Setup

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Eigenvalue problem $\quad \Delta u = -\lambda u$

Boundary condition: Dirichlet cond^n (DC): $u|_{\partial\Omega} = 0$

Neumann cond^n (NC): $\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0$, n is the outward normal to $\partial\Omega$

We first characterize eigenvalues as a minimum problem

We need Green's identities

Green's first identity: for $f \in C^1(\Omega)$, $g \in C^2(\Omega)$,

$$\iint_{\partial\Omega} f(\nabla g \cdot \hat{n})\, d\Omega = \int_\Omega (\nabla f)\cdot(\nabla g)\, dx + \int_\Omega f\, \Delta g\, dx \to ⓐ$$

$$\boxed{\begin{array}{l} \nabla f = (f_x, f_y) \\ \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \end{array}}$$

Pf follows from divergence theorem ($\iint_\Omega (\nabla \cdot F)\, dx = \iint_{\partial\Omega} (F\cdot\hat{n})\, dS$, for $F \in C^2(\Omega)$)

By product rule, $\nabla\cdot(f\nabla g) = \nabla f \cdot \nabla g + f\,\Delta g$

Integrating and using divergence th^m,

$$\int_\Omega \nabla\cdot(f\nabla g)\, dx = \iint_{\partial\Omega} f\,\nabla g\cdot\hat{n}\, dS$$

Result follows

Observe, for DC: $f = 0$ & NC: $\nabla g\cdot\hat{n} = \frac{\partial g}{\partial n} = 0$

$\therefore$ LHS $= 0$

$$\Rightarrow \int_\Omega (\nabla f)\cdot(\nabla g) = -\int_\Omega f\,\Delta g \to ①$$

Green's second identity for $f, g \in C^2(\Omega)$, $\iint_{\partial\Omega} (f\nabla g - g\nabla f)\cdot\hat{n}\, dS = \int_\Omega (f\,\Delta g - g\,\Delta f)\, dx$

Pf Interchanging f & g in ⓐ and subtracting it from ⓐ

Again observe if f, g satisfy either bd cond^n, then LHS $= 0$

$$\Rightarrow \int_\Omega f\,\Delta g = \int_\Omega g\,\Delta f \to ②$$

Dirichlet cond^n

1^{st} eigenvalue:

Define $A := \{ \omega \in C^2(\bar{\Omega}) \setminus 0 \mid \omega|_{\partial\Omega} = 0 \}$

and $Q : A \to \mathbb{R}$ $\quad Q[\omega] := \dfrac{\|\nabla \omega\|^2}{\|\omega\|^2} = \dfrac{\iint_\Omega | \omega_x^2 + \omega_y^2 | \, dx\,dy}{\iint_\Omega |\omega|^2 \, dx\,dy}$

Th^m 1 If $u \in A$ is a minimizer of Q and $m = Q[u]$, then $\lambda_1 = m$ and $\Delta u = -\lambda_1 u$ [ie, m is the smallest eigenvalue with eigen f^n]

Pf Let $v \in A$

Note $u + v\varepsilon \in A$ for $\varepsilon \in \mathbb{R}$

Define $f : \mathbb{R} \to \mathbb{R}$, $f(\varepsilon) := Q[u + \varepsilon v]$

$$f(\varepsilon) = \frac{\|\nabla(u + \varepsilon v)\|^2}{\|u + \varepsilon v\|^2} = \frac{\int ((\nabla u)^2 + 2\varepsilon \nabla u \cdot \nabla v + \varepsilon^2 \nabla v^2)}{\int u^2 + 2\varepsilon uv + \varepsilon^2 v^2}$$

f has min at $\varepsilon = 0 \Rightarrow f'(0) = 0$

$$f'(0) = \frac{(\int u^2)(\int 2 \nabla u \cdot \nabla v) - (\int (\nabla u)^2)(\int 2uv)}{(\int u^2)^2} = 0$$

$$\Rightarrow \int \nabla u \cdot \nabla v = \frac{\int (\nabla u)^2 \int uv}{\int u^2} = m \int uv \longrightarrow (**)$$

