Analytic regularization of quantum field theories on curved backgrounds

Antoine Géré

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Dipartimento di Matematica

Università degli studi di Genova

- rigorous study of **Quantum Field Theory**

  $\rightarrow$ **scalar field** ϕ

- **Interacting theory**

$$\mathrm{P}\phi + \mathrm{V}^{(1)}(\phi) = \left(\Box + \xi R + m^2\right)\phi + \mathrm{V}^{(1)}(\phi) = 0$$

$\rightarrow$ local / non linear potential, e.g : $\mathrm{V}(\phi) = \int_{\mathcal{M}} \mathrm{d}x \ \frac{\lambda}{4!} \ f(x) \ \phi(x)^4$

$\rightarrow$ **perturbative theory** : (expansion in series w.r.t. coupling constant λ)

Problems occuring :

1. ultraviolet (UV) divergences occurring at very high energy

2. infrared (IR) divergences occurring at very low energy

3. and the series usually do not converge in any rigorous way

Publications

A. Géré, P. Vitale, and J.-C. Wallet, “Quantum gauge theories on noncommutative 3-d space”, arxiv: 1312.6145 [hep-th], Physical Review D 90 (2014) 045019.

A. Géré, and J.-C. Wallet, “Spectral theorem in noncommutative field theories: Jacobi dynamics”, arxiv: 1402.6976 [math-ph], J. Phys. Conf. Ser. 634 (2014) 012006.

A. Géré, T. Jurić, and J.-C. Wallet, “Noncommutative gauge theories on $\mathbb{R}^3_\lambda$: Perturbatively finite models”, arxiv: 1507.08086 [hep-th], JHEP, 12 (2015) 1–29.

A. Géré, T.-P. Hack, and N. Pinamonti, “An analytic Regularization scheme on curved spacetimes with applications to cosmological spacetimes”, arxiv: 1505.00286 [math-ph], accepted in Classical and Quantum Gravity.

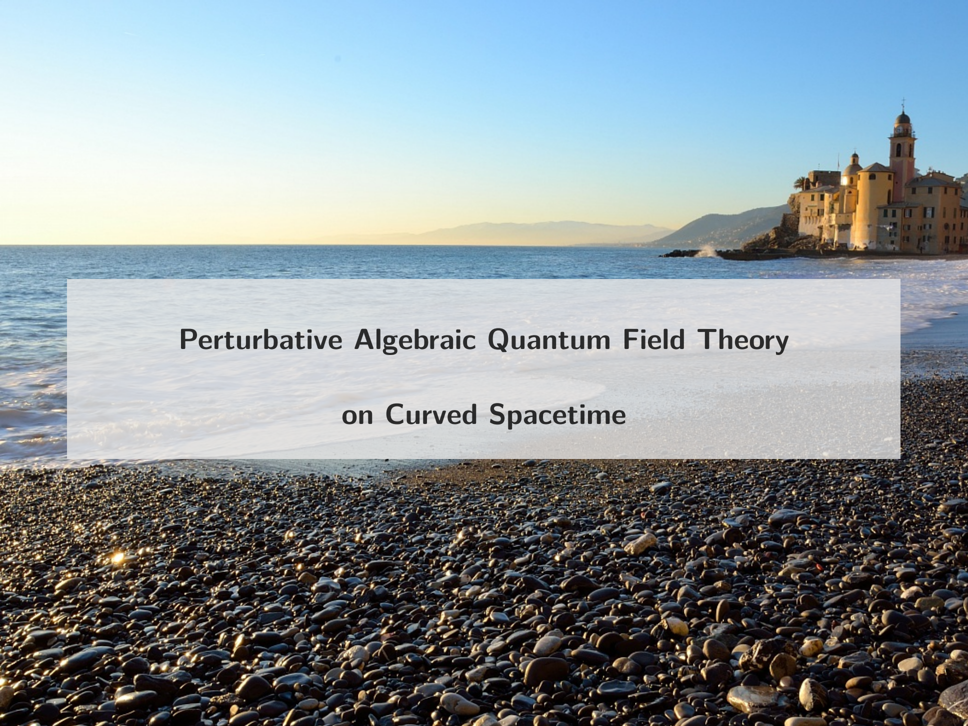
Story & plan

Motivation

- perturbative algebraic quantum field theory (pAQFT)
  $\rightarrow$ **conceptually well known**
  [Brunetti, Dütsch, Fredenhagen, Hollands, Köhler, Rejzner, Wald, ...]
- in pAQFT on curved spacetime (CST)
  $\rightarrow$ procedure **unconvenient for computations**
  [Brunetti & Fredenhagen, Hollands & Wald, Dang]
- desire to use framework of pAQFT for **cosmological model** !

What I am going to talk about

- **pAQFT** $\rightarrow$ in order to identify the regularization problem!
- a framework for an **analytic regularization** on CST
- **explicit computations** on a particular cosmological spacetime



Perturbative Algebraic Quantum Field Theory

on Curved Spacetime

Physical input

▸ details

- $(\mathcal{M}, \mathrm{g})$: **4 dimensional** spacetime

  $\rightarrow$ globally hyperbolic smooth manifold with Lorentzian metric

- **Off shell** configuration space : **real scalar field**

  $\rightarrow$ **space of real valued smooth maps** : $\phi \in \mathcal{C}^{\infty}(\mathcal{M}, \mathbb{R})$

- **Space of observables** : $\mathcal{F}(\mathcal{M})$

$$\mathsf{F} : \begin{cases} \mathcal{C}^{\infty}(\mathcal{M}, \mathbb{R}) & \rightarrow \quad \mathbb{C} \\ \phi & \mapsto \quad \mathsf{F}(\phi) \end{cases}$$

