Perturbatively finite gauge models on the noncommutative three-dimensional space $\mathbb{R}^3_\lambda$

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Plan

- **Noncommutative space $\mathbb{R}^3_\lambda$**
- **Family of gauge invariant action**
- **Finiteness to all orders**
- **Link to exactly solvable models**

Noncommutative space $\mathbb{R}^3_\lambda$

$$\boxed{\mathbb{R}^3_\lambda = \mathbb{C}\left[x_1, x_2, x_3, x_0\right] \setminus \mathcal{I}\left[\mathcal{R}_1, \mathcal{R}_2\right]}$$

- $\mathbb{C}\left[x_1, x_2, x_3, x_0\right]$: free algebra generated by coordinates x_1, x_2, x_3 and x_0
- $\mathcal{I}\left[\mathcal{R}_1, \mathcal{R}_2\right]$: two sided ideal generated by the relations

$$\mathcal{R}_1 : [x_\mu, x_\nu] = i\lambda\epsilon_{\mu\nu\rho}x_\rho \qquad \mathcal{R}_2 : x_0^2 + \lambda x_0 = \sum_{\mu=1}^{3} x_\mu^2$$

$\rightarrow$ center $\mathcal{Z}\left(\mathbb{R}^3_\lambda\right)$ generated by x_0

$\rightarrow$ unital $*$ algebra (involution : complex conjugation)

Element on $\mathbb{R}^3_\lambda = \left(\bigoplus_{j\in\frac{\mathbb{N}}{2}} \mathbb{M}_{2j+1}\left(\mathbb{C}\right), \cdot\right)$

$$\phi = \sum_{j\in\frac{\mathbb{N}}{2}} \sum_{-j\leq m,n\leq j} \phi^j_{mn} \, \boxed{v^j_{mn}} \rightsquigarrow \text{orthogonal basis : } \left\{v^j_{mn}, j \in \frac{\mathbb{N}}{2}, -j \leq m, n \leq j\right\}$$

Scalar product $\langle\phi, \psi\rangle = \text{Tr}\left(\phi^\dagger\psi\right)$

$$\text{Tr}\left(\phi\psi\right) = 8\pi\lambda^3 \sum_{j\in\frac{\mathbb{N}}{2}} w(j)\ \text{tr}_j\left(\phi^j\ \psi^j\right) = 8\pi\lambda^3 \sum_{j\in\frac{\mathbb{N}}{2}} w(j) \sum_{-j\leq m,n\leq j} \phi^j_{mn}\ \psi^j_{mn}$$

Differential calculus

- Lie algebra of real inner derivation

$$\mathcal{G} = \left\{ D_\mu \cdot = i\,[\theta_\mu, \cdot] \ , \quad \theta_\mu = \frac{x_\mu}{\lambda^2} \right\}$$

$$\text{with} \quad [D_\mu, D_\nu] = -\frac{1}{\lambda}\epsilon_{\mu\nu\rho} D_\rho, \quad \forall \mu, \nu, \rho = 1, 2, 3$$

- **Connection** on right module $\mathbb{M}$ over $\mathbb{R}^3_\lambda$: $\boxed{\nabla : \mathcal{G} \times \mathbb{M} \to \mathbb{M}}$

$\rightsquigarrow$ particular choice : $\mathbb{M} = \mathbb{R}^3_\lambda$

$$\nabla_{D_\mu}(a) := \nabla_\mu(a) = D_\mu a + A_\mu a \ , \quad \boxed{A_\mu = \nabla_\mu(\mathbb{I})} \ , \quad \boxed{A_\mu^\dagger = -A_\mu}$$

- **Curvature**

$$F(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

$$F(D_\mu, D_\nu) := \boxed{F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu + [A_\mu, A_\nu] + \tfrac{1}{\lambda}\epsilon_{\mu\nu\rho} A_\rho}$$

