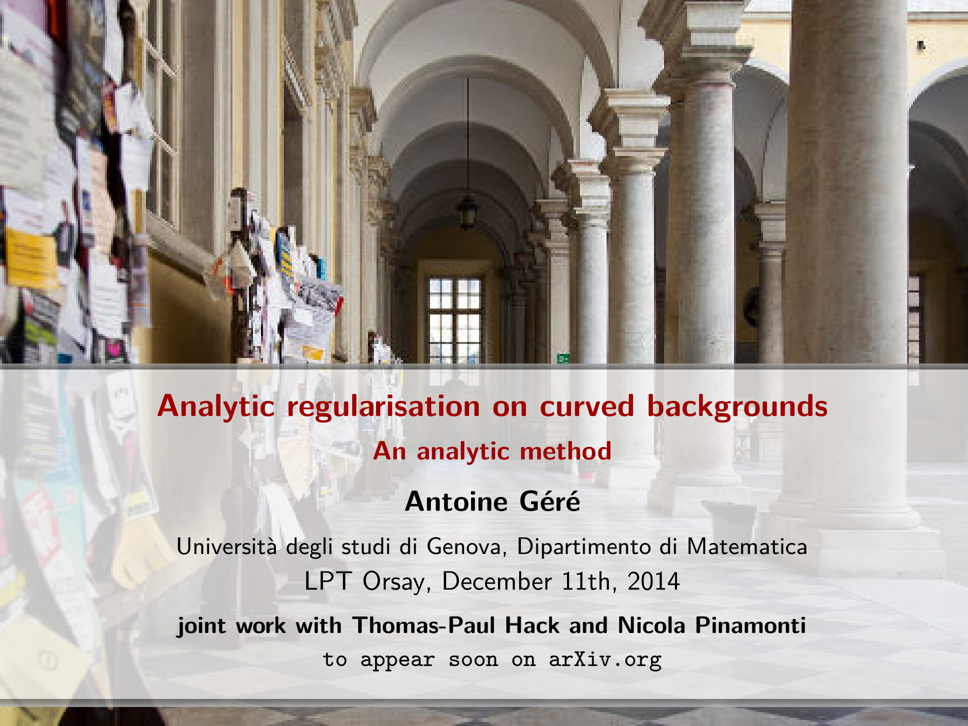

Analytic regularisation on curved backgrounds

An analytic method

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joint work with Thomas-Paul Hack and Nicola Pinamonti

`to appear soon on arXiv.org`

Global picture

QFT on CST $\rightarrow$ difficulty : apparent **non locality** of quantum physics

Even worst : "traditional QFT" **based on** several **non local concepts**

- vaccuum (defined as the state of lowest energy)
- particles (defined as irreducible representations of Poincaré group)
- ...

Fomulation of QFT based entirely on local concepts

$\rightsquigarrow$ **A**lgebraic **Q**uantum **F**ield **T**heory (AQFT)

we work at the **level** of the **algebra of observables**

$\rightarrow$ relation to the **Hilbert space formalism** is done via **GNS construction**

- **Quantization** : "formal deformation" (product as formal power series in $\hbar$)
- **Interactions** : via the time ordered product

Interacting theory can be build by **perturbing** the **quantum free theory** !

Motivation

- pertubative algebraic quantum field theory (pAQFT)

  $\rightarrow$ **conceptually well known**

  [Brunetti, Dütsch, Fredenhagen, Hollands, Köhler, Rejzner, Wald, ... ∼1996-2013]

- in pAQFT on curved spacetime (CST), regularisation uses ideas of Epstein and Glaser

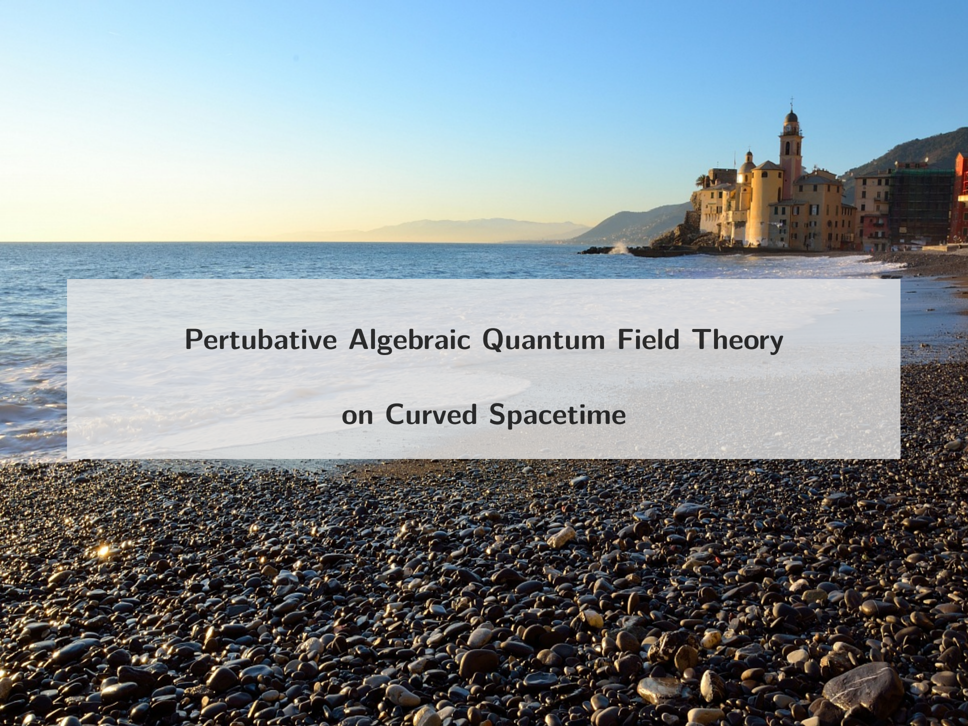
  $\rightarrow$ procedure **unconvenient for computations**

  [Brunetti & Fredenhagen 2000, Hollands & Wald 2002, Dang 2013]

- desire to use framework of pAQFT for **cosmological model** !