From (1),

$$\int (\Delta u) v = -m \int uv$$

$$\Rightarrow \int (\Delta u + mu) v = 0 \longrightarrow (*)$$

This holds for any $v \in A$, so $\Delta u + mv = 0$

Thus, m is an eigenvalue with eigenf^n u

Claim: it is the smallest, $m = \lambda_1$

Consider any eigenf^n v & eigenvalue $\lambda \Rightarrow \Delta v = -\lambda v$

$\because$ m is the minimum of $Q[\omega]$, $\omega \in A$

$$m \le Q[v] = \frac{\int (\nabla v)^2}{\int v^2} = \frac{-\int v(\Delta v)}{\int v^2} \quad \text{(from (1))}$$

$$= \lambda$$

n^{th} eigenvalue when smallest $n-1$ are known

Let $\lambda_1, \dots \lambda_{n-1}$ be the smallest eigenvalue with eigenf^n $v_1, \dots v_{n-1}$

Define $A_n := \{ w \in \mathcal{A} \mid \langle w, v_1 \rangle = \dots \langle w, v_{n-1} \rangle = 0 \}$

and $Q : A_n \longrightarrow \mathbb{R}$ $\quad Q[w] := \dfrac{\|\nabla w\|^2}{\|w\|^2}$

Th^m 2 If u is a minimizer of Q and $m = Q[u]$, then $\lambda_n = m$ and $\Delta u = -\lambda_n u$

Pf Same proof as th^m 1 with $\mathcal{A}_n$ instead of $\mathcal{A}$ until ⊛, gives us

$\int (\Delta u + m u) v = 0 \quad$ for $v \in A_n \rightarrow$ ⓐ

For $1 \le i \le n-1$, since $u \in A_n$, $\langle u, v_i \rangle = \int u v_i = 0$

$\Rightarrow u(\Delta v_i + m v_i) = (m - \lambda_i) u v_i \quad \Rightarrow \int u (\Delta v_i + m v_i) = 0$

From ②, $\int (\Delta u + m u) v_i = \int u (\Delta v_i + m v_i)$

$\therefore \int (\Delta u + m u) v_i = 0 \rightarrow$ ⓑ

Now let $h \in \mathcal{A}$

and define $\tilde{v} := h - \sum_{k=1}^{n-1} c_k v_k$, $\quad c_k = \dfrac{\langle h, v_k \rangle}{\langle v_k, v_k \rangle}$

Claim: $\langle \tilde{v}, v_i \rangle = 0$, $1 \le i \le n-1$ $\longrightarrow$ Eigenf^n are orthogonal

$\langle \tilde{v}, v_i \rangle = \langle h, v_i \rangle - c_i \langle v_i, v_i \rangle = 0$

$\therefore \tilde{v} \in \mathcal{A}_n$

Thus ⓐ holds for $\tilde{v} \Rightarrow \int (\Delta u + m u) \tilde{v} = 0 \rightarrow$ ⓒ

$\sum_{k=1}^{n-1} c_k$ ⓑ - ⓒ gives us

$\int (\Delta u + m u) h = 0$

This is true for any $h \in \mathcal{A}$

$\Rightarrow \Delta u + m u = 0$

So, m is an eigenvalue with eigenf^n u

Claim: $m = \lambda_n$

Consider any eigenvalue $\lambda \geq \lambda_{n-1}$ with eigenf^n w

Noting, $w \in A_n$, the proof is same as before

We get the following th^m if we take arbitrary f^n instead of eigenf^n

Th^m 3 (Maximin Principle) $(n \geq 2)$ fix $y_1, \dots y_{n-1}$ are piecewise-cont^n f^n

Define $A_n' := \{ \omega \in A \mid \langle \omega, y_1 \rangle = \dots \langle \omega, y_{n-1} \rangle = 0 \}$

and $Q : A_n' \longrightarrow \mathbb{R}$ $\quad Q[\omega] := \dfrac{\|\nabla \omega\|^2}{\|\omega\|^2}$

