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Algebraic and Noncommutative approaches to Quantum Field Theory

Ph.D. thesis

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Abstract

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Chapter 1

Spacetime

The starting block for a physical theory is the notion of spacetime, it is a set of points (events) located in time and space. Looked at space and time as an unique entity has been an important turning point in the understanding of “the laws of nature”. Newton’s physics treated them separately, but at the beginning of the last century Einstein and others introduced a completely new point of view of these two entities. In the theory of gravitation, the physical background (i.e. the spacetime) is an “active actor”. Indeed gravitation roughly speaking can be viewed as a deformation of the spacetime. Therefore we shall introduce the notion of spacetime starting from the very beginning.

1.1 From topology to manifold

The most fundamental way to define a space is to use the notion of topology. It is concerned with the intrinsec properties of spaces.

Definition 1 (Topological space)
Let X be a set[1]. A topology on X is a collection $\mathcal{T}$ of subsets satisfying the three following axioms,

- **conventions** : $\emptyset$, X $\in \mathcal{T}$;
- **arbitrary union** : $U_i \in \mathcal{T}$ for $i \in I \Longrightarrow \bigcup_{i\in I} U_i \in \mathcal{T}$, where I is an arbitrary index set ;
- **finite intersection** : $U_1, \dots, U_n \in \mathcal{T} \Longrightarrow U_1 \cap \dots \cap U_n \in \mathcal{T}$. ♦

The pair (X, $\mathcal{T}$) is called a topological space. The element of $\mathcal{T}$ are the open sets of X. We shall often omit to precise the topology $\mathcal{T}$, and simply say that X is a topological space. It is still general, for instance the operations between elments in X are not for now considered. Nonetheless it already characterizes maps between different topological spaces.

Definition 2 (Continous maps and homeomorphism)
Let X and Y be topological spaces. We consider a map f : X $\rightarrow$ Y. We say

- f is **continous** if $f^{-1}(U) \subset X$ is open for every open $U \subset$ Y ;
- f is a **homeomorphism** if f is bijective and both f and f^{-1} are continous. ♦

In the case we have two different continous functions f and g, which map a topological space X to another topological space Y, and a function h : X $\times [0, 1] \rightarrow$ Y, with $h(x, 0) = f(x)$ and $h(X, 1) = g(x)$ for all $x \in$ X, we say that h is an homotopy.

We can define the notion of distance (also called pseudometric) between two points on a topological space. This notion will appear to be useful to “build” topologies.

Definition 3 (Pseudometric)
Let $\mathcal{X}$ be a set. A pseudometric on $\mathcal{X}$ is a map $d : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}^+$ such that for all $x, y, z \in \mathcal{X}$,

- **separation** : $d(x, y) = 0 \Leftrightarrow x = y$;
- **symetry** : $d(x, y) = d(y, x)$;

[1] It is simply a(n) (in)finite collection of objects.

- **triangle inequality** : $d(x,y) \leq d(x,z) + d(z,y)$. ♦

A set $\mathcal{X}$ endowed with a distance d is called a pseudometric space, and denoted $(\mathcal{X}, d)$. If a pseudometric d satisfies

- **positivity** : $d(x,y) > 0$, for all $x \neq y$,

then d is called a metric. We can show that a pseudometric space is also a topological space for which the topology is induced by the pseudometric. Let us detail this in the following lemma.

Lemma 1 (Pseudometric topological space)
Every pseudometric space $(\mathcal{X}, d)$ is also a topological space for which the topology is induced by the collection of open sets in $\mathcal{X}(=$ X). We denote it by (X, $\mathcal{T}_d$). ▶

Proof 1
We shall consider the following collection of open subsets in $\mathcal{X}$.

$$\mathcal{T}_d = \{B_i\} = \left\{ B_i \,\middle|\, \forall x_i \in \mathcal{X},\ \exists \epsilon > 0,\ B_i = \{y \in \mathcal{X}, d(x_i, y) < \epsilon\} \subseteq \mathcal{X} \right\}$$

First observation is that $\emptyset$ and $\mathcal{X}$ are open and contained in $\mathcal{T}_d$. Second observation is that any arbitrary union or finite intersection of open sets B_i is still contained in $\mathcal{T}_d$. Therefore the metric space $(\mathcal{X}, d)$ is also a topological space for which the topology is induced by the collection of open sets B_i in $\mathcal{X}(=$ X). ■

Let us give some generic definition which shall appear to be useful later on. X shall always denote a topological space with topology $\mathcal{T}$, and Z a subspace of X.

- $\overline{\mathrm{Z}}$, the **closure** of Z $\subset$ X, is the intersection of all closed sets[2] containing Z.
- Z is **dense** in X if $\overline{\mathrm{Z}} =$ X.
- $\mathcal{B} \subset \mathcal{T}$ is a **topological basis** if every elements in $\mathcal{T}$ can be written as the union of elements of the basis $\mathcal{B}$.
- X is **second countable** if it has a countable[3] topological basis.
- X is **separable** if there exists a dense and countable subset.
- K $\subset$ X is **compact** if any covering of it admit a finite subcovering.
- X is **locally compact** if every point in X admit a neighborhood which has compact closure.
- X is **connected** if it cannot be written as a disjoint union of two nonempty open subsets.
- X is **Hausdorff** if every pair of points have disjoint neighborhood.
- X is **paracompact** if every open cover has a refinement covering that is locally finite[4].
- X is said to be **metrizable** if there is a metric on X for which the induced topology is $\mathcal{T}$.
- X is said to be **contractible** if the identity map X $\rightarrow$ X is homotopic to a constant map.

Let us illustrate these definitions with some examples. The firt one is the space $\mathbb{R}$ endowed with the order topology. The open sets are the sets U in which every element lies in an open interval contained in U. This topology can also be defined by the pseudometric $d(x,y) = |x-y|$. A subset of this topology is compact if and only if it is bounded and closed, and it is connected if and only if it is convex. A bit more advanced example is the space on $\mathbb{R}^n$ but this time endowed with the product topology. This topology can be induced by many different metrics but let us mention three of them,

$$d_1(x,y) \ = \ ||x-y||_1 \ = \ \textstyle\sum_k |x_k - y_k| \ ; \quad d_2(x,y) \ = \ ||x-y||_2 \ = \ \sqrt{\textstyle\sum_k |x_k - y_k|^2} \ ;$$
$$d_\infty(x,y) \ = \ ||x-y||_\infty \ = \ \max_k |x_k - y_k| \ .$$

In order to restrict ourselves to a less general picture, we require that topological spaces has to satisfy some separability (Hausdorff) and countability (second countable) conditions such that they look locally like $\mathbb{R}^n$. We have now enough background to introduce the notion of manifold. We start with topological manifolds and will implement differential structure in the next section.

[2] A set C $\subset$ X is **closed** if X$\backslash$C is open.
[3] A set is said to be **countable** if there exists a one to one correspondence between the set considered and the set of natural numbers.
[4] A cover $\{U_\alpha\}$ of X is **locally finitte** if every points in X has a neighborhood which has a nonempty intersection with a finite numbers of U_α.

Definition 4 (Topological manifold)
A topological manifold $\mathcal{M}$ of dimension n is a Hausdorff and second countable topological space in which every points admit an open neighborhood homeomorphic to a subset of $\mathbb{R}^n$. ♦

It is possible to show that instead of asking to $\mathcal{M}$ to be Hausdorff and second countable, it is equivalent to require $\mathcal{M}$ to be separable and metrizable. But these properties are global, what is important is that $\mathcal{M}$ admit locally the same topological properties as $\mathbb{R}^n$. In particular it is locally compact, connected, and contractible.

An important tool to pass from a local to a global point of view is the **partition of unity** of $\mathcal{M}$.

Definition 5 (Partition of unity)
Let X be a topological space. A partition of unity on X is a collection $\{g_i\}$ of continuous functions $g_i : \mathsf{X} \to [0,1]$ such that

- $\sum_i g_i(x) = 1$, for all $x \in$ X ,
- for every $x \in$ X, there is only a finite number of maps g_i such that $g_i(x) \neq 0$. ♦

A partition of unity defines an open cover of X. We call, $\{g_i\}_{i\in I}$, partition of unity subordinate to an open cover $U = (U_i)_{i\in I}$, if for all $i \in I$, the support of g_i is contained in U_i. And we have that a topological manifold $\mathcal{M}$ is paracompact if and only if it admit a partition of unity subordinate to every open cover of $\mathcal{M}$.

1.2 Lorentzian manifold

1.2.1 Smoothness

In the previous section we end up with the notion of topological manifold. Now we would like to implement differential structure on it in order to be able for instance to define tangent space. And in particular we shall focus ourselves on smooth manifolds.

Let $\mathcal{M}$ be a topological manifold of dimension n. We would like to be able to "localize" points on a manifold, it is done via the notion of a chart. A chart on $\mathcal{M}$ is a pair (U, ϕ), where U is an open subset on $\mathcal{M}$ and ϕ is a homeomorphism from U to an open subset $\phi(U) \subset \mathbb{R}^n$. We call U a coordinate neighborhood, ϕ a coordinate maps, and the component functions of ϕ local coordinates on U.

To give sense of smooth manifold, we need to implement an additional notion to the topology. Let (ϕ, U) and (ψ, V) be two charts on $\mathcal{M}$ such that $U \cap V \neq \emptyset$. We call transition map, the application

$$\psi \circ \phi^{-1} \; : \; \phi(U \cap V) \subset \mathbb{R}^n \; \to \; \psi(U \cap V) \subset \mathbb{R}^n \, .$$

It is a homeomorphism. We say (ϕ, U) and (ψ, V) are smoothly compatible if $U \cap V = \emptyset$ or if the transition map $\phi^{-1} \circ \psi$ is a diffeomorphism, i.e. bijective with smooth inverse.

We call an atlas of $\mathcal{M}$ a set of chart $\{(U, \phi)\}$ which covers $\mathcal{M}$. An atlas $\mathcal{A}$ is called smooth if any two charts in $\mathcal{A}$ are smoothly compatible. A smooth atlas $\mathcal{A}$ is called maximal if it is not contained in any stricly larger smooth atlas. We now can give the definition of a smooth manifold.

Definition 6 (Smooth manifold)
$\mathcal{M}$ is a smooth manifold if $\mathcal{M}$ is a topological manifold with a smooth maximal atlas $\{(U, \phi)\}$. ♦

One of the first characterization of a smooth manifold $\mathcal{M}$ that we can give is the orientability of a such maniflod. A smooth orientation of a smooth manifold is the choice of a maximal smooth oriented atlas. A smooth atlas $\{(U, \phi)\}$ is called oriented if the determinant of the derivatives of all transition maps is positive.

It shall appear useful to define smooth maps between manifolds. We shall also characterize real valued maps on smooth manifolds, and say on which conditions they are smooth and compactly supported.

Definition 7 (Smooth map)
A map $f : \mathcal{M} \to \mathcal{N}$ between two smooth manifolds is said to be smooth if there are two charts (U, ϕ) and (V, ψ) on $\mathcal{M}$ and $\mathcal{N}$ respectively, such that the transition function $\psi \circ f \circ \phi^{-1}$ is smooth. ♦

We notice that in particular if the two manifolds are of the same dimension then f is said to be a diffeomorphism.

And now let us define what is a real valued smooth (compactly supported) function.

Definition 8 (Smooth – compactly supported – function)
A function $f : \mathcal{M} \to \mathbb{R}$ is said to be **smooth** if and only if $f \circ \phi^{-1} : \phi(U) \subset \mathbb{R}^n \to f(U) \subset \mathbb{R}$ is smooth for each coordinate chart in the atlas.
A function $f : \mathcal{M} \to \mathbb{R}$ is said to be **compactly supported** if the support of $f : \mathcal{M} \to \mathbb{R}$ (i.e. the closure of the set where f does not vanish) is compact. ⧫

The set of all real valued smooth function on $\mathcal{M}$ is denoted by $\mathcal{E}(\mathcal{M})$, and the one of all real valued smooth compactly supported functions on $\mathcal{M}$ by $\mathcal{D}(\mathcal{M})$.

For now on $\mathcal{M}$ shall be understood as a smooth manifold of dimension n.

1.2.2 Lorentzian structure

Let us look locally to a generic manifold, and define a curve passing by x as $\gamma : [-1, 1] \to \mathcal{M}$ such that $\gamma(0) = x \in \mathcal{M}$. Then there is $\epsilon > 0$ small enough such that $\gamma([-1, 1]) \subset U$, for a coordinate neighborhood U of a chart (U, ϕ). We say that two curves γ and γ' are equivalent if

$$\lim_{t\to 0} \frac{1}{t} \left(\gamma(x+t) - \gamma(x)\right) = \lim_{t\to 0} \frac{1}{t} \left(\gamma'(x+t) - \gamma'(x)\right) \ .$$

We call tangent space at x, denoted by $T_x\mathcal{M}$, the equivalence class of the curves at x. An important observation is the equivalence relation is independent of the corrdinate system chosen on U. $T_x\mathcal{M}$ can be define in another way. Let consider the set of real valued smooth function on $\mathcal{M}$, $\mathcal{E}(\mathcal{M})$. We say that two functions $f, g \in \mathcal{E}(\mathcal{M})$ are equivalent on a coordinate neighborhood U if the restriction of f and g on U are equal for all points in U. The set of equivalence classes at a point x is denoted by $\mathcal{E}_x(\mathcal{M})$. Then we define a derivation V_x as a linear map from $\mathcal{E}_x(\mathcal{M})$ to $\mathbb{R}$ which satisfy the Leibniz rule,

$$V_x(fg) = f(x)V_x(g) + g(x)V_x(f) \ .$$

The tangent space at x is then the vector space of the derivation on $\mathcal{E}_x(\mathcal{M})$. We can notice that the equivalence relation defined on $\mathcal{E}$ is used to make $V(f)$ depend only on the value of f near x. The only thing we can now about f looking at $V(f)$ is its behaviour in a very thin neighborhood of x. Then the Leibniz rule assure that it depends at most on the first derivative of f. It can be shown that this two definitions are equivalent. A last remark on the tangent space to a manifold, it has the same dimension as the given manifold.

Before adding more structure on $T_x\mathcal{M}$ we shall introduce the notion of vetor bundle. A smooth real vector bundle is a triple $(E, \mathcal{M}, \pi)$, where E (total space) and $\mathcal{M}$ (base space of dimension n) are smooth manifolds and

$$\pi : E \to \mathcal{M},$$

is a smooth surjection such that

- $E_x = \pi^{-1}(\{x\})$, called the fibre of E at $x \in \mathcal{M}$, is a n dimensional vector space $\forall x \in \mathcal{M}$;
- $\exists$ an open $U \subseteq \mathcal{M}$, with $x \in U$ and a smooth diffeomorphism $\phi : \pi^{-1}(U) \to U \times E_x$, for all $y \in \mathbb{R}^n$, such that $(\pi \circ \phi)(x, y) = x$ and the map $y \mapsto \phi(x, y)$ is a linear isomorphism between the vector spaces $\mathbb{R}^n$ and E_x.

We shall omit to precise the corresponding triple when speaking of a vector bundle, we shall only precise the total space E, and is necessary the base space.

We call smooth section of a vector bundle a smooth map $s : \mathcal{M} \to E$, such that $\pi \circ s = \mathrm{id}$, the corresponding vector space is denoted $\Gamma(\mathcal{M})$. We shall later see the importance of the notion of section in physics.

There are important particular smooth real vector bundles, the **tangent bundle** and the **real line bundle**. This two notion which shall appear to be useful later on.

Definition 9 (Tangent and line bundle)
The **tangent bundle** is the triple $T\mathcal{M} = (T_x\mathcal{M}, \mathcal{M}, \pi_t)$ with $\pi_t : T_x\mathcal{M} \to \mathcal{M}$.
The real **line bundle** is the triple $(\mathbb{R}, \mathcal{M}, \pi_\ell)$ with $\pi_\ell : \mathbb{R} \to \mathcal{M}$. ⧫

The section of the tangent bundle is $v : \mathcal{M} \to T_x\mathcal{M}$, it is called **vector field**. And the section of the real line bundle is $f : \mathcal{M} \to \mathbb{R}$, it is called **real scalar field**.

Let us come back to the notion of tangent space. It is a vector space thereore we can consider the dual of it. It is called the cotangent space of $\mathcal{M}$ at x, and is denoted by $T^*_x\mathcal{M}$. We recall that the dual of vector space is the vector space of the set of linear applications from this space to $\mathbb{R}$, and it forms also a vector space. We can then consider the cotangent bundle, it is the triple $T^*\mathcal{M} = (T^*_x\mathcal{M}, \mathcal{M}, \pi^*_t)$ with $\pi^*_t : T^*_x\mathcal{M} \to \mathcal{M}$.

We need now to define what is a tensor metric, it permits to define a notion of distance on the manifold. It is a bilinear, symmetric, and non degenerate function g such that

$$g : \left\{ \begin{array}{lcl} E \times E & \to & \mathbb{R}\,, \\ (X,Y) & \mapsto & g_{ij}X^iY^j \end{array} \right. ,$$

with E a vector space. It defines a scalar product, (g_{ij}) is diagonal matrix. The signature of a scalar product is the pair (p,q) where p is the number of $+1$ on the diagonal, and q the number -1. On a manifold g is is a nondegenerate symmetric tensor of type $(0,2)$. we can show that smoothness implies that the sgnature is constant on any connected component of $\mathcal{M}$. Using the metric we can relate a vector space E to its dual E^*. Indeed $Y \mapsto g(X,Y) \in \mathbb{R}$ is an element of E^*. Since g is linear and non degenerate, it is a linear isomorphism. It follows that on a manifold we can use the metric g to define a linear isomorphism between vectors and one forms. We call g a rimannian metric if g has signature n, the dimension of $\mathcal{M}$, or a pseudo rimannian (or lorentzian) metric if the signature is equal to $n-2$. We shall use for now on only lorentzian metric g.

Definition 10 (Lorentzian manifold)
$(\mathcal{M}, g)$ is a Lorentzian manifold, where $\mathcal{M}$ is a n dimensional smooth manifold, and g is a Lorentzian metric. ◆

1.2.3 Integration

Let E_1, E_2, and F be vector spaces, and $E_1 \to E_1^*$ the "dual maping". The vector space of the multilinear forms from $E_1 \times E_2$ to F is called tensor product and denoted $E_1 \otimes E_2$. We can generalize this notion, therefore $E_1 \otimes \cdots \otimes E_r$ shall denote the tensor product of r vecor space, i.e the set of multilinear forms from $E_1 \times \cdots \times E_r$ to F. In the particular case where $E_1 = \cdots = E_{r_p} = E$ and $E_{r-p+1} = \cdots = E_r = E^*$, the the tensor product $E_1 \otimes \cdots \otimes E_r = E^{\otimes p} \otimes (E^*)^{\otimes q}$ is a tensor of type (p,q), p times covariant, and q times contravariant. A tensor product is said to be symmetric if its arguments are invariant under of permutation of the symmetric group, and antisymmetric if permutations can change the sign of the tensor.

A r differential form on $\mathcal{M}$ is a antisymmetric tensor of type $(0,r)$. We shall denote by $\Omega^r(\mathcal{M})$ the vecor space of the r differential forms on $\mathcal{M}$. Knowing $\mathcal{M}$ is of dimension n, $\Omega^r(\mathcal{M})$ with $r > n$ is equal to $\{0\}$. $\Omega^0(\mathcal{M})$ is the space of the smooth functions on $\mathcal{M}$.
We can define the exterior product of two differential forms, $\omega \in \Omega^r(\mathcal{M})$ and $\eta \in \Omega^s(\mathcal{M})$. These product is denoted as $\omega \wedge \eta \in \Omega^{r+s}(\mathcal{M})$.
If one have a smooth map F between two smooth manifolds M and N, and $\omega \in \Omega^r(\mathcal{N})$, then $F{*}\omega$ is a r differential form on $\mathcal{M}$. It is called the pull back of ω by F.
And last definition for $\Omega^r(\mathcal{M})$. The derivative of a differential form, denoted d, is defined as

$$d : \Omega^r(\mathcal{M}) \to \Omega^{r+1}(\mathcal{M})\,.$$

In particular d maps the smooth maps to the one differential forms.

As we now $\mathcal{M}$ looks locally like $\mathbb{R}^n$, it has been built in this purpose. Therefore wa can imagine that the measurability known for $\mathbb{R}^n$ can be in some way transfered to $\mathcal{M}$.

Let us first recall some elements of measure theory. First of all we introduce what is a σ algebra over a set X. It is a collection $\Sigma(X)$ of subsets of X such that,

- $X \in \Sigma(X)$;
- $E \in \Sigma(X)$ implies $X\backslash E \in \Sigma(X)$;
- and if $\{E_k\} \subset \Sigma(X)$ then $\bigcup\limits_{k\in\mathbb{N}} E_k \in \Sigma(X)$.

We can make few remarks.

- $\Sigma(X) \in X$ since $E \in X \Rightarrow X\backslash E = E^c \in X \Rightarrow E \cup E^c \in X$;
- $\emptyset \in X$ since $\Sigma(X) \in X \Rightarrow \Sigma(X)^c \in X \Rightarrow \Sigma(X) = \emptyset \in X$;
- X is closed under countable intersections, suppose $E_1, E_2, ... \in X$, then $\bigcap E_i = \bigcap (E_i^c)^c = (\bigcup E_i^c)^c \in X$.

A measurable space is a pair $(X, \Sigma(X))$, where X is a set and $\Sigma(X)$ a σ algebra on X. Then if $(X, \Sigma(X)$ and $(Y, \Sigma(Y)$ are two measurable spaces. A function $f : X \to Y$ is said to be measurable whenever $f^{(-1)}(E) \in \Sigma(X)$ for any $E \in \Sigma(Y)$. And for $(X, \mathcal{T})$ a topological space, the σ algebra on X generated by $\mathcal{T}$, denoted $\mathcal{B}(X)$, is called Borel σ algebra on X. A function $\mu : \Sigma(X) \to [0, +\infty]$ is said to be a measure if it satisfies the two following properties.

- $\mu(\emptyset) = 0;$
- $\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sum_{n\in\mathbb{N}} \mu(E_n)$, if $\{E_n\}_{n\in\mathbb{N}} \subset \Sigma(x)$, and $E_n \bigcap E_m = \emptyset$ if $n \neq m$.

The triple $(X, \Sigma(X), \mu)$ is called a measure space. A measure space $(X, \Sigma(X), \mu)$ and its measure μ are called are called Borel space and Borel measure respectively, if $\Sigma(X) = \mathcal{B}(X)$ with X locally compact Hausdorff space.

Let us consider a subset $\mathcal{O} \subset \mathcal{M}$. $\mathcal{O}$ is said to be Lebesgue measurable if around $x \in \mathcal{O}$ there is a chart (U, ϕ) such that $\phi(\mathcal{O} \cap U)$ is Lebesgue measurable. This definition does not depend on the coordinate system chosen. We set

$$\mathcal{L}(\mathcal{M}) = \{\mathcal{O} \subset \mathcal{M} \ \text{ and } \ \mathcal{O} \ \text{ is measurable }\} \ .$$

$\mathcal{L}(\mathcal{M})$ is a σ algebra over $\mathcal{M}$, the Lebesgue σ algebra of $\mathcal{M}$. It contains the Borel σ algebra $\mathcal{B}(\mathcal{M})$. We consider g a pseudo riemannian metric on $\mathcal{M}$ and (ϕ, U) a chart with $\phi = (x^1, \ldots, x^2)$. Then the Gram determinant $G = \det\,(g_{ij})$ is well defined, and $\sqrt{|G|} \in \mathcal{E}(\mathcal{O})$. Then for $\mathcal{O} \subset \mathcal{M}$ we define the volume form as

$$\text{vol}_g(\mathcal{O}) \ = \ \int_{\phi(\mathcal{O})} \phi * \sqrt{|G|}\, d\lambda_n \ = \ \int_{\phi(\mathcal{O})} \phi * \sqrt{|G|}\, dx \ . \tag{1.1}$$

It is independent of the chart. In the following se set $\lambda_{\mathcal{M},g} = \text{vol}_g(\mathcal{O})$.

Lemma 2 (Riemann Legesgue meseasure)
$\lambda_{\mathcal{M},g}$ is a Radon measure of $\mathcal{M}$. It is called the Riemann Legesgue meseasure. ▶

Proof 2
p. 404 Herbert & Escher. ■

We would like now define the integration of an r differential form ω over $\mathcal{M}$, which is assumed to be orientable. The idea is to map ω on $\mathbb{R}^n$, because we know how to integrate on $\mathcal{M}$. But we do not know how to do it in a global way. We need to use an atlas $\{(U, \phi)\}$ on $\mathcal{M}$, and map ω locally to $\mathbb{R}^n$. Using a subordinate partition of unity $\{g_i\}$ of $\mathcal{M}$, we can write $\omega = \sum g_i\omega$.On every open U_i the differential form $g_i\omega$ can be map to $\mathbb{R}^n$ thank to $\phi_i : U_i \to V_i \subset \mathbb{R}^n$. The integral of the differential form $\phi_i^{-1} * (g_i\omega)$ on V_i can be defined. We write

$$\int_{\mathcal{M}} \omega = \sum_i \int \phi_i^{-1} * (g_i\omega)$$

whenever ii is well defined. It does not depend of the choice of the coordinate system we choose to work with.

A r differential form which does not vanish on $\mathcal{M}$ is called a volume form. If $\mathcal{M}$ is orientable then it admits a volume form (the inverse implication is also true).

The integral of real valued function on $\mathcal{M}$ is defined as

$$\int_{\mathcal{M}} f\omega \ .$$

Fubini's theorem

1.2.4 Covariant derivative and curvature

A linear connection is a map

$$\nabla : \Gamma(\mathcal{M}) \times \Gamma(\mathcal{M}) \to \Gamma(\mathcal{M})$$

such that for every sections $X, Y, Z \in \Gamma(\mathcal{M})$ and any real valued smooth fucntion $f \in \mathcal{E}(\mathcal{M})$ we have

$$\begin{aligned}
&\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z \ ; \\
&\nabla_{fX} Y = f \nabla_X Y \ ; \\
&\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z \ ; \\
&\nabla_X (fY) = f \nabla_X Y + (X \cdot f) Y \ .
\end{aligned}$$

The torsion of ∇, T, is a tensor of type $(1,2)$ such that for every vector fields $X, Y \in \Gamma(T\mathcal{M})$

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \ .$$

The connexion ∇ is said to be torsion free if its torsion tensor is the zero tensor. We can also define the curvature R, it is a tensor of type $(1,3)$ such that for every vector fields $X, Y, Z \in \Gamma(T\mathcal{M})$

$$R(X,Y)Z = \left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \right) Z \ .$$

1.3 Causality

1.3.1 Futurs and pasts

We shall now work with the pair $(\mathcal{M}, g)$ which denotes a Lorentzian manifold of dimension $n \geq 2$ together with a Lorentzian metric g. We associate to each point x of the manifold its conresponding tangent space $T_x\mathcal{M}$. Considering a vector field $v \in T_x\mathcal{M}$, we can evaluate its Lorentzian scalar product with itself, using the metric g. It divides the tangent space in three different regions.

$$\begin{aligned}
g(v,v) &> 0, \ \text{then v is called timelike vector,} \\
g(v,v) &= 0, \ \text{then v is called null vector,} \\
g(v,v) &< 0, \ \text{then v is called spacelike vector.}
\end{aligned}$$

In every tangent space $T_x\mathcal{M}$ the set of timelike vectors, called light cone, consists of two connected components. A time orientation on $\mathcal{M}$ is a choice of one of these connected components. Then the light cone is refered as the union of the forward and backward lightcones,

$$\mathcal{V} = \mathcal{V}^+ \cup \mathcal{V}^- \ , \quad \text{with } \mathcal{V}^{\pm} = \left\{ x \in \mathcal{M} \mid x^2 > 0, \ \pm x^0 > 0 \right\} \ .$$

A vector $v \in T_x\mathcal{M}$ is **future** (respectively **past) directed** if v is a non spacelike vector and contained in $\mathcal{V}^+$ (respectively in $\mathcal{V}^-$).

A differentiable curve $\gamma(\lambda)$ is said to be

- a **future** (respectively **past) directed timelike curve** if at each point $x(\lambda) \in \gamma$ the tangent vector v is a future (respectively past) directed timelike vector ;
- a **future** (respectively **past) directed causal curve** if at each point $x(\lambda) \in \gamma$ the tangent vector v is either a future (respectively past) directed timelike or null vector.

The **chronological future** (respectively **past**) of $x \in \mathcal{M}$ is denoted by $I^+(x)$. It is defined as the sets of points which can be reached by a future (respectively past) directed timelike curve starting from x,

$$I^{\pm}(x) = \left\{ y \in \mathcal{M} \;\middle|\; \begin{array}{l} \text{There exists a future (respectively past) directed timelike curve } \lambda(t), \\ \text{with } \lambda(0) = x \text{ and } \lambda(1) = y \end{array} \right\} .$$

We define $I^+(S) = \bigcup_{x \in S} I^{\pm}(x)$ for any subset $S \subset \mathcal{M}$.

The causal future/past of a point of the spacetime is defined in a similar way as the chronological future/past of this point, using this time the notion of the causal curve.

The **causal future** (respectively **past**) of $x \in \mathcal{M}$, denoted by $J^+(x)$, is defined as the sets of points that can be reached by a future (respectively past) directed causal curve starting from x,

$$J^{\pm}(x) = \left\{ y \in M \;\middle|\; \begin{array}{l} \text{There exists a future (respectively past) directed causal curve } \lambda(t), \\ \text{with } \lambda(0) = x \text{ and } \lambda(1) = y \end{array} \right\} ,$$

We define $J^{\pm}(S) = \bigcup_{x\in S} J^{\pm}(x)$ for any subset $S \subset \mathcal{M}$.

A subset $S \subset M$ is said to be **achronal** if there do not exist $x, y \in S$ such that $y \in I^{+}(x)$, i.e., if $I^{+}(S) \cap S = \emptyset$.

We define the **future** (respectively **past**) domain of dependence of S, denoted by $D^{+}(S)$, by

$$D^{\pm}(S) = \left\{ x \in M \;\middle|\; \begin{array}{l} \text{Every past (respectively future) inextendible causal curve} \\ \text{through } x \text{ intersects } S \end{array} \right\}.$$

The (full) **domain of dependence** of S, denoted by $D(S)$, is defined as,

$$D(S) \;=\; D^{+}(S) \;\cup\; D^{-}(S).$$

The set S is a closed achronal set.

1.3.2 Global hyperbolicity

Definition 11 (Cauchy surface)
A closed achronal set Σ for which $D(\Sigma) = M$ is called a Cauchy surface. ♦

A spacetime $(\mathcal{M}, \mathrm{g})$ which possesses Cauchy surface is said to be globally hyperbolic. We invite the reader to look at the end of chapter 8 of for the equivalence of this definition of global hyperbolicity and the ones of Leray, Hawking, and Ellis. We have now enough background to define a curved spacetime.

