Dimensional regularisation on curved spacetime

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joint work with Thomas-Paul Hack and Nicola Pinamonti

`to appear soon on arXiv`

Motivation

- pertubative algebraic quantum field theory (pAQFT)

  $\rightarrow$ **conceptually well known**

  [Brunetti, Dütsch, Fredenhagen, Hollands, Köhler, Rejzner, Wald, ... $\sim$1996-2013]

- in pAQFT on curved spacetime (CST), regularisation uses ideas of Epstein and Glaser

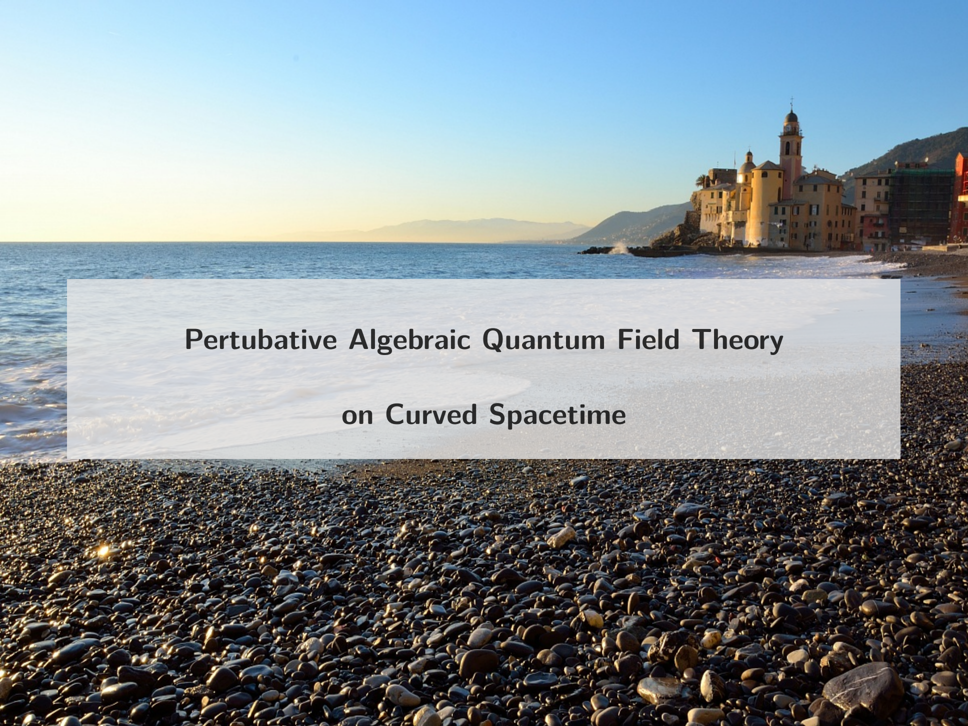
  $\rightarrow$ procedure **unconvenient for computations**

  [Brunetti & Fredenhagen 2000, Hollands & Wald 2002, Dang 2013]

- desire to use framework of pAQFT for **cosmological model**!

What I am going to talk about

- **pAQFT** $\rightarrow$ in order to identify the regularisation problem!
- a framework for a **dimensional regularisation** on CST
  $\rightarrow$ a general procedure more computationally friendly!
- explicit computations on spatially flat Friedmann Lemaître Robertson Walker spacetimes
  $\rightarrow$ finally a **bit of practice**!



