention to study this problem is to know how α & β must be distributed across assets to get the customer's utility maximised.

Thus individual's utility max problem is

$$\max_{\alpha, B > 0} \int U(\alpha Z + B)\, dF(Z) \cdot dF(S)$$

using expected utility form → because think about it! investing in an asset is like a lottery

utility is attained from return or outcome $X = \alpha Z + B$

distribution of safe asset → which is equal to 1

$$\text{st } \alpha + B = W$$

Budget constraint

<u>Now</u>

$\alpha + B = W$

$\Rightarrow B = W - \alpha$ — (1)

$\Rightarrow \alpha Z + B = x$ (outcome of assets)

$\Rightarrow \alpha Z + W - \alpha$

$\Rightarrow [W + \alpha(Z-1)]$

we can maximise this instead

$$\max \int U(W + \alpha(Z-1))\, dF(Z)$$

$$\text{st } 0 \le \alpha \le W.$$

<u>maximising for α^*</u>

$$\int \frac{\partial U(\,)}{\partial \alpha} \cdot [0 + (Z-1)]\, dF(Z) = \phi(\alpha^*)$$

$$\Rightarrow \int U'(Z-1)\, dF(Z) = \phi(\alpha^*)$$

$\boxed{\text{if } \alpha > 0 \text{ (interior optima)}}$

$\Rightarrow \phi(\alpha^*) > 0$

or if $\alpha < W$ then $\phi(\alpha^*) \le 0$

concavity

hence, if risk is actuarilly favourable, the risk averter will always accept at least a small of it.

Ex 6.C.2 continued (measure of risk, page 192)

Example 6.C.2 continued:

Suppose that we now have two individuals with Bernoulli utility functions $u_1(\cdot)$ and $u_2(\cdot)$, and denote by α_1^* and α_2^* their respective optimal investments in the risky asset. We will show that *if $u_2(\cdot)$ is more risk averse than $u_1(\cdot)$, then $\alpha_2^* < \alpha_1^*$*; that is, the second decision maker invests less in the risky asset than the first.

$$\max \int U_1(w-\alpha+z\alpha)\, dF(z)$$

st $0 \le \alpha \le w$

$\Rightarrow$ After maximisation

$$\int (z-1)\, U_1'(w+\alpha_1^*(z-1))\, dF(z) = \phi(\alpha_1^*)$$

similarly for α_2^*

$$\int (z-1)\, U_2'(w+\alpha_2^*(z-1))\, dF(z) = \phi_2(\alpha_2^*)$$

<u>Now</u> since $\phi(\alpha_2)$ is decreasing due to concavity of $U_2(\,)$,

now if U_2 is more risk averse than U_1

$U_2 = \psi(U_1(x))$ where ψ is a strict increasing concave transformation

??

6.D. Comparision of Payoff Distributions in Terms of Return & Risk

Like we compared utility functions (concave functions) of people, to see who is more risk averse than the other (arrow pratt measure of risk aversion). in 6.1) we compare for the payoffs ie the outcomes. or more specifically payoff distributions.

Unlike the comparision of utilities through $R_A(u)$, we compare the outcomes or the payoffs with 2 methods.

Level of returns (FOSD)

Dispersion of returns. (SOSD)

FOSD :- $F(\cdot)$ yielding unambigously higher returns than $G(\cdot)$

SOSD :- $F(\cdot)$ being unambigously Less risky than $G(\cdot)$

* $F(0) = 0$ & $F(x) = 1$ for some x

First-Order Stochastic Dominance (FOSD)

$F(\cdot)$ yields higher return than $G(\cdot)$

where $F(\cdot)$ & $G(\cdot)$ are distribution → we try to establish this through

2 points

① Every expected utility maximizer who values more to Less, prefers $F(\cdot)$ to $G(\cdot)$

② The probability of winning the outcome x in lottery $F(\cdot)$ > than winning the same outcome with Lottery $G(\cdot)$

↳ greater

Definition 6.D.1: The distribution $F(\cdot)$ *first-order stochastically dominates* $G(\cdot)$ if, for every nondecreasing function $u: \mathbb{R} \to \mathbb{R}$, we have

$$\overset{E_x U(F)}{\int u(x)\, dF(x)} \overset{\geq}{} \overset{E_x U(G)}{\int u(x)\, dG(x)}.$$

$E_x U(F)$ = Expected utility of Lottery F.