What I am going to talk about

- **pAQFT** $\rightarrow$ in order to identify the regularisation problem!
- a framework for a **dimensional regularisation** on CST
  $\rightarrow$ a general procedure more computationally friendly!
- explicit computations on spatially flat Friedmann Lemaître Robertson Walker spacetimes
  $\rightarrow$ finally a **bit of practice**!



Pertubative Algebraic Quantum Field Theory

on Curved Spacetime

Physical input

- $(\mathcal{M}, \mathrm{g})$: **4 dimensional** spacetime

    $\rightarrow$ globally hyperbolic Lorentzian manifold

- $\mathfrak{C}$: **off shell** configuration space

    $\rightarrow$ real scalar field $\phi \in \mathcal{C}^{\infty}(\mathcal{M}, \mathbb{R})$

- $\mathcal{F}$: **space of observables** $\mathsf{F} : \begin{cases} \mathfrak{C} & \rightarrow & \mathbb{C} \\ \phi & \mapsto & \mathsf{F}(\phi) \end{cases}$

Need to make restriction to have good working properties

$\rightarrow$ **support** properties

$\rightarrow$ **regularity** properties

Our **theory** is **described by**

$$\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{int}$$

$$e.g. \qquad \mathcal{L} = \left(\nabla\phi\nabla\phi + m^2\phi^2 + \xi R\phi^2\right) + \frac{\lambda}{4!}\,\phi^4, \quad \phi \text{ real smooth map.}$$

- **Free theory** $\rightarrow$ quantization well known

classical algebra, with **pointwise product**

$\downarrow$ Quantization $\downarrow$

quantum algebra, with **new products**

powers of distribution appear ! $\Rightarrow$ **Not always well defined !**

we **perturb** the free theory to **build** the **interacting theory**

- **Interacting theory**

via the famous "Bogoliubov's formula"

Free theory – A first look

($\mathcal{F}_{\text{reg}}$: regular functionals) $\qquad \mathrm{F}^*(\phi) = \int \mathrm{d}\mu_x \; \bar{f}(x) \; \phi(x)$

Free theory

Classical level		**Quantum level**
$\mathcal{A}_{\text{cl}} = (\mathcal{F}_{\text{reg}} \; , \; \cdot)$	$\leftarrow (*\ \text{algebra}) \rightarrow$	$\mathcal{A}_{\hbar} = (\mathcal{F}_{\text{reg}} \; , \; \star)$

$$(\mathrm{F} \cdot \mathrm{G})(\phi) = \mathrm{F}(\phi) \cdot \mathrm{G}(\phi)$$

$$(F \star G)(\phi) = F(\phi) G(\phi) + \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}(\phi), \Delta^{\otimes n} G^{(n)}(\phi) \right\rangle$$

$$\mathrm{F} \star \mathrm{G} \xrightarrow{\hbar \to 0} \mathrm{F} \cdot \mathrm{G}$$

e.g. linear functionals (fields themself),

in $\mathcal{A}_{\text{cl}}$: $\{F, G\} = \Delta(f, g)$ and in $\mathcal{A}_{\hbar}$: $[F, G] = i\hbar\Delta(f, g)$

- Given a $*$ algebra $\mathcal{A}$,

  a state ω is a positive, normalised, linear functional on $\mathcal{A}$.

- Often the algebra of observables $\mathcal{A}$, cannot be equipped with a norm

  BUT we have unital $*$ algebra

- **The GNS construction remains possible !**

  But in particular it is not garantee that selfadjoint elements of $\mathcal{A}$ are represented by selfadjoint Hilbert space operators !

Hilbert space formalism $\leftarrow$ **GNS construction** $\rightarrow$ **Algebraic formalism**

$\rightarrow$ **A state** ω on $\mathcal{A}$ vector is represented as a **"vacuum" vector**, and elements of $\mathcal{A}$ as **linear operators**.

$\rightarrow$ **Conversely**, any normalised **Hilbert space vector** is a **state on the algebra of linear operators** with the $*$ operation given by the Hermitian adjoint.

Functional approach – Support

Spacetime support of of F

$$\mathrm{supp}(\mathsf{F}) \doteq \left\{ x \in \mathcal{M} \;\middle|\; \begin{array}{l} \forall \text{ neighborhood } U \text{ of } x,\ \exists\ \phi, \psi \in \text{ smooth,} \\ \mathrm{supp}(\psi) \subset U,\ \text{such that } \mathsf{F}(\phi + \psi) \neq \mathsf{F}(\phi). \end{array} \right\}$$

Lemma

Usual properties for the spacetime support

Sum : $\mathrm{supp}(\mathsf{F} + \mathsf{G}) \subseteq \mathrm{supp}(\mathsf{F}) \cup \mathrm{supp}(\mathsf{G})$

Product : $\mathrm{supp}(\mathsf{F} \cdot \mathsf{G}) \subseteq \mathrm{supp}(\mathsf{F}) \cap \mathrm{supp}(\mathsf{G})$

We require that **all functionals** have **compact support**.

