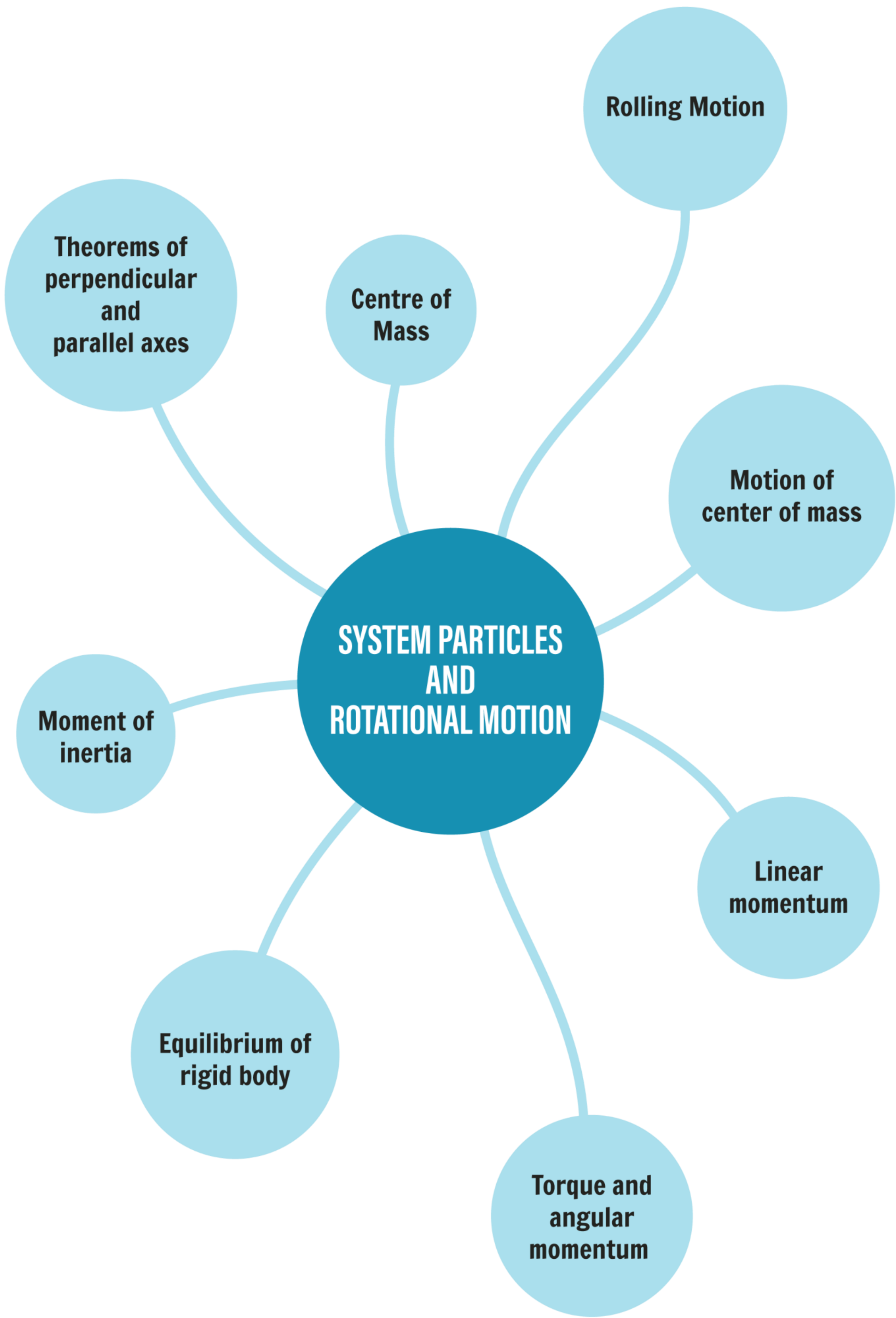
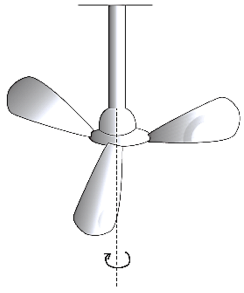
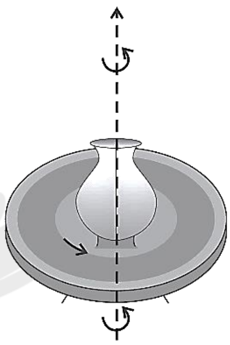

Rolling Motion
Theorems of perpendicular and parallel axes
Centre of Mass
Motion of center of mass
SYSTEM PARTICLES AND ROTATIONAL MOTION
Moment of inertia
Linear momentum
Equilibrium of rigid body
Torque and angular momentum

# System of Particles and Rotational Motion

We come across many objects that follow rotational movements. A ceiling fan or a potter's wheel, these rotating objects are a system of particles that consider the rotational motion. It is the combination of circular motion of large number of particles of a rigid system.



(a)

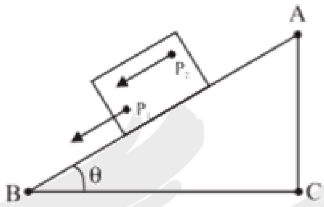


(b)

We know that force, energy and power are associated with rotational motion. These and other aspects of rotational motion are covered in this chapter. We shall see that all important aspects of rotational motion either have already been defined for linear motion.

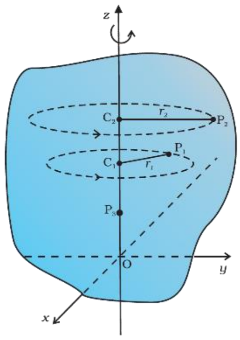
**Ideally a rigid body is a body with a perfectly definite and unchanging shape. The distances between all pairs of particles of such a body do not change.**

In **pure translational motion** at any instant of time, all particles of the body have the same velocity.




The line or fixed axis about which the body is rotating is its axis of **rotation**

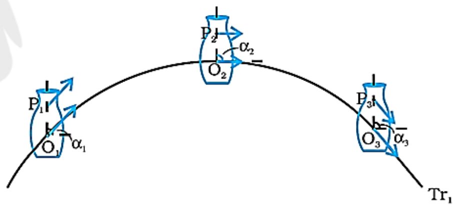
In rotation of a rigid body about a fixed axis, every particle of the body moves in a circle, which lies in a plane perpendicular to the axis and has its centre on the axis.




