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  `https://github.com/julianmak/academic-notes`

  The repository principally contains the compiled products rather than the source for size reasons.
- Associated Python code (as Jupyter notebooks mostly) will be held on the same repository. The source data however might be big, so I am going to be naughty and possibly just refer you to where you might get the data if that is the case (e.g. JRA-55 data). I know I should make properly reproducible binders etc., but I didn't...
- I do not claim the compiled products and/or code are completely mistake free (e.g. I know I don't write Pythonic code). Use the material however you like, but use it at your own risk.
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OCES 3301 :

basic Data Analysis in ocean sciences

Session 8: fairly basic time-series analysis

Outline

(Just overview here; for actual content see Jupyter notebooks)

- Fourier analysis

  $\rightarrow$ background and theory

  $\rightarrow$ power spectrum

  $\rightarrow$ Fourier analysis as a filtering tool

Signal analysis

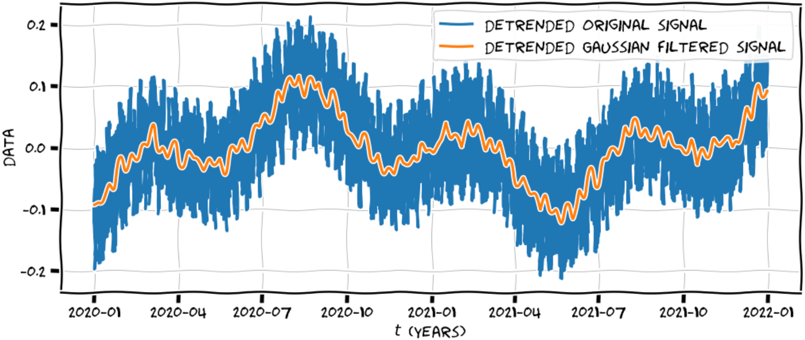


Figure: A detrended signal.

- low-passed signal still has some oscillations
- how to pick these out though?

  $\rightarrow$ can we say which period of oscillation is larger definitively?

Recall: waves

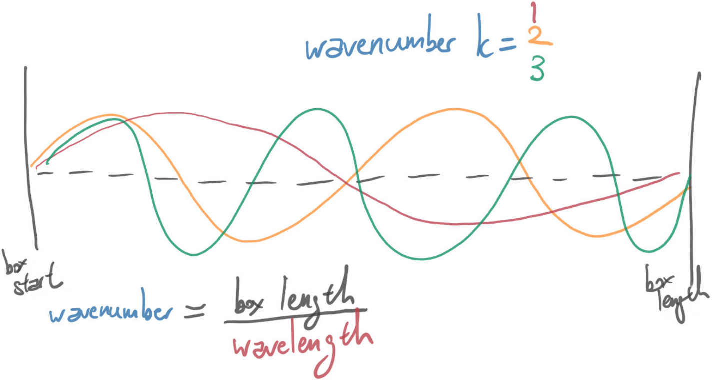


Figure: Schematic of wave features. Box length $L = 2\pi$ for simplicity.

- displacement η described by (could also be sine)

$$\eta \sim A\cos(2x), \qquad \eta \sim A\cos(1x), \qquad \eta \sim A\cos(3x)$$

Recall: waves

$$\gamma = \frac{v}{\lambda}, \qquad k = \frac{2\pi}{\lambda}$$

- γ the frequency (units: $\mathrm{s}^{-1} = \mathrm{Hz}$)

  $\rightarrow$ how quickly the wave oscillates
- $v = c_p$ the **phase** velocity

  $\rightarrow$ how fast the wave itself moves around
- λ the wavelength

  $\rightarrow$ how long the wave is
- k the wavenumber

  $\rightarrow$ intuitively how many waves can you fit in a box (so $k \sim \lambda^{-1}$)