Pertubative Algebraic Quantum Field Theory

on Curved Spacetime

Functional Approach

- $(\mathcal{M}, \mathrm{g})$: **4 dimensional** globally hyperbolic spacetime
- **off shell** configuration space $\rightarrow$ scalar field $\phi \in \mathcal{C}^{\infty}(\mathcal{M}, \mathbb{R})$
- $\mathcal{F}$: **space of observables** $\mathsf{F} : \begin{cases} \mathcal{C}^{\infty}(\mathcal{M}, \mathbb{R}) & \rightarrow & \mathbb{C} \\ \phi & \mapsto & \mathsf{F}(\phi) \end{cases}$

  Spacetime support of F

$$\mathrm{supp}(\mathsf{F}) \doteq \left\{ x \in \mathcal{M} \middle| \begin{array}{l} \forall \text{ neighborhood } U \text{ of } x,\ \exists\ \phi, \psi \in \text{ smooth,} \\ \mathrm{supp}(\psi) \subset U, \text{ such that } \mathsf{F}(\phi + \psi) \neq \mathsf{F}(\phi). \end{array} \right\}$$

wave front set

- **Regular** functionals $\mathcal{F}_{\text{reg}}$

$$\mathcal{F}_{\text{reg}} = \left\{ \mathsf{F} \mid \mathsf{F} \text{ smooth}, \mathsf{F}^{(n)} \text{ comp. sup., and } \mathrm{WF}(\mathsf{F}^{(n)}) = \emptyset \right\}$$

- **Microcausal** functionals $\mathcal{F}_{\mu c}$

$$\mathcal{F}_{\mu c} = \left\{ \mathsf{F} \mid \mathsf{F} \text{ smooth}, \mathsf{F}^{(n)} \text{comp. sup.}, \mathrm{WF}(\mathsf{F}^{(n)}) \cap \left(\mathcal{M}^n \times (\overline{V_+^n} \cup \overline{V_-^n}) \right) = \emptyset \right\}$$

$\rightarrow$ local interactions are a subset $\mathcal{F}_{\text{loc}} \subset \mathcal{F}_{\mu c}$

Interactions $\rightarrow$ **Local** functionals $\mathcal{F}_{\text{loc}}$

$$\mathcal{F}_{\text{loc}} = \left\{ \mathsf{F} \in \mathcal{F}_{\mu c} \mid \mathrm{supp}(\mathsf{F}^{(n)}) \subset \{(x, \ldots, x) \subset \mathcal{M}^n\} \right\}$$

Example: $\mathsf{F} \in \mathcal{F}_{\text{loc}}(\mathcal{M})$

$$\mathsf{F}(\phi) = \int_{\mathcal{M}} \mathrm{d}\mu \; f(x) \; \frac{\lambda}{4!} \phi(x)^4 \;, \text{ with } f \in \mathcal{C}_0^\infty(\mathcal{M}, \mathbb{R})$$

Poducts of Observables

"Deformation" of the **pointwise product** on $\mathcal{F}_{\mathrm{T}} \subset \mathcal{F}_{\mu c}$:

$\rightarrow$ $\star_{\mathrm{H}}$ product, defined with H_{+} (implement "quantum structure")

$\rightarrow$ $\cdot_{\mathrm{T}}$ time ordering product, defined with H_{f}

$$\mathrm{F} \cdot_{\mathrm{T}_{\mathrm{H}}} \mathrm{G} = \mathrm{F} \star_{\mathrm{H}} \mathrm{G}, \text{ if } \mathrm{supp}(\mathrm{F}) \geq \mathrm{supp}(G)$$

on a **normal convex neighborhood** of x

$$\mathrm{H}_{\sharp}(x,y) \doteq \lim_{\epsilon \downarrow 0} \frac{1}{8\pi^2} \left(\frac{u(x,y)}{\sigma_{\sharp}(x,y)} + v(x,y) \log(M^2 \sigma_{\sharp}(x,y)) + w(x,y) \right)$$

$$\sigma_{\pm}(x,y) = \sigma(x,y) \pm i\epsilon(t_x - t_y) + \epsilon^2 \qquad \sigma_{\mathrm{f}}(x,y) = \sigma(x,y) + i\epsilon$$

u, v, w : smooth, and $2\,\sigma$: squared geodesic distance.