Thus we can say that $F(\cdot)$ FOSD $G(\cdot)$

if

$$\underbrace{\int U(x)\, dF(x) \geq \int U(x)\, dG(x)}_{\text{VNM Expected utility Form}}$$

bernoulli's utility function.

Proposition 6.D.1: The distribution of monetary payoffs $F(\cdot)$ first-order stochastically dominates the distribution $G(\cdot)$ if and only if $F(x) \leq G(x)$ for every x.

$F(\cdot)$ FOSD $G(\cdot) \iff F(x) \leq G(x) \; \forall x \in X$

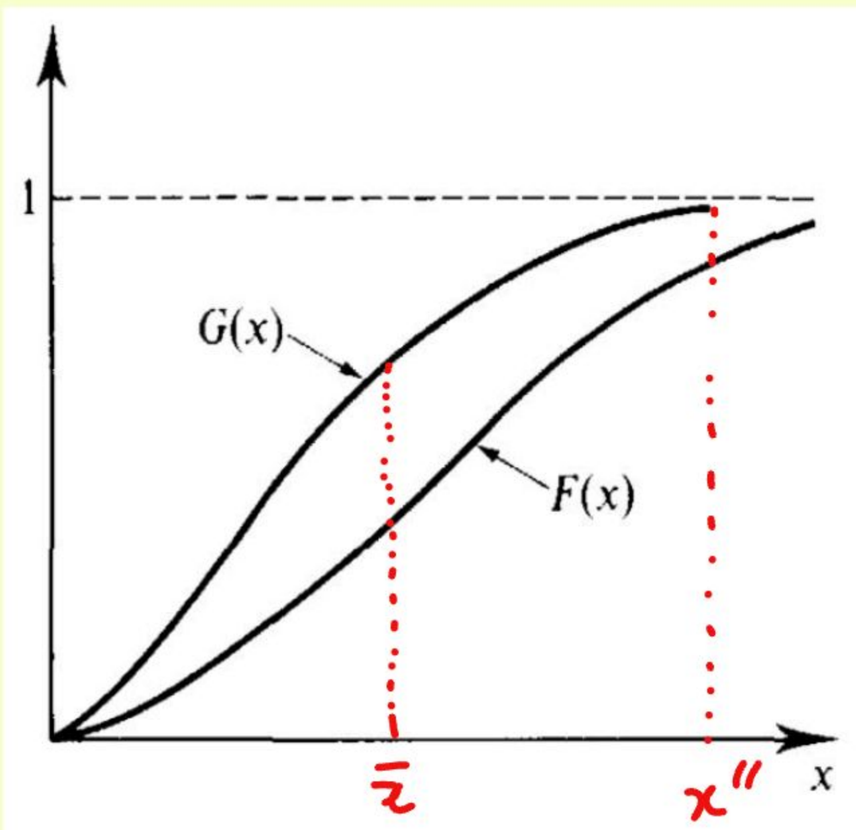
$\hookrightarrow F(x) - G(x) \leq 0$

What we are trying to prove here is

$F(X) = \int_{-\infty}^{x} f(x)\,dx = P(X \leq x)$ ie probability that lottery offers prize that is less than or equal to x

Now, If a consumer is strictly preffering more money x, and as we know that the consumer is a consequentialist, (does only care about the outcome, x and not the process of acheiving that outcome ie whether through Lottery or anything)

Then if $G(X) \geq F(X)$, the graphically



Observe; at $\bar{x}$, $F(X)$ gives distribution that is lower than $G(X)$, It mean, for a lower payoff $\bar{x}$, $G(X)$ is say "If you play this Lottery, there is a high chance, you will win $\bar{x}$". However contary to $G(X)$, $F(X)$ offers a lower chance to attain that probability, However In the long run, at maxima, of $G(X)$, where one could win x'', $F(X)$ still hasn't reached full potential. Now, since consumer's preference is of monotone, he will prefer a point $x''' > x''$, which could be at maxima of $F(X)$. Hence he will wait for probability to accumulate, choosing $F(X)$ as the choice of lottery than $G(X)$.