Let λ_n' be the minimum value of Q, then

$$\lambda_n = \max \lambda_n' , \text{ max taken over all } y_1, \dots y_{n-1} \in A$$

Pf Let $v_1, \dots v_n$ be the normalized eigenf^n for eigenvalues $\lambda_1, \dots \lambda_n$

Let $c_1, \dots c_n$ be the sol^n to the system of eq^n

$$\sum_{i=1}^{n} \langle v_i, y_k \rangle c_i = 0 \qquad 1 \leq k \leq n-1$$

(non-trivial sol^n exists as no. of eq^n $= n-1 < n =$ no. of unknowns)

Define $\omega : \Omega \to \mathbb{R}$, $\omega := \sum_{i=1}^{n} c_i v_i$

So, $\langle \omega, y_k \rangle = \sum_{i=1}^{n} c_i \langle v_i, y_k \rangle = 0$

$\Rightarrow \omega \in A_n'$

$$\Rightarrow \lambda_n' \leq \frac{\|\nabla \omega\|^2}{\|\omega\|^2} = \frac{\langle \nabla \omega, \nabla \omega \rangle}{\langle \omega, \omega \rangle} = \frac{\sum_{i,j} c_i c_j \int \nabla v_i \nabla v_j}{\sum_{i,j} c_i c_j \int v_i v_j}$$

$$= \frac{-\sum_{i,j} c_i c_j \int (\Delta v_i) v_j}{\sum_i c_i^2} \quad (\text{from } 1)$$

$$= \frac{\sum_i c_i^2 \lambda_i}{\sum_i c_i^2} = \lambda_i \leq \lambda_n$$

$\Rightarrow \max \lambda_n' \leq \lambda_n$ over $y_1, \dots, y_{n-1} \in A$

But from prev th^m, $\lambda_n' = \lambda_n$ when $y_1 = v_1, \dots y_{n-1} = v_{n-1}$

$\therefore \max \lambda_n' = \lambda_n$

Neumann condition

Also called free condition as we donot require any boundary cond^n on the admissable f^n and get the corresponding th^m for NC.

1st eigenvalue

Define $\mathcal{B} = \{ w \in C^2(\bar{\Omega}) \setminus 0 \}$ and $Q : \mathcal{B} \to \mathbb{R}$, $Q[w] = \frac{\|\nabla w\|^2}{\|w\|^2}$

Th^m 4 If $u \in \mathcal{B}$ is a minimizer of Q and $m = Q[u]$, then $\lambda_1 = m$ and $\Delta u = -\lambda_1 u$

Pf Same proof as th^m 1 with $\mathcal{B}$ instead of $\mathcal{A}$ until (**), gives us

$$\int \nabla u \cdot \nabla v = m \int uv \qquad \text{for } v \in \mathcal{B}$$

From Green's identity, $\int \nabla u \cdot \nabla v = \iint_{\partial\Omega} v \nabla u \cdot \hat{n}\, dS - \int v \Delta u$

$$\Rightarrow \int (\Delta u + mu) v = \iint_{\partial\Omega} v \nabla u \cdot \hat{n}\, dS \quad \text{for } v \in \mathcal{B}$$

Take v to be arbitrary inside Ω & 0 on $\partial\Omega$

$$\Rightarrow \int (\Delta u + mu) v = 0 \quad \Rightarrow \quad \Delta u + mu = 0 \; [v - \text{arbitrary}]$$

$$\Rightarrow \iint_{\partial\Omega} v \frac{\partial u}{\partial n} = 0$$

Take $v = \frac{\partial u}{\partial n}$ on $\partial\Omega$, $\Rightarrow \int \left(\frac{\partial u}{\partial n}\right)^2 = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial n} = 0$

So, m is an eigenvalue with eigenf^n u with NC

$m = \lambda_1$ follows as before

Th^m 5 $\mathcal{B}_n := \{ w \in \mathcal{B} \mid \langle w, v_1 \rangle = \dots \langle w, v_{n-1} \rangle = 0 \}$ instead of $\mathcal{A}_n$ in th^m 2

