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Mata Kuliah : Statistika 3

HOMEWORK — II

1. Suppose X is gamma random variable with $\alpha = 2$ and $\beta = 1$.
a) Find $P(X<3)$ by hands (manually, not using computer).

We know that the gamma distribution's density function is given by :

$$f(x,\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad \text{for } x>0$$

therefore by the parameter given in the problem $\alpha = 2$ and $\beta = 1$, we can calculate :

$$f(x,2,1) = \frac{1^2}{\Gamma(2)} x^{2-1} e^{-1 \cdot x} = \frac{1}{\Gamma(2)} x e^{-x}$$

we also know that $\Gamma(\alpha) = (\alpha-1)!$ therefore $\Gamma(2) = 1! = 1$.
hence,

$$f(x,2,1) = xe^{-x}$$ the probability density function (PDF)

To find $P(X<3)$, we need to integrate the density function from 0 to 3.

$$f_X(3) = P(X<3) = \int_0^3 xe^{-x}\, dx$$

To calculate that, we need to integrate by part :

Suppose $u = x$ $\quad du = dx$
$dv = e^{-x} dx$ $\quad v = -e^{-x}$

$$\begin{aligned} \int x \, dv &= uv - \int v\, du \\ &= x(-e)^{-x} - \int -e^{-x} dx \\ &= -xe^{-x} + (-e^{-x}) + c \\ &= -xe^{-x} - e^{-x} + c \end{aligned}$$

therefore

$$\begin{aligned} P(X<3) = \int_0^3 xe^{-x}\, dx &= -xe^{-x} - e^{-x} \Big|_0^3 \\ &= -3e^{-3} - e^{-3} - [0e^0 - e^0] \\ &= -4e^{-3} + 1 \\ &= 1 - 4e^{-3} \approx 0.800852 \end{aligned}$$

b) Use software to plot the PDF with area where $P(X<3)$ be shaded in blue colour.

This part is done in R, the result is shown separately.
Please look at the end of this file.

2. Given this function

$$f(x,y) = c(1-x)(1-y)$$
$$-1 \le x \le 1$$
$$-1 \le y \le 1$$

a) find the value of c that makes this function the join PDF of X and Y.

To make that function become a join PDF of X and Y, we need to ensure:

$$\int_{-1}^{1}\int_{-1}^{1} c(1-x)(1-y)\,dx\,dy = 1$$

Since c is a constant we can rewrite

$$c\int_{-1}^{1}\int_{-1}^{1} (1-x)(1-y)\,dx\,dy = 1$$

We solve the integral w.r.t dx first and treat $(1-y)$ as a constant.

$$c\int_{-1}^{1}(1-y)\left[\int_{-1}^{1}(1-x)\,dx\right]dy = 1$$

$$c\int_{-1}^{1}(1-y)\underbrace{\boxed{\left[x-\frac{1}{2}x^2\right]_{-1}^{1}}}_{=2}\,dy = 1$$

$$c\int_{-1}^{1}(1-y)\cdot 2\cdot dy = 1$$

$$2c\left[y-\frac{1}{2}y^2\right]_{-1}^{1} = 1$$

$$2c\left(\frac{1}{2}-\left(-\frac{3}{2}\right)\right) = 1$$

$$4c = 1$$

$$c = \frac{1}{4}$$

The value of c that makes $f(x,y) = c(1-x)(1-y)$ the join PDF of X and Y is $\frac{1}{4}$.

b) Find the conditional probability $f(x|Y=0{,}5)$.

