9.8 THE UNILATERAL LAPLACE TRANSFORM

In the preceding sections of this chapter we have dealt with a form of the Laplace transform referred to as the bilateral Laplace transform. A somewhat different form of the Laplace transform, referred to as the *unilateral Laplace transform*, plays a particularly important role in analyzing causal systems specified by linear constant-coefficient differential equations with initial conditions (i.e., which are not initially at rest).

The unilateral Laplace transform $\mathfrak{X}(s)$ of a signal $x(t)$ is defined as

$$\mathfrak{X}(s) \triangleq \int_{0+}^{\infty} x(t)e^{-st}\,dt \tag{9.120}$$

From comparison of eqs. (9.120) and (9.2) we see that the difference in the definition of the unilateral and bilateral Laplace transforms lies in the lower limit on the integral. The bilateral transform depends on the entire signal from $t = -\infty$ to $t = +\infty$, whereas the unilateral transform depends only on the signal from $t = 0+$ to ∞. Consequently, two signals which differ for $t < 0$ but which are identical for $t > 0$ will have different bilateral Laplace transforms but identical unilateral transforms. We note also that since the unilateral transform does not include $t = 0$, it does not incorporate any impulses or higher-order singularity functions $u_n(t)$, $n > 0$. Basically, the unilateral transform should not be thought of as a new transform. It is the *bilateral transform* of a signal whose values for $t < 0+$ have been set to 0. Thus, using Property 4 in Section 9.2 for right-sided signals, we see that the ROC for eq. (9.120) is always a right-half plane. To illustrate the unilateral Laplace transform, let us consider two examples.

Example 9.16

Consider the signal

$$x(t) = \frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \tag{9.121}$$

Since $x(t) = 0$, $t < 0$, and contains no singularities, the unilateral and bilateral transforms are identical. Thus, from Table 9.2,

$$\mathfrak{X}(s) = \frac{1}{(s+a)^n}, \qquad \mathcal{Re}\{s\} > -a \tag{9.122}$$

Example 9.17

Consider next

$$x(t) = e^{-a(t+1)}u(t+1) \tag{9.123}$$

The *bilateral* transform $X(s)$ for this example can be obtained from Example 9.1 and the time-shifting property (Section 9.5.2). Specifically,

$$X(s) = \frac{e^s}{s+a}, \qquad \mathcal{Re}\{s\} > -a \tag{9.124}$$

By contrast, the unilateral transform is

$$\begin{aligned}\mathfrak{X}(s) &= \int_{0+}^{\infty} e^{-a(t+1)}u(t+1)e^{-st}\,dt \\ &= \int_{0+}^{\infty} e^{-a}e^{-t(s+a)}\,dt \\ &= e^{-a}\frac{1}{s+a}, \qquad \mathcal{Re}\{s\} > -a\end{aligned} \tag{9.125}$$

Thus, for this example, the unilateral and bilateral Laplace transforms are distinctly different. In fact, we should recognize $\mathfrak{X}(s)$ as the bilateral transform not of $x(t)$ but of $x(t)u(t)$ consistent with our comment above that the unilateral transform *is* the bilateral transform of a signal whose values for $t < 0+$ have been set to zero.

Most of the properties of the unilateral transform are the same as for the bilateral transform. In fact, the initial and final value properties, eqs. (9.78) and (9.79), are more appropriately associated directly with the unilateral Laplace transform because they required for their validity that $x(t)$ be zero for $t < 0$ and contain no impulses or higher-order singularities. Under these conditions the bilateral Laplace transform is identical to the unilateral Laplace transform.

A particularly important difference between the properties of the unilateral and bilateral transforms is the differentiation property. Specifically, let $x(t)$ have unilateral Laplace transform $\mathfrak{X}(s)$. Then, integrating by parts, we find that the unilateral transform of $dx(t)/dt$ is given by

$$\begin{aligned}\int_0^{\infty} \frac{dx(t)}{dt}e^{-st}\,dt &= x(t)e^{-st}\Big|_{0+}^{\infty} + s\int_{0+}^{\infty} x(t)e^{-st}\,dt \\ &= s\mathfrak{X}(s) - x(0+)\end{aligned} \tag{9.126}$$

Similarly, a second application of this would yield the unilateral Laplace transform of $d^2x(t)/dt^2$,

$$s^2\mathfrak{X}(s) - sx(0+) - x'(0+) \tag{9.127}$$

where $x'(0+)$ denotes the derivative of $x(t)$ evaluated at $t = 0+$. Clearly, we can continue the procedure to obtain the unilateral transform of higher derivatives.