Pf same modification as before (when using ②)

Th^m 6 (Maximin Principle) $\mathcal{B}_n' = \{ w \in \mathcal{B} \mid \langle w, y_1 \rangle = \dots \langle w, y_{n-1} \rangle = 0 \}$ instead of $\mathcal{A}_n'$ in th^m 3

Pf same

Notation λ_n - eigenvalues with DC , μ_n - with NC

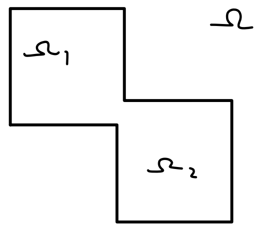
Th^m $\mu_n \leq \lambda_n \quad \forall n$

Pf In th^m 1 & 4, $A \subset B \Rightarrow \min_{w \in A} Q[w] \geq \min_{w \in B} Q[w]$

$\Rightarrow \mu_1 \leq \lambda_1$

Similarly, in th^m 3 & 5, for fixed $y_1, \dots y_{n-1}$, $A_n' \subset B_n'$

$\Rightarrow \mu_n' \leq \lambda_n' \quad \Rightarrow \max \mu_n' = \mu_n \leq \lambda_n = \max \lambda_n'$



Now we look at eigenvalues of subdomains

We consider $\Omega = \Omega_1 \cup \Omega_2$, where Ω_i are the squares

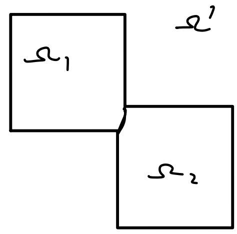
Denote the inc seq of the union of the eigenvalues on Ω_1 & Ω_2 (with multiplicity) by λ_n^* with DC & μ_n^* with NC

Th^m $\lambda_n \leq \lambda_n^*$

Pf In th^m 3, take $A^* = \{ w \in C^2(\Omega) \setminus 0 \mid w|(\partial\Omega \cup \partial\Omega_1 \cup \partial\Omega_2) = 0 \}$ instead of A

then solving the minimum problem gives us eigenvalue of the following Ω'

but this is λ_n^* [$\because$ set of eigenvalues of Ω' = union of set of eigenvalues of Ω_1, Ω_2]



$\because A^* \subset A \quad \Rightarrow \lambda_n \leq \lambda_n^*$

Th^m $\mu_n^* \leq \mu_n$

Pf Take $B^* = \{ w \in C^2(\Omega_1 \cup \Omega_2) \}$

With the same argument as previous

$\because B \subset B^*$ (for $w \in B$, w must be C^2 on the overlapping bd of Ω_1 & Ω_2)

$\Rightarrow \mu_n^* \leq \mu_n$

Cor $\mu_n^* \leq \mu_n \leq \lambda_n \leq \lambda_n^*$ or $\sum A_i(\lambda) \leq A(\lambda) \leq B(\lambda) \leq \sum B_i(\lambda)$

$A(\lambda)$ = no. of eigenvalue (with multiplicity) with NC on Ω $\leq \lambda$

$B(\lambda)$ = — DC on Ω

$A_i(\lambda)$ = — NC on Ω_i

$B_i(\lambda)$ = — DC on Ω_i

<u>Weyl's Law</u> $\quad \lim_{\lambda\to\infty} \frac{B(\lambda)}{\lambda} = \lim_{\lambda\to\infty} \frac{A(\lambda)}{4\pi} = \frac{\text{Area}(\Omega)}{4\pi}$

Pf We know, for Ω_i,

$$\lim_{\lambda\to\infty} \frac{B_i(\lambda)}{\lambda} = \lim_{\lambda\to\infty} \frac{A_i(\lambda)}{\lambda} = \frac{\text{Area}(\Omega_i)}{4\pi}$$