Need to make restriction to have good working properties

$\rightarrow$ **support** and **regularity** properties

Observables as functionals

- Observables : complex valued functionals

- **Compactly supported** functionals : $\mathcal{F}_0(\mathcal{M})$ ▸ definition

- **Smooth** functionals : possible to compute functional derivatives at all orders

  $\rightarrow$ Gâteaux derivatives ▸ functional derivative

Example

$$\mathrm{F}(\phi) = \int_{\mathcal{M}} \mathrm{d}x \, \frac{\lambda}{4!} \, f(x) \, \phi(x)^4 \, , \quad \mathrm{F}^{(1)}(\phi)[\psi] = \int_{\mathcal{M}} \mathrm{d}x \, \frac{\lambda}{3!} \, f(x) \, \phi(x)^3 \, \psi(x)$$

- functionals derivatives : distributions

  $\rightarrow$ need to have **"regularity" under control** $\rightarrow$ **wave front set**

Definition – Wave front set. *[Hörmander 1983]*

wave front set

The wave front set $\mathrm{WF}(u) \subset T^*\mathcal{M}^n$ of $u \in \mathcal{C}_0^\infty(\mathcal{M}^n)'$ as follows

(i) for every $x \in \mathcal{M}^n$ where u is singular, choose a non-vanishing test function $f \in \mathcal{C}_0^\infty(\mathcal{M}^n)$

(ii) $(x, k) \in \mathrm{WF}(u)$ iff $\hat{fu}(k)$ is **not** rapidly decreasing in the direction of $k \neq 0$ for some f.

$\rightarrow$ **local** and **covariant** under coordinate transformations.

Microlocal analysis. *[Hörmander 1983]*

The **pointwise product** of two distributions u, v is well defined if

$$\mathrm{WF}(u) \oplus \mathrm{WF}(v) \neq 0 \;\Rightarrow\; \exists!\; u.v \in \mathcal{C}_0^\infty(\mathcal{M}^n)'$$

- Equation of motion : **generalized Klein Gordon** equation

$$\mathrm{P}\phi = \left(\Box + \xi R + m^2\right)\phi = 0$$

$\mathcal{M}$: **globally hyperbolic** spacetime $\rightarrow$ Cauchy problem well posed

- **Fundamental solutions** $\Delta_{\mathrm{r/a}}$ $\rightarrow$ **causal propagator** of P

$$\Delta = \Delta_{\mathrm{r}} - \Delta_{\mathrm{a}}\ , \quad \text{thus} \qquad \mathrm{P}_x\Delta(x,y) = 0$$

- **Off shell** algebra

Definition (Classical free off shell algebra)

We define the classical free off shell algebra as follows

$$\mathcal{A}_{\mathrm{reg}}(\mathcal{M}) = (\mathcal{F}_{\mathrm{reg}}(\mathcal{M}),^{*},\cdot)\ , \quad \text{with}$$

$$\mathcal{F}_{\mathrm{reg}}(\mathcal{M}) = \left\{ \mathrm{F}(\phi) \;\middle|\; \mathrm{F}(\phi) \in \mathcal{F}_0(\mathcal{M}) \ \text{ is smooth, } \ \mathrm{F}^{(n)}(\phi) \in \mathcal{C}_0^{\infty}(\mathcal{M}^n) \right\} .$$

The involution is defined as $\mathrm{F}^*(\phi) = \overline{\mathrm{F}(\phi)}$, where $\bar{\cdot}$ is the complex conjugation.

Classical and **quantum observables** :

in the same vector space endowed with **different products** $\rightarrow$ different algebras

$$\mathcal{A}_{\text{reg}}(\mathcal{M})[[\hbar]] \quad \underset{\hbar\to 0}{\longrightarrow} \quad \mathcal{A}_{\text{reg}}(\mathcal{M}) \ .$$

"quantum regular observable" : $\mathcal{F}_{\text{reg}}(\mathcal{M})[[\hbar]]$

Definition (Quantum free regular off shell algebra)

We define it as a noncommutative, unital, associative $*$–algebra as follows

$$\mathcal{A}_{\text{reg}}(\mathcal{M})[[\hbar]] = (\mathcal{F}_{\text{reg}}(\mathcal{M})[[\hbar]], ^* , \star) \ , \quad \text{with}$$

$$(\mathsf{F} \star \mathsf{G})(\phi) = \mathsf{F}(\phi) \cdot \mathsf{G}(\phi) + \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} \left\langle \mathsf{F}^{(n)}, \Delta_+^{\otimes n} \mathsf{G}^{(n)} \right\rangle \ ,$$

where Δ_+ is a bidistribution, its antisymmetric part is the causal propagator, i.e. $i\Delta(x,y) = \Delta_+(x,y) - \Delta_+(y,x)$. This product is associative.

$\rightarrow$ **noncommutative product** : implements canonical commutation relations

$\rightarrow$ **freedom** : symmetric part of Δ_+ $\rightarrow$ isomorphic algebras

$\rightarrow$ **off shell**, but $\star$ product depends on the equation of motion with Δ_+

Hadamard condition *[Radzikowski 1996]*

$$\mathrm{WF}(\Delta_+) \;=\; \left\{ (x,k_x;y,-k_y) \in T^*\mathcal{M}^2 \backslash \{0\} \;\middle|\; (x,k_x) \sim (y,k_y),\;\; k_x \triangleright 0 \right\}$$

$\sim$: $\exists$ a null geodesic connecting x and x', and k' is the parallel transport of k.