Wave front set – Definition

- **Distribution.** $\big(X$: open set in $\mathbb{R}^n\big)$

u : linear form on $\mathcal{C}_0^\infty(X)$ such that $\forall$ comp. set $K \subset X$,

$$|u(\phi)| \;\leq\; C \sum_{|\alpha|\leq k} \sup|\partial^\alpha \phi| \;, \quad \phi \in \mathcal{C}_0^\infty(K).$$

- **Idea.**

$$v \in \mathcal{C}_0^\infty(\mathbb{R}^n) \;\Leftrightarrow\; |\hat{v}(k)| \;\leq\; C\;(1+|k|)^{-N}$$

Definition – Wave front set. *[Hörmander 1983]*

The wave front set $\mathrm{WF}(u) \in \mathbb{R}^n \times \mathbb{R}^n$ of $u \in \mathcal{C}_0^\infty(\mathbb{R}^n, \mathbb{R})'$ as follows

(i) for every $x \in \mathbb{R}^n$ where u is singular, choose a non vanishing test function $f \in \mathcal{C}_0^\infty(\mathbb{R}^n, \mathbb{R})$

(ii) $(x, k) \in \mathrm{WF}(u)$ iff $\hat{fu}(k)$ is **not** rapidely decreasing in the diection of $k \neq 0$ for some f.

Wave front set – Examples

- **Wave front set** : local and covariant under coordinate transformations.

  $\rightarrow$ It generalises to CST (in contrast to the Fourier transform).

- **Examples** :

$\rightsquigarrow$ $\mathrm{WF}(\delta) = \{(0, k) | k \in \mathbb{R}^n, k \neq 0\}$

Proof: The singular support of $\delta(x)$ is $\{0\}$ and $\hat{f\delta}(k) = f(0)$ is not fast decreasing if $f(0) = 0$.

$\rightsquigarrow$ $u(x) = \dfrac{1}{x^2 + i\epsilon}$, $\quad \mathrm{WF}(u) = \{(0; k) | k < 0\}$

Proof: By contour integration $\hat{u}(k) = -2i\pi\Theta(-k)$, thus

$$\left|\hat{fu}(k)\right| = \left|\frac{1}{2\pi} \int_{\mathbb{R}} dq\ \hat{f}(q)\ \hat{u}(k-q)\right| = \left|\int_k^\infty dq\ \hat{f}(q)\right|$$

Fourier transform of a test function, $\hat{f}(q)$, is fast decreasing for $q \geq 0$!!

- We would like to choose $\mathcal{F}$, space of observables, which gives us **"smooth functionals"**.
  - $\rightsquigarrow$ Need a careful definition of differentiability!

Definition

The derivative of F at ϕ w.r.t the direction ψ is defined as

$$\mathsf{F}^{(1)}(\phi)[\psi] \doteq \lim_{t\to 0} \frac{1}{t}\Big(\mathsf{F}(\phi + t\psi) - \mathsf{F}(\phi)\Big) \,.$$

whenever the limit exists. The functional F is said to be

$\rightarrow$ **differentiable** at ϕ if $\mathsf{F}^{(1)}(\phi)[\psi]$ exists for any ϕ

$\rightarrow$ **continuously differentiable** if it is differentiable for all directions and at all evaluations points, and $\mathsf{F}^{(1)}$ is a **jointly continuous map** from $\mathfrak{C} \times \mathfrak{C}$ to $\mathbb{C}$, i.e. $\mathsf{F} \in \mathcal{C}^{(1)}(\mathfrak{C}, \mathbb{C})$

- derivatives of functionals $\rightsquigarrow$ distribution

  Example:

$$\mathrm{F}(\phi) = \int_{\mathcal{M}} \mathrm{d}\mu_x \; f(x) \; \frac{\lambda}{4!}\phi(x)^4 \; , \quad \mathrm{F}^{(1)}(\phi) = \frac{\lambda}{3!} \; f(x) \; \phi(x)^3 \; \delta(x, y) \; , \quad \ldots$$

- **Important remark**: many of the useful results of calculus are still valid (Leibniz rule, First fundamental theorem calculus, . . .).

We will work with the following smooth observables.

Definition – Smooth functionals.

Our observables are all possible functionals F such that

- they are smooth, i.e. $\mathrm{F} \in \mathcal{C}^\infty(\mathfrak{C}, \mathbb{C})$
- k-th order derivatives $\mathrm{F}^{(k)}(\phi)$ are **distributions of compact support**.

Spaces of observables

- **Regular** functionals $\mathcal{F}_{\text{reg}}$

$$\mathcal{F}_{\text{reg}} = \left\{ \mathrm{F} \mid \mathrm{F} \text{ smooth}, \mathrm{F}^{(n)} \text{ comp. sup., and } \mathrm{WF}(\mathrm{F}^{(n)}) = \emptyset \right\}$$

- **Microcausal** functionals $\mathcal{F}_{\mu c}$

$$\mathcal{F}_{\mu c} = \left\{ \mathrm{F} \mid \mathrm{F} \text{ smooth}, \mathrm{F}^{(n)} \text{comp. sup.}, \mathrm{WF}(\mathrm{F}^{(n)}) \cap \left(\mathcal{M}^n \times (\overline{V_+^n} \cup \overline{V_-^n}) \right) = \emptyset \right\}$$

$\rightarrow$ local interactions are a subset $\mathcal{F}_{\text{loc}} \subset \mathcal{F}_{\mu c}$

Interactions $\rightarrow$ **Local** functionals $\mathcal{F}_{\text{loc}}$

$$\mathcal{F}_{\text{loc}} = \left\{ \mathrm{F} \in \mathcal{F}_{\mu c} \mid \mathrm{supp}(\mathrm{F}^{(n)}) \subset \{(x, \ldots, x) \subset \mathcal{M}^n\} \right\}$$

Example: $\mathrm{F} \in \mathcal{F}_{\text{loc}}(\mathcal{M})$

$$\mathrm{F}(\phi) = \int_{\mathcal{M}} \mathrm{d}\mu \; f(x) \; \frac{\lambda}{4!} \phi(x)^4 \; , \text{ with } f \in \mathcal{C}_0^\infty(\mathcal{M}, \mathbb{R})$$

- **Minkowski** : isometry group (Poincaré group) & spectrum condition

  $\Rightarrow$ **unique vacuum state**

On generic CST

isometry group & microlocal spectrum condition (μSC)

$\Rightarrow$ **Hadamard states**

- **Properties of Hadamard states** :

  (i) same UV behaviour as the Minkowski vacuum

  (ii) guarantee that quantum fluctuation of expectations values are finite

  (iii) ...