From the corollary,

$$A_1(\lambda) + A_2(\lambda) \le A(\lambda) \le B(\lambda) \le B_1(\lambda) + B_2(\lambda)$$

The result follows using sandwich theorem

For arbitrary domain

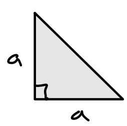
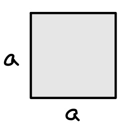
Th^m Ω is transformed ptwise to Ω' by eq^n: $x' = x + g(x,y)$, $y' = y + h(x,y)$
g, h - cont^n, piecewise cont^n 1st derivative & $|g|, |h|, |g_x|, |g_y|, |h_x|, |h_y| < \varepsilon$.
$\exists\ \eta_\varepsilon > 0$ st $\eta_\varepsilon \to 0$ as $\varepsilon \to 0$ & $\left|\frac{\lambda_n'}{\lambda_n} - 1\right| < \eta_\varepsilon$

Pf (we take normalized f^n)
Consider $Q[w] = \|\nabla w\|^2 = \int (w_x^2 + w_y^2)$
By change of variable $\left(F: A \to F(B),\ \int_{F(A)} f = \int_A f \circ F\ |\det DF|\right)$
$F: \Omega \to \Omega' \quad F(x,y) = (x', y') = (x + g(x,y),\ y + h(x,y))$
$$Q[w] = \int_\Omega w_x^2 + w_y^2\, dx\,dy = \int_{\Omega'} (w_x^2 + w_y^2) \circ F^{-1}\ |\det DF^{-1}|$$
$$= \int_{\Omega'} (w'_{x'}(1+g_x) + w'_{y'} h_x)^2 + (w'_{x'} g_y + w'_{y'}(1+h_y))^2\ M^{-1}$$

$$[w'(x', y') := w(x,y) \Rightarrow w = w' \circ F \Rightarrow [w_x \ \ w_y] = [w'_{x'} \ \ w'_{y'}] \begin{bmatrix} 1+g_x & g_y \\ h_x & 1+h_y \end{bmatrix}$$

$$Q'[w'] = \int_{\Omega'} w'^2_{x'} + w'^2_{y'}$$

$$Q[w] - Q'[w'] =$$

Lemma For $\Omega_1 =$  & $\Omega_2 =$  , $\lambda_n^{(1)} \geq \lambda_n^{(2)}$ or $N^{(1)}(\lambda) \leq N^{(2)}(\lambda)$
Pf set of eigenvalues for Ω_1 $\subset$ set of eigenvalues for Ω_2

Lemma For arbitrary right Δ, same
Pf change of variable $(x' = x,\ y' = \frac{ay}{b})$

Weyl's Law arbitrary bounded Ω
subdivide into squares. let n = no. of squares in the interior & m = intersecting $\partial\Omega$
then $\sum A_i(\lambda) \leq A(\lambda) \leq \sum B_i(\lambda) + \sum B_i'(\lambda)$
$$\frac{na^2}{4\pi} \leq \frac{A(\lambda)}{\lambda} \leq \frac{na^2}{4\pi} + \frac{mca^2}{4\pi}$$

$ma < k$ as a^2 can be made arbitrarily small (given $\delta > 0$, $\exists a$, $ma^2 < \delta$)

$$\text{Area}(\Omega) \leq n a^2 + m a^2 \quad \Rightarrow \quad \text{Area}(\Omega) - \delta \leq n a^2$$

$$\liminf \frac{A(\lambda)}{\lambda} \geq \liminf \frac{n a^2}{4\pi} \geq \liminf \frac{\text{Area}(\Omega) - \delta}{\lambda}$$

$$\limsup \frac{A(\lambda)}{\lambda} \leq \limsup \frac{n a^2}{4\pi} + c \limsup \frac{m a^2}{4\pi}$$

<u>Theorem</u> (Minimax Principle) fix $y_1, \dots y_n \in A$

Define $A_n^* = \text{span}\{y_1, \dots y_n\}$ and $Q : A_n^* \to \mathbb{R}$, $Q[w] = \frac{\|\nabla w\|^2}{\|w\|^2}$

