Avanzamento tesi

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joint work with Thomas-Paul Hack and Nicola Pinamonti

Our **theory** is **described by**

$$\mathcal{L} \;=\; \mathcal{L}_{free} \;+\; \mathcal{L}_{int}$$

$$e.g. \qquad \mathcal{L} \;=\; \left(\nabla\phi\nabla\phi + m^2\phi^2 + \xi R\phi^2\right) \;+\; \frac{\lambda}{4!}\,\phi^4, \quad \phi \text{ real smooth map.}$$

- **Free theory** $\rightarrow$ quantization well known

  we **perturbe** the free theory to **build** the **interacting theory**

  classical algebra, with **pointwise product**

  $\downarrow$ Quantization $\downarrow$

  quantum algebra, with **new products**

  powers of distribution appear ! $\Rightarrow$ **We will have to regularize it !**

- **Interacting theory**

  obtained via the famous "Bogoliubov's formula"

- pertubative algebraic quantum field theory (pAQFT)
  → **conceptually well known**
  [Brunetti, Dütsch, Fredenhagen, Hollands, Köhler, Rejzner, Wald, ... ∼1996-2013]

- in pAQFT on curved spacetime (CST), regularisation uses ideas of Epstein and Glaser
  → procedure **unconvenient for computations**
  [Brunetti & Fredenhagen 2000, Hollands & Wald 2002, Dang 2013]

- desire to use framework of pAQFT for **cosmological model**!

- $(\mathcal{M}, \mathrm{g})$: **4 dimensional** globally hyperbolic spacetime
- $\mathfrak{C}$: **off shell** configuration space $\rightarrow$ (non linear) scalar field $\phi \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$
- $\mathcal{F}$: **space of observables** $\mathsf{F} : \begin{cases} \mathcal{C}^\infty(\mathcal{M}, \mathbb{R}) & \rightarrow & \mathbb{C} \\ \phi & \mapsto & \mathsf{F}(\phi) \end{cases}$

Microcausal functionals $\mathcal{F}_{\mu\mathsf{c}}$

$$\mathcal{F}_{\mu\mathsf{c}} = \left\{ \mathsf{F} \; : \; \mathcal{C}^\infty(\mathcal{M}, \mathbb{R}) \; \rightarrow \; \mathbb{C} \; \middle| \; \begin{array}{l} \mathsf{F} \text{ smooth, } \mathsf{F}^{(n)} \text{comp. sup.,} \\ \mathrm{WF}(\mathsf{F}^{(n)}) \cap \left(\mathcal{M}^n \times (\overline{V_+^n} \cup \overline{V_-^n}) \right) = \emptyset \end{array} \right\}$$

Free theory

$$\mathcal{L}_0 = \frac{1}{2}\left(\nabla\phi\nabla\phi + m^2\phi^2 + \xi R\phi^2\right)$$

- **Classical level** $\rightarrow$ algebraic structure

$$\mathcal{A} = (\mathcal{F}_{\mu c}\ ,\ \cdot)$$

e.g linear functionals $\{F, G\} = \Delta(f, g)$.

- **Quantum level** $\rightarrow$ "formal deformation" of the pointwise product on $\mathcal{F}_{\mu c}$

$$(\mathrm{F}\ \star\ \mathrm{G})(\phi)\ =\ \mathrm{F}(\phi)\cdot\mathrm{G}(\phi) + \sum_{n=1}^{\infty}\frac{\hbar^n}{n!}\left\langle \mathrm{F}^{(n)}, \Delta_+^{\otimes n}\mathrm{G}^{(n)}\right\rangle$$

algebraic structure

$$\mathcal{A}_\hbar = (\mathcal{F}_{\mu c}\ ,\ \star)$$

e.g linear functionals $[F, G] = i\hbar\Delta(f, g)$.