  (iv) well suited for normal ordering !!

- **Hadamard states**

  $\rightarrow$ specify by **constrainig the singulary** of the two point function

Hadamard condition *[Radzikowski 1996]*

A state ω fulfils the **Hadamard condition** (μSC) iff

$$\mathrm{WF}(\omega_2) \;=\; \left\{ (x, k_x; y, -k_y) \in T^*\mathcal{M}^2 \backslash \{0\} \;\middle|\; (x, k_x) \sim (y, k_y),\;\; k_x \triangleright 0 \right\}$$

$\sim$: $\exists$ a null geodesic connecting x and x', and k' is the parallel transport of k.

$k_x \triangleright 0$: k_x is futur directed.

- **abstract definition** of Hadamard states

  $\rightarrow$ powerful BUT unconvenient for computations !

Hadamard form – Explicit Definition

- $\sigma(x,y)$: the half squared geodesic distance.
- $\mathcal{N}$: geodesically convex domain on $\mathcal{M}$.

Hadamard form

$\mathrm{H}_\sharp$ is said to be of Hadamard form on $\mathcal{N} \times \mathcal{N}$ iff $\exists$ u, v, and w such that

$$\mathrm{H}_\sharp(x,y) \doteq \lim_{\epsilon \downarrow 0} \frac{1}{8\pi^2} \left(\frac{u(x,y)}{\sigma_\sharp(x,y)} + v(x,y)\ \log(M^2 \sigma_\sharp(x,y)) + w(x,y) \right)$$

$$\sigma_\pm(x,y) = \sigma(x,y) \pm i\epsilon(t_x - t_y) + \epsilon^2 \qquad \sigma_\mathrm{f}(x,y) = \sigma(x,y) + i\epsilon$$

with u, v, w : smooth (Hadamard coefficients).

For the rest of the talk:

$\rightarrow$ H_+ (defined with σ_+) is the **positive frequency of the propagator**.

$\rightarrow$ H_f (defined with σ_f) is the **Feynman propagator**.

"Deformation" of the **pointwise product** on $\mathcal{F}_{\mu c}$

$\rightarrow$ $\star_{\mathrm{H}}$ product, defined with H_+ (implement "quantum structure")

the new product will encode the canonical commutation relation

$$(F \star G)(\phi) = F(\phi)G(\phi) + \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}(\phi), \Delta^{\otimes n} G^{(n)}(\phi) \right\rangle \ .$$

$\rightarrow$ powers of Δ well defined ?

$$\mathrm{WF}(\Delta) = \{(x, y; k_x, k_y) \in T^*\mathcal{M}\backslash\{0\} \mid (x, k_x) \sim (x, k_y)\}$$

Wave front set – Properties

Microlocal analysis. *[Hörmander 1983]*

The **pointwise product** of two distribution u, v is well defined if

$$\mathrm{WF}(u) \oplus \mathrm{WF}(v) \neq 0 \;\Rightarrow\; \exists!\; u.v \in \mathcal{D}'(\mathbb{R}^n).$$

For P a differential operator

$$\mathrm{WF}(\mathrm{P}u) \;\subset\; \mathrm{WF}(u)\ .$$

Examples

$(0, k)$ and $(0, -k) \in \mathrm{WF}(\delta) \;\Rightarrow\;$ powers of δ cannot be defined

$\mathrm{P}\Delta_{\mathrm{a}} = \delta \Rightarrow \mathrm{WF}(\mathrm{P}\Delta_{\mathrm{a}}) = \mathrm{WF}(\delta) \subset \mathrm{WF}(\Delta_{\mathrm{a}}) \Rightarrow$ powers of Δ_{a} cannot be defined

$(0; k_x, k_y)$ & $(0, -k_x, -k_y) \in \mathrm{WF}(\Delta) \Rightarrow$ **powers of Δ cannot be defined**

- The solution is to **spilt** Δ in a **positive** and **negative** frequency part.

$$i\Delta = \Delta_+ + \Delta_-$$

where Δ_+ and Δ_- are of **Hadamard form**.

"The quantum product"

$$(F \star G)(\phi) = F(\phi)G(\phi) + \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}(\phi), \Delta_+^{\otimes n} G^{(n)}(\phi) \right\rangle$$

$\rightarrow$ due to WF(Δ_+), powers of Δ_+ are well defined !

