A quick overview of few aspects in Algebraic and Noncommutative Quantum Field Theory

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Field theory

$$
\begin{tikzcd}
\boxed{\text{Classical Mechanics}} \arrow[r, "\text{Quantization}"] \arrow[d, "N \to \infty"'] & \boxed{\text{Quantum Mechanics}} \arrow[d, "N \to \infty"] \\
\boxed{\text{Classical Field Theory}} \arrow[r, "\text{Quantization}"'] & \boxed{\text{Quantum Field Theory}}
\end{tikzcd}
$$

What I am going to talk about

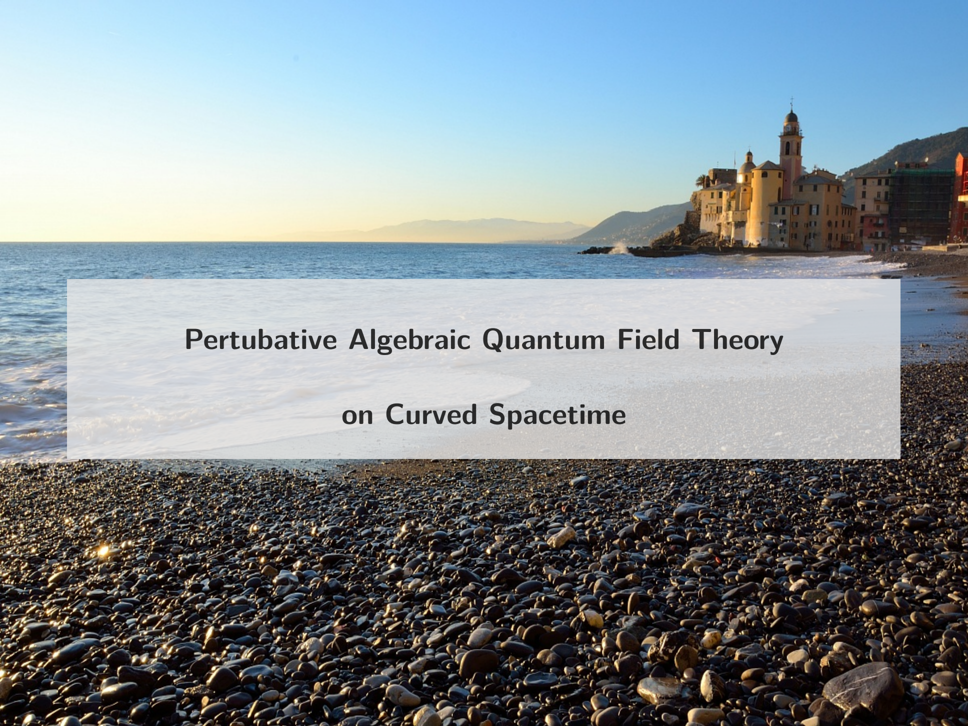
Algebraic approach

$\rightarrow$ general presentation

$\rightarrow$ a look at a particular problem and its treatment

Noncommutative approach

$\rightarrow$ a three dimensional gauge model with a nice property



Pertubative Algebraic Quantum Field Theory

on Curved Spacetime

Global picture

QFT on CST $\rightarrow$ difficulty : apparent **non locality** of quantum physics

Even worst : "traditional QFT" **based on** several **non local concepts**

e.g. vaccuum (defined as the state of lowest energy)

Fomulation of QFT based entirely on local concepts

$\rightsquigarrow$ **A**lgebraic **Q**uantum **F**ield **T**heory (AQFT)

- **Quantization** : "formal deformation" of the product
- **Interactions** : via introduction of a new product (the time ordered product)

Interacting theory can be build by **perturbing** the **quantum free theory** !

