Skills, Education, and Family Background: Solving a Dynamic Life Cycle Model of Education Choice

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1 Introduction

In their paper Education Policy and Intergenerational Transfers in Equilibrium, Abbot, Gallipoli, Meghir & Violante (2019) show the importance of budget constraints and transfers for educational attainment in a general equilibrium model. However their model is large and has many mechanisms through which binding budget constraints can affect education choice. As an exercise we try to set up a simpler model, that is able to capture the main mechanism in their paper. Furthermore we check whether we are still able to estimate some parameters of interest using simulated method of moments (SMM) on simulated data.

Our model is a smaller partial equilibrium model that uses a life-cycle framework. The model divides the life of an individual into two stages, one where the individual is studying followed by a stage where the individual works. The decision to transition from the study stage to the working stage is endogenous and therefore affected by the budget constraint and transfers received when studying. Furthermore it includes heterogeneous agents that differ in whether they have a rich or poor background and whether or not they have high cognitive abilities which affects the return to education.

To solve the model efficiently we use an Endogenous Grid Method (EGM) framework. EGM due to Carroll (2006) has two desirable properties, namely that it avoids computationally costly root finding and is very precise. However since our model has one continuous and one discrete choice variable in the study stage of life and two continuous variables in the working stage of life the original EGM algorithm cannot be applied. Instead we use two extensions of EGM that in most cases retain both desirable properties of the original EGM algorithm. Namely Dicrete Choice EGM (DC-EGM) due to Iskhakov et al. (2017) and a extension that is reminiscent of the method used in Druedahl and Jørgensen (2017) that exploits the additive separable utility function in the model.

Using Monte Carlo experiments and data moments on educational attainment we show that it is possible to estimate the shares of individuals with high cognitive abilities, conditional on observing their family background and transfers received in the study stage. When we try to estimate the transfers received by students with a rich family background jointly with the conditional distribution of cognitive abilities we cannot, however, identify the parameters with moments on educational attainment. This is because the share of rich agents with high cognitive abilities has the same effect on the moments used as transfers received.

2 Model

In this section we describe the economic environment of the model, the objective functions of the individuals and the major departures from the model in Abbott et al. (2019).

Overview The agent is initialized at age 20 and lives for $T = 45$ discrete time periods, each corresponding to a year. Each agent is characterized by his skill, $\theta_i \in \{\theta_{high}, \theta_{low}\}$, and his family background, $\phi_i \in \{\phi_{high}, \phi_{low}\}$, both of which remain fixed throughout his life-cycle. The agent first enters the education stage of life, throughout which he makes one discrete choice on whether to continue education or not and a continuous consumption choice. During the education stage, the agent's income consists of returns on his assets and a transfer from his parents. In the spirit of Abbott et al. (2019), individuals from a rich background receive income transfers from their parents during studying. The agent finishes his education once he chooses to do so or he reaches an exogenous maximum education level, after which he transitions to the absorbing working stage. Here, the agent's problem consists of two continuous choices, namely how much to consume and how much labor to supply. During this stage, his income consists of returns on assets and labor income. The wage depends on his educational attainment in interaction with his cognitive skill. At the terminal period $T = 45$, the agent retires. We incentivize savings for retirement by specifying a closed form continuation value as a function of assets in the terminal period.

Education stage In the education stage, the agent's state space Ω_t consists of his assets by the beginning of period t, m_t, his cognitive type θ_i, family type ϕ_i and his current accumulated education S_t.

We model the agent's utility of consumption, u_c as a CRRA utility function:

$$u_c(c_t) = \frac{c_t^{1-\rho}}{1-\rho}$$

Where c_t is consumption in period t and ρ governs the risk aversion of the agent.

Conditional on studying, the agent's problem is the following:

$$
\begin{aligned}
V_{t,study}(\Omega_t) &= \max_{c_t} \frac{c_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}_t \left[V_{t+1}(\Omega_{t+1}) \right] \\
s.t. \quad & m_{t+1} = (1+r)(m_t + t(\phi_i) - c_t) \\
& S_{t+1} = \sum_{\tau=0}^{t} s_t \\
& a_t \geq \underline{\mathrm{a}}
\end{aligned} \tag{1}
$$

Where $V_{t,study}(\cdot)$ is the choice specific value function conditional on studying, $\mathbb{E}(V_{t+1}(\cdot))$ is the expectation to the overall value function of the agent in period t+1 conditional on information available in period t, r is the real rental rate, $t(\phi_i)$ are the transfers received that depends on family background $\phi \in \{\phi_{low}, \phi_{high}\}$, S is accumulated years of education, s_t is an indicator equal to 1 if the agents decide to study in period t, a_t is end of period savings and $\underline{\mathrm{a}}$ is a credit constraint.

Working stage After finishing education, the agent enters the working stage of life, in which they choose continuous consumption and labor supply. We assume consumption and labor are additively separable in utility, such that utility is:

$$
u(c_t, \ell_t) = u_c(c_t) + u_\ell(\ell_t) = \frac{c_t^{1-\rho}}{1-\rho} - \vartheta \frac{\ell^{1+\nu}}{1+\nu}
$$

Where ℓ is labor supply, ϑ is a parameter that governs the disutility of working and ν is the inverse of the Frisch elasticity.

When working, the agent solves the following problem:

$$
\begin{aligned}
V_{t,work}(\Omega_t) &= \max_{c_t, \ell_t} \frac{c_t^{1-\rho}}{1-\rho} - \vartheta \frac{\ell^{1+\nu}}{1+\nu} + \beta \mathbb{E}_t \left[V_{t+1,work}(\Omega_{t+1}) \right] \\
s.t. \quad & m_{t+1} = (1+r)(m_t - c_t + w_t(S_t, \theta_i)\ell_t) \\
& \log(w_t(S_t, \theta_i)) = \log(\theta_i)\lambda_{S_t} + \epsilon_t \\
& a_t \geq \underline{\mathrm{a}} \\
& \epsilon \sim N(0, \sigma_\epsilon)
\end{aligned}
$$

Where agents receive a wage, w_t, which depends positively on their educational attainment and their cognitive skill in interaction, such that the return to education is higher for those of high cognitive skill. With the raw education premium given by λ_{S_t} and the raw return to cognitive skill given by $\log(\theta_i)$. The wage also contains a normally distributed stochastic component, ϵ_t, making the wage log-normally distributed conditional on education and skill. We assume for simplification that ϵ_t is independent over

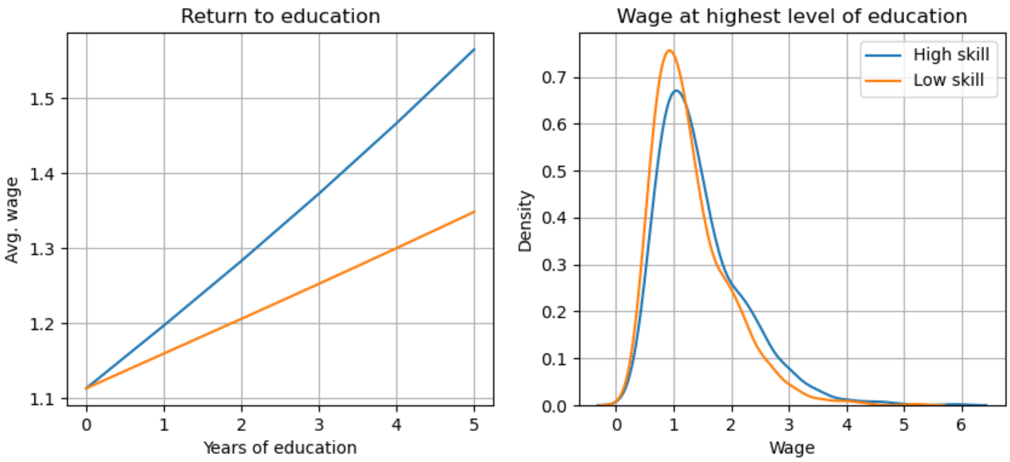


Figure 1: Wages and education

time. The exact functional form of the wage function is a simplified version of the wage in Abbott et al. (2019).

In the final period $T = 45$, the agent retires. While Abbott et al. (2019) explicitly models retirement behavior, we instead specify a continuation value capturing the value of retirement following the approach of Gourinchas and Parker (2002):

$$V_{T+1}(m_{T+1}) = \kappa \frac{m_{T+1}^{1-\rho}}{1-\rho}$$

Where κ is a parameter that governs the strength of the incentive to save for retirement. By including a retirement value, we ensure that agents still have incentive to save for retirement, even if we do not model consumption during retirement.

Bellman Equation The Bellman equation for the problem is:

$$V_t(\Omega_t) = \max_{s_t \in \mathcal{S}(\Omega_t)} s_t \left[V_{t,study}(\Omega_t) + \Delta_{t,study}\right] + (1 - s_t) \left[\mathbb{E}_\epsilon[V_{t,work}(\Omega_t)] + \Delta_{t,work}\right]$$

$$s.t. \quad \mathcal{S}(\Omega_t) = \begin{cases} \{0, 1\} & \text{if } s_{t-1} = 1 \text{ and } S_t < S_{max} \\ \{0\} & \text{else} \end{cases}$$

where $\Delta_{t,study}$ and $\Delta_{t,work}$ each are realizations of a i.i.d. type I extreme value distributed variable. These represent choice-specific tastes shocks. The expectation operator $\mathbb{E}_\epsilon$ over the value conditional on working is taken with respect to income shocks - in other words, we assume the agent does not observe the income shock ϵ before making the choice to enter the working stage.

While studying, the discrete choice set is $\{0, 1\}$, but upon entering the working stage, the choice set shrinks to $\{1\}$ - hence, the discrete choice essentially disappears and the working stage is absorbing.

2.1 Comparison with Abbott paper

With this model, we aim to replicate one of the many mechanisms in the paper by Abbott et al. (2019). Specifically, we are interested in how the interaction between different cognitive skills and different credit constraints affect educational attainment. To be able to focus on this specific mechanism, we have stripped away a number of components from the large model of Abbott et al. (2019).

Most notably, Abbott et al. (2019) build a general equilibrium model, where wages are determined on the labor market by the supply and demand for different educational groups. Their model is also one of overlapping generations, and children receive transfers depending on their parent's altruism for them. In our model, wages and transfers are completely exogenous, conditional on educational choices. While general equilibrium effects and parental altruism are detrimental for the incentives to study, we abstract away from them in order to better focus on partial equilibrium outcomes. We have also drastically simplified financial markets by introducing strict borrowing constraints for all individuals, regardless of their type.

Furthermore, we only model an education stage and a working stage, and the main incentive to study comes from the fact that the education raises wages in the working period. Abbott et al. (2019) also include a marriage stage and a retirement stage, with higher education giving advantages in both of these stages. We abstract away from marriage markets and only superficially model retirement, as we believe the main incentive to education lies in the working stage of life anyway.

Moreover, Abbott et al. (2019) include more sources of heterogeneity than we do, eg. through gender specific preference parameters and age specific wage profiles. We limit the sources of heterogeneity to family types, cognitive types and later in life, different levels of educational attainment. We also abstract away from different psychological costs of education, allowing us to focus only on incentives from current credit constraints and future wages. Finally, we have abstracted away from labor supply during study in order to amplify the importance of different transfer sizes between types.

In other words, we have attempted to create the simplest possible model which still includes some important mechanisms behind the results from Abbott et al. (2019).

3 Solving and simulating the model

3.1 EGM

To solve the model in the working stage, we rely on EGM as originally proposed by Carroll (2006). The method relies on reformulating the problem in terms of a post-decision state, which is a sufficient statistic for the next period's state. Unlike other solution methods such as value function iterations or time iterations, EGM relies on inter-temporal first order conditions, Euler-equations. If the Euler-equations are analytically invertible, EGM can be carried out without any root-finding operations, which significantly speeds up computation time.

Consider the following formulation of the working stage model:

$$
\begin{aligned}
V_{t,work}(\Omega_t, a_t) = \max_{c_t,\ell_t} & \frac{c_t^{1-\rho}}{1-\rho} - \vartheta\frac{\ell^{1+\nu}}{1+\nu} + \mathbb{E}_t\left[V_{t+1,work}(\Omega_{t+1}\right] \\
s.t. \quad & a_t = (1+r)a_{t-1} - c_t + w_t(S_t, \theta_i)\ell_t \\
& \log(w_t(S_t, \theta_i)) = \log(\theta)\lambda_{S_t} + \epsilon \\
& a_t \geq \underline{\mathrm{a}} \\
& \epsilon \sim N(0, \sigma_{epsilon})
\end{aligned}
$$

where $a_t = m_t + w_t(S_t, \theta) - c_t$, and $m_{t+1} = (1+r)a_t$. Here, the model is formulated in terms of the end of period state, a_t, rater than the beginning of period state m_t. Note, that since $m_{t+1} = (1+r)a_t$, a_t is a sufficient statistic for m_{t+1}, which allows ud to apply EGM as a solution method.

One complication is that contrary to the original model by Carroll (2006), the working stage model has two continuous choice variables - c_t and ℓ_t. We therefore rely on an extended EGM-method, which uses both the consumption Euler equation as well as the intratemporal first order condition to pin down c_t and ℓ_t. A similar method is used in eg. Druedahl and Jørgensen (2017).

The following sections presents the Euler equations for consumption and labor supply as well as the intratemporal first order condition for the working stage model and details how we apply EGM to solve the model.

3.1.1 Consumption and labor Euler-equations

During the working stage of the life cycle, the agent's behavior can be described by Euler equations for consumption and labor:

$$
c_t^{-\rho} = \beta(1+r)\mathbb{E}_t\left[c_{t+1}^{-\rho}\right] \tag{1}
$$

$$
\ell_t^{\nu} = \beta(1+r)\mathbb{E}_t\left[\frac{w_(S_t, \theta_i)}{w_{t+1}(S_{t+1}, \theta_i)}\ell_{t+1}^{\nu}\right] \tag{2}
$$

Furthermore, the agent substitutes between consumption and labor according to the intra-temporal first order condition:

$$
\frac{\vartheta}{w_t(S_t, \theta)}\ell_t^{\nu} = c_t^{-\rho} \tag{3}
$$

See appendix A for derivations. These equations yield two important realizations. Firstly, given that the Euler equations are sufficient conditions for the optimum, knowing consumption in one period pins down the entire consumption path throughout the life cycle through sequential period Euler equations. The same holds for labor. Secondly, the additive separability of consumption and labor in the utility function leads to the two respective Euler equations being independent of each other. This means that we can either solve the model by solving for each path separately, or we can solve for one and use the intra-temporal first order condition to pin down the other. We end up doing the latter, using the fact that we can express ℓ_t as a function of c_t in closed form (3).