**The motion of a rigid body which is not pivoted or fixed in some way is either a pure translation or a combination of translation and rotation.**
**The motion of a rigid body which is pivoted or fixed in some way is rotation.**

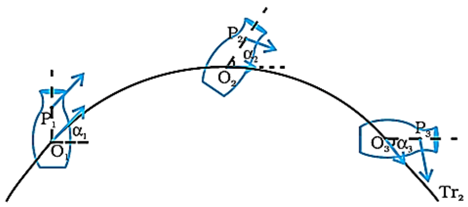
## Centre of Mass

Centre of mass is a very special point. The concept of centre of mass of a system enables us, in describing the overall motion of the system by replacing the system by an equivalent single point, where the entire mass of the body or system is supposed to be concentrated.




*(a) Motion of a rigid body which is pure translation.*

Now, suppose we have a system of n particles of masses $m_1, m_2, m_3 \ldots m_n$ respectively along a straight line at distances $x_1, x_2, x_3 \ldots x_n$ from the origin respectively. Then the centre of mass of the system is given by



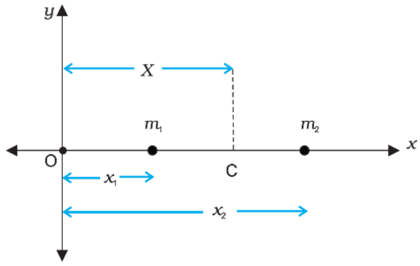

*(b) Motion of a rigid body which is a combination of translation and rotation.*

$$X_{cm} = \frac{m_1x_1 + m_2x_2 + \cdots m_nx_n}{m_1 + m_2 + \cdots m_n}$$

$$X_{cm} = \frac{m_1x_1 + m_2x_2 + \cdots m_nx_n}{M}$$

where M is the total mass of the system.

$$X = \frac{\sum m_ix_i}{M}$$




$$Y = \frac{\sum m_iy_i}{M}$$

and

$$Z = \frac{\sum m_iZ_i}{M}$$

For continuous Mass system –

$$\sum \Delta m_i \rightarrow \int dm = M$$

$$\sum (\Delta m_i)x_i \rightarrow \int xdm$$

$$\sum (\Delta m_i)y_i \rightarrow \int ydm$$

$$\sum (\Delta m_i)z_i \rightarrow \int zdm$$

**Q.** Find the centre of mass of three particles at the vertices of an equilateral triangle. The masses of the particles are 100 g, 150 g, and 200 g respectively. Each side of the equilateral triangle is 0.5 m long. **[NCERT Exercise]**

**Sol.** With the x-and y-axes chosen as shown in Figure, the coordinates of points O, A and B forming the equilateral triangle are respectively (0,0), (0.5,0), $(0.25, 0.25\sqrt{3})$. Let the masses 100 g, 150 g and 200 g be located at O, A and B be respectively. Then,

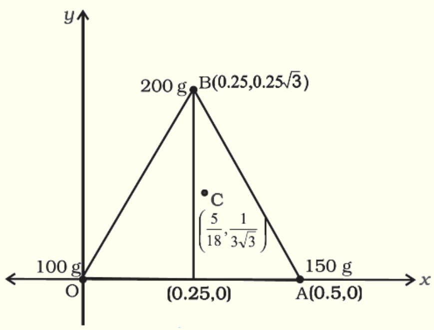
$$X = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$= \frac{100(0) + 150(0.5) + 200(0.25)gm}{(100 + 150 + 200)g}$$

$$= \frac{75 + 50}{450}m = \frac{125}{450}m = \frac{5}{18}m$$

$$Y = \frac{100(0) + 150(0) + 200(0.25\sqrt{3})gm}{450g}$$

$$= \frac{50\sqrt{3}}{450}m = \frac{\sqrt{3}}{9}m = \frac{1}{3\sqrt{3}}m$$



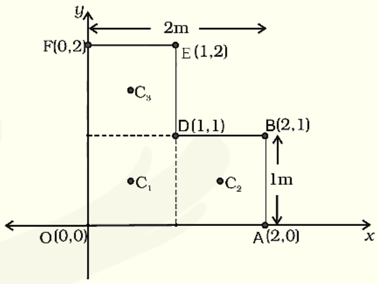
**Q.** Find the centre of mass of a uniform L-shaped lamina (a thin flat plate) with dimensions as shown. The mass of the lamina is 3 kg. **[NCERT Exercise]**

**Sol.** Choosing the $X$ and $Y$ axes as shown in Figure we have the coordinates of the vertices of the L-shaped lamina as given in the figure. We can think of the L-shape to consist of 3 squares each of length 1 m. The mass of each square is 1 kg, since the lamina is uniform. The centres of mass $C_1, C_2$ and $C_3$ of the squares are, by symmetry, their geometric centres and have coordinates (1/2,1/2), (3/2,1/2), (1/2,3/2) respectively. We take the masses of the squares to be concentrated at these points. The centre of mass of the whole L shape $(X, Y)$ is the centre of mass of these mass points.

Hence,

$$X = \frac{[1(1/2) + 1(3/2) + 1(1/2)]\text{kgm}}{(1+1+1)\text{kg}} = \frac{5}{6}\text{m}$$

$$Y = \frac{[1(1/2) + 1(1/2) + 1(3/2)]\text{kgm}}{(1+1+1)\text{kg}} = \frac{5}{6}\text{m}$$




## Motion of centre of mass

We can write equation of Centre of Mass for a system of particles as follow.

$$MR_{Cm} = m_1r_1 + m_2r_2 + m_3r_3 + \cdots + m_nr_n$$

Differentiating the two sides of the equation with respect to time, we get

$$M\frac{dR}{dt} = m_1\frac{dr_1}{dt} + m_2\frac{dr_2}{dt} + \cdots + m_n\frac{dr_n}{dt}$$

The rate of change of position is velocity. So we can replace dR/dt with $v_{cm}$ where $v_{cm}$ is the velocity of the centre of mass.

$$Mv_{cm} = m_1v_1 + m_2v_2 + \cdots + m_nv_n$$

$$M\frac{dv_{cm}}{dt} = m_1\frac{dv_1}{dt} + m_2\frac{dv_2}{dt} + \cdots + m_n\frac{dv_n}{dt}$$

Change in velocity is acceleration, so we get

$$Ma_{cm} = m_1a_1 + m_2a_2 + \cdots + m_na_n$$

where $a_1, a_2, \ldots . a_n$ are the acceleration of first, second, and ....n $^{th}$ particle respectively and $a_{cm}$ is the acceleration of the centre of mass of the system of particles.