  $\rightarrow$ does not necessarily have to be an integer

Fourier analysis

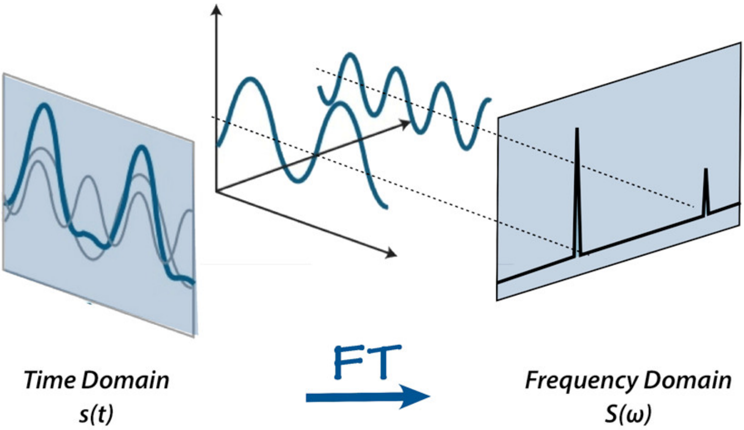


Figure: Schematic (?) of a Fourier transform.

Fourier analysis

- generically (!) a signal can **uniquely** be written as

$$\begin{aligned} f(t) &= a_0 + a_1\cos(t) + a_2\cos(2t) + \ldots + b_1\sin(t) + b_2\sin(2t) + \ldots \\ &= a_0 + \sum_{k=1}^{\infty} a_k\cos(kt) + \sum_{l=1}^{\infty} b_l\sin(lt), \qquad f \in L^2(P). \end{aligned}$$

- (k, l) are the wavenumbers

  $\rightarrow$ really angular frequencies here (since dealing with time)
- (a_k, b_l) are the amplitudes associated with the oscillations at (k, l)

magnitudes of (a_k, b_l) give you power of the oscillations at frequencies/wavenumbers at (k, l)

Fourier analysis: background

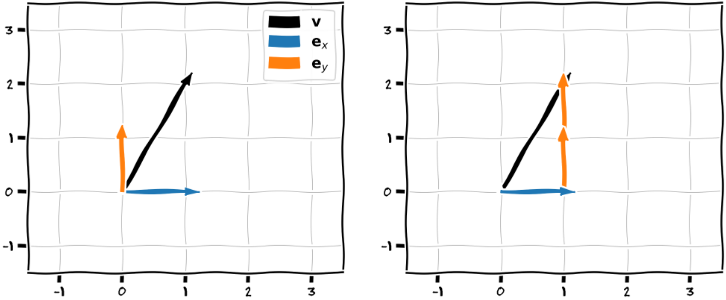


Figure: Demonstration of canonical basis in $\mathbb{R}^2$.

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1\mathbf{e}_x + 2\mathbf{e}_y = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Fourier analysis: background

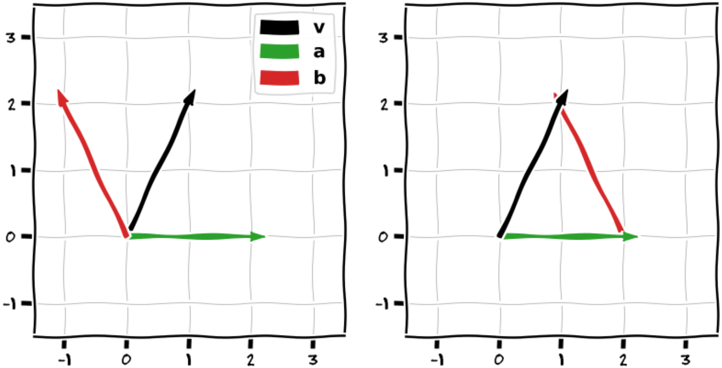


Figure: Demonstration of alternative basis in $\mathbb{R}^2$.

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 1\mathbf{a} + 1\mathbf{b}$$

Fourier analysis: background

- cf. change of units (e.g. 1 inch $\approx$ 2.5 cm)
- cf. papillon vs. Schmetterling vs. فراشة
- cf. change of co-ordinates (e.g. PCA in 04)
- cf. uses different Lego combination to make the same thing
- cf. below?