Our **theory** is **described by**

$$\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{int}$$

$$e.g. \qquad \mathcal{L} = \left(\nabla\phi\nabla\phi + m^2\phi^2 + \xi R\phi^2\right) + \frac{\lambda}{4!}\,\phi^4, \quad \phi \text{ real smooth map.}$$

- **Free theory** $\rightarrow$ quantization well known

classical algebra, with **pointwise product**

$\downarrow$ Quantization $\downarrow$

quantum algebra, with **new products**

powers of distribution appear ! $\Rightarrow$ **Not always well defined !**

- **Interacting theory**

  we **perturb** the free theory to **build** the **interacting theory**

  via the famous "Bogoliubov's formula" (la formula preferita di Nicolò ☺)

Physical input

- $(\mathcal{M}, \mathrm{g})$: **4 dimensional** spacetime

  $\rightarrow$ globally hyperbolic Lorentzian manifold

- $\mathfrak{C}$: **off shell** configuration space

  $\rightarrow$ real scalar field $\phi \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$

- $\mathcal{F}$: **space of observables** $\mathsf{F} : \begin{cases} \mathfrak{C} & \rightarrow & \mathbb{C} \\ \phi & \mapsto & \mathsf{F}(\phi) \end{cases}$

Need to make restriction to have good working properties

$\rightarrow$ **support** properties

$\rightarrow$ **regularity** properties

Functional approach – Support

Spacetime support of of F

$$\text{supp}(\mathsf{F}) \doteq \left\{ x \in \mathcal{M} \;\middle|\; \begin{array}{l} \forall \text{ neighborhood } U \text{ of } x,\ \exists\ \phi, \psi \in \text{ smooth}, \\ \text{supp}(\psi) \subset U,\ \text{such that } \mathsf{F}(\phi + \psi) \neq \mathsf{F}(\phi). \end{array} \right\}$$

$\rightarrow$ we require that **all functionals** have **compact support**.

$\rightarrow$ a way to characterise the influence that we see when we measure (in a laboratory ☺)

Wave front set – Definition

- **Distribution.** $\big(X$: open set in $\mathbb{R}^n\big)$

u : linear form on $\mathcal{C}_0^\infty(X)$ such that $\forall$ comp. set $K \subset X$,

$$|u(\phi)| \;\leq\; C \sum_{|\alpha|\leq k} \sup|\partial^\alpha \phi| \;,\quad \phi \in \mathcal{C}_0^\infty(K).$$

- **Idea.**

$$v \in \mathcal{C}_0^\infty(\mathbb{R}^n) \;\Leftrightarrow\; |\hat{v}(k)| \;\leq\; C\;(1+|k|)^{-N}$$

Definition – Wave front set. *[Hörmander 1983]*

The wave front set $\mathrm{WF}(u) \in \mathbb{R}^n \times \mathbb{R}^n \backslash \{0\}$ of $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)'$ is defined as follow

(i) for every $x \in \mathbb{R}^n$ where u is singular

(ii) $(x,k) \in \mathrm{WF}(u)$ **if and only if**

$\hat{fu}(k)$ is **not** rapidely decreasing in the diection of $k \neq 0$

- **Wave front set** : local $\rightarrow$ It generalises to CST.

- **Examples** :

$\rightsquigarrow$ $\mathrm{WF}(\delta) = \{(0, k) | k \in \mathbb{R}^n, k \neq 0\}$

Proof: The singular support of $\delta(x)$ is $\{0\}$ and $\hat{f\delta}(k) = f(0)$ is not fast decreasing if $f(0) \neq 0$.

$\rightsquigarrow$ $u(x) = \dfrac{1}{x^2 + i\epsilon}, \quad \mathrm{WF}(u) = \{(0; k) | k < 0\}$

Proof: By contour integration $\hat{u}(k) = -2i\pi\Theta(-k)$, thus

$$\left|\hat{fu}(k)\right| = \left|\frac{1}{2\pi}\int_{\mathbb{R}} dq\ \hat{f}(q)\ \hat{u}(k-q)\right| = \left|\int_k^{\infty} dq\ \hat{f}(q)\right|$$

The Fourier transform of a test function, $\hat{f}(q)$, is fast decreasing for $q \geq 0$!!

Functional approach – Regularity I

- We choose to work with **"smooth functionals"**.

The derivative of F at ϕ w.r.t the direction ψ is defined as

$$\mathrm{F}^{(1)}(\phi)[\psi] \doteq \lim_{t\to 0} \frac{1}{t}\Big(\mathrm{F}(\phi + t\psi) - \mathrm{F}(\phi)\Big) .$$

whenever the limit exists.