3.1.2 The EGM algorithm

The EGM algorithm solves the model by looping backwards in time and over an exogenous grid of post-decision states, conditional on discrete types. Note that we use capital letter X_t to denote a grid over the variable x_t. Given an exogenous end of period state a_t, the next period state m_{t+1} follows directly and next period consumption c_{t+1} can be interpolated from the next period policy function. By inverting the consumption Euler equation (1) we can compute the current consumption c_t from c_{t+1}, and next current labor supply from the intratemporal FOC (3). Finally, the solution allows us to compute endogenous beginning of period state $m_t^{endo} = a_t + c_t - w_t(S_t, \theta_i)\ell_t$. Doing this over a grid A_t over exogenous end of period states yields a grid of beginning of period states M_t^{endo} and corresponding grids over consumption C_t^{endo} and labor L_t^{endo}. The fact that the consumption Euler equation is analytically invertible and labor can be expressed as a closed form function of consumption means that the approach can be carried out completely without root finding, making the solution very fast.

Imposing the budget constraint The above approach assumes an interior solution by relying on the Euler equations and does not hold if agents are credit constrained or otherwise in a corner solution. EGM generally handles budget constraints well, particularly in the standard single-choice model. By letting the first grid point in the exogenous end-of-period state grid A_t be equal to the credit constraint $\underline{a}$, the first grid point in the endogenous grid M_t^{endo} gives the exact point in the beginning-of-period state where the agent lets go of the credit constraint, m_t^*. For any state $m_t < m_t^*$, optimal consumption is therefor just to consume everything, and in this segment the policy function is linear.

Our introduction of labor as a continuous choice, however, complicates matters. Specifically, since agents can now work to gain income, we can no longer guarantee that agents are credit constrained, even when $m_t = \underline{m}$ - they could just work enough to finance both consumption and some savings. Moreover, we can no longer rely on linear interpolation if the budget constraint binds, since the agent still faces a trade-off between leisure and consumption. To handle this, we do two things.

Firstly, between each iteration in the EGM algorithm, we interpolate the results found over the endogenous grid M_t^{endo} back to a pre-specified exogenous grid M_t, shown above in the extra step in step 8. We do this because the value m_t^* that makes agents credit constrained differs between periods, and is only sometimes positive. In periods where m_t^* is positive, we would traditionally add grid points to the grid to cover the credit constrained segment. We choose to instead interpolate back to an exogenous grid. By doing this, we avoid having to add grid points in some periods but not in others, keeping all solution grids within the working period of the same size.

Secondly, we compute an endogenous end-of-period state a_t^{endo}, which we use to inspect if agents are credit constrained. If $a_t^{endo} < 0$, we can no longer use the Euler equations to pin down consumption and labor. However, the intratemporal first order condition must still hold. In these cases, we find the optimal choice by simultaneously imposing the intratemporal first order condition and letting the budget constraint bind:

$$
\begin{aligned}
c_t &= m_t + w(S_t, \theta)\ell_t \\
\ell_t &= \left(\frac{w_t(S_t, \theta)}{\vartheta} c_t^{-\rho} \right)^{\frac{1}{\nu}}
\end{aligned}
$$

This system of equations simultaneously pins down c_t and ℓ_t in the cases where the budget constraint binds. We solve the system using a root finder.

The exact algorithm for solving the working stage is described below:

1. Fix an exogenous grid over post-decision states, A_t as well as an exogenous grid over beginning of period states M_t.
2. Solve the final period T for each element m_t in M_t by conventional methods.
3. Set $t = T - 1$.
4. For each quadrature node over income shocks ϵ_t, do:
    - For each grid point a_t in the exogenous grid A_t, do:
        - Compute $m_{t+1} = (1 + r)a_t$
        - Linearly interpolate consumption in the following period, $\check{c}_{t+1}$.
        - Back out present consumption from the Euler equation (1), $c_t^{endo} = \left[\beta(1+r)\mathbb{E}_t(\check{c}_{t+1}^{-\rho})\right]^{-\frac{1}{\rho}}$, where the expectation is taken with respect to next period income shocks using Gauss-Hermite quadrature.
        - Use the intra-temporal first order condition (3) to pin down labor $\ell_t^{endo} = \left(\frac{w_t(S_t,\theta)}{\vartheta} c_t^{endo^{-\rho}}\right)^{\frac{1}{\nu}}$
        - Compute the endogenous grid point $m_t^{endo} = a_t + c_t^{endo} - w_t^{endo}(S_t, \theta)\ell_t$
    - *Extra step*: For each point m_t in the exongenous grid, M_t, do:
        - Linearly interpolate consumption c_t from the EGM solution C_t^{endo}.
        - Linearly interpolate consumption ℓ_t from the EGM solution L_t^{endo}.
        - Compute endogenous end of state assets, $a_t = m_t + w_t(S_t, \theta_i)\ell_t - c_t$.

– If $a_t < 0$, do:
  * Set $a_t = 0$
  * Impose budget constraint by solving $c_t = w(S_t, \theta_i)\ell_t$ and $\ell_t = \left(\frac{w_t(s_t,\theta_i)}{\vartheta} c_t^{-\rho}\right)^{\frac{1}{\nu}}$

- Store solution grids, C_t, L_t, A_t and M_t
- Compute the value $V_t = \frac{C_t^{1-\rho}}{1-\rho} + \vartheta \frac{L_t^{1+\nu}}{1+\nu} + \beta \mathbb{E}_t[V_{t+1}]$, where the expectation is over next period income shocks, computed using Gauss-Hermite quadrature.

5. Set $t = t - 1$ and return to step 4. Repeat until the beginning period is reached.

3.2 DC-EGM

Due to the speed of the EGM algorithm, we also want to use it to solve the study stage part of the model. However because the study stage involves a discrete choice, the value function of a student will in general not be globally concave in the direction of the consumption choice in the periods before the last possible study period. The implication of this is that the Euler equation for students is no longer a sufficient condition but only a necessary condition for global maximum. Thus for some levels of start of period assets multiple consumption levels can be local maximands of the student's value function. To handle this problem we use a version of the DC-EGM algorithm due to Iskhakov et al. (2017). DC-EGM introduces another step to the EGM algorithm that finds the global extrema of the value function and the associated level of consumption.

Since all the solution grids in all periods for the choices of the worker are found using the algorithm described above, DC-EGM is only used to solve for the solution grids conditional on studying. This section therefore first describes the Euler equation of the student, conditional on choosing to study. Then the section describes the problem caused by having a discrete choice and how to solve the problem before describing the entire DC-EGM algorithm used in this paper.

3.2.1 The Euler equation of the student

The intertemporal Euler equation for the student, conditional on studying is given by:

$$c_{t,study}^{-\rho} = \beta(1+r)\left(c_{t+1,study}^{-\rho} P_{t+1}(study|\Omega_t) + E_t\left[c_{t+1,work}^{-\rho}\right] P_{t+1}(work|\Omega_t)\right) \quad (4)$$

Where $c_{t+1,study}$ is the consumption level in period $t+1$ conditional on studying, and $c_{t+1,working}$ is the consumption level in period $t+1$ conditional on working, and $c_{t,study}$ is the consumption level of the student conditional on studying in period in period t. Note that since income when working is stochastic, so is $c_{t+1,work}^{-\rho}$, and hence it is its expectation which enters the Euler equation. Furthermore the conditional probability of the student choosing to study in the next period is given by

$$P_{t+1}(study|\Omega_{t+1}) = \exp\left(\frac{V_{t+1,study}(\Omega_{t+1})}{\sigma_\Delta}\right)\left(\exp\left(\frac{\mathbb{E}[V_{t+1,work}(\Omega_{t+1})]}{\sigma_\Delta}\right) + \exp\left(\frac{V_{t+1,study}(\Omega_{t+1})}{\sigma_\Delta}\right)\right)^{-1}$$

And the choice probability of a student choosing to work in the next period is given by

$$P_{t+1}(work|\Omega_{t+1}) = \exp\left(\frac{\mathbb{E}[V_{t+1,work}(\Omega_{t+1})]}{\sigma_\Delta}\right)\left(\exp\left(\frac{\mathbb{E}[V_{t+1,work}(\Omega_{t+1})]}{\sigma_\Delta}\right) + \exp\left(\frac{V_{t+1,study}(\Omega_{t+1})}{\sigma_\Delta}\right)\right)^{-1}$$

Derivations can be found in appendix B. (4) is sometimes refereed to as a smoothed Euler Equation as the extreme value type one distributed taste shocks make the decision to enter the work force probabilistic rather than deterministic, which makes the right hand side a weighted average of conditional marginal utilities of consumption.

3.2.2 The propagation of kinks backwards in time

To understand why we need to introduce another step to the EGM algorithm when there is a discrete choice, it is a good idea to think about the model as if it had no taste shocks, that is $\sigma_\Delta \to 0$ and look at the model from the last period where it is possible to study, $T^{s_{max}}$.

In period $T^{s_{max}}$ the agent knows that he cannot continue to study in the next period, $T^{s_{max}}+1$, so $P_{t+1}(work|\Omega_{t+1}) = 1$ and $P_{t+1}(study|\Omega_{t+1}) = 0$. However the agent can still choose whether or not to study within the period $T^{s_{max}}$. In this case normal EGM will not have any problems solving for the choice specific consumption functions $c_{t,study}(\Omega_t)$, which could then be used to find the decision specific value function $V_{t,study_t}(\Omega_t)$.

After finding the decision specific value function, one would then be able to find the intersection between $V_{t,study}(\Omega_t)$ and $V_{t,work}(\Omega_t)$ by interpolation, after which one can find the overall value function $V_t(\Omega_t)$ by taking the upper envelope over the two choice specific value functions. The consumption function $c_t(\Omega_t)$ would likewise be defined by either $c_{t,study}(\Omega_t)$ or $c_{t,work}(\Omega_t)$ dependent on whether m_t is to the right or to the left of the intersection between the choice specific value functions. The important thing to understand, is that the intersection between the two choice specific value functions generates a kink in the overall value function referred to as a primary kink which propagates back in time and causes problems that requires the introduction of an additional step.

Say that the primary kink point in $T^{s_{max}}$ happens at $\bar{m}$ and that the level of end of period assets in period $T^{s_{max}} - 1$ that ensures $\bar{m}$ in period $T^{s_{max}}$ is given by $\bar{a} = \frac{\bar{m}}{1+r}$. Then in period $t = T^{s_{max}} - 1$ when the EGM goes through the elements of the exogenous end of period assets grid A_t, it will generate the choice specific consumption function by inverting the Euler equation above but with $P_{t+1}(work, \Omega_{t+1}) = 1$ when $a^j <= \bar{a}$ and $P_{t+1}(study, \Omega_{t+1}) = 1$ when $a^j > \bar{a}$ with j marking the relative position of an element in the exogenous grid. The result is that the Euler changes for the points following $\bar{a}$. Assuming that students have a higher marginal utility of consumption, reflecting a lower level of consumption while studying, this can generate a drop in consumption, that is $c_t^{j+1} < c_t^j$, which also implies that the endogenous grid can become non-monotonic, that is $m_t^{j+1} < m_t^j$

Figure 2 sketches the situation for the choice specific value function of the student. The numbers indicate the relative ordering of the points. The number 4 for an example indicates the consumption level that ensures end of period assets equal to the fifth element in A_t. Until the point $j = 3$ EGM will generate a monotonically increasing consumption function and a monotonically increasing endogenous grid. However as soon as $a_t^j > \bar{a}$ the consumption level will drop and the endogenous grid point m_t^4 will decrease compared to m_t^3 with the implication that the graph bends backwards. From $j = 4$ EGM will again generate monotonically increasing consumption and endogenous grid. The implication is that EGM will generate a consumption correspondence rather than a consumption function. Intuitively this happens because for some m_t^j the agent faces a trade off between high consumption and working in the next period or low consumption such that the agent has enough savings to continue to study in the next period.

However as we have assumed a concave utility function then theoretically the savings function $a_{t,study}(\Omega_t) = m_{t,study}(\Omega_t) - c_{t,study}(\Omega_t)$ should be non-decreasing. This is not the case if the endogenous grid is non-monotonic, which implies that the solution which will be found by EGM will not characterize optimal behavior.

Furthermore it is important to note that what is described above only holds for $T^{s^{max}} - 1$. When the EGM algorithm goes further back in time there will an accumulation of kinks with the implication that the consumption path will bend backwards several times.

3.2.3 The Secondary upper envelope

To get a solution that can be said to characterize the behavior of a rational economic agent we need to find out which solution to the Euler represents the global optima. That is we need to turn the consumption correspondence generated by EGM and illustrated above into a consumption function.

To do this we call another upper envelope algorithm, referred to as the secondary upper envelope which is reminiscent to the algorithm presented in Iskhakov et al. (2017) but different in the sense that it does not remove any points in the endogenous grid. This algorithm is described below.

