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Weyl algebra and field quantization

Contents

1 Weyl Algebras

1.1 Weyl *-algebra

Let E be a real vector space [Weib], and σ a symplectic form ($\sigma : E \times E \to \mathbb{R}$). Then we call (E, σ) a real symplectic space [Weid].

Definition 1.1. A symplectic form [Weic] is a 2-form which is, for $\forall x, y, a, b \in E$, and $\alpha, \beta \in \mathbb{R}$, bilinear $\big(\sigma(x, \alpha a + \beta b) = \alpha\sigma(x, a) + \beta\sigma(x, b)\big)$, antisymetric $\big(\sigma(x, y) = -\sigma(y, x)\big)$, and non degenerate $\big(\sigma(x, y) \neq 0 \quad \text{if} \quad x \neq y, \quad \text{and} \quad \sigma(x, 0) = 0 = \sigma(0, x)\big)$.

If instead to require that σ has to be non-degenrate, we require that σ has to be degenerate ($\forall x \in E$, $\exists y \in E$ such that $\sigma(x, y) = 0$), then we will call (E, σ) a real pre-symplectic space.

We now are going to define what is a $*$ - algebra, cf. [Lan98, definition 2.1.9].

Definition 1.2. An involution map on an algebra $\mathbb{A}$ [Weia] is a real linear map $A \to A^*$, such that for all $A, B \in \mathbb{A}$ and $\alpha, \beta \in \mathbb{C}$ one has,

$$(\alpha A + \beta B)^* = \bar{\alpha} A^* + \bar{\beta} B^* \quad \text{(antilinearity)}, \tag{1.1}$$
$$(AB)^* = B^* A^*, \tag{1.2}$$
$$\text{and} \quad A^{**} = A \quad \text{(involutivity)}. \tag{1.3}$$

A $*$ - algebra is an algebra with an involution.
A homomorphism of $*$ - algebras $h : \mathbb{A}_1 \to \mathbb{A}_2$ is a $*$ - homomorphism if it preserves the involution, $h(x^{*_1}) = h(x)^{*_2}$ for any $x \in \mathbb{A}_1$ ($*_1$ is the involution of $\mathbb{A}_1$ and $*_2$ the involution in $\mathbb{A}_2$), and a $*$ - homomorphism is a $*$ - isomorphism if it is additionally bijective.

The next step is to define what is a Weyl $*$ - algebra.

Definition 1.3. A $*$ - algebra is called a Weyl $*$ - algebra of (E, σ), denoted by $\mathcal{W}(E, \sigma)$, if there exists a family $\{W(u)\}_{u \in E}$ of non-zero elements, called the generators, such that,

(i) Weyl's (commutation) relations hold,

$$W(u)W(v) = \exp\left(\frac{i\hbar}{2}\sigma(u, v)\right) W(u + v), \quad W(u)^* = W(-u), \quad \forall u, v \in E, \tag{1.4}$$

(ii) $\mathcal{W}(E, \sigma)$ is generated by $\{W(u)\}_{u \in E}$, i.e. $\mathcal{W}(E, \sigma)$ coincides with the linear span of finite combinations of finite products of $\{W(u)\}_{u \in E}$.

Lemma 1.1. Any Weyl $*$ - algebra $\mathcal{W}(E,\sigma)$ has a unit $\mathbb{I}$, and

$$W(0) = \mathbb{I} \quad W(u)^* = W(-u) = W(u)^{-1}, \quad u \in E. \tag{1.5}$$

The generators $\{W(u)\}_{u\in E}$ are linearely independent, so in particular $W(u) \neq W(v)$ if $u \neq v$.

Proof 1.2. In fact, for $u \in E$, $W(u)W(0) = W(u) = W(0)W(u)$, and $W(u)W(-u) = W(0) = \mathbb{I} = W(-u)W(u)$, then $W(0) = \mathbb{I}$ and $W(-u) = W(u)^* = W(u)^{-1}$.

We will show now that a real symplectic space (E,σ) determines a unique Weyl $*$ - algebra up to $*$ - isomorphisms.

Theorem 1.3. If $\mathcal{W}(E,\sigma)$, generated by $\{W(u)\}_{u\in E}$, and $\mathcal{W}'(E,\sigma)$, generated by $\{W'(u)\}_{u\in E}$, are Weyl $*$ - Weyl algebras of (E,σ), there is a unique $*$ - isomorphism $\alpha : \mathcal{W}(E,\sigma) \to \mathcal{W}'(E,\sigma)$, which is determined by imposing,

$$\alpha\big(W(u)\big) = W'(u), \quad \forall u \in E. \tag{1.6}$$

Proof 1.4. The Weyl generators are linearely independent, and the product of two is a complex multiple of a generator, whence generators form a basis for the Weyl $*$ - algbra.

