Mapping class groups

Curves

(Study linear transf by action on vectors $\leftrightarrow$ Mod(S) by action on homotopy classes of simple closed curves)

Classification of surfaces

Closed connected oriented $\cong$ connect sum of S^2 & g tori (genus)

$\hookrightarrow$ cmpt without boundary

$$\text{closed} \xrightarrow[\text{open disks}]{\text{remove } b} \text{cmpt} \qquad b = \text{no. of boundary comp}$$

$$\text{cmpt} \xrightarrow[\text{pts}]{\text{remove } n} \text{non-cmpt} \qquad n = \text{no of punctures} \sim \text{marked pts}$$

So, surface is determined by (g, b, n)

$\chi(S) = 2 - 2g - (b+n)$

Hyperbolic plane $\mathbb{H}^2$ $\left(\mathbb{H}^2, \frac{dx^2+dy^2}{y^2}\right)$ $= \frac{dz\,d\bar{z}}{\mathrm{Im}(z)^2}$ or $\left(D, 4\frac{dx^2+dy^2}{(1-r^2)^2}\right)$ open disk in $\mathbb{C}$

$\nwarrow$ Riemannian metric on $\mathbb{H}$ - upper half plane

- $g_{\mathbb{H}^2,(x,y)}(v,w) = \frac{v^1w^1+v^2w^2}{y^2} = \frac{1}{y^2}\langle v,w\rangle$ (Euclidean), $v,w \in T_{(x,y)}\mathbb{H}^2$
- $\|v\|_{\mathbb{H}^2} = (\langle v,v\rangle_{\mathbb{H}^2})^{1/2} = \frac{1}{y}\sqrt{\langle v,v\rangle}$

$$l(\gamma) = \int_0^1 \|\gamma'(t)\|\,dt = \int_0^1 \frac{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}}{y(t)}\,dt$$

$d(p,q) = \inf l(\gamma)$, $p,q \in \mathbb{H}^2$, γ - piecewise C^∞ paths joining p,q

- d is a metric

Thm $\mathrm{Isom}(\mathbb{H}^2) = PSL(2,\mathbb{R})$

Lemma All mobius transf are isometries of $\mathbb{H}^2$

Thm $\chi(S) < 0$, then S admits a hyperbolic metric, ie, $\exists$ a complete, finite-area Riemannian metric of constant curvature -1, ∂S is totally geodesic (geodesics in ∂S are geodesics in S)

$\chi(S) = 0$, then S admits an Euclidean or flat metric, ie, $\exists$ a complete, finite-area Riemannian metric of constant curvature 0 and totally geodesic boundary

Closed curve cont map $S^1 \longrightarrow S$
Simple if injective
Essential if not homotopic to a pt, puncture or boundary component

Geodesic representative S - hyperbolic surface, α - closed curve in S not homotopic to a nbhd of a puncture, then α is homotopic to a unique geodesic closed curve γ
Prop if α is simple then γ is simple

Intersection number
Algebraic $\hat{i}(a, b) =$
Geometric $i(a, b) =$

Bigon criterion 2 transverse simple closed curves are in minimal position if and only if they do not form a bigon
Corollary distinct simple closed geodesics in hyperbolic surface are in minimal pos^n

Isotopy bet^n SC curves α, β is a homotopy from α to β such that $H_t : S^1 \to S$ is SC curve $\forall t \in [0,1]$

Theorem α, β - essential SC curve in S, α is isotopic to β iff homotopic

Isotopy of surface homotopy $H: S \times I \to I$ st $H_t : S \to S$ is a homeo

Extension of isotopy $F: S^1 \times I \to S$ - smooth isotopy of SC curves, $\exists$ isotopy $H: S \times I \to S$ st $H|_{S \times 0} = id$ & $H|_{F(S^1 \times 0) \times I} = F$

Change of coordinate principle ($\leftrightarrow$ change of basis)

Cut surface of α $S_\alpha = S \setminus N_\alpha$. N_α - annular nbhd of α

Topology - type of α is the homeomorphism - type of cut surface S_α

Theorem $\exists$ orientation - preserving homeo of a surface taking SC curve to another iff their cut surfaces are homeo

Theorem S - cmpt, f, g - homotopic homeomorphisms of S, then they are isotopic (except or - reversing homeo on D^2 and or - reversing fixing S^1 on $A = S^1 \times I$ - closed annulus)

Theorem S - cmpt, homeo of S are isotopic to diffeo of S

Mapping Class Groups

$\mathrm{Mod}(S) = \mathrm{Homeo}^+(S, \partial S) \,/\, \mathrm{Homeo}_0(S, \partial S)$

O-P homeo on S identity on ∂S ; connected comp of id in $\mathrm{Homeo}^+(S, \partial S)$

