Kinetic energy, $T = \frac{1}{2} ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2)$

Potential energy, $V_i = mgl\cos\theta_i$ (due to gravity)

$\Rightarrow$ for small angles, $\cos\theta \approx 1 - \frac{\theta^2}{2}$

$\Rightarrow V_i = mgl\left(1 - \frac{\theta_i^2}{2}\right)$

The term mgl is a constant, and can be ignored

$\Rightarrow V_i \approx \frac{1}{2} mgl\,\theta_i^2$

$\Rightarrow V = \frac{1}{2} mgl(\theta_1^2 + \theta_2^2 + \theta_3^3)$

$\therefore$ We have the Lagrangian, $\mathcal{L} = T - V$

* This is missing the spring energy term

$\therefore V_{spring} = \frac{1}{2} k[(\theta_1 - \theta_2)^2 + (\theta_2 - \theta_3)^2]$

$\therefore \mathcal{L} = T - V - V_{spring}$

$$\mathcal{L} = \frac{1}{2} ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2) - \frac{1}{2} mgl(\theta_1^2 + \theta_2^2 + \theta_3^2) - \frac{1}{2} k[(\theta_1 - \theta_2)^2 + (\theta_2 - \theta_3)^2]$$

Now, we can derive the equations of motion

* for θ_1:

$$\frac{\partial L}{\partial \theta_1} = -\frac{1}{2} mgl \times 2\theta_1 - \frac{1}{2} k(\theta_1-\theta_2)\times 2$$

$$\Rightarrow \frac{\partial L}{\partial \theta_1} = -mgl\theta_1 - k(\theta_1-\theta_2) \quad \text{—— (i)}$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = \frac{1}{2} ml^2 2\dot{\theta}_1 = ml^2\dot{\theta}_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = ml^2\ddot{\theta}_1 \quad \text{— (ii)}$$

Using Euler-Langrange equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0$$

$$\Rightarrow ml^2\ddot{\theta}_1 - (-mgl\theta_1 - k(\theta_1-\theta_2)) = 0$$

$$\Rightarrow \ddot{\theta}_1 = -\left[\frac{mgl\theta_1 + k(\theta_1-\theta_2)}{ml^2}\right] \quad \text{—— (iii)}$$

* for θ_2:

$$\frac{\partial L}{\partial \theta_2} = -\frac{1}{2} mgl\times 2\theta_2 - \frac{1}{2}k\left[2(\theta_1-\theta_2)(-1) + 2(\theta_2-\theta_3)\right]$$

$$\frac{\partial L}{\partial \theta_2} = -mgl\theta_2 - k\left[(\theta_2-\theta_1)+(\theta_2-\theta_3)\right] \quad \text{— (iv)}$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = ml^2\dot{\theta}_2 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = ml^2\ddot{\theta}_2$$

$$\therefore \ddot{\theta}_2 = -\left(\frac{mgl\theta_2 + k(2\theta_2 - \theta_1 - \theta_3)}{ml^2}\right) \quad \text{— (v)}$$

For θ_3:

$$\frac{\partial L}{\partial \theta_3} = -\frac{1}{2}mgl\cdot 2\theta_3 - \frac{1}{2}k\, 2(\theta_2-\theta_3)(-1)$$

$$\frac{\partial L}{\partial \theta_3} = -mgl\theta_3 - k(\theta_3-\theta_2) \quad \text{— (vi)}$$

$$\frac{\partial L}{\partial \dot{\theta}_3} = ml^2\dot{\theta}_3 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_3}\right) = ml^2\ddot{\theta}_3$$

$$\Rightarrow \ddot{\theta}_3 = -\frac{mgl\theta_3 + k(\theta_3-\theta_2)}{ml^2} \quad \text{— (vii)}$$

$\therefore$ we have the following equations of motion:

$$\ddot{\theta}_1 = -\left[\frac{mgl\theta_1 + k(\theta_1-\theta_2)}{ml^2}\right]$$

$$\ddot{\theta}_2 = -\left[\frac{mgl\theta_2 + k(2\theta_2-\theta_1-\theta_3)}{ml^2}\right]$$

$$\ddot{\theta}_3 = -\left[\frac{mgl\theta_3 + k(\theta_3-\theta_2)}{ml^2}\right]$$

Rewriting the in matrix form, we get $\Rightarrow$

$$\begin{pmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\\ \ddot{\theta}_3\end{pmatrix} = -\frac{1}{ml^2}\begin{pmatrix} mgl+k & -k & 0 \\ -k & mgl+2k & -k \\ 0 & -k & mgl+k\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix}$$

which equivalent to

$$\boxed{\ddot{\theta} = A\theta}$$