**The centre of mass of a system of particles moves as if all the mass of the system was concentrated at the centre of mass and all the external forces were applied at that point.**

## Linear Momentum of a System of Particles

The linear momentum of a particle is defined as

$$P = mv$$

and according to Newton's second law,

$$F = \frac{dP}{dt}$$

write

$$Mv_{cm} = \sum_{i=1}^{n} m_i v_i$$

$$\sum_{i=1}^{n} P_i = \sum_{i=1}^{n} m_i v_i$$

center of mass of the system.

$$P = Mv_{cm}$$

Differentiating the above equation w.r.t. time, we get

$$\frac{dP}{dt} = M\frac{dv_{cm}}{dt} = ma_{cm} = F_{ext}$$

velocity.

Now, if the net external force on the system is zero, the linear momentum of the system, is conserved and the centre of mass will move with constant velocity.

## Rigid Body

A rigid body is a collection of large number of particles, moving in a constrained manner. The constraint is that separation between any two particles of the system does not change.

## Motion of Rigid Body

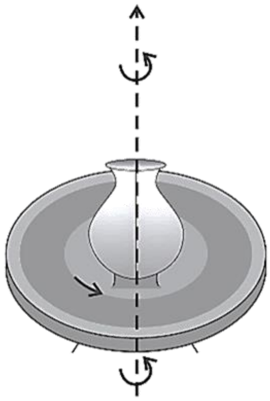
**(i) Translation**

If any line drawn on the rigid body remains parallel to itself throughout the motion, then the body is said to be in pure translation. In the particles of the body have equal velocity and acceleration at all instants and they cover equal distance and displacement in equal time.

**(ii) Rotation**

If any line drawn on the rigid body does not remain parallel to itself throughout its motion, then the body is said to be rotating. For example, the ceiling fan, bicycle wheel or a football rolling on ground.

**(iii) Axis of Rotation**



An imaginary line drawn perpendicular to the plane of motion of different points of the body and passing through the stationary point is called the axis of rotation.

**(iv) Angle of Rotation ($\theta$)**

When the object rotates, its configuration changes, the angle by which any line drawn on the object rotates during the change in angle of rotation.

While the body rotates, every point of the body moves in a circle, whose centre lies on axis of rotation, and every point experience the same angular displacement during a particular time interval.

**(v) Angular Velocity (w)**

$$\omega = \frac{d\theta}{dt}$$

The unit of the angular velocity is rad/s.

**(v) Angular Acceleration (A)**

The angular acceleration is defined as

$$\alpha = \frac{d\omega}{dt}$$

The unit of the angular acceleration is rad/s$^2$.

## Dot Product and Cross Product

**(i) The Dot Product of Two Vectors**

The scalar product or dot product of any two vectors $\vec{A}$ and $\vec{B}$, denoted as $\vec{A} \cdot \vec{B}$ (read as $\vec{A}$ dot $\vec{B}$) is defined as

$$\vec{A} \cdot \vec{B} = AB\cos\ \theta$$

Where A & B are magnitudes of vectors $\vec{A}$ and $\vec{B}$ respectively and $\theta$ is the smaller angle between them is is product is and cos $\theta$ are scalars. Both vectors have a direction but their scalar product does not have a direction.

**Properties**

- Dot product is commutative

  $A \cdot B = B \cdot A$
- Dot product is distributive

  $A \cdot (B + C) = A \cdot B + A \cdot C$
- Dot product of a vector with itself gives square of its magnitude

  $A \cdot A = AA\cos\ \theta = A$
- $A \cdot (\lambda B) = \lambda(A \cdot B)$

  where $\lambda$ is a real number
- $\hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot \hat{k} = \hat{k} \cdot \hat{\imath} = 0$
- $\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = 1$

**(ii) The Cross Product of Two Vectors**

The vector product or cross product of any two vectors $\vec{A}$ and $\vec{B}$, denoted as $\vec{A} \times \vec{B}$( read as $\vec{A}$ cross $\vec{B}$) is defined as $\vec{A} \times \vec{B} = AB\sin\ \theta$

Where A & B are magnitudes of vectors $\vec{A}$ and $\vec{B}$ respectively and $\theta$ is the smaller angle between them. Cross product is called vered and sin $\theta$ are scalars. Both vectors have a direction and their vector product has a same direction.

**Properties**

- The vector product is do not have Commutative Property.

  $$A \times B = -(B \times A)$$
- The following property holds true in case of vector multiplication

  $$(kA) \times B = k(A \times B) = A \times (kB)$$
- If the given vectors are collinear then

  $$A \times B = 0$$
- Following the above property, We can say that the vector multiplication of a vector with itself would be $A \times A = |A||A|\sin\ 0\hat{n} = 0$
- Also in terms of unit vector notation

  $$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$$
- From the above discussion it also follows that

  $$\hat{\imath} \times \hat{\jmath} = \hat{k} = -\hat{\jmath} \times \hat{\imath}$$
  $$\hat{\jmath} \times \hat{k} = \hat{\imath} = -\hat{k} \times \hat{\jmath}$$
  $$\hat{k} \times \hat{\imath} = \hat{\jmath} = -\hat{\imath} \times \hat{k}$$

## Relation between Angular Acceleration and Linear Acceleration

Angular Acceleration is given by

$$\alpha = \frac{d\omega}{dt}$$

Linear Acceleration is given by

$$a = \frac{dv}{dt}$$

From eq (1) and (2),

$$a = \frac{d(r \times \omega)}{dt}$$
$$a = r \times \frac{d\omega}{dt}$$
$$a = r \times \alpha$$

## Relation between Angular Velocity and Linear Velocity

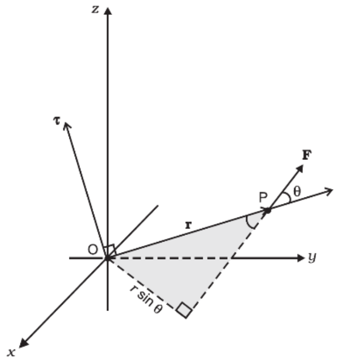
Let, any rigid body is rotating about any rotational axis with angular velocity ( $\omega$ ). If distance of a particle at a perpendicular be r from the fixed axis and linear velocity be v of any particle of them rigid system then relation between them is given by

$$\vec{v} = \vec{r} \times \vec{\omega}$$

## Torque

The tendency of a force to rotate the body to which it is applied is called torque. The torque, specified with regard to the axis of rotation, is equal to the magnitude of the component of the force vector lying in the plane perpendicular to the axis, multiplied by the shortest distance between the axis and the direction of the force component.