Figure 6.D.1
$F(\cdot)$ first-order stochastically dominates $G(\cdot)$.

Proof

Part I Given $F(X)$ & $G(X)$; and $F(X) \leq G(X)$

prove $F(x)$ FOSD $G(X)$

or $F(X) - G(X) \leq 0$

Let $H(X) = F(x) - G(X)$

then it must be that $H(x) \leq 0$

we show it through contradiction that must hold.

Suppose $H(\bar{X}) > 0$ for some $\bar{X} \Rightarrow F(\bar{X}) > G(\bar{X})$

then,

Let $U(X) = \begin{cases} 1, & X > \bar{X} \\ 0, & X \leq \bar{X} \end{cases}$ = step function

$\downarrow$ non decreasing

Then Expected Utility from $H(X)$ Lottery is

$\Rightarrow \int U(X)\, dH(x)$ or $\int U(X) \cdot dF(X) - \int U(X)\, dG(X)$

$\Rightarrow$ (Some outcome) < 0 (But How is this possible?)

as we assumed $H(\bar{X}) > 0$ for some $\bar{X}$, while $F(X) \leq G(x)\ \forall x$

then it must be that $\bar{x} \in X$ Like all other xs, hence it must be For all x, $F(X) \leq G(X)$ ie $H(\bar{x}) < 0$ where $\bar{x}$ is included in all x as well.

Thus $F(X)$ doest FOSD $G(X)$ untill $F(X) \leq G(X)$. (proved)

Part II

Given $F(X)$ FOSD $G(X)$, show $F(X) \leq G(X)$

Given $U(x)$ is an strictly increasing function

and $H(X) = F(X) - G(X)$

$$\int \overset{u}{U(x)} \cdot d\overset{v}{H(x)}$$

integrating by parts.

$$\Rightarrow \quad U(x)\left[H(X)\right]_0^{\infty} - \int U'(X)\left[H(X)\right]_0^{\infty} dx.$$

$$\Rightarrow \quad U(X)\left[H(X)\right]_0^{\infty} - \int U'(x)\left[H(X)\right.$$

at ∞, $CDF = 0 \therefore H(X) = 0$

at 0, $CDF = 0$, $H(X) = 0$

<u>First term</u>

$$U(X)\left[H(\infty) - H(0)\right] - \int U'(X)\, H(X)$$

$$\Rightarrow \underbrace{U(X)\left[0\right]}_{0} - \int U'(X)\, H(X)$$

$$\Rightarrow \quad -\int U'(x)\, H(X) \leq 0 \text{ if, } H(X) < 0$$

as $U'(X)$ is a increasing function.

$\Rightarrow$ as $H(X) < 0$

$\Rightarrow \quad G(X) \geq F(X)$ (proved)

$F(x)$ FOSD $G(X)$ means,

$\int x\,dF(x) \geq \int x\,dG(X)$ ie mean of x under $F(\cdot)$ is greater than mean of x under $G(x)$.

A Ranking of means hence doesn't matter when comparing FOSD, it's the entire Distribution that matters.

Ex-6.D.1

① First stage:- A random value x is the outcome of a lottery $G(\cdot)$

② Second Stage:- The selected value x gets an "upward Probabilistic shift" by adding another random amount Z with it's own distribution or lottery $H(Z)$ x specific to x. ie $x + Z = t$

Note:- Z is alway non negative $\Rightarrow$ final outcome is atleast x with probability 1

Let the final outcome probability dist $= F(\cdot)$, then for any non-decreasing utility,

$$\int U(t)\cdot dF(t) = \int \Big[\underbrace{\int U(x+Z)\,dH_x(Z)}_{\text{second stage Expected utility}} \Big] dG(x) \geq \int U(x)\,G(x)$$

Diagram.