- water, 35 L; carbon, 20 kg; ammonia, 4 L; lime, 1.5 kg; phosphorus, 800 g; salt, 250 g; saltpeter, 100 g; sulfur, 80 g; fluorine, 7.5 g; iron, 5 g; silicon, 3 g; trace amounts of 15 other elements

Figure: "Ed...ward..." (sorry not sorry)

Fourier analysis: background

A basis of (say) $\mathbb{R}^2$ is a set $\{\mathbf{v}_1, \mathbf{v}_2\}$, $v_i \in \mathbb{R}^2$ where

- the elements themselves are linear independent of each other, i.e.

$$\sum_i a_i \mathbf{v}_i = \mathbf{0} \qquad \Leftrightarrow \qquad a_i = 0 \quad \forall i$$

- the set spans $\mathbb{R}^2$, i.e.,

$$\forall\, \mathbf{u} \in \mathbb{R}^2,\ \exists\, \{b_i\} : \boldsymbol{u} = \sum_i b_i \boldsymbol{v}_i.$$

- note **uniqueness** follows from the two definitions
- no criteria about v_i being orthogonal to each other

  $\rightarrow$ elements of canonical basis $\{\mathbf{e}_i\}$ are mutually orthogonal and unit length (i.e. orthonormal)

Fourier analysis

Claim:

- $\{\cos(kt), \sin(lt)\}$ is a basis of the space of functions $L^2(P)$, where the space is equipped with the inner product

$$\langle f, g \rangle = \int_I f(t) g(t)\ \mathrm{d}t$$

- with respect to that inner product, the Fourier basis is an **orthogonal basis**, i.e.,

$$\int_I \sin(kt) \cos(lt)\ \mathrm{d}t = \int_I \sin(kt) \sin(lt)\ \mathrm{d}t = \int_I \cos(kt) \cos(lt)\ \mathrm{d}t = 0.$$

Fourier analysis

- a Fourier transform takes $f(t) \in L^2(P)$ into the co-ordinates (a_k, b_l)
  $\rightarrow$ e.g. by orthogonality,

$$\int_I f(t)\cos(t)\,\mathrm{d}t = \int_I \cos(t)\left[a_0 + \sum_{k=1}^{\infty} a_k \cos(kt) + \sum_{l=1}^{\infty} b_l \sin(lt)\right]\,\mathrm{d}t$$
$$= a_1 \int_I \cos^2(t)\,\mathrm{d}t,$$

  where we can in principle compute the integrals (numerically or otherwise)
  $\rightarrow f(t)$ is in time-domain or real space
  $\rightarrow (a_k, b_l)$ is in frequency-domain or spectral space

- `scipy.fft.fft/rfft` uses the **Fast Fourier Transform** to do it (Cooley–Tukey 1965 algorithm, although was actually known to Gauss 1876)

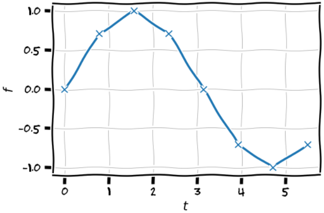
Fourier analysis

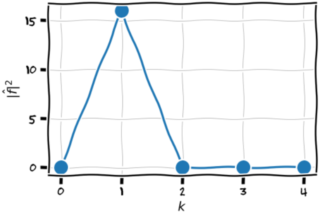
- a trivial example: taking $I = [0, 2\pi]$, $f(t) = \sin(t)$
  $\rightarrow$ choice of I means wavenumbers $(k, l) \in \mathbb{Z}$
  $\rightarrow b_1 = 1$, everything else zero
- transform is (up to **normalisation**)

$$f(t) = (f(t_0), f(t_1), \ldots) \quad \leftrightarrow \quad \hat{f} \sim (a_0,\ a_1 + \mathrm{i}b_1,\ a_2 + \mathrm{i}b_2, \ldots),$$

  where $\mathrm{i} = \sqrt{-1}$, and $(a_k, b_l) \in \mathbb{R}$
- power spectrum as a function of k is