$$\vec{\tau} = \vec{r} \times \vec{F}$$




## Relation between Torque and Angular Velocity

$$\vec{L} = \vec{r} \times \vec{P}$$

Differentiating w.r.t. t, we get

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{P}}{dt} + \frac{d\vec{r}}{dt} \times \vec{P}$$
$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} + \vec{v} \times m\vec{v}$$
$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} + (\vec{v} \times \vec{v})$$
$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} + 0 \dots [\because \vec{v} \times \vec{v} = 0]$$
$$\frac{d\vec{L}}{dt} = \tau$$

**Q.** Find the torque of a force $7\hat{i} + 3\hat{j} - 5\hat{k}$ about the origin. The force acts on a particle whose position vector is $\hat{i} - \hat{j} + \hat{k}$. **[NCERT Exercise]**

**Sol.** Here $r = \hat{i} - \hat{j} + \hat{k}$ and $F = 7\hat{i} + 3\hat{j} - 5\hat{k}$.

We shall use the determinant rule to find the torque $\tau = r \times F$

$$\tau = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 7 & 3 & -5 \end{vmatrix} = (5-3)\hat{i} - (-5-7)\hat{j} + (3-(-7))\hat{k}$$

or $\tau = 2\hat{i} + 12\hat{j} + 10\hat{k}$

## EQUILIBRIUM OF A RIGID BODY

We are now going to concentrate on the motion of rigid bodies rather than on the motion of general systems of particles.

A rigid body is said to be in mechanical equilibrium, if both its linear momentum and angular momentum are not changing with time, or equivalently, the body has neither linear acceleration nor angular acceleration. This means

(1) the total force, i.e. the vector sum of the forces, on the rigid body is zero;

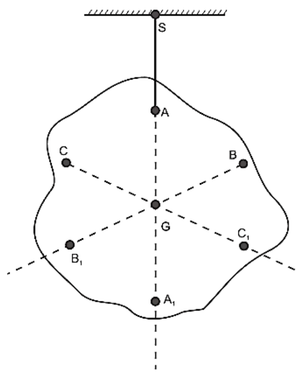
$$F_1 + F_2 + \cdots + F_n = \sum_{i=1}^{n} F_i = 0$$

If the total force on the body is zero, then the total linear momentum of the body does not change with time. Eq. (7.30a) gives the condition for the translational equilibrium of the body.

(2) The total torque, i.e. the vector sum of the torques on the rigid body is zero,

$$\tau_1 + \tau_2 + \cdots + \tau_n = \sum_{i=1}^{n} \tau_i = 0$$

## Centre of Gravity




Centre of Gravity gravity of a body as that point where the total gravitational torque acting on the body is zero.

Now, suppose we have a system of n particles of masses $w_1, w_2, w_3 \ldots w_n$ respectively along a straight line at distances $x_1, x_2, x_3 \ldots x_n$ from the origin respectively. Then the centre of gravity of the system is given by

$$X_{cd} = \frac{w_1x_1 + w_2x_2 + \cdots w_nx_n}{w_1 + w_2 + \cdots w_n}$$

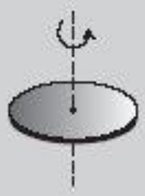
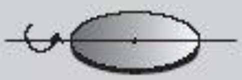
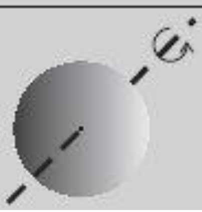
## Moment of Inertia

When any object rotate about any axis then it has tendency to resist its motion, this tendency of resistance is called moment of inertia. It is denoted by I and it's SI unit is $kg/m^2$.

Moment of Inertia of any object is defined as product of mass of that object and square of perpendicular distance of rotational axis.

$$I = MR^2$$

The **radius of gyration** of a body about an axis may be defined as the distance from the axis of a mass point whose mass is equal to the mass of the whole body and whose moment of inertia is equal to the moment of inertia of the body about the axis.

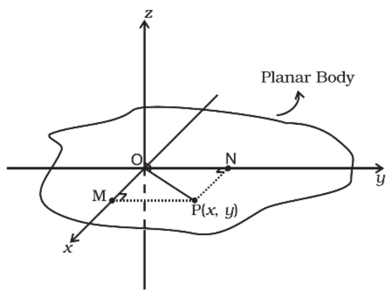
| Z | Body | Axis | Figure | I |
|---|---|---|---|---|
| (1) | Thin circular ring, radius $R$ | Perpendicular to plane, at centre |  | $MR^2$ |
| (2) | Thin circular ring, radius $R$ | Diameter |  | $MR^2/2$ |
| (3) | Thin rod, length L | Perpendicular to rod, at mid point |   | $ML^2/12$ |
| (4) | Circular disc, radius $R$ | Perpendicular to disc at centre |  | $MR^2/2$ |
| (5) | Circular disc, radius $R$ | Diameter |  | $MR^2/4$ |
| (6) | Hollow cylinder, radius $R$ | Axis of cylinder |  | $MR^2$ |
| (7) | Solid cylinder, radius $R$ | Axis of cylinder |  | $MR^2/2$ |
| (8) | Solid sphere, radius $R$ | Diameter |  | $2MR^2/5$ |

## Theorems of Perpendicular and Parallel Axes

### (i) Theorems of Perpendicular Axes

It states that the moment of inertia of a planar body (lamina) about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two perpendicular axes concurrent with perpendicular axis and lying in the plane of the body.

Consider a lamina in $x - y$ plane as shown in the figure and suppose that it consists of n particles of masses $m_1, m_2, m_3 \ldots m_n$ at perpendicular distances $r_1, r_2, r_3 \ldots r_n$ respectively from the axis OZ and suppose the corresponding perpendicular distances of these particles from the OY are $x_1, x_2, x_3 \ldots x_n$ and from the axis OX are $y_1, y_2, y_3 \ldots y_n$ respectively.




Let $I_x, I_y,$, and $I_z$ be the moment of inertia of the lamina about axes OX, OY and OZ respectively.

Now, $\quad I_x = \sum_{i=1}^{n} m_i y_i^2$

Similarly $\quad I_y = \sum_{i=1}^{n} m_i x_i^2$

And $\quad I_z = \sum_{i=1}^{n} m_i r_i^2$

Adding equations (i) and (ii), we get

$$I_x + I_y = \sum_{i=1}^{n} m_i (y_i^2 + x_i^2)$$

From the figure, we can see

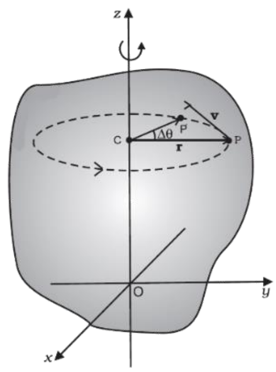
$$r_i^2 = x_i^2 + y_i^2$$

$$I_x + I_y = \sum_{i=1}^{n} m_i r_i^2 = I_z$$

$$I_z = I_x + I_y$$

### (ii) Theorems of Parallel Axes

It states that the moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through its centre of mass and the product of its mass and the square of the distance between the two parallel axes. Consider a rigid body, as shown in the figure and suppose we know the moment of inertia of the body about axis AB, and want to find the moment of inertia of the body about EF which is at a perpendicular distance d from AB.