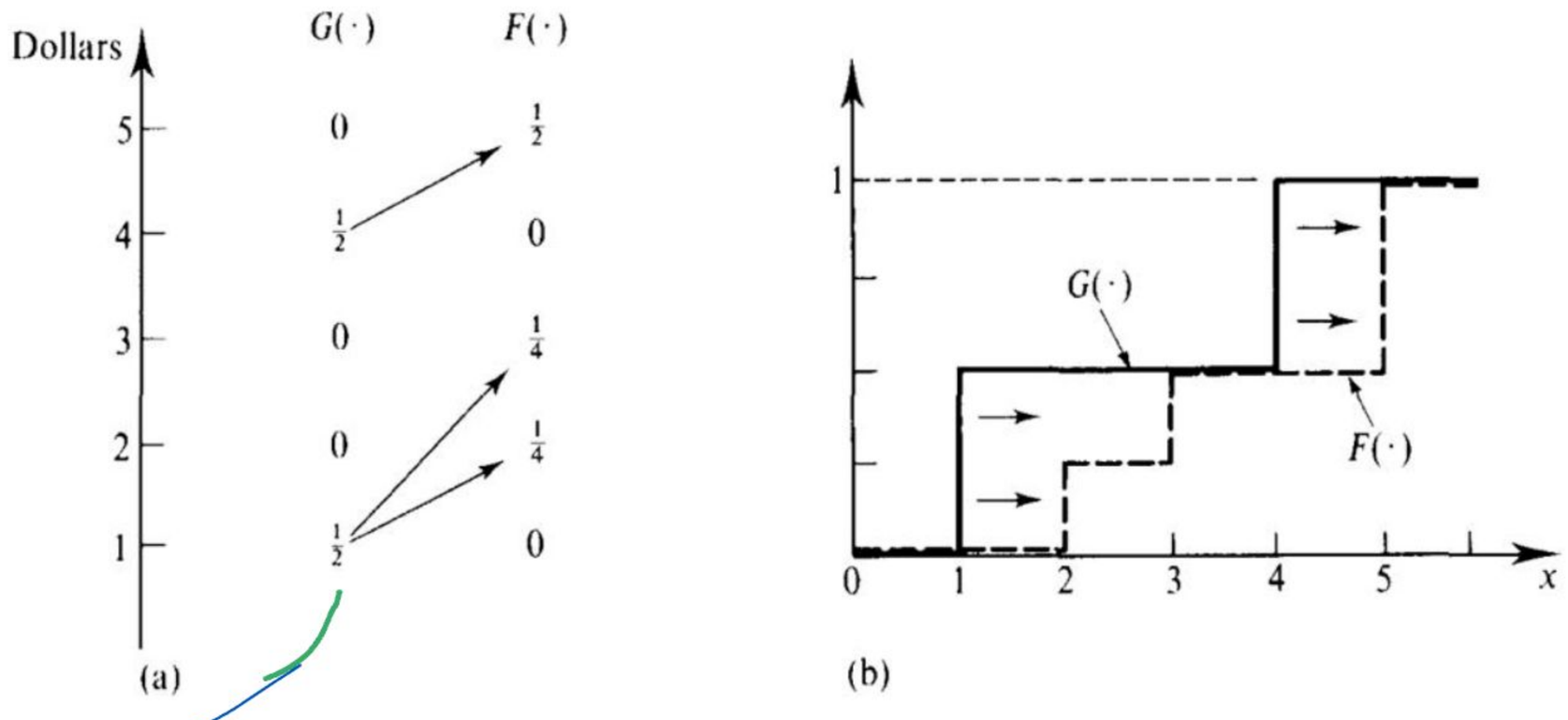


Figure 6.D.2
$F(\cdot)$ first-order stochastically dominates $G(\cdot)$.

Initial Lottery $G(\cdot)$

- A 50% chance of getting \$1 & 50% chance of \$4

The New Adjusted Lottery $F(\cdot)$

- A random shift Z ensures that every outcome of $G(\cdot)$ is pushed upwards by a +ve amount
- $F(\cdot)$ provides outcome that is always atleast as good as ($\geq$) $G(\cdot)$

The CDF of $F(\cdot)$ lies to left (increase in direction of x) of $G(\cdot)$ (above), confirming $F(\cdot)$ FOSD $G(\cdot)$

End note:- FOSD means high/better vs Lower/worse.