Definition 12 (Curved spacetime)
A pair $(\mathcal{M}, g)$ is a curved space time if $\mathcal{M}$ is a $n \geq 2$ dimensional Lorentzian manifold, endowed with a Lorentzian metric of signature $(- + \cdots +)$. The spacetime is required to be orientable, time orientable, and globally hyperbolic. ♦

A set $\mathcal{O}_x \subset \mathcal{M}$ is called a **geodesically starshaped** with respect to $x \in \mathcal{M}$ if there is an open subset $\mathcal{O}'_x$ in T_xM which is starshaped with respect to $0 \in T_x$M such that $\exp_x \;:\; \mathcal{O}'_x \;\to\; \mathcal{O}_x$ is a diffeomorphism. $\mathcal{O} \subset \mathcal{M}$ is **geodesically convex** if it is starshaped with all its points. This entails in particular that each point x, y in $\mathcal{O}$ are connected by a unique geodesic which is completely contained in $\mathcal{O}$.

Chapter 2

Free theory

We shall in this chapter speak about quantum field theory (QFT) on curved background. In the last decades QFT has been tested with very sophisticated experimetnation, and until now the predictions made by the theory were always correct with a very high precision. The only block missing to this robust framework is to implement gravitation. And QFT on curved spacetime is a first step in that direction. For simplicity we will restarict ourselves to the case of scalar field. We shall focus here only on the free theory, and treat interaction perturbatively in the next chapter.

We shall first describe the mathematical elements necessary to describe quantum field theories on a curved spacetime $\mathcal{M}$, i.e the notion of fields and observables within this appoach, then introduce the classical field theory, and finally present the quantization procedure used.

2.1 Off shell configuration space

We assigne to the spacetime the configuration space $\mathfrak{C}(\mathcal{M})$ of fields defined on it. In the general case $\mathfrak{C}(\mathcal{M})$ can be defined as the space of sections of some vector bundle E over $\mathcal{M}$, i.e. $\mathfrak{C}(\mathcal{M}) = \Gamma(\mathcal{M})$. Therefore $\mathfrak{C}(\mathcal{M})$ shall be the space of maps

$$\phi : \mathcal{M} \to E \ .$$

We would like to characterized the space $\mathfrak{C}(\mathcal{M})$, i.e. defined its topology and the notion of convergence in it. The coordinates maps ϕ are defined on $\mathcal{M}$ and take value on a subset $\Omega \subset \mathbb{R}^n$. Therefore we start by looking at the space of functions defined as

$$f : \Omega \subset \mathbb{R}^n \to \mathbb{R} \ .$$

Using L. Schwartz's notation we deonte the space of real valued smooth functions on Ω by $\mathcal{E}(\Omega)$. For later purposes we shall also introduce the space of real valued compactly supported smooth functions on Ω, denoted by $\mathcal{D}(\Omega)$.

Let us first introduce some notions in order to be as presize as possible. A vector space together with a topology such that the single point sets are closed, and the vector space operations are continuous with respect to this topology, is a topological vector space **(tvs)**. Therefore the topology is translation invariant, and it is completely determined by any neighborhood of the origin. Thus we call a **local base** of a tvs X a collection $\mathcal{B}$ of neighborhood $\mathcal{B}$ of the origin such that every neighborhoods of the origin contains an element of $\mathcal{B}$. The open sets of X are the union of translated sets of $\mathcal{B}$. If there is s a local base in the tvs X whose members are convex[1], then X is a locally convex topological vector space **(lctvs)**.

In practise lctvs are associated to families of **seminorms**. First let us recall what is a seminorm on a vector space E. It is a real valued function $p : E \to \mathbb{R}$ such that

- $p(x) \leq 0$ for any $x \in E$;
- $p(\lambda x) = |\lambda|\, p(x)$ for any $\lambda \in \mathbb{R}$ and $x \in E$;
- $p(x + y) \geq p(x) + p(y)$ for any $x, y \in E$.

If furthermore we have $p(x) = 0 \ \Rightarrow \ x = 0$ for any $x \in E$, then p is called a **norm**. We shall require a property to family of seminorms $\mathcal{P}$. It can be useful to work with separating family of seminorms on X, i.e. when for each

[1] A subspace Y of a vector space X is called convex, if for $a_1, a_2 \in \mathbb{R}$, such that $a_1 + a_2 = 1$, and $y_1, y_2 \in$ Y , it implies $a_1 y_1 + a_2 y_2 \in$ Y.

non zero $x \in$ X corresponds at least one seminorm $p \in \mathcal{P}$ with $p(x) \neq 0$. And then it is possible to show that if the topology of a tvs is induced from a separating family of seminorms it is a lctvs.

It is possible in some case to relate the notion of pseudometric to the one of seminorms. Indeed a lctvs (X, $\mathcal{T}$) is metrizable if and only if the topology $\mathcal{T}$ can be defined by a countable separating family of seminorms $\mathcal{P} = \{p_n(x-y),\ n \in \mathbb{N}\}$. Then one equip X with a pseudometric which is compatible with $\mathcal{T}$. For instance a lctvs can be equipped with the following pseudometric

$$d(x,y) = \sum_{n\in\mathbb{N}} \left(\frac{1}{2}\right)^n \frac{p_n(x-y)}{1+p_n(x-y)} \ .$$

Therefore a sequence $(x_k)_{k\in\mathbb{N}} \subset$ X is Cauchy for a distance d if and only if it is Cauchy for every seminorm p_n generating the topology. For any $\epsilon > 0$ there is $N_\epsilon \in \mathbb{R}$ such that $p_n(x_k - x_q) < \epsilon$ whenever $k, q > N_\epsilon$.

A lctvs X whose topology is Hausdorff, induced by a finite number of seminorms, and complete, i.e. every Cauchy sequence converges, is called a **Frechet space**.

Let us look at the space of all smooth functions on Ω

$$\mathcal{E}(\Omega) = \{f : \Omega \subset \mathbb{R}^n \to \mathbb{R}\ ,\ \text{ with } f \text{ smooth}\}\ .$$

We equipe $\mathcal{E}(\Omega)$ with the following family of seminorm

$$\mathcal{P} = \{p_{K,r}(f)\ ,\ \text{ with compact subset } K \subset \Omega\ ,\ \text{and } r \in \mathbb{N}\}\ .$$

The seminorms are defined as

$$p_{K,r}(f) = \sup\{|\partial^\alpha f(x)| \ \text{ with } \ x \in K \ \text{ and } \ |\alpha| \leq r\}\ ,$$
$$\text{where } \ \alpha \in \mathbb{N}^n,\ |\alpha| = \sum_{i=1}^n \alpha_i, \ \text{ and } \ \partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\ .$$

The family of seminorms $\mathcal{P}$ endow $\mathcal{E}(\Omega)$ with a locally convex topology. On this space a sequence (f_n) is convergent with limit f means $\forall\alpha$ and $\forall K$, $(\partial^\alpha f_n)$ converge uniformly to $(\partial^\alpha f)$, i.e. $\forall\epsilon > 0$ there is $N_\epsilon \in \mathbb{N}$ such that $\forall x \in K$ and $\forall n \geq N_\epsilon$, we have $|\partial^\alpha f_n(x) - \partial^\alpha f(x)| < \epsilon$. This space is a lctvs which appears to be a Frechet space.

Now let us come back to the case of real valued functions defined on $\mathcal{M}$. We consider the space of smooth functions defined as follow

$$\mathcal{E}(\mathcal{M}, E) = \{\phi : \mathcal{M} \to E \ \text{ and } \ \phi \ \text{ smooth}\}\ .$$

We choose to work with the atlas $\{(U_i, \Phi_i)\}$ on $\mathcal{M}$. Then we can define

$$\Psi : \mathcal{E}(\mathcal{M}, E) \to \underset{i}{\times}\, \mathcal{E}\left(\Phi_i(U_i) \subset \mathbb{R}^n, \mathbb{R}\right)^p\ ,$$

with p the dimension of the vector bundle E. We know now how to define the topology on every single members of the Cartesian product of the right hand side. It is due to fact that $\Phi_i(U_i)$ is a subset of $\mathbb{R}^n$. The topology of the all Cartesian product is the product topology, for which the open sets are $\emptyset$ and the union of Cartesian products $\times_i W_i$ with W_i subset of the topology in $\mathcal{E}\left(\Phi_i(U_i), \mathbb{R}\right)$. The topology of the left hand side is the induced topology. Let set $\gamma = (i, K, r)$, with i to index the opens U_i, $K \subset \Phi_i(U_i)$ compact, and $r \in \mathbb{N}$. We define the seminorm p_γ on $\mathcal{E}(\mathcal{M}, E)$ as follow

$$p_\gamma(\phi) = p_{K,r}\left(\Psi\left(\phi_{|U_i}(x)\right)^p\right)\ ,$$

with $\phi \in \mathcal{E}\left(\mathcal{M}, E\right)$. It defines the locally convex topology on $\mathcal{E}\left(\mathcal{M}, E\right)$. A sequence $(\phi_n)_{n\geq 1}$ converges to ϕ with respect to this topology if and only if for any chart (U, ϕ), and any compact subset $K \subset U$, $\partial_K^\alpha \phi_m$ converges uniformly on K to $\partial_K^\alpha \phi$.

For later puposes we define real valued smooth compactly supported functions defined as

$$\mathcal{D}(\mathcal{M}, E) = \{\phi : \mathcal{M} \to E \ \text{ with } \ \phi \ \text{ smooth and compactly supported}\}\ .$$

It is endowed with the locally convex topology implemented as follow

$$\mathcal{D}(\mathcal{M}, E) = \bigcup_K \mathcal{D}_K(\mathcal{M}, E)\ ,$$

where in the right hand side it is the union over all compact set $K \subset \mathcal{M}$. $\mathcal{D}_K(\mathcal{M}, E) \subset \mathcal{E}(\mathcal{M}, E)$ is the space of all smooth functions supported in K, endowed with the topology induced from $\mathcal{E}(\mathcal{M}, E)$. On $\mathcal{D}(\mathcal{M}, E)$ we have the inductive limit topology. It is a lctvs but non metrizable, therefore it is not a Frechet space.

Moreover we shall work only with real scalar fields, it means the vector bundle chosen is the real line bundle. We shall for now on denote $\mathfrak{C}(\mathcal{M})$ as $\mathcal{E}(\mathcal{M})$ if we consider smooth functions otherwise if we choose to work with compactly supported smooth functions we shall write $\mathcal{D}(\mathcal{M})$. We shall work off shell, i.e. we do not implement any dynamic, therefore we do not not put any further restriction on the field configurations.

Definition 13 (Off shell configuration space)
The off shell configuration space over $\mathcal{M}$ is the space of real valued smooth maps, $\phi \in \mathcal{C}^\infty(\mathcal{M})$. We denote it by $\mathcal{E}(\mathcal{M})$. For ϕ real valued, smooth, and compactly supported, the configuration space is denoted by $\mathcal{D}(\mathcal{M}) \subset \mathcal{E}(\mathcal{M})$. Both spaces are endowed with the locally convex topology. ♦

2.2 Functional view of observables

We need now to define what is an observable in this framework. Roughly speaking an observable will give us a way to measure physical quantities. Therefore it shall map field of the configuration space to numbers.

We defined an observable as a functional in the following way

$$\mathsf{F} : \left\{ \begin{array}{ccc} \mathcal{E}(\mathcal{M}) & \to & \mathbb{C} \\ \phi & \mapsto & \mathsf{F}(\phi) \end{array} \right. .$$

!!!! PRECISE THE TOPOLOGY OF THE SPACE OF FUNCTIONALS !!!!

Due to the fact that we shall have to consider functionals which will not be defined for all fields configuration, we need a concept which permit to localize functionals in certain region of spacetime.

Definition 14 (Spacetime support)
The spacetime support of an observable F is

$$\mathsf{supp}(\mathsf{F}) \doteq \left\{ x \in \mathcal{M} \,\middle|\, \forall\, U \ni x,\ \exists\, \phi, \psi \in \mathcal{E}(\mathcal{M}),\ \mathsf{supp}(\psi) \subset U,\ \text{such that } \mathsf{F}(\phi + \psi) \neq \mathsf{F}(\phi). \right\} .$$

It is a closed set. ♦

Therefore the spacetime support of an observable isis the set of points on the spacetime on which the observable does "feel" the influence of the fields configuration.

We denote by $\mathcal{F}_0(\mathcal{M})$ the functionals with compact spacetime support over $\mathcal{M}$. We follow and endow $\mathcal{F}_0(\mathcal{M})$ with the following algebraic structure.

- Sum : $(\mathsf{F} + \mathsf{G})(\phi) = \mathsf{F}(\phi) + \mathsf{G}(\phi)$;
- Multiplication by a scalar $z \in \mathbb{C}$: $(z \cdot \mathsf{F})(\phi) = z\mathsf{F}(\phi)$;
- Pointwise product : $(\mathsf{F} \cdot \mathsf{G})(\phi) = \mathsf{F}(\phi) \cdot \mathsf{G}(\phi)$;
- Involution : $\mathsf{F}^*(\phi) = \overline{\mathsf{F}(\phi)}$;
- Unit : $\mathbb{I} = \mathsf{F}(\phi) = 1$.

A direct consequence is that $\mathcal{F}_0(\mathcal{M})$ is a commutative unital $*$-algebra. And we can check that these algebraic operations do not modify the spacetime support

Lemma 3 ("Rigidity" of the spacetime support)
The above algebraic relations do preserve the spacetime support of a functional. In particular we have

- Sum : $\mathsf{supp}(\mathsf{F} + \mathsf{G}) \subseteq \mathsf{supp}(\mathsf{F}) \cup \mathsf{supp}(\mathsf{G})$;
- Pointwise product : $\mathsf{supp}(\mathsf{F} \cdot \mathsf{G}) \subseteq \mathsf{supp}(\mathsf{F}) \cap \mathsf{supp}(\mathsf{G})$. ▶

Proof 3
(blablabla) ■

As we already said $\mathcal{E}(\mathcal{M})$ and $\mathcal{D}(\mathcal{M})$ are infinite dimensional spaces, therefore we need to precisely define the notion of derivative of objects taking value on these spaces.

Definition 15 (Functional derivative)
Let U and W be two locally convex topological vector spaces, and $V \subseteq U$ an open subset. The functional derivative (or Gâteau derivative) of a map $\mathsf{F} : V \to W$ at $\phi \in V$ in the direction $\psi \in U$ is defined as the map $\mathsf{F}^{(1)} : V \times U \to W$,

$$\mathsf{F}^{(1)}(\phi)[\psi] = \lim_{t \to 0} \frac{1}{t}\Big(\mathsf{F}(\phi_n + t\psi) - \mathsf{F}(\phi)\Big) .$$

The map F is called differentiable (or Gâteau differentiable) at $\phi \in V$ if the limit exists for all $\psi \in U$, and continously differentiable if $F^{(1)}$ is jointly continous on the product space $V \times U$.
The generalization to n-th functional derivative of F at $\phi \in V$ with respect to the directions $\psi_1, \dots, \psi_n \in U$ is defined as a map $\mathsf{F} : V \times U^{\otimes n} \to W$,

$$\mathsf{F}^{(n)}(\phi)[\psi_1, \dots, \psi_n] = \lim_{t \to 0} \frac{1}{t}\Big(\mathsf{F}^{(n-1)}(\phi_n + t\psi)[\psi_1, \dots, \psi_{n-1}] - \mathsf{F}^{(n-1)}(\phi)[\psi_1, \dots, \psi_{n-1}]\Big) .$$

The map F is said to be smooth at $\phi \in V$ if the limit exists for all $\psi_1, \dots, \psi_n \in U$, and if $\mathsf{F}^{(n)}$ is jointly continuous on the product space $V \times U^{\otimes n}$. ♦

Let us precise what is a jointly continous map on a product space. A map $f : V \times U \to W$ at $(x, y) \in V \times U$ is jointly continous if for each neighborhood W' of $f(x, y)$ there exists a product of open sets $U' \times V' \subseteq U \times V$ containing (x, y) such that $f(U' \times V') \subseteq W'$.

If insteaf of taking generic locally convex topological vector space U in the previous definition, we take $\mathcal{E}(\mathcal{M})$ or $\mathcal{D}(\mathcal{M})$, then we have a precise definition of observables defined as functionals.

We illustrate this definition via a simple example.

Example 1
Here is the first two derivatives of a "functional potential" ϕ^4.

$$\begin{aligned}
&\mathsf{V}(\phi) = \int \mathrm{d}x \, \sqrt{|\mathsf{det}(\mathsf{g})|} \, \frac{\lambda(x)}{4!}\phi(x)^4 , \\
&\mathsf{V}^{(1)}(\phi) = \frac{\lambda(x)}{3!}\phi(x)^3 , \qquad \mathsf{V}^{(2)}(\phi) = \frac{\lambda(x)}{2!}\phi(x)^2\delta(x, y) .
\end{aligned}$$ ♦

We can show that the following properties still hold in this framework.

Lemma 4
Let F and G be two functionals at least continously differentiable, and let $\phi, \psi_\sharp$ contained in a locally convex topological vector space.

- fundamental theorem of calculus

$$\mathsf{F}(\phi + \psi) - \mathsf{F}(\phi) = \int_0^1 \mathrm{d}t \, \mathsf{F}^{(1)}(\phi + t\psi)[\phi]$$

- Taylor's formula

$$\mathsf{F}(\phi + \psi) = \mathsf{F}(\phi) + \mathsf{F}^{(1)}(\phi)[\psi] + \dots + \frac{1}{n!}\mathsf{F}^{(n)}(\phi)[\psi_1, \dots, \psi_n] + \frac{1}{n!}\int_0^1 \mathrm{d}t \, (1-t)^n \, \mathsf{F}^{(n+1)}(\phi + t\psi)[\psi^{\otimes n}]$$

- Leibniz formula

$$(\mathsf{F} \cdot \mathsf{G})^{(n)}(\phi)[\psi_1, \dots, \phi_n] = \sum_{k=0}^{n} \binom{n}{k} \mathsf{F}^{(k)}(\phi)[\psi_1, \dots, \psi_k] \, \mathsf{G}^{(n-k)}(\phi)[\psi_1, \dots, \psi_{n-k}] .$$

Lemma 5 (Smooth functional)
Let us consider a complex valued functional $\mathsf{F} : \mathcal{E}(\mathcal{M}) \to \mathbb{C}$. If F is smooth, then for $\phi \in \mathcal{E}(\mathcal{M})$ the distribution $\mathsf{F}^{(1)}(\phi)$ is of compact support. ►

It has been proved in that the spacetime support of a functional can be described by its first derivatives.

Lemma 6 ("Characterization" of the spacetime support)
If the first derivative of $\mathsf{F} \in \mathcal{F}_0(\mathcal{M})$ exists, then

$$\mathsf{supp}\,(\mathsf{F}) = \overline{\bigcup_{\phi \in \mathcal{E}(\mathcal{M})} \mathsf{supp}\left(\mathsf{F}^{(1)}(\phi)\right)}\ ,$$

with $\mathsf{supp}\left(\mathsf{F}^{(1)}(\phi)\right)$ the usual support of the distribution $\mathsf{F}^{(1)}(\phi)$. ►

Proof 4
(blablabla) ■

2.3 Classical field theory

- actions
- euler lagrange
- klein gordon equation
- adv - ret fund. sol.
- cauchy problem
- propagator
- poisson algebra

After having introduce the fucntional approach which will be used here, we formulate the clasical field theory. We work with scalar fields on curved spacetime, therefore we have as equation of motion the generalised Klein Gordon eqation.

$$\mathsf{P}\phi = \left(\Box + \xi \mathsf{R} + m^2\right)\phi = 0\ , \tag{2.1}$$

with m the (positive real) mass of the theory, $\xi \in \mathbb{R}$, and R the scalar curvature. We required in the case of vanishing curvature eqrefeq:klein-gordon reduces to the Klein Gordon equation of the free scalar field theory on Minkowski spacetime. The case $\xi = 0$ is called minimal coupling, and $\xi = \frac{1}{6}$ the conformally coupling .

The spacetime $\mathcal{M}$ we considere is globally hyperbolic therefore the differential equation eqrefeq:klein-gordon admit unique solution once we give sufficient data condition. It has been shown in that the operator P has unique retarded and advanced fundamental solutions. We will denote by H_a (respectively H_r) the fundamental advanced solution (respectively the retarded solution).

$$\mathsf{supp}\left(\mathsf{H}_{\mathsf{a}/\mathsf{r}} f\right) \subset J^{\pm}\left(\mathsf{supp}\,(f)\,, \mathcal{M}\right)\ ,\quad f \in \mathcal{C}_0^\infty(\mathcal{M})\ .$$

Definition 16 (Action)
A map $\mathcal{S}$ such that

$$\mathcal{S} : \mathcal{D}(\mathcal{M}) \to \mathcal{F}_{\mathsf{loc}}(\mathcal{M})$$

is an action if it fulfills the following rquirements.

- $f \mapsto S[f]$ is linear ;

- $S[f]$ is real ;
- $S[f]^* = S[f^*]$;
- $\mathrm{supp}\,(S[f]) \subset \mathrm{supp}\,(f)$. ♦

Two action $\mathcal{S}_1$ and $\mathcal{S}_2$ will be called equivalent when

$$\mathrm{supp}\,(S_1[f] - S_2[f]) \subset \mathrm{supp}\,(\mathrm{d}f)$$

2.4 Quantization via formal deformation

- definition (formal power series of functionals) $\mathcal{F}_\sharp[[\hbar]]$
- the noncommutative algebra
- definition (Hadamard two point functions)
- definition ($\star$ product)
- definition ($\star$ algebra of off shell observables)
- equivalent $\star$ product / algebra

Chapter 3

Interacting quantum field theory

(blablabla)

3.1 Basic definitions

We recall the perturbative construction of an interacting quantum field theory on a generic curved spacetime in the framework of **perturbative algebraic quantum field theory (pAQFT)** recently developed in In this construction, the basic object of the theory is an algebra of observables which is realised as a suitable set of functionals on field configurations equipped with a suitable product. In order to implement the perturbative constructions following the ideas of Bogoliubov and others, the field configurations ϕ are assumed to be off–shell. Namely, $\phi \in \mathcal{E}(\mathcal{M}) = C^\infty(\mathcal{M})$ is a smooth function on the globally hyperbolic spacetime $(\mathcal{M}, g)$ and observables are modelled by functionals $F : \mathcal{E}(\mathcal{M}) \to \mathbb{C}$ satisfying further properties. In particular all the functional derivatives exist as distributions of compact support, where we recall that the functional derivative of a functional F is defined for all $\psi_1, \dots, \psi_n \in \mathcal{D}(\mathcal{M}) = C_0^\infty(\mathcal{M})$ as

$$F^{(n)}(\phi)(\psi_1 \otimes \dots \otimes \psi_n) := \frac{d^n}{d\lambda_1 \dots d\lambda_n} F(\phi + \lambda_1\psi_1 + \dots \lambda_n\psi_n)\bigg|_{\lambda_1 = \dots = \lambda_n = 0} \in \mathcal{E}'(\mathcal{M}^n).$$

The set of these functionals is indicated by $\mathcal{F}$. Further regularity properties are assumed for the construction of an algebraic product. In particular, the set of local functionals $\mathcal{F}_{\mathsf{loc}} \subset \mathcal{F}$ is formed by the functionals whose n–th order functional derivatives are supported on the total diagonal $d_n = \{(x, \dots, x), x \in \mathcal{M}\} \subset \mathcal{M}^n$. Furthermore, their singular directions are required to be orthogonal to d_n, namely $\mathsf{WF}(F^{(n)}) \subset \{(x, k) \in T^*\mathcal{M}^n, x \in d_n, k \perp Td_n\}$ where WF denotes the wave front set. A generic local functional is a polynomial $P(\phi)(x)$ in ϕ and its derivatives integrated against a smooth and compactly supported tensor. The functionals whose functional derivatives are compactly supported smooth functions are instead called **regular functionals** and indicated by $\mathcal{F}_{\mathsf{reg}}$.
The quantum theory is specified once a product among elements of $\mathcal{F}_{\mathsf{loc}}$ and a $*$–operation (an involution on $\mathcal{F}$) are given. For the case of free (linear) theories the product can be explicitly given by a **$\star$–product**

$$F \star_H G = \sum_n \frac{\hbar^n}{n!} \left\langle F^{(n)}, H_+^{\otimes n} G^{(n)} \right\rangle , \tag{3.1}$$

where H_+ is a Hadamard distribution of the linear theory we are going to quantize, namely a distribution whose antisymmetric part is proportional to the commutator function $\Delta = \Delta_R - \Delta_A$ and whose wave front set satisfies the Hadamard condition, see e.g. for further details and Section for our propagator conventions. Owing to the properties of H_+, iterated $\star_H$–products of local functionals $F_1 \star_H \dots \star_H F_n$ are well defined and $\star_H$ is associative. In a normal neighbourhood of $(\mathcal{M}, g)$, a Hadamard distribution H_+ is of the form

$$H_+(x, y) = \frac{1}{8\pi^2} \left(\frac{u(x, y)}{\sigma_+(x, y)} + v(x, y) \log(M^2 \sigma_+(x, y)) \right) + w(x, y), \tag{3.2}$$

where $\sigma_+(x, y) = \sigma(x, y) + i\epsilon(t(x) - t(y)) + \epsilon^2/2$ with t a time–function, i.e. a global time-coordinate, $2\sigma(x, y)$ is the squared geodesic distance between x and y and M is an arbitrary mass scale. The Hadamard coefficients u and v are purely geometric and thus state–independent, whereas w is smooth and state–dependent if $H_+(x, y)$ is the two–point function of a quantum state.
For the perturbative construction of interacting models we further need a **time–ordered product** $\cdot_{T_H}$ on local functionals. This product is characterised by **symmetry** and the **causal factorisation property**, which requires that

$$F \cdot_{T_H} G = F \star_H G \quad \text{if} \quad F \gtrsim G\,, \tag{3.3}$$

where $F \gtrsim G$ indicates that F is later than G, i.e. there exists a Cauchy surface Σ of $(\mathcal{M}, g)$ such that $\mathrm{supp}(F) \subset J^+(\Sigma)$ and $\mathrm{supp}(G) \subset J^-(\Sigma)$. However, the causal factorisation fixes uniquely only the time–ordered products among regular functionals, in which case

$$F \cdot_{T_H} G = \sum_n \frac{\hbar^n}{n!} \left\langle F^{(n)}, H_F^{\otimes n} G^{(n)} \right\rangle , \tag{3.4}$$

where H_F is the time–ordered (Feynman) version of H_+, i.e. $H_F = H_+ + i\Delta_A$ with Δ_A the advanced propagator of the free theory, cf. Section . For local functionals, is only correct up to the need to employ a non–unique renormalisation procedure, cf. Section . This renormalisation can be performed in such a way that iterated $\cdot_{T_H}$–products of local functionals $F_1 \cdot_{T_H} \cdots \cdot_{T_H} F_n$ are well defined with $\cdot_{T_H}$ being associative. Moreover, $\star_H$–products of such time–ordered products of local functionals are well–defined as well, cf. . Consequently, we may consider the algebra $\mathcal{A}_0$ $\star_H$–generated by iterated $\cdot_{T_H}$–products of local functionals. This algebra contains all observables of the free theory which are relevant for perturbation theory.
In the perturbative construction of interacting models, namely when the free action is perturbed by a non–linear local functional V, the observables associated with the interacting theory are represented on the free algebra $\mathcal{A}_0$ by means of the **Bogoliubov formula**. This is given in terms of the local S–matrix, i.e., the time–ordered exponential

$$S(V) = \sum_{n=0}^{\infty} \frac{i^n}{n!\hbar^n} \underbrace{V \cdot_{T_H} \cdots \cdot_{T_H} V}_{n \text{ times}} , \tag{3.5}$$

where V is the interacting Lagrangean. In particular, for every interacting observable F the corresponding representation on the free algebra $\mathcal{A}_0$ is given by

$$\mathcal{R}_V(F) = S^{-1}(V) \star_H (S(V) \cdot_{T_H} F) , \tag{3.6}$$

where $S^{-1}(V)$ is the inverse of $S(V)$ with respect to the $\star_H$–product. The problem in using $\mathcal{R}_V(F)$ as generators of the algebra of interacting observables lies in the construction of the time–ordered product which a priori is an ill–defined operation.
This problem can be solved using ideas which go back to Epstein and Glaser, see e.g. , by means of which the time–ordered product among local functionals is constructed recursively. The time–ordered products can be expanded in terms of distributions smeared with compactly supported smooth functions which play the role of coupling constants (multiplied by a spacetime–cutoff). At each recursion step the causal factorisation property permits to construct the distributions defining the time–ordered product up to the total diagonal. The extension to the total diagonal can be performed extending the distributions previously obtained without altering the scaling degree towards the diagonal. In this procedure there is the freedom of adding finite local contributions supported on the total diagonal. This freedom is the well known renormalisation freedom. In addition to the properties already discussed, the renormalised time–ordered product is required to satisfy further physically reasonable conditions. We refer to for details on these properties and the proof that they can be implemented in the recursive Epstein–Glaser construction.
In spite of the theoretical clarity of this construction, the Epstein–Glaser renormalisation is quite difficult to implement in practise. The aim of this paper is to discuss a renormalisation scheme which is suitable for practical computations.

3.2 Particular spaces of observables

In the procedure of quantization we shall introduce a special product between observables. In particular we shall consider observables which have conditions on their derivatives in order to have something well defined. These conditions will be imposed using of wave front set. Roughly speaking the wave front set of a distribution is the set of points (x, k) where x specifies the location of the singularity on the spacetime, k the direction of the propagation of this singularity in the cotangent space at x.

Let us look in more details to the notion of wave front set of a distribution.

!!!! WAVE FRONT SET !!!!

We now have all the tools to carefully identify the space of functionals which have "good" working property. The simplest space is the regular space $\mathcal{F}_{\mathrm{reg}}(\mathcal{M})$, it is the space of all smooth functionals, with compactly sumported derivatives and having an empty wave front set.

Definition 17 (Space of regular functionals)
We define the space of regular functional as follow

$$\mathcal{F}_{\text{reg}}(\mathcal{M}) = \left\{ \mathsf{F}(\phi) \,\middle|\, \mathsf{F}(\phi) \in \mathcal{F}^{\infty}(\mathcal{M}),\ \mathsf{F}(\phi)^{(n)} \in \mathcal{E}'(\mathcal{M}^{\otimes n}),\ \text{and}\ \mathsf{WF}(\mathsf{F}(\phi)^{(n)}) = \emptyset \right\} ,$$

with ϕ a test function, i.e. element of $\mathcal{E}(\mathcal{M})$. ♦

However it does not contain the interaction functionals, those functionals that we would like to work with. Therefore we have to impose a less restrictive condition on the wave front set, we set that the wave front set of $F^{(n)}$ does not intersect the set $\mathcal{M} \times (\overline{V_+^n} \cup \overline{V_-^n})$ where $\overline{V_\pm}$ denotes the closed forward and backward light cone, respectively. It forms the space of microcausal functional $\mathcal{F}_{\mu c}(\mathcal{M})$.