- derivatives of functionals $\rightsquigarrow$ distribution

  Example:

$$\mathrm{F}(\phi) = \int_{\mathcal{M}} \mathrm{d}\mu_x \; f(x)\, \frac{\lambda}{4!}\phi(x)^4 \; , \quad \mathrm{F}^{(1)}(\phi) = \frac{\lambda}{3!}\; f(x)\; \phi(x)^3\; \delta(x,y)\; , \quad \ldots$$

Spaces of observables

- **Regular** functionals $\mathcal{F}_{\text{reg}}$

$$\mathcal{F}_{\text{reg}} = \left\{ \mathsf{F} \mid \mathsf{F} \text{ smooth}, \mathsf{F}^{(n)} \text{ comp. sup., and } \mathrm{WF}(\mathsf{F}^{(n)}) = \emptyset \right\}$$

- **Microcausal** functionals $\mathcal{F}_{\mu c}$

$$\mathcal{F}_{\mu c} = \left\{ \mathsf{F} \mid \mathsf{F} \text{ smooth}, \mathsf{F}^{(n)}\text{comp. sup.}, \mathrm{WF}(\mathsf{F}^{(n)}) \cap \left(\mathcal{M}^n \times (\overline{V_+^n} \cup \overline{V_-^n}) \right) = \emptyset \right\}$$

$\rightarrow$ local interactions are a subset $\mathcal{F}_{\text{loc}} \subset \mathcal{F}_{\mu c}$

Interactions $\rightarrow$ **Local** functionals $\mathcal{F}_{\text{loc}}$

$$\mathcal{F}_{\text{loc}} = \left\{ \mathsf{F} \in \mathcal{F}_{\mu c} \mid \mathrm{supp}(\mathsf{F}^{(n)}) \subset \{(x, \dots, x) \subset \mathcal{M}^n\} \right\}$$

Example: $\mathsf{F} \in \mathcal{F}_{\text{loc}}(\mathcal{M})$

$$\mathsf{F}(\phi) = \int_{\mathcal{M}} \mathrm{d}\mu \; f(x) \; \frac{\lambda}{4!} \phi(x)^4 \; , \text{ with } f \in \mathcal{C}_0^\infty(\mathcal{M}, \mathbb{R})$$

Classical level

$$(\mathsf{F} \cdot \mathsf{G})(\phi) = \mathsf{F}(\phi) \cdot \mathsf{G}(\phi)$$

Quantum level

$$(F \star G)(\phi) = F(\phi)G(\phi) + \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}(\phi), \Delta_+^{\otimes n} G^{(n)}(\phi) \right\rangle$$

$\rightarrow$ powers of Δ_+ well defined, due to $\mathrm{WF}(\Delta_+)$!

Microlocal analysis. *[Hörmander 1983]*

$$\mathrm{WF}(u) \oplus \mathrm{WF}(v) \neq 0 \;\Rightarrow\; \exists! \; u.v \in \mathcal{D}'(\mathbb{R}^n).$$

Time ordered product

to **implement interactions**, we need **another product**

$\rightarrow$ time ordering product, defined with Δ_f

$$\mathsf{F} \cdot_T \mathsf{G} = \mathsf{F} \star \mathsf{G}, \text{ if } \mathrm{supp}(\mathsf{F}) \geq \mathrm{supp}(G)$$

"The interaction product"

$$(\mathsf{F} \ \cdot_T \ \mathsf{G})(\phi) \ = \ \mathsf{F}(\phi) \cdot \mathsf{G}(\phi) + \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} \left\langle \mathsf{F}^{(n)}, \Delta_f^{\otimes n} \mathsf{G}^{(n)} \right\rangle$$

$\rightarrow$ powers of Δ_f ill defined !!