SOSD

↳ talks about riskiness or dispersion of returns

Definition 6.D.2: For any two distributions $F(x)$ and $G(\cdot)$ with the same mean, $F(\cdot)$ *second-order stochastically dominates* (or *is less risky than*) $G(\cdot)$ if for every nondecreasing concave function $u: \mathbb{R}_+ \to \mathbb{R}$, we have

$$\int u(x)\, dF(x) \geq \int u(x)\, dG(x).$$

Example 6.D.2 introduces an alternative way to characterize the second-order stochastic dominance relation.

- $F(x)$ & $G(x)$ with same means $\Rightarrow \int x\,dF = \int x\,dG$
- $F(x)$ SOSD $G(x) \rightarrow F(x)$ is less risky than $G(\cdot)$
- $F(x) = G(x)$ here.
- mean preserving spread = A new dist obtained by spreading the outcomes of $F(\cdot)$ st mean stay same. Here $G(\cdot)$ is mean preserving spread of $F(\cdot)$ iff mean of $G(\cdot)$ = mean of $F(\cdot)$

1) x is the outcome of original lottery $F(\cdot)$
2) Add a randomised Z to x, with dist of Z as $H_x(z)$ and mean $\int Z\, dH_x(z) = 0$
3) Adjustment to x by adding Z, doesn't affect mean of x if $\int (x+z)\, dH = \int (x+z)\, dF$

mean of $G(\cdot)$ = mean of $F(\cdot)$

ie

$$\int u(x)\, dG(\cdot) = \int \left[\int u(x+z)\, dH_x(z) \right] dF(x)$$

→ x is the outcome of $F(X)$, then a spread function $G(\cdot)$ will create a new outcome $(X+Z)$, such that outcome $(X+Z)$ has same mean as x

For eg

we have 2 dist.

original dist $F(\cdot)$

- outcomes are \$2 and \$3
- probabilities: equal probabilities between both outcomes ie 50% for \$2 & 50% for \$3

$$F(x) = \begin{cases} 0, & x < 2 \\ \frac{1}{2}, & 2 \le x < 3 \\ 1, & x > 3 \end{cases}$$

New dist $G(\cdot)$

- outcomes spread and are: \$1, \$2, \$3, \$4
- probabilities :—

$$G(x) = \begin{cases} 0 & , x < 1 \\ 0.25 & , 1 \le x < 2 \\ 0.5 & , 2 \le x < 3 \\ 0.75 & , 3 \le x < 4 \\ 1 & , x \ge 4 \end{cases}$$

Here, there is a distribution of probabilities on outcomes

st mean of $G(\cdot)$ = mean of $F(\cdot)$

ie mean of $F(\cdot) = 0.5 \times 2 + 0.5 \times 3 = 2.5$

mean of $G(\cdot) = 0.25 \times 1 + 0.25 \times 2 + 0.25 \times 3 + 0.25 \times 4$

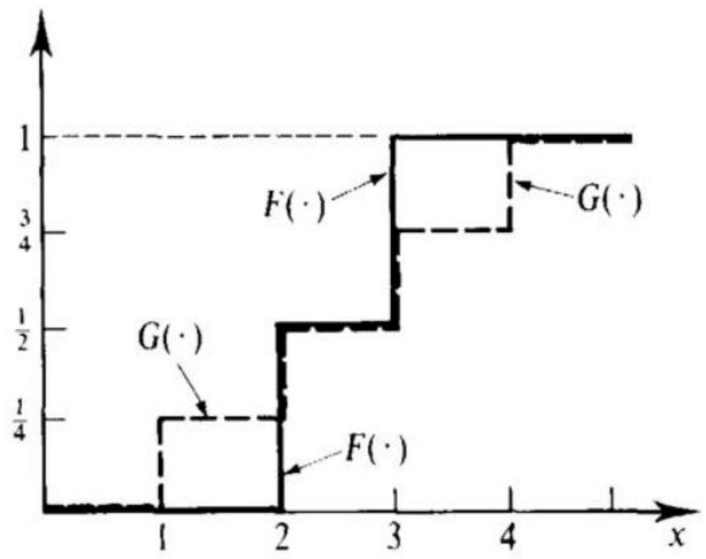
$= (0.25)(10) = 2.5$

<u>How is $G(\cdot)$ riskier than F?</u>

$F(\cdot)$'s outcomes is tightly concentrated around mean

$G(\cdot)$'s outcomes are spread across mean, meaning more variablity in data.