$$(\mathrm{F} \sharp \mathrm{G})(\phi) \doteq \mathrm{F}(\phi) \cdot \mathrm{G}(\phi) + \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} \left\langle \mathrm{F}^{(n)}, \mathrm{H}_{\sharp}^{\otimes n} \mathrm{G}^{(n)} \right\rangle$$

details

Algebra of Observables

- algebraic structure, with 2 products

$$\mathcal{A}_{\mathrm{H}} = (\mathcal{F}_{\mathrm{T}}, \star_{\mathrm{H}}, \cdot_{\mathrm{T_H}})$$

- We work with interaction functionals!

  for $\mathrm{F} \in \mathcal{A}_{\mathrm{H}}$, $\quad \alpha_{\mathrm{H}_+}(\mathrm{F}) = e^{-\hbar \Gamma_{\mathrm{H}_+}} \mathrm{F}$, $\quad$ with $\Gamma_{\mathrm{H}_+} = \left\langle \mathrm{H}_+, \frac{\delta^2}{\delta \phi^2} \right\rangle$

  e.g. $\alpha_{\mathrm{H}_+}(\phi^2(x)) = \lim\limits_{x \to y} \left(\phi(x)\phi(y) - \mathrm{H}_+(x, y)\right)$

- maps of the products

  for $F, G \in \mathcal{F}_{\mathrm{T}}$, defining $d = \mathrm{H}'_+ - \mathrm{H}_+ = \mathrm{H}'_{\mathrm{f}} - \mathrm{H}_{\mathrm{f}} \simeq \frac{1}{8\pi}(w' - w)$

$$\begin{aligned} \mathrm{F} \star_{\mathrm{H}'} \mathrm{G} &= \alpha_d \left(\alpha_{-d}(\mathrm{F}) \star_{\mathrm{H}} \alpha_{-d}(\mathrm{G})\right) \\ \mathrm{F} \cdot_{\mathrm{T}_{\mathrm{H}'}} \mathrm{G} &= \alpha_d \left(\alpha_{-d}(\mathrm{F}) \cdot_{\mathrm{T_H}} \alpha_{-d}(\mathrm{G})\right) \end{aligned}$$

$\mathcal{A}_{\mathrm{H}'} \sim \mathcal{A}_{\mathrm{H}}$, the two algebras are isomorphic

$\Rightarrow$ thus we will choose a particular representation $\mathcal{A}_{\mathrm{H}}$

Pertubative picture

- **Local S matrix**: $\mathsf{S}(F) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \underbrace{\mathsf{F} \cdot_{\mathsf{T}} \dots \cdot_{\mathsf{T}} \mathsf{F}}_{n}$

Microlocal Analysis – Important property *[Hörmander 1983]*

$u, v \in \mathcal{D}'$,

$$\mathrm{WF}(u) \oplus \mathrm{WF}(v) \not\ni \{0\} \Rightarrow \exists!\ u.v \in \mathcal{D}'$$

- **Bogoliubov formula** *[Brunetti, Fredenhagen 2009]*

$$\mathsf{S}_{\mathsf{v}}(\mathsf{F}) = \mathsf{S}(\mathsf{V})^{\star -1} \star (\mathsf{S}(\mathsf{V}) \cdot_{\mathsf{T}} \mathsf{F})$$

$\rightarrow$ transition from the free action to the one with additional interaction term V.

Computational trick

for computational convenience, useful to work with $\mathcal{A}_{\Delta_+}$

$\rightarrow \Delta_+$: the Wightman 2pt of the free field state Ω_0, $d = \Delta_+ - \mathrm{H}_+$

Framework for a Dimensional Regularisation on Curved Spacetime

The regularisation problem

Causality: $\mathrm{F} \cdot_{\mathrm{T}} \mathrm{G} = \mathrm{F} \star \mathrm{G}$, if $\mathrm{supp}(\mathrm{F}) \geq \mathrm{supp}(G)$