to implement interactions $\rightarrow$ $\cdot_T$ time ordering product

$\star$ and $\cdot_T$ products **coincide** when the **product is time ordered**

$$\mathsf{F} \cdot_T \mathsf{G} = \mathsf{F} \star \mathsf{G}, \text{ if } \mathrm{supp}(\mathsf{F}) \geq \mathrm{supp}(G)$$

$$(\mathsf{F} \ \cdot_T \ \mathsf{G})(\phi) \ = \ \mathsf{F}(\phi) \cdot \mathsf{G}(\phi) + \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} \left\langle \mathsf{F}^{(n)}, \Delta_f^{\otimes n} \mathsf{G}^{(n)} \right\rangle$$

algebraic structure

$$\mathcal{A}_\hbar^\circ = (\mathcal{F}_{\mu c} \ , \ \star, \ \cdot_T)$$

Microlocal Analysis – Important property *[Hörmander 1983]*

$u, v \in \mathcal{D}'$, $\qquad \mathrm{WF}(u) \oplus \mathrm{WF}(v) \not\ni \{0\} \Rightarrow \exists! \ u.v \in \mathcal{D}'$

$$\Delta_f(x, y)^n \ \text{ ill defined in general if } x = y$$

- **Local** S **matrix**:

$$\mathsf{S}(F) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \underbrace{\mathsf{F} \cdot_{\mathsf{T}} \ \ldots \ \cdot_{\mathsf{T}} \mathsf{F}}_{n}$$

- **Bogoliubov formula** *[Brunetti, Fredenhagen 2009]*

$$\mathsf{B_v}(\mathsf{F}) = \mathsf{S}(\mathsf{V})^{\star -1} \star (\mathsf{S}(\mathsf{V}) \cdot_{\mathsf{T}} \mathsf{F})$$

$\rightarrow$ transition from the free action to the one with additional interaction term V.

- **Algebraic structure**

$$\mathcal{A}^{\mathsf{i}}_{\hbar} \ = \ \left(\mathcal{F}^{\mathsf{i}} \ , \ \star, \ \cdot_{\mathsf{T}}\right), \qquad \text{with} \ \ \mathcal{F}^{\mathsf{i}} = \left\{\mathsf{B_v}(\mathsf{F}), \ \mathsf{F} \in \mathcal{F}_{\mu c}\right\}$$

$$\mathcal{A}^{\mathsf{i}}_{\hbar} \xrightarrow{\mathsf{B_v}} \mathcal{A}^{\circ}_{\hbar}$$

Causality:

$$\mathrm{F}\cdot_{\mathrm{T}}\mathrm{G} = \mathrm{F}\star\mathrm{G}, \quad \text{if} \;\; \mathrm{supp}(\mathrm{F}) \geq \mathrm{supp}(G)$$

$$\rightarrow \quad \mathrm{H}_{\mathrm{f}}(x,y)^n \;\; \text{ill defined in general if } x = y$$

on a **normal convex neighborhood** of x

$$\mathrm{H}_{\mathrm{f}}(x,y) \;=\; \lim_{\epsilon\downarrow 0} \frac{1}{8\pi^2}\left(\frac{u(x,y)}{\sigma_{\mathrm{f}}(x,y)} + v(x,y)\log(M^2\sigma_{\mathrm{f}}(x,y)) + w(x,y)\right)$$

$$\sigma_{\mathrm{f}}(x,y) = \sigma(x,y) + i\epsilon$$

u, v, w : smooth, and $2\ \sigma$: squared geodesic distance.

Most singular part:

$$\text{powers of } \; \frac{1}{\sigma_{\mathrm{f}}}$$

Epstein Glaser induction *[Brunetti, Fredenhagen 2000]*

$F_{1.T}$ ... F_n can be defined up to $\mathrm{Diag}(\mathcal{M}^n) = \{\mathbf{x} \in \mathcal{M}^n | x_1 = \cdots = x_n\}$

Scaling degree: For $t \in \mathcal{D}'(\mathbb{R}^4)$ *[Steinmann 1971]*

$$\mathrm{sd}(t) = \inf\{\omega \in \mathbb{R} \mid \lim_{\rho \to 0} \rho^{\omega} t(\rho x) = 0\}$$

"A divergence criterion", $t \in \mathcal{D}'(\mathbb{R}^4 \backslash \{0\})$ *[Brunetti, Fredenhagen 2000]*

- **if** $\mathrm{sd}(t) < 4$,

  then t has a unique extension $\dot{t} \in \mathcal{D}'(\mathbb{R}^4)$ with the same scaling degree
- **for** $4 \leq \mathrm{sd}(t) < \infty$, $\dot{t}$ has non unique extensions (with same sd)

Scaling degree: Fix x and set the distribution $t(y) = \sigma_{\mathrm{f}}^{-n}(x, y)$,

then for $x \to y$, $\mathrm{sd}(t(y)) = 2n$.

