Chapter 5

5.1 The Lorentz Force Law

Empirically, for a moving charge in a magnetic field, we have

$$F_{mag} = Q(v \times B)$$

The generalized magnetic force is

$$F_{mag} = \int I \times B dl = I \int dl \times B$$

The magnetic force does *NO* work.

Currents

Line Currents: $I = v\lambda$
Surface Currents Density: $K = \sigma v$, $K = \frac{dI}{dl_\perp}$
(for solenoids of n winds per length, $K = nI\vec{\phi}$)
Volume Current Density: $J = \rho v$, $J = \frac{dI}{da_\perp}$

Conservation of Charge

$$\int_{S(V)} J \cdot da = \frac{d}{dt}\int_V \rho d\tau$$

$$\nabla \cdot J = \frac{\partial \rho}{\partial t}$$

5.2 Biot-Savart Law

For $\frac{\partial J}{\partial t} = 0$, we have,

$$B = \frac{\mu_0}{4\pi} I \int \frac{dl' \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0}{4\pi}\int \frac{J(r') \times B}{\mathscr{r}^2} d\tau'$$

Field of a straight wire: $B = \frac{\mu_0 I}{2\pi d}$
Field of a circular wire: $B = \frac{\mu_0 I}{2} \frac{R^2}{(R^2+z^2)^{3/2}}$

5.3 The Divergence and Curl of B

The Divergence

$$\nabla \cdot B = 0$$

The Curl: Ampere's Law

$$\nabla \times B = \mu_0 \vec{J}$$

$$\oint B \cdot dl = \mu_0 I_{enc}$$

Ampere's law is particularly useful for

1. Infinite straight wires
2. Infinite planes
3. Infinite solenoids
4. Toroids

Field of a solenoid: $B = \begin{cases} \mu_0 n I \hat{z} & \text{inside} \\ 0 & \text{outside} \end{cases}$

Field of a toroid: $B = \begin{cases} 0 & \text{inside the coil} \\ \frac{\mu_0 N I}{2\pi s} & \text{outside the coil} \end{cases}$

5.4 Magnetic Vector Potential

$$B = \nabla \times A$$

$$\nabla \times B = \nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A = \mu_0 \vec{J}$$

(3 Poisson's equations corres. x,y,z) $\quad \nabla^2 A = \mu_0 \vec{J}$

$$A(r) = \frac{\mu_0}{4\pi}\int \frac{J}{\mathscr{r}^2} d\tau' = \frac{\mu_0}{4\pi}\int \frac{K}{\mathscr{r}^2} da' = \frac{\mu_0}{4\pi}\int \frac{I}{\mathscr{r}^2} dl'$$

By using $B = \nabla \times A$, Ampere's law can also be applied on A,

$$\oint A \cdot dl = \Phi = \int B \cdot da$$

Typically, the direction of A mimics that of currents.

Boundary Conditions

$$\begin{aligned} B_{above} - B_{below} &= \mu_0(\mathbf{K} \times \hat{n}) \\ A_{above} - A_{below} &= 0 \quad \text{(continuous potential)} \\ \frac{\partial A_{above}}{\partial n} - \frac{\partial A_{below}}{\partial n} &= -\mu_0 K \end{aligned}$$

Multipole Expansion on Vector Potential

$$\frac{1}{\mathscr{r}} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'cos\alpha}} = \frac{1}{r}\sum_{n=0}^{\infty}(\frac{r'}{r})^n P_n(cos\alpha)$$

$$A(r) = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint r'^n P_n(cos\alpha) dl'$$

where the first magnetic monopole term always vanishes as $\oint dl' = 0$. The dipole term is

$$A_{dip}(r) = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \hat{r}}{r^2}$$

$$B_{dip}(r) = \frac{\mu_0}{4\pi}\frac{3(\mathbf{m} \cdot r)\hat{r} - \mathbf{m}}{r^3}$$

where $\mathbf{m}$ is the magnetic moment

$$\mathbf{m} = I \int da = Ia$$

Chapter 6