$k_x \triangleright 0$: k_x is future directed

- **Microcausal** functionals $\mathcal{F}_{\mu c}(\mathcal{M})$

$$\mathcal{F}_{\mu c}(\mathcal{M}) = \left\{ \mathrm{F}(\phi) \;\middle|\; \begin{array}{l} \mathrm{F}(\phi) \in \mathcal{F}_0(\mathcal{M}),\;\; \mathrm{F}^{(n)}(\phi) \in \mathcal{E}'(\mathcal{M}^{\otimes n}), \\ \mathrm{WF}(\mathrm{F}^{(n)}) \cap \left(\mathcal{M}^n \times (\overline{V_+^n} \cup \overline{V_-^n})\right) = \emptyset \end{array} \right\}$$

$\rightsquigarrow$ **pointwise product** of Δ_+ : **well defined !**

$$\mathrm{WF}(\Delta_+) \oplus \mathrm{WF}(\Delta_+) \neq 0$$

- **Interactions** $\rightarrow$ **Local** functionals $\mathcal{F}_{\mathrm{loc}}(\mathcal{M})$

$$\mathcal{F}_{\mathrm{loc}}(\mathcal{M}) := \left\{ \mathrm{F}(\phi) \in \mathcal{F}_{\mu c}(\mathcal{M}) \;\middle|\; \mathrm{supp}\left(\mathrm{F}^{(n)}(\phi)\right) \subset d_n = \{(x,\dots,x) \subset \mathcal{M}^n\} \right\}$$

Perturbation of the free theory to **build** the **interacting theory**

Causal factorization property

$$F \cdot_T G = \begin{cases} F \star G & \text{if } \operatorname{supp}(F) \text{ is later than } \operatorname{supp}(G) \\ G \star F & \text{if } \operatorname{supp}(G) \text{ is later than } \operatorname{supp}(F) \end{cases}$$

Definition (Regular time ordered product)

The time ordered product on $\mathcal{F}_{reg}(\mathcal{M})[[\hbar]]$ is defined as follows

$$(F \cdot_T G)(\phi) = F(\phi) \cdot G(\phi) + \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}, \Delta_f^{\otimes n} G^{(n)} \right\rangle ,$$

where Δ_f is a time ordered version of Δ_+, i.e. :

$\Delta_f(x, y) = \Theta(t_x - t_y)\Delta_+(x, y) + \Theta(t_y - t_x)\Delta_+(y, x).$

- **Bogoliubov formula**

$\rightarrow$ **represents** the **interacting algebra** on the **free algebra**

$$\mathsf{R}_\mathsf{V}(\mathsf{F}) = S(\mathsf{V})^{\star -1} \star (S(\mathsf{V}) \cdot_\mathsf{T} \mathsf{F})$$

$$\text{with} \quad S(\mathsf{V}) = \exp_\mathsf{T}(\mathsf{V}) = \sum_{n=0}^{\infty} \frac{i^n}{n!\ \hbar^n} \underbrace{\mathsf{V}(\phi) \cdot_\mathsf{T} \cdots \cdot_\mathsf{T} \mathsf{V}(\phi)}_{n \text{ times}}$$

$\rightarrow$ "interacting products" in terms of the "free products"

$$\mathsf{F} \star_{\mathrm{int}} \mathsf{G} = \mathsf{R}_\mathsf{V}^{-1}\left(\mathsf{R}_\mathsf{V}(\mathsf{F}) \star_{\mathrm{free}} \mathsf{R}_\mathsf{V}(\mathsf{G})\right)$$

$\rightarrow$ $\mathsf{R}_\mathsf{V}\left(\mathsf{F}_{\mathrm{lin}}\left(\mathsf{P}\phi + \mathsf{V}^{(1)}(\phi)\right)\right) = \mathsf{F}_{\mathrm{lin}}(\mathsf{P}\phi), \quad \mathsf{F}_{\mathrm{lin}}(\phi) = \int_\mathcal{M} \mathrm{d}x\ \phi(x)\ f(x)$

- **"Causal functionals"** $\mathcal{F}_\mathsf{T}(\mathcal{M})$: time ordered products of $\mathcal{F}_{\mathrm{loc}}(\mathcal{M})$

Proposition (Interacting quantum algebra)

$$\mathsf{R}_\mathsf{V} \; : \; \mathcal{R}_\mathsf{T}(\mathcal{M})[[\hbar]] \; \rightarrow \; \mathcal{A}_\mathsf{T}(\mathcal{M})[[\hbar]]$$

$$\textit{with} \quad \mathcal{A}_\mathsf{T}(\mathcal{M})[[\hbar]] = (\mathcal{F}_\mathsf{T}(\mathcal{M})[[\hbar]],^*, \star\cdot_\mathsf{T})$$

First insight of our future problem

Time ordered product

$$(F \cdot_T G)(\phi) = F(\phi) \cdot G(\phi) + \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}(\phi), \Delta_f^{\otimes n} G^{(n)}(\phi) \right\rangle$$

$\rightarrow$ powers of Δ_f **ill defined**

$\rightsquigarrow$ **looking for** a time ordered product satisfying a **set of axioms**

- T1 – Initial condition.
- T2 – Symmetry.
- T3 – Unitarity.
- T4 – Causal factorization.
- T5 – Field independence.
- T6 – Locality and covariance.
- T7 – Microlocal spectrum condition.
- ...