Given the solution arrays for the choice specific consumption and value functions conditional on studying from the normal EGM the secondary upper envelope algorithm does:

- Sort the endogenous grid in ascending order to get the refined endogenous grid $M^r_{t,study}$.
- Sort the consumption grid and the value function grid such that the relative order of the elements, given by i, corresponds to the relative order of their associated value of $m_t^i \in M^r_{t,study}$. Yielding

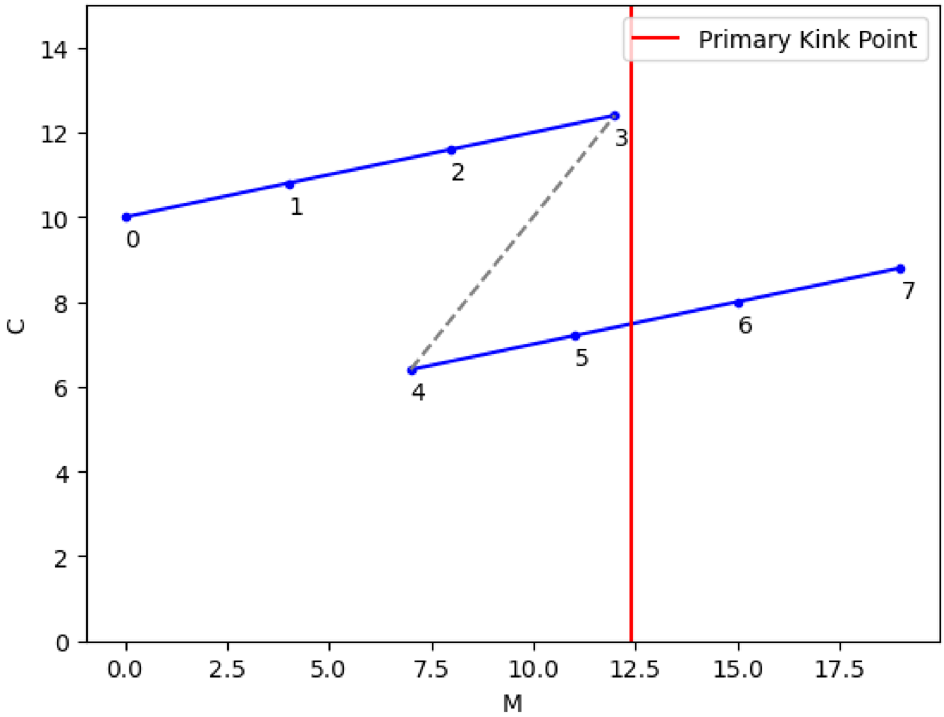


Figure 2: Naive EGM

Note: This is only a sketch of the problem, it does not come from actual code

$C^r_{t,study}$ and $V^r_{t,study}$

- Starting from the j=0 element in the non refined endogenous grid $M_{t,study}$:
  - Fix m^j_t and m^{j+1}_t
  - For all i such that $m^j_t \leq m^i_t \leq m^{j+1}_t$
    * Calculate $\check{c}_t = c^j_t + \frac{c^{j+1}_t - c^j_t}{m^{j+1}_t - m^j_t} * (m^i_t - m^j_t)$
    * Calculate $\check{v}_t$ by inserting $\check{c}_t$ into (1) alongside m^i_t
    * If $\check{v}_t > v^i_t \in V^r_{t,study}$ then replace v^i_t and $c^i_t \in C^r_{t,study}$ with $\check{v}_t$ and $\check{c}_t$ respectively
- return $C^r_{study,t}, V^r_{study,t}, M^r_{study,t}$

The algorithm is illustrated in figure 3 for period $t = T^{s_{max}} - 1$. The graph in the top left part of the figure shows the choice specific value function that could be generated by EGM. It itself also being a correspondence. However notice that the two lines of the choice specific value function intersect. This point is called a secondary kink point and it is a direct consequence of the primary kink in period $T^{S_{max}}$. At the secondary kink point the two solutions to the Euler equation are equally good, since the two value functions have the same value, while to the left the solution to the Euler that stems from $P_{t+1}(work|\Omega_{t+1}) = 1$ dominates and to the right it is the solution that stem from $P_{t+1}(study|\Omega_{t+1}) = 1$ that dominates. This is for example seen in the top right graph of the figure. Notice that for m^1_t and m^2_t of the unrefined grid, there is a point, m^4_t which falls in between their numerical values. However when looking at the top left graph it can be seen that the point v^4_t is dominated by the line segment between

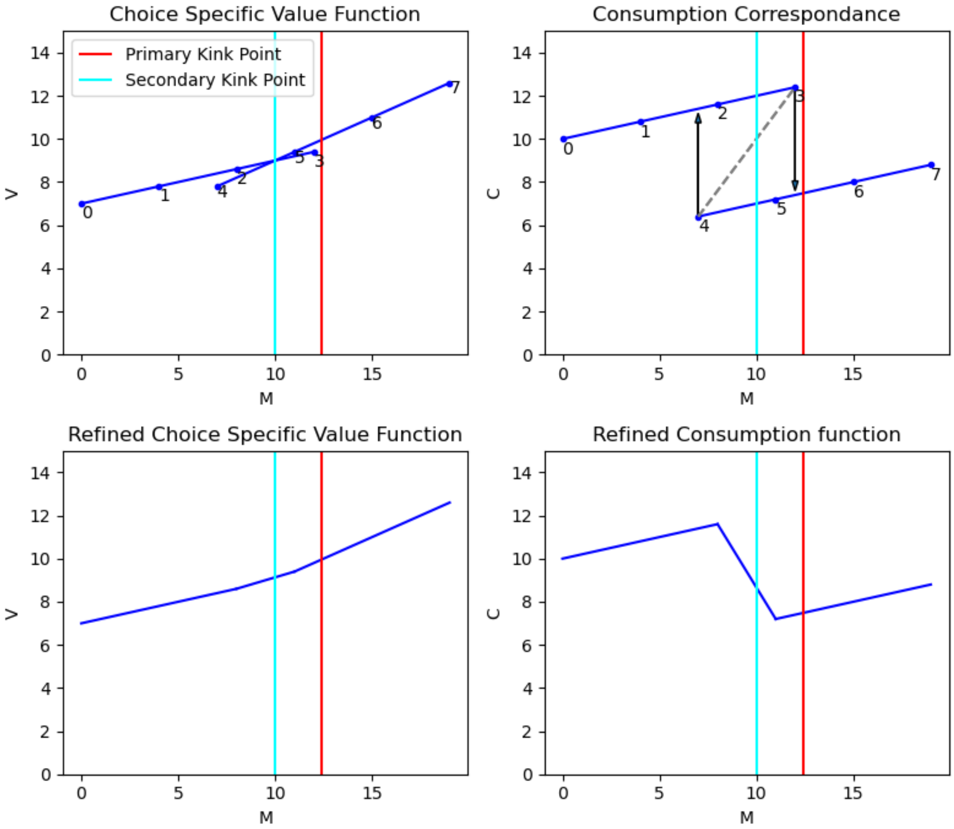


Figure 3: The Secondary Upper Envelope

Note: This is only a sketch of the problem, it does not come from actual code

v_t^1 and v_t^2. Therefore to refine the solution a new consumption value is calculated for the value of the endogenous grid m_t^4 that falls on the line segment between the consumption levels c_t^1 and c_t^2. A similar situation is illustrated to the right of the secondary kink in the top right graph.

The result of running the algorithm is shown in the two bottom graphs of figure 3. It can be seen that both are now proper functions. It should be noted that the true solution only has a single kink in the value function at the point of the secondary kink and a single discrete drop in the consumption function, also at the secondary kink. However these graphs are only shown for a fairly sparse grid, and with more grid points the solution comes closer to the true solution as the exogenous grid becomes denser around the primary and secondary kinks.

3.2.4 The effect of Type I extreme value distributed shocks on the DC-EGM algorithm

While the type I extreme value distributed shocks are included in the model as econometric error terms that captures non-observable variables that affect decisions they also serve as a modeling tool as they make the model easier to solve.

One way to see that is that having $\sigma_\Delta > 0$ makes the discrete choice probabilistic which smooths out the primary kinks and removes the need to calculate the upper envelope over the overall value function and also removes the need to find the actual primary kink point. The intuition for this, is that the agent can never fully anticipate a certain action to take place in the next period given the end of period asset level. Instead one can rely on interpolating the expected choice specific value functions. Assuming taste shocks follow a type I extreme value distribution, we can then use the logit formulation to compute the conditional choice probabilities for each discrete choice in the next period rather than keeping track of primary kinks. There may still be non-concavities due to partially smoothed secondary kinks, so the model is not globally concave in general, and we still have to run the secondary upper envelope over the choice specific value function of the student. However it should be noted that with enough smoothing the problem will eventually become globally concave and in that case the need for a secondary upper envelope disappears.

3.2.5 The DC-EGM algorithm with taste shocks

The DC-EGM algorithm used for solving the studying stage does the following for period $T^{s_{max}}$ and backward:

1. Set $t = T^{s_{max}}$
2. Fix an exogenous grid over the post-decision states, A_t
3. For each grid point in the exogenous grid do:
   - Compute $m_{t+1,study} = (1+r)a_t + t(\phi_i)$ and $m_{t+1,work} = (1+r)a_t$
   - Interpolate $v_{t+1,study}$ and $\mathbb{E}[v_{t+1,work}]$ using the solution for period $t = t+1$ and calculate choice probabilities
   - Interpolate $c_{t+1,study}$ and $\mathbb{E}[c_{t+1,work}]$ using the solution for period $t = t+1$
   - Calculate the expected marginal utility as the right hand side of (4).
   - Back out consumption by isolating $c_{t,study}$ in (4)
   - Compute $m_t = a_t + c_t$
4. If $t < T^s_{max}$ pass $C_{t,study}, V_{t,study}$ and $M_{t,study}$ to the secondary upper envelope algorithm which returns $C^r_{t,study}, V^r_{t,study}$ and $M^r_{t,study}$
5. Handle the budget constraint by
   - Insert n points in the refined solution grid for $M^r_{t,study}$ equally spaced between 0 and the first element
   - Insert the same n points in the refined solution grids for consumption $C^r_{t,study}$
   - Calculate the value of the student at those levels of consumption and insert it in the start of the refined solution grid for the value function $V^r_{t,study}$
6. Set $t = t - 1$
7. If $t < 0$ return the refined solution grids, else repeat 2 through 6

3.3 Solving and simulating the model

3.3.1 Choosing parameters

When parametrizing the model, we rely as much as possible on calibrated and estimated parameters from Abbott et al. (2019). Since the parameters from Abbott et al. (2019) are estimated under a different framework than ours, we do not expect that a full estimation of our model would yield the same estimates for all parameters as reported by Abbott et al. (2019). However, assuming that Abbott et al. (2019) have succeeded in identifying deep parameters correctly (such as preference for leisure, returns to education), their parameter estimates will be suitable as a baseline for our model.

Re-using estimates from Abbott et al. (2019) is not possible for all parameters, however. Some parameters that are exogenous to our model are determined endogenously in Abbott et al. (2019), such as the level of transfers doing the education stage. Similarly, some parameters in our model do not have a clear parallel in Abbott et al. (2019) due to out simplifications and different modeling choices. Such parameters include the scale of the taste and income shocks, σ_Δ and σ_ϵ. Of course, we would have liked to estimate such parameters - however, in the context of this paper we have settled on testing different combinations of parameters in order to get the model to generate reasonably plausible behavior. Our targets have been primarily to get individuals with high cognitive skill to get a higher education than those with low cognitive skill and those with a rich family background to get higher education than those with poor background. Simultaneously, we kept an eye on comparable parameters in Abbott et al. (2019).

To calibrate return to education λ_S, we choose the wage premium to highest education, $_{S_{max}}$ to match that of Abbott et al. (2019). Next, we choose the remaining λ_S as a an increasing function with diminishing marginal return.

We calibrate type parameters loosely based on the comparable parameters in Abbott et al. (2019). The authors measure cognitive type as a continuous variable based on test scores quantiles. We define $\theta_{low} = 1.33$ and $\theta_{high} = 1.66$, representing one plus the first and second tercile.

We deviate from Abbott et al. (2019) when setting the variance of income shocks, σ_Δ. For simplicity, we abstract away from persistent income shocks, making our income process not directly comparable to that of Abbott et al. (2019). For that reason, we have had to set a relatively high variance of the transitory income shock, σ_ϵ, to be able to still generate sufficient income risk in the working stage for it to affect choices in the study stage [1].

Finally, we have taken the liberty to choose transfers $t(\phi_{low})$, $t(\phi_{high})$ as well as initial assets in the simulation freely to generate plausible behavior. In a different context, initial assets could likely be

[1] Lower values of σ_ϵ resulted in education choices being determined almost exclusively by ϕ_i and not varying much between different values of θ_i.

Parameter	Value	Source
Preference parameters		
β	0.975	Abbott et al. (2019)
ρ	1.5	Abbott et al. (2019)
ν	3.0	Abbott et al. (2019)
ϑ	0.0415	Abbott et al. (2019)
κ	1.0	Hand calibration
σ_{Δ}	0.3	Hand calibration
Education and income parameters		
S_{max}	6	Hand calibration
$\lambda_{S_{max}}$	0.797	Abbott et al. (2019)
$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$	0, 0.143, 0.280, 0.413, 0.543, 0.671	Hand calibration
σ_{ϵ}	0.5	Hand calibration
r	0.018	Hand calibration
Type parameters		
$\theta_{high}, \theta_{low}$	1.33, 1.66	Hand calibration (Abbott et al. (2019))
ϕ_{high}, ϕ_{low}	5, 1	Hand calibration
Simulation		
Initial assets	3	Hand calibration
Type distribution	Equally large shares of population	Hand calibration

Table 1: Externally set parameters

estimated externally from data on wealth holdings when initiating education. Ideally, we would also like to estimate transfers ϕ_{high} and ϕ_{low} internally. Instead, we fix ϕ_{low} at a value 1 - a value which could otherwise be normalized to reflect universal education transfers or a similar income parameter. We choose ϕ_{high} high enough that the credit constraint for rich types never binds when they study. We will later explore whether ϕ_{high} could plausibly be estimated by a simulated method of moments approach.

3.3.2 Code performance

We solve the model over a 200-point grid of assets and 5-node quadrature grid over the income shock ϵ. Average solution time is 100.99 sec., cf. tabel 2. The main bottleneck in the solution is the optimization in the last period, which takes up 97.3 pct. of computation time. This is because of the fact that the non-zero continuation value in the final period requires us to perform an optimization for each point in the grid over assets, income shocks, education levels, and discrete types - a total of 24,000 optimizations. We have attempted to mitigate this by eg. computing analytical gradients of the last period problem and by experimenting with parallelization.

After having solved the terminal period, the rest of the solution is fast due to EGM and DC-EGM

Average solution time	100.99 sec
Time spend solving...	% of time
Last period	97.3
Remaining working stage	1.7
- checking budget constraint	52.2
Study stage	0.5

Table 2: Computation time for solving the model

making use of inverted Euler equations which all have an analytical solution. Solving the working stage occasionally requires us to solve a root finding problem whenever the budget constraint binds, but apart from this every other step in the solution relies solely on analytical methods.

3.3.3 Simulation

Next, we simulate 10,000 agents from the model. We initiate agents by drawing their type, assuming each of the four types are equally probable - this nests the assumption that there is no correlation between cognitive type and family type. We initiate each agent in period $t = 0$ with assets of $m_0 = 3$, excluding the transfer they receive if they study. The simulation results in life cycle profiles of average consumption, labor supply and assets as reported in figure 4.

First thing to notice is the fact that consumption is higher for agents from a rich background than those from a poor background at all times, regardless of their cognitive skill. This is the case even though the types only differ through their transfers during the education stage. Note also that while poor agents gradually increase their consumption in the beginning of the life cycle as they exit the study stage, rich agents smooth consumption evenly between the two stages.