- **Nomal ordering**. $\left(\mathrm{F} \in \mathcal{A}_\hbar,\ \alpha_{\Delta_+}(\mathrm{F}) = \mathrm{e}^{-\hbar \Gamma_{\Delta_+}} \mathrm{F},\ \text{with } \Gamma_{\Delta_+} = \left\langle \Delta_+, \frac{\delta^2}{\delta\phi^2} \right\rangle \right)$

$$:\phi^2(x): = \alpha_{\Delta_+}(\phi^2(x)) = \lim_{x \to y} \left(\phi(x)\phi(y) - \Delta_+(x,y)\right)$$

$$:\phi(x)^2: :\phi(y)^2: = \phi(x)^2 \star \phi(y)^2$$

$$= :\phi(x)^2\phi(y)^2: + 4\hbar :\phi(x)\phi(y): \Delta_+(x,y) + 2\hbar^2 \Delta_+^2(x,y)$$

- to **implement interactions**, we need **another product**

  $\rightarrow$ time ordering product, defined with Δ_{f}

$$\mathrm{F} \cdot_{\mathrm{T_H}} \mathrm{G} = \mathrm{F} \star_{\mathrm{H}} \mathrm{G}, \text{ if } \mathrm{supp}(\mathrm{F}) \geq \mathrm{supp}(G)$$

"The interaction product"

$$(\mathrm{F} \ \cdot_{\mathrm{T}} \ \mathrm{G})(\phi) \ \doteq \ \mathrm{F}(\phi) \cdot \mathrm{G}(\phi) + \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} \left\langle \mathrm{F}^{(n)}, \Delta_{\mathrm{f}}^{\otimes n} \mathrm{G}^{(n)} \right\rangle$$

$\rightarrow$ powers of Δ_{f} ill defined

because $P\Delta_{\mathrm{f}} = \delta \Rightarrow \mathrm{WF}(\Delta_{\mathrm{f}}) \supset \mathrm{WF}(\delta)$

- "**Standard formulation**"

$$\begin{aligned} \mathrm{T}\left(:\phi(x)^2: :\phi(y)^2:\right) &= \phi(x)^2 \cdot_{\mathrm{T}} \phi(y)^2 \\ &= :\phi(x)^2\phi(y)^2: + 4\hbar :\phi(x)\phi(y): \Delta_{\mathrm{f}}(x,y) + 2\hbar^2 \Delta_{\mathrm{f}}^2(x,y) \end{aligned}$$

- **Algebraic structure**, with 2 products

$$\mathcal{A}_{\mathrm{H}} \doteq \mathcal{A}^{0}_{\hbar} = (\mathcal{F}_{\mu c}\ ,\ \star, \cdot_{\mathrm{T}})$$

- maps of the products

  for $F, G \in \mathcal{F}_{\mu c}$, defining $d = \mathrm{H}'_{+} - \mathrm{H}_{+} = \mathrm{H}'_{\mathrm{f}} - \mathrm{H}_{\mathrm{f}} \simeq \frac{1}{8\pi}(w' - w)$

$$\begin{aligned} \mathrm{F} \star_{\mathrm{H}'} \mathrm{G} &= \alpha_{d}\left(\alpha_{-d}(\mathrm{F}) \star_{\mathrm{H}} \alpha_{-d}(\mathrm{G})\right) \\ \mathrm{F} \cdot_{\mathrm{T}_{\mathrm{H}'}} \mathrm{G} &= \alpha_{d}\left(\alpha_{-d}(\mathrm{F}) \cdot_{\mathrm{T}_{\mathrm{H}}} \alpha_{-d}(\mathrm{G})\right) \end{aligned}$$

$\mathcal{A}_{\mathrm{H}'} \sim \mathcal{A}_{\mathrm{H}}$, the two algebras are isomorphic

$\Rightarrow$ thus we will choose a particular representation $\mathcal{A}_{\mathrm{H}}$

Interacting picture

- **Local** S **matrix**: $\mathsf{S}(F) = \sum_{n=0}^{\infty} \frac{i^n}{\hbar^n n!} \underbrace{\mathsf{F}\cdot_{\mathsf{T}} \ \dots \ \cdot_{\mathsf{T}}\mathsf{F}}_{n}$

- **Bogoliubov formula** *[Brunetti, Fredenhagen 2009]*

$$\mathsf{B_v}(\mathsf{F}) = \mathsf{S}(\mathsf{V})^{\star-1} \star (\mathsf{S}(\mathsf{V})\cdot_{\mathsf{T}}\mathsf{F})$$

$\rightarrow$ transition from the free action to the one with additional interaction term V.

- **Algebraic structure** $\mathcal{A}^I_\hbar = (\mathcal{F}_I, \star, \cdot_{\mathsf{T}})$

$$\mathcal{F}_I \xrightarrow{\mathsf{B_v}} \mathcal{F}_{\mu c}$$

- "**Gell Mann and Low theorem**". $(\phi_I(x) = \mathsf{B_v}(\phi(x)))$

  In the Minkowski vacuum state Ω_0, the expectation value of $\phi_I(x)$

$$\langle \phi_I(x) \rangle_{\Omega_0} = \left\langle S(V)^{-1} T(\mathsf{S}(V)\ \phi(x)) \right\rangle_{\Omega_0} = \frac{\left\langle \mathsf{T}\left(e^{\frac{i}{\hbar}V} \phi(x) \right) \right\rangle_{\Omega_0}}{\langle S(V) \rangle_{\Omega_0}}$$

Compuations of $\langle \phi_v(x) \star_\Delta \phi_v(y) \rangle_\Omega$

the two point function of the Interacting field in ϕ^4 up to second order in λ

Ω : Gaussian Hadamard state of the free field

$(\mu(x) = 3\lambda\ d(x,x))$

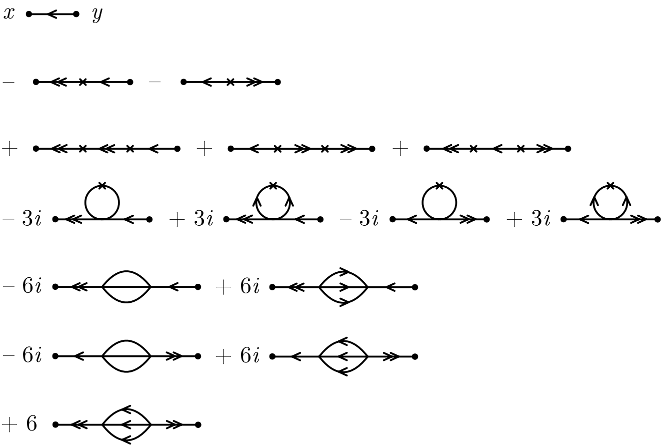
$$\Delta_F(x,y) = x \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\ y = \Delta_F(y,x)$$

$$\Delta_+(x,y) = x \bullet\!\!\longleftarrow\!\!\bullet\ y = \Delta_-(y,x)$$

$$\Delta_R(x,y) = x \bullet\!\!\twoheadleftarrow\!\!\bullet\ y = \Delta_A(y,x)$$

$$\times = \lambda$$

$$-\!\!\!\times\!\!\!- = \mu(x)$$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Framework for a Dimensional Regularisation on Curved Spacetime