We know that for continuous random variable

$$f(x|Y=y) = \frac{f(x,y)}{f_Y(y)}$$

where $f(x,y)$ is the value of joint PDF at $Y=y$

$f_Y(y)$ is the marginal PDF of Y.

hence we calculate them first,

$$f_Y(y) = \int_{-1}^{1} \frac{1}{4}(1-x)(1-y)\,dx$$

$$= \frac{1}{4}(1-y)\int_{-1}^{1}(1-x)\,dx$$

$$= \frac{1}{4}(1-y)\left[x-\frac{1}{2}x^2\right]_{-1}^{1}$$

$$= \frac{1}{4}(1-y)\cdot 2$$

$$= \frac{1}{2}(1-y) \longrightarrow f_Y(0{,}5) = \frac{1}{2}(1-0{,}5)$$

$$= \frac{1}{4}$$

$$f(x,\tfrac{1}{2}) = \frac{1}{4}(1-x)(1-\tfrac{1}{2})$$

$$= \frac{1}{8}(1-x)$$

therefore :

$$f(x|Y=0{,}5) = \frac{f(x,\frac{1}{2})}{f_Y(y)}$$

$$= \frac{\frac{1}{8}(1-x)}{\frac{1}{4}}$$

$$= \frac{1}{2}(1-x)$$

The conditional probability $f(x|Y=0{,}5) = \frac{1}{2}(1-x)$

KIKY

3. Explain why $Cov(X, Y) = 0$ when X and Y are independent.

First we know that for two random variable X and Y

$$Cov(X,Y) = E(XY) - \mu_X \mu_Y$$

and when X and Y are independent, it means that knowing the value of X provides no information about Y.
That is why, when X and Y are independent :

$$E(XY) = E(X) . E(Y)$$

Where by definition

$$E(X) = \sum_x x . P(x) \rightarrow \text{the weighted average of all possible values that } X \text{ can take on}$$

$E(X) = \mu_x$
the same for $E(Y) = \mu_y$.

Hence, we can rewrite the covariance in :

$$\begin{aligned} Cov(X,Y) &= E(XY) - \mu_x \mu_y && \text{since } X \text{ and } Y \text{ are independent} \\ &= E(X)E(Y) - \mu_x \mu_y && \text{by property of } E(XY) = E(X)\,E(Y) \\ &= \mu_x \mu_y - \mu_x \mu_y \\ &= 0 \end{aligned}$$

Statistics 3 - Homework 2

By: Andreas Syaloom Kurniawan (552751)