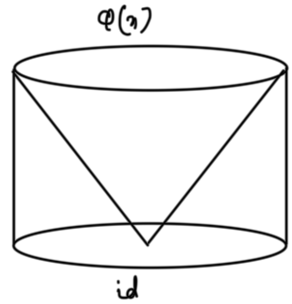
or, the group of isotopy classes of $\mathrm{Homeo}^+(S,\partial S)$, with isotopies fixing boundary pointwise (isotopies relative to boundary)

$$\begin{aligned}\mathrm{Mod}(S) &= \mathrm{Homeo}^+(S,\partial S)/\text{isotopy} \\ &= \mathrm{Homeo}^+(S,\partial S)/\text{homotopy} \\ &= \mathrm{Diff}^+(S,\partial S)/\text{smooth isotopy or smooth homotopy}\end{aligned}$$

- we consider maps that leave the set of marked points invariant

Examples

$\mathrm{Mod}(D^2) = 0$ (Alexander lemma) $\left[f_t(x) = \begin{cases} t\varphi(x/t) & 0 \le |x| \le t \\ x & t \le |x| \le 1 \end{cases} \right]$



$\mathrm{Mod}(D^2 \setminus p) = 0$ [take p to lie at origin]

$\mathrm{Mod}(S_{0,1}) = 0$ [$S_{0,1} \cong \mathbb{R}^2$ & then straight-line homotopy]

$\mathrm{Mod}(S_{0,3}) = S_3$ [by action on set of marked pts]

inj: α- curve from p to q & φ- homeo

Lemma: ess simple arcs with same endpt are isotopic

So, α & $\varphi(\alpha)$ are isotopic & φ can be extended to a map that fixes α pointwise (cld come from α)

Cut along α to get a disk with 1 MP, φ here is homotopic to identity (from eg 2) which give a homotopy on $S_{0,3}$

$\mathrm{Mod}(S_{0,2}) = \mathbb{Z}_2$ [similarly as above]

$\mathrm{Mod}(S^2) = 0$ [similarly as above]

$\mathrm{Mod}(A) = \mathbb{Z}$

$f: \mathrm{Mod}(A) \to \mathbb{Z}$

$[\varphi] \in \mathrm{Mod}(A)$ $[\varphi] \mapsto \tilde{\varphi}_1(0)$, $\tilde{\varphi}$ is the lift of φ to $\mathbb{R}^1 \times [0,1]$

and $\tilde{\varphi}_1$ is the restriction to $\mathbb{R} \times \{1\}$, so $\tilde{\varphi}_1$ being the lift of id on $S^1 \times \{1\}$ is an integer translation

surj: $M = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ is equivariant wrt group of deck transf, so descends

to a homeo φ of A $[\tilde{\varphi}(x,y) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}]$
$\tilde{\varphi}_0(x,1) = \begin{pmatrix} x+n \\ 1 \end{pmatrix} \Rightarrow \rho(\varphi) = n$

inj: $\rho(\varphi) = 0$, st-line homotopy from $\tilde{\varphi}$ to id, being equivariant, descends, and equivariant because (?) $\tilde{\varphi}(\tau \cdot x) = \varphi_*(\tau) \cdot \tilde{\varphi}(x)$

$\mathrm{Mod}(T^2) = SL(2,\mathbb{Z})$
φ induces φ_* on $H_1(T^2) = \mathbb{Z}^2$, φ_* is an auto $\Rightarrow \varphi_* \in \mathrm{Aut}(\mathbb{Z}^2) = GL(2,\mathbb{Z})$

Alexander method
Lemma $\varphi \in \mathrm{Homeo}^+(S, \partial S)$, $\alpha_1, \dots \alpha_n$ and $\beta_1, \dots \beta_n$ of ESC curve and arc satisfying
γ_i are pairwise (1) in minimal position and (2) non isotopic
(3) for distinct i,j,k, at least one of $\alpha_i \cap \alpha_j$, $\alpha_i \cap \alpha_k$, $\alpha_k \cap$; $= \phi$, for β...
If α_i is iso to β_i $\forall i$, then there is a relative isotopy of S that takes $\bigcup \alpha_i$ to $\bigcup \beta_i$

Structure graph $\Gamma_{\{\alpha_i\}}$ is the graph with vertices at intersection pts

Alexander method $\{\alpha_i\}$ fills S — complement of $\bigcup \alpha_i$ is a disjoint union of disks & once-marked disks
i.
φ induces auto φ_* of Γ
ii. if φ_* is trivial then φ_* is iso to id

Dehn Twists

Properties

1. α, β - ESC curve, $k \in \mathbb{Z}$

$$i(T_a^k(b), b) = |k|\, i(a,b)^2$$

2. $T_a = T_b \iff a = b$

$(\Rightarrow)$ $a \neq b$, $\exists c$ st $i(a,c) = 0$, $i(b,c) \neq 0$

$$i(T_a(c), c) = i(a,c)^2 = 0 \neq i(b,c)^2 = i(T_b(c), c)$$

3. $f \in \mathrm{Mod}(S)$, $T_{f(a)} = f T_a f^{-1}$

4. $f T_a = T_a f \iff f(a) = a$

$$f T_a = T_a f \iff f T_a f^{-1} = T_a = T_{f(a)} \iff a = f(a)$$

5. $i(a,b) = 0 \iff T_a(b) = b \iff T_a T_b = T_b T_a$

Braid relation $i(a,b) = 1$

$\iff T_a T_b T_a = T_b T_a T_b$, equivalently $T_a T_b (a) = b$