$\rightarrow \mathrm{H_f}(x, y)^n$ ill defined if $x = y$

Epstein Glaser induction *[Brunetti, Fredenhagen 2000]*

▸ Sketch of the proof

if $\mathrm{F}_1, \ldots, \mathrm{F}_n \in \mathcal{F}_{\mathrm{loc}}(\mathcal{M}^n)$,

then $\mathrm{F}_1 \cdot_{\mathrm{T}} \ \ldots \ \mathrm{F}_n$ can be defined up to $\mathrm{Diag}(\mathcal{M}^n) = \{\mathbf{x} \in \mathcal{M}^n | x_1 = \cdots = x_n\}$

Scaling degree: For $t \in \mathcal{D}'(\mathbb{R}^4)$ *[Steinmann 1971]*

$$\mathrm{sd}(t) = \inf\{\omega \in \mathbb{R} \mid \lim_{\rho \to 0} \rho^\omega t(\rho x) = 0\}$$

"A divergence criterion", $t \in \mathcal{D}'(\mathbb{R}^4 \backslash \{0\})$ *[Brunetti, Fredenhagen 2000]*

- **if** $\mathrm{sd}(t) < 4$,

  then t has a unique extension $\dot{t} \in \mathcal{D}'(\mathbb{R}^4)$ with the same scaling degree
- **for** $4 \leq \mathrm{sd}(t) < \infty$, $\dot{t}$ has non unique extensions (with same sd)

The reason of our problem:

pointwise powers of Δ_f: ill defined, because contain σ_f^{-n} with n too large.

Fix x and set the distribution $t(y) = \sigma_f^{-n}(x, y)$,

then for $x \to y$, $\mathrm{sd}(t(y)) = 2n$.

$\Rightarrow$ $t(y)$ has an unique extension only for $n < 2$

"Main steps of the recipe"

(i) Deform n to $n + \alpha$, with $\alpha \in \mathbb{C}$

Result: the map $\alpha \mapsto \sigma_f^{-(n+\alpha)}$ is meromorphic in α.

▸ Sketch of the proof

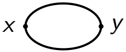
(ii) compute the **Laurent series** with respect to α

$\to$ play with the identities fulfilled by σ

(iii) subtract the principal part

(iv) take the limit $\alpha \to 0$

The fish diagram (2 vertices)



$$\longrightarrow \quad \Delta_{\mathrm{f}}^2(x,y) = \frac{1}{8\pi^2}\left(\frac{u^2(x,y)}{\sigma_{\mathrm{f}}^2(x,y)} + \text{“well defined for } x = y\text{”}\right)$$

- regularize only $\sigma_{\mathrm{f}}^{-(2+\alpha)}$
- use σ identities : $\quad \Box\sigma = 4 + f\sigma$
- $\alpha \mapsto \sigma_{\mathrm{f}}^{-(2+\alpha)}$ (weakly) meromorphic in α.

  $\rightarrow$ Laurent series w.r.t α

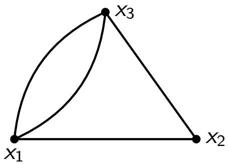
  $\rightarrow$ subtract the principal part and take the limit $\alpha \rightarrow 0$

$$\left(\frac{1}{\sigma_{\mathrm{f}}^2}\right)_{\mathrm{reg}} = \lim_{\alpha\to 0}\,(1 - \mathrm{pp})\,\frac{1}{\sigma_{\mathrm{f}}^{2+\alpha}}$$

$$\Longrightarrow \quad \left(\Delta_{\mathrm{f}}^2\right)_{\mathrm{reg}}$$

▸ details

General case (N vertices) I



$\mathcal{G}$: graph with N vertices $x_1, \ldots, x_N$

x_k: a reference point arbitrarily chosen

$$\sigma_{ij} = \sigma_{\mathrm{f}}(x_i, x_j), \quad \sigma_i = \sigma_{\mathrm{f}}(x_k, x_j), \quad \sigma_{ij}^a = \nabla_i^a \sigma_{ij}$$

$$\Rightarrow \; t^\alpha(x_1, \ldots, x_N) \; = \; \prod_{i<j} \frac{1}{\sigma_{ij}^{n_{ij}+\alpha_{ij}}}, \quad \alpha_{ij} \in \mathbb{C}$$