Definition 18 (Space of microcausal functional)
We define the space of microcausal functional as follow

$$\mathcal{F}_{\mu c}(\mathcal{M}) = \left\{ \mathsf{F}(\phi) \,\middle|\, \begin{array}{c} \mathsf{F}(\phi) \in \mathcal{F}^{\infty}(\mathcal{M}),\ \mathsf{F}(\phi)^{(n)} \in \mathcal{E}'(\mathcal{M}^{\otimes n}) \\ \text{and}\ \mathsf{WF}(\mathsf{F}^{(n)}(\phi)) \cap \left(\mathcal{M}^n \times (\overline{V_+^n} \cup \overline{V_-^n})\right) = \emptyset \end{array} \right\} .$$

This space contains the interactions functionals but not only. For instance the regular functionals are still contained in it. The space which contains only the interaction functionals is called the local space $\mathcal{F}_{\text{loc}}$. We define it as the space of microcausal functionals having as support for their derivatives the small diagonal, $d_n = \{(x, \dots, x) \subset \mathcal{M}^n\}$.

Definition 19 (Space of local functional)
The local functionals are a subspace of microcausal functionals $\mathcal{F}_{\mu c}(\mathcal{M})$ defined as follow

$$\mathcal{F}_{\text{loc}}(\mathcal{M}) = \left\{ \mathsf{F}(\phi) \in \mathcal{F}_{\mu c}(\phi) \,\middle|\, \text{supp}\left(\mathsf{F}(\phi)^{(n)}\right) \subset d_n = \{(x, \dots, x) \subset \mathcal{M}^n\} \right\} \subset \mathcal{F}_{\mu c}(\mathcal{M}) .$$

We can define $\mathcal{F}_{\text{loc}}(\mathcal{M})$ by imposing the additivity property. And in this case the definition becomes natural.

Definition 20 (Additivity)
A functional $\mathsf{F}(\phi) \in \mathcal{F}_0(\mathcal{M})$ is said to be additive if for all $\phi, \psi, \chi \in \mathcal{E}(\mathcal{M})$ and $\text{supp}(\phi) \cap \text{supp}(\chi) = \emptyset$ we have

$$\mathsf{F}(\phi + \psi + \chi) = \mathsf{F}(\phi + \psi) - \mathsf{F}(\psi) + \mathsf{F}(\psi + \chi) .$$

From this definition it follows

Lemma 7 (Locality via the additivity condition)
If Fϕ) is additive, then

$$\mathsf{F}(\phi + \psi + \chi)^{(n)}[\gamma_1, ..., \gamma_n] = \mathsf{F}(\phi + \psi)^{(n)}[\gamma_1, ..., \gamma_n] - \mathsf{F}(\psi)^{(n)} + \mathsf{F}(\psi + \chi)^{(n)}[\gamma_1, ..., \gamma_n] .$$

and in particular if furthermore WF $\left(\mathsf{F}(\phi)^{(n)}\right) \perp Td_n$, we have that the derivatives $F(\phi)^{(n)}$ have support on the small diagonal d_n. ►

Proof 5
(blablabla) ■

An interesting property for additive functional is the following one

Lemma 8 (Decomposition of additive functionals)
Any additive functional F(ϕ) can be decomposed as a finite sum of additive functionals with arbitrarily small spacetime support. ►

Proof 6
proof ■

The study of these additive functionals is motivated by the fact that the renormalization freedom will correspond to this type of term.

3.3 Relation to the standard formulation of perturbative QFT

In this subsection we outline the relation of the pAQFT framework to the standard formulation of perturbative QFT. As an example, we demonstrate how the two-point (Wightman) function of the interacting field in ϕ^4 theory on a four–dimensional curved spacetime is computed, where we assume that the quantum state of the interacting field is just the state of free field modified by the interacting dynamics. We further assume that the free field is in a pure and Gaussian Hadamard state.
Let us recall the relevant formulae in perturbative algebraic quantum field theory where we shall always try to write expressions both in the pAQFT and in the more standard notation, indicating the latter by a $\doteq$. Given a local action V, such as $V = \int_{\mathcal{M}} d^4x\sqrt{-g}\;\frac{\lambda}{4}\phi(x)^4$ in ϕ^4–theory, the corresponding S–matrix, which is loosely speaking the "S-matrix in the interaction picture", is defined by and corresponds to $S(V) \doteq Te^{\frac{i}{\hbar}V}$.
The interacting field, i.e. "the field in the interaction picture" $\phi_I(x)$, is defined by the Bogoliubov formula

$$\phi_I(x) = \mathcal{R}_V(\phi(x)) = S(V)^{-1} \star_H (S(V) \cdot_{T_H} \phi(x)) \doteq T(e^{\frac{i}{\hbar}V})^{-1}T(e^{\frac{i}{\hbar}V}\phi(x)) \tag{3.7}$$

similarly to , where by unitarity $S(V)^{-1} = S(V)^*$. Interacting versions of more complicated expressions in the field, e.g. polynomials at different and coinciding points, are defined analogously. A thorough discussion of the relation between the Bogoliubov formula and the more common formulation of observables in the interaction picture may be found e.g. in . We only remark that, in the Minkowski vacuum state Ω_0, the expectation value of the Bogoliubov formula can be shown to read (also for more general expressions in the field)

$$\langle\phi_I(x)\rangle_{\Omega_0} \doteq \left\langle T(e^{\frac{i}{\hbar}V})^{-1}T(e^{\frac{i}{\hbar}V}\phi(x))\right\rangle_{\Omega_0} = \frac{\left\langle T(e^{\frac{i}{\hbar}V}\phi(x))\right\rangle_{\Omega_0}}{\left\langle T(e^{\frac{i}{\hbar}V})\right\rangle_{\Omega_0}}\,,$$

which is the theorem of Gell–Mann and Low, see for details.
In the algebraic formulation one usually cuts off the interaction in order to avoid infrared problems by replacing $\lambda \to \lambda f(x)$ with a compactly supported smooth function f and considers the adiabatic limit $f \to 1$ in the end when computing expectation values. As our aim is to compute expectation values in this section, we shall write the results in the adiabatic limit keeping in mind that proving the absence of infrared problems, i.e. the convergence of the spacetime integrals, is non–trivial and may depend on the state of the free field chosen. Note that the so-called "in–in–formalism" often used in perturbative QFT on cosmological spacetimes corresponds to considering a cutoff function f of the form $f(t,\vec{x}) = \Theta(t-t_0)$, i.e. f is a step function in time and the parameter t_0 corresponds to the time where the interaction is switched on.
Our choice for the quantum state Ω of the interacting field implies that e.g. the interacting two-point function

$$\langle\phi_I(x) \star_H \phi_I(y)\rangle_\Omega \doteq \langle\phi_I(x)\phi_I(y)\rangle_\Omega$$

is computed by writing ϕ_I in terms of the free field ϕ and computing the expectation value of the resulting observable of the free field in the pure, Gaussian, homogeneous and isotropic Hadamard state of the free field which we may thus denote by the same symbol Ω. The interacting vacuum state in Minkowski spacetime is of this form, whereas interacting thermal states in flat spacetime do not belong to this class, as they roughly speaking require to take into account both the change of dynamics and the change of spectral properties induced by V
The functionals in the functional picture of pAQFT correspond to Wick–ordered quantities of the free field in the sense we shall explain now. To this avail we recall the form of the (quantum) $\star_H$–product and (time–ordered) $\cdot_{T_H}$–product in and which are defined by means of a Hadamard distribution H_+ and its Feynman–version $H_F = H_+ + i\Delta_A$. Up to renormalisation of the time–ordered product, these products computed for the special case of the functional $\phi^2(x)$ give

$$\phi(x)^2 \star_H \phi(y)^2 = \phi(x)^2\phi(y)^2 + 4\hbar\phi(x)\phi(y)H_+(x,y) + 2\hbar^2 H_+^2(x,y)\,,$$
$$\phi(x)^2 \cdot_{T_H} \phi(y)^2 = \phi(x)^2\phi(y)^2 + 4\hbar\phi(x)\phi(y)H_F(x,y) + 2\hbar^2 H_F^2(x,y)\,.$$

This example shows that the $\star_H$–product ($\cdot_{T_H}$–product) implements the Wick theorem for normal–ordered (time–ordered) fields, and thus the previous formulae can be interpreted in more standard notation as

$$:\phi(x)^2:_H:\phi(y)^2:_H = :\phi(x)^2\phi(y)^2:_H + 4\hbar:\phi(x)\phi(y):_H H_+(x,y) + 2\hbar^2 H_+^2(x,y)\,,$$
$$T\left(:\phi(x)^2:_H:\phi(y)^2:_H\right) = :\phi(x)^2\phi(y)^2:_H + 4\hbar:\phi(x)\phi(y):_H H_F(x,y) + 2\hbar^2 H_F^2(x,y)\,,$$

where

$$:A:_H := \alpha_{-H_+}(A) := e^{-\hbar\left\langle H_S(x,y),\frac{\delta}{\delta\phi(x)}\otimes\frac{\delta}{\delta\phi(y)}\right\rangle}A\,, \tag{3.8}$$
$$H_S(x,y) := \frac{1}{2}(H_+(x,y) + H_+(y,x))\,,$$

e.g.

$$:\phi(x)^2:_H = \lim_{x\to y}\left(\phi(x)\phi(y) - H_+(x,y)\right)\,.$$

The Wick theorem relates (time–ordered) products of Wick–ordered quantities to sums of Wick–ordered versions of contracted products, where the definition of "Wick–ordering" and "contraction" are directly related, they both depend on the Hadamard distribution H_+ chosen. Thus, if we choose a particular H_+ to define $\star_H$ and $\cdot_{T_H}$ in pAQFT, we immediately fix the interpretation of all functionals in terms of expressions Wick–ordered with respect to H_+.
For the algebraic formulation the choice of H_+ is not important, indeed choosing a different H'_+ with the same properties, one has that $w := H'_+ - H_+ = H'_F - H_F$, because the advanced propagator Δ_A is unique and thus universal. Moreover, w is real, smooth and symmetric and

$$A\star_{H'} B = \alpha_w\left(\alpha_{-w}(A)\star_H \alpha_{-w}(B)\right)\,,\qquad A\cdot_{T_{H'}} B = \alpha_w\left(\alpha_{-w}(A)\cdot_{T_H}\alpha_{-w}(B)\right)\,,$$

with α defined as in and thus the algebras associated to $\star_H$, $\cdot_{T_H}$ and $\star_{H'}$, $\cdot_{T_{H'}}$ are isomorphic via

$$\alpha_w : \mathcal{A}_0 \to \mathcal{A}'_0\,,$$

where we recall that $\mathcal{A}_0$ is algebra $\star_H$–generated by $\cdot_{T_H}$–products of local functionals.
Hence, one may choose a suitable H_+ according to ones needs. However, since $\alpha_d(A) \neq A$ for functionals containing multiple field powers, statements like "the potential is ϕ^4" are ambiguous in pAQFT, and in fact also in the standard treatment of QFT. They become non–ambiguous only if one says "the potential is $:\phi^4:_H$, i.e. ϕ^4 Wick–ordered with respect to H_+". In pAQFT the corresponding non–ambiguous statement would be "the potential is the functional ϕ^4 in the algebra $\mathcal{A}_0$ constructed by means of H_+". If one then passes to the algebra $\mathcal{A}'_0$ constructed by means of H'_+, the potential picks up quadratic and c–number terms as we shall compute explicitly below. Alternatively, this ambiguity may be seen to correspond to the renormalisation ambiguity of tadpoles in Feynman diagrams.
Given a Gaussian and Hadamard free field state Ω, a convenient choice or representation of the algebra is to take $H_+ = \Delta_+$, where $\Delta_+(x,y) = \langle\phi(x)\star_\Delta\phi(y)\rangle_\Omega \doteq \langle\phi(x)\phi(y)\rangle_\Omega$ is the two-point function of the free field in the state Ω. This corresponds to standard normal–ordering and consequently in this representation the expectation values of all expressions which contain non-trivial powers of the field vanish, i.e.

$$\langle A\rangle_\Omega = A|_{\phi=0} \doteq \langle :A:_\Delta\rangle_\Omega\,. \tag{3.9}$$

Keeping the state Ω fixed, but passing on to a representation of the algebra with arbitrary H_+, the expectation value is computed as

$$\langle A\rangle_\Omega = \alpha_w(A)|_{\phi=0} \doteq \langle :A:_H\rangle_\Omega\,,\qquad w = \Delta_+ - H_+\,,$$

for instance

$$\langle\phi^2(x)\rangle_\Omega = \alpha_w(\phi^2(x))|_{\phi=0} = \left(\phi^2(x) + w(x,x)\right)|_{\phi=0} = w(x,x) \doteq \langle :\phi^2(x):_H\rangle_\Omega\,,$$

which in more standard terms would be computed as

$$\langle :\phi^2(x):_H\rangle_\Omega = \lim_{x\to y}\langle\phi(x)\phi(y) - H_+(x,y)\rangle_\Omega = \lim_{x\to y}\left(\Delta_+(x,y) - H_+(x,y)\right) = w(x,x)\,.$$

In QFT in curved spacetimes normal–ordering is in principle problematic, because (pointlike) observables should be defined in a local and generally covariant way, i.e. they should only depend on the spacetime in an arbitrarily small neighbourhood of the observable localisation . This is not satisfied for e.g. field polynomials Wick–ordered with $\Delta_+(x,y)$, because this distribution satisfies the Klein-Gordon equation and thus it encodes non–local information on the curved spacetime . It is still possible to compute in the convenient normal–ordered representation in the following way. In the example of ϕ^4–theory, one defines the potential $\frac{\lambda}{4}\phi(x)^4$ as a local and covariant observable by identifying it with the corresponding monomial in a representation of the algebra furnished by a purely geometric H_+, i.e. a H_+ of the form with $w = 0$.
In other words, we set once and for all in the H_+–representation

$$V_H = \int_{\mathcal{M}} d^4x\sqrt{-g}\,\frac{\lambda}{4}\phi(x)^4 \doteq \int_{\mathcal{M}} d^4x\sqrt{-g}\,\frac{\lambda}{4}:\phi(x)^4:_H\,.$$

This does not fix V uniquely, because H depends on the scale M inside of the logarithm, but the freedom in defining V_H, and analogously the free/quadratic part of Klein–Gordon action, as above corresponds to the usual freedom in choosing the "bare mass" m, "bare coupling to the scalar curvature" ξ, "bare cosmological constant" Λ, "bare Newton constant" G, as well as the "bare coefficients" β_1, β_2 of higher–derivative gravitational terms in the extended Einstein–Hilbert–Klein–Gordon action

$$\mathcal{S}(\phi, g_{ab}) = \int_{\mathcal{M}} d^4x\sqrt{-g}\left(\frac{R-2\Lambda}{16\pi G} + \beta_1 R^2 + \beta_2 R_{ab}R^{ab} - \frac{(\nabla\phi^2)}{2} - \frac{(m^2+\xi R)\phi^2}{2} - \frac{\lambda}{4}\phi^4\right)\,.$$

In order to switch to the normal-ordered representation, we use the map α_w defined in where $w = \Delta_+ - H_+$ is the state-dependent part of the Hadamard distribution Δ_+ whose dependence on the choice of M in H_+ corresponds to the above-mentioned freedom in the definition of the Wick-ordered Klein-Gordon action. That is, we have in the normal-ordered representation in the state Ω

$$V := V_\Delta = \alpha_w(V_H) = \int_\mathcal{M} d^4x\sqrt{-g}\,\frac{\lambda}{4}\phi(x)^4 + \frac{3\lambda}{2}w(x,x)\phi(x)^2 + \frac{3\lambda}{4}w(x,x)^2 \tag{3.10}$$

$$\doteq \int_\mathcal{M} d^4x\sqrt{-g}\,\frac{\lambda}{4}:\phi(x)^4:_\Delta + \frac{3\lambda}{2}w(x,x):\phi(x)^2:_\Delta + \frac{3\lambda}{4}w(x,x)^2 \tag{3.11}$$

We observe that the combination of the requirements that the interaction potential is a local and covariant observable and that, in order to compute expectation values in the state Ω, one would like to compute in the convenient normal-ordered representation with respect to Ω, leads to the introduction of an effective spacetime-dependent and state-dependent (squared) mass term $\mu(x) = 3\lambda w(x,x)$ in the interaction potential which of course leads to additional Feynman graphs in perturbation theory, cf. Figures and . The field-independent term $\frac{3\lambda}{4}w(x,x)^2$ plays no role for computations of quantities which do not involve functional derivatives of the extended Einstein-Hilbert-Klein-Gordon action with respect to the metric (an example where it does play a role is the stress-energy tensor), just as the modification of the free action by the change of representation plays no role for the computation of such quantities. A similar phenomenon as in occurs in thermal quantum field theory on Minkowski spacetime, where the effective mass generated by changing from the normal-ordered picture with respect to the free vacuum state to the normal-ordered picture with respect to the free thermal state is termed "thermal mass", cf. for details. After these general considerations, we can proceed to compute as an example the two-point function of the interacting field ϕ_I in ϕ^4 up to second order in λ, whereby ϕ_I is assumed to be in a state induced by a Gaussian Hadamard state of the free field. To this avail, we shall exclusively compute in the associated normal-ordered representation and thus omit the subscripts on the star product, and the time-ordered product, $\star := \star_\Delta$, $\cdot_T := \cdot_{T_\Delta}$. We start from the Bogoliubov formula and compute (from now on $\hbar = 1$)

$$S(V) = 1 + iV - \frac{1}{2}V \cdot_T V + O(\lambda^3)$$

$$S(V)^{\star-1} = 1 - iV + \frac{1}{2}V \cdot_T V - V \star V + O(\lambda^3)$$

$$\phi_I = \phi - iV \star \phi + iV \cdot_T \phi + \frac{1}{2}(V \cdot_T V) \star \phi - V \star V \star \phi - \frac{1}{2}V \cdot_T V \cdot_T \phi + V \star (V \cdot_T \phi) + O(\lambda^3)\,.$$

It remains to compute the $\star$-product of $\phi_I(x)$ and $\phi_I(y)$ and to set $\phi = 0$ in the remaining expression in order to obtain the expectation value in the state Ω. The result can as always be conveniently expressed in terms of Feynman diagrams, where we use the Feynman rules depicted in Figure

In the computation of $\langle\phi_I(x)\phi_I(y)\rangle_\Omega$, many expressions can be shortened considerably by using the relation $\Delta_F - \Delta_+ = i\Delta_A$, in particular this holds for the external legs of the appearing Feynman diagrams. The resulting Feynman diagrams are depicted in Figure

Chapter 4

The renormalization problem

4.1 Extension of distributions

4.2 The microlocal framework

4.3 The Epstein Glaser procedure

Chapter 5

A regularisation sheme

As discussed above, the main problem in using the Bogoliubov formula

$$\mathcal{R}_V(F) = \frac{\hbar}{i}\frac{d}{d\lambda} S(V)^{-1} \star_H S(V+\lambda F)\Big|_{\lambda=0}$$

for constructing interacting fields perturbatively is that it is given in terms of the S–matrix, which is the time–ordered exponential Unfortunately, the time–ordered product defined in terms of a "deformation" the singularities present in H_F forbid their application to more general functionals.
In order to proceed there is the need of employing a renormalisation procedure to construct the time–ordered products. In this work we discuss the use of certain analytic methods to solve this problem. The procedure we shall pursue is the following. We deform the Feynman propagator by means of complex parameter α with values in the neighbourhood of the origin obtaining a function with distributional values $\alpha \mapsto H_F^{(\alpha)}$. The deformation we are looking for needs to be such that in the limit $\alpha \to 0$ we recover the ordinary Feynman propagator. Furthermore, when α is non–vanishing, but sufficiently small, pointwise powers of $H_F^{(\alpha)}$ and integral kernels of more complicated loop diagrams should be well–defined. If this is the case, since the corresponding distributions obtained in the limit $\alpha \to 0$ are well defined outside of the total diagonal, the poles of $\alpha \mapsto H_F^{(\alpha)}$ and more complicated loop expressions are supported on the total diagonal. The idea, similar to what happens in dimensional regularisation, is that it is possible to renormalise these distributions by simply removing the poles.

5.1 Analytic regularistion of time–ordered products and the minimal subtraction scheme

In order to discuss the analytic regularisation of time–ordered products, we employ the notation used e.g. in which efficiently encodes the full combinatorics of Feynman diagrams in a compact form. Namely, the time–ordered product of n local functionals $V_1, \ldots, V_n$ can be formally defined in the following way[1]

$$V_1 \cdot_{T_H} \cdots \cdot_{T_H} V_n := \mathcal{T}_n(V_1 \otimes \cdots \otimes V_n) := m \circ T_n(V_1 \otimes \cdots \otimes V_n)\,, \tag{5.1}$$

where m denotes the pointwise product $m(F_1 \otimes \cdots \otimes F_n)(\phi) = F_1(\phi) \ldots F_n(\phi)$ and the operator T_n is written in terms of an exponential

$$T_n = \exp\left(\sum_{1\le i<j\le n} \Delta_{ij}\right) = \prod_{1\le i<j\le n} \sum_{l_{ij}\ge 0}^{\infty} \frac{\Delta_{ij}^{l_{ij}}}{l_{ij}!} \tag{5.2}$$

with

$$\Delta_{ij} := \left\langle H_F, \frac{\delta^2}{\delta\phi_i \delta\phi_j} \right\rangle. \tag{5.3}$$

Here the functional derivative $\frac{\delta}{\delta\phi_i}$ acts on the i–th element of the tensor product $V_1 \otimes \cdots \otimes V_n$ and $H_F = H_+ + i\Delta_A$ is the time–ordered version of the Hadamard distribution H_+ entering the construction of the free algebra $\mathcal{A}_0$ via $\star_H$. The exponential admits the usual representation in terms of Feynman graphs. More precisely, it can be written as a sum over all graphs Γ in $\mathcal{G}_n$, the set of all graphs with vertices $V(\Gamma) = \{1, \ldots, n\}$ and l_{ij} edges $e \in E(\Gamma)$

[1] In fact, in view of locality and covariance a better definition of the time–ordered product is $\mathcal{T}_1(V_1) \cdot_{T_H} \cdots \cdot_{T_H} \mathcal{T}_1(V_n) := \mathcal{T}_n(V_1 \otimes \cdots \otimes V_n)$ where $\mathcal{T}_1 : \mathcal{F}_{\text{loc}} \to \mathcal{F}_{\text{loc}} \subset \mathcal{A}_0$ plays the role of identifying local and covariant (smeared) Wick polynomials as particular elements of the free algebra $\mathcal{A}_0$, cf. . As we shall not touch upon this point in our renormalisation scheme, we choose to omit $\mathcal{T}_1$ in our formulas for simplicity.

joining the vertices i, j. Furthermore, in this construction, there are no tadpoles $l_{ii} = 0$ (cf. Section for details on why these are absent) and the edges are not oriented $l_{ij} = l_{ji}$. With this in mind

$$T_n = \sum_{\Gamma \in \mathcal{G}_n} \frac{1}{N(\Gamma)} \left\langle \tau_\Gamma, \frac{\delta^{2|E(\Gamma)|}}{\prod_{i \in V(\Gamma)} \prod_{E(\Gamma) \ni e \supset i} \delta\phi_i(x_i)} \right\rangle, \tag{5.4}$$

where $N(\Gamma) = \prod_{i<j} l_{ij}!$ is a numerical factor counting the possible permutations among the lines joining the same two vertices, the second product $\prod_{e \supset i}$ is over the edges having i as a vertex and x_i is a point in $\mathcal{M}$ corresponding to the vertex i. Moreover, τ_Γ is a distribution which is well–defined outside of all partial diagonals, namely on $\mathcal{M}^n \setminus D_n$, where

$$D_n := \{x_1, \dots, x_n \,|\, x_i = x_j \text{ for at least one pair } (i,j),\ i \neq j\} \tag{5.5}$$

and τ_Γ has the form

$$\tau_\Gamma = \prod_{e=(i,j) \in E(\Gamma)} H_F(x_i, x_j) = \prod_{1 \le i < j \le n} H_F(x_i, x_j)^{l_{ij}}. \tag{5.6}$$

The a priori restricted domain of τ_Γ is the reason why T_n defined as above is not a well–defined operation on $\mathcal{F}_{\text{loc}}^{\otimes n}$. In order to complete the construction we need to extend the obtained distributions to the diagonals D_n. This is not a straightforward limit because the singular structure of the Feynman propagator H_F contains the one of the δ–distribution and because pointwise products of the latter distribution are ill–defined. Consequently, a renormalisation procedure needs to be implemented in order to extend τ_Γ to the full $\mathcal{M}^n$. This extension is in general not unique, but subject to renormalisation freedom.
Here we shall discuss a procedure to extend the distributions τ_Γ to D_n called **minimal subtraction (MS)**, which makes use of an analytic regularisation $\Delta_{ij}^{\alpha_{ij}}$ of Δ_{ij} given in terms of a family of deformations $H_F^{\alpha_{ij}}$ of the Feynman propagator H_F parametrised by complex parameters α_{ij} contained in some neighbourhood of $0 \in \mathbb{C}$. To this end, we follow and call $t^{(\alpha)}$ an analytic regularisation of a distribution t defined outside of a point $x_0 \in \mathcal{M}$ if for all $f \in \mathcal{D}(\mathcal{M})$ $\langle t^{(\alpha)}, f\rangle$ is a meromorphic function in α for α in some neighbourhood of 0 which is analytic for $\alpha \neq 0$. Moreover $t^{(\alpha)}$ may be extended to x_0 for $\alpha \neq 0$ whereas $\lim_{\alpha \to 0} t^{(\alpha)} = t$ on $\mathcal{M} \setminus \{x_0\}$.
We shall introduce an analytic regularisation of the Feynman propagator H_F in the following section, but the basic idea of the MS–scheme is independent of the details of the analytic regularisation. Namely, given any analytic regularisation $H_F^{(\alpha)}$ of H_F, we repeat the formal construction of T_n presented above by replacing H_F by $H_F^{(\alpha)}$ in and Δ_{ij} by the induced $\Delta_{ij}^{\alpha_{ij}}$ in . Proceeding in this way we define

$$T_n^{(\boldsymbol{\alpha})} := e^{\sum_{i<j} \Delta_{ij}^{\alpha_{ij}}} \qquad \text{with} \qquad \boldsymbol{\alpha} := \{\alpha_{ij}\}_{i<j}\,,$$

and the corresponding integral kernels $\tau_\Gamma^{(\boldsymbol{\alpha})}$ of Feynman graphs Γ in analogy to . We expect that the distributions $\tau_\Gamma^{(\boldsymbol{\alpha})}$ are multivariate meromorphic functions which have poles at the origin for some of the α_{ij}. Hence, in order to obtain well–defined distributions in the limit α_{ij} to 0 and consequently a renormalised time–ordered product $\cdot_{T_H}$, all these poles need to be subtracted.
The properties of the analytically regularised Feynman propagator imply that $\tau_\Gamma^{(\boldsymbol{\alpha})}$ is well–defined on $\mathcal{M}^n \setminus D_n$ even if all α_{ij} are vanishing. Since $\tau_\Gamma^{(\boldsymbol{\alpha})}$ is a multivariate meromorphic function in $\boldsymbol{\alpha}$ which is analytic if restricted to $\mathcal{M}^n \setminus D_n$, we may deduce that the principal part of $\tau_\Gamma^{(\boldsymbol{\alpha})}$ for some α_{ij} must be supported on a partial diagonal of $\mathcal{M}^n$. In fact, in order for the time–ordered products to fulfil the factorisation property , the subtraction of the principal parts of $\tau_\Gamma^{(\boldsymbol{\alpha})}$ needs to be done in such a way that at each step only local terms are subtracted. However, the previous discussion only implies that the support of the principal parts is contained in D_n, i.e. the union of all the partial diagonals in $\mathcal{M}^n$. In order to satisfy the causal factorisation property, the principal parts need to be removed in a recursive way starting from the partial diagonals corresponding to two vertices and proceeding with the partial diagonals corresponding to an increasing number $m \le n$ of vertices $\mathfrak{d}_I := \{(x_1, \dots, x_n) \in \mathcal{M}^n, x_i = x_j, i, j \in I \subset \{1, \dots, n\}, |I| = m\}$.
For every subset $I \subset \overline{n}$ we indicate by R_I the operator which extracts the principal part with respect to α_I of a multivariate meromorphic function $f(\{\alpha_{ij}\}_{i<j})$, where for every $i, j \in I$, $\alpha_{ij} = \alpha_I$, and multiplies it with -1:

$$R_I f := -\text{pp} \lim_{\alpha_{ij} \to \alpha_I \forall i,j \in I} f(\{\alpha_{ij}\}_{i<j}). \tag{5.7}$$

We complement this definition by setting $R_{\{\}}$ to be the identity.
Given all these data, we define the renormalised time–ordered product in the MS–scheme as in e.g. by

$$\mathcal{T}_n = (\mathcal{T}_n)_{\text{ms}} := \lim_{\boldsymbol{\alpha} \to 0} m \circ \left(\sum_{F \in \mathfrak{F}_{\overline{n}}} \prod_{I \in F} R_I \right) \circ T_n^{(\boldsymbol{\alpha})}, \tag{5.8}$$

where, in the product over $I \in F$, R_I appears before R_J if $I \subset J$. Furthermore, for each graph Γ, the limit $\boldsymbol{\alpha} = \{\alpha_{ij}\}_{i<j} \to 0$ is computed by setting $\alpha_{ij} = \alpha_\Gamma$ for every $i < j$ before taking the sum over the forests and finally considering the limit α_Γ to 0. In this context we recall that, for every element of the sum over $\mathfrak{F}_{\overline{n}}$, part of the limit $\alpha_{ij} \to \alpha_\Gamma$ is already taken by applying R_I, see
Given the renormalised $\mathcal{T}_n$ in the MS–scheme, the corresponding local S–matrix may be constructed as

$$S(V) = \sum_{n=0}^{\infty} \frac{i^n}{\hbar^n n!} \mathcal{T}_n(V \otimes \cdots \otimes V)$$

for any local interaction Lagrangean V.
In order to implement the minimal subtraction scheme as outlined above we first need to specify an analytic regularisation $H_F^{(\alpha)}$ of the Feynman propagator H_F on generic curved spacetimes. Afterwards we have to demonstrate that for all graphs $\Gamma \in \mathcal{G}_n$ the analytically regularised integral kernels

$$\tau_\Gamma^{(\boldsymbol{\alpha})} = \prod_{e=(i,j)\in\Gamma} H_F^{\alpha_{ij}}(x_i, x_j) = \prod_{1\le i<j\le n} \left(H_F^{\alpha_{ij}}(x_i, x_j)\right)^{l_{ij}} . \tag{5.9}$$

appearing in

$$T_n^{(\boldsymbol{\alpha})} = \sum_{\Gamma\in\mathcal{G}_n} \frac{1}{N(\Gamma)} \left\langle \tau_\Gamma^{(\boldsymbol{\alpha})}, \frac{\delta^{2|E(\Gamma)|}}{\prod_{i\in V(\Gamma)} \prod_{e\ni i} \delta\phi_i(x_i)} \right\rangle \tag{5.10}$$

satisfy the properties necessary for the implementation of the MS–scheme. In particular we need to demonstrate that the distribution $\tau_\Gamma^{(\boldsymbol{\alpha})}$, which is a priori defined only on $\mathcal{M}^n \setminus D_n$, can be uniquely extended to the full $\mathcal{M}^n$ without renormalisation, where the uniqueness of this extension is important in order to obtain a definite renormalisation scheme. Moreover, we need to show that this distribution $\tau_\Gamma^{(\boldsymbol{\alpha})} \in \mathcal{D}'(\mathcal{M}^n)$ is weakly meromorphic in $\boldsymbol{\alpha}$ in a neighbourhood of 0, where in view of the forest formula it is only necessary to show that, setting $\alpha_{ij} = \alpha_I$ for all $i, j \in I$, $\tau_\Gamma^{(\boldsymbol{\alpha})}$ is weakly meromorphic in α_I. Additionally, we need to prove that, if τ_Γ prior to regularisation is well–defined outside of the partial diagonal d_I, then the pole of $\tau_\Gamma^{(\boldsymbol{\alpha})}$ with $\alpha_{ij} = \alpha_I$ for all $i, j \in I$ in α_I is supported on d_I and thus local. Finally, we need to prove that our MS–scheme satisfies all properties given in which a physically meaningful renormalisation scheme on curved spacetimes should satisfy, and we need to provide means to explicitly compute the minimal subtraction, which after all is the main motivation for this work.
Our plan to construct the mentioned quantities and to prove their required properties is as follows.