Next thing to notice is the fact that agents from a poor background on average supply more labor in the first periods, reflecting that they exit the study stage earlier than those from a rich background. Simultaneously, agents of low cognitive skill increase labor supply slightly earlier than those of high cognitive skill - reflecting that the return to education depends on cognitive skill.

Finally, the life cycle asset profile for all types has the distinctive hump-shape that is characteristic for life cycle models. Rich agents receive income during studying that is high enough to facilitate savings, while poor agents only begin to save after entering the working stage. After a certain point, agents begin to dissave, reflecting the fact that the precautionary savings motive weakens as the end of the working stage approaches. This is also indicated by how consumption increases and labor supply decreases over the course of the life cycle for all four types.

Figure 5 shows the distribution of educational attainment between types. Note how rich agents reach

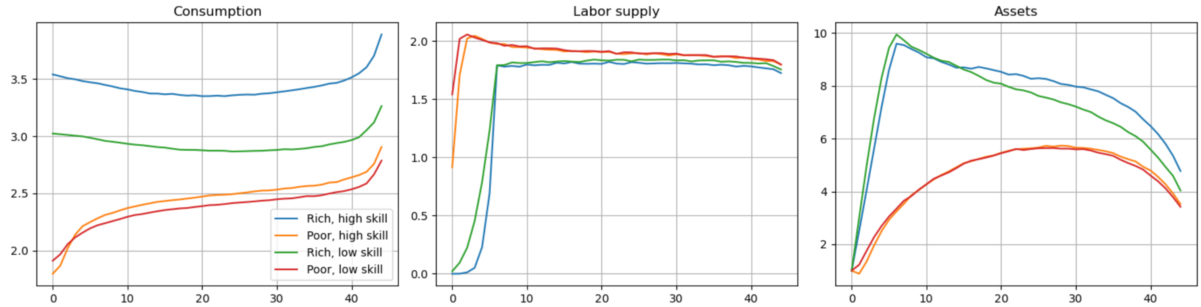


Figure 4: Simulated life cycle profiles

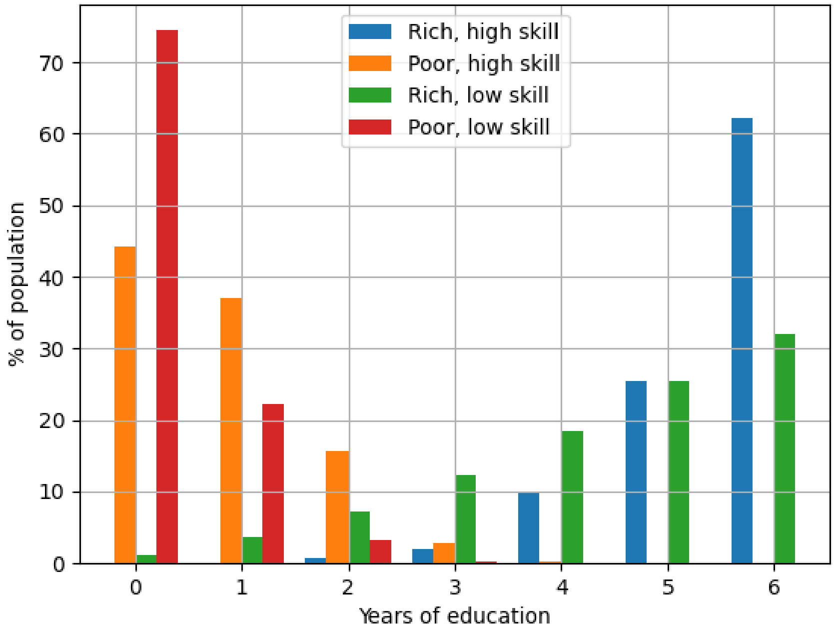


Figure 5: Educational attainment across types

high levels of education, while the opposite is the case for those from a poor background. Even after this fact, agents with high cognitive skill become more highly educated than those with low skill. Thus, our model succeeds in generating the type of behavior that we targeted.

With the way we parameterized the model, the decisive factor for an agent's educational attainment is their family's financial status rather than their own skill. Whether this is realistic or not is, of course, debatable. Our motivation for this particular set of parameters stems from the estimation experiments, we aim to perform.

3.3.4 Validating the solution

The model solution consists of a number of type-specific policy functions, dictating the optimal choice for any given state, as well as the conditional probability of choosing to exit the studying stage in the next period for any given state. Figure 6 shows as an example the solution for an agent in period $t = 4$ from a poor background with low skill and with four years of accumulated education. The blue graphs show the solutions conditional on the agent studying in $t = 4$, while the orange plots show the solution conditional on working.

While studying, the agent is credit constraint for cash on hand lower than approximately 2. This is evident from the linearity of the consumption function in this segment, as well as the constancy of the probability of working in the next period over this segment.

While working, the agent is credit constrained for a slightly lower value of cash-on-hand. Note however, that even under the credit constraint, consumption is higher than the level of assets, due to the agent adjusting labor supply to compensate. Appendix C contains more detailed examples of policy functions for varying types and periods.

We validate the solution by computing errors in the Euler equations. We compute Euler errors of consumption and labor for each individual in each period according to:

$$\delta^c_{i,t} = c_{it} - \left(\beta(1+r)\mathbb{E}_t[c^{-\rho}_{t+1}]\right)^{-\frac{1}{\rho}}$$
$$\delta^\ell_{i,t} = \ell_{it} - \left(\beta(1+r)\mathbb{E}_t[\ell^{\nu}_{t+1}]\right)^{\frac{1}{\nu}}$$

where the expectation is computed the same way as in the model solution - that is, using conditional choice probabilities to take expectation with respect to next period's discrete choice, and Gauss-Hermite quadrature to take expectations with respect to next period's income shock.

We then report average Euler errors as

$$\Delta^c_t = \frac{\sum_{i=1}^N \mathbf{1}_{a_{i,t}>\eta}\delta^c_{i,t}}{\sum_{i=1}^N \mathbf{1}_{a_{i,t}>\eta}}$$
$$\Delta^\ell_t = \frac{\sum_{i=1}^N \mathbf{1}_{a_{i,t}>\eta}\delta^\ell_{i,t}}{\sum_{i=1}^N \mathbf{1}_{a_{i,t}>\eta}}$$

Where $\eta \approx 0$ is a small number which we include to ensure we are only considering cases where the budget constraint does not bind. Down to interpolation error and machine precision, the Euler errors should be equal to 0.

Moreover, we compute average relative absolute Euler errors according to:

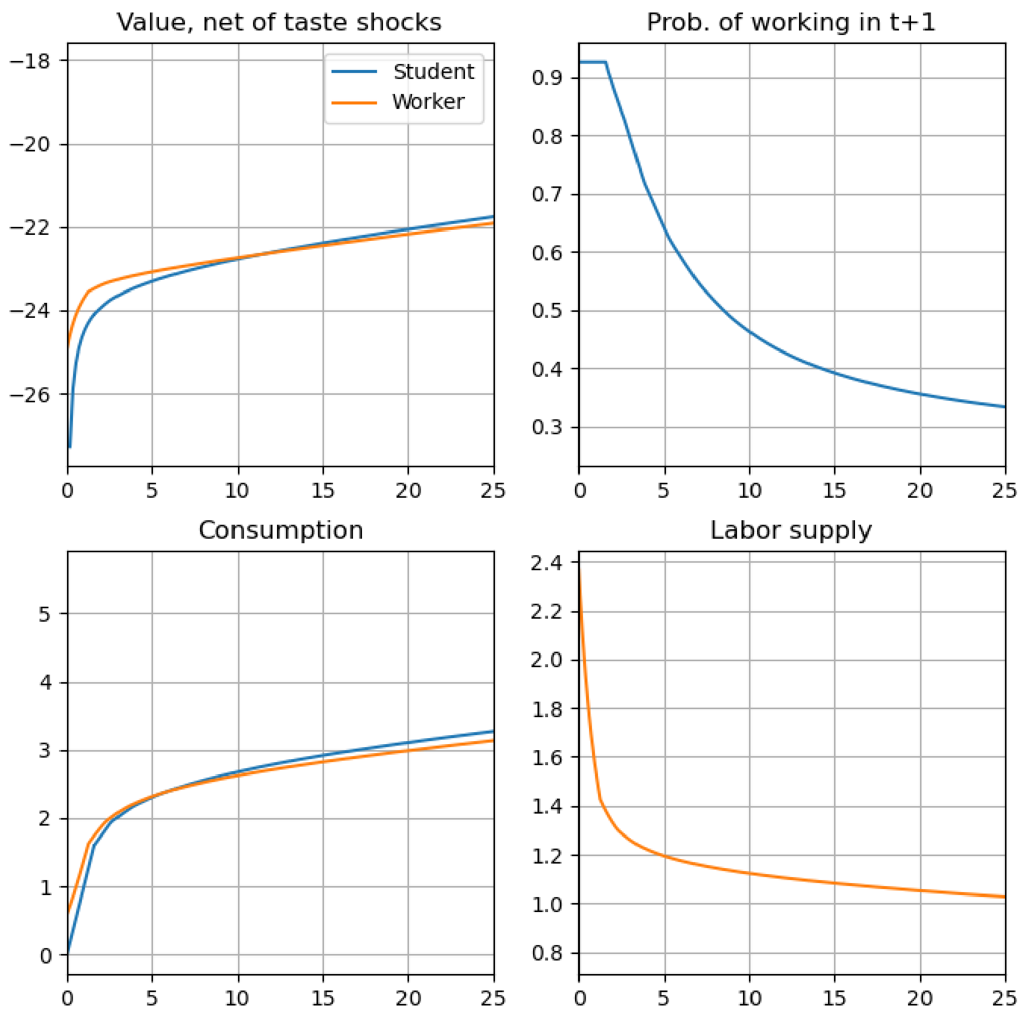


Figure 6: Solution for agent of type $(\theta_{low}, \phi_{low})$ with $S_t = 4$ in period $t = 4$

$$\varepsilon_t^c = \frac{\sum_{i=1}^N \mathbf{1}_{a_{i,t}>\eta} \log_{10}(|\delta_{i,t}^c / c_{i,t}|)}{\sum_{i=1}^N \mathbf{1}_{a_{i,t}>\eta}}$$
$$\varepsilon_t^\ell = \frac{\sum_{i=1}^N \mathbf{1}_{a_{i,t}>\eta} \log_{10}(|\delta_{i,t}^\ell / \ell_{i,t}|)}{\sum_{i=1}^N \mathbf{1}_{a_{i,t}>\eta}}$$

The absolute relative Euler errors show how large Euler errors are in relation to actual consumption or labor supply and are therefore more easily interpreted than the average Euler errors.

Euler errors are reported in figures 7a and 7b, showing that in each period, the average Euler errors in both consumption and labor are between 0.01% and 0.0001% of consumption and labor, respectively. While there are some spikes in the average Euler error - notably around $T^{S_{max}}$ and T - the average absolute relative errors are small enough that we are not concerned about lack of precision.

Finally, we verify that the intratemporal first order condition holds by computing:

$$\Delta^{MRS} = \vartheta / w_t \ell_t^\nu - c_t^{-\rho} \tag{5}$$

This should be equal to zero down to numerical precision and interpolation errors, even in periods where the budget constraint binds. These errors are reported in figure 7c. There is consistently a small, positive error, but we deem it sufficiently close to zero to be satisfactory.

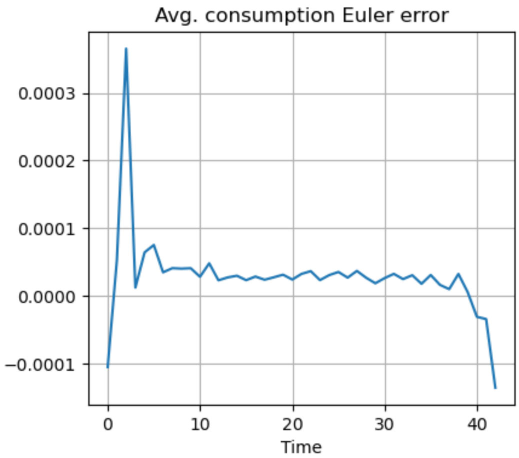
4 Estimation

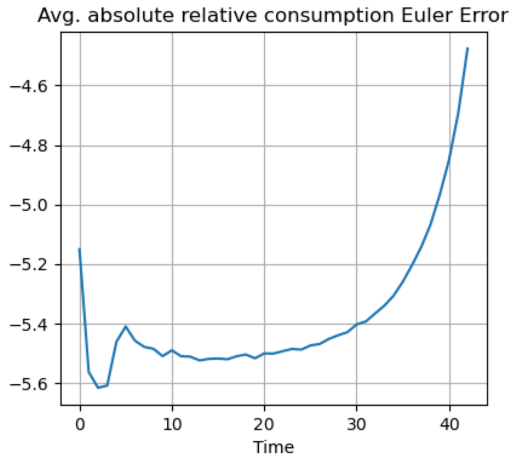
In order for a model such as this one to generate policy relevant predictions, it is important that model parameters are set accurately. Here, parameter estimation becomes important.

In this section, we explore how well parameters related to the type distribution and transfers can be identified. Specifically, we perform two Monte Carlo experiments, first trying to estimate the probability of being high skill conditional on family type, and next trying to estimate the same parameters jointly with the size of transfers received by rich individuals.

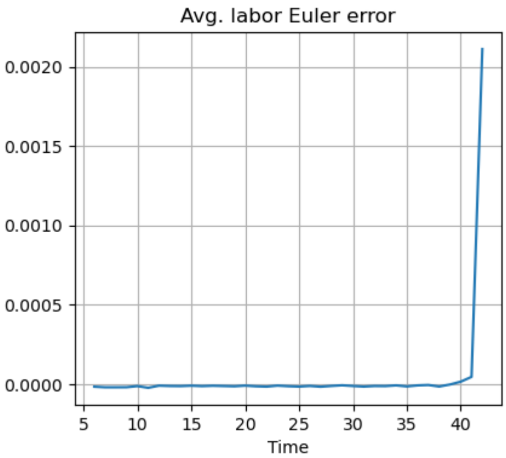
We will assume, we can observe agent's family type, $\phi \in \{\phi_{low}, \phi_{high}\}$, as well as their educational attainment, but not their cognitive type. We will then use the share of individuals from each family type that reach 0, 1, 2,...6 years of education as well as the average years of education as moments in a simulated method of moments estimation. We intend this as an experiment that potentially can cast some light on how well model parameters could be identified from data that would reasonably be available.