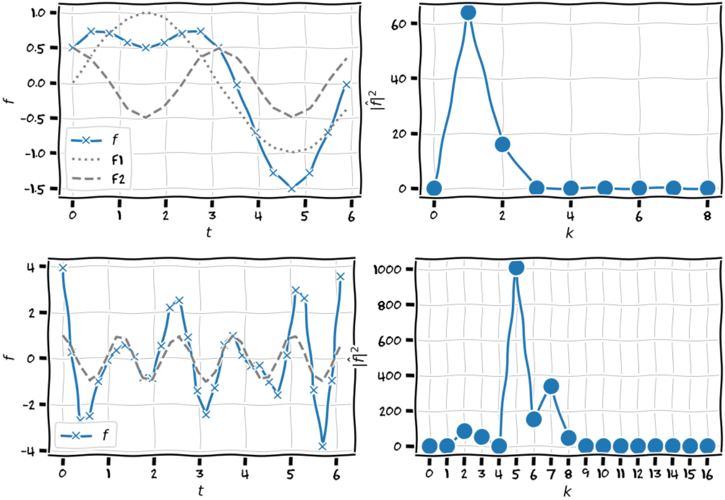
$$|\hat{f}|^2 \sim (a_0^2,\ |a_1|^2 + |b_1|^2,\ |a_2|^2 + |b_2|^2, \ldots)$$





Fourier analysis

- still pure waves (sines and cosines) on $I = [0, 2\pi]$

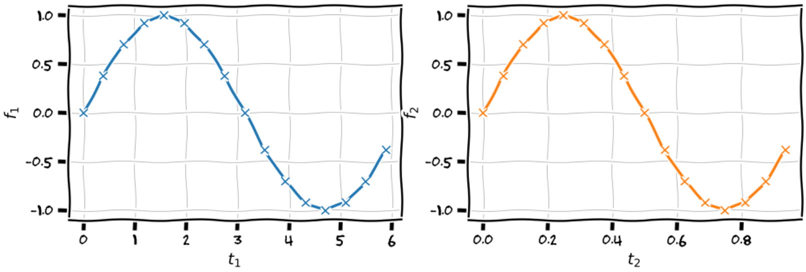


Fourier analysis: change of domain

- still pure waves (sines and cosines) but now on $I = [0, L]$
- take instead

$$\{\cos(kt), \sin(lt)\} \quad \leftrightarrow \quad \left\{\cos\left(\frac{2\pi k}{L}t\right), \sin\left(\frac{2\pi l}{L}t\right)\right\}$$

- below is when $L = 1$, showing $f_2(t) = \sin(2\pi t)$ (so $l = 1$)



Fourier analysis: change of domain

- still pure waves (sines and cosines) but now on $I = [0, L]$
- it's just a stretch/contraction, so just need to re-scale the definition of (k, l), as

$$k(\in \mathbb{Z}) \quad \rightarrow \quad \frac{2\pi k}{L}$$

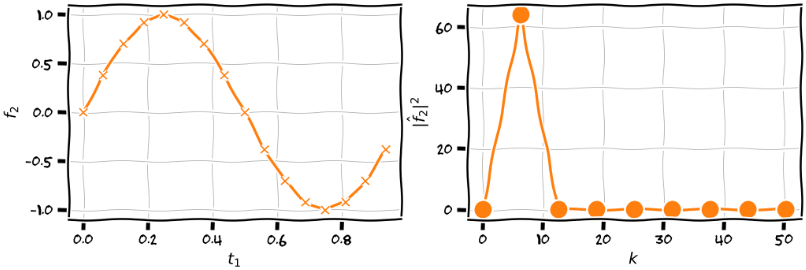


Figure: For $I = [0, 1]$. Compare with result on $I = [0, 2\pi]$ four slides ago.