- Script ini bisa juga diakses di Github

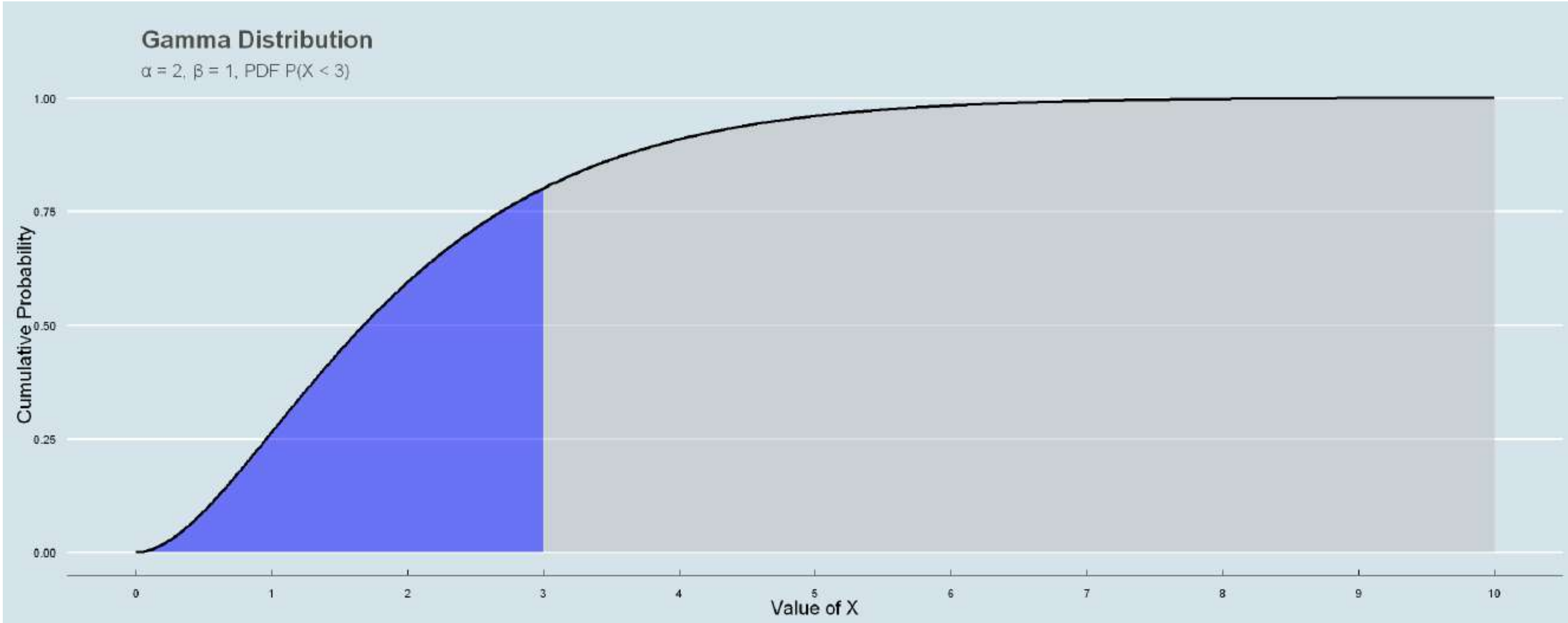
Problem 1

Suppose X is a gamma random variable with $\alpha = 2$ and $\beta = 1$.

- Use software to plot the pdf with area where $P(X < 3)$ be shaded in blue colour.

In [5]:

```
####################################################################################
## PLOTTING GAMMA CUMMULATIVE DISTRIBUTION FUNCTION
####################################################################################

library(ggplot2) #untuk visualisasi ggplot secara keseluruhan --> dipakai mulai dari script 'ggplot'
library(dplyr) # untuk mengolah data di bagian dataframe -> dipakai mulai dari script 'mutate'
library(ggthemes) # untuk pilihan jenis-jenis background theme di R nya --> dipakai untuk 'theme_economist'

#Pendefinisian parameter
alpha <- 2
theta <- 1

# Membuat data frame untuk CDF dari distribusi Gamma
data_frame <- data.frame(x = seq(0, 10, length.out = 1000)) %>%
  mutate(prob = pgamma(x, shape = alpha, scale = theta),
         Interval = ifelse(x <= 3, "Up to 3", "Other"))

options(repr.plot.width=20, repr.plot.height=8) # this is just to make the plots wider (extend to the end of laptop c

# Plot CDF dengan area yang diwarnai dan garis yang menonjol
ggplot(data_frame, aes(x = x, y = prob)) +
  geom_area(aes(fill = Interval), alpha = 0.5) + # Setengah transparan untuk area
  geom_line(color = "black", size = 1.2) + # Garis CDF yang jelas
  scale_fill_manual(values = c("Up to 3" = "blue", "Other" = "grey")) +
  scale_x_continuous(breaks = seq(0, 10, by = 1)) + # Menampilkan sumbu X dari 0 hingga 10 dengan langkah 1
  labs(title = "Gamma Distribution ",
       subtitle = "α = 2, β = 1, PDF P(X < 3)",
       x = "Value of X",
       y = "Cumulative Probability") +
  theme_economist() +
  theme(legend.position = "none",
        plot.title = element_text(size= 22, hjust=0.01, color = "#4e4d47", margin = margin(b = -0.1, t = 0.4, l = 2,
        plot.subtitle = element_text(size= 16, hjust=0.01, color = "#4e4d47", margin = margin(b = -0.1, t = 0.43, l =
        axis.title.x = element_text(size = 18), # Perbesar teks sumbu X
        axis.title.y = element_text(size = 18)) # Perbesar teks sumbu Y)
```



In [7]:
```
#Validating the CDF result
x = 3
pgamma(q = x, shape = alpha, scale = theta)

# Di gambar terlihat bahwa saat X sama dengan 3, ordinat Y berada di 0.800 ( di atas 0.75)
```

0.800851726528544

In [100…
```
##################################################################### END ############################################
```