[Hollands & Wald]

A Covariant Regularization Scheme

Definition

- $\mathring{u} \in \mathcal{D}'(\mathcal{M}^n)$: **extension** of $u \in \mathcal{D}'\left(\mathcal{M}^n \setminus d_n\right)$ **if** $\mathring{u}(\phi) = u(\phi)$
- $\left\{u^{(\alpha)} \in \mathcal{D}'(\mathcal{M}^n)\right\}$: **analytic regularization** of a distribution, s.t. $u^{(\alpha)}$ weakly analytic w.r.t $\alpha \in \Omega \setminus \{0\} \subset \mathbb{C}$, and $\lim\limits_{\alpha \to 0} u^{(\alpha)} = u$

$\to$ **total diagonal** : $d_n = \{(x, \ldots, x) \in \mathcal{M}^n\}$

Extension using an analytic regularization

$$\mathring{u} = \lim_{\alpha \to 0} (1 - \mathrm{pp})\, u^{(\alpha)}$$

$\to$ $\mathrm{pp}\left(u^{(\alpha)}\right)$ is supported in d_n, and poles in α of finite orders

- The **scaling degree** of u towards d_n is defined as

▸ scaling degree

$$\mathrm{sd}(u) \;:=\; \inf\left\{\omega \in \mathbb{R} \;\middle|\; \lim_{\lambda \downarrow 0} \lambda^{\omega}\, u_\lambda \;=\; 0\right\}$$

Corollary *[Brunetti & Fredenhagen (2000)]*

If $\mathrm{sd}(u) < 4(n-1)$, **then** $\exists!\ \mathring{u}$ towards d_n with **same scaling degree**.

Problem : **extending** $\cdot_{\mathrm{T}}$ to **local functionals**

$$\mathcal{T}_n \; : \; \begin{cases} \mathcal{F}_{\mathrm{loc}}(\mathcal{M})[[\hbar]]^{\otimes n} & \rightarrow \quad \mathcal{F}_{\mathrm{T}}(\mathcal{M})[[\hbar]] \\ \mathrm{F}_1 \otimes \; \dots \; \otimes \mathrm{F}_n & \mapsto \quad \mathrm{m_n} \circ \mathrm{T}_n\Big(\mathrm{F}_1 \otimes \; \dots \; \otimes \mathrm{F}_n\Big) \end{cases}$$

Graphical view :

$\rightarrow$ $\mathcal{G}_n$: set of graphs γ with n vertices, with $\ell_{ii} = 0$

ℓ_{ij} : number of edges between vertices x_i and x_j

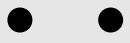
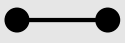
details

$$\mathrm{T}_n = \sum_{\gamma \in \mathcal{G}_n} \mathrm{T}_\gamma \; , \quad \text{with} \quad \mathrm{T}_\gamma = \frac{1}{\mathrm{N}(\gamma)} \; \langle \mathrm{t}_\gamma \; , \; \delta_\gamma \rangle$$

$$\mathrm{t}_\gamma = \prod_{(i,j)\in\gamma} \Delta_{\mathrm{f}}(x_i, x_j)^{\ell_{ij}} \quad \text{powers of} \quad \Delta_{\mathrm{f}} \quad : \textbf{ ill defined!}$$

$$\mathrm{sd}\left(\Delta_{\mathrm{f}}(x_i, x_j)^{\ell_{ij}}\right) = 2\ell_{ij}, \text{ unique extension for only } \ell_{ij} < 2$$

Example

$\mathrm{F} \cdot_{\mathrm{T}} \mathrm{G} =$  $+ \hbar$  $+ \hbar^2$  $+ \hbar^3$  $+ \mathcal{O}(\hbar^4)$

Strategy

- t_γ : well defined **outside** of all **partial diagonals**, namely on $\mathcal{D}(\mathcal{M}^n \setminus D_n)$

$$D_n = \left\{ x_1, \ldots, x_n \;\middle|\; x_i = x_j \;\text{ for at least one pair }\; (i,j), \;\text{ with }, \; i \neq j \right\}$$

Problem

Extend t_γ to $\mathcal{D}(\mathcal{M}^n)$

- **Regularized expression**

$$\mathrm{t}_\gamma^{(\boldsymbol{\alpha})} = \prod_{(i,j)\in\gamma} \Delta_{\mathrm{f}}(x_i, x_j)^{(\alpha_{ij}, \ell_{ij})}$$

- **Principal part** : supported in partial diagonals D_n

  $\rightarrow$ subtraction in a recursive way : **Epstein Glaser forest formula**

  starting from D_2 and proceeding with an increasing number of vertices

- Extension : **method to compute principal parts**

Extension

$$(\mathrm{t}_\gamma)_{\mathrm{ms}} = \lim_{\boldsymbol{\alpha}\to 0} (1 - \mathrm{pp})\, \mathrm{t}_\gamma^{(\boldsymbol{\alpha})}$$

Forest formula & Minimal Subtraction

Theorem – Regularized time ordered product in the MS scheme *[Keller, ...]*

$$\mathcal{T}_n = (\mathcal{T}_n)_{\mathrm{ms}} = \lim_{\boldsymbol{\alpha}\to 0} \mathrm{m}_n \circ \left(\sum_{F\in\mathfrak{F}_{\overline{n}}} \prod_{I\in F} \mathrm{R}_I \right) \circ \mathrm{T}_n^{(\boldsymbol{\alpha})}$$