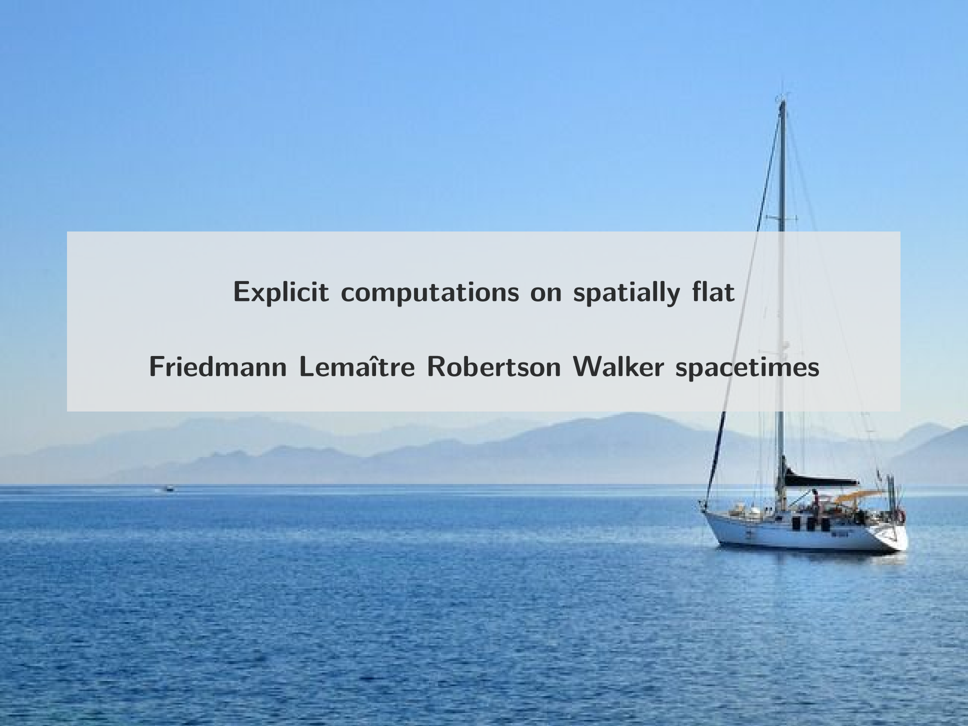
- Need a way to isolate the poles in $\{\alpha\}$

$$\mathrm{R} \doteq \nabla_a^i \sigma_i^a \;\rightarrow\; \text{“analogue” of } \Box\sigma + \ldots \text{ that we use in the case } N = 2$$

$$\mathrm{t}^\alpha \;=\; \frac{1}{\mathcal{P}(N, n_{ij}, \alpha_{ij})} \left(\mathrm{R}^N \, \mathrm{t}^\alpha + \text{ “corr.” } \right)$$

General case (N vertices) II

- then **apply** R as many times as **needed**, to have **sufficiently small** sd
- the terms in $\log(\sigma)$ obtained by **deriving** w.r.t α_{ij}
- subtract the pole **order by order** as explained or use the **forest formula** *[Keller 2010, Dütsch, Fredenhagen, Keller, Rejzner 2013]*
- **if** $(\mathcal{M}, g)$ has isometries and Δ_{f} (i.e. the state Ω_0) is invariant w.r.t. these, **then** the regularisation scheme preserves this invariance



Explicit computations on spatially flat Friedmann Lemaître Robertson Walker spacetimes