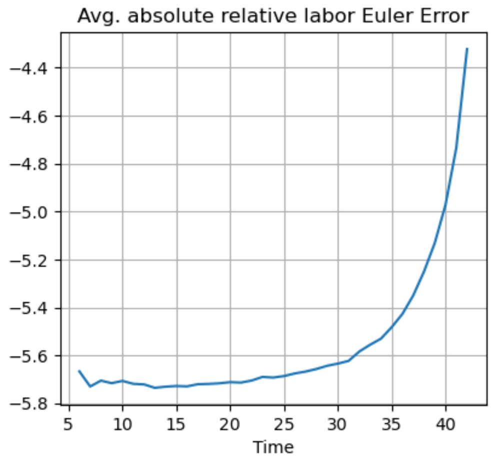
(a) Consumption Euler errors





(b) Labor Euler errors





(c) Intratemporal FOC errors

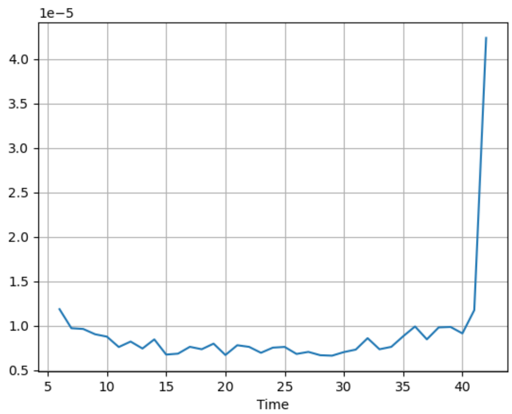


Figure 7: Solution precision

4.1 Simulated method of moments

We will use the simulated method of moments estimator, which estimates parameters by minimizing the distance between a set of empirical moments and moments simulated from the model. Instead of using actual empirical data, we will simulate a data set of 10,000 individuals from the model given the chosen parameter specification.

The moments targeted in the data are the share of agents within each family type who obtain each number of years of education:

$$\lambda^d_{\phi,S} = \frac{\sum_{i=1}^{10,000} \mathbf{1}_{\phi_i=\phi}\mathbf{1}_{S_i=S}}{\sum_{i=1}^{10,000} \mathbf{1}_{\phi_i=\phi}}$$

which we compute for each combination of $\phi \in \{\phi_{high}, \phi_{low}\}$ and $S_i \in \{0, 1, 2, 3, 4, 5, 6\}$. Additionally, we add the average educational attainment for rich and low types respectively, $\lambda^{avg}_{\phi_{low}}$ and $\lambda^{avg}_{\phi_{high}}$. This gives us a vector of 16 moments. We denote moments computed from the base data $\Lambda^d = (\lambda^d_{\phi_{low},0}, \lambda^d_{\phi_{low},1}, \cdots \lambda^d_{\phi,5}, \lambda^d_{\phi,6}, \lambda^{avg}_{\phi_{low}}, \lambda^{avg}_{\phi_{high}})'$.

For a given set of parameters, ψ, we can solve, simulate and compute the same moments from our model. We simulate $N = 10,000$ agents $S = 10$ times and compute the average simulated moments:

$$\lambda^m_{\phi,S}(\psi) = \frac{1}{S}\sum_{j=1}^{S} \frac{\sum_{i=1}^{N} \mathbf{1}_{\phi_i=\phi}\mathbf{1}_{S_i=S}}{\sum_{i=1}^{N} \mathbf{1}_{\phi_i=\phi}}$$

Each moment is a function of the parameters ψ. We have chosen a relatively high number of simulated individuals due to the fact that our model involves a discrete choice that may be taken with a very small probability. A large number of simulation increases the probability that the simulated choices match the underlying probability distribution.

Denoting the vector of simulated moments

$$\Lambda^m(\psi) = (\lambda^m_{\phi_{low},0}(\psi), \lambda^m_{\phi_{low},1}(\psi), \cdots, \lambda^m_{\phi_{high},5}(\psi), \lambda^m_{\phi_{high},6}(\psi), \lambda^{avg}_{\phi_{low}}(\psi), \lambda^{avg}_{\phi_{high}}(\psi))$$

we define the criterion function:

$$Q(\psi) = \left(\Lambda^m(\psi) - \Lambda^d\right)' W \left(\Lambda^m(\psi) - \Lambda^d\right)$$

where W is a weighting matrix determining the weight put on each moment. For simplicity, we set W equal to the identity matrix, putting equal weight on all moments.

The SMM estimator minimizes the criterion function:

$$\hat{\psi}_{SMM} = \arg\min_{\psi} Q(\psi)$$

Because the criterion function is not necessarily smooth it implies that we cannot use numerical optimizes that relies on the Jacobian of the criterion function. Instead we use the Nelder-Mead algorithm for the optimization.

4.2 Experiment 1: Can we identify the type distribution?

A central prediction from Abbott et al. (2019) is that some individuals obtain more education than what is efficient due to transfers from parents, while others are restricted by binding budget constraints. In this context it is relevant to know the distribution over types, particularly the correlation between family types and cognitive types. In this section we try to estimate the share of individuals who have high cognitive abilities conditional on coming from a rich background, $p_{high} = p(\theta_i = \theta_{high}|\phi_i = \phi_{high})$, and on coming from a poor background, $p_{low} = p(\theta_i = \theta_{high}|\phi_i = \phi_{low})$, setting the vector of parameters in the SMM estimation algorithm $\psi = (p_{high}, p_{low})'$. If we are able to estimate these parameters, it would also be a sign that our model is able to capture the mechanism in Abbott et al. (2019) which we ultimately cared about when setting up the model, namely the effect of the interaction between the budget constraint and cognitive skill on education attainment. That the parameters are identified are not given a priori since as shown, for our parameters, the main variable that explains education choice is the transfer received during the study stage.

We perform a Monte Carlo experiment, simulating 50 sets of 10.000 individuals and performing the estimation on each of them. Each estimation is based on $S = 10$ simulations of $N = 10.000$ individuals. When carrying out the estimation, we exploit the fact that the parameters we are estimating are only related to the distribution of types in the simulation, not to the model solution. Hence the model solution does not change each time we guess on a new set of parameters, and we only need to re-do the simulation between each evaluation of the criterion function. This speeds up estimation immensely.

Results are reported in table 3. On average, p_{high} and p_{low} are estimated at 0.4987 and 0.5049 - both well within a standard deviation of their true values of 0.5. To further explore how well the parameters are identified, we plot the contour of the criterion function for one of the estimations in figure 8. The function appears to have a clearly defined minimum without being noticeably flat in any direction and without clear correlations that may cause trouble for identification. We deem the experiment a success and conclude that these parameters are well identified conditional on observing family background and transfers.

Parameter	Mean	Std.	True value
p_{high}	0.49867	0.0202	0.500
p_{low}	0.50493	0.0253	0.500
Criterion at optimum	1.1343	0.7419	
Est. time	494.14 sec.	43.22 sec.	
No. estimations	50		
Simulation N	10,000		
Simulation S	10		
Starting val.	$(0.2, 0.8)$		
Optimizer	Nelder-Mead		

Table 3: Monte Carlo Estimation of the distribution of cognitive skills

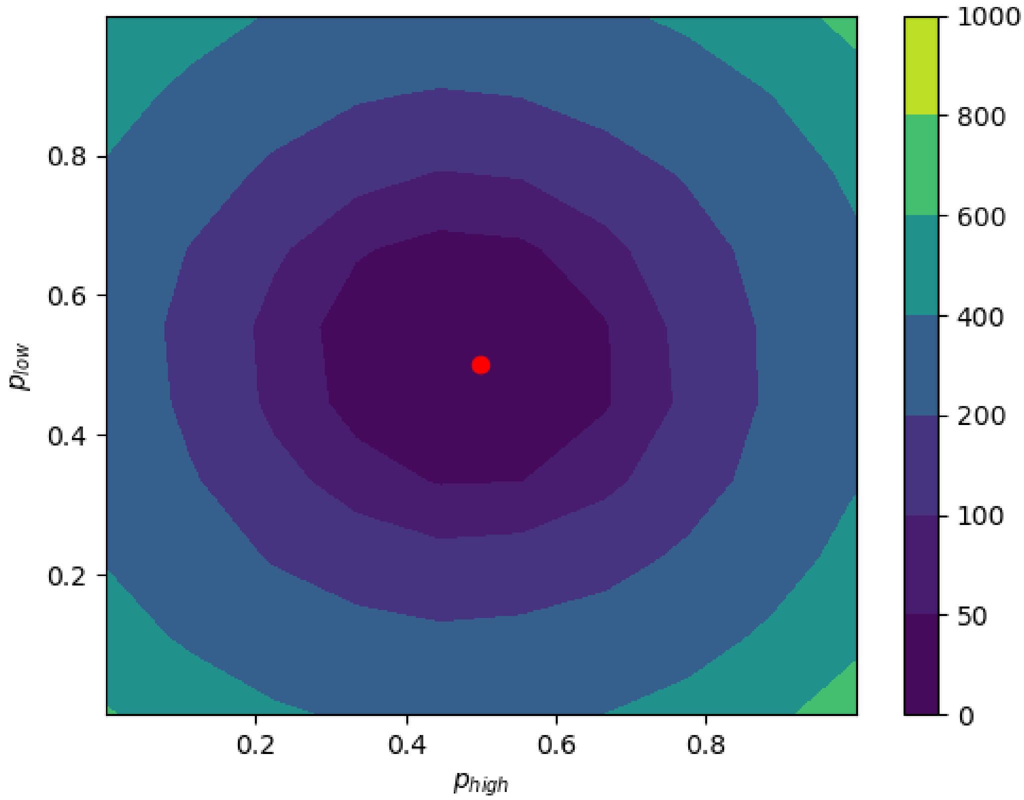


Figure 8: Criterion function for estimation experiment 1

4.3 Experiment 2: Can we identify the size of transfers?

The behavior generated in this model depends heavily on the transfers the different types receive during the education stage. In the original paper by Abbott et al. (2019), transfers are set by modeling the parental generation's altruism towards their children, and the authors estimate a preference parameter measuring said altruism. We have opted not to model parent's behavior and instead chosen transfers as an exogenous parameter. Up until now we have set transfers for rich types high enough that they will never be budget constrained while studying. However, it would be preferable if we could estimate the size of transfers internally. In this section, we explore whether it is possible to jointly identify the type distribution conditional on family background and the size of transfers for rich agents, using the same estimation technique as described before and setting $\psi = (p_{high}, p_{low}, t(\phi_{high}))'$. We suspect that this may be difficult - as mentioned above, rich individuals' incentive to study consist of both the return to education through wages and the transfer received while studying. Since p_{high} increases aggregate educational attainment through the first incentive and $t(\phi_{high})$ through the second incentive, we suspect that a large probability of being high skill together with a low transfer would yield the same educational distribution among rich individuals as a low probability of being high skill combined with large transfers. If this is the case, it will not be possible to jointly identify both p_{high} and $t(\phi_{high})$.

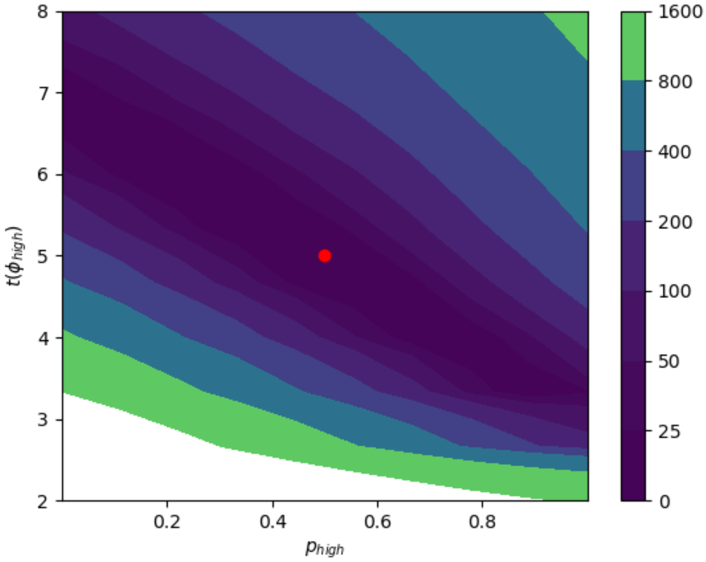
Contrary to the first experiment, we are trying to estimate a parameter that enters into the agent's optimization problem, namely transfers $t(\phi_{high})$, and therefore we need to solve the model between each evaluation of the criterion function. However, as transfers only affect behavior during the study stage, the working stage solution does not change with the transfer size. This means that we can get away with re-using the working stage solution and only solving the study stage between criterion evaluations - most importantly, we do not need to re-do the costly optimization in the last period. As solving the studying stage only takes up a fraction of solution time, this means that we can perform this estimation rather quickly.

Before running the experiment, we plot the criterion function for one of the estimations. In figure 9a, we fix p_{low} at its true value of 0.5 and plot the criterion function for varying values of p_{high} and $t(\phi_{high})$. Note how the function has an almost completely flat section along increasing values of p_{high} and decreasing values of $t(\phi_{high})$. This is indicative of the potential identification problem and a negative correlation between the estimators of the two parameters.

In figure 9b we fix p_{high} at its true value of 0.5 and instead vary p_{low} and $t(\phi_{high})$. Here, we do not see any signs of correlations between parameter estimators, which is to be expected as p_{low} only affects agents from a poor background while $t(\phi_{high})$ only affects agents from a rich background, and there are no interactions between the two types of agents.

Once again, we run a Monte Carlo experiment of 50 estimations using the same estimation technique as outline above. Results are reported in table4.