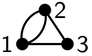
- R_I : "**principal part operator**" $\rightarrow$ if $I \subset J$, R_I applied before R_J

$$\mathrm{R}_I \mathrm{T}_n^{(\boldsymbol{\alpha})} = -\mathrm{pp}_{\boldsymbol{\alpha}_I} \mathrm{T}_n^{(\boldsymbol{\alpha})} \ , \quad \text{with} \quad \boldsymbol{\alpha}_I = \{\alpha_{ij}\}_{i,j\in I} \ , \quad \text{if} \quad I = \emptyset \ , \quad \mathrm{R}_\emptyset = \mathbb{I}$$

- before applying R_I we set $\alpha_{ij} = \alpha_I$ for $i,j \in I$
- $\alpha_I = \alpha_F$ for $I \in F$ before taking the sum over all forests

definition

- then $\alpha_F \to 0$



$\mathrm{t}_\gamma^{(\boldsymbol{\alpha})} = \Delta_{\mathrm{f}}(x_1,x_2)^{(\alpha_{12},2)} \ \Delta_{\mathrm{f}}(x_1,x_3) \ \Delta_{\mathrm{f}}(x_2,x_3)$

divergent contributions : $\{1,2\}$ and $\{1,2,3\}$

relevant forests : $\{\emptyset\}$, $\{\{1,2\}\}$, $\{\{1,2,3\}\}$, $\{\{1,2\},\{1,2,3\}\}$

regularized t_γ : $(\mathrm{t}_\gamma)_{\mathrm{ms}} = \lim_{\boldsymbol{\alpha}\to 0} (1 + \mathrm{R}_{12} + \mathrm{R}_{123} + \mathrm{R}_{123}\mathrm{R}_{12})\, \mathrm{t}_\gamma^{(\boldsymbol{\alpha})}$

Explicit forms

Hadamard form

$$\Delta_{\mathrm{f}}(x,y) = \lim_{\epsilon\downarrow 0} \frac{1}{8\pi^2}\left(\frac{u(x,y)}{\sigma_{\mathrm{f}}(x,y)} + v(x,y)\,\log(M^2\sigma_{\mathrm{f}}(x,y)) + w(x,y)\right)$$

$$\sigma_{\mathrm{f}}(x,y) = \sigma(x,y) + i\epsilon$$

$\rightarrow$ $\sigma(x,y)$: half squared geodesic distance

$\rightarrow$ u, v, w smooth (Hadamard coefficients)

we recall $\mathrm{t}^{(\boldsymbol{\alpha})}_{\gamma}$:

$$\mathrm{t}^{(\boldsymbol{\alpha})}_{\gamma} = \prod_{(i,j)\in\gamma} \Delta_{\mathrm{f}}(x_i,x_j)^{(\alpha_{ij},\ell_{ij})}$$

$$\Delta^{\alpha}_{\mathrm{f}} = \lim_{\epsilon\downarrow 0} \frac{1}{8\pi^2}\left(\frac{u}{\sigma_{\mathrm{f}}^{1+\alpha}} + \frac{v}{\alpha}\left(1-\frac{1}{\sigma_{\mathrm{f}}^{\alpha}}\right)\right) + w$$

Relevant part in $\mathrm{t}^{(\boldsymbol{\alpha})}_{\gamma}$:

$$\mathrm{A}^{\boldsymbol{\alpha}}_{\gamma} = \prod_{(i,j)\in\gamma} \frac{1}{\sigma_{\mathrm{f}}^{\ell_{ij}(1+\alpha_{ij})}}$$

Building blocks :

$$\frac{1}{\sigma_{\mathrm{f}}^{\alpha}}$$

Proposition

$\mathcal{N} \subset \mathcal{M}^2$ geodesically convex space, $\alpha \in \mathbb{C}$, $\phi \in \mathcal{C}_0^\infty(\mathcal{N})$

$$\left\langle \frac{1}{\sigma_{\mathrm{f}}^\alpha}, \phi \right\rangle = \lim_{\epsilon \to 0^+} \int_{\mathcal{M}^2} \mathrm{d}x \, \mathrm{d}y \, \frac{1}{(\sigma(x,y) + i\epsilon)^\alpha} \, \phi(x,y)$$

- *It is a distribution which is analytic in α on $\mathcal{C}_0^\infty(\mathcal{N} \setminus d_2)$*
- *It is homogeneous of degree -2α* ▸ details
- *It is a well defined distribution on $\mathcal{C}_0^\infty(\mathcal{N})$ for $2\alpha - 4 \notin \mathbb{N}$*
- *Its smearing product is analytic for $2\alpha - 4 \notin \mathbb{N}$, and meromorphic in α with simple poles at $2\alpha - 4 \in \mathbb{N}$*

Proposition

$\left\langle \mathrm{A}_\gamma^{(\boldsymbol{\alpha})}, \phi \right\rangle$ *is an **analytic** function in every α_{ij}, with $\boldsymbol{\alpha} = \{\alpha_{ij}\}$ and $\phi \in \mathcal{C}_0^\infty(\mathcal{M}^n \setminus D_n)$*

Theorem

$\mathrm{A}_\gamma^{(\boldsymbol{\alpha})}$ can be extended from $\mathcal{M}^n \setminus D_n$ to $\mathcal{M}^n$