- **Local** S **matrix**: $\mathsf{S}(F) = \sum_{n=0}^{\infty} \frac{i^n}{\hbar^n n!} \underbrace{\mathsf{F} \cdot_{\mathsf{T}} \dots \cdot_{\mathsf{T}} \mathsf{F}}_{n}$

- **Bogoliubov formula** *[Brunetti, Fredenhagen 2009]*

$$\mathsf{B}_{\mathsf{v}}(\mathsf{F}) = \mathsf{S}(\mathsf{V})^{\star -1} \star (\mathsf{S}(\mathsf{V}) \cdot_{\mathsf{T}} \mathsf{F})$$

$\rightarrow$ transition from the free action to the one with additional interaction term V.

Framework for a Dimensional Regularisation on Curved Spacetime

- **Causality** : $\mathsf{F}\cdot_{\mathsf{T}}\mathsf{G} = \mathsf{F}\star\mathsf{G}$, if $\mathrm{supp}(\mathsf{F}) \geq \mathrm{supp}(G)$

  The time ordered product of local functionals is well defined if their support are pairwise disjoint

  $\rightarrow$ $\mathsf{H}_{\mathsf{f}}(x,y)^n$ ill defined if $x = y$

- **Regularisation problem** : extend time ordered products to local functionals

  $\rightarrow$ extend on the full space $\mathsf{H}_{\mathsf{f}}(x,y)^n$

Epstein Glaser induction *[Brunetti, Fredenhagen 2000]*

if $\mathsf{F}_1, \ldots, \mathsf{F}_n \in \mathcal{F}_{\mathrm{loc}}(\mathcal{M}^n)$,

then $\mathsf{F}_1 \cdot_{\mathsf{T}} \ \ldots\ \mathsf{F}_n$ can be defined up to $\mathrm{Diag}(\mathcal{M}^n) = \{\mathbf{x} \in \mathcal{M}^n | x_1 = \cdots = x_n\}$

Scaling degree: For $t \in \mathcal{D}'(\mathbb{R}^4)$ *[Steinmann 1971]*

$$\mathrm{sd}(t) = \inf\{\omega \in \mathbb{R} \mid \lim_{\rho \to 0} \rho^\omega t(\rho x) = 0\}$$

"A divergence criterion", $t \in \mathcal{D}'(\mathbb{R}^4 \backslash \{0\})$ *[Brunetti, Fredenhagen 2000]*

- **if** $\mathrm{sd}(t) < 4$,

  then t has a unique extension $\dot{t} \in \mathcal{D}'(\mathbb{R}^4)$ with the same scaling degree
- **for** $4 \leq \mathrm{sd}(t) < \infty$, $\dot{t}$ has non unique extensions (with same sd)

Example

- $t(x) = |x|^{-1} \in \mathcal{D}'(\mathbb{R}\backslash\{0\}) \to \mathrm{sd}(t) = 1$: $\exists$ ext. (non unique)

$$\Rightarrow \quad \dot{t}_1(x) = \mathcal{P}\frac{1}{x} \,, \quad \dot{t}_2(x) = \lim_{\epsilon \downarrow 0} \frac{1}{x + i\epsilon}$$

The reason of our problem:

pointwise powers of Δ_{f}: ill defined, because contain σ_{f}^{-n} with n too large.

Fix x and set the distribution $t(y) = \sigma_{\mathrm{f}}^{-n}(x, y)$,

then for $x \to y$, $\mathrm{sd}(t(y)) = 2n$.

$\Rightarrow t(y)$ has an unique extension only for $n < 2$

“Main steps of the recipe”

(i) Deform n to $n + \alpha$, with $\alpha \in \mathbb{C}$

Result: the map $\alpha \mapsto \sigma_{\mathrm{f}}^{-(n+\alpha)}$ is meromorphic in α.