(a) Fixing $p_{low} = 0.5$ (True value)



(b) Fixing $p_{high} = 0.5$ (True value)

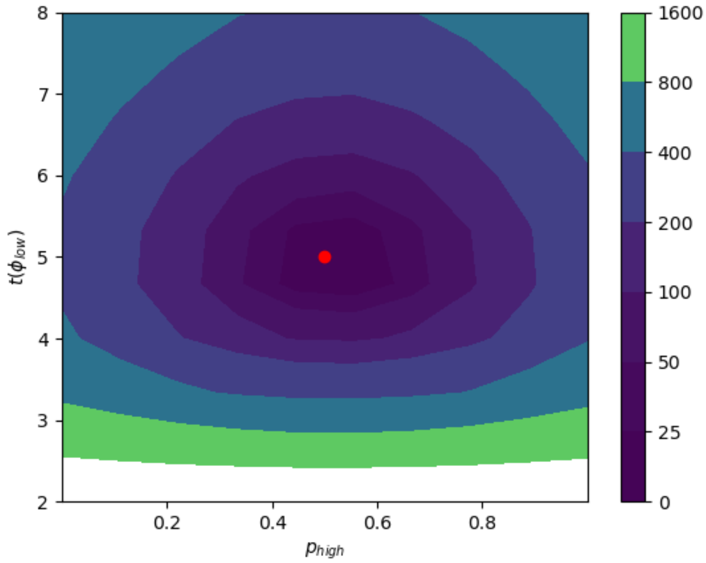


Figure 9: Criterion function for estimation experiment 2

As expected, the estimate of p_{low} does not differ much from that of the first experiment. However, p_{high} is on average estimated too low while $t(\phi_{high})$ is estimated too high, and the standard deviations on both estimates are large compared to the first estimation experiment, indicating that estimation is considerably more imprecise.

To further inspect what goes wrong, we plot estimates of p_{low} and $t(\phi_{high})$ against each other in a scatter plot, cf. figure 10. The plot reveals two clusters of estimate sets - one around the true values and around $(p_{high}, t(\phi_{high}) = (0.2, 0.625)$. This could be explained by the criterion function possible having multiple local minima, and the optimizer getting caught up in the wrong minimum - a possibility we have not handled while performing the optimization. This problem could potentially have been mitigated by using a more advanced optimization strategy, eg. by using multistart approaches. This is, however, outside the scope of this paper, and we conclude that we have not succeeded in identifying the size of transfers $t(\phi_{high})$ jointly with type distribution parameters p_{low} and p_{high}.

This estimation experiment in many ways represents the ideal conditions for estimation of parameters. Data is simulated directly from the model, hence we know the data generating process is inline with the model and the only noise is due to the stochastic shocks. Moreover, with the way we have parametrized the model, the size of transfers is highly informative about educational attainment, cf. figure 5, and hence we expect it to be relatively easily identified - if we had set the true value of transfers lower, we would expect it to be less influential in determining educational outcomes. Nevertheless, we fail to identify the size of transfers jointly with type distributions based solely on moments related to educational attainment. Identifying model parameters from real data would therefore likely be tricky with this approach.

This sparks the discussion of how we could otherwise estimate the size of transfers. The main issue regarding joint identifiability of transfers and cognitive types stems from not being able to observe who has high and low cognitive skill. Hence, any way in which high skill types differ from low skill types apart from educational attainment would likely be informative about the type distribution. Cf. figure 4, high skill types on average have higher consumption than low skill types at all stages of the life cycle. Given availability of data on consumption, we would suggest adding average consumption at some age conditional on family type to the set of moments targeted in estimation. This would hopefully provide enough information on the probability of being high skill given family type for us to be able to identify type distributions jointly with transfers.

Another way in which cognitive types differ from each other is, of course, by their returns to education in the from if wages. Cf. figure 1, wage increases more steeply with education for high skill types than for low skill types. Conditional on educational attainment, wages are therefore directly informative about cognitive ability - and importantly, independent of family type. Given access to data, we could add average wages conditional on educational attainment and parental background to the set of targeted moments.

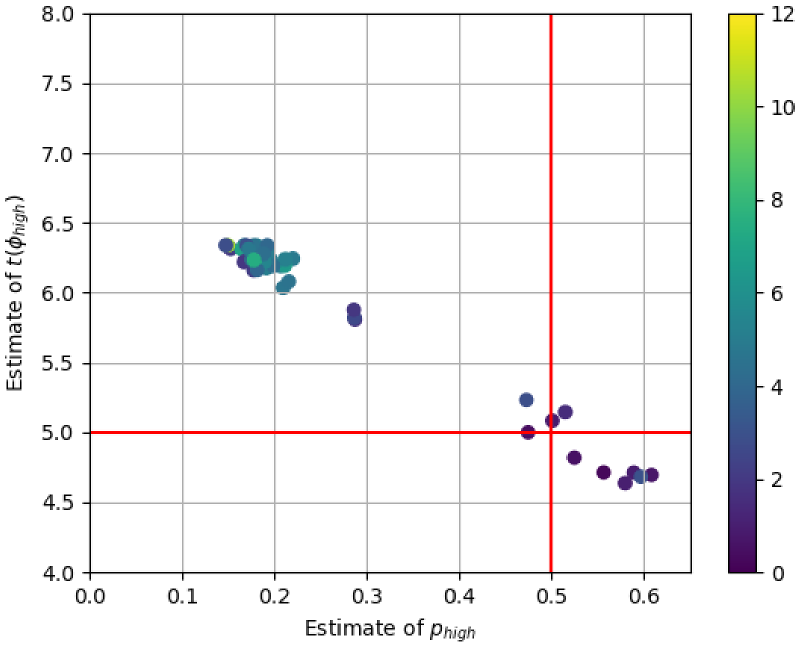


Figure 10: Correlation between estimates of p_{high} and $t(\phi_{high})$.

Note: Color scaling indicates the criterion value at the estimates. The intersection of the red lines indicates the true parameter values

And because wages are independent of family type once we condition on educational attainment, we trust we would be able to estimate the probability of being of high and low skill conditional on family type, while also estimating transfers received by agents from a rich background. The empirical relevance of this experiment would, however, depend on the assumption that conditional on education, wages are informative about cognitive types - which may not be as clearly the case empirically as in our relatively simple model. Moreover, in the present model, wages are independent of family background once we condition on education. This may also not be true empirically, as different types of network effects and limited social mobility can induce intergenerational correlations in income. This would create the correlation between parental type and wages, that made it difficult to estimate transfers jointly with type distribution in the first place. Nevertheless, this estimation technique would be more likely to succeed than the present one.

5 Discussion and conclusion

In this paper, we build, solve, and simulate a model inspired by that of Abbott et al. (2019) to explore how different credit constraints for students due to different familial backgrounds can affect educational attainment. We solve the model using a variety of techniques in continuous and discrete-continuous choice modeling, including DC-EGM and EGM extended to two choice variables. We are able to qualitatively replicate the result from Abbott et al. (2019), namely that budget constraints matter for educational attainment and can dominate the incentives from high returns to education due to high cognitive skills.

Parameter	Mean	Std.	True value
p_{high}	0.2842	0.1743	0.500
p_{low}	0.5073	0.0226	0.500
$t(\phi_{high})$	5.8545	0.6725	5.0
Criterion at optimum	3.5934	2.1661	
Est. time	798.89 sec	172.31 sec.	
No. estimations	50		
Simulation N	10,000		
Simulation S	10		
Starting val.	$(0.2, 0.8, 3)$		
Optimizer	Nelder-Mead		

Table 4: Monte Carlo Estimation of the distribution of cognitive skills and transfer size

We run two estimation experiments to explore whether the probability of being high skill conditional on family background can be identified by targeting moments related to educational attainment conditional on family background.

In the first experiment, we assume we know the size of the transfer received by agents from a rich background, and we only estimate the probability of being of high cognitive skill, conditional on background. We succeed in identifying both parameters.

In the second experiment, we assume we do not know the size of transfers received by rich agents and instead estimate the transfers size jointly with the two conditional probabilities. We suspected that transfers and probability of being high skill would not be identified jointly, as both incentivize edcucation and thus have the same effect on aggregate educational outcomes. Indeed, we could not precisely estimate transfers and probability of being high skill for rich individuals by our chosen approach. We discuss whather it would be possible to mitigate this issue by expanding the set of moments targeted or adjusted the optimization technique.

In building our model, we have made a large number of departures from the original model by Abbott et al. (2019), mainly due to the scope of this paper being narrower. We do expect some of these deviations to have implications for model implications. Here, we focus on two deviations.

Firstly, Abbott et al. (2019) allow for endogenous labor supply even in the study stage. This mechanism may be crucial, as positive labor supply provides the agent with a tool to escape an otherwise binding credit constraint, thus allowing for agents who otherwise could not afford education to continue their studies, and blurring one of the central mechanisms in our model. Moreover, positive labor supply while studying clearly takes place empirically. Estimating transfer sizes in the current model from real data where students have positive labor supply and hence income from both transfers and wage income may therefore be difficult. Including labor supply in the studying stage could therefore be a relevant extension of the present model.

Secondly, we have assumed away persistent income shocks, and in our model all income uncertainty is with respect to a completely transitory wage shock. We have done this to limit the number of continuous state variables and keep the model as simple as possible. However, persistence in income shocks is a more realistic assumption and better in line with the model proposed by Abbott et al. (2019). Particularly, we believe that persistent income shocks would increase precautionary behavior, as getting a bad shock realization would have lasting effects on income - in our model, they only affect present income. If agents partly educate themselves to insure themselves against future income shocks, we believe the inclusion of persistent income shocks could affect model predictions regarding educational attainment. This is particularly the case if we were to follow the approach of Abbott et al. (2019) and make income variance education-specific.

Nevertheless, we believe that our model, though simple, captures some central mechanisms behind differences in educational attainment, and our estimation experiments cast light on potential identification issues.

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A Derivation of working stage Euler equations for consumption and labor supply

The first order conditions of the working stage Bellman equation with respect to consumption are:

$$\begin{aligned}\frac{\partial V_{t,work}}{\partial c_t} &= c^{-\rho} + \beta \mathbb{E}_t \left[\frac{\partial V_{t+1,work}}{\partial m_{t+1}} \frac{\partial m_{t+1}}{\partial c_t} \right] = c^{-\rho} - \beta(1+r)\mathbb{E}_t \left[\frac{\partial V_{t+1,work}}{\partial m_{t+1}} \right] = 0 \\ &\iff c^{-\rho} = \beta(1+r)\mathbb{E}_t \left[\frac{\partial V_{t+1,work}}{\partial m_{t+1}} \right] \end{aligned} \tag{6}$$

$$\begin{aligned}\frac{\partial V_{t,work}}{\partial \ell_t} &= -\vartheta \ell_t^{\nu} + \beta E_t \left[\frac{\partial V_{t+1,work}}{\partial m_{t+1}} \frac{\partial m_{t+1}}{\partial \ell_t} \right] = -\vartheta \ell_t^{\nu} + \beta(1+r) w_t(S_t,\theta)\mathbb{E}_t \left[\frac{\partial V_{t+1}}{\partial m_{t+1}} \right] = 0 \\ &\iff \vartheta \ell_t^{\nu} = \beta(1+r) w_t(S_t,\theta)\mathbb{E}_t \left[\frac{\partial V_{t+1,work}}{\partial m_{t+1}} \right] \end{aligned} \tag{7}$$

By additive separativity, the first order condition for consumption is independent of the level of labor, except through the marginal continuation value, and vice versa. By combining (6) and (7), we obtain the intratemporal first order condition:

$$c_t^{-\rho} = \frac{\vartheta}{w_t(S_t,\theta)} \ell_t^{\nu} \iff \ell_t = \left(\frac{w_t(S_t,\theta)}{\vartheta} c_t^{-\rho} \right)^{\frac{1}{\nu}} \tag{8}$$

Equations (6) and (7) can be further simplified, if we can find a closed form expression for the derivative of the value function. By applying the envelope theorem, we see that the derivative of the value function with respect to assets m_t is equal to the derivative of the continuation value:

$$\frac{\partial V_{t,work}}{\partial m_t} = \beta \mathbb{E}_t \left[\frac{\partial V_{t+1,work}}{\partial m_{t+1}} \frac{\partial m_{t+1}}{\partial m_t} \right] = \beta(1+r)\mathbb{E}_t \left[\frac{\partial V_{t+1,work}}{\partial m_{t+1}} \right] = c_t^{-\rho}$$

Where the final equality follows from (6). Combining this with (6), (7) and (8) finally results in the Euler equations for consumption and labor:

$$\begin{aligned} c_t^{-\rho} &= \beta(1+r)\mathbb{E}_t \left[c_{t+1}^{-\rho} \right] \\ \ell_t^{\nu} &= \beta(1+r)\mathbb{E}_t \left[\frac{w_t}{w_{t+1}} \ell_{t+1}^{\nu} \right] \end{aligned}$$

B Derivation of study stage Euler equation for consumption

B.0.1 The Euler equation of the student

For the study stage we only need to derive the euler equation for the student conditional on choosing to study in period t as the solution of the worker's problem is solved above.

Because the taste shocks in the value functions are type one extreme value distributed the value function conditional on being a student can be rewritten as:

$$V_{t,study}(\Omega_t) = \underset{c_t}{max} \frac{c_t^{1-\rho}}{1-\rho} + \beta\sigma_\Delta \log\left\{\exp\left(\frac{\mathbb{E}_t[V_{t+1,work}(\Omega_{t+1})]}{\sigma_\Delta}\right) + \exp\left(\frac{V_{t+1,study}(\Omega_{t+1})}{\sigma_\Delta}\right)\right\} \tag{9}$$

Where it is used that the expectation over the max of the choice specific value functions in the next period can be written as a log-sum due to the type one extreme value distributed taste shocks. $\mathbb{E}_t$ is in this case only the expectation over the wage shocks, and σ_Δ is the scale parameter of the taste shocks.