$\therefore$ More variance $\Rightarrow$ more risk.



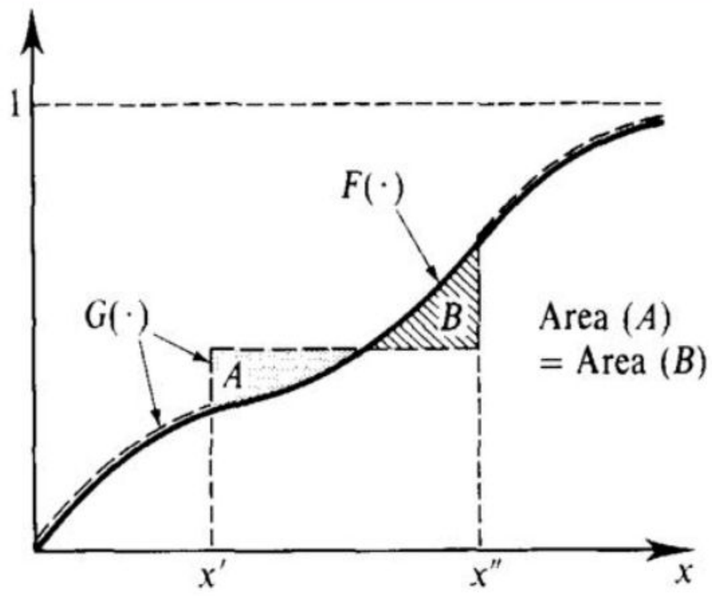


Figure 6.D.3 (left)
$G(\cdot)$ is a mean-preserving spread of $F(\cdot)$.

Figure 6.D.4 (right)
$G(\cdot)$ is an elementary increase in risk from $F(\cdot)$.

Prop - 6.D.2

If F & G are 2 dist with same mean,

then

- $F(\,)$ SOSD $G(\,)$
- $G(\,)$ is mean preserving

Leftover parts 6.C

Def 6.C.4

Definition 6.C.4: The Bernoulli utility function $u(\cdot)$ for money exhibits *decreasing absolute risk aversion* if $r_A(x, u)$ is a decreasing function of x.

It means, that individual becomes wealthy It's attitude towards risk will decrease than compared to a relatively poor individual. ie $U(\cdot)$ is a increasing function of x which is our outcome, however, as $U(\cdot)$ is strictly or concave function the gain per additional dollar won is Less than the loss of a additional dollar. This implies as wealth increase over a period of time, Utility from winning x is Less. It implies

If U_2 & U_1 are 2 individuals and U_1 is wealthier than U_2, then

$$r_A(U_1, x) \leq r_A(U_2, x)$$

hence, from proposition 6.C.2, ① holds.

This also satisfies the ① of proposition 6.C.3

Proposition 6.C.3: The following properties are equivalent:

(i) The Bernoulli utility function $(\cdot)$ exhibits decreasing absolute risk aversion.

(ii) Whenever $x_2 < x_1$, $u_2(z) = u(x_2 + z)$ is a concave transformation of $u_1(z) = u(x_1 + z)$.

(iii) For any risk $F(z)$, the certainty equivalent of the lottery formed by adding risk z to wealth level x, given by the amount c_x at which $u(c_x) = \int u(x + z)\, dF(z)$, is such that $(x - c_x)$ is decreasing in x. That is, the higher x is, the less is the individual willing to pay to get rid of the risk.

(iv) The probability premium $\pi(x, \varepsilon, u)$ is decreasing in x.

(v) For any $F(z)$, if $\int u(x_2 + z)\, dF(z) \geq u(x_2)$ and $x_2 < x_1$, then $\int u(x_1 + z)\, dF(z) \geq u(x_1)$.

Relative Risk aversion

The Application of (DARA) ie Decreasing Absolute risk Aversion, is actually not economically applicable extensively by it is not convinient to establish risk aversion in absolute terms. A better, stronger assumption called (non-decreasing relative risk aversion) is applied.