- **spatially flat Friedmann Lemaître Robertson Walker** spacetimes
$$g = a(\tau)^2 \ \left(-d\tau^2 + d\vec{x}^2\right)$$
- choosing a **free field state** Ω_0 (inv. under FLWR sym.) of the quantized free Klein Gordon field on spatially flat FLWR spacetimes

  $\Delta_\pm$, Δ_f : can be written using **spatial Fourier transform** in terms of **temporal modes** $\chi_k(\tau)$
$$\left(\partial_\tau^2 + k^2 + m^2 a^2 + \left(\xi - \frac{1}{6}\right) R a^2\right) \chi_k(\tau) = 0$$
$$\chi_k \partial_\tau \overline{\chi_k} - \overline{\chi_k} \partial_\tau \chi_k = i$$
- **in principle** we can compute all quantities in terms of $\vec{k}$- and τ-integrals using our regularisation procedure

The conformal trick

- Even for spatially flat FLRW spacetimes,

  σ, u, v are **not explicitly known** neither in **position space**, nor in $\vec{k}$**-space**,

  $\rightarrow$ but we need to know the $\vec{k}$-space version of e.g $\left(\Delta_{\mathrm{f}}^{2}\right)_{\mathrm{reg}}$

- **Trick**: $\Delta_{\mathrm{f}} = \Delta_{\mathrm{f},0} + \delta\Delta_{\mathrm{f}}$, with

  $\rightarrow$ $\Delta_{\mathrm{f},0}$ contains sufficiently many singular contributions

  $\rightarrow$ $\Delta_{\mathrm{f},0}$ explicitly known in position and $\vec{k}$-space

  $\rightarrow$ $(\Delta_{\mathrm{f},0})^{n}$ can be regularized using our procedure

- $\Delta_{\mathrm{f},0} \to$ the **Feynman propagator** of the **massless**, **conformally coupled** ($\xi = \frac{1}{6}$) **Klein Gordon field** in the conformal vacuum state

$$\Delta_{\mathrm{f},0}(x_1, x_2) = \frac{1}{8\pi^2 a(\tau_1) a(\tau_2)} \frac{1}{\sigma_{\mathbb{M}}(x_1, x_2) + i\epsilon}$$

- Fish diagram

$$(\Delta_{\mathrm{f}})^2_{\mathrm{reg}} = (\Delta_{\mathrm{f},0})^2_{\mathrm{reg}} + 2\delta\Delta_{\mathrm{f}}\Delta_{\mathrm{f},0} + (\delta\Delta_{\mathrm{f}})^2$$

$$\begin{aligned} (\Delta_{\mathrm{f},0})^2_{\mathrm{reg}} &= \lim_{\alpha\to 0} (1 - \mathrm{pp}) \frac{1}{M^{2\alpha}} (\Delta_{\mathrm{f},0})^{2+\alpha} \\ &= -\frac{1 + 2\log(a)}{16\pi^2 a^4} i\delta_{\mathbb{M}} - \frac{1}{2(8\pi^2)^2 a^2 \otimes a^2} (\Box_{\mathbb{M}} \otimes 1) \frac{\log M^2 \sigma_{\epsilon,\mathbb{M}}}{\sigma_{\epsilon,\mathbb{M}}} \end{aligned}$$

and then compute the Fourier transform . . .

Summary

BEFORE

$\rightarrow$ conceptual well understanding of pAQFT on CST

Problem $\rightarrow (\Delta_\mathrm{f})^n, \ldots$

regularisation procedure $\rightarrow (\Delta_\mathrm{f})^n \simeq (\sigma_\mathrm{f})^{-n} + \ldots$

with $(\sigma_\mathrm{f}^{-n})_\mathrm{reg} = \lim_{\alpha \to 0} (1 - \mathrm{pp})\, (\sigma_\mathrm{f})^{-(n+\alpha)}, \ \alpha \in \mathbb{C}$

NOW

$\rightarrow$ computations accessible!

Danke schön.