1. In Section we construct an analytic regularisation $H_F^{(\alpha)}$ of the Feynman propagator based on the observation that locally H_F is of the form up to considering instead of σ_+ the half squared geodesic with the Feynman ϵ–prescription $\sigma_F := \sigma + i\epsilon$. Motivated by the fact that the singular structure of H_F originates from the form in which σ_F appears, we set locally

$$H_F^{(\alpha)} := \lim_{\epsilon\to 0^+} \frac{1}{8\pi^2} \left(\frac{u}{M^{2\alpha}\sigma_F^{1+\alpha}} + \frac{v}{\alpha}\left(1 - \frac{1}{M^{2\alpha}\sigma_F^{\alpha}}\right)\right) + w, \tag{5.11}$$

   where we use the (arbitrary but fixed) mass scale M present in also for preserving the mass dimension of H_F in the regularisation.

2. In Proposition we then prove that the relevant distributions

$$t_\Gamma^{(\boldsymbol{\alpha})} := \prod_{1\le i<j\le n} \frac{1}{\sigma_F^{l_{ij}(1+\alpha_{ij})}} \in \mathcal{D}'(\mathcal{M}^n \setminus D_n) \tag{5.12}$$

   are multivariate analytic functions. The distribution only displays the most singular contribution of $\tau_\Gamma^{(\boldsymbol{\alpha})}$, but the subleading contributions are clearly of the same form up to replacing some of the factors $(1 + \alpha_{ij})$ in the exponents by α_{ij} or 0.

3. In order to show that $t_\Gamma^{(\boldsymbol{\alpha})}$ can be uniquely extended from $\mathcal{M}^n \setminus D_n$ to $\mathcal{M}^n$ in a weakly meromorphic fashion, i.e. that the singularities relevant for the forest formula are poles of finite order, we follow a strategy similar to the one used in and consider a scaling expansion with respect to a suitable scaling transformation. We first argue in Proposition that an analytically regularised distribution $t^{(\alpha)} \in \mathcal{D}'(\mathcal{M}^n \setminus d_n)$, which can be written as a sum of homogeneous terms with respect to this scaling transformation plus a sufficiently regular remainder, can be extended to $\mathcal{M}^n$ in a weakly meromorphic way, were the uniqueness of the extension follows from its weak meromorphicity. In Proposition , we give a sufficient condition for the existence of such a homogeneous expansion and we demonstrate in Proposition that the distributions $t_\Gamma^{(\boldsymbol{\alpha})}$ satisfy this condition.

4. The above–mentioned results are proved by means of generalised Euler operators which can be written abstractly in terms of a scaling transformation, but also in terms of covariant differential operators whose explicit form can be straightforwardly computed as we argue in Section . In Proposition we use these operators in order to demonstrate how the full relevant pole structure of $t_\Gamma^{(\boldsymbol{\alpha})}$ can be computed, thus showing the practical feasibility of the MS–scheme. We find that our renormalisation scheme corresponds in fact to a particular form of differential renormalisation and expand on this by computing a few examples in Section

5. Finally, in Proposition we prove that the MS–scheme satisfies the axioms of for time–ordered products and in addition preserves invariance under any spacetime isometries present.

Remark 1
The local Hadamard expansion of H_F and correspondingly the analytically continued $H_F^{(\alpha)}$ defined in are only meaningful on normal neighbourhoods $\mathcal{N}$ of $(\mathcal{M}, g)$. In order to define $H_F^{(\alpha)}$ and the induced distributions $\tau_\Gamma^{(\boldsymbol{\alpha})}$ globally, we may employ suitable partitions of unity. Rather than providing general and cumbersome formulas, we prefer to illustrate the idea at the example of the triangular graph

$$\tau_\Gamma = H_{F,13} H_{F,23} H_{F,12}^2. := H_F(x_1,x_3) H_F(x_2,x_3) H_F(x_1,x_2)^2$$

the renormalisation of which is discussed in detail in Section . We define the sets

$$\mathcal{N}_{12} := \bigcup_{x_1 \in \mathcal{M}} \{x_1\} \times \mathcal{N}_{x_1} \subset \mathcal{M}^2 \,, \qquad \mathcal{N}_{123} := \bigcup_{x_1 \in \mathcal{M}} \{x_1\} \times \mathcal{N}_{x_1}^2 \subset \mathcal{M}^3$$

where $\mathcal{N}_{x_1}$ is an arbitrary normal neighbourhood of x_1 in $(\mathcal{M}, g)$. We call sets of the form $\mathcal{N}_{12}$ and $\mathcal{N}_{123}$ a **normal neighbourhood of the total diagonal**.
Setting $\sigma_{ij} := \sigma(x_i, x_j)$, we observe that σ_{12} is well–defined on $\mathcal{N}_{12}$, and that σ_{12} , σ_{13} and σ_{23} are well–defined on $\mathcal{N}_{123}$. We now consider smooth and compactly supported functions $\chi_{12} \in \mathcal{D}(\mathcal{N}_{12})$, $\chi_{123} \in \mathcal{D}(\mathcal{N}_{123})$ which are such that $\chi_{12} = 1$ on $d_2 \subset \mathcal{N}_{12}$ and $\chi_{123} = 1$ on $d_3 \subset \mathcal{N}_{123}$. Note that by construction χ_{12} and χ_{123} vanish outside of $\mathcal{N}_{12}$ and $\mathcal{N}_{123}$ respectively. We may now define the analytically regularised distribution $\tau_\Gamma^{(\boldsymbol{\alpha})}$ by setting

$$\begin{aligned}\tau_\Gamma^{(\boldsymbol{\alpha})} := {} & H_{F,13}^{(\alpha_{13})} H_{F,23}^{(\alpha_{23})} \left(H_{F,12}^{(\alpha_{12})}\right)^2 \chi_{12}\chi_{123} + H_{F,13} H_{F,23} H_{F,12}^2 (1-\chi_{12}) \\ & + H_{F,13} H_{F,23} \left(H_{F,12}^{(\alpha_{12})}\right)^2 \chi_{12}(1-\chi_{123}) \,,\end{aligned}$$

where the Feynman propagators are regularised as in . By construction, $\tau_\Gamma^{(\boldsymbol{\alpha})}$ is globally well–defined and the analysis outlined above and performed in the following sections implies that it can be uniquely extended to a weakly meromorphic distribution on the full $\mathcal{M}^3$. Moreover, the local pole contributions corresponding to $\alpha_{12} = \alpha_I$ with $I = \{1,2\}$ and $\alpha_{12} = \alpha_{13} = \alpha_{23} = \alpha_J$ with $J = \{1,2,3\}$ are clearly independent of the choice of χ_{12}, χ_{123} and $\mathcal{N}_{12}$, $\mathcal{N}_{123}$ such that the MS–regularised amplitude $(\tau_\Gamma)_{\mathrm{ms}}$ is both globally well–defined and independent of the quantities entering the global definition of the analytic regularisation.
Keeping this approach to define global analytically regularised quantities in mind, we shall for simplicity work only with local quantities in the following. ♦

5.2 Analytic regularisation of the Feynman propagator on curved spacetimes

Following the plan outlined in Section , we would like to define an analytic regularisation $H_F^{(\alpha)}$ of H_F by . To this end, we start our analysis by constructing the distribution $1/\sigma_F^{1+\alpha}$ in $\mathcal{M}^2$ for $\alpha \in \mathbb{C} \setminus \mathbb{N}$. As anticipated in Section we shall make use of scaling properties of $1/\sigma_F^{1+\alpha}$ and the induced quantities $t_\Gamma^{(\boldsymbol{\alpha})}$ with respect to a particular geometric scaling transformation.
For every pair of points x_1, x_i in a normal neighbourhood $\mathcal{N} \subset (\mathcal{M}, g)$ there exists a unique geodesic γ connecting x_1 and x_i. We shall assume that $\gamma : \lambda \mapsto x_i(\lambda)$ is affinely parametrised and that $x_i(0) = x_1$ whereas $x_i(1) = x_i$. For all $\lambda \geq 0$ and all $f \in \mathcal{D}(\mathcal{N}_n)$ with $\mathcal{N}_n \subset \mathcal{M}^n$ a normal neighbourhood of the total diagonal d_n (cf. Remark), the geometric scaling transformation we shall consider is

$$f_\lambda := \lambda^{4(n-1)} f(x_1, x_2(\lambda), \ldots, x_n(\lambda)) \prod_{i=2}^{n} \frac{\sqrt{g(x_i(\lambda))}}{\sqrt{g(x_i)}} \,, \tag{5.13}$$

where $g(x)$ is the absolute value of the determinant of the metric expressed in normal coordinates. For $\lambda > 1$ it may happen that $x_i(\lambda)$ lies outside of $\mathcal{N}_n$ and is thus not well–defined in general. In this case we set $f_\lambda = 0$ which is well–defined because $f = 0$ outside of $\mathcal{N}_n$. For later purposes, we recall that the determinant of the metric computed in normal coordinates centred at x_1 is such that

$$\sqrt{g(x_i)} = \frac{1}{u^2(x_1, x_i)} ,$$

where u is the Hadamard coefficient in and u^2 is the van Vleck–Morette determinant, see e.g.
By means of this transformation, relevant information about the behaviour of a distribution in the neighbourhood of the total diagonal d_n can be obtained. We recall two definitions which we shall use in the following. The first one is taken from and adapted to our case.

Definition 21
The **scaling degree** of a distribution $t \in \mathcal{D}'(\mathcal{M}^n)$ or $t \in \mathcal{D}'(\mathcal{M}^n \setminus d_n)$ towards d_n is defined as

$$\mathrm{sd}(t) := \inf \left\{ w \in \mathbb{R} \,\Big|\, \lim_{\lambda \to 0^+} \lambda^w \langle t, f_{1/\lambda} \rangle = 0 \;\; \forall f \in \mathcal{D}(\mathcal{N}_n \setminus d_n) \right\} .$$

♦

If a distribution has scaling degree lower than the total dimension of the scaled coordinates $4(n-1)$, then it possesses a unique extension towards d_n with the same scaling degree, see e.g. Theorem 5.2 and 5.3 of . The scaling degree towards a partial diagonal may be defined in analogy to Definition . The same geometric transformation can be used to introduce relevant homogeneity properties of a distribution.

Definition 22
A distribution $t \in \mathcal{D}'(\mathcal{M}^n)$ or $t \in \mathcal{D}'(\mathcal{M}^n \setminus d_n)$, which satisfies the equality

$$\lambda^\delta \langle t, f_\lambda \rangle = \langle t, f \rangle \qquad \forall \lambda > 0$$

under transformations of the form for all $f \in \mathcal{D}(\mathcal{N}_n \setminus d_n)$ and for a $\delta \in \mathbb{C}$, is called **homogeneous of degree** δ. ♦

These definitions imply that a distribution which is homogeneous of degree δ has scaling degree $-\Re(\delta)$. We further recall that homogeneous distributions $t \in \mathcal{D}(\mathcal{M}^n \setminus d_n)$ possess unique extensions to $\mathcal{M}^n$ with the same degree of homogeneity δ if $-(\delta + 4(n-1)) \notin \mathbb{N}$, see e.g. Theorem 3.2.3 in

Proposition 1
Consider a normal neighbourhood $\mathcal{N}_2 \subset \mathcal{M}^2$ of d_2 (cf. Remark) and the following expression for $\alpha \in \mathbb{C}$ and $f \in \mathcal{D}(\mathcal{N}_2)$

$$\left\langle \frac{1}{\sigma_F^\alpha}, f \right\rangle := \lim_{\epsilon \to 0^+} \int_{\mathcal{M}^2} \frac{1}{(\sigma(x,y) + i\epsilon)^\alpha} f(x,y) d\mu_g(x) d\mu_g(y) .$$

Then the following statements hold.

1. $1/\sigma_F^\alpha$ restricted to $\mathcal{D}(\mathcal{N}_2 \setminus d_2)$ is a distribution which is weakly analytic in α.
2. $1/\sigma_F^\alpha$ is homogeneous of degree -2α with respect to transformations of the form for $f \in \mathcal{D}(\mathcal{N}_2 \setminus d_2)$.
3. $1/\sigma_F^\alpha$ is well–defined as a distribution on $\mathcal{N}_2$ for $2\alpha - 4 \notin \mathbb{N}$. Furthermore, for all $f \in \mathcal{D}(\mathcal{N}_2)$ $\langle 1/\sigma_F^\alpha, f \rangle$ is analytic for $2\alpha - 4 \notin \mathbb{N}$ and meromorphic for $\alpha \in \mathbb{C}$ with simple poles at $2\alpha - 4 \in \mathbb{N}$. ▶

Proof 7
$a)$ For every $x \in \mathcal{M}$ we fix a normal coordinate system $\xi_x : y \to \mathbb{R}^4$ in order to parametrise points y in a normal neighbourhood of x. Consequently, on $\mathcal{N}_2$ the squared geodesic distance divided by 2 can be easily expressed as

$$\sigma(x,y) = \frac{1}{2} \eta(\xi_x(y), \xi_x(y)) = \frac{1}{2} \xi_x^a \xi_{xa} ,$$

where η is the standard Minkowski metric given in Cartesian coordinates. Furthermore,

$$\left\langle \frac{1}{\sigma_F^\alpha}, f \right\rangle = \lim_{\epsilon \to 0^+} \int_{\mathcal{M}} \int_{\mathbb{R}^4} \frac{2^\alpha}{(\xi_x^a \xi_{xa} + i\epsilon)^\alpha} f(x, \xi_x) \sqrt{g(\xi_x)}\; d^4\xi \; d\mu_g(x). \tag{5.14}$$

which is well defined for $f \in \mathcal{D}(\mathcal{N}_2)$.
Observe that $1/(\xi^a \xi_a)^\alpha$ for $\xi^a \in \{z \in \mathbb{C}^4 \,|\, \Im(z) \in V^\pm\}$, where $V^\pm$ is the forward or past light cone with respect to the Minkowski metric, is analytic both in ξ and α. Furthermore, in the limit $\epsilon \to 0^+$, $1/(\xi^a \xi_a + i\epsilon)^\alpha$ can be seen as the boundary value of that analytic function. Since this function grows at most polynomially for large $1/\Im(\xi^a \xi_a)$ its boundary value defines a distribution, see e.g.
$b)$ The transformation defined in acts on points parametrised by normal coordinates as $\xi \to \lambda \xi$. Furthermore, $1/(\xi^a \xi_a)^\alpha$ on $A \subset \mathbb{C}^4$ is homogenous of degree 2α with respect to the transformation $\xi \to \lambda \xi$. The statement follows from this observation, taking into account
$c)$ Theorem 3.2.3 in ensures that the distribution $1/\sigma_F^\alpha \in \mathcal{D}'(\mathcal{N}_2 \setminus d_2)$ has a unique extension to d_2 preserving the degree of homogeneity for every $2\alpha - 4 \notin \mathbb{N}$. The other parts of the statement can be shown in the same way. ■

The previous proposition guarantees that $1/\sigma_F^\alpha$ is weakly meromorphic in α with simple poles at $2\alpha - 4 \in \mathbb{N}$. This property is preserved under taking linear combinations and multiplication by smooth functions. Consequently, the analytically regularised Feynman propagator $H_F^{(\alpha)}$ defined by is well-defined on a normal neighbourhood of the diagonal and weakly meromorphic in α.

Proposition 2
Consider a normal neighbourhood $\mathcal{N}_2$ of the diagonal $d_2 \in \mathcal{M}^2$. The following statements hold for the analytically continued Feynman propagator $H_F^{(\alpha)} \in \mathcal{D}'(\mathcal{N}_2)$ defined in .

1. $\lim_{\alpha\to 0} H_F^{(\alpha)} = H_F$.
2. $\mathsf{WF}(H_F^{(\alpha)}) \subset \mathsf{WF}(H_F)$.
3. The scaling degree of $H_F^{(\alpha)}$ tends to $-\infty$ when the real part of α tends to ∞. ▶

Proof 8
The proof of this proposition follows from the properties of $\sigma_F^{1+\alpha}$ obtained in Proposition . In particular, $a)$ and $c)$ can be directly obtained from the weak analyticity, while $b)$ follows from the fact that the distribution $1/\sigma_F^\alpha$ is well defined on $\mathcal{N}_2 \setminus d_2$ where it coincides either with $1/\sigma_+^\alpha$ or with $1/\sigma_-^\alpha$. In order to analyse the wave front sets of $1/\sigma_\pm^\alpha$, we pass to a normal coordinate system and obtain $1/\sigma_\pm^\alpha = 2/(\xi^a\xi_a \pm i\epsilon\xi^0)$. This distribution can be extended to a tempered distribution for every α and thus its Fourier transform can be directly computed. One finds that for $1/\sigma_\pm^\alpha$, only the null future/past directed directions do not decay rapidly, consequently $H_F^{(\alpha)}$ restricted to $\mathcal{N}_2 \setminus d_2$ has $\mathsf{WF}(H_F^{(\alpha)}) \subset \mathsf{WF}(H_F)$. Finally, we observe that the extension of $H_F^{(\alpha)}$ to $\mathcal{N}_2$ may possess further singularities supported on the diagonal with singular directions orthogonal to d_2. Hence, $\mathsf{WF}(H_F^{(\alpha)}) \subset \mathsf{WF}(H_F)$ still holds for $H_F^{(\alpha)} \in \mathcal{D}'(\mathcal{N}_2)$. ■

We are now able to discuss the analytical regularisation $\tau_\Gamma^{(\boldsymbol{\alpha})}$ of the distributions τ_Γ given in which appear in the graph expansion of the time-ordered products $\mathcal{T}_n$. As anticipated in Section , owing to the form of $H_F^{(\alpha)}$ given in the relevant distributions which need to be discussed are $t_\Gamma^{(\boldsymbol{\alpha})}$ introduced in and analysed in the following proposition.

Proposition 3
The operation

$$\left\langle t_\Gamma^{(\boldsymbol{\alpha})}, f\right\rangle := \int_{\mathcal{M}^n} \prod_{1\le i<j\le n} \frac{1}{\sigma_F(x_i,x_j)^{l_{ij}(1+\alpha_{ij})}} f dx_1 \dots dx_n \tag{5.15}$$

defined for $f \in \mathcal{D}(\mathcal{M}^n \setminus D_n \cap \mathcal{N})$ where $\mathcal{N}$ is a normal neighbourhood of the union of all partial diagonals D_n (cf. Remark) has the following properties.

1. $t_\Gamma^{(\boldsymbol{\alpha})}$ is distribution on $\mathcal{M}^n \setminus D_n \cap \mathcal{N}$.
2. $\left\langle t_\Gamma^{(\boldsymbol{\alpha})}, f\right\rangle$ is a continuous function for $\boldsymbol{\alpha} = \{\alpha_{ij}\}_{i<j} \in \mathbb{R}^{n(n-1)/2}$.
3. $\left\langle t_\Gamma^{(\boldsymbol{\alpha})}, f\right\rangle$ is analytic for every α_{ij} with $i<j$ and thus a multivariate analytic function. ▶

Proof 9
$a)$ The domain $\mathcal{M}^n \setminus D_n \cap \mathcal{N}$ is a disjoint union of connected components. On every connected component $\mathcal{C}$ $\sigma_F(x_i,x_j)$ equals either $\sigma_+(x_i,x_j)$ or $\sigma_+(x_j,x_i)$ depending on the causal relation between x_i and x_j which is fixed in $\mathcal{C}$. Hence, on $\mathcal{C}$, the wave front set of $\sigma_F(x_i,x_j)^{-1}$ is contained either in $\mathcal{V}_+$ or $\mathcal{V}_-$, where $\mathcal{V}_{+/-} = \{(x,x',k,k') \in T^*\mathcal{M}^2 \setminus 0, (x,k) \sim (x',-k'), k \triangleleft / \triangleright 0\}$. Consequently, $\sigma_F(x_i,x_j)^{-1}$ satisfies the Hadamard condition up to a permutation of the arguments. The very same holds for the distributions $\sigma_F(x_i,x_j)^{l_{ij}(1+\alpha_{ij})}$ for every l_{ij} and every α_{ij} which have been discussed in Proposition
Owing to the form of their wave front set, the pointwise products of these distributions present in $t_\Gamma^{(\boldsymbol{\alpha})}$ are well-defined because the Hörmander-criterion for multiplication of distributions is satisfied. In fact, up to some fixed permutation of the arguments $(x_1,\dots x_n)$, $t_\Gamma^{(\boldsymbol{\alpha})}$ satisfies the micro local spectrum condition introduced in Hence $t_\Gamma^{(\boldsymbol{\alpha})}$ is a well-defined distribution on every connected component $\mathcal{C}$ of $\mathcal{M}^n \setminus D_n \cap \mathcal{N}$ and thus it is well-defined also on $\mathcal{M}^n \setminus D_n \cap \mathcal{N}$.
$b)$ In order to check continuity for $\boldsymbol{\alpha} = \{\alpha_{ij}\}_{i<j} \in \mathbb{R}^{n(n-1)/2}$ in a fixed point $\overline{\boldsymbol{\alpha}}$ we may analyse the distribution on a fixed connected component $\mathcal{C}$ of the domain of $t_\Gamma^{(\boldsymbol{\alpha})}$ and factorize the distribution in two parts. In fact, due to the wave front set of $t_\Gamma^{(\boldsymbol{\alpha})}$ on $\mathcal{C}$ the factorisation $t_\Gamma^{(\boldsymbol{\alpha})} = t_\Gamma^{(\overline{\boldsymbol{\alpha}})} \cdot \tau_\Gamma^{(\boldsymbol{\beta})}$ is unique where the integral kernel of $\tau_\Gamma^{(\boldsymbol{\beta})}$

is $\prod_{1\leq i<j\leq n}\frac{1}{\sigma_F(x_i,x_j)^{\beta_{ij}}}$. For $\boldsymbol{\beta}$ in a sufficiently small neighbourhood of 0, $\tau_\Gamma^{(\boldsymbol{\beta})}$ is an integrable function which is differentiable for $\boldsymbol{\beta}=0$ as can be obtained by dominated convergence. Finally, the continuity is preserved by pointwise multiplication with $t_\Gamma^{(\overline{\boldsymbol{\alpha}})}$.

c) For an arbitrary but fixed pair of indices i,j, α_{ij} appears in the product displayed in We shall thus interpret $t_\Gamma^{(\boldsymbol{\alpha})}$ as a composition of distributions, namely as $1/\sigma_F^{\alpha_{ij}}\circ z$ where z is an operator which maps $\mathcal{D}(\mathcal{M}^n\setminus D_n\cap\mathcal{N})$ to $\mathcal{D}'(\mathcal{M}^2\setminus D_2\cap\mathcal{N}_2)$ for a suitable $\mathcal{N}_2\supset D_2=d_2$. The ϵ–regularised integral kernel of z corresponds to the product present in with the factor $1/\sigma_F^{\alpha_{ij}}$ removed. Because of the singular structure of z, for every $f\in\mathcal{D}(\mathcal{M}^n\setminus D_n\cap\mathcal{N})$, $\langle z,f\rangle$ is in fact a compactly supported smooth function supported on $\mathcal{M}^2\setminus D_2\cap\mathcal{N}_2$. Hence, the analysis of its composition with $1/\sigma_F^{\alpha_{ij}}$ is straightforward. These considerations imply separate analyticity of $t_\Gamma^{(\boldsymbol{\alpha})}$ in each α_{ij} whereas joint analyticity follows from the continuity proved in b). ■

5.3 Generalised Euler operators and principal parts of homogeneous expansions

The next step in the strategy outlined at the end of Section is to extend the distributions $t_\Gamma^{(\boldsymbol{\alpha})}$, which are a priori defined only outside of (a normal neighbourhood) of the union of all partial diagonals D_n to D_n and to show that this extension is weakly meromorphic in α_I upon setting $\alpha_{ij}=\alpha_I$ for all $i,j\in I\subset\{1,\dots,n\}$. As anticipated, we shall prove this by using particular homogeneity properties of $t_\Gamma^{(\boldsymbol{\alpha})}$ with respect to the scaling transformations . Even if $t_\Gamma^{(\boldsymbol{\alpha})}$ is not homogeneous in the strong sense of Definition , it has weaker homogeneity properties which are still strong enough in order to obtain the wanted results. In this section we analyse analytically regularised distributions satisfying this weaker homogeneity condition, provide sufficient conditions for this weaker homogeneity to hold and show how the principal part of a distribution of this type can be efficiently computed.

To this avail, we consider a normal neighbourhood $\mathcal{N}_n$ of the total diagonal d_n (cf. Remark) and define the **generalised Euler operator** $E_p:\mathcal{D}(\mathcal{N}_n)\to\mathcal{D}(\mathcal{N}_n)$ by

$$E_pf(x_1,\dots,x_n):=(-1)^p\;\lambda^{p+4(n-1)}\frac{d^p}{d\lambda^p}\left(\lambda^{-4(n-1)}f_\lambda(x)\right)\Big|_{\lambda=1},\tag{5.16}$$

where the scaling transformation is used. We then consider a family of distributions $t^{(\alpha)}\in\mathcal{D}'(\mathcal{N}_n\setminus d_n)$ defined for α in some neighbourhood $\mathcal{O}$ of $0\in\mathbb{C}$ and assume that $t^{(\alpha)}$ can be expanded as

$$t^{(\alpha)}=\sum_{k=0}^m t_k^{(\alpha)}+r^{(\alpha)}.$$

where $t_k^{(\alpha)}$ are homogeneous with degree with degree $a_k=-\delta_\alpha+k$ whose real part is smaller or equal to $-4(n-1)$ and a remainder $r^{(\alpha)}\in\mathcal{D}'(\mathcal{N}_n\setminus d_n)$ which has scaling degree smaller than $4(n-1)$ and can thus be uniquely extended to d_n for every $\alpha\in\mathcal{O}$ by . Owing to its homogeneity, every $t_k^{(\alpha)}$ can be rewritten by means of the generalised Euler operator E_p as

$$\left\langle t_k^{(\alpha)},f\right\rangle=\frac{1}{\prod_{j=0}^{p-1}(a_k+j+4(n-1))}\left\langle t_k^{(\alpha)},E_pf\right\rangle.\tag{5.17}$$

Note that, $E_pf(x_1,\dots x_n)$ is smooth and vanishes for $y=(x_1,\dots x_n)\to x=(x_1,\dots,x_1)$ as $C|y-x|^p$, i.e. it is in the class $O(|y-x|^p)$. For this reason, if p is chosen sufficiently large as $p>-a_k-4(n-1)$, $t_k^{(\alpha)}\circ E_p$ possesses a unique extension to d_n. We recall that, in order to renormalise $t^{(\alpha)}$ for $\alpha=0$ in the MS–scheme, we have to subtract its principal part before computing the limit of vanishing α

$$\langle(t_k)_{\mathsf{ms}},f\rangle:=\lim_{\alpha\to0}\left(\left\langle t_k^{(\alpha)},f\right\rangle-\mathsf{pp}\left\langle t_k^{(\alpha)},f\right\rangle\right).$$

However, if we use the representation of $t_k^{(\alpha)}$ provided by the right hand side of equation , its poles are manifestly exposed and can be easily subtracted. We recall that, since the original distribution $t_k^{(\alpha)}$ is well defined on $\mathcal{N}_n\setminus d_n$ even for $\alpha=0$, the principal part we are subtracting can only be supported on d_n. We summarise this discussion in the following proposition.

Proposition 4

Consider a normal neighbourhood $\mathcal{N}_n$ of the total diagonal d_n and a distribution $t\in\mathcal{D}'(\mathcal{N}_n\setminus d_n)$. Assume that $t^{(\alpha)}\in\mathcal{D}'(\mathcal{N}_n\setminus d_n)$ is an analytic regularisation of t, i.e. $t^{(\alpha)}$ is weakly analytic for α in a neighbourhood $\mathcal{O}$ of the origin of $\mathbb{C}$ and $\lim_{\alpha\to0}t^{(\alpha)}=t$. Moreover, assume that $t^{(\alpha)}$ can be decomposed as

$$t^{(\alpha)}=\sum_{k=0}^m t_k^{(\alpha)}+r^{(\alpha)}$$

where $t_k^{(\alpha)}$ are weakly analytic distributions which scale homogeneously under transformations of the form with degree $a_k = -\delta_\alpha + k$ and $r_k^{(\alpha)}$ is a weakly analytic distribution whose scaling degree towards d_n is strictly smaller than $4(n-1)$. Then the following statements hold.