Fourier analysis: change of units

Taking t to have units of seconds:

- k has units of radians per second (angular frequency here)
- $k = 2\pi\nu$, so frequency $\nu = k/(2\pi)$ and has units of per second, or **Hertz** (Hz)
- $\nu = 1/T$, so period $T = 2\pi/k$, here having units of seconds

- again, start with $k \in \mathbb{Z}$ and re-scale accordingly

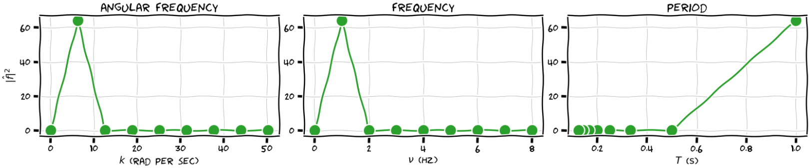
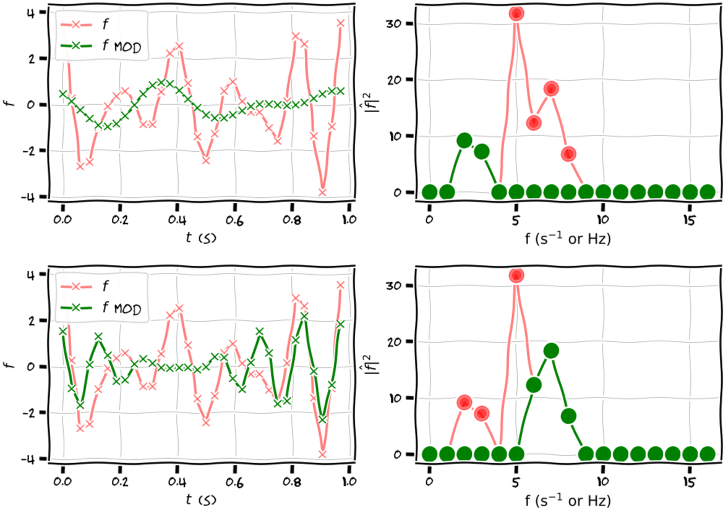


Figure: For $I = [0, 1]$, as previous slide, but in different units.

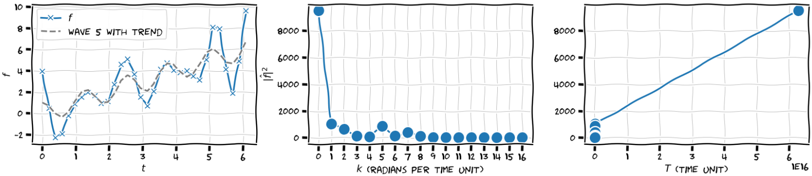
Fourier analysis: filtering

- modify the spectrum and do inverse Fourier transform



Things to be careful: signals with trends / non-zero

- a signal with some trend



$\rightarrow$ massive power in the $k = 0$ mode and in one with massive period

- non-zero mean so $a_0 \neq 0$, but this thing has no period

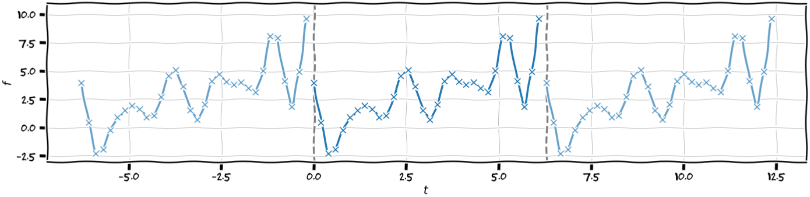
  $\rightarrow$ if you know there is a trend then detrend it

  $\rightarrow$ you could also get rid of the mean

  $\rightarrow$ or ignore / don't plot it, because the mean has no defined period of oscillation anyway

Things to be careful: non-periodic signal

- signal is not periodic (i.e. not strictly in $L^2(P)$)