The Wave Front Set

$u : \mathcal{C}_0^\infty(\mathbb{R}^n) \to \mathbb{C}$: **distribution** $(u \in \mathcal{D}'(\mathbb{R}^n))$

Singular support: $\mathrm{singsupp}(u) \doteq \{x \in \mathbb{R}^n | \nexists U_x \ni x \text{ s.t. } u|_{U_x} \in \mathcal{C}^\infty(U_x)\}$

Example: $\mathrm{singsupp}(\delta) = \{0\}$

Wave front set:

$\mathrm{WF}(u) \doteq \{(x,k) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\}) \mid x \in \mathrm{singsupp}(u) \text{ and } k \in \Sigma_x(u)\}$

with $\Sigma_x(u) \doteq \{k \in \mathbb{R}\backslash\{0\} \text{ s.t. } |\hat{u}(\phi)| \text{ does not decay rapidly in direction } k\}$

Example: $\mathrm{WF}(\delta) = \{(0,k) | k \in \mathbb{R}^n, k \neq 0\}$

Microlocal analysis: $WF(u) \oplus WF(v) \neq 0 \Rightarrow \exists!\ u.v \in \mathcal{D}'(\mathbb{R}^n)$.

$(u, v \in D'(\mathbb{R}^n))$ P diff. op. $\Rightarrow \mathrm{WF}(\mathrm{P}u) \subset \mathrm{WF}(u)$

Example: $(0,k)$ and $(0,-k) \in \mathrm{WF}(\delta) \Rightarrow$ powers of δ cannot be defined

$\mathrm{P}\Delta_{\mathrm{f}} = \delta \Rightarrow \mathrm{WF}(\mathrm{P}\Delta_{\mathrm{f}}) = \mathrm{WF}(\delta) \subset \mathrm{WF}(\Delta_{\mathrm{f}})$

back

$\mathrm{F}, \mathrm{G} \in \mathcal{F}_{\mathrm{T}}(\mathcal{M})$

The $\star$ product

$$(\mathrm{F} \star \mathrm{G})(\phi) = \mathrm{F}(\phi) \cdot \mathrm{G}(\phi) + \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} \left\langle \mathrm{F}^{(n)}, \mathrm{H}_+^{\otimes n} \mathrm{G}^{(n)} \right\rangle$$

$$\mathrm{WF}(\mathrm{H}_+) = \left\{ (x, k_x; y, k_y) \in T^*\mathcal{M}^2 \backslash \{0\} \mid (x, k_x) \sim (x, k_y),\ \ k_x \in (\overline{\mathrm{V}}_+)_x \right\}$$

$\rightarrow$ powers of H_+ well defined

The time ordered product

$$(\mathrm{F} \cdot_{\mathrm{T}} \mathrm{G})(\phi) = \mathrm{F}(\phi) \cdot \mathrm{G}(\phi) + \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} \left\langle \mathrm{F}^{(n)}, \mathrm{H}_{\mathrm{f}}^{\otimes n} \mathrm{G}^{(n)} \right\rangle$$

back

back

Induction Basis: For F, G $\in \mathcal{F}_{\mathrm{T}}(\mathcal{M})$ let

$$\mathrm{T}_0(\mathrm{F}) = \mathbb{I}, \qquad \mathrm{T}_1(\mathrm{F}) = \mathrm{F}, \qquad \mathrm{T}_2(\mathrm{F} \otimes \mathrm{G}) = \mathrm{F} \cdot_{\mathrm{T}} \mathrm{G}$$

Induction Hypothesis: Let $\forall k < n$ the maps T_k

- be well defined on the whole $\mathcal{M}^k$
- be symmetric
- fulfill following condition:

  Let $I \subset \{1, \dots, k\}$ with complement $I^c \neq \emptyset$, and $\mathrm{F}_1, \dots \mathrm{F}_k \in \mathcal{F}_{\mathrm{T}}(\mathcal{M})$