- $\mathrm{A}_\gamma^{(\boldsymbol{\alpha})}$: sum of **homogeneous distributions** plus a **remainder** towards the total diagonal d_n
$$\mathrm{A}_\gamma^{(\boldsymbol{\alpha})} = \sum_{k=1}^{m} \mathrm{A}_{\gamma,k}^{(\boldsymbol{\alpha})} + r_\gamma^{(\boldsymbol{\alpha})}$$
$\rightarrow$ $r_\gamma^{(\boldsymbol{\alpha})}$ has lower scaling degree than $\mathrm{A}_\gamma^{(\boldsymbol{\alpha})}$
- isolate the poles

details

$$\rho \mathrm{A}_\gamma^{(\boldsymbol{\alpha})} = -\sum_{(i,j)\in\gamma} 2\ell_{ij}(1+\alpha_{ij})\mathrm{A}_\gamma^{(\boldsymbol{\alpha})} + r_\gamma^{(\boldsymbol{\alpha})}$$

with ρ a differential operator built in terms of σ and ∇

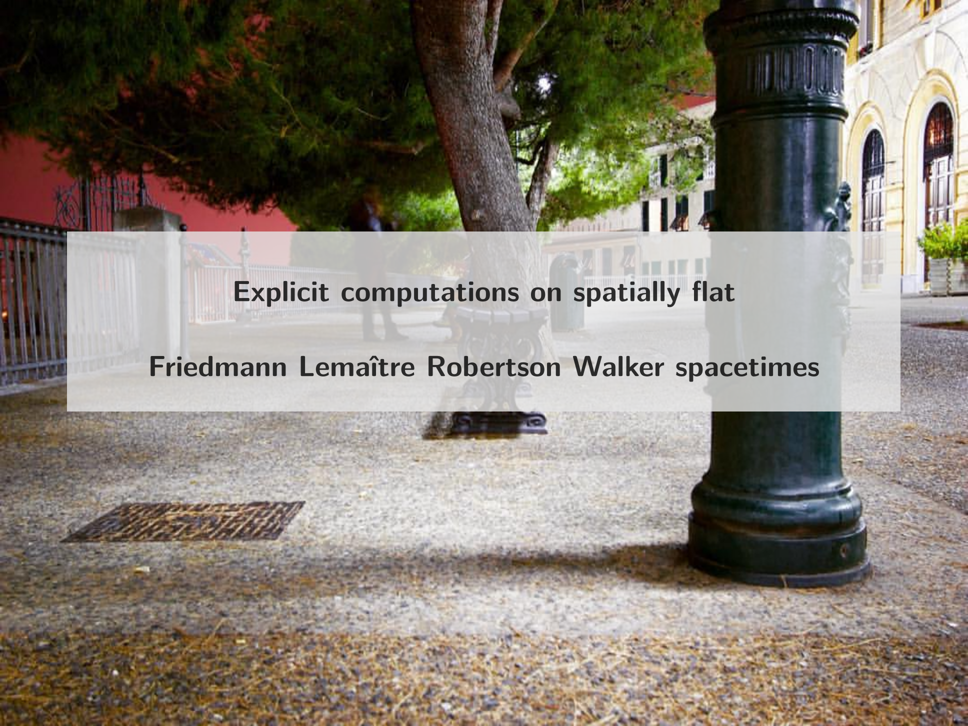
example

Theorem

This regularization scheme satisfies the axioms we asked before.

In particular

- $\mathcal{T}_n$ satisfies the causal factorization
- $\mathcal{T}_n$ is local and covariant
- If the spacetime $\mathcal{M}$ has non trivial isometries and if Δ_f is chosen such as to be invariant under these isometries, then $\mathcal{T}_n$ is invariant under these isometries as well



Explicit computations on spatially flat Friedmann Lemaître Robertson Walker spacetimes

Spatially flat FLRW spacetime

- **spatially flat Friedmann Lemaître Robertson Walker** spacetime

$$g = a(\tau)^2 \left(-d\tau^2 + d\vec{x}^2\right)$$

$\rightarrow$ our scheme is invariant under the isometries of FLRW spacetime

Goal :

Compute the fish diagram :  $\rightarrow$ $(\Delta_\mathrm{f})^2$

- **in principle** we can compute all quantities in spatial and momentum space
- however even for spatially flat FLRW spacetimes, σ, u, v, w are **not explicitly known** neither in **position space**, nor in **momentum space**
- **Idea**: $\Delta_\mathrm{f} = \Delta_{\mathrm{f},0} + \delta\Delta_\mathrm{f}$, with

  $\rightarrow$ $\Delta_{\mathrm{f},0}$ contains sufficiently many singular contributions

  $\rightarrow$ $\Delta_{\mathrm{f},0}$ explicitly known in position and momentum space

Fish diagram for FLRW

- $\Delta_{\mathrm{f},0} \to$ the **Feynman propagator** of the **massless**, **conformally coupled** $(\xi = \frac{1}{6})$ **Klein Gordon field** in the conformal vacuum

$$\Delta_{\mathrm{f},0}(x_1, x_2) = \frac{1}{8\pi^2 a(\tau_1) a(\tau_2)} \frac{1}{\sigma_{\mathbb{M}}(x_1, x_2) + i\epsilon}$$

- Fish diagram

$$(\Delta_{\mathrm{f}})^2_{\mathrm{ms}} = (\Delta_{\mathrm{f},0})^2_{\mathrm{ms}} + 2\delta\Delta_{\mathrm{f}}\Delta_{\mathrm{f},0} + (\delta\Delta_{\mathrm{f}})^2$$

$$\begin{aligned}
(\Delta_{\mathrm{f},0})^2_{\mathrm{ms}} &= \lim_{\alpha \to 0} (1 - \mathrm{pp}) \frac{1}{M^{2\alpha}} (\Delta_{\mathrm{f},0})^{2+\alpha} \\
&= -\frac{1 + 2\log(a)}{16\pi^2 a^4} i\delta_{\mathbb{M}} - \frac{1}{2(8\pi^2)^2 a^2 \otimes a^2} (\Box_{\mathbb{M}} \otimes 1) \frac{\log M^2 \sigma_{\epsilon,\mathbb{M}}}{\sigma_{\epsilon,\mathbb{M}}}
\end{aligned}$$

and then compute the Fourier transform . . .