Absolute risk Aversion - Concerned about absolute gain & Loss. However we evalute risk is relative or percentage terms as well. ∴ Relative risk Aversion is used.

RRA (Relative risk Aversion)

Suppose the measure of Loss or gain of wealth is denoted by $t > 0$. on the outcome x

then utility derieved is $U(tx) = \tilde{U}(t)$
$= t\,U(x)$

Let $t = 1$ mean no change in wealth. (Intial wealth intact)

if change occurs around 1, then degree of risk aversion (Arrow pratt measure) $= \gamma_A(\tilde{U}, x) = -\dfrac{\tilde{U}''(t)}{\tilde{U}'(t)}$

$U(tx)$

$\dfrac{\partial U(tx)}{\partial tx} \cdot \dfrac{\partial tx}{\partial t}$

$\Rightarrow x U'(tx)$

$\Rightarrow x$

$$= \frac{\tilde{U}''(1)}{\tilde{U}'(1)}$$

we can hence write, that

$$\frac{\tilde{U}''(t)}{\tilde{U}(t)} = tx\,\frac{U''(x)}{U'(x)}$$

if $t \approx 1$ {small risk}

$$\frac{\tilde{U}''(1)}{\tilde{U}'(1)} = 1 \times \frac{U''(x)}{U'(x)}$$

$$= \underbrace{-x\,\frac{U''(x)}{U'(x)}}$$

$$\Rightarrow \underbrace{\underbrace{\gamma_r}_{\text{relative}}(\tilde{U}, x)}_{\text{relative measure of risk}} = \underbrace{x}_{\text{coefficient of risk Aversion}} \underbrace{\gamma_A(U, x)}_{\text{absolute measure of risk}}$$

The above eq^n means

The risk measure varies due to WEALTH x.

Property of non increasing risk Aversion states that individuals become less risk averse as $x \uparrow$

NOTE

A risk-averse individual with decreasing relative risk will exhibit decreasing absolute risk aversion. (but reverse is not true)

Insurance

A person is considered risk Averse. Then, suppose he fears that there is a risk of loss. Let that loss amount be D. For this loss, not to occur, the person tries to Insure himself by paying a risk premia. The risk premia or insurance pays 1 dollar if loss occurs. Then if α units of Insurance is bought at Q per unit cost, then total price of the insurance is αQ.

Let wealth of Individual is W

If no loss occurs: $W - \alpha Q$

If loss occurs: $W - \underbrace{\alpha Q}_{\text{Fee}} - \underbrace{D}_{\text{Loss}} + \underbrace{\alpha}_{\text{premia.}}$

Suppose, there is now a chance that Loss might occur with probability π

$$W - \alpha Q - \pi D + \pi \alpha$$

$$\Rightarrow \quad W - \pi D + \alpha(\pi - Q).$$

$\therefore$ person's now decision is to how much α is he going to buy such that he is free from risk of D. Since person is a risk averter, he is hence a Expected utility maximiser.

$$\max (1-\pi) U(W - \alpha Q) + \pi U(W - \alpha Q - D + \alpha) \text{ wrt } \alpha.$$

or $W - D + \alpha(1-Q)$

$$\boxed{-(1-\pi) Q U(\cdot) + \pi(1-Q) U(\cdot) = 0}$$

$\hookrightarrow$ from this we obtain α^*

now suppose, the expected cost of Insuarce & the cost of 1 unit of Insuarce = equal.

$$\therefore \pi = Q$$

then from FOC

$$-(1-\pi)Q\,U'(\cdot) + \pi(1-Q)\,U'(\cdot) \leq 0$$

or

$$\pi(1-\pi)\,U'(\cdot) - (1-\pi)\,\pi\,U'(\cdot) \leq 0$$

since $Q = \pi$

$$\Rightarrow \quad U'(w - \alpha Q) = U'(w - \alpha Q - D + \alpha)$$

$$\Rightarrow \quad \cancel{w} - \alpha \cancel{Q} = \cancel{w} - \alpha \cancel{Q} - D + \alpha$$

$$\Rightarrow \quad \boxed{\alpha = D}$$

$\therefore$ If insuarance is actuarily fair the decision maker or risk averter must completly insure himself.