The first order condition of (9) can be found as

$$\begin{aligned}\frac{\partial V_{t,study}(\Omega_t)}{\partial c_t} &= c_t^{-\rho} + \beta\sigma_\Delta \left\{\frac{1}{\sigma_\Delta}\frac{\partial V_{t+1,study}(\Omega_{t+1})}{\partial m_{t+1}}\frac{\partial m_{t+1}}{\partial c_t}\exp\left(\frac{V_{t+1,study}(\Omega_{t+1})}{\sigma_\Delta}\right) + \right.\\ &\left.\frac{1}{\sigma_\Delta}\mathbb{E}_t\left[\frac{\partial V_{t+1,work}(\Omega_{t+1})}{\partial m_{t+1}}\right]\frac{\partial m_{t+1}}{\partial c_t}\exp\left(\frac{\mathbb{E}_t[V_{t+1,work}(\Omega_{t+1})]}{\sigma_\Delta}\right)\right\} *\\ &\left\{\exp\left(\frac{\mathbb{E}_t[V_{t+1,work}(\Omega_{t+1})]}{\sigma_\Delta}\right) + \exp\left(\frac{V_{t+1,study}(\Omega_{t+1})}{\sigma_\Delta}\right)\right\}^{-1}\\ &= 0\end{aligned}$$

Using the fact that the taste shocks are type one extreme value distributed, the choice probability of a student continuing to study in period t+1 is given by

$$P_{t+1}(study|\Omega_{t+1}) = \exp\left(\frac{V_{t+1,study}(\Omega_{t+1})}{\sigma_\Delta}\right)\left(\exp\left(\frac{\mathbb{E}[V_{t+1,work}(\Omega_{t+1})]}{\sigma_\Delta}\right) + \exp\left(\frac{V_{t+1,study}(\Omega_{t+1})}{\sigma_\Delta}\right)\right)^{-1}$$

And the choice probability of a student choosing to work in the next period is given by

$$P_{t+1}(work|\Omega_{t+1}) = \exp\left(\frac{\mathbb{E}[V_{t+1,work}(\Omega_{t+1})]}{\sigma_\Delta}\right)\left(\exp\left(\frac{\mathbb{E}[V_{t+1,work}(\Omega_{t+1})]}{\sigma_\Delta}\right) + \exp\left(\frac{V_{t+1,study}(\Omega_{t+1})}{\sigma_\Delta}\right)\right)^{-1}$$

These two choice probabilities can be inserted into the first order condition yielding

$$\begin{aligned}\frac{\partial V_{t,study}(\Omega_t)}{\partial c_t} &= c_t^{-\rho} + \beta \frac{\partial V_{t+1,study}(\Omega_{t+1})}{\partial m_{t+1}} \frac{\partial m_{t+1}}{\partial c_t} P_{t+1}(study|\Omega_{t+1}) + \\ &\mathbb{E}_t \left[\frac{\partial V_{t+1,work}(\Omega_{t+1})}{\partial m_{t+1}} \right] \frac{\partial m_{t+1}}{\partial c_t} P_{t+1}(work|\Omega_{t+1}) \\ &= 0\end{aligned}$$

Where it has been used that $\frac{1}{\sigma_\Delta}$ cancels.

Using that $m_{t+1} = (1+r)(m_t - c_t)$ we have that $\frac{\partial m_{t+1}}{\partial c_t} = -(1+r)$. Inserting this and putting the term outside the parentheses the first order condition can be rewritten to:

$$\begin{aligned}\frac{\partial V_{t,study}(\Omega_t)}{\partial c_t} &= c_t^{-\rho} - \beta(1+r) \left[\frac{\partial V_{t+1,study}(\Omega_{t+1})}{\partial m_{t+1}} P_{t+1}(study|\Omega_{t+1}) + \right. \\ &\left. \mathbb{E}_t \left[\frac{\partial V_{t+1,work}(\Omega_{t+1})}{\partial m_{t+1}} \right] P_{t+1}(work|\Omega_{t+1}) \right] \\ &= 0 \\ &\Leftrightarrow \\ c_t^{-\rho} &= \beta(1+r) \left[\frac{\partial V_{t+1,study}(\Omega_{t+1})}{\partial m_{t+1}} P_{t+1}(study|\Omega_{t+1}) + \right. \\ &\left. \mathbb{E}_t \left[\frac{\partial V_{t+1,work}(\Omega_{t+1})}{\partial m_{t+1}} \right] P_{t+1}(work|\Omega_{t+1}) \right]\end{aligned}$$

To reduce this further first find the first order condition of the choice specific value functions:

$$\begin{aligned}\frac{\partial V_{t,d}(\Omega_t)}{\partial c_{t,d}} &= c_{t,d}^{-\rho} - \beta(1+r) E_t \left[\frac{\partial V_{t+1}(\Omega_t)}{\partial m_{t+1}} \right] = 0 \\ &\Leftrightarrow \\ c_{t,d}^{-\rho} &= \beta(1+r) E_t \left[\frac{\partial V_{t+1}(\Omega_t)}{\partial m_{t+1}} \right]\end{aligned}$$

for $d \in \{study, work\}$

Since the envelope theorem yields

$$\frac{\partial V_{t,d}(\omega_t)}{\partial M_t} = \beta(1+r) E_t \left[\frac{\partial V_{t+1}(\Omega_t)}{\partial m_{t+1}} \right]$$

Then for all periods it holds that

$$c_t^{-\rho} = \frac{\partial V_{t,d}(\omega_t)}{\partial M_t}$$

Using this we can insert for $t + 1$ in the first order condition above to get:

$$c_t^{-\rho} = \beta(1 + r) \left(c_{t+1,study}^{-\rho} P_{t+1}(study|\Omega_t) + E_t \left[c_{t+1,work}^{-\rho} \right] P_{t+1}(work|\Omega_t) \right) \quad (10)$$

Where it has been used, that there are no uncertainty in the study stage conditional on choosing to study. This is the consumption Euler for the agent when studying, and it relates the marginal utility of consumption in period t, to the expected marginal utility of consumption in the next period and is what the DC-EGM uses in its EGM step to find the initial choice specific consumption function of the student.