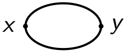
(ii) compute the **Laurent series** with respect to α

$\rightarrow$ play with the identities fulfilled by σ

(iii) subtract the principal part

(iv) take the limit $\alpha \to 0$

The fish diagram (2 vertices)



$$\longrightarrow \quad \Delta_{\mathrm{f}}^2(x,y) = \frac{1}{8\pi^2}\left(\frac{u^2(x,y)}{\sigma_{\mathrm{f}}^2(x,y)} + \text{“well defined for } x=y\text{”}\right)$$

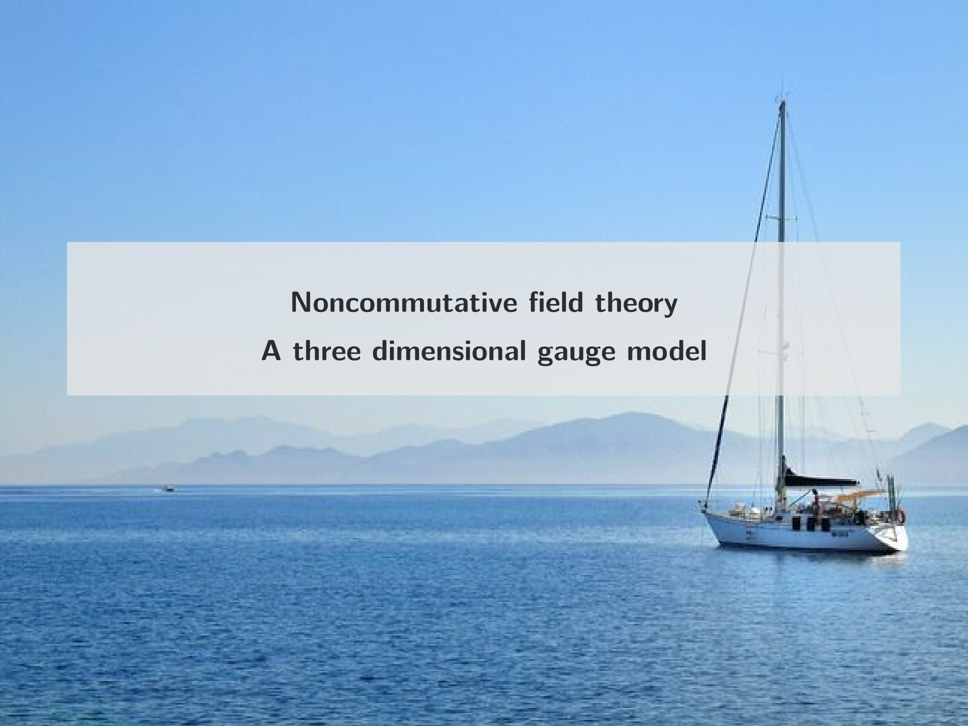
- regularize only $\sigma_{\mathrm{f}}^{-(2+\alpha)}$
- use σ identities : $\Box\sigma = 4 + f\sigma$
- $\alpha \mapsto \sigma_{\mathrm{f}}^{-(2+\alpha)}$ (weakly) meromorphic in α.

  $\to$ Laurent series w.r.t α

  $\to$ subtract the principal part and take the limit $\alpha \to 0$

$$\left(\frac{1}{\sigma_{\mathrm{f}}^2}\right)_{\mathrm{reg}} = \lim_{\alpha\to 0}\,(1-\mathrm{pp})\,\frac{1}{\sigma_{\mathrm{f}}^{2+\alpha}}$$

$$\Longrightarrow \left(\Delta_{\mathrm{f}}^2\right)_{\mathrm{reg}}$$



Noncommutative field theory
A three dimensional gauge model

An euclidean space

$$\mathbb{R}^3_\lambda = \mathbb{C}[x_0, x_1, x_2, x_3] \backslash \mathcal{I}\left[\mathcal{R}_1, \mathcal{R}_1\right]$$

where $\mathbb{C}[x_0, x_1, x_2, x_3]$ is the free algebra generated by $x_{i=1,2,3}$ and x_0 and $\mathcal{I}\left[\mathcal{R}_1, \mathcal{R}_1\right]$ is the two sided ideal generated by the relations

$$\mathcal{R}_1 : [x_i, x_j] = i\lambda\epsilon_{ijk}x_k \ , \quad \mathcal{R}_2 : x_0^2 + \lambda x_0 = \sum_{i=1}^{3} x_i^2$$