1. $t^{(\alpha)}$ can be extended to $\dot{t}^{(\alpha)} \in \mathcal{D}'(\mathcal{N}_n)$ for every $\alpha \in \mathcal{O} \setminus \{0\}$.
2. $\dot{t}^{(\alpha)}$ is weakly meromorphic for $\alpha \in \mathcal{O}$ with possible poles for $\alpha = 0$ and it is the unique weakly meromorphic extension of $t^{(\alpha)}$.
3. The pole of $\dot{t}^{(\alpha)}$in 0 is supported on d_n.
4. The limit $\alpha \to 0$ can be considered after subtracting the pole part, namely

$$\langle t_{\mathsf{ms}}, f\rangle := \lim_{\alpha\to 0}\left(\left\langle \dot{t}^{(\alpha)}, f\right\rangle - \mathsf{pp}\left\langle \dot{t}^{(\alpha)}, f\right\rangle\right)$$

is well–defined for all $f \in \mathcal{D}(\mathcal{N}_n)$ and t_{ms} is an extension of t which preserves the scaling degree. ▶

Proof 10
The proof of $a)$ and $b)$ is an application of to every $t_k^{(\alpha)}$. Furthermore, since the scaling degree of $r^{(\alpha)}$ is strictly smaller than $4(n-1)$, $r^{(\alpha)}$ possesses an unique extension towards d_n, cf.
In order to prove $c)$ we note that the original distribution $t^{(\alpha)}$ defined on $\mathcal{N}_n \setminus d_n$ is weakly analytic and that an explicit construction of the weakly meromorphic extension $\dot{t}^{(\alpha)}$ to $\mathcal{N}_n$ is provided by , choosing for every component $t_k^{(\alpha)}$ a sufficiently large p and using . Hence, the poles of $\dot{t}^{(\alpha)}$ can only be supported on d_n. For this reason, after subtracting the principal part of the distribution the limit $\alpha \to 0$ can be safely taken. The such obtained distribution prior to considering the limit $\alpha \to 0$ coincides with $t^{(\alpha)}$ on $\mathcal{N}_n \setminus d_n$ and the same holds in the limit $\alpha \to 0$. Consequently t_{ms} is an extension of t. Finally, $\mathsf{sd}(t_{\mathsf{ms}}) = \mathsf{sd}(t)$, because our assumptions and the above analysis imply that $\mathsf{sd}(\dot{t}^{(\alpha)}) = \mathsf{sd}(t^{(\alpha)}) = \Re(\delta_\alpha)$, $\mathsf{sd}(\mathsf{pp}(\dot{t}^{(\alpha)})) \le \Re(\delta_\alpha)$ and $\lim_{\alpha\to 0}\Re(\delta_\alpha) = \mathsf{sd}(t)$. ■

We now discuss how equation can be used in order to regularise the most singular part of a distribution $t^{(\alpha)}$ which is known to be of the form $t^{(\alpha)} = \sum_{k=0}^m t_k^{(\alpha)} + r^{(\alpha)}$ but where the distributions $t_k^{(\alpha)}$ are not explicitly known. To this end, observe that equation implies

$$\left\langle t^{(\alpha)}, E_p f\right\rangle = \sum_{k=0}^m \left(\prod_{j=0}^{p-1}(a_k + j + 4(n-1))\right)\left\langle t_k^{(\alpha)}, f\right\rangle + \left\langle r^{(\alpha)}, E_p f\right\rangle .$$

Moreover, we may assume without loss of generality as in Proposition that the homogeneity degrees a_k of $t_k^{(\alpha)}$ are of the form $a_k = -\delta_\alpha + k$ where $\Re(\delta_\alpha)$ is the scaling degree of $t^{(\alpha)}$. Consequently, $t_0^{(\alpha)}$ is the contribution with the highest scaling degree which may be extracted by introducing the coefficients

$$c_k := \prod_{j=0}^{p-1}(a_k + j + 4(n-1))$$

and considering

$$\left\langle t^{(\alpha)}, E_p f\right\rangle - c_0\left\langle t^{(\alpha)}, f\right\rangle = \sum_{k=1}^m (c_k - c_0)\left\langle t_k^{(\alpha)}, f\right\rangle + \left\langle r^{(\alpha)}, E_p f\right\rangle - c_0\left\langle r^{(\alpha)}, f\right\rangle , \tag{5.18}$$

where the distribution on the right hand side has a scaling degree smaller than $\Re(\delta_\alpha) = -\Re(a_0)$. Hence, although in general the distribution $t^{(\alpha)}$ does not scale homogeneously, equation still holds up to distributions with a lower scaling degree. Knowing the decreasing degree of homogeneity of the components in the expansion of $t^{(\alpha)}$, we may use a recursive procedure in order to expose the pole part of this distribution. In fact, the previous discussion straightforwardly implies the validity of the following proposition.

Proposition 5
We consider a distribution $t^{(\alpha)}$ with the properties assumed in Proposition and set

$$u_0 := t^{(\alpha)}, \qquad u_{k+1} := c_k u_k - u_k \circ E_{p_k}, \qquad 0 \le l < m$$

where p_k are the smallest natural numbers chosen in such a way that $p_k + \Re(a_k) + 4(n-1) > 0$ and $c_k := \prod_{j=0}^{p_k-1}(a_k + j + 4(n-1))$. Then, in order to expose the poles of $t^{(\alpha)}$, we may invert the recursive definition of u_k obtaining

$$t^{(\alpha)} = \frac{1}{c_0}\left(u_0 \circ E_{p_0} + \frac{1}{c_1}\left(u_1 \circ E_{p_1} + \cdots + \frac{1}{c_n}\left(u_n \circ E_{p_n} + u_{n+1}\right)\right)\right). \tag{5.19}$$ ▶

In order to be able use the previous results for our purposes, we provide in the next proposition a criterion which is sufficient to ensure that a generic distribution can be decomposed into the sum of a homogeneous distribution and a remainder with lower scaling degree. We shall use this criterion in order to prove that the distributions $t_\Gamma^{(\boldsymbol{\alpha})}$ defined in Propositionhave the desired property.

Proposition 6
Let $\mathcal{N}_n$ be a normal neighbourhood of the total diagonal d_n and suppose that $t \in \mathcal{D}'(\mathcal{N}_n)$ has scaling degree s_1 towards d_n under transformations of the form and that there exists an α with $-\Re(\alpha) = s_1$ such that $t \circ (E_1 + \alpha + 4(n-1))$ has scaling degree $s_2 < s_1$. Then t can be decomposed into the sum of a homogeneous distribution with degree α and a remainder with scaling degree smaller than or equal to s_2. ▶

Proof 11
We start by observing that, for every test function $f \in \mathcal{D}(\mathcal{N}_n)$,

$$F(\lambda, f) := \langle t, f_{1/\lambda} \rangle$$

is a continuous linear functional of f which is smooth in λ for $\lambda > 0$. Moreover, since the scaling degree of t is s_1, $\lambda^a F(\lambda, f)$ vanishes in the limit $\lambda \to 0$ for every $a > s_1$ and for every $f \in \mathcal{D}(\mathcal{N}_n)$. Let us now consider

$$G(\lambda, f) := \langle (-E_1 + \alpha + 4(n-1))t, f_{1/\lambda} \rangle .$$

$G(\lambda, \cdot)$ is again a family of distributions on $\mathcal{N}_n$ which depends smoothly on λ for positive λ. Furthermore, $\lambda^a G(\lambda, f)$ vanishes in the limit $\lambda \to 0$ for every $a > s_2$ and every $f \in \mathcal{D}(N_n)$. Hence, $\lambda^{\alpha-1} G(\lambda, \cdot)$ tends to 0 in $\mathcal{D}'(\mathcal{N}_0)$ for $\lambda \to 0$ and, additionally, the Banach–Steinhaus theorem implies that

$$|\lambda^a G(\lambda, f)| \leq C \sum_{\alpha \leq k} |\partial^\alpha f|, \tag{5.20}$$

for every $a > s_2$, uniformly for f supported in a compact set $K \subset \mathcal{N}_n$ and for suitable C and k which do not depend on λ.
After these preparatory considerations, we observe that G and F are related by the generalised Euler operator in the following way

$$G(\lambda, f) = \lambda^{-\alpha+1} \frac{d}{d\lambda} \lambda^\alpha F(\lambda, f).$$

We can invert this relation to obtain

$$F(\lambda, f) = \frac{C(f)}{\lambda^\alpha} + \frac{1}{\lambda^\alpha} \int_0^\lambda \tilde{\lambda}^{\alpha-1} G(\tilde{\lambda}, f) d\tilde{\lambda},$$

where, $C(f)$ is a suitable constant which depends on f. We want to prove that $C(\cdot)$ is in fact a distribution. To this end, we note that, owing to the bound , the integral in $\tilde{\lambda}$ can be performed and the result of this integration is a distribution for every $\lambda > 0$ because $\Re(\alpha) > s_2$. This implies that

$$C(f) = F(1, f) - \int_0^1 \tilde{\lambda}^{\alpha-1} G(\tilde{\lambda}, f) d\tilde{\lambda}$$

is a distribution because it is a linear combination of distributions. By construction $F(1, f_{1/\lambda}) = F(\lambda, f)$ and $C \circ (E_1 + \alpha) = 0$, hence C is a homogeneous distribution of degree α. By means of a direct computation we also find that the scaling degree of the remainder $F(1, f) - C(f)$ is smaller than or equal to the scaling degree of G which is s_2. ■

5.3.1 The differential form of generalised Euler operators and homogeneous expansions of Feynman amplitudes

In order to make the previous discussion operative, we have to analyse the action of the generalised Euler operators E_p appearing in on test functions. In fact, we shall see that E_p corresponds to a particular geometric partial differential operator. To this end, we observe that $E_p = (E_1 - (p-1))E_{p-1}$. Hence, knowing the differential form of the generalised Euler operator E_1, it is possible to construct recursively every E_p.
Regarding the differential form of E_1, we note that it can be written in terms of the geodesic distance and the van Vleck–Morette determinant[2] u^2 as

$$E_1 f(x_1, \ldots, x_n) = \sum_{j=2}^{n} \left(\sigma^a(x_j) \nabla_a^{x_j} - (2\sigma^a(x_j) \nabla_a^{x_j} \log(u(x_j, x_1))) \right) f(x_1, \ldots, x_n) ,$$

[2] Recall that the square-root of the van Vleck–Morette determinant coincides with the Hadamard coefficient u appearing in

where $\nabla_a^{x_j}$ indicates the a–th component of the covariant derivative computed in x_j and $\sigma^a(x_j) := \nabla^{x_j\,a}\sigma(x_1,x_j)$. Considering the adjoint $E_p^\dagger$ of E_p, we have $t\circ E_p = E_p^\dagger t$ where, using the relation $\Box\sigma + 2\sigma^a\nabla_a\log(u) = 4$, we find for $p=1$

$$\begin{aligned} E_1^\dagger t(x_1,\dots,x_n) &= \sum_{j=2}^{n}\left(-\nabla_a^{x_j}\sigma^a(x_j) - 2\sigma^a(x_j)\left(\nabla_a^{x_j}\log(u(x_j,x_1))\right)\right)t(x_1,\dots,x_n) \\ &= -\left(4(n-1)+\sum_{j=2}^{n}\sigma^a(x_j)\nabla_a^{x_j}\right)t(x_1,\dots,x_n). \end{aligned} \tag{5.21}$$

We finally observe that the recursive identity for E_p implies that also $E_p^\dagger$ can be constructed recursively starting from $E_1^\dagger$ as

$$E_p^\dagger = E_{p-1}^\dagger(E_1^\dagger - (p-1)).$$

We proceed by showing that upon applying $E_1^\dagger$ introduced in to a distribution $t_\Gamma^{(\boldsymbol{\alpha})}$ of the form

$$t_\Gamma^{(\boldsymbol{\alpha})} = \prod_{1\le i<j\le n}\frac{1}{\sigma_F(x_i,x_j)^{l_{ij}(1+\alpha_{ij})}}$$

which has scaling degree $\mathsf{sd}(t_\Gamma^{(\boldsymbol{\alpha})}) = \sum_{i<j}2l_{ij}(1+\Re(\alpha_{ij}))$ towards the thin diagonal d_n, the result is a term proportional to $t_\Gamma^{(\boldsymbol{\alpha})}$ plus a remainder which has lower scaling degree as foreseen in . Hence, Proposition implies that $t_\Gamma^{(\boldsymbol{\alpha})}$ can be written as a homogeneous distribution plus a remainder with lower scaling degree. If the scaling degree of the remainder is not sufficiently low, we reiterate the procedure in order to obtain a full almost homogeneous expansion of the desired form.

In order to analyse this issue we shall only consider the relevant differential operator on $\mathcal{M}^n$ appearing in $E_1^\dagger$, namely,

$$\rho := -\sum_{j=2}^{n}\sigma^a(x_j)\nabla_a^{x_j}. \tag{5.22}$$

We start by analysing the action of ρ on $\sigma(x_2,x_3)$ for x_2,x_3 in a normal neighbourhood of the point x_1.

Lemma 9
Let $\mathcal{N}_{x_1}$ be a normal neighbourhood of the point x_1 and let $x_2,x_3\in\mathcal{N}_{x_1}$. Then,

$$\rho\sigma(x_2,x_3) = 2\sigma(x_2,x_3) + G(x_1,x_2,x_3)$$

where G is a smooth function which vanishes in the limit $x_2,x_3\to x_1$ as a monomial of order 4 in the normal coordinates of x_2 and x_3 centred in x_1. ▶

Proof 12
Using the notation in the proof of Proposition we write the action of ρ on $\sigma_{23} := \sigma(x_2,x_3)$ as

$$\rho\sigma_{23} = \xi_a(x_2)\sigma_{23}^a + \xi_{b'}(x_3)\sigma_{23}^{b'}.$$

Recall that σ_{23}^a is the covector in $T_{x_2}^*\mathcal{M}$ cotangent to the unique geodesic joining x_2 and x_3, that $-\sigma_{23}^{b'}$ is equal to the parallel transport of σ_{23}^a from x_2 to x_3 along the geodesic γ joining the two points, and that $\xi^c(x_i) := \sigma^c(x_1,x_i)$. Let us parametrise the image of γ with an affine parameter λ such that $x(0)=x_2$ and $x(1)=x_3$. In order to simplify the notation, we indicate by $t(\lambda)$ the tangent vector of the geodesic in $x(\lambda)$. As argued before, we have

$$t^a(0) = \sigma_{23}^a, \qquad \text{and} \qquad t^{b'}(1) = -\sigma_{23}^{b'}.$$

Consequently,

$$\begin{aligned} \rho\sigma_{23} &= \xi^a t_a(0) - \xi^b t_b(1) = -\int_0^1 \frac{d}{d\lambda}(\xi^a t_a)(\lambda)d\lambda \\ &= -\int_0^1 t^a\nabla_t\xi_a d\lambda = \int_0^1 t^a t^b\sigma_{ab}(x(\lambda),x_1)d\lambda\,, \end{aligned}$$

where $\sigma_{ab} := \nabla_a\nabla_b\sigma$. If we now consider the covariant Taylor expansion of $\sigma_{ab}(x(\lambda),x_1)$ around $x(\lambda)$ (see e.g.), we find that $E_{ab}(x,x_1) := \sigma_{ab}(x,x_1) - g_{ab}(x)$ is a smooth function that vanishes for $x\to x_1$ as $O(\sigma(x,x_1))$, hence

$$\rho\sigma_{23} = \int_0^1 t^a t^b g_{ab}(x(\lambda))d\lambda + \int_0^1 t^a t^b E_{ab}(x(\lambda),x_1)d\lambda = 2\sigma(x_2,x_3) + G(x_1,x_2,x_3)\,,$$

where the remainder is smooth because of the smoothness of the metric g and can be further expanded as

$$\begin{aligned} G(x_1,x_2,x_3) &= \int_0^1 t^a(\lambda)t^b(\lambda)\left(\sigma_{ab}(x(\lambda),x_1)-g_{ab}(x(\lambda))\right)d\lambda \\ &= \int_0^1 t^a(\lambda)t^b(\lambda)t^c(\lambda)t^d(\lambda)R_{acbd}(x(\lambda))d\lambda+\cdots=O(|\xi(x_2)|^4+|\xi(x_3)|^4)\,, \end{aligned} \tag{5.23}$$

where the absolute value of the normal coordinates $|\xi(x_i)|$ of x_i, $i=2,3$ is intended in the Euclidean sense. ■

We are now in position to analyse the action of ρ on the distribution $t_\Gamma^{(\boldsymbol{\alpha})}$ introduced in .

Proposition 7
The distribution $t_\Gamma^{(\boldsymbol{\alpha})}$ introduced in can be written as a sum of homogeneous distributions with respect to scaling towards the total diagonal d_n plus a remainder. The degrees of homogeneity of these homogeneous distributions are contained in the following set

$$\left\{k-\sum_{1\le i<j\le n}2l_{ij}(1+\alpha_{ij}),k\in\mathbb{N}\cup\{0\}\right\}.$$

▶

Proof 13
We perform this analysis with ϵ in σ_F taken to be strictly positive. We start by applying ρ given in to $t_\Gamma^{(\boldsymbol{\alpha})}$. Thanks to the results stated in Lemma we have

$$\rho t_\Gamma^{(\boldsymbol{\alpha})}=Ct_\Gamma^{(\boldsymbol{\alpha})}+r_\Gamma^{(\boldsymbol{\alpha})},$$

where the constant C is

$$C=-\sum_{1\le i<j\le n}2l_{ij}(1+\alpha_{ij}).$$

Furthermore, Lemma and in particular implies that the remainder $r_\Gamma^{(\boldsymbol{\alpha})}$ has a scaling degree towards d_n which is lower than the one of $t_\Gamma^{(\boldsymbol{\alpha})}$ by at least two,

$$\mathsf{sd}(r_\Gamma^{(\boldsymbol{\alpha})})\le\mathsf{sd}(t_\Gamma^{(\boldsymbol{\alpha})})-2=\sum_{1\le i<j\le n}2l_{ij}(1+\Re(\alpha_{ij}))-2\,. \tag{5.24}$$

Proposition then implies that the distribution $t_\Gamma^{(\boldsymbol{\alpha})}$ can be written as a homogeneous distribution of degree C plus a remainder with lower scaling degree.
In order to finalise the proof we need to control the recursive application of ρ, therefore we discuss the application of ρ on $\rho^n t_\Gamma^{(\boldsymbol{\alpha})}$ for an arbitrary n. Let us start with $n=1$. In this case, we observe that the relevant contribution is the one given by the remainder $\rho r_\Gamma^{(\boldsymbol{\alpha})}$, which reads

$$r_\Gamma^{(\boldsymbol{\alpha})}=\sum_{1\le i<j\le n}l_{ij}(1+\alpha_{ij})\frac{G(x_1,x_i,x_j)}{\sigma_F(x_i,x_j)}t_\Gamma^{(\boldsymbol{\alpha})}.$$

Note that for every $i<j$, $\sigma_F(x_i,x_j)t_\Gamma^{(\boldsymbol{\alpha})}$ has the same structure like $t_\Gamma^{(\boldsymbol{\alpha})}$, but the scaling degree $\mathsf{sd}(t_\Gamma^{(\boldsymbol{\alpha})})+2$, whereas $G(x_1,x_i,x_j)$ defined in is a smooth function whose Taylor expansion for x_i,x_j around x_1 starts with components of order 4. Hence, if we apply ρ to $r_\Gamma^{(\boldsymbol{\alpha})}$ we obtain a constant multiple of $r_\Gamma^{(\boldsymbol{\alpha})}$ plus a remainder which has scaling degree lower or equal to $\mathsf{sd}(r_\Gamma^{(\boldsymbol{\alpha})})-1$, where the difference with respect to stems from the fact that G can be expanded as a polynomial in $\sigma_a(x_i)$ whose lowest components are monomials of degree 4 multiplied by curvature tensors. These monomials are homogeneous and thus contribute to the degree of homogeneity of $\rho r_\Gamma^{(\boldsymbol{\alpha})}$, while the contributions in G with degree higher or equal to five influence the scaling degree of the remainder. Repeating this analysis for a generic n, we find that similar results hold when ρ is applied recursively to the remainder.
Consequently, an iterated application of Proposition implies that the distribution $t_\Gamma^{(\boldsymbol{\alpha})}$ can be written as a finite sum of homogeneous distributions plus a remainder. Furthermore, since the scaling degree of these distributions is always finite, the degree of homogeneity of these components is finite as well. ■

As outlined at the end of Section , we can use Proposition in conjunction with the propositions and in order to extend the distributions $t_\Gamma^{(\boldsymbol{\alpha})}$ in a unique and weakly meromorphic fashion to a normal neighbourhood of the union of all partial diagonal D_n and in order to compute the relevant pole part of this extension as used in the forest formula, cf. , and . To this avail, we stress that Proposition holds in particular for any subgraph Γ_I, $I\subset\{1,\dots,n\}$ of Γ and the corresponding distribution $t_{\Gamma_I}^{(\boldsymbol{\alpha})}$ which is obtained by omitting all factors in $t_\Gamma^{(\boldsymbol{\alpha})}$ which correspond to edges not contained in Γ_I. Finally, the recursive structure of the forest formula implies that we are not dealing only with expressions of the form $t^{(\boldsymbol{\alpha})}_{\Gamma_I}|_{\alpha_{ij}=\alpha_I \forall i,j \in I}$, but also with expressions which are of this form up to a subtraction of their principal part. However, our above analysis and in particular the discussion in the proof of Proposition implies that the propositions and also hold in this case.

Remark 2
Proposition and the above analysis imply that our renormalisation scheme is in fact a particular form of differential renormalisation. Notwithstanding, the advantage of formulating this scheme in terms of analytic regularisation and minimal subtraction is the ability to define the renormalisation scheme in a closed form at all orders by means of the forest formula ♦

5.4 Properties of the minimal subtraction scheme

We conclude the general analysis of the renormalisation scheme introduced in this work by demonstrating that this scheme satisfies – up to one property we shall mention at the end of this section – all axioms ofwhich, as argued in these works, any physically meaningful scheme to renormalise time–ordered products should satisfy. We refer to these works for a detailed formulation and discussion of these axioms. In addition to showing these properties of the scheme, we also argue that it preserves invariance under any spacetime isometries present.

Proposition 8
The time–ordered product $\mathcal{T}_n$ defined by means of , where the quantities appearing in this formula are defined by means of , , and , and were we recall Remark , have the following properties.

1. $\mathcal{T}_n$ is symmetric and satisfies the causal factorisation condition.
2. $\mathcal{T}_n$ is unitary.
3. $\mathcal{T}_n$ is local and covariant.
4. $\mathcal{T}_n$ satisfies the microlocal spectrum condition.
5. $\mathcal{T}_n$ is ϕ–independent.
6. $\mathcal{T}_n$ satisfies the Leibniz rule.
7. $\mathcal{T}_n$ satisfies the Principle of Perturbative Agreement for perturbations of the generalised mass term μ in the free Klein–Gordon equation $P\phi := (-\Box + \mu)\phi = 0$.
8. If the spacetime $(\mathcal{M}, g)$ has non–trivial isometries and if the Feynman propagator H_F is chosen such as to be invariant under these isometries, then $\mathcal{T}_n$ is invariant under these isometries as well. ▶

Proof 14
$a)$ holds because we constructed the renormalised time–ordered product by means of the forest formula and because, as implied by Proposition , all counterterms subtracted in the forest formula are local.
$b)$ Unitarity holds because the operation of extracting the relevant principal part of a regularised amplitude $\tau^{(\boldsymbol{\alpha})}_\Gamma$ commutes with complex conjugation (even if $\boldsymbol{\alpha}$ is not real).
$c)$ The regularised amplitudes $\tau^{(\boldsymbol{\alpha})}_\Gamma$ satisfy locality and covariance. Upon setting $\alpha_{ij} = \alpha_I$ for $i, j \in I \subset \{1, \dots, n\}$, $\tau^{(\boldsymbol{\alpha})}_\Gamma$ is weakly meromorphic in α_I. Thus locality and covariance holds for each term in the corresponding Laurent series and consequently also after subtracting the principal part of this series.
$d)$ As argued in the proof of Proposition, the distributions $t^{(\boldsymbol{\alpha})}_\Gamma$ defined in satisfy the microlocal spectrum condition, i.e. they have the correct wave front set. Consequently, the regularised amplitudes $\tau^{(\boldsymbol{\alpha})}_\Gamma$ have the correct wave front set as well. As $\tau^{(\boldsymbol{\alpha})}_\Gamma$ is weakly meromorphic in the sense recalled in the proof of $c)$, each term in the corresponding Laurent series has a wave front set bounded by the wave front set of $\tau^{(\boldsymbol{\alpha})}_\Gamma$. Consequently the microlocal spectrum condition holds after subtracting the principal part and considering the limit of vanishing regularisation parameters.
$e)$ This property follows directly from the construction. In particular the subtraction of counterterms is defined in terms of numerical distributions and independent of the field ϕ.
$f)$ In analogy to $b)$, the Leibniz rule holds because the operation of extracting the relevant principal part of a regularised amplitude $\tau^{(\boldsymbol{\alpha})}_\Gamma$ commutes with all partial differential operators.
$g)$ The Principle of Perturbative Agreement for perturbations of the generalised mass term μ demands essentially that upon setting $\mu = \mu_0 + \mu_1$, the renormalisation of $\mathcal{T}_n$ commutes with the operation of perturbatively expanding quantities in μ_1 around μ_0. A Feynman propagator H_F depends on μ only via the Hadamard coefficients v and w in However, in the definition of the analytically regularised H^α_F in and the corresponding regularised amplitudes

$\tau_\Gamma^{(\boldsymbol{\alpha})}$ defined in , these coefficients are not altered but only the σ–dependent terms multiplying these coefficients are modified. Consequently, the analytic regularisation and minimal subtraction scheme we consider commutes with a perturbative expansion in μ_1 around μ_0.
$h)$ As recalled in $g)$ all operations in our analytic regularisation and minimal subtraction scheme act directly on quantities defined entirely in terms of the geometric quantity σ. As σ is invariant under any spacetime isometries present, the renormalisation scheme preserves this invariance. ■

Remark 3
Note that the Principle of Perturbative Agreement (PPA) as introduced in also poses conditions on $\mathcal{T}_1$, i.e. the renormalisation of local and covariant Wick polynomials, which we omitted in our analysis, cf. Footnote on page . However, given $\mathcal{T}_n$ for $n > 1$, $\mathcal{T}_1$ can be adjusted in order to satisfy the PPA for changes of μ by using e.g. . Moreover, the PPA as introduced in further demands that, setting $g = g_0 + g_1$, the renormalisation also commutes with perturbatively expanding quantities in g_1 around an arbitrary but fixed background metric g_0. Since σ depends on g, it is not easy to check whether a perturbative expansion in g_1 commutes with our analytic regularisation and minimal subtraction scheme and thus it might well be that the renormalisation scheme discussed in the present work fails to satisfy this part of the PPA. However, if this is the case, the scheme can be modified according to the construction in in order to satisfy also this condition while preserving the other properties in Proposition , including the invariance under any spacetime isometries present. ♦

Remark 4
We have omitted the explicit dependence of renormalised quantities on the mass scale M appearing in the analytically regularised Feynman propagator $H_F^{(\alpha)}$, but our analysis implies that the dependence of these quantities on M is such that all renormalised quantities are polynomials of (derivatives of) $\log\left(M^2\sigma_F(x_i,x_j)\right)$, see also the examples in the next section. Thus, the renormalisation group flow with respect to changes of M may be easily computed. ♦

5.5 Examples

In this section we illustrate the method developed in Section to explicitly compute renormalised quantities in our scheme by considering first the example of the fish graph and the sunset graph, i.e. Δ_F^n for $n = 2,3$. These pointwise powers of the Feynman propagator are the only ones occurring in renormalisable scalar field theories in four spacetime dimensions. Afterwards we will consider a triangular graph in Section in order to illustrate the method in the case of more than two vertices. Recalling Remark , we shall work only on subsets of the spacetime where the geodesic distance is well–defined without loss of generality.
In the special case of Δ_F^n, we are dealing with distributions which are already defined on $\mathcal{M}^2 \setminus d_2$ and have to be extended to $\mathcal{M}^2$. In order to accomplish this task we shall use in order to expose the poles before subtracting them. In this context, we note that $E_1^\dagger$ given in applied to a distribution t whose integral kernel $t(\sigma_F)$ depends on x, y only via $\sigma_F := \sigma(x,y)$, can be further simplified. In particular, introducing $t_1(\sigma_F)$ such that $\nabla^a t_1(\sigma_F) = \sigma^a t(\sigma_F)$, we have

$$\begin{aligned} E_1^\dagger t(\sigma_F) &= -\left(4+\sigma^a\nabla_a\right)t(\sigma) = -\nabla_a\sigma^a t - 2\sigma^a(\nabla_a\log(u))t(\sigma_F) \\ &= -\Box t_1(\sigma_F) - 2\frac{\nabla_a u}{u}\nabla^a t_1(\sigma_F)\,, \end{aligned} \tag{5.25}$$

where x is considered to be arbitrary but fixed and all the covariant derivatives are taken with respect to y.

5.5.1 Computation of the renormalised fish and sunset graphs in our scheme

5.5.2 Alternative computation of the renormalised fish and sunset graphs

As a preparation towards the application of our renormalisation scheme to QFT in cosmological spacetimes, we shall now derive an alternative way to compute $\left(\Delta_F^2\right)_{\mathsf{ms}}$ and $\left(\Delta_F^3\right)_{\mathsf{ms}}$, which is better suited for practical computations. We start by stating and proving a few distributional identities.

Lemma 10
The following distributional identities hold.

1. For any continuous F_0 and any twice continuously differentiable F_2,

$$\sigma F_0\delta = 0\,,\qquad \sigma_a F_0\delta = 0\,,\qquad F_0\nabla_{\nabla\sigma}\delta = -[F_0\Box\sigma]\delta\,,$$

$$F_2\Box\delta = [\Box F_2]\delta + \Box[F_2]\delta - 2\nabla^a[\nabla_a F_2]\delta\,.$$

2.
$$(\Box + f)\frac{1}{\sigma_F} = 8\pi^2 i\delta \qquad (\Box + 2f)(\Box + f)\frac{1}{\sigma_F} = 8\pi^2 i\left(\Box - \frac{R}{3}\right)\delta$$

3. For all $n_1, n_2, n_3 \in \mathbb{N}_0$ and $n_4, n_5, n_6 \in \{0,1\}$ with $n_2 - n_3 + n_4 \geq -1$,
$$\log^{n_1}(\sigma_F)\,(\sigma_F^a)^{n_4}\sigma_F^{n_2}\left(\frac{1}{\sigma_F^{n_3}}\right)_{\mathsf{ms}} = \log^{n_1}(\sigma_F)\,(\sigma_F^a)^{n_4}\sigma_F^{n_2-n_3}\,,$$
$$\Box\log(\sigma_F) = \frac{\Box\sigma - 2}{\sigma_F}\,, \qquad \nabla_a\frac{\log^{n_5}(\sigma_F)}{\sigma_F^{n_6}} = \frac{(n_5 - n_6\log^{n_5}(\sigma_F))\,\nabla_a\sigma}{\sigma_F^{n_6+1}}\,.$$

4.
$$\sigma_F\left(\frac{1}{\sigma_F^3}\right)_{\mathsf{ms}} = \left(\frac{1}{\sigma_F^2}\right)_{\mathsf{ms}}$$

Proof 15
$a)$ These identities follow from $B\delta = [B]\delta$ for any continuous bitensor B, $[\sigma] = 0$, $[\sigma_a] = 0$ and the definition of weak derivatives.
$b)$ The first identity holds in Minkowski spacetime because $1/(8\pi^2\sigma_F)$ is the Feynman propagator of the massless vacuum state. In curved spacetimes imply that $(\Box + f)1/\sigma_F$ vanishes outside of the origin and thus must be a sum of derivatives of δ distributions. Because σ depends smoothly on the metric, the coefficients in this sum must be smooth functions of the metric with appropriate mass dimension and thus $(\Box + f)1/\sigma_F = c\delta$ with a constant c that can be fixed in Minkowski spacetime. The second identity follows from the first and
$c)$ The distributions on both sides of each equation, considered as distributions in y for fixed x, have the same scaling degree < 4 for $y \to x$ and agree outside of the diagonal. Thus they agree also on the diagonal as unique extensions.
$d)$ As in the proof of a) we observe that the potential local correction term on the right hand side must be a sum of derivatives of δ with coefficients that depend smoothly on the metric because σ does. Thus the correction term must be of the form $c\delta$ with a constant c that can be computed in Minkowski spacetime. This computation may be performed by using the previous statements of this lemma, and the following identities which are valid in Minkowski spacetime for any function F s.t. $F(\sigma_F)$ is a distribution
$$\sigma_F\Box F(\sigma_F) = \Box\sigma_F F(\sigma_F) - 4F(\sigma_F) - 2\nabla_{\nabla\sigma_F}F(\sigma_F)\,,$$
$$\sigma_F\Box^2 F(\sigma_F) = \Box^2\sigma_F F(\sigma_F) - 4\Box F(\sigma_F) - 4\Box\nabla_{\nabla\sigma_F}F(\sigma_F)\,,$$
whereby one finds that $c = 0$. ■

These identities can be used to compute $\left(\Delta_F^2\right)_{\mathsf{ms}}$ and $\left(\Delta_F^3\right)_{\mathsf{ms}}$ in an alternative way under certain conditions.