Grazie.

Definition (Curved spacetime)

A pair $(\mathcal{M}, g)$ is a **curved spacetime** if $\mathcal{M}$ is a 4 dimensional **Lorentzian manifold**, endowed with a Lorentzian metric of signature $(- + \cdots +)$, required to be orientable, paracompact, time orientable, and **globally hyperbolic**.

Definition (Off shell configuration space)

The off shell configuration space over the spacetime $\mathcal{M}$ is the **space of real valued smooth maps**, $\phi \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$. It is equipped with the locally convex topology.

back

Spacetime support of F

$$\mathrm{supp}(\mathrm{F}) \doteq \left\{ x \in \mathcal{M} \,\middle|\, \begin{array}{l} \forall \text{ neighborhood } U \text{ of } x,\ \exists\ \phi, \psi \in \text{ smooth,} \\ \mathrm{supp}(\psi) \subset U,\ \text{such that } \mathrm{F}(\phi + \psi) \neq \mathrm{F}(\phi). \end{array} \right\}$$

Definition (Smooth functional)

The n-th derivative of a functional F at $\phi \in \mathrm{U}$ with respect to the directions $\psi_1, \ldots, \psi_n \in \mathcal{C}^\infty(\mathcal{M})$ is defined as a map $\mathrm{F} : \mathrm{U} \times \mathcal{C}^\infty(\mathcal{M})^{\otimes n} \to \mathcal{C}^\infty(\mathcal{M})$,

$$\mathrm{F}^{(n)}(\phi)[\psi_1, \ldots, \psi_n] = \lim_{t \to 0} \frac{1}{t} \left(\mathrm{F}^{(n-1)}(\phi + t\psi_n) \ - \ \mathrm{F}^{(n-1)}(\phi) \right) [\psi_1, \ldots, \psi_{n-1}] \ .$$

The map F is said to be **smooth** at $\phi \in \mathrm{U}$ if

details

- the limit exists for all $\psi_1, \ldots, \psi_n \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$,
- and if $\mathrm{F}^{(n)}$ is jointly continuous on the product space $\mathrm{U} \times \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})^{\otimes n}$.

back

We recall that a map

$$\mathrm{f} : \mathrm{U} \times \mathrm{X} \to \mathrm{Y}$$

is **jointly continuous** at $(x, y) \in \mathrm{X} \times \mathrm{U}$, if

for each neighborhood Y' of $\mathrm{f}(x, y)$,

$\exists$ a product of open sets $\mathrm{U}' \times \mathrm{X}' \subseteq \mathrm{U} \times \mathrm{X}$ containing (x, y) ,

such that $\mathrm{f}(\mathrm{U}' \times \mathrm{X}') \subseteq \mathrm{Y}'$.

back

Examples :

$\rightsquigarrow$ $\mathrm{WF}(\delta) = \{(0,k) | k \in \mathbb{R}^n, k \neq 0\}$

Proof: The singular support of $\delta(x)$ is $\{0\}$ and $\hat{f\delta}(k) = f(0)$ is not fast decreasing if $f(0) = 0$.

$\rightsquigarrow$ $u(x) = \dfrac{1}{x^2 + i\epsilon}$, $\quad \mathrm{WF}(u) = \{(0;k) | k < 0\}$

Proof: By contour integration $\hat{u}(k) = -2i\pi\Theta(-k)$, thus

$$\left|\hat{fu}(k)\right| = \left|\frac{1}{2\pi}\int_{\mathbb{R}} dq\ \hat{f}(q)\ \hat{u}(k-q)\right| = \left|\int_k^{\infty} dq\ \hat{f}(q)\right|$$

Fourier transform of a test function, $\hat{f}(q)$, is fast decreasing for $q \geq 0$!!

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Scaling degree

- $\exists!$ geodesic γ_i connecting x_1 and x_i

$$\gamma_i : \lambda \mapsto \gamma_i(\lambda) = x_i(\lambda) \ , \quad \text{with} \quad x_i(0) = x_1, \ x_i(1) = x_i$$

- **geometric scaling transformation**

$$\phi_\lambda = \lambda^{4(n-1)} \, \phi\left(x_1(\lambda), x_2(\lambda), \ldots, x_n(\lambda)\right) \prod_{i=2}^{n} \sqrt{\left| \frac{g(x_i(\lambda))}{g(x_i)} \right|}$$

where $g(x)$ is the determinant of the metric g

- **scaled distributions**

$$\langle u_\lambda, \phi \rangle := \left\langle u, \phi_{1/\lambda} \right\rangle$$

- $\mathrm{WF}(u)$ is **transversal** to d_n if

$$\overline{\mathrm{WF}(u)} \cap \{(x_1, \ldots, x_n, k, 0, \ldots, 0) \in T^*\mathcal{M}^n, \forall k \neq 0\} = \emptyset$$

- Definition of $\mathcal{D}'_\Gamma(\mathcal{M}^n \setminus d_n)$

$$\mathcal{D}'_\Gamma(\mathcal{M}^n \setminus d_n) = \left\{ u \in \mathcal{D}'(\mathcal{M}^n \setminus d_n) \ , \ \mathrm{WF}(u) \subset \Gamma \right\}$$