- the FFT routines don't actually care, and will go ahead and return a spectrum

Things to be careful: non-periodic signal

- can force signal to be periodic through a window function

(cf. lec 07)

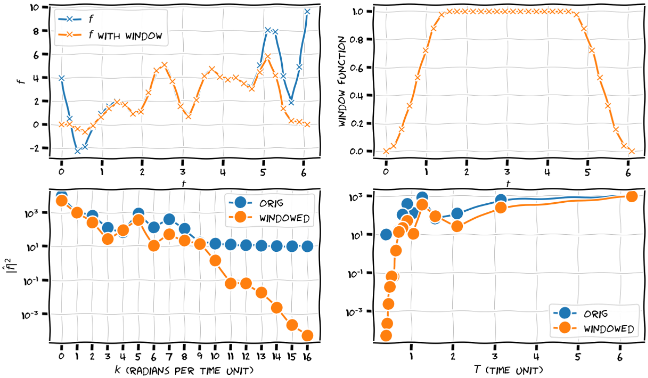


Figure: Example with Tukey window.

- different window functions looks different in real space, and have different responses in spectral space

Things to be careful: non-smooth signal

- Fourier transform actually still works

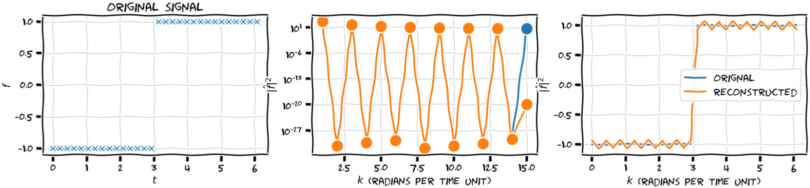


Figure: Classic example of square wave with Gibbs phenomenon.

- power is now everywhere and decays slowly with increasing wavenumber
- minor modification in spectrum leads to saw-tooth pattern

$\rightarrow$ Gibbs phenomenon or Gibbs oscillations

Things to be careful: non-smooth signal

- signal with noise

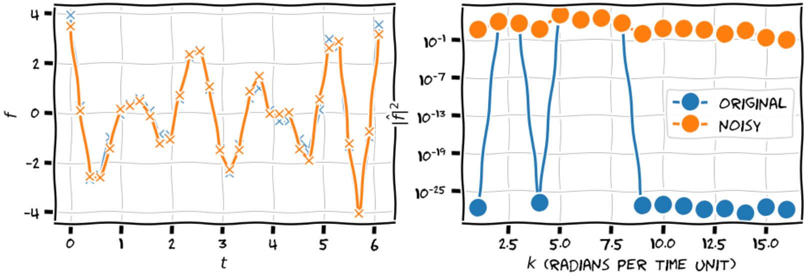


Figure: Power spectrum of signal, and signal with a bit of noise.

- power is now everywhere and decays slowly with increasing wavenumber

A real example: tides (see OCES 2003, lec 18)

Figure: High (or flood) and low (or ebb) tide at Tobermory, Isle of Mull, Scotland, using the pastel pink and red house as references. Modified images from `www.thechaoticscot.com` (left) and from myself (right).

A real example: tides (see OCES 2003, lec 18)

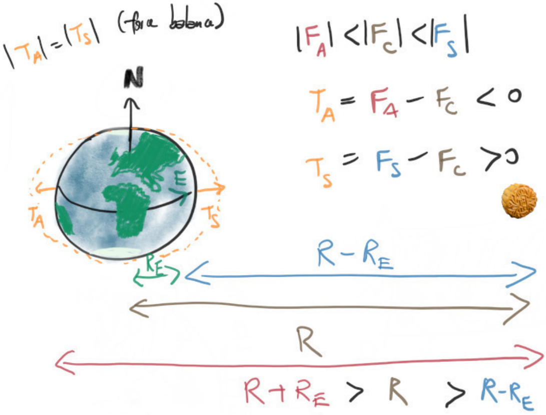


Figure: Schematic of tidal forcing by an astronomical body. Assume instantaneous response (“equilibrium theory”). No rotation is assumed here.