Basis

$$\phi \in \mathbb{R}^3_\lambda \ , \qquad \phi = \sum_{j\in\frac{\mathbb{N}}{2}} \left(\sum_{m,n=-j}^{j} \phi^j_{mn} v^j_{mn} \right)$$

with $\phi^j_{mn} \in \mathbb{C}$ and $\{v^j_{mn}\}$ the natural orthogonal basis of $\mathbb{R}^3_\lambda$

Scalar product

$$\langle \Phi, \Psi \rangle = \mathrm{Tr}\left(\Phi^\dagger \Psi\right) = 8\pi\lambda^3 \sum_{j\in\frac{\mathbb{N}}{2}} (j+1) \sum_{m,n=-j}^{j} \bar{\phi}^j_{nm}\ \psi^j_{mn}$$

$\rightarrow$ we need a notion of "noncommutative connection" to define gauge field

Lie algebra of real inner derivations of $\mathbb{R}^3_\lambda$

$$\mathcal{G} = \left\{ D_i \cdot = i \left[\frac{x_i}{\lambda^2}, \cdot \right], \ \forall i = 1, 2, 3 \right\}$$

connection on $\mathbb{R}^3_\lambda$ can be defined as linear map $\nabla : \mathcal{G} \times \mathbb{R}^3_\lambda \rightarrow \mathbb{R}^3_\lambda$, with

$$\nabla_{D_i}(a) = D_i(a) + A_i a \ , \quad \text{with} \quad A_i = \nabla_{D_i}(\mathbb{I}) \ \rightarrow \ \text{gauge field !}$$

gauge transformation: $A_i^g = g^\dagger A_i g + g^\dagger D_i g$

$$\text{invariant gauge connection : } \nabla^{inv}_{D_i}(a) = -ia\tfrac{x_i}{\lambda^2}$$

$$\Phi_i = i(\nabla^{inv}_{D_i} - \nabla_{D_i}) \ \rightarrow \ \Phi_i^g = g^\dagger \Phi_i g$$

The classical actions

$$\mathrm{S}(\Phi) = \frac{1}{g^2} \mathrm{Tr} \left(\alpha \Phi_i \Phi_j \Phi_j \Phi_i + \beta \Phi_i \Phi_j \Phi_i \Phi_j + i\xi \epsilon_{ijk} \Phi_i \Phi_j \Phi_k + (M + \mu x^2) \Phi_i \Phi_i \right)$$

$\rightarrow$ particular choice of the parameter in the action (ta make life easier ☺)

Gauge fixing (fixe the freedom whci remains) : BRST

we choose $\Phi_3 = x_3$

. . . (after a bit of computations) . . .

$$S(\Phi) = \frac{1}{g^2}\mathrm{Tr}\left(\Phi_1 K \Phi_1 + \Phi_2 K \Phi_2\right) + \frac{4}{g^2}\mathrm{Tr}\left(\left(\Phi_1^2 + \Phi_2^2\right)^2\right)$$

Propagator of the full theory

$$P = \frac{1}{(j+1)\left(M + \mu\lambda^2 j(j+1) + \lambda^{-2}(k^2+l^2)\right)}$$

Propagator of the troncated theory (whitout the field Φ_3)

$$Q = \frac{1}{(j+1)\left(M + \mu\lambda^2 j(j+1)\right)}$$

Theory finite at all orders

Truncated theory

$\rightarrow$ we look if the amplitude of general diagram is always finite

$$\left|\mathcal{T}^j\right| \leq K.(j+1)^{vertex}.(2j+1)^{loop}.Q^{intern\ \ lines} \leq K' \frac{(j+1)^{V-I}.(2j+1)^{\ell}}{(M'+j^2)^I}$$

- $j = 0$: finite !
- $j \rightarrow \infty$: finite because we have $3I - V - \ell \geq 0$.

Full theory

the following estimate holds true

$$0 \leq P \leq Q$$

therefore

$$\left|\mathcal{F}^j\right| \leq \sum_{k_1,\ldots,k_n=-j}^{j} \prod_{intern\ \ lines} \left|P^j\right| \left|f^j(\delta)\right| \leq \sum_{k_1,\ldots,k_n=-j}^{j} \prod_{intern\ \ lines} \left|Q^j\right| \left|f^j(\delta)\right| \leq \infty$$


Grazie.