Proposition 9
Let $(\mathcal{M}, g)$ be such that $\mathcal{M}$ is a normal neighbourhood and let Δ_F be a distribution on $\mathcal{M}^2$ of Feynman-Hadamard form Then the following identities hold.

1. If Δ_F^α is a well-defined distribution which is weakly meromorphic in α, then
$$(\Delta_F^2)_{\mathsf{ms}} = \lim_{\alpha\to 0}\left(\frac{1}{M^{2\alpha}}\Delta_F^{2+\alpha} - \mathsf{pp}\frac{1}{M^{2\alpha}}\Delta_F^{2+\alpha}\right) + \frac{i\log(8\pi^2)}{16\pi^2}\delta\,.$$

2. If Δ_F^α is a well-defined distribution which is weakly meromorphic in α, then
$$(\Delta_F^2\log\left(M^{-2}\Delta_F\right))_{\mathsf{ms}} = \lim_{\alpha\to 0}\left(\frac{d}{d\alpha}\frac{1}{M^{2\alpha}}(\Delta_F)^{2+\alpha} - \mathsf{pp}\frac{d}{d\alpha}\frac{1}{M^{2\alpha}}\Delta_F^{2+\alpha}\right) - \frac{i\log^2(8\pi^2)}{32\pi^2}\delta\,.$$

3. If Δ_F^α is a well-defined distribution which is weakly meromorphic in α and $[v] = 0$, then
$$(\Delta_F^3)_{\mathsf{ms}} = \lim_{\alpha\to 0}\left(\frac{1}{M^{2\alpha}}\Delta_F^{3+\alpha} - \mathsf{pp}\frac{1}{M^{2\alpha}}\Delta_F^{3+\alpha}\right) + \frac{i\left((1 + 2\log(8\pi^2))R + 192\pi^2[w]\right)}{48(8\pi^2)^2}\delta\,.$$

Proof 16
$a)$ Setting $h = 8\pi^2\sigma_F\Delta_F$ and $k = \sqrt{8\pi^2/h}$, we obtain

$$\frac{1}{M^{2\alpha}}\Delta_F^{2+\alpha} = \frac{h^2}{(8\pi^2)^2}\frac{1}{(Mk)^{2\alpha}}\frac{1}{\sigma_F^{2+\alpha}}\,.$$

Using , $[h^2] = [u^2] = 1$ and Lemma a), b) & c) we may compute

$$\begin{aligned}
&\lim_{\alpha\to 0}\left(\frac{1}{M^{2\alpha}}\Delta_F^{2+\alpha} - \mathsf{pp}\frac{1}{M^{2\alpha}}\Delta_F^{2+\alpha}\right)\\
=\;& \frac{h^2}{(8\pi^2)^2}\lim_{\alpha\to 0}\left(\frac{1}{(Mk)^{2\alpha}}\frac{1}{\sigma_F^{2+\alpha}} - \mathsf{pp}\frac{1}{(Mk)^{2\alpha}}\frac{1}{\sigma_F^{2+\alpha}}\right)\\
=\;& \frac{h^2}{(8\pi^2)^2}\left(\left(\frac{1}{\sigma_F^2}\right)_{\mathsf{ms}} - \frac{\log(k^2)}{2}(\Box + f)\frac{1}{\sigma_F}\right) = (\Delta_F^2)_{\mathsf{ms}} - \frac{i\log(8\pi^2)}{16\pi^2}\delta\,.
\end{aligned}$$

$b)$ In analogy to $a)$, we may compute

$$\begin{aligned}
&\lim_{\alpha\to 0}\left(\frac{d}{d\alpha}\frac{1}{M^{2\alpha}}\Delta_F^{2+\alpha} - \mathsf{pp}\frac{d}{d\alpha}\frac{1}{M^{2\alpha}}\Delta_F^{2+\alpha}\right)\\
=\;& \frac{h^2}{(8\pi^2)^2}\left(-\left(\frac{\log\left(M^2\sigma_F\right)}{\sigma_F^2}\right)_{\mathsf{ms}} - \log\left(\frac{8\pi^2}{h^2}\right)\left(\frac{1}{\sigma_F^2}\right)_{\mathsf{ms}} + \frac{\log^2\left(\frac{8\pi^2}{h^2}\right)}{4}(\Box + f)\frac{1}{\sigma_F}\right)\\
=\;& \left(\Delta_F^2\log\left(M^{-2}\Delta_F\right)\right)_{\mathsf{ms}} + \frac{i\log^2(8\pi^2)}{32\pi^2}\delta\,.
\end{aligned}$$

$c)$ This can be proven in analogy to a) and b), whereby one also needs Lemma d) and the fact that $[v] = 0$ implies by means of the covariant expansion of bitensors near the diagonal (see e.g. cite[Section 5]Poisson:2011nh) that

$$v = [v] + ([\nabla_a v] - \nabla_a[v])\sigma^a + \mathsf{R}_v = [\nabla_a v]\sigma^a + \mathsf{R}_v\,,$$

where the remainder term R_v vanishes towards the diagonal fast than σ_a. Thus, the assumption $[v] = 0$ implies that the term in Δ_F^3 proportional to $\sigma_F^{-2}\log M^2\sigma_F$ does not need to be renormalised, which is crucial for the present proof. The correction term arises from the $\log h/(8\pi^2)$ term in the expansion of

$$\frac{1}{(Mk)^{2\alpha}}\frac{1}{\sigma_F^{3+\alpha}}$$

whose contribution may be computed as

$$\begin{aligned}
&\frac{h^3\log\left(\frac{h}{8\pi^2}\right)}{8(8\pi^2)^3}(\Box + 2f)(\Box + f)\frac{1}{\sigma_F} = -\frac{i\left(2[h^3 f]\log(8\pi^2) - [\Box h^3\log(h)]\right)\delta}{8(8\pi^2)^2} =\\
&= \frac{i\left(2[\Box u]\log(8\pi^2) + [\Box u + 8\pi^2 w\Box\sigma]\right)\delta}{8(8\pi^2)^2} = \frac{i\left((1 + 2\log(8\pi^2))R + 192\pi^2[w]\right)}{48(8\pi^2)^2}\delta\,,
\end{aligned}$$

where again Lemma a) & b) prove to be useful. ■

5.5.3 A more complicated graph

In order to show how the proposed renormalisation scheme works for graphs which have more than two vertices we discuss the renormalisation of the following triangular graph

$$\tau_\Gamma := \Delta_{F,13}\Delta_{F,23}\Delta_{F,12}^2\,,$$

where $\Delta_{F,ij} := \Delta_F(x_i, x_j)$. In order to apply the forest formula to renormalise this graph, we note that the forests which correspond to divergent contributions are

$$\{12\}\,,\quad \{123\}\,,\quad \{12, 123\}\,.$$

The renormalisation of τ_Γ thus reads

$$(\tau_\Gamma)_{\mathsf{ms}} = (1 + R_{12} + R_{123} + R_{123}R_{12})\tau_\Gamma^{(\boldsymbol{\alpha})} = (1 + R_{123})(1 + R_{12})\tau_\Gamma^{(\boldsymbol{\alpha})}.$$

In order to illustrate the explicit form of the R, we consider only the most singular contribution to $\tau_\Gamma^{(\boldsymbol{\alpha})}$, namely

$$t_{\Gamma,0}^{(\boldsymbol{\alpha})} := \frac{1}{\sigma_{13}^{1+\alpha_{13}}} \frac{1}{\sigma_{12}^{2(1+\alpha_{12})}} \frac{1}{\sigma_{23}^{1+\alpha_{23}}} ,$$

where $\sigma_{ij} := \sigma_F(x_1, x_j)$. Note that, with obvious notation, $(8\pi^2)^{-4} u_{13} u_{12}^2 u_{23} t_{\Gamma,0}$ is in fact the only contribution to τ_Γ which needs to be renormalised. The application of $1 + R_{12}$ to $t_{\Gamma,0}^{(\boldsymbol{\alpha})}$ has already been discussed in the preceding sections and corresponds to the renormalisation of the fish graph. Indeed, after setting α_{12}, α_{23} and α_{13} to $\alpha = \alpha_I$ for $I = \{1, 2, 3\}$ we obtain

$$t_{\Gamma,1}^{(\alpha)} := \lim_{\alpha_{ij} \to \alpha} (1 + R_{12}) t_{\Gamma,0}^{(\boldsymbol{\alpha})} = \left(\left(\frac{1}{\sigma_{12}^2} \right)_{\mathsf{ms}} + O(\alpha) \right) \frac{1}{(\sigma_{13})^{1+\alpha}} \frac{1}{(\sigma_{23})^{1+\alpha}} .$$

The distribution $(1/\sigma_{12}^2)_{\mathsf{ms}}$ is a homogeneous distribution of degree $\delta = -4$ under scaling of x_2 towards x_1, consequently, $t_{\Gamma,1}^{(\alpha)}$ has scaling degree $8 + 4\alpha$.

Owing to Proposition , we know that $t_{\Gamma,1}^{(\alpha)}$ can be decomposed into the sum of a homogeneous distribution of degree $-8 - 4\alpha$ and a remainder. Hence, in order to expose the poles of $t_{\Gamma,1}^{(\alpha)}$, we can directly apply Proposition with $m = 1$ and $c_0 = -4\alpha$. To this end, we set $u_0 := t_{\Gamma,1}^{(\alpha)}$ and find

$$u_1 := -4\alpha u_0 - E_1^\dagger u_0 = \left(\left(\frac{1}{\sigma_{12}^2} \right)_{\mathsf{ms}} + O(\alpha) \right) \frac{1}{(\sigma_{13})^{1+\alpha}} \frac{1}{(\sigma_{23})^{2+\alpha}} \, G ,$$

where $G = G(x_1, x_2, x_3)$ is the smooth function introduced in Lemma From eqrefeq:expose-poles we can infer that the principal part of $t_{\Gamma,1}^{(\alpha)}$ is

$$\mathsf{pp}\, t_{\Gamma,1}^{(\alpha)} = -\frac{1}{4\alpha} \left(E_1^\dagger + \frac{G}{\sigma_{23}} \right) \left(\left(\frac{1}{\sigma_{12}^2} \right)_{\mathsf{ms}} \frac{1}{\sigma_{13}} \frac{1}{\sigma_{23}} \right) ,$$

whereas the constant regular part can be easily computed as well. Consequently, the renormalised distribution

$$(t_{\Gamma,0})_{\mathsf{ms}} = \lim_{\alpha \to 0} \left(t_{\Gamma,1}^{(\alpha)} - \mathsf{pp}\, t_{\Gamma,1}^{(\alpha)} \right)$$

can be straightforwardly computed in explicit terms.

Chapter 6

Exercises time

6.1 Explicit computations in cosmological spacetimes

The aim of this section is provide prêt-à-porter formulae for doing perturbative computations in the renormalisation scheme devised in the previous sections for the special case of Friedmann–Lemaître–Robertson–Walker (FLRW) spacetimes. We thus consider spacetimes $(\mathcal{M}, g)$ of the form $\mathcal{M} = I \times \mathbb{R}^3 \subset \mathbb{R}^4$ and, in comoving coordinates,

$$g = -dt^2 + a(t)^2 d\vec{x}^2 = a(\tau)^2 \left(-d\tau^2 + d\vec{x}^2\right).$$

Here, t is cosmological time and τ is conformal time related to t by $dt = a d\tau$ and

$$H := \partial_t \log(a) = \frac{\partial_\tau a}{a^2} =: \frac{\mathcal{H}}{a}, \qquad R = 6(\partial_t H + 2H^2) = \frac{\partial_\tau^2 a}{a^3}. \tag{6.1}$$

We consider here the spatially flat FLRW case for simplicity. Note that these spacetimes are normal neighbourhoods so that can be considered as a global expression and all Feynman amplitudes can be analytically regularised without the need of introducing partitions of unity such as in Remark .

6.1.1 Propagators in Fourier space

In comoving coordinates with conformal time, the Klein-Gordon operator reads

$$P = -\Box + \xi R + m^2 = \frac{1}{a(\tau)^3}\left(\partial_\tau^2 - \vec{\nabla}^2 + \left(\xi - \frac{1}{6}\right) Ra^2 + m^2 a^2\right) a(\tau).$$

It is convenient to employ Fourier transformations with respect to the spatial coordinates in order to expand quantities in QFT on FLRW spacetimes in terms of mode solutions of the free Klein-Gordon equation

$$\phi_{\vec{k}}(\tau, \vec{x}) = \frac{\chi_k(\tau) e^{i\vec{k}\vec{x}}}{(2\pi)^{\frac{3}{2}} a(\tau)},$$

where the temporal modes $\chi_k(\tau)$ satisfy

$$\left(\partial_\tau^2 + k^2 + m^2 a^2 + \left(\xi - \frac{1}{6}\right) Ra^2\right) \chi_k(\tau) = 0 \tag{6.2}$$

and the normalisation condition

$$\chi_k \partial_\tau \overline{\chi_k} - \overline{\chi_k} \partial_\tau \chi_k = i\,. \tag{6.3}$$

Here, $k := |\vec{k}|$ and $\bar{\cdot}$ denotes complex conjugation.

In particular, we can use the mode expansion in order to give explicit expressions for the various propagators of the free Klein-Gordon quantum field in a pure, Gaussian, homogeneous and isotropic state Ω (see for associated technical conditions on the mode functions). To this avail, we define

$$\Delta_\sharp(x_1, x_2) =: \lim_{\epsilon \downarrow 0} \frac{1}{8\pi^3 a(\tau_1) a(\tau_2)} \int_{\mathbb{R}^3} d^3k\, \widehat{\Delta_\sharp}(\tau_1, \tau_2, k)\, e^{i\vec{k}(\vec{x}_1 - \vec{x}_2) - \epsilon k}\,, \tag{6.4}$$

where $\Delta_\sharp$ stands for either Δ_+ (two-point function), $\Delta_{R/A}$ (retarded/advanced propagator) or Δ_F (Feynman propagator). See Section for our conventions for these propagators and their relations. Recall that our renormalisation

scheme preserves invariance under spacetime isometries and thus we know that renormalised powers of the Feynman propagator may also be written in the form .
The Fourier versions of the single propagators read whereas by the convolution theorem, we have the following Fourier versions of products and convolutions of multiple propagators, provided those products and convolutions are well-defined. Defining we have

$$\widehat{\prod_{i=1}^{n} \Delta_{\sharp_i}}(\tau_1,\tau_2,k) = \frac{1}{\left((2\pi)^3 a(\tau_1)^2 a(\tau_2)^2\right)^{n-1}} \left[\widehat{\Delta_{\sharp_1}} *_3 \cdots *_3 \widehat{\Delta_{\sharp_n}}\right](\tau_1,\tau_2,k)\,, \tag{6.5}$$

$$\Delta_{\sharp_1} \widehat{*_4 \cdots *_4} \Delta_{\sharp_n} = \widehat{\Delta_{\sharp_1}} *_1 \cdots *_1 \widehat{\Delta_{\sharp_n}}\,. \tag{6.6}$$

Choosing a pure, Gaussian, homogeneous and isotropic state Ω of the quantized free Klein-Gordon field on a spatially flat FLRW spacetimes amounts to choosing a solution of and for each k. In order for Ω to be a Hadamard state the temporal modes χ_k have to satisfy certain conditions in the limit of large k which are difficult to formulate precisely. Heuristically, a necessary but not sufficient condition is that the dominant part of χ_k for large k, when the mass and curvature terms in are dominated by k^2, is $\frac{1}{\sqrt{2k}}e^{-ik\tau}$, i.e. a positive frequency solution. Note that the retarded and advanced propagators are state-independent and thus $\widehat{\Delta_{R/A}}(\tau_1,\tau_2,k)$ is independent of the particular χ_k chosen for each k.

6.1.2 The renormalised fish and sunset graphs in Fourier space

In perturbative calculations at low orders we encounter (pointwise) powers of $\Delta_\pm$ and Δ_F. While the powers of $\Delta_\pm$ are well-defined if Ω is a Hadamard state on account of the wave front set properties of these distributions, we need to renormalise the powers of Δ_F by means of the scheme developed in the previous sections. In order to be useful for explicit computations in FLRW spacetimes, we have to develop a spatial Fourier–space version of this scheme. Having in mind the application to ϕ^4 theory, we shall compute $\widehat{(\Delta_F)^n_{\mathsf{ms}}}(\tau_1,\tau_2,k)$ for $n=2,3$. The difficulty in achieving this is that, to our knowledge, despite of the large symmetry of flat FLRW spacetimes, neither σ nor the Hadamard coefficients u, v and w may written in a tractable form which can be Fourier–transformed easily. Our strategy to circumvent this problem is the following. **Computational strategy**

1. For a general mass m and coupling to the scalar curvature ξ and a general homogeneous and isotropic, pure and Gaussian Hadamard state Ω, split Δ_F as

$$\Delta_F = \Delta_{F,0} + d\,, \qquad d := \Delta_F - \Delta_{F,0}\,, \tag{6.7}$$

   where $\Delta_{F,0}$ must satisfy the following conditions.

   - $\Delta_{F,0}$ is explicitly known in position space and Fourier space.
   - $\Delta_{F,0}$ is of the form

$$\Delta_{F,0} = \frac{1}{8\pi^2}\left(\frac{u_0}{\sigma_F} + v_0 \log\left(M^2\sigma_F\right)\right) + w_0\,,$$

     with $u_0 = u$, i.e. it agrees with Δ_F in the most singular term but not necessarily in the subleading singularities. .This is crucial for preserving the explicit knowledge of $\Delta_{F,0}$ in position space in the renormalisation procedure, so that one may hope to compute the Fourier transforms of the renormalised powers.

2. With these assumptions on $\Delta_{F,0}$ it follows that the renormalised fish and sunset graphs may be computed as

$$(\Delta_F)^2_{\mathsf{ms}} = (\Delta_{F,0})^2_{\mathsf{ms}} + 2\Delta_{F,0}d + d^2 \tag{6.8}$$

$$(\Delta_F)^3_{\mathsf{ms}} = (\Delta_{F,0})^3_{\mathsf{ms}} + 3\left(\Delta^2_{F,0}d\right)_{\mathsf{ms}} + 3\Delta_{F,0}d^2 + d^3$$

   because the non–renormalised terms in the above formulae are distributions with scaling degree < 4 for $y \to x$ and thus can be directly and uniquely extended to the diagonal.

3. $\left(\Delta^2_{F,0}\right)_{\mathsf{ms}}$ and $\left(\Delta^3_{F,0}\right)_{\mathsf{ms}}$ may be computed with Proposition as anticipated. In order to compute $\left(\Delta^2_{F,0}d\right)_{\mathsf{ms}}$, we further split d as

$$d = d_1 + d_2\,, \qquad d_1 := -\frac{[v]\log\left(M^{-2}\Delta_{F,0}\right)}{8\pi^2}\,, \qquad d_2 := d - d_1 \tag{6.9}$$

Because $v = [v] + O(\sigma_a)$, d_1 contains the leading logarithmic singularity in d (and thus Δ_F) which is the only logarithmic singularity relevant for the renormalisation of the sunset graph. Consequently

$$\left(\Delta_{F,0}^2 d\right)_{\mathsf{ms}} = -\frac{[v]}{8\pi^2}\left(\Delta_{F,0}^2 \log\left(M^{-2}\Delta_{F,0}\right)\right)_{\mathsf{ms}} + d_2\left(\Delta_{F,0}^2\right)_{\mathsf{ms}}\,, \tag{6.10}$$

and thus Proposition can be applied again.

4. Due to the symmetry of FLRW spacetimes and the assumption that the pure and Gaussian Hadamard state Ω is invariant under this symmetry, $[v]$ and $[w]$ do not depend on the spatial coordinates. Given that one succeeds to compute the spatial Fourier transforms of $\log\left(M^{-2}\Delta_{F,0}\right)$, $\left(\Delta_{F,0}^2\right)_{\mathsf{ms}}$, $\left(\Delta_{F,0}^2\log\left(M^{-2}\Delta_{F,0}\right)\right)_{\mathsf{ms}}$ and $\left(\Delta_{F,0}^3\right)_{\mathsf{ms}}$, $\widehat{(\Delta_F)^n_{\mathsf{ms}}}(\tau_1,\tau_2,k)$ for $n \in \{2,3\}$ may be computed by means of the convolution identities , since the Fourier transforms of $\Delta_{F,0}$, d_1 and d_2 are known by construction.

In order to follow the computational strategy outlined above, we first compute $[v]$ and $[w]$. Indeed, the coinciding point limit of the Hadamard coefficient v reads (see e.g.

$$[v] = \frac{m^2 + \left(\xi - \frac{1}{6}\right)R}{2}\,. \tag{6.11}$$

Moreover, using the method of to compute a spatial Fourier representation of the Hadamard parametrix H_F – here considered as with $w = 0$ – in FLRW spacetimes, one can compute (see the review in and a related method in for the conformally coupled case)

$$\begin{aligned}
[w] &= \lim_{x\to y}\left(\Delta_F(x,y) - H_F(x,y)\right)\\
&= \frac{1}{(2\pi)^3 a^2}\int_{\mathbb{R}^3} d^3k\, |\chi_k(\tau)|^2 - \frac{1}{2\sqrt{k^2 + a^2m^2 + a^2\left(\xi-\frac{1}{6}\right)R}}\\
&\quad + \frac{1}{16\pi^2}\left(m^2 + \left(\xi - \frac{1}{6}\right)R\right)\left(2\gamma - 1 + \log\left(\frac{m^2 + \left(\xi-\frac{1}{6}\right)R}{2M^2}\right)\right) - \frac{R}{36(8\pi^2)}\,,
\end{aligned} \tag{6.12}$$

where γ is the Euler-Mascheroni constant and H_F is taken with the mass scale M inside of the logarithm of σ[1]. As anticipated we see that $[v]$ and $[w]$ are functions of time only (recall . Moreover, we see that $[v] = 0$ for a conformally coupled ($\xi = \frac{1}{6}$) massless scalar field. Thus, in order to pursue our computational strategy, we should look for a candidate for $\Delta_{F,0}$ among the Feynman propagators in suitable states of this theory. In fact, choosing the conformal vacuum state of the massless conformally coupled scalar field does the job. The conformal vacuum is given by choosing the modes $\chi_k(\tau) = e^{-ik\tau}/\sqrt{2k}$, and thus the Feynman propagator $\Delta_{F,0}$ in this state is of the form

$$\Delta_{F,0}(x_1,x_2) = \frac{1}{8\pi^2 a(\tau_1)a(\tau_2)}\frac{1}{\sigma_{F,\mathbb{M}}(x_1,x_2)}\,,\qquad \widehat{\Delta_{F,0}}(\tau_1,\tau_2,k) = \frac{e^{-ik|\tau_1-\tau_2|}}{2k}\,. \tag{6.13}$$

Here, and in the following, the index $\mathbb{M}$ indicates quantities in Minkowski spacetime, in particular $\sigma_{\mathbb{M}}(x_1,x_2) = \frac{1}{2}(\vec{x}_1 - \vec{x_2})^2 - \frac{1}{2}(\tau_1-\tau_2)^2$. $\Delta^\alpha_{F,0}$ is weakly meromorphic in α because the massless vacuum Feynman propagator in Minkowski spacetime has this property and the conformal rescaling by a does not violate it. Thus, we may follow our computational strategy and compute $\left(\Delta_{F,0}^2\right)_{\mathsf{ms}}$, $\left(\Delta_{F,0}^2\log\left(M^{-2}\Delta_{F,0}\right)\right)_{\mathsf{ms}}$ and $\left(\Delta_{F,0}^3\right)_{\mathsf{ms}}$ by means of Proposition . This is easily done using for $\sigma_{F,\mathbb{M}}$ rather than σ_F and $h = \sqrt{8\pi^2 a(\tau_1)a(\tau_2)} = \sqrt{8\pi^2 a\otimes a}$. The results are

$$\begin{aligned}
(\Delta_{F,0})^2_{\mathsf{ms}} &= \lim_{\alpha\to 0}\left(\frac{1}{M^{2\alpha}}(\Delta_{F,0})^{2+\alpha} - \mathsf{pp}\frac{1}{M^{2\alpha}}(\Delta_{F,0})^{2+\alpha}\right) + \frac{i\log(8\pi^2)}{16\pi^2}\delta\\
&= \lim_{\alpha\to 0}\frac{1}{(8\pi^2)^2 a^2\otimes a^2}\left(\frac{1}{(M\sqrt{8\pi^2 a\otimes a})^{2\alpha}}\frac{1}{\sigma^{2+\alpha}_{F,\mathbb{M}}} - \mathsf{pp}\frac{1}{(M\sqrt{8\pi^2 a\otimes a})^{2\alpha}}\frac{1}{\sigma^{2+\alpha}_{F,\mathbb{M}}}\right) + \frac{i\log(8\pi^2)}{16\pi^2}\delta\\
&= -\frac{1+2\log(a)}{16\pi^2 a^4}i\delta_{\mathbb{M}} - \frac{1}{2(8\pi^2)^2 a^2\otimes a^2}\Box_{\mathbb{M}}\frac{\log\left(M^2\sigma_{F,\mathbb{M}}\right)}{\sigma_{F,\mathbb{M}}}\,,
\end{aligned}$$

[1] Note that one may take instead of the function $F(k) = 1/(2\sqrt{k^2 + a^2m^2 + a^2\left(\xi-\frac{1}{6}\right)R})$ in any distribution $F'(k)$ such that $F'(k) - F(k)$ is $O(k^{-5})$ for large k and integrable. By taking e.g. $F'(k) = 1/(2k) - \Theta(k - am)(a^2m^2 + a^2\left(\xi-\frac{1}{6}\right)R)/(4k^3)$ one may cancel the $\log R$ term outside of the integral.

$$\left(\Delta_{F,0}^2 \log\left(M^{-2}\Delta_{F,0}\right)\right)_{\mathsf{ms}} = \lim_{\alpha\to 0}\left(\frac{d}{d\alpha}\frac{1}{M^{2\alpha}}(\Delta_{F,0})^{2+\alpha} - \mathsf{pp}\frac{d}{d\alpha}\frac{1}{M^{2\alpha}}(\Delta_{F,0})^{2+\alpha}\right) - \frac{i\log^2(8\pi^2)}{32\pi^2}\delta$$
$$= \frac{2+2\log(a^2 8\pi^2)+\log^2(a^2)}{32\pi^2 a^4} i\delta_{\mathbb{M}} + \frac{1}{4(8\pi^2)^2 a^2\otimes a^2}\Box_{\mathbb{M}}\frac{\log^2\left(M^2\sigma_{F,\mathbb{M}}\right)}{\sigma_{F,\mathbb{M}}}$$
$$+ \frac{1+\log(8\pi^2)a\otimes a}{2(8\pi^2)^2 a^2\otimes a^2}\Box_{\mathbb{M}}\frac{\log\left(M^2\sigma_{F,\mathbb{M}}\right)}{\sigma_{F,\mathbb{M}}},$$

and

$$(\Delta_{F,0})^3_{\mathsf{ms}} = \lim_{\alpha\to 0}\left(\frac{1}{M^{2\alpha}}(\Delta_{F,0})^{3+\alpha} - \mathsf{pp}\frac{1}{M^{2\alpha}}(\Delta_{F,0})^{3+\alpha}\right) + \frac{i\left((1+2\log(8\pi^2))R + 192\pi^2[w]\right)}{48(8\pi^2)^2}\delta$$
$$= -\frac{(15+12\log(a))\Box_{\mathbb{M}} + 6(\Box_{\mathbb{M}}\log(a)) + 2(\partial_\tau^2 a)/a}{48(8\pi^2)^2 a^6} i\delta_{\mathbb{M}} - \frac{1}{8(8\pi^2)^3 a^3\otimes a^3}\Box^2_{\mathbb{M}}\frac{\log\left(M^2\sigma_{F,\mathbb{M}}\right)}{\sigma_{F,\mathbb{M}}}.$$

where we have used $\delta = \delta_{\mathbb{M}}/a^4$, $f_{\mathbb{M}} = 0$ and the fact that, by , $8\pi^2[w_0] = -R/36$ for the conformal vacuum state of the massless, conformally coupled scalar field.
Using these results as well as the Fourier representation of $1/\sigma_{\mathbb{M},\epsilon}$ and $\log\left(M^2\sigma_{F,\mathbb{M}}\right)$, and convolution identities, we can finally obtain the Fourier versions of the renormalised powers of $\Delta_{F,0}$. For instance, we find for $\left(\Delta_{F,0}^2\right)_{\mathsf{ms}}$ where the appearing renormalisation of $1/p^3$ is defined in . Note that the $\vec{p}$-integral has no convergence problems for large p because one may write the potentially dangerous $-i|\tau_1-\tau_2|e^{-2ip|\tau_1-\tau_2|}/p$ contribution as $\partial_p(e^{-2ip|\tau_1-\tau_2|}/(2p^2))$ plus an $O(p^{-3})$ term. Regarding the convergence for small p we observe that the integral is manifestly convergent if $k\neq 0$, thus yielding a well-defined distribution in $\vec{k}$ on $\mathbb{R}^3\setminus\{0\}$. The scaling degree of this distribution is easily seen to be $1<3$ and thus a unique extension towards the origin exists. In practical terms this means that the integral for $k=0$ may be computed as a limit $k\to 0$ of the integral with nonvanishing k without any renormalisation.