A real example: tides (see OCES 2003, lec 18)

symbol	period (in solar hrs)	rel. amp (to M_2)	name
M_2	12.42	1	principal lunar (semi-diurnal)
K_1	23.93	0.58	luni-solar (diurnal)
S_2	12.00	0.47	principal solar (semi-diurnal)
O_1	25.82	0.42	principal lunar (diurnal)
N_2	12.66	0.19	larger lunar elliptic (semi-diurnal)
$\vdots$	$\vdots$	$\vdots$	$\vdots$
Mf	327.85 ($\approx$ 14 days)	0.09	lunar fortnightly
Mm	661.30 ($\approx$ 28 days)	0.05	lunar monthly
SSa	4382.86	0.04	solar semi-annual

Table: Some sample tidal forcings sorted by relative amplitude to the M_2 tide (which is the largest forcing for Earth). Subset of Table 6.2 given in Wunsch (2015). The last few entries are weak and long term but they are there.

- M_2 and K_1 the dominant ones
  $\rightarrow$ usually do include these two in numerical models
- notice the periods are close to multiples of 12 hours

A real example: tides (see OCES 2003, lec 18)

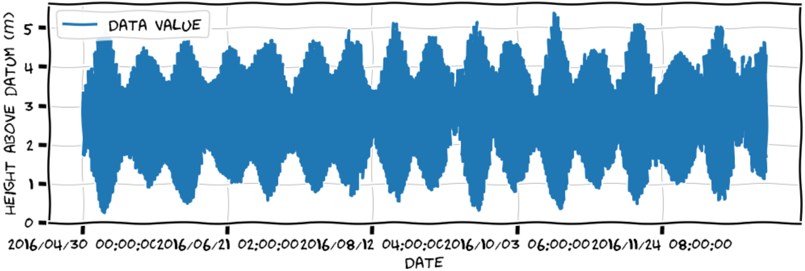


Figure: Data segment from BODC of sea level above datum at Tobermory.

- short-ish period, ignore trend with rising sea level
  $\rightarrow$ going to be ignoring $k = 0$ mode anyway

A real example: tides (see OCES 2003, lec 18)

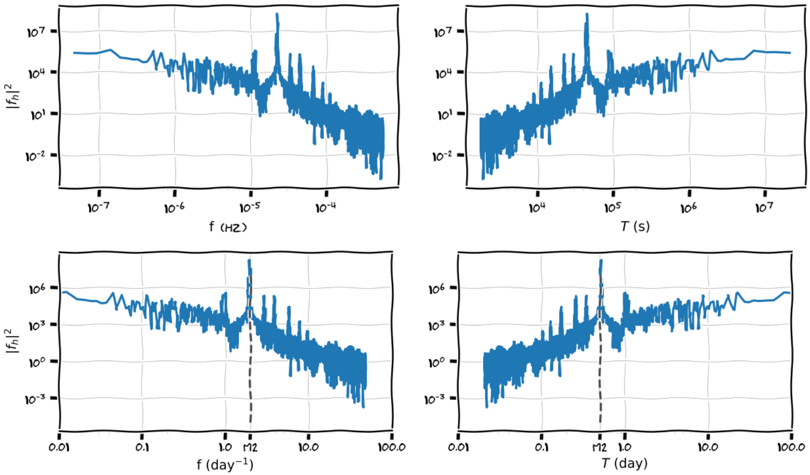


Figure: Power spectrum with respect to different quantities in different units, and denoting the M2 tide.

Jupyter notebook

go to 08 Jupyter notebook to get some code practise

- try something similar for the El-Niño 3.4 data

  $\rightarrow$ be careful of trends

  $\rightarrow$ be careful of units (time units is in **years**)

Note: none of the content I introduced in 'times series' are exclusive to 'time', and works just as well for 'space' too (see lec 09 and 10)