C Model solution

C.1 Policy function conditional on studying

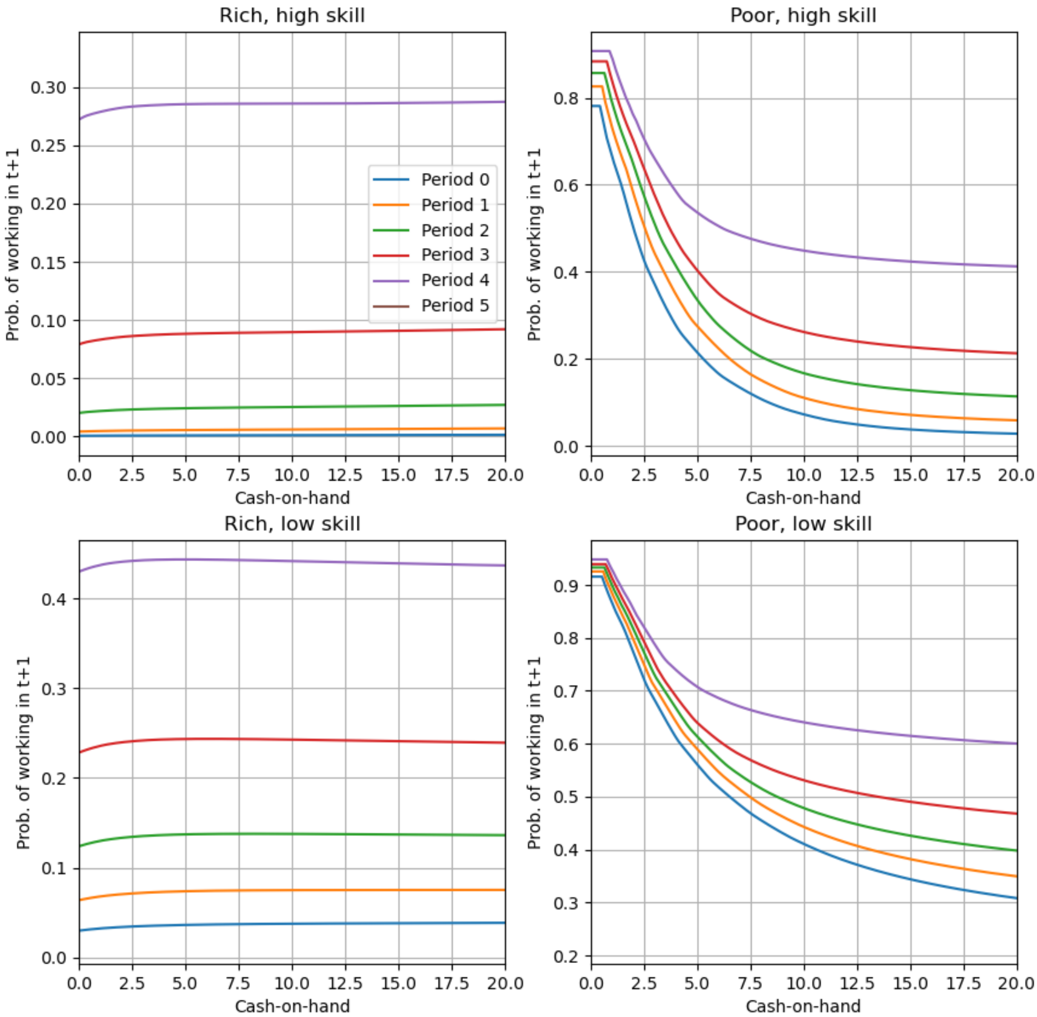

Figure 11: Conditional choice probabilities in studying stage

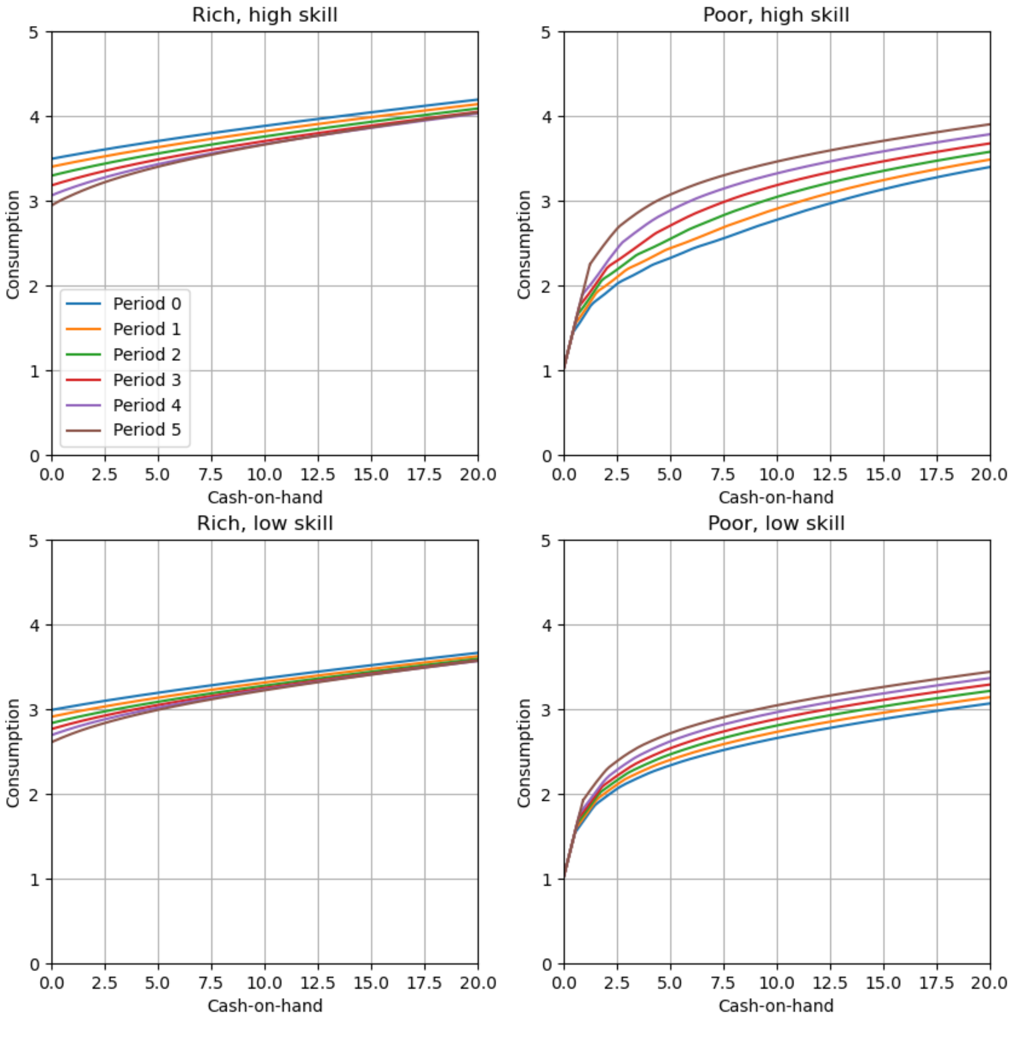

Figure 12: Consumption in studying stage

C.2 Policy functions conditional on working

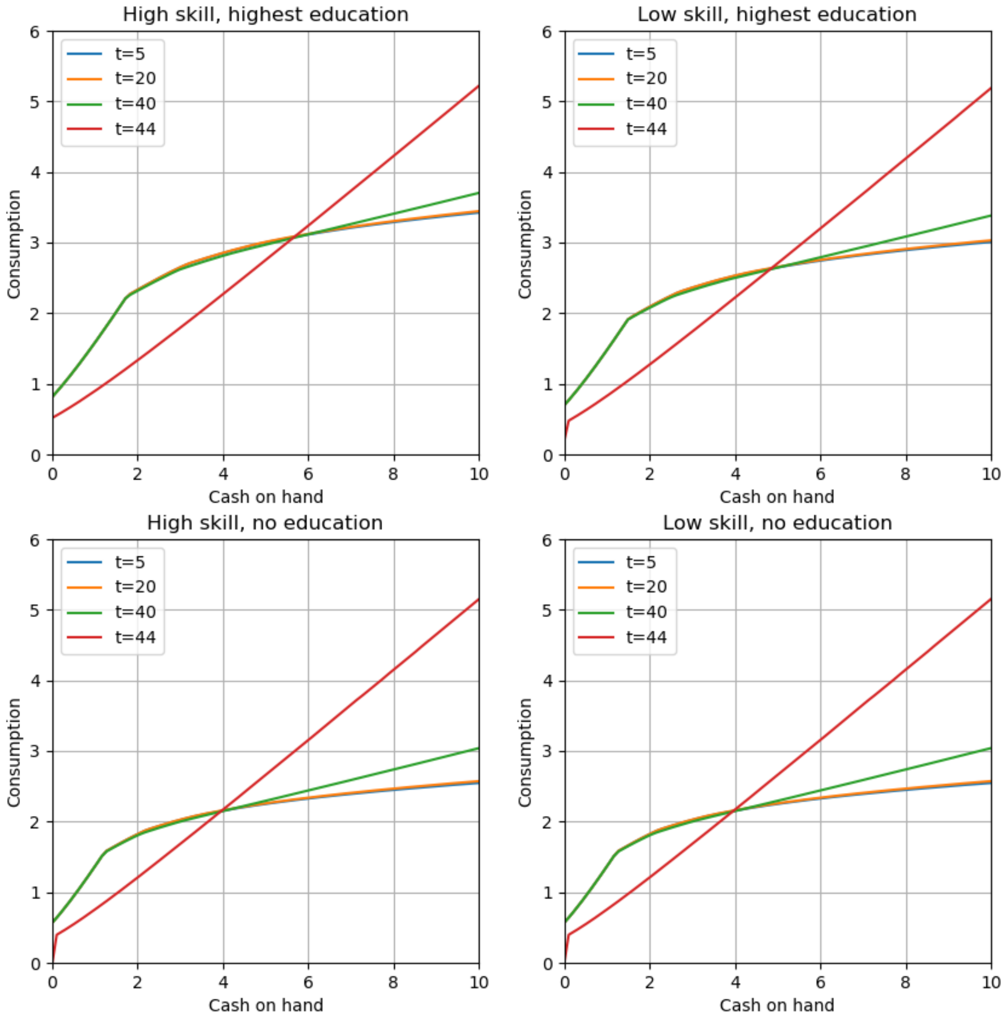

Figure 13: Consumption function in working stage

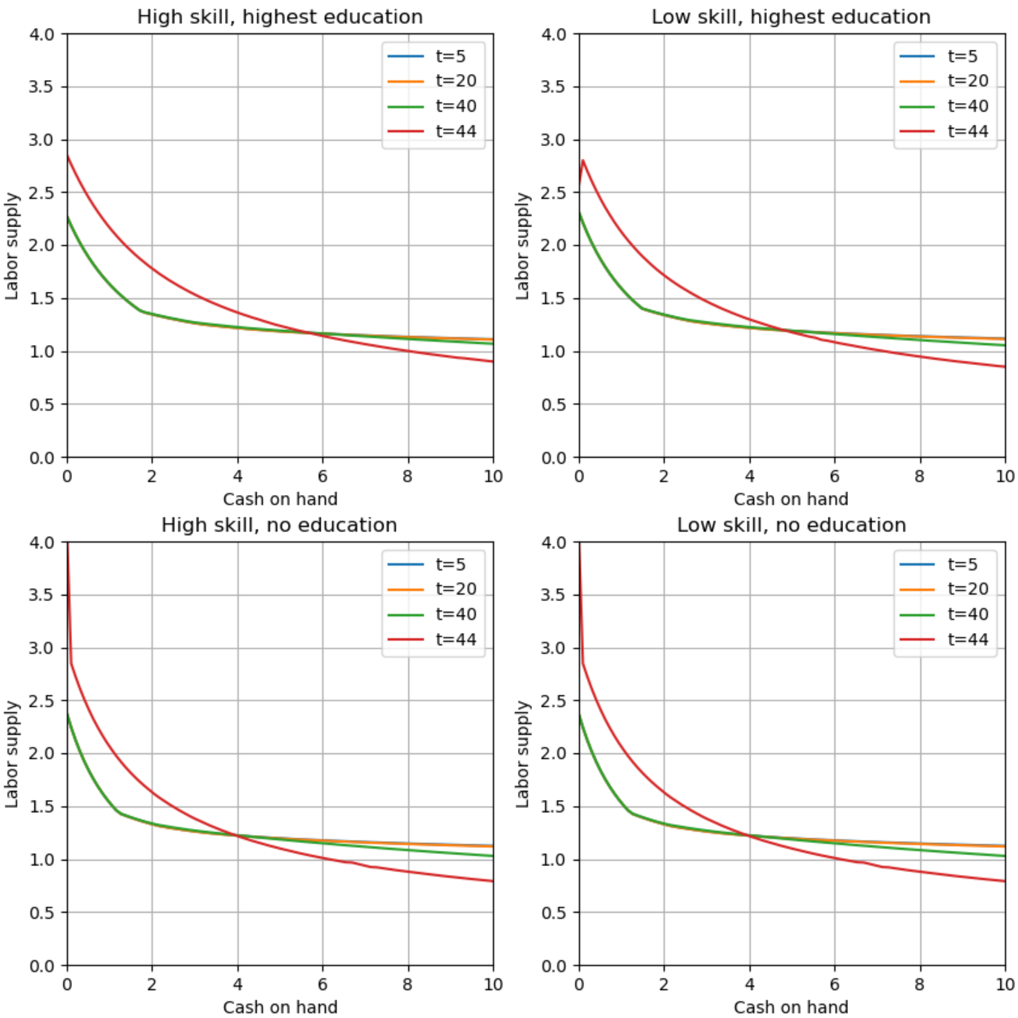

Figure 14: Labor function in working stage

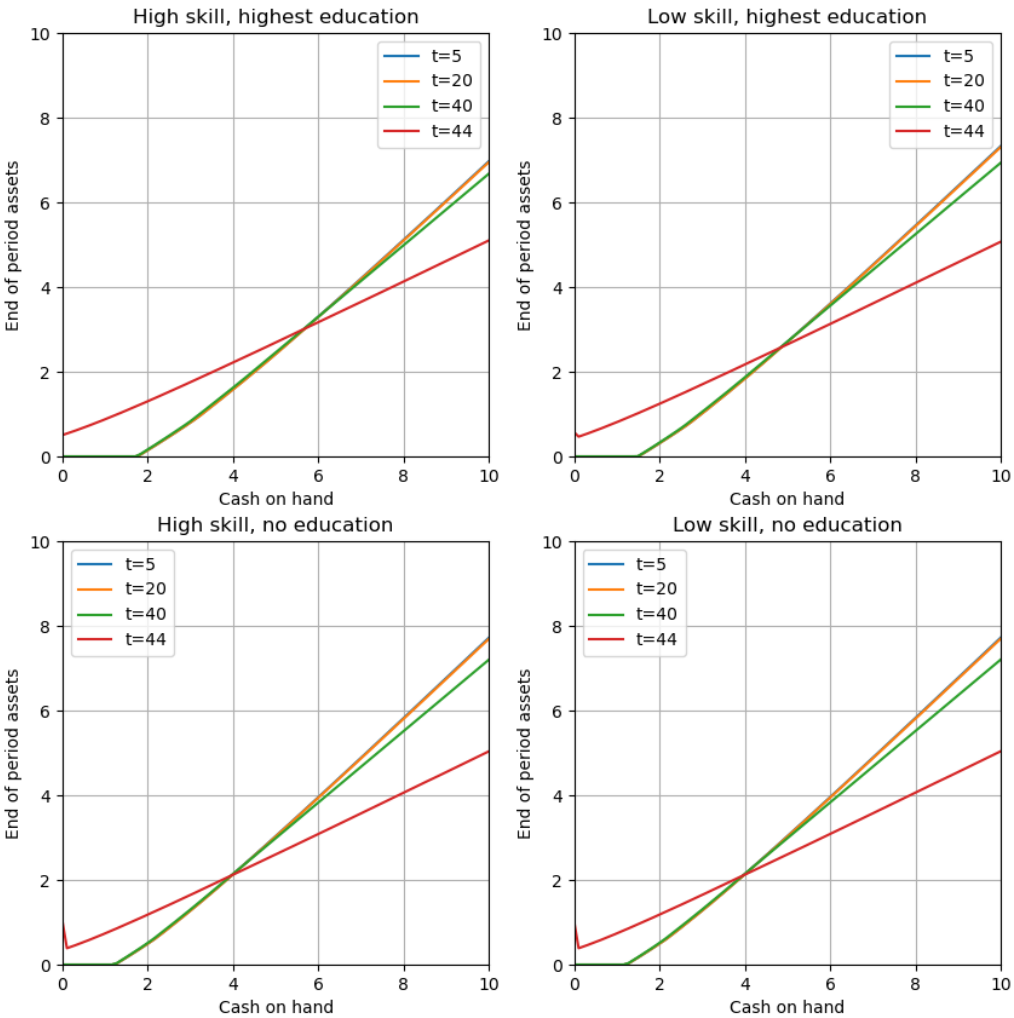

Figure 15: End-of-period savings in working stage

C.3 Model solution in period t = 0

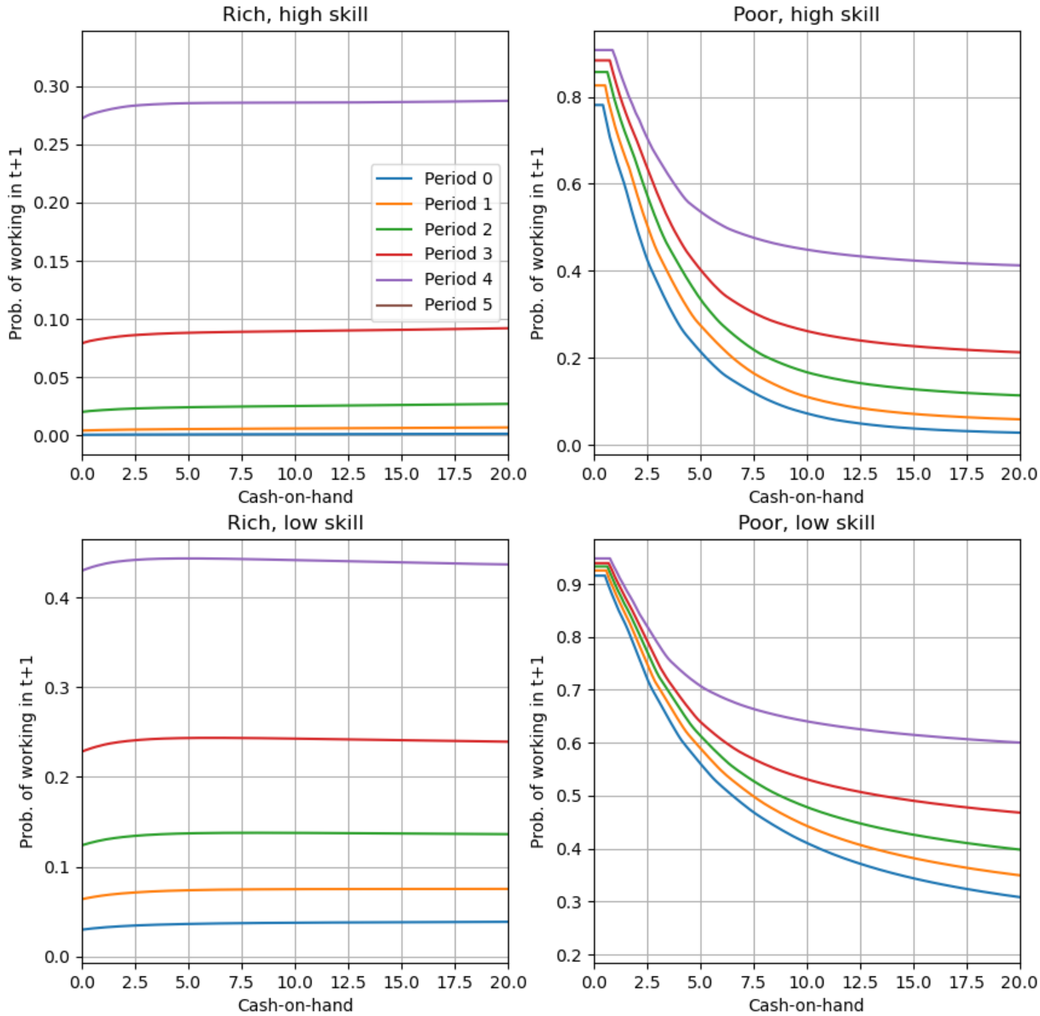

Figure 16: Consumption, t=0

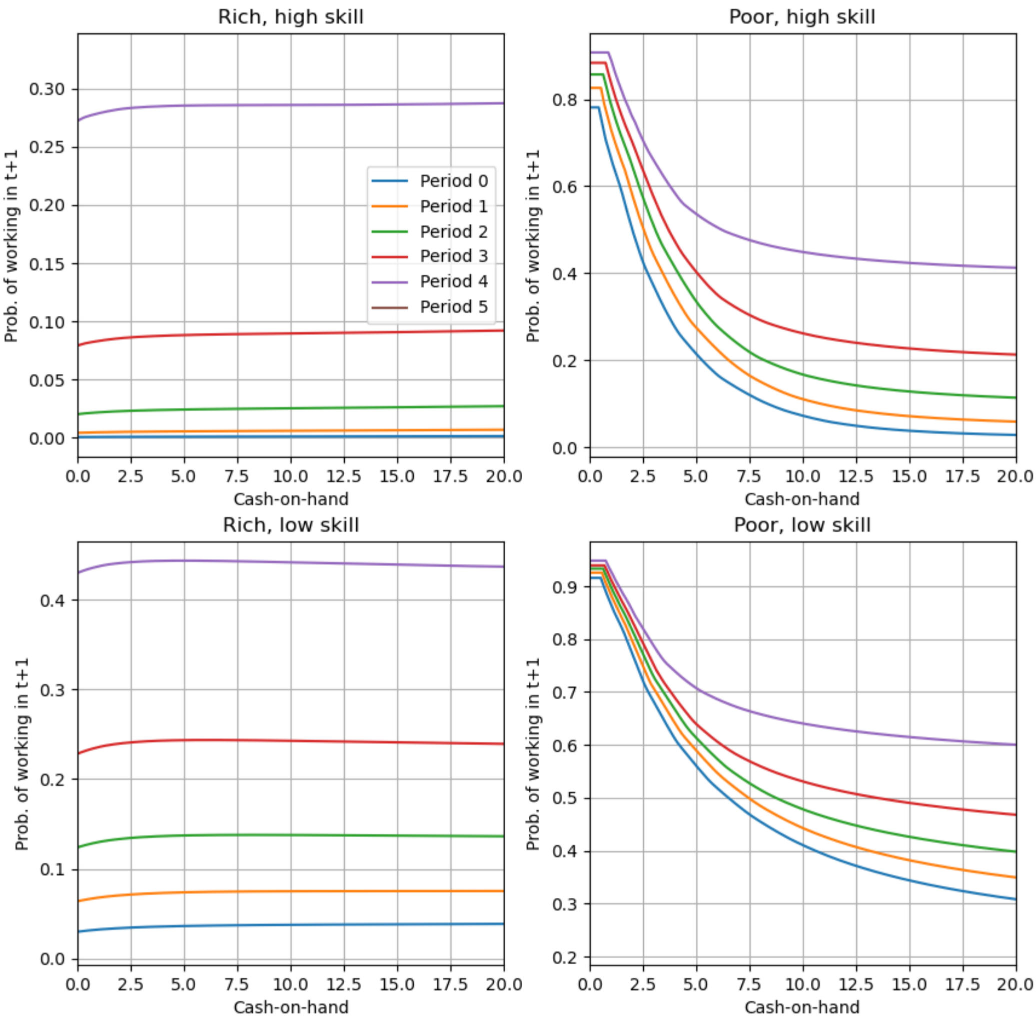

Figure 17: Value, net of taste shocks, t=0