Demonstration of the Parallel-Axis Theorem

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The period of small-amplitude oscillations of a physical pendulum (Fig. 1) is

$$T_o = 2\pi \sqrt{\frac{I}{MgR}} \quad (1)$$

where I is the moment of inertia about the axis, M is the mass, g is the acceleration due to gravity, and R is the distance from the center of mass to the axis.[1] The moment of inertia about the axis is related to the moment of inertia I_{cm} about the parallel axis through the center of mass by the *parallel-axis theorem*:[2]

$$I = I_{cm} + MR^2 \quad (2)$$

Although Eq. (2) is readily derived, students can always benefit from a demonstration of the validity of a general relationship. An apparatus to achieve this for Eq. (2) is presented here.

For an arbitrary physical pendulum represented in Fig. 1, the period (1) and parallel-axis theorem (2) imply that the period is the same for all mutually parallel axes along the circle of radius R, because the distance to the center of mass and the moment of inertia remain the same. That the moment of inertia remains the same is not obvious, but can be demonstrated in the simple case of a rectangular plate with a concentric hole, where the plate is supported by a knife edge on the rim of the hole. The period is predicted to be independent of the location of the knife edge along the rim, as long as the plate remains perpendicular to the knife edge. For a lecture demonstration, it is convenient to employ two such plates that are identical but that have different orientations (Fig. 2). It should be noted that these are physically distinct pendulums. It is advisable to first demonstrate that the pendulums have the same period when the orientations are the same, and then to alter the orientation of one by 90°. In both cases, the two pendulums are set oscillating in phase with the same amplitude, which can be accomplished by directly moving the pendulums by hand, or by carefully rocking the knife edge and supports back and forth at the natural frequency of the pendulums. The oscillations are observed to remain in phase, which shows that the periods are indeed the same. Other orientations can also be demonstrated (for example, one pendulum vertical or horizontal and the other rotated 45° with respect to the first).

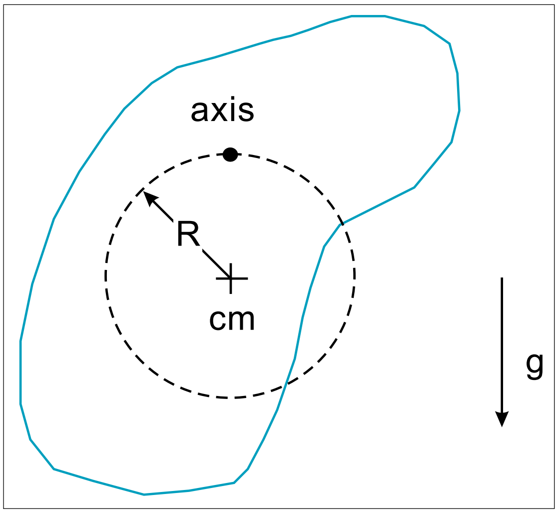

Fig. 1. Arbitrary physical pendulum, where axis is perpendicular to plane of figure. Center of mass is labeled "cm." According to parallel-axis theorem, period is the same for all mutually parallel axes along circle.

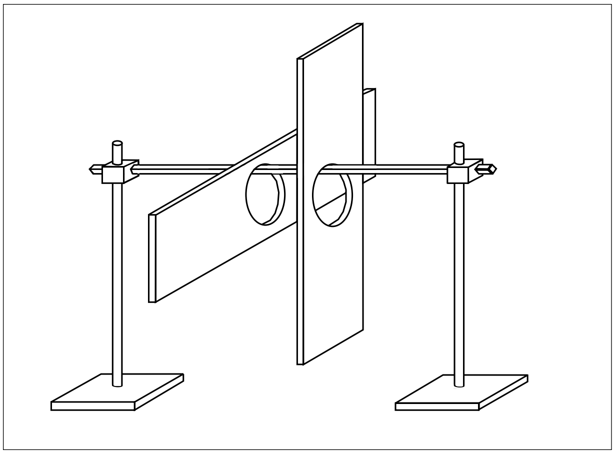

Fig. 2. Apparatus to demonstrate validity of parallel-axis theorem. The pendulums have the same period.

For a lecture demonstration, the amplitude of oscillation should be substantial so that the motion can be easily observed. The actual period T will then be greater than the small-amplitude period T_o in Eq. (1). In addition to the amplitude, however, T is a function of only T_o because this quantity is what occurs in the exact equation of motion $d^2\theta/dt^2 + (4\pi^2/T_o^2)\sin(\theta) = 0$, where θ is the angular displacement from equilibrium. Hence, as long as the two pendulums in Fig. 2 have the same amplitude, their periods will be the same.

An alternative to the use of two identical pendulums as in Fig. 2 is to employ a single pendulum and measure the time for many (e.g., 10) oscillations. The time should be registered with a large stopclock or on a large display, so that it can be seen by the audience. The pendulum is then adjusted so that the axis is in contact with a different point of the rim of the hole. The measurement is repeated, yielding the same time.

In a previous note on an assortment of challenging physical pendulums,[3] we described how a rectangular plate with a concentric hole can be employed in the teaching laboratory, where students compare the measured period for small-amplitude oscillations with the calculated value, Eq. (1). The calculation involves the superposition principle in the subtraction of the mass and moment of inertia corresponding to the hole. So that a mistake in this calculation will clearly show in the comparison of measured and calculated values, the aspect ratio of the rectangle should not be large; otherwise, the effect of the hole is small. For lecture demonstration purposes, however, the aspect ratio should be large so that the different orientations are clearly distinct. The dimensions of our laboratory pendulums are 10.0 by 6.0 in (25.4 by 15.2 cm) with a 4.5-in (11.4-cm) diameter hole, whereas the dimensions of our demonstration pendulums are 15.0 by 4.0 in (38.1 by 10.2 cm) with a 3.0-in (7.6-cm) diameter hole. Both sets of pendulums are precision machined to ± 0.002 in (± 0.05 mm) from aluminum plates of thickness 3/8 in (1.0 cm). Rods with square cross sections rotated 45° serve as knife edges. The laboratory rods are made of 1/2-in (1.3-cm) wide Delrin™ (similar to nylon), which is advantageous because it cannot damage the pendulums. In addition, only small amplitudes can occur without slipping, which prevents students from obtaining large-amplitude data for which Eq. (1) does not apply. Milled brass 3/8-in (1.0-cm) wide is used for the demonstration rod because it tends not to damage the pendulums and because substantial amplitudes can occur without slipping.

In this demonstration and experiment, the plates should be perpendicular to the support rod. In fact, violating this leads to an interesting demonstration with the apparatus in Fig. 2. For simplicity, it is advisable to begin with the plates having the same orientation. They are arranged to be perpendicular to the support rod, and then one is rotated roughly 5° about a vertical axis. If both plates are set oscillating with the same amplitude and phase, the rotated plate slowly gains phase relative to the perpendicular plate. The corresponding small decrease in period is due to the moment of inertia of the rotated plate being slightly reduced. Large rotations (45°, for example) lead to the plate having a substantially smaller period than the perpendicular plate.

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References

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2. Ref. 1, pp. 248–250.
3. Bruce Denardo and Richard Masada, "A not-so-simple pendulum experiment," *Phys. Teach.* **28**, 51–52 (1990).