We will represent this Weyl $*$ - algebra $\mathcal{W}(E,\sigma)$ on $\mathfrak{B}(H)$, the set of all bounded operator on the Hillbert space H, that we will choose. We will show that we can always find a norm on this representation, and to have a more general result we will proof that it doesn't on our choice on how represent the Weyl $*$ - algebra.

Definition 1.4. Given a $*$ - algebra $\mathbb{A}$ and a Hilbert space H, a $*$ - homomorphism $\pi : \mathbb{A} \to \mathfrak{B}(H)$ is called a representation of $\mathbb{A}$ on H.

$$\pi : \mathbb{A} \to \mathfrak{B}(H) \tag{1.7}$$

Let's choose as a Hilbert space $L^2(E,\mu)$, where μ is the counting measure [1].
We represent $\mathcal{W}(E,\sigma)$ on $\mathfrak{B}(L^2(E,\mu)$.

$$\pi : \mathcal{W}(E,\sigma) \to \mathfrak{B}(L^2(E,\mu)) \tag{1.8}$$

We defined $W(u) \in \mathfrak{B}(L^2(E,\mu))$ by,

$$\big(W(u)f\big)(v) = \exp\left(i\sigma(u,v)\right) f(u+v), \quad \psi \in L^2(E,\mu), \quad u,v \in E. \tag{1.9}$$

It's not so long to show that $W(u)$ represented in this way fullfilled the Weyl's relations. Thus we have well defined a Weyl $*$ algebra.

1.2 Weyl C^*-Algebra

Our goal is now to define a C^* Weyl algebra. To do that we will first define what we call C^* algebra.

Definition 1.5. A norm on a vector space V is a map

1. $\|v\|_V \leq 0 \ \forall v \in V$,
2. $\|v\|_V = 0$ if and only if $\|v\|_V = 0$,
3. $\|\lambda v\|_V = |\lambda|\|v\|_V$,
4. $\|v + w\|_V \leq \|v\|_V + \|w\|_V$ (triangle inequality).

A norm on V defines a metric d on V by $d(v,w) := v - w$. A vector space with a norm which is complete in the associated metric (in the sense that every Cauchy sequence converges) is called a Banach space. We will denote a Banach space by the symbol B.

Definition 1.6. Let X and Y be two normed vector spaces. Then $\mathcal{B}(X,Y)$ is the space comprising all bounded linear operators. For $T \in \mathcal{B}(X,Y)$ we define the operator norm in the following way,

$$\|T\|_{op} = \sup_{\substack{x \in X \\ \|x\|_X \leq 1}} \|Tx\|_X. \tag{1.10}$$

1. $\mu(\mathrm{S}) = \Big\{\text{number of elements of S}\Big\}$, with $\mu(\mathrm{S}) = \infty$ if S is infinite.

Definition 1.7. A bounded operator on a Banach space B, is a linear map $A : B \to B$ for which

$$\|A\|_{op} < \infty. \tag{1.11}$$

The set of all the bounded operators on a Banach space B is denoted by $\mathfrak{B}$.

Definition 1.8. A Banach algebra is a Banach space B which is at the same time an algebra, in which for all $a, b \in B$ one has

$$\|ab\| \leq \|a\|\|b\| \tag{1.12}$$

Definition 1.9. A C $*$ - algebra is a Banach $*$ - algebra $\mathbb{B}$ such that for all $a \in B$ one has

$$\|a^* a\| = \|a\|^2. \tag{1.13}$$

A such norm is called a C^* norm.

We are going to see that we can always find a C^* - norm on $\mathcal{W}(E, \sigma)$, and that it is the unique one, and finally we will see why it is necessary to consider the conpletion of the $\mathcal{W}(E, \sigma)$ to obtain a C^* -algebra.

Theorem 1.5. We still consider E as a real vector space, but now $\sigma : E \times E \to \mathbb{R}$ is a weakly non-degenerate symplectic form. There exists a norm $\|.\|_{op}$ on $\mathcal{W}(E, \sigma)$, which is our Weyl $*$ - algebra, and it satisfys the C^* property, $\|A^* A\|_{op} = \|A\|^2_{op}$, for any A $\in \mathcal{W}(E, \sigma)$.

Definition 1.10. A bilinear form $\sigma : E \times E \to \mathbb{R}$ is said to be weakly nondegenerate if

$$\big\{\sigma(x, y) = 0 | \forall y \in E\big\} \Rightarrow x = 0. \tag{1.14}$$

Proof 1.6. ....