Suppose the rigid body is made up of n particles of masses $m_1, m_2, m_3 \dots m_n$ at perpendicular distances $r_1, r_2, r_3, \dots r_n$ respectively from the axis AB passing through the centre of mass C of the body. If ri is the perpendicular distance of the particle from the axes AB; then

$$I = \sum_{i=1}^{n} m_i r_i^2$$

$$I = \sum_{i=1}^{n} m_i (R + a)^2$$

$$I = \sum_{i=1}^{n} m_i (R^2 + a^2 + 2Ra)$$

$$I = \sum_{i=1}^{n} m_i R^2 + \sum_{i=1}^{n} m_i a^2 + \sum_{i=1}^{n} m_i 2Ra$$

$$I = \sum_{i=1}^{n} m_i R^2 + \sum_{i=1}^{n} m_i a^2 + 0$$

$$I = I_{cm} + \sum_{i=1}^{n} m_i R^2$$

**Q.** What is the moment of inertia of a rod of mass M, length l about an axis perpendicular to it through one end? **[NCERT Exercise]**

**Sol.** For the rod of mass M and length l, $I = ML^2/12$. Using the parallel axes theorem, $I' = I + Ma^2$ with $a = l/2$ we get,

$$I' = M\frac{l^2}{12} + M\left(\frac{l}{2}\right)^2 = \frac{Ml^2}{3}$$

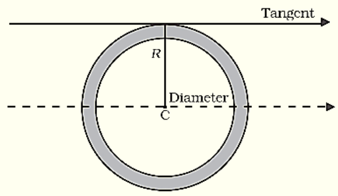
We can check this independently since I is half the moment of inertia of a rod of mass 2M and length 2l about its midpoint,

$$I' = 2M \cdot \frac{4l^2}{12} \times \frac{1}{2} = \frac{Ml^2}{3}$$

**Q.** What is the moment of inertia of a ring about a tangent to the circle of the ring?

**Sol.** The tangent to the ring in the plane of the ring is parallel to one of the diameters of the ring. **[NCERT Exercise]**

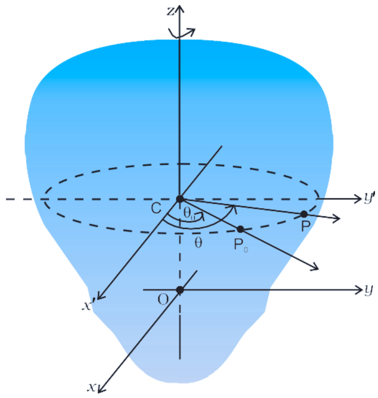
The distance between these two parallel axes is R, the radius of the ring. Using the parallel axes theorem,




$$I_{tangent} = I_{dia} + MR^2 = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2.$$

## KINEMATICS OF ROTATIONAL MOTION ABOUT A FIXED AXIS

The kinematical quantities in rotational motion, angular displacement ($\theta$), angular velocity ($\omega$) and angular acceleration ($\alpha$) respectively are analogous to kinematic quantities in linear motion, displacement (x), velocity (v) and acceleration (A).




$$v = v_0 + at$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

$$v^2 = v_0^2 + 2ax$$

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

and $$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

**Q.** The angular speed of a motor wheel is increased from 1200 rpm to 3120 rpm in 16 seconds. (i) What is its angular acceleration, assuming the acceleration to be uniform? (ii) How many revolutions does the engine make during this time? **[NCERT Exercise]**

**Sol.** (i) We shall use $\omega = \omega_0 + \alpha t$

$\omega_0 =$ initial angular speed in rad/s

$= 2\pi \times$ angular speed in rev /s

$= \dfrac{2\pi \times \text{ angular speed in rev /min}}{60\text{s/min}}$

$= \dfrac{2\pi \times 1200}{60}\text{rad/s} = 40\pi\text{rad/s}$

Similarly $\omega =$ final angular speed in rad/s

$= \frac{2\pi\times 3120}{60}\text{rad/s} = 2\pi \times 52\text{rad/s} = 104\pi\text{rad/s}$

$\therefore$ Angular acceleration

$\alpha = \frac{\omega - \omega_0}{t} = 4\pi\text{rad/s}^2$

The angular acceleration of the engine $= 4\pi\text{rad/s}^2$

(ii) The angular displacement in time t is given by

$\theta = \omega_0 t + \dfrac{1}{2}\alpha t^2$

$= \left(40\pi \times 16 + \frac{1}{2} \times 4\pi \times 16^2\right)\text{rad} = (640\pi + 512\pi)\text{rad} = 1152\pi\text{rad}$

Number of revolutions $= \frac{1152\pi}{2\pi} = 576$

## DYNAMICS OF ROTATIONAL MOTION ABOUT A FIXED AXIS

| | Linear Motion | Rotational Motion about a Fixed Axis |
|---|---|---|
| 1 | Displacement x | Angular displacement $\theta$ |
| 2 | Velocity $v = dx/dt$ | Angular velocity $\omega = d\theta/dt$ |
| 3 | Acceleration $a = dv/dt$ | Angular acceleration $\alpha = d\omega/dt$ |
| 4 | Mass M | Moment of inertia I |
| 5 | Force $F = Ma$ | Torque $\tau = I\alpha$ |
| 6 | Work $dW = F\,ds$ | Work $W = \tau d\theta$ |
| 7 | Kinetic energy $K = Mv^2/2$ | Kinetic energy $K = I\omega^2/2$ |
| 8 | Power $P = Fv$ | Power $P = \tau\omega$ |
| 9 | Linear momentum $p = Mv$ | Angular momentum $L = I\omega$ |

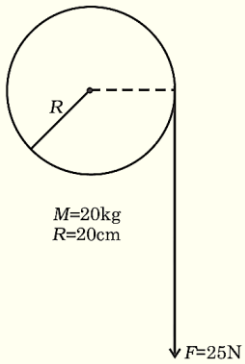
**Q.** A cord of negligible mass is wound round the rim of a fly wheel of mass 20 kg and radius 20 cm. A steady pull of 25 N is applied on the cord as shown in Figure. The flywheel is mounted on a horizontal axle with frictionless bearings. **[NCERT Exercise]**

(A) Compute the angular acceleration of the wheel.