- **Causality** : $\mathsf{F} \cdot_{\mathsf{T}} \mathsf{G} = \mathsf{F} \star \mathsf{G}$, if $\mathrm{supp}(\mathsf{F}) \geq \mathrm{supp}(G)$

  The time ordered product of local functionals is well defined if their support are pairwise disjoint

  $\rightarrow$ $\mathsf{H}_{\mathsf{f}}(x, y)^n$ ill defined if $x = y$

- **Regularisation problem** : extend time ordered products to local functionals

  $\rightarrow$ extend on the full space $\mathsf{H}_{\mathsf{f}}(x, y)^n$

Epstein Glaser induction *[Brunetti, Fredenhagen 2000]*

if $\mathsf{F}_1, \ldots, \mathsf{F}_n \in \mathcal{F}_{\mathrm{loc}}(\mathcal{M}^n)$,

then $\mathsf{F}_1 \cdot_{\mathsf{T}} \ \ldots \ \mathsf{F}_n$ can be defined up to $\mathrm{Diag}(\mathcal{M}^n) = \{\mathbf{x} \in \mathcal{M}^n | x_1 = \cdots = x_n\}$

- *Main theorem of Renormalization* *[Brunetti, Dütsch, Fredenhagen, 2009]*

  S_1, S_2 renormalized S matrices $\Rightarrow \exists!$ a map $\mathsf{Z} : \mathcal{F}_{\mathrm{loc}} \rightarrow \mathcal{F}_{\mathrm{loc}}$, such that

$$\mathsf{S}_1 = \mathsf{S} \circ \mathsf{Z} \ .$$

Scaling degree: For $t \in \mathcal{D}'(\mathbb{R}^4)$ *[Steinmann 1971]*

$$\mathrm{sd}(t) = \inf\{\omega \in \mathbb{R} \mid \lim_{\rho \to 0} \rho^\omega t(\rho x) = 0\}$$

"A divergence criterion", $t \in \mathcal{D}'(\mathbb{R}^4 \backslash \{0\})$ *[Brunetti, Fredenhagen 2000]*

- **if** $\mathrm{sd}(t) < 4$,

  then t has a unique extension $\dot{t} \in \mathcal{D}'(\mathbb{R}^4)$ with the same scaling degree
- **for** $4 \leq \mathrm{sd}(t) < \infty$, $\dot{t}$ has non unique extensions (with same sd)

Examples

- $t(x) = \log|x| \in \mathcal{D}'(\mathbb{R}\backslash\{0\}) \to \mathrm{sd}(t) = 0 \, : \, \exists! \text{ ext.} \Rightarrow \dot{t}(x) = \lim_{\epsilon \downarrow 0} \log|x + i\epsilon|$
- $t(x) = |x|^{-1} \in \mathcal{D}'(\mathbb{R}\backslash\{0\}) \to \mathrm{sd}(t) = 1 \, : \, \exists \text{ ext. (non unique)}$

$$\Rightarrow \quad \dot{t}_1(x) = \mathcal{P}\frac{1}{x} \, , \quad \dot{t}_2(x) = \lim_{\epsilon \downarrow 0} \frac{1}{x + i\epsilon}$$

- Feynman propagator (Minkowski)

$$\Delta_f(x) = \frac{1}{(2\pi)^d} \int \mathrm{d}^d p \, \frac{e^{ipx}}{p^2 - m^2 + i\epsilon} \to \mathrm{sd}(\Delta_f) = d - 2 \Rightarrow \text{ in dim. } 4 \, \exists! \text{ ext.}$$

The reason of our problem:

pointwise powers of Δ_{f}: ill defined, because contain σ_{f}^{-n} with n too large.

Fix x and set the distribution $t(y) = \sigma_{\mathrm{f}}^{-n}(x, y)$,

then for $x \to y$, $\mathrm{sd}(t(y)) = 2n$.

$\Rightarrow$ $t(y)$ has an unique extension only for $n < 2$

"Main steps of the recipe"

(i) Deform n to $n + \alpha$, with $\alpha \in \mathbb{C}$

Result: the map $\alpha \mapsto \sigma_{\mathrm{f}}^{-(n+\alpha)}$ is meromorphic in α.

(ii) compute the **Laurent series** with respect to α

$\rightarrow$ play with the identities fulfilled by σ

(iii) subtract the principal part

(iv) take the limit $\alpha \to 0$

Meromorphicity in α of $\alpha \mapsto \left\langle \sigma^{-(n+\alpha)}, \phi \right\rangle$ **?**

with ϕ smooth and comp. supp.