Let $\lambda_n^* = \max_{w \in A_n^*} Q[w]$, then $\lambda_n = \min \lambda_n^*$ over all possible $y_1, \dots y_n$

<u>Th^m</u> (Monotonicity of Dirichlet eigenvalues)

$\Omega \subset \tilde{\Omega} \Rightarrow \lambda_n \geq \tilde{\lambda}_n$

<u>Pf</u> For fixed $y_1, \dots y_n$, let $y = \sum_{i=1}^{n} c_i y_i$ be the maximizer of $Q : A_n^* \to \mathbb{R}$

ie, $\lambda^* = Q[y]$

Define $\tilde{y}_i := y_i$ on Ω & 0 on $\tilde{\Omega} \setminus \Omega$ & $\tilde{y} := \sum c_i \tilde{y}_i$

then $Q[y] = \tilde{Q}[\tilde{y}]$

Now, $\lambda_n = \min_{y_i \in A} \max_{w \in \langle y_i \rangle} Q[w] = \min_{\tilde{y}_i \in \tilde{A}} \max_{\tilde{w} \in \langle \tilde{y}_i \rangle} \tilde{Q}[\tilde{w}] \geq \min_{y_i \in \tilde{A}} \max_{\tilde{w} \in \langle y_i \rangle} \tilde{Q}[\tilde{w}] = \tilde{\lambda}_n$

$\hookrightarrow$ $\tilde{y}_i$ are the one extended from $y_i \in A$ $\qquad$ $\hookrightarrow$ all $y_i \in \tilde{A}$

<u>Def^n</u> (Jordan measurable) $\Omega \subset \mathbb{R}^2$ is Jordan measurable if

$\text{area}(\Omega) = \sup_{\Omega_1 \subset \Omega} \text{area}(\Omega_1) = \inf_{\Omega \subset \Omega_1} \text{area}(\Omega_2)$, Ω_i are union of open squares

<u>Th^m</u> Weyl's Law holds for Jordan measurable domains

<u>Pf</u> Given $\varepsilon > 0$, $\exists\, \Omega_1, \Omega_2$ - finite union of rectangles st $\Omega_1 \subset \Omega \subset \Omega_2$ and

$\text{area}(\Omega) + \varepsilon \geq \text{area}(\Omega_2)$ & $\text{area}(\Omega) - \varepsilon \leq \text{area}(\Omega_1)$

Note $\Omega_1 \subset \Omega \subset \Omega_2 \Rightarrow \lambda_n^{(1)} \geq \lambda_n \geq \lambda_n^{(2)} \Rightarrow N^{(1)}(\lambda) \leq N(\lambda) \leq N^{(2)}(\lambda)$

Now, $\limsup_{\lambda \to \infty} \frac{N(\lambda)}{\lambda} \leq \limsup_{\lambda \to \infty} \frac{N^{(2)}(\lambda)}{\lambda} = \frac{\text{area}(\Omega_2)}{4\pi} \leq \frac{\text{area}(\Omega) + \varepsilon}{4\pi}$

$\because \varepsilon$ is arbitrary, $\limsup_{\lambda \to \infty} \frac{N(\lambda)}{\lambda} \leq \frac{\text{area}(\Omega)}{4\pi}$ $\qquad$ $[a \leq b + \varepsilon \;\; \forall \varepsilon > 0 \Rightarrow a \leq b]$

and $\liminf_{\lambda \to \infty} \frac{N(\lambda)}{\lambda} \geq \liminf_{\lambda \to \infty} \frac{N^{(1)}(\lambda)}{\lambda} = \frac{\text{area}(\Omega_1)}{4\pi} \geq \frac{\text{area}(\Omega) - \varepsilon}{4\pi}$

$\|^{ly}$, $\liminf_{\lambda \to \infty} \frac{N(\lambda)}{\lambda} \geq \frac{\text{area}(\Omega)}{4\pi}$

$\therefore \lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda} = \frac{\text{area}(\Omega)}{4\pi}$