(B) Find the work done by the pull, when 2 m of the cord is unwound.

(C) Find also the kinetic energy of the wheel at this point. Assume that the wheel starts from rest.

(D) Compare answers to parts (B) and (C).




**Sol.** (A) We use the $I\alpha = \tau$

torque $\tau = FR = 25 \times 0.20\text{Nm}(\text{ as } R = 0.20\text{m}) = 5.0 \text{ Nm}$

I = Moment of inertia of flywheel about its

$$\text{axis } = \frac{MR^2}{2} = \frac{20.0 \times (0.2)^2}{2} = 0.4 \text{ kg m}^2$$

$$\alpha = \text{ angular acceleration} = 5.0 \text{ N m}/0.4 \text{ kg m}^2 = 12.5 \text{ s}^{-2}$$

(B) Work done by the pull unwinding 2 m of the cord = $25\text{N} \times 2\text{m} = 50\text{J}$

(C) Let $\omega$ be the final angular velocity. The kinetic energy gained $= \frac{1}{2} I\omega^2$,

since the wheel starts from rest.

Now, $\omega^2 = \omega_0^2 + 2\alpha\theta,\ \omega_0 = 0$

The angular displacement $\theta$ = length of unwound string/radius of wheel

$= 2 \text{ m}/0.2 \text{ m} = 10 \text{ rad}$

$\omega^2 = 2 \times 12.5 \times 10.0 = 250(\text{rad/s})^2$

$\therefore$ K.E. gained $= \frac{1}{2} \times 0.4 \times 250 = 50 \text{ J}$

(D) The answers are the same, i.e. the kinetic energy gained by the wheel = work done by the force. There is no loss of energy due to friction.

## Conservation of angular momentum

If the external torque is zero,

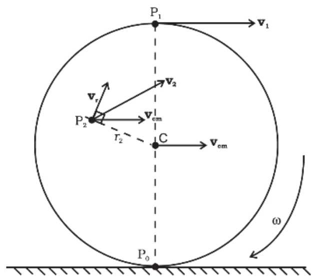
$L = I\omega = \text{constant}$

Sit on a swivel chair (a chair with a seat, free to rotate about a pivot) with your arms folded and feet not resting on, i.e., away from, the ground. Ask your friend to rotate the chair rapidly. While the chair is rotating with considerable angular speed stretch your arms horizontally.

Your angular speed is reduced. If you bring back your arms closer to your body, the angular speed increases again. This is a situation where the principle of conservation of angular momentum is applicable.

## Rolling motion

Rolling Motion of a body is a combination of both translational and rotational motion of a round shaped body placed on a surface. When a body is set in rolling motion, every particle of body has two velocities - one due to its rotational motion and the other due to its translational motion, and the resulting effect is the vector sum of both velocities at all particles.




Rolling Motion is classified in two categories - Pure Rolling and Rolling with Sliding. Pure rolling is a case when there is no relative motion at point of contact of rolling body and the surface; and body is considered to be rotating about this point of contact frame.

For a rolling disc –

$V_{cm}$ – Velocity of centre of mass of disc (Translational velocity of Disc)

$V_r$ – Velocity on account of Rotation

$V_r = rw$

For the condition of rolling

$V_{cm} = rw$

## Rotational Kinetic Energy

If the mass of $i^{th}$ particle is $m_i$ and its speed is $v_i$, its kinetic energy is

$$K_i = \frac{1}{2}m_i v_i^2$$

$$K_i = \frac{1}{2}m_i r_i^2 \omega^2$$

$$K_i = \frac{1}{2}(m_i r_i^2)\omega^2$$

$$K_i = \frac{1}{2}I\omega^2$$

KE of a rolling Body $\quad K = KE_R + KE_T$

$$K = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$$

$$K = \frac{1}{2}\frac{mk^2v_{cm}^2}{R^2} + \frac{1}{2}mv_{cm}^2$$

$$\text{or } K = \frac{1}{2}mv_{cm}^2\left(1 + \frac{k^2}{R^2}\right)$$

**Q.** Three bodies, a ring, a solid cylinder and a solid sphere roll down the same inclined plane without slipping. They start from rest. The radii of the bodies are identical. Which of the bodies reaches the ground with maximum velocity? **[NCERT Exercise]**

**Sol.** We assume conservation of energy of the rolling body, i.e. there is no loss of energy due to friction etc. The potential energy lost by the body in rolling down the inclined plane (= mgh) must, therefore, be equal to kinetic energy gained. (See Figure) Since the bodies start from rest the kinetic energy gained is equal to the final kinetic energy of the bodies. From

Eq. (7.49 b), $K = \frac{1}{2}mv^2\left(1+\frac{k^2}{R^2}\right)$, where v is the final velocity of (the centre of mass of) the body. Equating K and mgh.

$$mgh = \frac{1}{2}mv^2\left(1+\frac{k^2}{R^2}\right)$$

$$\text{or } v^2 = \left(\frac{2gh}{1+k^2/R^2}\right)$$

Note is independent of the mass of the rolling body;

For a ring, $k^2 = R^2$

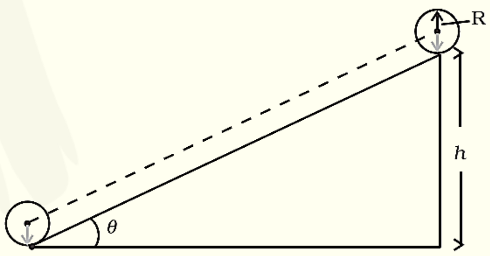
$$v_{ring} = \sqrt{\frac{2gh}{1+1}}$$

$$= \sqrt{gh}$$

For a solid cylinder $k^2 = R^2/2$

$$v_{disce} = \sqrt{\frac{2gh}{1+1/2}}$$

$$= \sqrt{\frac{4gh}{3}}$$




For a solid sphere $k^2 = 2R^2/5$

$$v_{sphere} = \sqrt{\frac{2gh}{1+2/5}} = \sqrt{\frac{10gh}{7}}$$

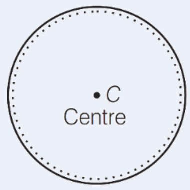
From the results obtained it is clear that among the three bodies the sphere has the greatest and the ring has the least velocity of the centre of mass at the bottom of the inclined plane.