Gauge transformation

- group of **unitary elements** $\mathcal{U}\left(\mathbb{R}^3_\lambda\right)$ with **left action**
  $\rightarrow$ for any $\phi \in \mathbb{R}^3_\lambda$ and $g \in \mathcal{U}\left(\mathbb{R}^3_\lambda\right)$

$$g^\dagger g = 1 \ , \quad \phi^g = g\phi \ , \quad \nabla^g_\mu = g^\dagger \nabla_\mu \circ g$$

thus

$$\boxed{A^g_\mu = g^\dagger A_\mu \ g + g^\dagger D_\mu \ g \ , \quad \text{and} \quad F^g_{\mu\nu} = g^\dagger F_{\mu\nu} \ g}$$

- $\exists$ **gauge invariant connection** and **curvature**

$$\nabla^{inv}_\mu(a) = D_\mu a - i\theta_\mu a = -ia\theta_\mu \quad \text{and} \quad F^{inv}_{\mu\nu} = 0$$

- **Covariant coordinates**

$$\nabla_\mu - \nabla^{inv}_\mu := \boxed{\mathcal{A}_\mu = A_\mu + i\theta_\mu} \quad \text{and} \quad \boxed{\mathcal{A}^\dagger_\mu = -\mathcal{A}_\mu}$$

then

$$F_{\mu\nu} = [\mathcal{A}_\mu, \mathcal{A}_\nu] + \frac{1}{\lambda}\epsilon_{\mu\nu\rho}\mathcal{A}_\rho$$

Family of gauge invariant classical action I

Convenient to work with hermitean fields

$$\mathcal{A}_\mu = i\Phi_\mu \quad \rightsquigarrow \quad \Phi_\mu^\dagger = \Phi_\mu$$

gauge-invariant functional (classical) **actions**
$\rightarrow$ **trace** of **gauge-covariant polynomial** in the covariant coordinates

$$S_{inv}(\Phi_\mu) = \mathrm{Tr}\,(P(\Phi_\mu))$$

Natural requirement for the gauge-invariant functional are:

1. $P(\Phi_\mu)$ is **at most quartic** in Φ_μ,
2. $P(\Phi_\mu)$ **does not involve linear term** in Φ_μ
   $\rightarrow$ (no tadpole at the classical order)
3. the **kinetic operator** is **positive**

$\rightsquigarrow$ **gauge-invariant harmonic term** $\sim \mathrm{Tr}(x^2\Phi_\mu\Phi_\mu)$

$$x^2 := \sum_{\mu=1}^{3} x_\mu x_\mu \in \mathcal{Z}(\mathbb{R}^3_\lambda)$$

Family of gauge invariant classical action II

Requirements 1 and 2 give :

$$S(\Phi) = \frac{1}{g^2}\mathrm{Tr}\left(2(\Omega+1)\Phi_\mu\Phi_\nu\Phi_\nu\Phi_\mu + 2(\Omega-1)\Phi_\mu\Phi_\nu\Phi_\mu\Phi_\nu + i\zeta\epsilon_{\mu\nu\rho}\Phi_\mu\Phi_\nu\Phi_\rho + (M+\mu x^2)\Phi_\mu\Phi_\mu\right)$$

$S(\Phi)$ is positive when

$\Omega \geq 0,\ \mu > 0,\ \zeta = 0,\ M > 0$ or $\Omega \geq 0,\ \mu > 0,\ \zeta = \frac{4}{\lambda},\ M > \frac{2}{\lambda^2}$

thus

$$S_\Omega = \frac{1}{g^2}\mathrm{Tr}\Big((F_{\mu\nu} - \frac{i}{\lambda}\epsilon_{\mu\nu\rho}\Phi_\rho)^\dagger(F_{\mu\nu} - \frac{i}{\lambda}\epsilon_{\mu\nu\rho}\Phi_\rho) + \Omega\{\Phi_\mu,\Phi_\nu\}^2 + (M+\mu x^2)\Phi_\mu\Phi_\mu\Big)$$

Equation of motion

$$4(\Omega+1)(\Phi_\rho\Phi_\mu\Phi_\mu + \Phi_\mu\Phi_\mu\Phi_\rho) + 8(\Omega-1)\Phi_\mu\Phi_\rho\Phi_\mu + 2(M+\mu x^2)\Phi_\rho = 0$$