6.1.3 Example: the two-point function for a quartic potential up to second order

In order to compute the analytic expressions corresponding to the graphs in Figure , we may use the Fourier versions of the appearing propagators , and the analogous expressions for $\widehat{\left(\Delta_{F,0}^2\log\left(M^{-2}\Delta_{F,0}^2\right)\right)}_{\mathsf{ms}}(\tau_1,\tau_2,k)$ and $\widehat{\left(\Delta_{F,0}^3\right)}_{\mathsf{ms}}(\tau_1,\tau_2,k)$ the explicit form of $\mu(x) = 3\lambda w(x,x)$ in , as well as and the identities for products and convolutions . Note that $\mu(x)$ is in fact only time-dependent because Ω was chosen homogeneous and isotropic. Thus the integrals with μ-vertices can be computed partly with the above-mentioned identities by means of

$$\widehat{(1\otimes\mu)\Delta_\sharp}(\tau_1,\tau_2,k) = \mu(\tau_2)\widehat{\Delta_\sharp}(\tau_1,\tau_2,k), \qquad \widehat{(\mu\otimes 1)\Delta_\sharp}(\tau_1,\tau_2,k) = \mu(\tau_1)\widehat{\Delta_\sharp}(\tau_1,\tau_2,k).$$

Similarly, the bubbles in the third line of Figure contribute only time-dependent vertex factors which can be computed as

$$h_\sharp(\tau) := \int_{\mathcal{M}} d\tau_1 d^3x_1\; a(\tau_1)^4\mu(\tau_1)\Delta_\sharp(\tau,\tau_1,\vec{x}-\vec{x}_1) = \frac{1}{a(\tau)}\int_I d\tau_1\; a(\tau_1)^3\mu(\tau_1)\widehat{\Delta_\sharp}(\tau,\tau_1,0)$$

where $\Delta_\sharp$ is either Δ_+^2 or $\left(\Delta_F^2\right)_{\mathsf{ms}}$.
With these preparations, we can compute e.g. the first graphs of the fourth and fifth line in Figure in Fourier space as

$$\widehat{\Delta_R *_4 ((h_F\otimes 1)\Delta_+)} = \widehat{\Delta_R} *_1 (\widehat{(h_F\otimes 1)\Delta_+})$$
$$= \int_{I^2} d\tau_3\, d\tau_4\; a(\tau_3)a(\tau_4)^3\mu(\tau_4)\widehat{\Delta_R}(\tau_1,\tau_3,k)\widehat{\Delta_+}(\tau_3,\tau_2,k)\widehat{(\Delta_F^2)_{\mathsf{ms}}}(\tau_3,\tau_4,0)$$

and

$$\widehat{\Delta_R *_4 (\Delta_F)^3_{\mathsf{ms}} *_4 \Delta_+} = \widehat{\Delta_R} *_1 \widehat{(\Delta_F)^3_{\mathsf{ms}}} *_1 \widehat{\Delta_+}$$
$$= \int_{I^2} d\tau_3\, d\tau_4\; a(\tau_3)^2 a(\tau_4)^2\widehat{\Delta_R}(\tau_1,\tau_3,k)\widehat{(\Delta_F^3)_{\mathsf{ms}}}(\tau_3,\tau_4,k)\widehat{\Delta_+}(\tau_4,\tau_2,k).$$

6.1.4 More complicated graphs on cosmological spacetimes

In order to compute the Fourier transforms of more complicated graphs on FLRW spacetimes, one can use a strategy generalising the one employed in Section . Namely, one again decomposes the Feynman propagator

Δ_F into several pieces which capture the relevant singularities and can be expressed in terms of the conformal vacuum Feynman propagator $\Delta_{F,0}$ whose explicit form in position and Fourier space is well–known in contrast to the form of σ itself. The corresponding decomposition of general Feynman amplitudes τ_Γ is straightforward. The only non–trivial step is to generalise Proposition to the case of general amplitudes, i.e. to compute the difference between the minimal subtraction scheme used in conjunction with either analytically regularising powers of σ directly or analytically regularising powers of the full propagator $\Delta_{F,0}$. However, we do not foresee any problems in obtaining such a generalisation by proving versions of Lemma and Proposition for $\Delta_{F,0}$ rather than σ.
In fact, one can also skip this last step by taking a rather pragmatic approach and working directly with the renormalisation scheme consisting of decomposition in $\Delta_{F,0}$, analytic regularisation of powers of this propagator and minimal subtraction of the principal parts. This scheme, clearly applicable only to conformally flat spacetimes, satisfies all properties proved in Proposition , with two exceptions. It is not obvious whether the Principle of Perturbative agreement with respect to generalised mass perturbations holds for this scheme, whereas locality and covariance of course only hold in the sense restricted to conformally flat spacetimes. In this respect it is essential that the Feynman propagator of the conformal vacuum $\Delta_{F,0}$ on conformally flat spacetimes is manifestly "geometric", because the corresponding propagator of the massless Minkowski vacuum has this property.

6.2 Stress energy tensor

(blablabla)

Part II

Noncommutative approach to field theory

Chapter 7

Noncommutative gauge theories on 3-dimensional noncommutative space

7.1 Main definitions and properties

The algebra $\mathbb{R}^3_\lambda$ has been first introduced in and further considered in various works . Besides, a characterization of a natural basis has been given in . We refer to these references for more details. Here[1], it will be convenient to view $\mathbb{R}^3_\lambda$ as

$$\mathbb{R}^3_\lambda = \mathbb{C}\left[x_1, x_2, x_3, x_0\right]/\mathcal{I}[\mathcal{R}_1, \mathcal{R}_2] \ , \tag{7.1}$$

where $\mathbb{C}\left[x_1, x_2, x_3, x_0\right]$ is the free algebra generated by the 4 (hermitean) elements (coordinates) $\{x_{\mu=1,2,3},\ x_0\}$ and $\mathcal{I}\left[\mathcal{R}_1, \mathcal{R}_2\right]$ is the two-sided ideal generated by the relations

$$\mathcal{R}_1 :\ [x_\mu, x_\nu] = i\lambda\varepsilon_{\mu\nu\rho}x_\rho \ , \quad \mathcal{R}_2 :\ x_0^2 + \lambda x_0 = \sum_{\mu=1}^{3} x_\mu^2,\ \forall\mu,\nu,\rho = 1,2,3 \tag{7.2}$$

with $\lambda \neq 0$. $\mathbb{R}^3_\lambda$ is a unital $*$-algebra, with complex conjugation as involution and center $\mathcal{Z}(\mathbb{R}^3_\lambda)$ generated by x_0 and satisfying the following strict inclusion $\mathbb{R}^3_\lambda \supsetneq U(\mathfrak{su}(2))$, where $U(\mathfrak{su}(2))$ is the universal enveloping algebra of the Lie algebra $\mathfrak{su}(2)$. Alternative (equivalent) presentations can be found in e.g As shown in , any element $\phi \in \mathbb{R}^3_\lambda$ has the following blockwise expansion

$$\phi = \sum_{j\in\frac{\mathbb{N}}{2}} \ \sum_{-j\leq m,n\in\mathbb{N}\leq j} \phi^j_{mn}\, v^j_{mn} \ , \tag{7.3}$$

where $\phi^j_{mn} \in \mathbb{C}$, and the family $\{v^j_{mn}\ ,\ j \in \frac{\mathbb{N}}{2}\ ,\ -j \leq m, n \leq j\}$ is the natural orthogonal basis of $\mathbb{R}^3_\lambda$ introduced in , stemming from the direct sum decomposition

$$\mathbb{R}^3_\lambda = \bigoplus_{j\in\frac{\mathbb{N}}{2}} \mathbb{M}_{2j+1}(\mathbb{C}). \tag{7.4}$$

For fixed j, the corresponding subfamily is simply related to the canonical basis of the matrix algebra $\mathbb{M}_{2j+1}(\mathbb{C})$. The following fusion relation and conjugation hold true

$$v^{j_1}_{mn} v^{j_2}_{qp} = \delta^{j_1 j_2}\delta_{nq}\, v^{j_1}_{mp}\ ,\ \ (v^j_{mn})^\dagger = v^j_{nm}\ ,\ \ \forall j \in \frac{\mathbb{N}}{2}\ ,\ -j \leq m,n,q,p \leq j\ . \tag{7.5}$$

The orthogonality among the v^j_{mn}'s is taken with respect to the usual scalar product $\langle a, b\rangle := \mathrm{tr}(a^\dagger b)$, for any $a, b \in \mathbb{R}^3_\lambda$. Here, the trace functional tr can be defined for any $\Phi, \Psi \in \mathbb{R}^3_\lambda$ as

$$\mathrm{tr}(\Phi\Psi) := 8\pi\lambda^3 \sum_{j\in\frac{\mathbb{N}}{2}} w(j)\ \mathrm{tr}_j(\Phi^j\Psi^j) \tag{7.6}$$

with $w(j)$ is a center-valued weight factor to be discussed below, tr_j denotes the canonical trace of $\mathbb{M}_{2j+1}(\mathbb{C})$, and Φ^j (resp. Ψ^j) an element of $\mathbb{M}_{2j+1}(\mathbb{C})$ is simply defined from the expansion of Φ by the $(2j+1)\times(2j+1)$

[1] To simplify the notations, the associative $\star$-product for $\mathbb{R}^3_\lambda$ is understood everywhere in any product of elements of the algebra. Besides, summation over repeated indices is understood everywhere, unless explicitly stated.

matrix $\Phi^j := (\phi^j_{mn})_{-j\leq m,n\leq j}$ (resp. $\Psi^j := (\psi^j_{qp})_{-j\leq q,p\leq j}$). Therefore we have

$$\mathrm{tr}(\Phi\Psi) = 8\pi\lambda^3 \sum_{j\in\frac{\mathbb{N}}{2}} w(j) \left(\sum_{-j\leq m,n\leq j} \phi^j_{mn}\psi^j_{nm} \right), \tag{7.7}$$

and

$$\mathrm{tr}_j(v^j_{mn}) = \delta_{mn} \ , \ \ \langle v^{j_1}_{mn}, v^{j_2}_{pq}\rangle = 8\pi\lambda^3 \sum_{j_1\in\frac{\mathbb{N}}{2}} w(j_1)\ \delta^{j_1 j_2}\delta_{mp}\delta_{nq} \ . \tag{7.8}$$

Eqn. defines a family of traces depending on the weight factor $w(j)$. Recall that the particular choice

$$w(j) = j + 1 \tag{7.9}$$

leads to a trace that reproduces the expected behavior[2] for the usual integral on $\mathbb{R}^3$ once the (formal) commutative limit is applied . For a general discussion on this point based on a noncommutative generalization of the Kustaanheimo-Stiefel map We define $x_\pm := x_1 \pm ix_2$. Other useful relations that will be needed for computations in the ensuing analysis are

$$\begin{aligned} x_+\, v^j_{mn} &= \lambda\, \mathcal{F}(j,m)\, v^j_{m+1,n} & v^j_{mn}\, x_+ &= \lambda\, \mathcal{F}(j,-n)\, v^j_{m,n-1} \\ x_-\, v^j_{mn} &= \lambda\, \mathcal{F}(j,-m)\, v^j_{m-1,n} & v^j_{mn}\, x_- &= \lambda\, \mathcal{F}(j,n)\, v^j_{m,n+1} \\ x_3\, v^j_{mn} &= \lambda\, m\, v^j_{mn} & v^j_{mn}\, x_3 &= \lambda\, n\, v^j_{mn} \\ x_0\, v^j_{mn} &= \lambda\, j\, v^j_{mn} & v^j_{mn}\, x_0 &= \lambda\, j\, v^j_{mn}\ , \end{aligned} \tag{7.10}$$

where

$$\mathcal{F}(j,m) := \sqrt{(j+m+1)(j-m)}\ . \tag{7.11}$$

7.2 Differential calculus on noncommutative spaces

The construction of noncommutative gauge models can be conveniently achieved by using the general framework of the noncommutative differential calculus based on the derivations of an algebra which has been introduced a long ago . The general framework can actually be viewed as a noncommutative generalization the Koszul approach of differential geometry . Mathematical details and some related applications to NCFT can be found in In the present paper, we consider as in the differential calculus generated by the Lie algebra of real inner derivations of $\mathbb{R}^3_\lambda$

$$\mathcal{G} := \left\{ D_\mu := Ad_{\theta_\mu} = i\,[\theta_\mu, \cdot] \right\}\ ,\ \ \theta_\mu := \frac{x_\mu}{\lambda^2}\ ,\ \ \forall \mu = 1,2,3\ , \tag{7.12}$$

where the inner derivation D_μ satisfy the following commutation relation

$$[D_\mu, D_\nu] = -\frac{1}{\lambda}\epsilon_{\mu\nu\rho}D_\rho,\ \forall \mu,\nu,\rho = 1,2,3\ . \tag{7.13}$$

Denoting, for any $n\in\mathbb{N}$, by $\Omega^n_{\mathcal{G}}$ the space of $n-(\mathcal{Z}(\mathbb{R}^3_\lambda))$-linear) antisymmetric maps $\omega : \mathcal{G}^n \to \mathbb{R}^3_\lambda$, the corresponding $\mathbb{N}$-graded differential algebra is $(\Omega^\bullet_{\mathcal{G}} = \oplus_{n\in\mathbb{N}}\Omega^n_{\mathcal{G}},\ d,\ \times)$, with nilpotent differential $d : \Omega^n_{\mathcal{G}} \to \Omega^{n+1}_{\mathcal{G}}$ and product $\times$ on $\Omega^\bullet_{\mathcal{G}}$ defined for any $\omega\in\Omega^p_{\mathcal{G}}$ and $\rho\in\Omega^q_{\mathcal{G}}$ by

$$\begin{aligned} d\omega(X_1,...,X_{p+1}) \ &= \ \sum_{k=1}^{p+1}(-1)^{k+1}X_k\omega(X_1,...,\vee_k,...,X_{p+1}) \\ &+ \ \sum_{1\leq k<l\leq p+1}(-1)^{k+l}\omega([X_k,X_l],...,\vee_k,...,\vee_l,...,X_{p+1}), \end{aligned} \tag{7.14}$$

$$\omega\times\rho(X_1,...,X_{p+q}) = \frac{1}{p!q!}\sum_{\sigma\in\mathfrak{S}_{p+q}}|\sigma|\omega(X_{\sigma(1),...,X_{\sigma(p)}})\rho(X_{\sigma(p+1),...,X_{\sigma(p+q)}}), \tag{7.15}$$

where the X_i's are elements of $\mathcal{G}$ and $|\sigma|$ is the signature of the permutation $\sigma\in\mathfrak{S}_{p+q}$.
Let $\mathbb{M}$ denotes a right-module over $\mathbb{R}^3_\lambda$. Recall that a connection on $\mathbb{M}$ can be defined as a linear map $\nabla : \mathcal{G}\times\mathbb{M}\to\mathbb{M}$ with

$$\nabla_X(ma) = \nabla_X(m)a + mXa\ ,\ \ \nabla_{zX}(a) = z\nabla_X(a)\ ,\ \ \nabla_{X+Y}(a) = \nabla_X(a) + \nabla_Y(a)\ ,$$

[2] For instance, observe that one easily obtains from the expected volume of a sphere of radius λN with $\Phi^j = \Psi^j = \mathbb{I}_j$ and summing up to $j = \frac{N}{2}$. Namely, one obtains $8\pi\lambda^3\sum_{k=0}^{N}\left(\frac{k}{2}\right)(k+1) \simeq \frac{4}{3}\pi\,(\lambda N)^3$.

for any $a \in \mathbb{R}^3_\lambda$, any $m \in \mathbb{M}$, $z \in \mathcal{Z}(\mathbb{R}^3_\lambda)$ and any $X, Y \in \mathcal{G}$.
As we are interested by noncommutative versions of $U(1)$ gauge theories, we assume from now on $\mathbb{M} = \mathbb{C} \otimes \mathbb{R}^3_\lambda$ which can be viewed as a noncommutative analog of the complex line bundle relevant for abelian ($U(1)$) commutative gauge theories. We further restrict ourself to hermitean connections[3] for the canonical hermitean structure given by $h(a_1, a_2) = a_1^\dagger a_2$, $a_1, a_2 \in \mathbb{R}^3_\lambda$.
A mere application of the above definition yields

$$\begin{aligned} \nabla_{D_\mu}(a) &:= \nabla_\mu(a) = D_\mu a + A_\mu a \ , \\ A_\mu &:= \nabla_\mu(\mathbb{I}) \ , \quad \text{with } A_\mu^\dagger = -A_\mu \ , \end{aligned} \tag{7.16}$$

for $a \in \mathbb{R}^3_\lambda$ and $\mu = 1, 2, 3$. The definition of the curvature

$$F(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \ , \quad \forall X, Y \in \mathcal{G} \ ,$$

yields

$$F(D_\mu, D_\nu) := F_{\mu\nu} = [\nabla_\mu, \nabla_\nu] - \nabla_{[D_\mu, D_\nu]} = D_\mu A_\nu - D_\nu A_\mu + [A_\mu, A_\nu] + \frac{1}{\lambda}\epsilon_{\mu\nu\rho} A_\rho \ . \tag{7.17}$$

The group of gauge transformations, defined as the group of automorphisms of the module compatible with both hermitean and right-module structures, is easily found to be the group of unitary elements of $\mathbb{R}^3_\lambda$, $\mathcal{U}(\mathbb{R}^3_\lambda)$, with left action of $\mathbb{R}^3_\lambda$. For any $g \in \mathcal{U}(\mathbb{R}^3_\lambda)$ and $\phi \in \mathbb{R}^3_\lambda$, one has $g^\dagger g = g g^\dagger = \mathbb{I}$, $\phi^g = g\phi$. From the definition of the gauge transformations of the connection given by $\nabla^g_\mu = g^\dagger \nabla_\mu \circ g$, for any $g \in \mathcal{U}(\mathbb{R}^3_\lambda)$, one infers

$$A^g_\mu = g^\dagger A_\mu \, g + g^\dagger D_\mu \, g \ , \ \text{ and } \quad F^g_{\mu\nu} = g^\dagger F_{\mu\nu} \, g \ . \tag{7.18}$$

The existence of a canonical gauge invariant connection, denoted hereafter by ∇^{inv}, stems from the existence of inner derivations in the Lie algebra of derivations that generates the differential calculus. See for a general analysis. In the present case, one finds

$$\nabla^{inv}_\mu(a) = D_\mu a - i\theta_\mu a = -ia\theta_\mu \ , \quad \forall a \in \mathbb{R}^3_\lambda \ , \tag{7.19}$$

with curvature $F^{inv}_{\mu\nu} = 0$. A natural gauge covariant tensor 1-form is then obtained by forming the difference between ∇^{inv}_μ and any arbitrary connection. The corresponding components, sometimes called covariant coordinates, are given by

$$\mathcal{A}_\mu := \nabla_\mu - \nabla^{inv}_\mu = A_\mu + i\theta_\mu \ , \quad \forall i = 1, 2, 3 \ , \tag{7.20}$$

and one has $\mathcal{A}^\dagger_\mu = -\mathcal{A}_\mu$, $\mu = 1, 2, 3$ ($A^\dagger_\mu = -A_\mu$). By using , one obtains

$$F_{\mu\nu} = [\mathcal{A}_\mu, \mathcal{A}_\nu] + \frac{1}{\lambda}\epsilon_{\mu\nu\rho}\mathcal{A}_\rho \ . \tag{7.21}$$

One easily verifies that for any $a \in \mathbb{R}^3_\lambda$, and $g \in \mathcal{U}(\mathbb{R}^3_\lambda)$, the following gauge transformations hold true

$$(\nabla^{inv}_\mu(a))^g = \nabla^{inv}_\mu(a) \ , \quad \mathcal{A}^g_\mu = g^\dagger \mathcal{A}_\mu \, g \ , \quad \forall \mu = 1, 2, 3 \ . \tag{7.22}$$

Define the real invariant 1-form $\Theta \in \Omega^1_\mathcal{G}$ by

$$\Theta \in \Omega^1_\mathcal{G} \ : \ \Theta(D_\mu) = \Theta(Ad_{\theta_\mu}) = \theta_\mu \ . \tag{7.23}$$

By making use of and , one easily check that

$$d(-i\Theta) + (-i\Theta)^2 = 0 \ , \tag{7.24}$$

reflecting $F^{inv}_{\mu\nu} = 0$.
The form Θ related to the 1-form invariant canonical connection supports an interesting interpretation. Recall that a natural noncommutative analog of a symplectic form is defined as a real closed 2-form ω such that for any element a in the algebra, there exists a derivation Ham(a) (the analog of Hamiltonian vector field) verifying $\omega(X, \text{Ham}(a)) = X(a)$ for any derivation X. One then observes that $\omega := d\Theta \in \Omega^2_\mathcal{G}$ can be viewed as the natural symplectic form on the algebra $\mathbb{R}^3_\lambda$ in the setting of with Ham$(a) = Ad_{ia}$ for any $a \in \mathbb{R}^3_\theta$ as the noncommutative analog of Hamiltonian vector field and

$$\{a, b\} := \omega\left(\text{Ham}(a), \text{Ham}(b)\right) = -i\,[a, b] \tag{7.25}$$

the related (real) Poisson bracket.

[3] Given a hermitean structure, says $h : \mathbb{M} \times \mathbb{M} \to \mathbb{R}^3_\lambda$, ∇ is hermitean if $Xh(m_1, m_2) = h(\nabla_X(m_1), m_2) + h(m_1, \nabla_X(m_2))$, for any $X \in \mathcal{G}$, $m_1, m_2 \in \mathbb{M}$.

7.3 A family of gauge invariant classical actions

Families of gauge-invariant functional (classical) actions can be easily obtained from the trace of any gauge-covariant polynomial functional in the covariant coordinates $\mathcal{A}_\mu$, namely $S_{inv}(\mathcal{A}_\mu) = \text{tr}\,(P(\mathcal{A}_\mu))$. Here, we will assume that the relevant field variable is $\mathcal{A}_\mu$, akin to a matrix model formulation of gauge theories on $\mathbb{R}^3_\lambda$, thus proceeding in the spirit of . Natural requirement for the gauge-invariant functional are:

1. $P(\mathcal{A}_\mu)$ is at most quartic in $\mathcal{A}_\mu$,
2. $P(\mathcal{A}_\mu)$ does not involve linear term in $\mathcal{A}_\mu$ (not tadpole at the classical order),
3. the kinetic operator is positive.

Set from now on

$$x^2 := \sum_{\mu=1}^{3} x_\mu x_\mu.$$

We observe that gauge theories on $\mathbb{R}^3_\lambda$ can accommodate a gauge-invariant harmonic term $\sim \text{tr}(x^2\mathcal{A}_\mu\mathcal{A}_\mu)$. This property simply stems from the fact that $x^2 \in \mathcal{Z}(\mathbb{R}^3_\lambda)$ combined with the gauge-invariance of the 1-form canonical connection whose components in the module are given by

$$\nabla^{inv}(\mathbb{I})_\mu := A^{inv}_\mu = -i\theta_\mu \tag{7.26}$$

as it can be readily obtained from and . One easily checks that

$$(A^{inv}_\mu)^g = (-i\theta_\mu)^g = -i\theta_\mu, \tag{7.27}$$

as a mere combination of

$$\sum_{\mu=1}^{3}(-i\theta_\mu)(-i\theta_\mu) = -\frac{1}{\lambda^4}x^2 = -\frac{1}{\lambda^4}(x_0^2 + \lambda x_0)\ , \tag{7.28}$$

in which the LHS is obviously gauge-invariant since holds true while the RHS belongs to $\mathcal{Z}(\mathbb{R}^3_\lambda)$ as a polynomial in x_0. Hence, the gauge-invariant object $\sum_{\mu=1}^{3}(-i\theta_\mu)^2$ belongs to the center of $\mathbb{R}^3_\lambda$. Therefore, by using the cyclicity of the trace, one can write (summation over repeated α indice understood)

$$\begin{aligned}
\text{tr}(\sum_{\mu=1}^{3}(-i\theta_\mu)^g(-i\theta_\mu)^g(\mathcal{A}^g_\alpha\mathcal{A}^g_\alpha)) &= \text{tr}(g\sum_{\mu=1}^{3}(-i\theta_\mu)(-i\theta_\mu)g^\dagger(\mathcal{A}_\alpha\mathcal{A}_\alpha)) \\
&= \text{tr}(\sum_{\mu=1}^{3}(-i\theta_\mu)(-i\theta_\mu)(\mathcal{A}_\alpha\mathcal{A}_\alpha))
\end{aligned} \tag{7.29}$$

where we used $\sum_\mu(-i\theta_\mu)(-i\theta_\mu) \in \mathcal{Z}(\mathbb{R}^3_\lambda)$ to obtain the last equality. Note that such a gauge-invariant harmonic term cannot be built in the case of gauge theories on the Moyal space $\mathbb{R}^4_\theta$ simply because, says $x^2_{\nu=1,2,3,4}$, while still related to a gauge invariant object (a canonical gauge-invariant connection still exists, see e.g It is convenient to work with hermitean fields. Thus, we set from now on

$$\mathcal{A}_\mu = i\Phi_\mu$$

so that $\Phi_\mu^\dagger = \Phi_\mu$ for any $\mu = 1,2,3$. The above observation, combined with the requirements i) and ii) given above points towards the following general expression for a gauge-invariant action

$$\begin{aligned}
S(\Phi) &= \frac{1}{g^2}\text{tr}\big(\kappa\Phi_\mu\Phi_\nu\Phi_\nu\Phi_\mu + \eta\Phi_\mu\Phi_\nu\Phi_\mu\Phi_\nu + i\zeta\epsilon_{\mu\nu\rho}\Phi_\mu\Phi_\nu\Phi_\rho + (M+\mu x^2)\Phi_\mu\Phi_\mu\big) \\
&= \frac{1}{g^2}\text{tr}\big((\frac{\eta-\kappa}{4})[\Phi_\mu,\Phi_\nu]^2 + (\frac{\eta+\kappa}{4})\{\Phi_\mu,\Phi_\nu\}^2 + i\zeta\epsilon_{\mu\nu\rho}\Phi_\mu\Phi_\nu\Phi_\rho \\
&+ \ (M+\mu x^2)\Phi_\mu\Phi_\mu\big),
\end{aligned} \tag{7.30}$$

where from now on Einstein summation convention is used, the trace is still given by and g^2, κ, η, ζ, M and μ are real parameters. The corresponding mass dimensions are

$$[\kappa] = [\eta] = 0,\ [g^2] = [\zeta] = 1,\ [M] = 2,\ [\mu] = 4 \tag{7.31}$$

so that the action is dimensionless, assuming that the “engineering” dimension 3 of the noncommutative space is the relevant dimension.

We will mainly focus on sub-families involving positive actions obtained from , we set

$$\kappa = 2(\Omega + 1),\ \eta = 2(\Omega - 1), \tag{7.32}$$

where the real parameter Ω is dimensionless, thus fixing for convenience the overall normalization of the term $\sim [\Phi_\mu, \Phi_\nu]^2$ in . This latter action can be rewritten as

$$\begin{aligned} S(\Phi) &= \frac{1}{g^2}\mathrm{tr}\big((F_{\mu\nu} - \frac{i}{\lambda}\epsilon_{\mu\nu\rho}\Phi_\rho)^\dagger(F_{\mu\nu} - \frac{i}{\lambda}\epsilon_{\mu\nu\rho}\Phi_\rho) + \Omega\left\{\Phi_\mu, \Phi_\nu\right\}^2 + i\zeta\epsilon_{\mu\nu\rho}\Phi_\mu\Phi_\nu\Phi_\rho \\ &+ (M + \mu x^2)\Phi_\mu\Phi_\mu\big) \\ &= \frac{1}{g^2}\mathrm{tr}\big(F^\dagger_{\mu\nu}F_{\mu\nu} + \Omega\left\{\Phi_\mu, \Phi_\nu\right\}^2 + i\zeta'\epsilon_{\mu\nu\rho}\Phi_\mu\Phi_\nu\Phi_\rho + \left(M' + \mu x^2\right)\Phi_\mu\Phi_\mu\big), \end{aligned} \tag{7.33}$$

with

$$\zeta = \zeta' + \frac{4}{\lambda};\ \ M = M' + \frac{2}{\lambda^2}. \tag{7.34}$$

We note that the first two terms in the gauge-invariant action $S(\Phi)$ are formally similar to those occurring in the so-called induced gauge theory on $\mathbb{R}^4_\theta$ $S(\Phi)$ is positive when

$$\Omega \geq 0,\ \mu > 0,\ \zeta = 0,\ M > 0 \tag{7.35}$$

or

$$\Omega \geq 0,\ \mu > 0,\ \zeta = \frac{4}{\lambda},\ M > \frac{2}{\lambda^2}, \tag{7.36}$$

as it can be realized respectively from the 1st and 2nd equality in (see also section and the appendix for the positivity of the kinetic operator).
In the rest of this paper, we will focus on the family of actions fulfilling the first condition

$$S_\Omega = \frac{1}{g^2}\mathrm{tr}\big((F_{\mu\nu} - \frac{i}{\lambda}\epsilon_{\mu\nu\rho}\Phi_\rho)^\dagger(F_{\mu\nu} - \frac{i}{\lambda}\epsilon_{\mu\nu\rho}\Phi_\rho) + \Omega\left\{\Phi_\mu, \Phi_\nu\right\}^2 + (M + \mu x^2)\Phi_\mu\Phi_\mu\big). \tag{7.37}$$

The equation of motion for given by

$$4(\Omega + 1)(\Phi_\rho\Phi_\mu\Phi_\mu + \Phi_\mu\Phi_\mu\Phi_\rho) + 8(\Omega - 1)\Phi_\mu\Phi_\rho\Phi_\mu + 2(M + \mu x^2)\Phi_\rho = 0, \tag{7.38}$$

one infers that $\Phi_\rho = 0$ is the absolute minimum of [4].
In the section , we will show that one class of gauge-invariant models pertaining to the families , yields after gauge-fixing to a finite theory at all orders in perturbation. This stems from the conjunction of the gauge-invariant harmonic term in $\sim \mu x^2\Phi_\mu\Phi_\mu$, the orthogonal sum structure of $\mathbb{R}^3_\lambda$ and the existence of a bound on the (absolute value of) the propagator for Φ_μ. This will be discussed at the end of the paper. Notice that in the Moyal case only the term $\sim M$ is allowed by gauge invariance.