## *NCERT Practice Questions*

**1.** For which of the following does the centre of mass lie outside the body ?

(A) A pencil (B) A shotput

(C) A dice (D) A bangle

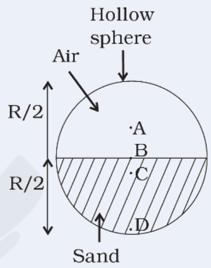



**Sol.** **(D)**

A bangle is in the form of a ring as shown in the adjacent diagram. The centre of mass lies at the centre, which is outside the body (boundary).

**2.** Which of the following points is the likely position of the centre of mass of the system shown in Figure?

(A) A (B) B (C) C (D) D




**Sol.** **(C)**

Centre of mass of a system lies towards the part of the system, having bigger mass. In the above diagram, lower part is heavier, hence CM of the system lies below the horizontal diameter.

**3.** A particle of mass m is moving in yz-plane with a uniform velocity v with its trajectory running parallel to +ve y-axis and intersecting z-axis at z = a (Figure). The change in its angular momentum about the origin as it bounces elastically from a wall at y = constant is:

(A) moa $\hat{e}_x$ (B) $2mva\hat{e}_x$ (C) $ymv\hat{e}_x$ (D) $2ymv\hat{e}_x$

**Sol.** **(B)**

The initial velocity is $v_i = v\hat{e}_y$ and after reflection from the wall, the final velocity is $v_f = -v\hat{e}_y$. The trajectory is described as position vector $r = y\hat{e}_y + a\hat{e}_z$.

Hence, the change in angular momentum is $r \times m(v_f - v_j) = 2mva\hat{e}_x$.

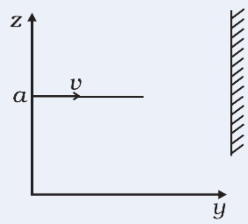
**4.** When a disc rotates with uniform angular velocity, which of the following is not true?

(A) The sense of rotation remains same.

(B) The orientation of the axis of rotation remains same.

(C) The speed of rotation is non-zero and remains same.

(D) The angular acceleration is non-zero and remains same.




**Sol.** **(D)**

We know that angular acceleration $\alpha = \frac{d\omega}{dt}$ given $\omega =$ constant

where $\omega$ is angular velocity of the disc $\Rightarrow \alpha = \frac{d\omega}{dt} = \frac{0}{dt} = 0$

Hence, angular acceleration is zero.

**5.** Choose the correct alternatives:

(A) For a general rotational motion, angular momentum L and angular velocity ω need not be parallel.

(B) For a rotational motion about a fixed axis, angular momentum L and angular velocity ω are always parallel.

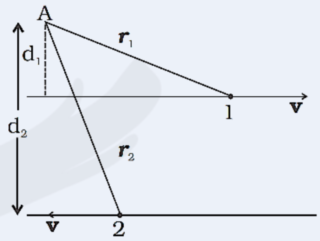
(C) For a general translational motion, momentum p and velocity v are always parallel.

(D) For a general translational motion, acceleration a and velocity v are always parallel.

**Sol.** **(A, C)**

For a general rotational motion, where axis of rotation is not symmetric. Angular momentum L and angular velocity ω need not be parallel. For a general translational motion momentum $p = mv$, hence, p and v are always parallel.

**6.** Figure shows two identical particles 1 and 2, each of mass m, moving in opposite directions with same speed v along parallel lines. At a particular instant, $r_1$ and $r_2$ are their respective position vectors drawn from point A which is in the plane of the parallel lines. Choose the correct options:




(A) Angular momentum $l_1$ of particle 1 about A is $l_1 = mvd_1$

(B) Angular momentum $l_2$ of particle 2 about A is $l_2 = mur_2$

(C) Total angular momentum of the system about A is $l = mv(r_1 + r_2)$

(D) Total angular momentum of the system about A is $l = mv(d_2 - d_1) \otimes$

represents a unit vector coming out of the page.

$\otimes$ represents a unit vector going into the page.

**Sol.** **(A, B)**

The angular momentum $L$ of a particle with respect to origin is defined to be $L = r \times p$ where, $r$ is the position vector of the particle and $p$ is the linear momentum. The direction of $L$ is perpendicular to both $dr$ and $p$ by right hand rule.

For particle 1, $I_1 = r_1 \times mv$, is out of plane of the paper and perpendicular to $r_1$ and $p(mv)$

Similarly $I_2 = r_2 \times m(-v)$ is into the plane of the paper and perpendicular to $r_2$ and $-p$.

Hence, total angular momentum $l = l_1 + l_2 = r_1 \times mv + (-r_2 \times mv)$

$|l| = mvd_1 - mvd_2$ as $d_2 > d_1$, total angular momentum will be inward

Hence, $I = m \vee (d_2 - d_1) \otimes$

**7.** The net external torque on a system of particles about an axis is zero. Which of the following are compatible with it?

(A) The forces may be acting radially from a point on the axis.

(B) The forces may be acting on the axis of rotation.

(C) The forces may be acting parallel to the axis of rotation.

(D) The torque caused by some forces may be equal and opposite to that caused by other forces.

**Sol.** **(A, B, C, D)**

We know that torque on a system of particles $\tau = r \times F = F\sin\,\theta\hat{n}$

where, $\theta$ is angle between $r$ and $F$, and $\hat{n}$ is a unit vector perpendicular to both $r$ and $F$.

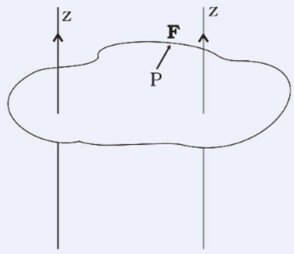
(A) When forces act radially, $\theta = 0$ hence $|\tau| = 0$ [from Eq. (i)]

(B) When forces are acting on the axis of rotation, $r = 0, |\tau| = 0$ [from Eq. (i)]

(C) When forces acting parallel to the axis of rotation $\theta = 0°, |\tau| = 0$ [from Eq. (i)]

(D) When torque by forces are equal and opposite, $\tau_{\text{net}} = \tau_1 - \tau_2 = 0$

**8.** Figure shows a lamina in $x - y$ plane. Two axes $z$ and $z'$ pass perpendicular to its plane. A force $F$ acts in the plane of lamina at point $P$ as shown. Which of the following are true?




(The point $P$ is closer to $Z'$-axis than the $Z$-axis.)

(A) Torque $\tau$ caused by $F$ about $z$ axis is along $-\hat{k}$.

(B) Torque $\tau'$ caused by $F$ about $z'$ axis is along $-\hat{k}$.

(C) Torque $\tau$ caused by $F$ about $z$ axis is greater in magnitude than that about $z$ axis.