$\Phi_\rho = 0$ is the absolute minimum

Kinetic operator of the classical action

We have

$$S_\Omega(\Phi) = S_{Kin}(\Phi) + \frac{1}{g^2}\mathrm{Tr}\left((F_{\mu\nu} - \frac{i}{\lambda}\epsilon_{\mu\nu\rho}\Phi_\rho)^\dagger(F_{\mu\nu} - \frac{i}{\lambda}\epsilon_{\mu\nu\rho}\Phi_\rho) + \Omega\left\{\Phi_\mu, \Phi_\nu\right\}^2\right)$$

Kinetic term of the classical action S_Ω :

$$\begin{aligned} S_{Kin}(\Phi) &= \frac{1}{g^2}\mathrm{Tr}\left(\Phi_\mu(M + \mu x^2)\Phi_\mu\right) \\ &= \frac{1}{g^2}\mathrm{Tr}\left(\Phi_\mu G \Phi_\mu\right) \end{aligned}$$

with the **positive self-adjoint operator** written in the basis

$$G^{j_1 j_2}_{mn;kl} = \frac{8\pi\lambda^3}{g^2} w(j_1)\ \left(M + \lambda^2\mu j_1(j_1+1)\right)\delta^{j_1 j_2}\delta_{nk}\delta_{ml}$$

Gauge fixing I

- **BRST operation δ_0**

$$\delta_0 \Phi_\mu = i[C, \Phi_\mu]$$

  - C : the ghost field
  - δ_0 acts as antiderivation w.r.t. grading

- **Fixing the gauge symmetry :**

$$\Phi_3 = \theta_3 \quad \text{thus} \quad \delta_0 \bar{C} = b \quad \delta_0 b = 0$$

  - where $\bar{C}$: the antighost field
  - and b : the Stückelberg field

- **BRST invariant gauge-fixing term**

$$S_{fix} = \delta_0 \mathrm{Tr}\Big(\bar{C}(\Phi_3 - \theta_3)\Big) = \mathrm{Tr}\Big(b(\Phi_3 - \theta_3) - i\bar{C}[C, \Phi_3]\Big)$$

Integration over the Stüeckelberg field b $\rightarrow$ constraint $\Phi_3 = \theta_3$

Gauge fixing II

- **Gauge-fixed action**

$$S_\Omega^f = S_2 + S_4$$

with

$$\begin{aligned} S_4 &= \frac{4}{g^2}\mathrm{Tr}\left(\Omega(\Phi_1^2+\Phi_2^2)^2 + (\Omega-1)(\Phi_1\Phi_2\Phi_1\Phi_2 - \Phi_1^2\Phi_2^2)\right) \\ S_2 &= \frac{1}{g^2}\mathrm{Tr}\left((\Phi_1,\Phi_2)\begin{pmatrix} Q & 0 \\ 0 & Q\end{pmatrix}\begin{pmatrix}\Phi_1 \\ \Phi_2\end{pmatrix}\right) \\ Q &= G + 8\Omega L(\theta_3^2) + i4(\Omega-1)L(\theta_3)D_3 \end{aligned}$$

where $G = M + \mu x^2$ and $L(X)$ is the left multiplication by X.

- **Particular case :** $\Omega = 1$
  - Kinetic operator : $K = G + 8\Omega L(\theta_3^2)$
  - Interaction term : $S_4 = \frac{4}{g^2}\mathrm{Tr}\left(\left(\Phi_1^2+\Phi_2^2\right)^2\right)$

Gauge-fixed action at $\Omega = 1$

$$S^f_{\Omega=1} = \frac{1}{g^2}\mathrm{Tr}\left((\Phi_1,\Phi_2)\begin{pmatrix} K & 0 \\ 0 & K\end{pmatrix}\begin{pmatrix}\Phi_1 \\ \Phi_2\end{pmatrix}\right) + \frac{4}{g^2}\mathrm{Tr}\left(\left(\Phi_1^2+\Phi_2^2\right)^2\right)$$

