1 The Dirac delta function is the derivative of the Heaviside step function

1.1 An integral representation of the Heaviside step function

First, prove that the following integral representation of the Heaviside step function is valid:

$$H(x) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\tau - i\epsilon} \mathrm{e}^{ix\tau} d\tau$$

First we define the following closed curve C_+ to be the contour from $-R$ to R along the real axis, (C_0), followed by a semicircular contour in the upper half plane from $+R$ to $-R$ in the anticlockwise direction (C_R). Note that C_+ contains the pole at $+i\epsilon$.

Similarly, we define C_- to be the same construction, but with the contour closed in the lower half plane in the clockwise direction. Note that C_- does *not* contain the pole at $+i\epsilon$.

Next we define the following complex integral:

$$\begin{aligned}
I_+ &= \frac{1}{2\pi i} \oint_{C_+} \frac{1}{\omega - i\epsilon} \mathrm{e}^{i\omega x} d\omega \\
&= \frac{1}{2\pi i} \left(\int_{C_0} + \int_{C_R} \right) \frac{1}{\omega - i\epsilon} \mathrm{e}^{i\omega x} d\omega
\end{aligned}$$

From the Residue Theorem we know that

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \mathrm{res}_{z=z_k} f(z)$$

where $f(z)$ is some analytic function in the region R except for a finite number of poles at $z_1, ..., z_n$ and C is a simple closed curve that encircles the poles anticlockwise.

Using the Residue Theorem with $x > 0$ then, we can calculate I_+ as

$$\begin{aligned}
I_+ &= \frac{1}{2\pi i} 2\pi i \mathrm{res}_{\omega = i\epsilon} f(\omega) \\
&= (\omega - i\epsilon) \frac{1}{\omega - i\epsilon} \mathrm{e}^{ix(i\epsilon)} \\
&= \mathrm{e}^{-\epsilon x} \\
\lim_{\epsilon \to 0^+} \mathrm{e}^{-\epsilon x} &= 1
\end{aligned}$$

For $x < 0$ instead we immediately know that $I_- = 0$ since the contour C_- contains no poles and is therefore zero by Cauchy's theorem.

Now, from Jordan's lemma, we know that

$$\lim_{R\to\infty}\int_{C_R}\frac{1}{\omega-i\epsilon}\mathrm{e}^{i\omega x}d\omega=0$$

which leaves us with just the integral over C_0:

$$I_+=\frac{1}{2\pi i}\int_{-R}^{R}\frac{1}{\omega-i\epsilon}\mathrm{e}^{i\omega x}d\omega$$

So we can see that in the limit as $R\to\infty$ we have that

$$H(x)=\begin{cases}I_+=1 & x>0\\ I_-=0 & x<0\end{cases}$$

which is the definition of the step function, with step at $x=0$.

1.2 Derivative of the integral step function representation

Let us take the derivative w.r.t x of the integral representation of the step function as proved above.

$$\partial_x H(x)=\lim_{\epsilon\to0^+}\int_{-\infty}^{\infty}\frac{1}{\tau-i\epsilon}\cdot i\tau\mathrm{e}^{ix\tau}d\tau$$

Now use a change of variables:

$$\begin{aligned}z&=\tau-i\epsilon\\ i\tau&=i(z+i\epsilon)\\ d\tau&=dz\end{aligned}$$

Hence we get that

$$\begin{aligned}&=\lim_{\epsilon\to0^+}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{z}i(z+i\epsilon)\mathrm{e}^{ix(z+i\epsilon)}dz\\ &=\lim_{\epsilon\to0^+}\frac{\mathrm{e}^{-\epsilon x}}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{ixz}dz+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{-\epsilon}{z}\mathrm{e}^{ixz}\mathrm{e}^{-\epsilon}dz\\ &=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{ixz}dz\\ &:=\delta(x)\end{aligned}$$

1.3 Fourier representation of the Dirac delta function

Finally, we conclude by showing that the integral representation of the Dirac delta function that we use to complete the proof above satisfies the required properties of a delta function.

In particular, consider the following Fourier and inverse Fourier transform of some function $f(x)$:

$$\begin{aligned}
f(x) &= \frac{1}{2\pi}\int \tilde{f}(k)\mathrm{e}^{-ikx}dk \\
&= \frac{1}{2\pi}\int\int \mathrm{e}^{ikx'}dx' f(x')\mathrm{e}^{-ikx} \\
&= \frac{1}{2\pi}\int\int dx' f(x')\mathrm{e}^{ik(x'-x)}dk \\
&= \int dx' f(x')\delta(x'-x)
\end{aligned}$$

Thus we see that the characteristic property of the Dirac delta function is demonstrated.

To finish, we need to show that the integral of the delta function over real space is unity:

$$\begin{aligned}
\int_{-\infty}^{\infty}\delta(x)dx &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{ixz}dxdz \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{ix}\mathrm{e}^{ixz}dx\bigg|_{z=-\infty}^{z=\infty} \\
&= \lim_{\epsilon\to 0^+}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dx}{x-i\epsilon}\mathrm{e}^{ixz}\mathrm{e}^{\epsilon z}\bigg|_{z=-\infty}^{z=\infty} \\
&= \lim_{\epsilon\to 0^+}\frac{2\pi i}{2\pi i}\mathrm{e}^{-\epsilon z}\mathrm{e}^{\epsilon z}\bigg|_{z=-\infty}^{z=\infty} \\
&= 1
\end{aligned}$$