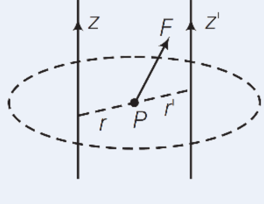
(D) Total torque is given be $\tau = \tau + \tau'$.

**Sol.** **(B, C)**

(A) Consider the adjacent diagram, where $r > r'$ Torque $\tau$ about $z$-axis $\tau = r \times F$ which is along $\hat{k}$

(B) $\tau' = r' \times F$ which is along $-\hat{k}$

(C) $|\tau|_z = Fr_\perp$ = magnitude of torque about z-axis where $r_\perp$ is perpendicular distance between $\bar{F}$ and z-axis.




Similarly, $|\tau|'_z = Fr'_\perp$

Clearly $r_\perp > r'_\perp \Rightarrow |\tau|_{z>}|\tau|_{z'}$

(D) We are always calculating resultant torque about a common axis.

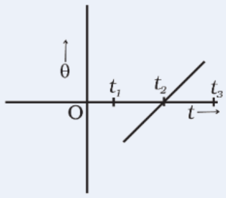
Hence, total torque $\tau \neq \tau + \tau'$, because $\tau$ and $\tau'$ are not about common axis.

**9.** The centre of gravity of a body on the earth coincides with its centre of mass for a 'small' object whereas for an 'extended' object it may not. What is the qualitative meaning of 'small' and 'extended' in this regard? For which of the following the two coincides? A building, a pond, a lake, a mountain?

**Sol.** When the vertical height of the object is very small as compared to the earth's radius, we call the object small, otherwise it is extended.
(i) Building and pond are small objects.
(ii) A deep lake and a mountain are examples of extended objects.

**10.** Why does a solid sphere have smaller moment of inertia than a hollow cylinder of same mass and radius, about an axis passing through their axes of symmetry?

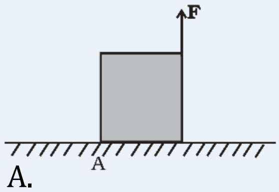
**Sol.** The moment of inertia of a body is given by $I = \Sigma m_i r_i^2$ [sum of moment of inertia of each constituent particles] All the mass in a cylinder lies at distance R from the axis of symmetry but most of the mass of a solid sphere lies at a smaller distance than R.




**11.** The variation of angular position θ, of a point on a rotating rigid body, with time t is shown in Figure. Is the body rotating clock-wise or anti-clockwise?

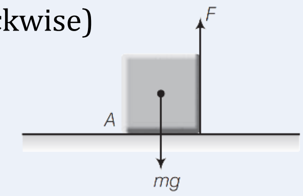
**Sol.** As the slope of $\theta - t$ graph is positive and positive slope indicates anti-clockwise rotation which is traditionally taken as positive.

**12.** A uniform cube of mass m and side a is placed on a frictionless horizontal surface. A vertical force F is applied to the edge as shown in Figure. Match the following (most appropriate choice):




| | |
|---|---|
| (A) $mg/4 < F < mg/2$ | (i) Cube will move up. |
| (B) $F > mg/2$ | (ii) Cube will not exhibit motion. |
| (C) $F > mg$ | (iii) Cube will begin to rotate and slip at A. |
| (D) $F = mg/4$ | (iv) Normal reaction effectively at a/3 from A, no motion. |

**Sol.** Consider the below diagram




Moment of the force F about point A, $\tau_1 = F \times a$ (anti-clockwise)
Moment of weight mg of the cube about point A, $\tau_2 = mg \times \frac{a}{2}$ (clockwise)
Cube will not exhibit motion, if $\tau_1 = \tau_2$
($\because$ In this case, both the torque will cancel the effect of each other)
$\therefore F \times a = mg \times \dfrac{a}{2} \Rightarrow F = \dfrac{mg}{2}$
Cube will rotate only when, $\tau_1 > \tau_2 \Rightarrow F \times a > mg \times \frac{a}{2} \Rightarrow F > \frac{mg}{2}$
Let normal reaction is acting at $\frac{a}{3}$ from point A, then
$mg \times \dfrac{a}{3} = F \times a$ or $F = \dfrac{mg}{3}$ (For no motion)
When $F = \frac{mg}{4}$ which is less than $\frac{mg}{3}$, $\left(F < \frac{mg}{3}\right)$ there will be no motion.
$\therefore$ (A) → (ii) (B) → (iii) (C) → (i) (D) → (iv)

**13.** The vector sum of a system of non-collinear forces acting on a rigid body is given to be non-zero. If the vector sum of all the torques due to the system of forces about a certain point is found to be zero, does this mean that it is necessarily zero about any arbitrary point?

**Sol.** No, not necessarily.

Given,

$$\sum_i F_i \neq 0$$

The sum of torques about a certain point O,

$\sum_i r_i \times F_i = 0$

The sum of torques about any other point O′

$$\sum_i (r_i - a) \times F_i = \sum_i r_i \times F_i - a \times \sum_i F_i$$

Here, the second term need not vanish.

Therefore, sum of all the torques about any arbitrary point need not be zero necessarily.

**14.** A wheel in uniform motion about an axis passing through its centre and perpendicular to its plane is considered to be in mechanical (translational plus rotational) equilibrium because no net external force or torque is required to sustain its motion. However, the particles that constitute the wheel do experience a centripetal acceleration directed towards the centre. How do you reconcile this fact with the wheel being in equilibrium? How would you set a half-wheel into uniform motion about an axis passing through the centre of mass of the wheel and perpendicular to its plane? Will you require external forces to sustain the motion?

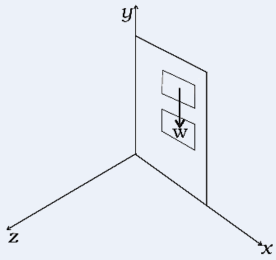
**Sol.** Wheel is a rigid body. The particles that constitute the wheel do experience a centripetal acceleration directed towards the centre. This acceleration arises due to internal elastic forces, which cancel out in pairs.

In a half wheel, the distribution of mass about its centre of mass (through which axis of rotation passes) is not symmetrical.

Therefore, the direction of angular momentum of the wheel does not coincide with the direction of its angular velocity.

Hence, an external torque is required to maintain the motion of the half wheel.

**15.** A door is hinged at one end and is free to rotate about a vertical axis (Figure). Does its weight cause any torque about this axis? Give reason for your answer.




Consider the diagram, where weight of the door acts along negative y-axis.

A force can produce torque only along a direction normal to itself as $\tau = r \times F$. So, when the door is in the xy-plane, the torque produced by gravity can only be along $\pm$ z direction, never about an axis passing through y-direction.

# Ab Phod do !!!!