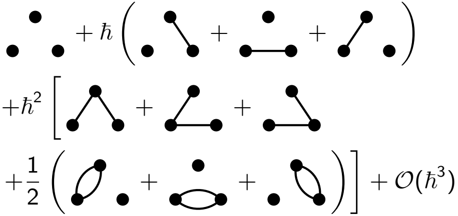
back

Graphical analysis

$$\mathsf{T}_n = \sum_{\gamma \in \mathcal{G}_n} \mathsf{T}_\gamma \ , \qquad \text{with} \quad \mathsf{T}_\gamma = \frac{1}{\mathsf{N}(\gamma)} \ \langle \mathsf{t}_\gamma \ , \ \delta_\gamma \rangle$$

$$\mathsf{N}(\gamma) = \hbar^{-|\mathsf{E}(\gamma)|} \prod_{(i,j)\in\gamma} |\ell_{ij}|! \ , \quad \delta_\gamma = \frac{\delta^{2|E(\gamma)|}}{\prod_{(i,j)\in\gamma} \delta\phi_i^{|\ell_{ij}|}} \ , \quad \mathsf{t}_\gamma = \prod_{(i,j)\in\gamma} \Delta_{ij}^{|\ell_{ij}|}$$

$$(\mathsf{F} \cdot_{\mathsf{T}} \mathsf{G} \cdot_{\mathsf{T}} \mathsf{K})(\phi) \quad =$$



back

Epstein Glaser forest formula

(this method does not depend on the graph expansion)

- **set of indices**

$$\overline{n} := \{1, \dots, n\}$$

- **forest** F : a collection of subsets of $\overline{n}$

$$F = \{I_1, \dots, I_k\} \ , \qquad I_j \subset \overline{n} \ , \qquad \text{and} \qquad |I_j| \geq 2$$

we require

$$I_i \cap I_j = \emptyset \ , \qquad \text{or} \qquad I_i \subset I_j \ , \qquad \text{or} \qquad I_j \subset I_j$$

$\mathfrak{F}_{\overline{n}}$: the **set of all forests** of n indices together with the empty forest $\{\emptyset\}$.

back

Definition

A distribution $u \in \mathcal{D}'(\mathcal{M}^n)$ or $u \in \mathcal{D}'(\mathcal{M}^n \setminus d_n)$, which satisfies the equality

$$\lambda^{\delta} \langle u, \phi_\lambda \rangle = \langle u, \phi \rangle \ , \quad \forall \lambda > 0 \ ,$$

under geometric scaling transformations for all $\phi \in \mathcal{D}(\mathcal{N}_n \setminus d_n)$ and for a $\delta \in \mathbb{C}$, is called **homogeneous of degree** δ.

$$\inf \left\{ \omega \in \mathbb{R} \ \middle| \ \lim_{\lambda \downarrow 0} \lambda^{\omega} \left\langle u, \phi_{1/\lambda} \right\rangle = \lim_{\lambda \downarrow 0} \lambda^{\omega + \mathrm{Re}(\delta)} \lambda^{i \mathrm{Im}(\delta)} \langle u, \phi \rangle = 0 \right\} = -\mathrm{Re}(\delta) \ ,$$

back

Euler operator

$$u^{(\alpha)} = \sum_{k=0}^{m} u_k^{(\alpha)} + r^{(\alpha)}$$

where $u_k^{(\alpha)}$ are homogeneous, with degree with degree $a_k = -\delta_\alpha + k$

$$\mathrm{sd}(u_k^{(\alpha)}) = -\mathrm{Re}(a_k) = \mathrm{Re}\,(\delta_\alpha) - k \geq 4(n-1)$$

Euler operator

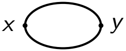
$$\mathrm{E}_p : \begin{cases} \mathcal{D}(\mathcal{N}_n) & \rightarrow \quad \mathcal{D}(\mathcal{N}_n) \\ \phi(x_1, \dots, x_n) & \mapsto \quad (-1)^p \; \lambda^{p+4(n-1)} \, \dfrac{d^p}{d\lambda^p} \left(\lambda^{-4(n-1)} \phi_\lambda(x) \right) \Big|_{\lambda=1} \end{cases}$$

we have

$$\left\langle u_k^{(\alpha)}, \phi \right\rangle = \frac{1}{\prod_{j=0}^{p-1} \left(a_k + j + 4(n-1)\right)} \left\langle u_k^{(\alpha)}, \mathrm{E}_p \phi \right\rangle$$

back

The fish diagram (2 vertices)



$$\longrightarrow \quad \Delta_{\mathrm{f}}^2(x,y) = \frac{1}{8\pi^2}\left(\frac{u^2(x,y)}{\sigma_{\mathrm{f}}^2(x,y)} + \text{“well defined for } x = y\text{”}\right)$$

- regularize only $\sigma_{\mathrm{f}}^{-(2+\alpha)}$
- use σ identities : $\quad \Box\sigma = 4 + f\sigma$
- $\alpha \mapsto \sigma_{\mathrm{f}}^{-(2+\alpha)}$ (weakly) meromorphic in α.

  $\rightarrow$ Laurent series w.r.t α

  $\rightarrow$ subtract the principal part and take the limit $\alpha \rightarrow 0$

$$\left(\frac{1}{\sigma_{\mathrm{f}}^2}\right)_{\mathrm{ms}} = \lim_{\alpha\to 0}\ (1 - \mathrm{pp})\,\frac{1}{\sigma_{\mathrm{f}}^{2+\alpha}}$$

$$\Longrightarrow \left(\Delta_{\mathrm{f}}^2\right)_{\mathrm{ms}}$$

back