Kinetic operator

$$K = G + 8\Omega L(\theta_3^2)$$
$$K^{j_1 j_2}_{mn;kl} = \frac{8\pi\lambda^3}{g^2} w(j_1)\left(M + \mu\lambda^2 j_1(j_1+1) + \frac{4}{\lambda^2}(k^2+l^2)\right)\delta^{j_1 j_2}\delta_{ml}\delta_{nk}$$

It verifies

$$K^{j_1 j_2}_{mn;kl} = K^{j_1 j_2}_{lk;nm} = K^{j_1 j_2}_{mn;lk}$$

reflecting reality of the functional action and the self-adjointness of K.

Inverse of K

$$\sum_{j_2,k,l} K^{j_1 j_2}_{mn;lk} P^{j_2 j_3}_{kl;rs} = \delta^{j_1 j_3}\delta_{ms}\delta_{nr} \qquad \sum_{j_2,n,m} P^{j_1 j_2}_{rs;mn} K^{j_2 j_3}_{nm;kl} = \delta_{j_1 j_3}\delta_{rl}\delta_{sk}$$

$$\rightsquigarrow \quad P^{j_1 j_2}_{mn;kl} = \frac{g^2}{8\pi\lambda^3}\frac{1}{w(j_1)(M + \lambda^2\mu j_1(j_1+1) + \frac{4}{\lambda^2}(k^2+l^2))}\delta^{j_1 j_2}\delta_{ml}\delta_{nk}$$

One loop 2-point function

Contribution to the quadratic part at one-loop

$$\Gamma_2^1(\Phi_\alpha) = \frac{32\pi\lambda^3}{g^2} \sum_{-j\leq m,n,r,p\leq j} (\phi_\alpha)^j_{mn} \left(\sigma^{NP\ j}_{pr;mn} + \sigma^{P\ j}_{pr;nm}\right) (\phi_\alpha)^j_{kl}$$

with

$$\begin{aligned}
\sigma^{NP\ j}_{pr;mn} &= w(j)P^j_{pr;mn} \sim \frac{1}{(M + \lambda^2\mu j(j+1) + \frac{4}{\lambda^2}(m^2+n^2))} \\
\sigma^{P\ j}_{pr;nm} &= 3\delta_{mp} \sum_{m=-j}^{j} w(j)P^j_{rm;mn} \sim \sum_{m=-j}^{j} \frac{1}{(M + \lambda^2\mu j(j+1) + \frac{4}{\lambda^2}(m^2+n^2))}
\end{aligned}$$

- σ^{NP} is finite for $j=0$ and $j\to\infty$
- σ^{P} is also finite for $j=0$ and $j\to\infty$

$$\sum_{m=-j}^{j} w(j)P_{rm;mn} \leq \frac{2j+1}{(M+\lambda^2\mu j(j+1))}$$

Finiteness – "Truncated model" I

- gauge choice : $\Phi_3 = 0$
- propagator of the truncated theory :

$$(G^{-1})^{j_1 j_2}_{mn;kl} = \delta^{j_1 j_2} \delta_{mn} \delta_{kl} \frac{\Pi(M, j_1)}{w(j_1)}$$

with

$$\Pi(M, j) := \frac{g^2}{8\pi\lambda^3} \frac{1}{(M + \lambda^2 \mu j(j+1))}$$

- Loop built from from any N-point sub-diagram

$$\mathbb{A}_{m_3, n_3, \ldots, m_N, n_N} = \sum_{-j \leq m_1, n_1, m_2, n_2 \leq j} \mathcal{A}_{m_1, n_1, \ldots, m_N, n_N} (G^{-1})^{j}_{m_1 n_1; m_2 n_2}$$

where

$$\mathcal{A}_{m_1, n_1, \ldots, m_N, n_N} = F_N(j) \prod_{p=1}^{N} \delta_{m_p n_{\sigma(p)}}$$

and

- $\sigma \in \mathfrak{S}_N$ is some permutation of $\{1, 2, \ldots, N\}$
- $F_N(j)$ is some function depending on j and the other parameters of the model