(i) $\sigma \to \sigma_{\mathrm{f}} = \sigma + i\epsilon \;\; : \;\; {\sigma_{\mathrm{f}}}^{-(n+\alpha)} = \exp\Big(-(n+\alpha)\log(\sigma + i\epsilon)\Big)$

(ii) $\alpha = a + ib, \quad a, b \in \mathbb{R}$

(iii) ${\sigma_{\mathrm{f}}}^{-(n+\alpha)} = {\sigma_{\mathrm{f}}}^{-(n+a)}\, {\sigma_{\mathrm{f}}}^{-ib}$,

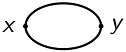
$$\frac{\partial}{\partial a}\left\langle {\sigma_{\mathrm{f}}}^{-(n+\alpha)}, \phi \right\rangle \overset{?}{=} \left\langle \frac{\partial}{\partial a}{\sigma_{\mathrm{f}}}^{-(n+\alpha)}, \phi \right\rangle$$

$$\frac{\partial}{\partial b}\left\langle {\sigma_{\mathrm{f}}}^{-(n+\alpha)}, \phi \right\rangle \overset{?}{=} \left\langle \frac{\partial}{\partial b}{\sigma_{\mathrm{f}}}^{-(n+\alpha)}, \phi \right\rangle$$

$\rightarrow$ **equalities hold!**

(iv) Cauchy Riemann equations hold!

$$\Rightarrow \; \alpha \mapsto \left\langle {\sigma_{\mathrm{f}}}^{-(n+\alpha)}, \phi \right\rangle \; \textbf{meromorphic in } \alpha$$



$$\longrightarrow \quad \Delta_{\mathrm{f}}^2(x,y) = \frac{1}{8\pi^2}\left(\frac{u^2(x,y)}{\sigma_{\mathrm{f}}^2(x,y)} + \text{“well defined for } x=y\text{”}\right)$$

- regularize only $\sigma_{\mathrm{f}}^{-(2+\alpha)}$
- use σ identities : $\quad \square\sigma = 4 + f\sigma$
- $\alpha \mapsto \sigma_{\mathrm{f}}^{-(2+\alpha)}$ (weakly) meromorphic in α.

  $\rightarrow$ Laurent series w.r.t α

  $\rightarrow$ subtract the principal part and take the limit $\alpha \rightarrow 0$

$$\left(\frac{1}{\sigma_{\mathrm{f}}^2}\right)_{\mathrm{reg}} = \lim_{\alpha\rightarrow 0}\,(1-\mathrm{pp})\,\frac{1}{\sigma_{\mathrm{f}}^{2+\alpha}}$$

$$\Longrightarrow \left(\Delta_{\mathrm{f}}^2\right)_{\mathrm{reg}}$$

The fish diagram – details

Fix y, and consider for $x \in \mathcal{N}$

$$\mathrm{t}^{\alpha}(x) = \frac{1}{M^{2\alpha}} \frac{1}{\sigma_{\mathrm{f}}(x,y)^{2+\alpha}} \,, \qquad (M \,:\, \text{to correct the change of dimension})$$

Using $\Box\sigma = 4 + f\sigma$ we have

$$\frac{1}{\sigma_{\mathrm{f}}^{2+\alpha}} = \frac{1}{2\alpha(1+\alpha)} \left(\Box_x + (1+\alpha)f\right) \frac{1}{\sigma_{\mathrm{f}}^{1+\alpha}}$$

then

$$\mathrm{t}(x) = \frac{1}{2}(\Box + f)\left(\frac{1}{\alpha\sigma_{\mathrm{f}}} - \frac{\log(M^2\sigma_{\mathrm{f}})}{\sigma_{\mathrm{f}}}\right) - \Box\frac{1}{2\sigma_{\mathrm{f}}} + \mathcal{O}(\alpha)$$

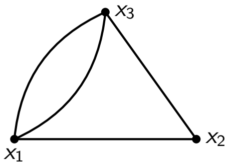
therefore

$$\dot{\mathrm{t}}(x) \;=\; \lim_{\alpha\to 0}\,(1-\mathrm{pp})\;\mathrm{t}^{\alpha}(x) \;=\; -\frac{1}{2}(\Box_x + f)\frac{\log M^2\sigma_F}{\sigma_{\mathrm{f}}} - \Box_x\frac{1}{2\sigma_{\mathrm{f}}}$$

Remark :

$$(\Box + f)\,\frac{1}{\sigma_{\mathrm{f}}} \;=\; 8\pi^2 i\,\delta$$

General case (N vertices) I



$\mathcal{G}$: graph with N vertices $x_1, \ldots, x_N$

x_k: a reference point arbitrarily chosen

$$\sigma_{ij} = \sigma_{\mathrm{f}}(x_i, x_j), \quad \sigma_i = \sigma_{\mathrm{f}}(x_k, x_j), \quad \sigma_{ij}^a = \nabla_i^a \sigma_{ij}$$

$$\Rightarrow \; t^\alpha(x_1, \ldots, x_N) \; = \; \prod_{i<j} \frac{1}{\sigma_{ij}^{n_{ij}+\alpha_{ij}}}, \quad \alpha_{ij} \in \mathbb{C}$$