Theorem 1.7. If we set, for any $a \in \mathcal{W}(X, \sigma)$:

$$\|a\|_c := sup\big\{p(a) \quad | \quad p : \mathcal{W}(X, \sigma) \to [0, +\infty) \text{ is a } C^* norm\big\} \ ,$$

then $\|.\|_c$ is a C^* norm.

Proof 1.8. .....

Theorem 1.9. For $\mathcal{W}(X, \sigma)$ a Weyl $*$ -algebra associated to (X, σ), we denote by $\overline{\mathcal{W}(X, \sigma)}$ the C^* completion of $\mathcal{W}(X, \sigma)$ in the norm $\|.\|_c$. Then $\overline{\mathcal{W}(X, \sigma)}$ is simple: it does not admit two-sided closed ideals invariant under the involution other than $\{0\}$ and $\overline{\mathcal{W}(X, \sigma)}$ itself.

Proof 1.10. ...

Theorem 1.11. A $*$ - homomorphism $\pi : A \to B$ of C^* - algebras with unit is continuous, for $\|\pi(a)\|_A \leq \|a\|_B$, for any $a \in A$. Furthermore π is one-to-one if and only if isometric, i.e. $\|\pi(a)\| = \|a\|$ for any $a \in A$.

Proof 1.12. ...

Theorem 1.13. There exist a unique norm on $\mathcal{W}(X, \sigma)$ satisfying the C^* property: $\|a^* a\| = \|a\|^2$ for any $a \in \mathcal{W}(X, \sigma)$.

Proof 1.14. ....

Theorem 1.15. Let $\overline{W(X, \sigma)}$ be the C^* - algebra completion of $W(X, \sigma)$ for the C^* norm. If $W(X, \sigma)$ is another Weyl σ -algebra associated to the same space (X, σ) and $\|.\|$ the unique C^* norm, call $\overline{W(X, \sigma)}$ the corresponding C^* - algebra with unit. Then there is a unique isometric $*$ - isomorphism $\gamma : \overline{W(X, \sigma)} \to \overline{W(X, \sigma)}$ such that

$$\gamma(W(f)) = W(f), \quad f \in E \tag{1.15}$$

where $W(f)$, $W(f)$ are generators of the Weyl $*$ - algebras $W(X, \sigma)$, $W(X, \sigma)$.

Proof 1.16. ....

2 Strict Quantization

Definition 2.1. A Poisson algebra $\big(\mathcal{P}, \{.,.\}\big)$ is an associative algebra $\mathbb{A}$ over a field $\mathbb{K}$ with a linear braket $\{.,.\} : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ such that,

$$\{f,g\} = -\{g,h\} \quad \text{(Antisymetry)} \tag{2.1}$$

$$\{f,gh\} = g\{f,h,+\}\,h\{f,g\} \quad \text{(Leibniz rule)} \tag{2.2}$$

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0 \quad \text{(Jacobi Identity)} \tag{2.3}$$

forall $f, g, h \in \mathbb{A}$.

Example 2.1. ...

Definition 2.2. A strict quantization $(\mathcal{A}^\hbar, \mathcal{Q}_\hbar)$ of the Poisson algebra $\big(\mathcal{P}, \{.,.\}\big)$ consists for each value $\hbar \in I$ of a linear, $*$ - preserving quantization map $\mathcal{Q}_\hbar : \mathcal{P} \to \mathcal{A}^\hbar$, where $\mathcal{A}^\hbar$ is a linear C^* - algebra with norm $\|.\|_\hbar$ [2], such that Q_0 is the identical embedding of $\mathcal{P}$ into $\mathcal{A}^0$, and such that for all $A, B \in \mathcal{P}$ the following conditions are satisfied:

[**Dirac's condition**] The $\hbar$ - scaled commutator $[X,Y] := \frac{i}{\hbar}(XY - YX)$ approaches the Poisson bracket as $\hbar \to 0$,

$$\lim_{\hbar\to 0} \|[Q^\hbar(A), Q^\hbar(B)] - Q^\hbar(\{A,B\})\| = 0. \tag{2.4}$$

[**von Neumann's condition**] In the limit $\hbar \to 0$ one has the asymptotic behaviour for the product,

$$\lim_{\hbar\to 0} \|Q^\hbar(A)Q^\hbar(B) - Q^\hbar(AB)\| = 0. \tag{2.5}$$

[**Rieffel's condition**] $I \ni \hbar \to \|Q^\hbar(A)\|$ is continuous.

Theorem 2.1. ...

Proof 2.2. ...

3 Field-theoretic Weyl quantization

[blablabla]

4 References

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2. this norm is an operator norm.