Finiteness – "Truncated model" II

One obtains

$$\mathbb{A}_{m_3,n_3,\ldots,m_N,n_N} = \frac{F_N(j)\Pi(j,M)}{w(j)} \sum_{-j\leq n_1,n_2\leq j} \left(\prod_{p=3}^{N} \delta_{m_p n_{\sigma(p)}}\right) \delta_{n_{\sigma(1)} n_1} \delta_{n_{\sigma(2)} n_2}$$

If $\sigma(1) = 1$ and $\sigma(2) = 2$, then

$$\mathbb{A}_{m_3,n_3,\ldots,m_N,n_N} = (2j+1)^2 \frac{F_N(j)\Pi(j,M)}{w(j)} \left(\prod_{p=3}^{N} \delta_{m_p n_{\sigma(p)}}\right)$$

Contribution from summations over the indices of any loop give

$$(2j+1)^{\varepsilon} \quad \text{with} \quad \varepsilon \leq 2$$

loop summations decouple from the propagators in the computation of diagram amplitudes

General ribbon diagram

General ribbon diagram $\mathcal{D}$

$$\begin{array}{ccc} m_1 & \text{———————} & n_1 \\ m_2 & \text{———————} & n_2 \end{array}$$

- a ribbon carries 4 bounded indices
- conservation of the indices along each line
- characterized by a set of positive integer (V, I, F, B)
  - V : number of vertices
  - I : number of internal ribbons
  - F : number of faces
  - B : number of boundaries, equal to the number of closed lines with external legs
- $\mathcal{L}$: number of ribbon loops, given by

$$\mathcal{L} = F - B$$

- $g \in \mathbb{N}$: genus of the Riemann surface on which $\mathcal{D}$ can be drawn

$$2 - 2g = V - I + F$$

Finiteness – "Truncated model" III

Amplitude $\mathbb{A}^{\mathcal{D}}$ for a general ribbon diagram :

- V vertex factors
  $\rightarrow$ each vertex contributing to $w(j)$
- I propagators
  $\rightarrow$ each propagator contribute to

$$G^{-1} \sim \frac{\Pi(M,j)}{w(j)}$$

- summations over indices corresponding to $F-B$ loops which give an overall factor bounded by

$$(2j+1)^{2(F-B)}$$

$$\mathbb{A}^{\mathcal{D}} \leq K w(j)^{V-I} \Pi(M,j)^{I} (2j+1)^{2(F-B)} = K' \frac{w(j)^{V-I}(2j+1)^{2(F-B)}}{(j^2+\rho^2)^{I}}$$

where

- K and K' are finite constants and $\rho^2 = \frac{M}{\lambda\mu^2}$
- $w(j) = j+1$

Finiteness – "Truncated model" IV

$$\mathbb{A}^{\mathcal{D}} \leq K' \frac{w(j)^{V-I}(2j+1)^{2(F-B)}}{(j^2+\rho^2)^I}$$

It is a (positive) function of j, finite and non singular for $j=0$

$$\text{we set} \quad w(j) \sim j, \quad \text{for } j \to \infty$$

Thus we have the condition

$$\omega(\mathcal{D}) = I + 2B + 2(2g-2) + V \geq 0$$

- For $g \geq 1$, one has $\omega(\mathcal{D}) > 0$.
- For $g = 0$

$$\omega(\mathcal{D}) = I + 2B + V - 4 \geq 0$$

  - $V \geq 2$: $\omega(\mathcal{D}) > 0$
  - $V = 1$: 2-point function for the truncated model $\to$ finite

the truncated model is finite to all orders in perturbation.