$\Rightarrow$ $t(y)$ has an unique extension only for $n < 2$

“Main steps of the recipe”

(i) Deform n to $n+\alpha$, with $\alpha \in \mathbb{C}$

Result: the map $\alpha \mapsto \sigma_{\mathrm{f}}^{-(n+\alpha)}$ is meromorphic in α.

(ii) compute the **Laurent series** with respect to α

$\rightarrow$ play with the identities fulfilled by σ

$$\Box\sigma = 4 + f\sigma$$

$$\Box\sigma^{p} = (2p(p+1) \; + \; p\,\sigma\,f)\;\sigma^{p-1}$$

(iii) subtract the principal part with respect to α

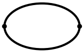
example of pole w.r.t α: $\Box\sigma^{-1} \sim \delta$

(iv) take the limit $\alpha \rightarrow 0$

$\Rightarrow$ well defined regularized distribution

PhD progress

Fish diagram

x  y $\longrightarrow$

$$\Delta_{\mathrm{f}}^2(x,y) = \frac{1}{8\pi^2}\left(\frac{u^2(x,y)}{\sigma_{\mathrm{f}}^2(x,y)} + \text{“well defined for } x=y\text{”}\right)$$

- regularize only $\sigma_{\mathrm{f}}^{-(2+\alpha)}$
- use σ identities : $\Box\sigma = 4 + f\sigma$
- $\alpha \mapsto \sigma_{\mathrm{f}}^{-(2+\alpha)}$ (weakly) meromorphic in α.

$\rightarrow$ Laurent series w.r.t α

$$\frac{1}{\sigma_{\mathrm{f}}^{2+\alpha}} = \frac{1}{2\alpha(1+\alpha)}\left(\Box_x + (1+\alpha)f\right)\frac{1}{\sigma_{\mathrm{f}}^{1+\alpha}}$$

$\rightarrow$ subtract the principal part and take the limit $\alpha \rightarrow 0$

$$\left(\frac{1}{\sigma_{\mathrm{f}}^2}\right)_{\mathrm{reg}} \doteq \lim_{\alpha\rightarrow 0}(1-\mathrm{pp})\,\frac{1}{M^{2\alpha}}\frac{1}{\sigma_{\mathrm{f}}^{2+\alpha}} = -\frac{1}{2}(\Box_x + f)\frac{\log M^2\sigma_F}{\sigma_{\mathrm{f}}} - \Box_x\frac{1}{2\sigma_{\mathrm{f}}}$$

$$\Longrightarrow \left(\Delta_{\mathrm{f}}^2\right)_{\mathrm{reg}}$$

BEFORE

$\to$ conceptual well understanding of pAQFT on CST

Problem $\to (\Delta_{\mathrm{f}})^n, \ldots$

regularisation procedure $\to (\Delta_{\mathrm{f}})^n \simeq (\sigma_{\mathrm{f}})^{-n} + \ldots$

with $(\sigma_{\mathrm{f}}^{-n})_{\mathrm{reg}} = \lim_{\alpha \to 0} (1 - \mathrm{pp})\, (\sigma_{\mathrm{f}})^{-(n+\alpha)}, \ \alpha \in \mathbb{C}$

NOW

$\to$ computations accessible!

Publication, preprint, in writing process

A. Géré, P. Vitale, J.-C. Wallet,

"*Quantum gauge theories on noncommutative three-dimensional space*",

Physical Review D, **2014**, (90) 045019, `arXiv:hep-th [1312.6145]`.

A. Géré, J.-C. Wallet,

"*Spectral theorem in noncommutative field theories I: Jacobi dynamics*",

`arXiv:math-ph [1402.6976]`.

A. Géré, T.-P. Hack, N. Pinamonti,

"*Dimensional/Analytic regularisation on curved spacetime*",

in progress, to appear soon on `arxiv`.

Scientific activities

- **Talks**
  - November 15th, 2014.
    "Dimensional regularisation on curved spacetime",
    35th Local Quantum Pathroads Workshop, Goslar (Germany).
  - December 11th, 2014.
    Seminar of Mathematical Physics, LPT, Orsay (France).

- **Conferences Attended**
  - November 14th – 15th, 2014,
    35th Local Quantum Pathroads Workshop, Goslar (Germany).
  - October 29th – November 2th, 2014,
    Quantum Mathematical Physics, Regensburg (Germany).
  - April 25th – 26th, 2014,
    34th Local Quantum Pathroads Workshop, Erlangen (Germany).