  Then $\forall i \in I$, $\forall j \in I^c$, $\mathrm{supp}(F_i) \cap \mathrm{supp}(F_j) = \emptyset$

  it follows that $\mathrm{T}(\mathrm{F}_1 \otimes \cdots \otimes \mathrm{F}_n) = \mathrm{T}\left(\bigotimes_{i \in I} F_i\right) \cdot_{\mathrm{T}} \mathrm{T}\left(\bigotimes_{j \in I^c} F_j\right)$

Lemma: Let T_k fulfill the induction hypothesis $\forall k < n$,
then T_n is uniquely defined for all functionals $\sum \mathrm{F}_1 \otimes \dots \mathrm{F}_n \in \mathcal{F}_{\mathrm{T}}(\mathcal{M})$ with

$$\mathrm{supp}\left(\sum \mathrm{F}_1 \otimes \dots \mathrm{F}_n\right) \cap \mathrm{Diag}(\mathcal{M}^n) = \emptyset$$

Sketch of the proof of the meromorphicity in α

- **meromorphicity** in α of $\alpha \mapsto \left\langle \sigma^{-(n+\alpha)}, \phi \right\rangle$ **?**

  with ϕ smooth and comp. supp.
- $\sigma \to \sigma_{\mathrm{f}} = \sigma + i\epsilon \;\; : \;\; \sigma_{\mathrm{f}}^{-(n+\alpha)} = \exp\Big(-(n+\alpha)\, \log(\sigma + i\epsilon) \Big)$
- $\alpha = a + ib, \quad a, b \in \mathbb{R}$
- $\sigma_{\mathrm{f}}^{-(n+\alpha)} = \sigma_{\mathrm{f}}^{-(n+a)} \, \sigma_{\mathrm{f}}^{-ib}$,

$$\frac{\partial}{\partial a} \left\langle \sigma_{\mathrm{f}}^{-(n+\alpha)}, \phi \right\rangle \stackrel{?}{=} \left\langle \frac{\partial}{\partial a} \sigma_{\mathrm{f}}^{-(n+\alpha)}, \phi \right\rangle$$
$$\frac{\partial}{\partial b} \left\langle \sigma_{\mathrm{f}}^{-(n+\alpha)}, \phi \right\rangle \stackrel{?}{=} \left\langle \frac{\partial}{\partial b} \sigma_{\mathrm{f}}^{-(n+\alpha)}, \phi \right\rangle$$

$\rightarrow$ **equalities hold!**

- **Cauchy Riemann equations hold!**

$$\Rightarrow \;\; \alpha \mapsto \left\langle \sigma_{\mathrm{f}}^{-(n+\alpha)}, \phi \right\rangle \;\; \textbf{meromorphic in} \;\; \alpha$$

back

Fish diagram

- Let consider

$$t^{\alpha}(x,y) = \frac{1}{\sigma_{\mathrm{f}}(x,y)^{2+\alpha}}$$

- identities fulfilled by σ

$$\Box_x \sigma \;=\; 4 + f\sigma,$$

then

$$\frac{1}{\sigma_{\mathrm{f}}^{2+\alpha}} = \frac{1}{2\alpha(1+\alpha)}\left(\Box_x + (1+\alpha)f\right)\frac{1}{\sigma_{\mathrm{f}}^{1+\alpha}}$$

- Thus

$$\left(\frac{1}{\sigma_{\mathrm{f}}^2}\right)_{\mathrm{reg}} \doteq \lim_{\alpha\to 0}\left(1-\mathrm{pp}\right)\frac{1}{M^{2\alpha}}\frac{1}{\sigma_{\mathrm{f}}^{2+\alpha}} = -\frac{1}{2}(\Box_x + f)\frac{\log M^2\sigma_F}{\sigma_{\mathrm{f}}} - \Box_x\frac{1}{2\sigma_{\mathrm{f}}}$$

back