[4] There are also other nontrivial solutions of the equation of motion related to . Namely, there is one more solution belonging to the center $\mathcal{Z}(\mathbb{R}^3_\lambda)$ given by $\Phi_\mu\Phi_\mu = -\frac{M+\mu x^2}{2(\kappa+\eta)}$. We found also solution outside the center given by $\Phi_i = f x_i$, where $f = \frac{-\eta\lambda \pm \sqrt{\eta^2\lambda^2 - 32\left[x^2(\kappa+\eta) - \eta\lambda^2\right](M+\mu x^2)}}{8\left[x^2(\kappa+\eta) - \eta\lambda^2\right]}$. The corresponding quantum field theories are still under investigation.

Chapter 8

Perturbative analysis

8.1 Gauge-fixing

We set

$$\Phi_\mu = \sum_{j,m,n} (\phi_\mu)^j_{mn} v^j_{mn}\ ,\ \ \forall \mu = 1,2,3. \tag{8.1}$$

The kinetic term of the classical action S_Ω is given by

$$S_{Kin}(\Phi) = \frac{1}{g^2}\mathrm{tr}(\Phi_\mu(M+\mu x^2)\Phi_\mu) \tag{8.2}$$

$$= \frac{8\pi\lambda^3}{g^2}\sum_{j,m,n} w(j)(M+\lambda^2\mu j(j+1))|(\phi_\mu)^j_{mn}|^2 \tag{8.3}$$

where $w(j)$ is the center-valued weight introduced in

$$x_0 = \lambda\sum_{j,m} j\ v^j_{mm},\ x^2 = \lambda^2\sum_{j,m} j(j+1)\ v^j_{mm}, \tag{8.4}$$

stemming from . Recall that we have assumed that the condition holds true. We assume for the moment that $w(j)$ is a polynomial function of j, thus insuring a suitable decay of the related propagators at large indices. We will specialize to the cases $w(j)=1$ and $w(j)=j+1$ in a while.
Now, defining the kinetic operator by

$$S_{Kin}(\Phi) = \sum_{j,m,n,k,l} (\phi_\mu)^{j_1}_{mn} G^{j_1 j_2}_{mn;kl} (\phi_\mu)^{j_2}_{kl},$$

one can write

$$G^{j_1j_2}_{mn;kl} = \frac{8\pi\lambda^3}{g^2} w(j_1)\ \left(M+\lambda^2\mu j_1(j_1+1)\right)\delta^{j_1j_2}\delta_{nk}\delta_{ml}. \tag{8.5}$$

The relation defines a positive self-adjoint operator. The corresponding details are collected in the appendix
The gauge-invariance of S_Ω can be translated into invariance under a nilpotent BRST operation δ_0 defined by the following structure equations

$$\delta_0\Phi_\mu = i[C,\Phi_\mu]\ ,\ \ \delta_0 C = iCC \tag{8.6}$$

where C is the ghost field. Recall that δ_0 acts as an antiderivation with respect to the grading given by (the sum of) the ghost number (and degree of forms), modulo 2. C (resp. Φ_i) has ghost number $+1$ (resp. 0). Fixing the gauge symmetry can be conveniently done by using the gauge condition

$$\Phi_3 = \theta_3. \tag{8.7}$$

This can be implemented into the action by enlarging

$$\delta_0\bar{C} = b\ ,\ \ \delta_0 b = 0 \tag{8.8}$$

where $\bar{C}$ and b are respectively the antighost and the Stückelberg field (with respective ghost number -1 and 0) and by adding to S_Ω a BRST invariant gauge-fixing term given by

$$S_{fix} = \delta_0\mathrm{tr}\big(\bar{C}(\Phi_3-\theta_3)\big) = \mathrm{tr}\big(b(\Phi_3-\theta_3) - i\bar{C}[C,\Phi_3]\big). \tag{8.9}$$

Integrating over the Stüeckelberg field b yields the constraint $\Phi_3 = \theta_3$ into , while the ghost part can be easily seen to decouple[1].
Now, we define the kinetic operator by

$$K := G + 8\Omega L(\theta_3^2). \tag{8.10}$$

where $G = M + \mu x^2$ and $L(\theta_3^2)$ is the left multiplication by θ_3^2. The resulting gauge-fixed action can be written (up to an unessential constant term) as

$$S_\Omega^f = S_2 + S_4, \tag{8.11}$$

with

$$\begin{aligned} S_2 &= \frac{1}{g^2}\mathrm{tr}((\Phi_1, \Phi_2)\begin{pmatrix} Q & 0 \\ 0 & Q\end{pmatrix}\begin{pmatrix}\Phi_1 \\ \Phi_2\end{pmatrix}), \\ Q &= K + i4(\Omega - 1)L(\theta_3)D_3\ , \end{aligned} \tag{8.12}$$

$$S_4 = \frac{4}{g^2}\mathrm{tr}\left(\Omega(\Phi_1^2 + \Phi_2^2)^2 + (\Omega - 1)(\Phi_1\Phi_2\Phi_1\Phi_2 - \Phi_1^2\Phi_2^2)\right). \tag{8.13}$$

The gauge-fixed action is thus described by a rather simple NCFT with "flavor diagonal" kinetic term (see and quartic interaction terms. We find also convenient to introduce the complex fields

$$\Phi = \frac{1}{2}(\Phi_1 + i\Phi_2),\ \Phi^\dagger = \frac{1}{2}(\Phi_1 - i\Phi_2), \tag{8.14}$$

so that the gauge-fixed action S_Ω^f can be expressed alternatively into the form

$$S_\Omega^f = \frac{2}{g^2}\mathrm{tr}\left(\Phi Q\Phi^\dagger + \Phi^\dagger Q\Phi\right) + \frac{16}{g^2}\mathrm{tr}\left((\Omega + 1)\Phi\Phi^\dagger\Phi\Phi^\dagger + (3\Omega - 1)\Phi\Phi\Phi^\dagger\Phi^\dagger\right). \tag{8.15}$$

At this level, some comments are in order.

- The action bears some similarity with the (matrix model representation of) the action describing the family of complex LSZ models
- For $\Omega = 1/3$, the quartic interaction potential depends only on $\Phi\Phi^\dagger$, so that the action is formally similar to the action describing an exactly solvable LSZ-type model investigated in . Only the respective kinetic operators are different. It turns out that the partition function for $S^f_{\Omega=\frac{1}{3}}$ can be actually related to τ-functions of integrable hierarchies. More precisely, due to the orthogonal decomposition of $\mathbb{R}^3_\lambda$, the partition function can be expressed as a product of factors labelled by $j \in \frac{\mathbb{N}}{2}$, each one being expressible as a τ-function for a 2-d Toda hierarchy. Note that each factor can be actually interpreted as the partition function for the reduction of the gauge-fixed theory on the matrix algebra $\mathbb{M}_{2j+1}(\mathbb{C})$. The corresponding analysis will be presented in a separate publication
- For $\Omega = 1$, the kinetic operator in simplifies while the interaction term takes a more symmetric form, as it is apparent e.g from . We will find that the corresponding theory is finite to all orders in perturbation.

8.2 Gauge-fixed action

In this subsection, we will assume $\Omega = 1$. The corresponding action is

$$S^f_{\Omega=1} = \frac{1}{g^2}\mathrm{tr}((\Phi_1, \Phi_2)\begin{pmatrix} K & 0 \\ 0 & K\end{pmatrix}\begin{pmatrix}\Phi_1 \\ \Phi_2\end{pmatrix}) + \frac{4}{g^2}\mathrm{tr}((\Phi_1^2 + \Phi_2^2)^2). \tag{8.16}$$

The kinetic term is expressed as

$$S^f_{2,\Omega=1} = \frac{8\pi\lambda^3}{g^2}\sum_{j,m,n} w(j)(M + \mu\lambda^2 j(j+1) + \frac{8}{\lambda^2}n^2)|(\phi_{1\mu})_{mn}|^2 + (1 \to 2), \tag{8.17}$$

where we used

$$x_3^2 = \lambda^2\sum_{j,m} m^2 v^j_{mm}. \tag{8.18}$$

[1]Recall it amounts to consider an "on-shell" formulation for which nilpotency of the BRST operation (and corresponding BRST-invariance of the gauge-fixed action) is verified modulo the ghost equation of motion.

Accordingly, the "matrix elements" of the kinetic operator can be written as

$$K^{j_1j_2}_{mn;kl} := \frac{8\pi\lambda^3}{g^2}w(j_1)(M+\mu\lambda^2 j_1(j_1+1)+\frac{4}{\lambda^2}(k^2+l^2))\delta^{j_1j_2}\delta_{ml}\delta_{nk}. \tag{8.19}$$

Note that

$$K^{j_1j_2}_{mn;kl} = K^{j_1j_2}_{lk;nm} = K^{j_1j_2}_{mn;lk} \tag{8.20}$$

reflecting reality of the functional action and the self-adjointness of K (see appendix ; recall we use the natural Hilbert product $\langle a,b\rangle = \text{tr}(a^\dagger b)$).
The inverse of (i.e the matrix elements of the propagator) $P^{j_1j_2}_{mn;kl}$ is then defined by

$$\sum_{j_2,k,l} K^{j_1j_2}_{mn;lk}P^{j_2j_3}_{kl;rs} = \delta^{j_1j_3}\delta_{ms}\delta_{nr},\quad \sum_{j_2,n,m} P^{j_1j_2}_{rs;mn}K^{j_2j_3}_{nm;kl} = \delta_{j_1j_3}\delta_{rl}\delta_{sk}, \tag{8.21}$$

leading to

$$P^{j_1j_2}_{mn;kl} = \frac{g^2}{8\pi\lambda^3}\frac{1}{w(j_1)(M+\lambda^2\mu j_1(j_1+1)+\frac{4}{\lambda^2}(k^2+l^2))}\delta^{j_1j_2}\delta_{ml}\delta_{nk}. \tag{8.22}$$

We will start the perturbative analysis by computing the 2-point (connected) correlation function at the first (one-loop) order. To prepare the discussion, we introduce sources variables for the Φ_α's, namely $J_\alpha = \sum_{j,m,n}(J_\alpha)^j_{mn}v^j_{mn}$, for any $\alpha=1,2$. Then, a standard computation yields the free part of the generating functional of the connected correlation functions $W_0(J)$ given (up to an unessential prefactor) by

$$\begin{aligned} e^{W_0(J)} &= \int\prod_{\alpha=1}^{2}\mathcal{D}\Phi_\alpha e^{-(S^f_{2\Omega=1}+\text{tr}(\Phi_\alpha J_\alpha))} = \int\prod_{\alpha=1}^{2}\mathcal{D}\Phi_\alpha e^{-\sum((\phi_\alpha)^{j_1}_{mn}K^{j_1j_2}_{mn;kl}(\phi_\alpha)^{j_2}_{kl}+(\mathcal{J}_\alpha)^j_{mn}(\phi_\alpha)^j_{nm})} \\ &= \exp(\frac{1}{4}\sum(\mathcal{J}_\alpha)^{j_1}_{mn}P^{j_1j_2}_{mn;kl}(\mathcal{J}_\alpha)^{j_2}_{kl}), \end{aligned} \tag{8.23}$$

where we have defined for further convenience

$$(\mathcal{J}_\alpha)^j := 8\pi\lambda^3 w(j)(J_\alpha)^j,\ -j\le m,n\le j \tag{8.24}$$

for any $j\in\frac{\mathbb{N}}{2}$. To obtain , one simply uses the generic field redefinition among the fields components given by

$$(\phi_\alpha)^j_{mn} = (\phi'_\alpha)^j_{mn} - \frac{1}{2}P^j_{nm;kl}(\mathcal{J}_\alpha)^j_{kl} = (\phi'_\alpha)^j_{mn} - \frac{1}{2}(\mathcal{J}_\alpha)^j_{rs}P_{rs;nm}.$$

Correlation functions involving modes $(\phi_\alpha)^j_{mn}$ will be obtained from the successive action of the corresponding functional derivatives $\frac{\delta}{\delta(\mathcal{J}_\alpha)^j_{nm}}$ on the full generating functional. We use

$$e^{-S_4(\Phi_1,\Phi_2)}e^{-\text{tr}(J_\alpha\Phi_\alpha)} = e^{-S_4(\frac{\delta}{\delta\mathcal{J}_1},\frac{\delta}{\delta\mathcal{J}_2})}e^{-\sum(\mathcal{J}_\alpha)^j_{mn}(\phi_\alpha)^j_{nm}} \tag{8.25}$$

where

$$S_4(\frac{\delta}{\delta\mathcal{J}_1},\frac{\delta}{\delta\mathcal{J}_2}) = \sum\frac{8\pi\lambda^3}{g^2}w(j)S^j_4(\frac{\delta}{\delta\mathcal{J}}) \tag{8.26}$$

in which S^j_4 denotes the tr_j part of the interaction term in the action . We then write

$$e^{W(\mathcal{J})} = e^{-S_4(\frac{\delta}{\delta\mathcal{J}_1},\frac{\delta}{\delta\mathcal{J}_2})}\, e^{W_0(\mathcal{J})}$$

to obtain

$$W(\mathcal{J}) = W_0(\mathcal{J}) + \ln\left[1+e^{-W_0(\mathcal{J})}\left(e^{-S_4(\frac{\delta}{\delta\mathcal{J}_1},\frac{\delta}{\delta\mathcal{J}_2})}-1\right)e^{W_0(\mathcal{J})}\right], \tag{8.27}$$

where S_4 is defined by . The expansion of both the logarithm and e^{S_4} then gives rise to the perturbative expansion.

8.3 One-loop 2-point and 4-point functions

The computational details of the one-loop contribution to the 2-point function are collected in the appendix . From , it can be realized that the quadratic part of the classical action receives a 1st order (one-loop) contribution $\Gamma^1_2(\Phi_\alpha)$ given by

$$\begin{aligned} \Gamma^1_2(\Phi_\alpha) = \frac{32\pi\lambda^3}{g^2}\sum_{j\in\frac{\mathbb{N}}{2}} \Bigg[& \sum_{-j\le m,n,r,p\le j}(\phi_\alpha)^j_{pr}\left(w(j)P^j_{rm;np}\right)(\phi_\alpha)^j_{mn} \\ & + \sum_{-j\le p,r,n\le j} 3(\phi_\alpha)^j_{pr}\left(\sum_{m=-j}^{j}w(j)P^j_{rm;mn}\right)(\phi_\alpha)^j_{np}\Bigg], \end{aligned} \tag{8.28}$$

in which the 1st (resp. 2nd) term corresponds to the non-planar (resp. planar) contribution. Writing generically $\Gamma^1_2(\Phi_\alpha)=\frac{32\pi\lambda^3}{g^2}\sum(\phi_\alpha)^j_{mn}\sigma^j_{mn;kl}(\phi_\alpha)^j_{kl}$, we have explicitly

$$\sigma^{NP\,j}_{pr;mn} = w(j)P^j_{pr;mn} \tag{8.29}$$

$$\sigma^{P\,j}_{pr;nm} = 3\delta_{mp}\sum_{m=-j}^{j} w(j)P^j_{rm;mn}. \tag{8.30}$$

One can easily verify that are always finite, even for $j=0$ and $j\to\infty$ and without any singularity whenever $M>0$, which is assumed here. This is obvious for . For the planar contribution, one simply observes that the summation over m, which corresponds to an internal ribbon loop, satisfies the estimate

$$\begin{aligned}\sum_{m=-j}^{j} w(j)P_{rm;mn} &= \delta_{nr}\sum_{m=-j}^{j}\frac{g^2}{8\pi\lambda^3}\frac{1}{(M+\lambda^2\mu j(j+1)+\frac{4}{\lambda^2}(m^2+n^2))}\\ &\leq \delta_{nr}\frac{g^2}{8\pi\lambda^3}\frac{2j+1}{(M+\lambda^2\mu j(j+1))}\end{aligned} \tag{8.31}$$

which is always finite for any $j\in\frac{\mathbb{N}}{2}$. Note that no dangerous UV/IR mixing shows up in the computation of the one-loop 2-point function.
Eqn.reflects simply the existence of an estimate obeyed by the propagator . This can be used in the subsection to show the finitude of the theory to all orders in perturbation. Indeed, we have from

$$0\leq P^{j_1j_2}_{mn;kl}\leq\frac{\Pi(M,j_1)}{w(j_1)}\delta_{j_1j_2}\delta_{ml}\delta_{nk}, \tag{8.32}$$

for any $j_1,j_2\in\frac{\mathbb{N}}{2},\ -j_1\leq m,n,k,l\leq j_1,$, where

$$\Pi(M,j):=\frac{g^2}{8\pi\lambda^3}\frac{1}{(M+\lambda^2\mu j(j+1))}. \tag{8.33}$$

A similar analysis can be carried out for the 1-loop contributions to the 4-point function showing that those contributions are again finite. For instance, consider the vertex functional for one specie Φ_α, written generically as (no sum over α)

$$\Gamma^1_4(\Phi_\alpha)=\sum_{m_i,n_i,r_i,s_i}V_{m_1,m_2,n_1,n_2,r_1,r_2,s_1,s_2}(\phi_\alpha)^j_{m_1m_2}(\phi_\alpha)^j_{n_1n_2}(\phi_\alpha)^j_{r_1r_2}(\phi_\alpha)^j_{s_1s_2}. \tag{8.34}$$

Typical planar contributions to the vertex functional are of the form

$$\begin{aligned}\Gamma^{P\,1}_4 \sim &\sum\Big(\sum_{-j\leq p,q\leq j}w^2(j)P^j_{n_1p;qr_2}P^j_{pm_2;s_1q}\delta_{m_1n_2}\Big)\\ &\times\delta_{s_2r_1}(\phi_\alpha)^j_{m_1m_2}(\phi_\alpha)^j_{n_1n_2}(\phi_\alpha)^j_{r_1r_2}(\phi_\alpha)^j_{s_1s_2},\end{aligned} \tag{8.35}$$

where the factor $w^2(j)$ comes from the 2 vertex contributions to the loop. One can easily check that

$$\sum_{-j\leq p,q\leq j}w^2(j)P^j_{n_1p;qr_2}P^j_{pm_2;s_1q}\leq\delta_{n_1r_2}\delta_{s_1m_2}(2j+1)\Pi(M,j)^2, \tag{8.36}$$

which is finite for any value of j and decays to 0 as j^{-3} when $j\to\infty$.
Other planar 1-loop contributions to the vertex function can be checked to be finite by using a similar argument. There are 3 species of non-planar contributions with typical respective contributions being of the form

$$\Gamma^1_{14}\sim\sum\Big(w^2(j)P^j_{m_1n_2;s_1r_2}P^j_{n_1m_2;r_1s_2}\Big)(\phi_\alpha)^j_{m_1m_2}(\phi_\alpha)^j_{n_1n_2}(\phi_\alpha)^j_{r_1r_2}(\phi_\alpha)^j_{s_1s_2}, \tag{8.37}$$

$$\begin{aligned}\Gamma^1_{24}\sim\sum\Big(&\sum_p w^2(j)P^j_{m_1p;s_1r_2}P^j_{pn_2;r_1s_2}\delta_{m_2n_1}\Big)\\ &\times(\phi_\alpha)^j_{m_1m_2}(\phi_\alpha)^j_{n_1n_2}(\phi_\alpha)^j_{r_1r_2}(\phi_\alpha)^j_{s_1s_2},\end{aligned} \tag{8.38}$$

$$\begin{aligned}\Gamma^1_{34}\sim\sum\Big(&\sum_{p,q} w^2(j)P^j_{pm_2;qs_2}P^j_{n_1p;s_1q}\delta_{m_1n_2}\delta_{s_2r_1}\Big)\\ &\times(\phi_\alpha)^j_{m_1m_2}(\phi_\alpha)^j_{n_1n_2}(\phi_\alpha)^j_{r_1r_2}(\phi_\alpha)^j_{s_1s_2},\end{aligned} \tag{8.39}$$

where obvious summations are not explicitly written. By further performing the summations over p and q in thanks to the delta functions in the propagators $P^j_{mn;kl}$, we arrive easily at the following estimates:

$$
\begin{aligned}
w^2(j)P^j_{m_1n_2;s_1r_2}P^j_{n_1m_2;r_1s_2} &\leq \Pi(M,j)^2\delta_{m_1r_2}\delta_{n_2s_1}\delta_{n_1s_2}\delta_{m_2r_1} && (8.40)\\
\sum_p w^2(j)P^j_{m_1p;s_1r_2}P^j_{pn_2;r_1s_2} &\leq \Pi(M,j)^2\delta_{m_1r_2}\delta_{r_1n_2} && (8.41)\\
\sum_{p,q} w^2(j)P^j_{pm_2;qs_2}P^j_{n_1p;s_1q} &\leq \Pi(M,j)^2\delta_{s_1s_2}\delta_{m_2n_1}, && (8.42)
\end{aligned}
$$

leading to finite non-planar contributions to the vertex functional . A similar conclusion holds true for the other non-planar contribution. Notice, by the way that the RHS of each of the relations decay to zero as j^{-4} for $j\to\infty$. As for the 2-point function, the diagram amplitudes for the 4-point function are finite, thanks to the existence of the bound for the propagator together with the fact that loop summation indices are bounded by $\pm j$. Summarizing the above 1-loop analysis, a simple inspection shows that no singularity can occur for $j=0$ within the present model (recall $M>0$) while the only source for divergence might come from the limit $j\to\infty$. But such divergences are prevented to occur thanks to the upper bound and the decay of $\Pi(M,j)$ at large j, namely $\Pi(M,j)\sim j^{-2}$ for $j\to\infty$ so that the model is finite at the one-loop order. In the next subsection, we will show that this property extends to any order of perturbation.

8.4 Finitude of the diagram amplitudes to all orders

We first observe that is related obviously to the propagator for the "truncated" gauge model obtained by simply dropping the field Φ_3 in the action . Notice that this latter *formally* may be viewed as resulting from the gauge choice $\Phi_3=0$ in instead of $\Phi_3=\theta_3$. For convenience, we quote here the expression for the propagator of the truncated theory which can be simply read off from the RHS of

$$(G^{-1})^{j_1j_2}_{mn;kl}=\delta^{j_1j_2}\delta_{mn}\delta_{kl}\frac{\Pi(M,j_1)}{w(j_1)} \tag{8.43}$$

which depends only on a single $j\in\frac{\mathbb{N}}{2}$, says j_1.
The "truncated model" belongs to one particular class of NCFT on $\mathbb{R}^3_\lambda$ among those which have been investigated in where it was shown that the models in this class are finite to all orders in perturbation. We first discuss useful property of this model. The key observation is that the amplitude of any ribbon diagram depends only on one $j\in\frac{\mathbb{N}}{2}$. Indeed, observe e.g the $\delta^{j_1j_2}$ in the propagator plus its j-dependence and the delta functions in any quartic vertex. These $\delta^{j_mj_k}$'s all boil down to a single one in the computation of any amplitude.
Since the propagator depends on the bounded indices $m,n,...$ only through Kronecker delta's, the summations over the indices of any loop can be exactly carried out so that any ribbon loop contributes to a factor

$$(2j+1)^\varepsilon,\ \varepsilon\leq 2 \tag{8.44}$$

to a given amplitude. This can be understood from a simple inspection of the Kronecker delta's and the summations over the indices for a ribbon loop built from any N-point sub-diagram $\mathcal{A}_{m_1,n_1,...,m_N,n_N}$ and a propagator that can be taken to be $(Q^{-1})^j_{m_1n_1;m_2n_2}$ without loss of generality. Namely, one has

$$\mathbb{A}_{m_3,n_3,...,m_N,n_N}=\sum_{-j\leq m_1,n_1,m_2,n_2\leq j}\mathcal{A}_{m_1,n_1,...,m_N,n_N}(Q^{-1})^j_{m_1n_1;m_2n_2}. \tag{8.45}$$

There are 4 summed (internal) indices related to the product of N delta's coming from the N-point sub-diagram by the 2 delta's of the propagator depending only on internal indices. Two summations can be trivially performed leading to N remaining delta functions. There are a priori 3 possibilities depending how the 2 remaining summed indices are distributed among the delta's: either a single delta depends only on one internal index, or one get a product of two such deltas, one of each internal index, or the 2 summations combine 2 deltas among the N one leading to $N-2$ remaining deltas. The details are given in the appendix . Notice that the value $\varepsilon=2$ is obtained from purely algebraic and combinatorial arguments and represents actually the maximal power of the factor $2j+1$ any loop can contribute. A refinement of this analysis by taking into account indices conservation may well lower the maximal value of this exponent by one unit. Nevertheless, it turns out that the use of this somewhat crude maximal value in the ensuing analysis is sufficient to prove the finitude of arbitrary amplitudes. Summarizing the above discussion, it appears that the loop summations decouple from the related propagators in the computation of diagram amplitudes for the truncated model, so that any loop simply contribute by a power of $(2j+1)$ given by . This leads to a major simplification in the analysis of amplitudes of arbitrary order, as it will be shown in a while.

To end up with perturbative considerations within the truncated model, consider now a general ribbon diagram $\mathcal{D}$ related to this model[2] . Any ribbon diagram built from the quartic vertices is characterized by a set of positive integer (V, I, F, B). V is the number of vertices, I the number of internal ribbons. F is the number of faces. Recall that F is obtained by closing the external lines of a diagram and counting the number of closed *single* lines. Finally, B is the number of boundaries which is equal to the number of closed lines with external legs. The number of ribbon loops if given by

$$\mathcal{L} = F - B. \tag{8.46}$$

Let $g \in \mathbb{N}$ be the genus of the Riemann surface on which $\mathcal{D}$ can be drawn. Recall that g is determined by the following relation

$$2 - 2g = V - I + F. \tag{8.47}$$

Now consider the amplitude $\mathbb{A}^{\mathcal{D}}$ for a diagram characterized by the parameters (V, I, F, B). It is a (positive) function of j, obviously finite and non singular for $j = 0$, built from the product of V vertex factors, each vertex contributing to $w(j)$ up to unessential finite factor, I propagators with summations over indices corresponding to $F - B$ loops which, by the decoupling argument discussed above, give a net overall factor bounded by $(2j+1)^{2(F-B)}$. Therefore, we can write

$$\mathbb{A}^{\mathcal{D}} \leq K w(j)^{V-I} \Pi(M, j)^I (2j+1)^{2(F-B)} = K' \frac{w(j)^{V-I}(2j+1)^{2(F-B)}}{(j^2+\rho^2)^I} \tag{8.48}$$

where K and K' are finite constants and $\rho^2 = \frac{M}{\lambda \mu^2}$ and we have isolated the factor $w(j)$. Recall that the choice $w(j) = j + 1$ as given in leads to a trace reproducing at the formal commutative limit the expected behavior for the usual integral on $\mathbb{R}^3$. The natural choice $w(j) = 1$ is related to a functional trace built from all the canonical traces of the components $\mathbb{M}_{2j+1}(\mathbb{C})$ occurring in the decomposition of $\mathbb{R}^3_\lambda$, . To study both cases when taking the $j \to \infty$ of the RHS of

$$w(j) \sim j^\alpha, \ \alpha = 0, 1, \ \text{for } j \to \infty. \tag{8.49}$$

The RHS of is always finite for $j = 0$ while it is also finite for $j \to \infty$ provided

$$\omega(\mathcal{D}) = \alpha I + 2B + 2(2g - 2) + V(2 - \alpha) \geq 0, \tag{8.50}$$

where we used and one has still $\alpha = 0, 1$. For $g \geq 1$, one has $\omega(\mathcal{D}) > 0$. The case $g = 0$, for which the finitude condition becomes $\omega(\mathcal{D}) = \alpha I + 2B + V(2 - \alpha) - 4 \geq 0$ requires a closer analysis. In fact, when $V = 2$ a simple inspection shows that holds true for $\alpha = 0, 1$. The case $V = 1$ corresponds to the 2-point function for the truncated model whose finitude when $j \to \infty$ is almost apparent from the rightmost quantity in . Note that this can be obtained from simple topological consideration for the planar and non planar contributions to this 2-point function. One obtains $B = 2$ and $B = 1$ respectively so that holds true whenever $V = 1$ for $\alpha = 0, 1$. Summarizing the above analysis, we conclude that the truncated model in finite to all orders in perturbation.
Let us go back to the gauge model . As far as finitude of the diagrams is concerned[3] one observes that differs from the truncated model only through the propagator. Hence, for a given diagram $\mathcal{D}$, the amplitude computed within the gauge model $\mathfrak{A}^j_{\mathcal{D}}$ satisfies

$$|\mathfrak{A}^j_{\mathcal{D}}| \leq |\mathbb{A}^j_{\mathcal{D}}|, \tag{8.51}$$

thanks to the estimate . Indeed, by using the general expression for any ribbon amplitudes of NC ϕ^4 theory, one infers $\mathfrak{A}^j_{\mathcal{D}}$ has the generic structure

$$\mathfrak{A}^j_{\mathcal{D}} = \sum_{\mathcal{I}} \prod_{\lambda} P^j_{m_\lambda(\mathcal{I}) n_\lambda(\mathcal{I}); k_\lambda(\mathcal{I}) l_\lambda(\mathcal{I})} F^j(\delta)_{m_\lambda(\mathcal{I}) n_\lambda(\mathcal{I}); k_\lambda(\mathcal{I}) l_\lambda(\mathcal{I})}, \tag{8.52}$$

where $\mathcal{I}$ is some set of (internal) indices, all belonging to $\{-j, ...j\}$ so that all the sums in $\sum_{\mathcal{I}}$ are finite, λ labels the internal lines of $\mathcal{D}$, $P^j_{mn;kl}$ is the (positive) propagator given in and $F^j(\delta)_{mn;kl}$ collects all the delta's plus vertex weights depending only on j. One has

$$|\mathfrak{A}^j_{\mathcal{D}}| \quad \leq \quad \sum_{\mathcal{I}} \prod_{\lambda} \left| (G^{-1})^j_{m_\lambda(\mathcal{I}) n_\lambda(\mathcal{I}); k_\lambda(\mathcal{I}) l_\lambda(\mathcal{I})} \right| \, \left| F^j(\delta)_{m_\lambda(\mathcal{I}) n_\lambda(\mathcal{I}); k_\lambda(\mathcal{I}) l_\lambda(\mathcal{I})} \right|. \tag{8.53}$$

From , one then obtains

$$|\mathfrak{A}^j_{\mathcal{D}}| \leq K' \frac{w(j)^{V-I}(2j+1)^{2(F-B)}}{(j^2+\rho^2)^I} < \infty \tag{8.54}$$

where the last inequality stems from which has been shown to hold true.
One concludes that all the ribbon amplitudes stemming from are finite so that $S^f_{\Omega=1}$ is perturbatively finite to all orders.

[2]Recall that any ribbon in such a diagram is made of two lines each carrying 2 bounded indices, says $m, n \in \{-j, ..., j\}$. Thus, a ribbon carries 4 bounded indices (as the propagator . Notice that there is a conservation of the indices along each line, as it can be seen by observing the delta function in the expression of the propagator , each delta defining the indices affected to one line. For more details, see .

[3]We consider only the finitude of the loop contributions and not the nature of the various vertices generated by loop corrections (i.e external legs) which simply amounts to analyze planar and non-planar contribution for a ϕ^4 theory either with propagator or with

Conclusion

Acknowledgements

(blablabla)

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