Finiteness to all orders I

Back to our gauge model

- differs from the truncated model only through the propagator

$$P^{j_1 j_2}_{mn;kl} = \frac{g^2}{8\pi\lambda^3} \frac{1}{w(j_1)\left(M + \lambda^2 \mu j_1(j_1+1) + \boxed{\frac{4}{\lambda^2}(k^2+l^2)}\right)} \delta^{j_1 j_2} \delta_{ml} \delta_{nk}$$

- generic structure of $\mathfrak{A}^j_{\mathcal{D}}$

$$\mathfrak{A}^j_{\mathcal{D}} = \sum_{\mathcal{I}} \prod_{\lambda} P^j_{m_\lambda(\mathcal{I}) n_\lambda(\mathcal{I}); k_\lambda(\mathcal{I}) l_\lambda(\mathcal{I})} \; F^j(\delta)_{m_\lambda(\mathcal{I}) n_\lambda(\mathcal{I}); k_\lambda(\mathcal{I}) l_\lambda(\mathcal{I})}$$

where

- $\mathcal{I}$: set of (internal) indices $\subset \{-j, ...j\}$ so that all sums $\sum_{\mathcal{I}}$ are finite
- λ : labels the internal lines of $\mathcal{D}$
- $P^j_{mn;kl}$: (positive) propagator
- $F^j(\delta)_{mn;kl}$ collects all the delta's plus vertex weights depending only on j

Finiteness to all orders II

- One has the following estimate

$$|\mathfrak{A}^j_{\mathcal{D}}| \leq \sum_{\mathcal{I}} \prod_{\lambda} \left| (G^{-1})^j_{m_\lambda(\mathcal{I})n_\lambda(\mathcal{I});k_\lambda(\mathcal{I})l_\lambda(\mathcal{I})} \right| \left| F^j(\delta)_{m_\lambda(\mathcal{I})n_\lambda(\mathcal{I});k_\lambda(\mathcal{I})l_\lambda(\mathcal{I})} \right|$$

- From the previous condition

$$\omega(\mathcal{D}) = \alpha I + 2B + V(2-\alpha) - 4 \geq 0$$

we have

$$|\mathfrak{A}^j_{\mathcal{D}}| \leq K' \frac{w(j)^{V-I}(2j+1)^{2(F-B)}}{(j^2+\rho^2)^I} < \infty$$

Finiteness to all orders

All ribbon amplitudes in our gauge theory $(\Omega = 1)$ are finite so that $S^f_{\Omega=1}$ is perturbatively finite to all orders. $\rightsquigarrow$ generalized to $\Omega \neq 1$

1. a sufficient rapid decay of the propagator at large indices (large j) so that correlations at large separation indices disappear
2. the special role played by j, the radius of the fuzzy sphere components act as a cut-off
3. the existence of an upper bound for the propagator that depends only of the cut-off

Solvability

We rewrite the action

$$S_{\Omega}^{f} = \frac{2}{g^2}\mathrm{Tr}\left(\Phi Q\Phi^{\dagger} + \Phi^{\dagger}Q\Phi\right) + \frac{16}{g^2}\mathrm{Tr}\left((\Omega+1)\Phi\Phi^{\dagger}\Phi\Phi^{\dagger} + (3\Omega-1)\Phi\Phi\Phi^{\dagger}\Phi^{\dagger}\right)$$

with the complex fields

$$\Phi = \frac{1}{2}(\Phi_1 + i\Phi_2) \qquad \Phi^{\dagger} = \frac{1}{2}(\Phi_1 - i\Phi_2)$$

- **Particular case :** $\Omega = 1/3$ (Nucl.Phys.B 2016, `[arxiv:1603.05045]`)
  - Kinetic operator :
    $$Q = K - \frac{8i}{3}L(\theta_3)D_3$$
  - Interaction term :
    $$S_4 = \frac{64}{3g^2}\mathrm{Tr}\left(\Phi\Phi^{\dagger}\Phi\Phi^{\dagger}\right)$$
    $\rightarrow$ depends only on $\Phi\Phi^{\dagger}$
  - action is formally similar to the action describing an exactly solvable LSZ-type model
  - partition function for $S^{f}_{\Omega=\frac{1}{3}}$ can be related to τ-functions of integrable hierarchies

Thank you.