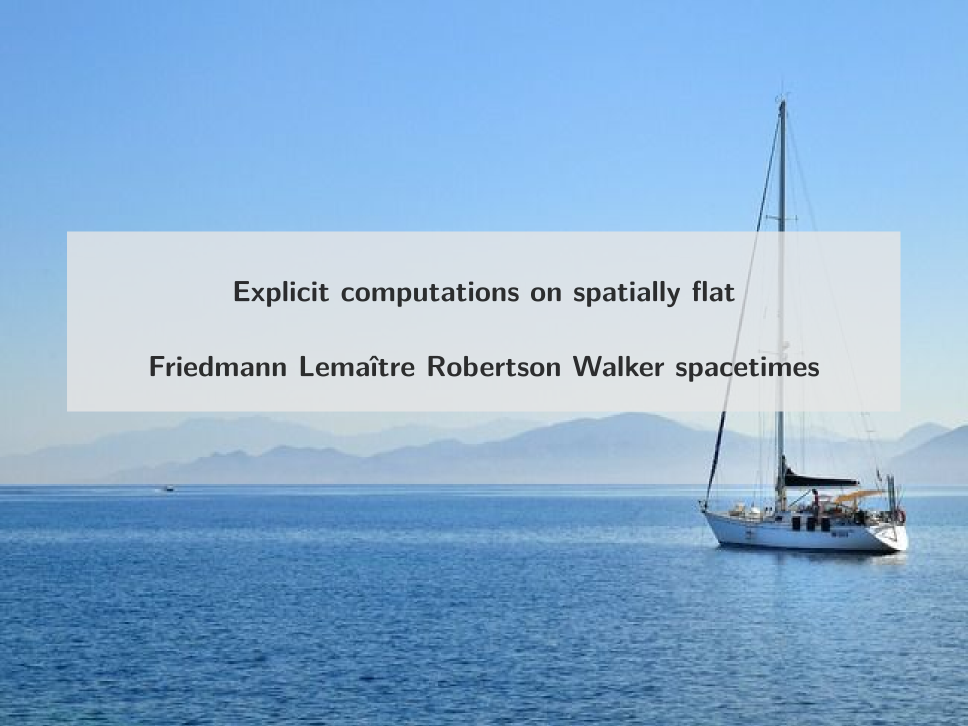
- Need a way to isolate the poles in $\{\alpha\}$

$$\mathrm{R} \doteq \nabla_a^i \sigma_i^a \;\; \rightarrow \;\; \text{“analogue” of } \square\sigma + \ldots \text{ that we use in the case } N = 2$$

$$\mathrm{t}^\alpha \; = \; \frac{1}{\mathcal{P}(N, n_{ij}, \alpha_{ij})} \; \left(\mathrm{R}^N \; \mathrm{t}^\alpha + \; \text{“corr.”} \; \right)$$

General case (N vertices) II

- then **apply** R as many times as **needed**, to have **sufficiently small** sd
- the terms in $\log(\sigma)$ obtained by **deriving** w.r.t α_{ij}
- subtract the pole **order by order** as explained or use the **forest formula**
  [Keller 2010, Dütsch, Fredenhagen, Keller, Rejzner 2013]
- **if** $(\mathcal{M}, g)$ has isometries and Δ_{f} (i.e. the state Ω_0) is invariant w.r.t. these, **then** the regularisation scheme preserves this invariance


Explicit computations on spatially flat
Friedmann Lemaître Robertson Walker spacetimes

Spiatially flat FLWR spacetimes

- **spatially flat Friedmann Lemaître Robertson Walker** spacetimes

$$g = a(\tau)^2 \ \left(-d\tau^2 + d\vec{x}^2\right)$$

- choosing a **free field state** Ω_0 (inv. under FLWR sym.) of the quantized free Klein Gordon field on spatially flat FLWR spacetimes

  $\Delta_\pm$, Δ_f : can be written using **spatial Fourier transform** in terms of **temporal modes** $\chi_k(\tau)$

$$\left(\partial_\tau^2 + k^2 + m^2 a^2 + \left(\xi - \frac{1}{6}\right) R a^2\right) \chi_k(\tau) = 0$$

$$\chi_k \partial_\tau \overline{\chi_k} - \overline{\chi_k} \partial_\tau \chi_k = i$$

- **in principle** we can compute all quantities in terms of $\vec{k}$- and τ-integrals using our regularisation procedure

The conformal trick

- Even for spatially flat FLRW spacetimes,
  σ, u, v are **not explicitly known** neither in **position space**, nor in $\vec{k}$**-space**,
  $\rightarrow$ but we need to know the $\vec{k}$-space version of e.g $\left(\Delta_{\mathrm{f}}^2\right)_{\mathrm{reg}}$

- **Trick**: $\Delta_{\mathrm{f}} = \Delta_{\mathrm{f},0} + \delta\Delta_{\mathrm{f}}$, with
  $\rightarrow$ $\Delta_{\mathrm{f},0}$ contains sufficiently many singular contributions
  $\rightarrow$ $\Delta_{\mathrm{f},0}$ explicitly known in position and $\vec{k}$-space
  $\rightarrow$ $(\Delta_{\mathrm{f},0})^n$ can be regularized using our procedure

- $\Delta_{f,0} \to$ the **Feynman propagator** of the **massless**, **conformally coupled** ($\xi = \frac{1}{6}$) **Klein Gordon field** in the conformal vacuum state

$$\Delta_{f,0}(x_1, x_2) = \frac{1}{8\pi^2 a(\tau_1) a(\tau_2)} \frac{1}{\sigma_{\mathbb{M}}(x_1, x_2) + i\epsilon}$$

- Fish diagram

$$(\Delta_f)^2_{\text{reg}} = (\Delta_{f,0})^2_{\text{reg}} + 2\delta\Delta_f \Delta_{f,0} + (\delta\Delta_f)^2$$

$$\begin{aligned}
(\Delta_{f,0})^2_{\text{reg}} &= \lim_{\alpha \to 0} (1 - \text{pp}) \frac{1}{M^{2\alpha}} (\Delta_{f,0})^{2+\alpha} \\
&= -\frac{1 + 2\log(a)}{16\pi^2 a^4} i\delta_{\mathbb{M}} - \frac{1}{2(8\pi^2)^2 a^2 \otimes a^2} (\Box_{\mathbb{M}} \otimes 1) \frac{\log M^2 \sigma_{\epsilon,\mathbb{M}}}{\sigma_{\epsilon,\mathbb{M}}}
\end{aligned}$$

and then compute the Fourier transform ...


Merci.