Noncommutative Poincare recurrence and multiple recurrence

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C*-algebra

Definition (Banach algebra)

A *Banach algebra* A is an algebra over $\mathbb{C}$ with a norm with respect to which it is a normed space and is sub-multiplicative: for all $x, y \in A$, $\|xy\| \le \|x\| \, \|y\|$.

Definition (Involution)

An *involution* on algebra A is a map $A \ni x \mapsto x^* \in A$ such that

$$(x+y)^* = x^* + y^*, \quad (\lambda x)^* = \overline{\lambda} x^*, \quad (xy)^* = y^* x^*, \quad x^{**} = x.$$

x^* is called *adjoint* of x.

Definition (C*-algebra)

A *C*-algebra* is a Banach algebra with an involution that satisfies the C*-condition:

$$\|x^* x\| = \|x\|^2 .$$

C*-algebra

Example

1. $\mathcal{B}(\mathcal{H})$ on a Hilbert space H
2. Norm-closed *-subalgebras of $\mathcal{B}(\mathcal{H})$ are called 'concrete' C*-algebras.
3. $L^{\infty}(\mathbb{R})$ with pointwise operations and involution $(f \mapsto \bar{f})$.
4. $L^{1}(\mathbb{R})$ with convolution and above involution is not a C*-algebra.
5. $C_0(X)$ on locally compact Hausdorff space X and by Gelfand-Naimark theorem, every commutative C*-algebra is isometrically isomorphic to some $C_0(X)$.

Positive elements

$a \in \mathcal{B}(\mathcal{H})$

Definition

Resolvent $\rho(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \text{ is invertible}\}$.

Spectrum $\sigma(a) = \mathbb{C} \setminus \rho(A)$.

Definition (Positive elements)

$a \in \mathcal{B}(\mathcal{H})$ is *positive* if $a^* = a$ and $\sigma(a) \geq 0$.

Theorem

The following are equivalent:

(i) *a is positive.*

(ii) *$a = b^2$ for some $b \in \mathcal{B}(\mathcal{H})$.*

(iii) *$a = x^*x$ for some $x \in \mathcal{B}(\mathcal{H})$.*

(iv) *$\langle x\xi, \xi \rangle \geq 0$ for all $\xi \in \mathcal{H}$.*

Positive elements span $\mathcal{B}(\mathcal{H})$.

States

Definition

Positive functional A functional $\phi : A \to \mathbb{C}$ if $\phi(a) > 0$ when $a > 0$.

States A positive linear functional of norm 1.

Example

$0 < \xi \in H$, define $\phi : \mathcal{B}(H) \to \mathbb{C},\ \phi(x) = \langle x\xi, \xi \rangle$ is positive. ϕ is a state if $\|\xi\| = 1$.

Theorem

(i) *Positive linear functionals are continuous.*

(ii) *If e_λ is an approximate identity on A, then $\|\phi\| = \lim \phi(e_\lambda)$.*

(iii) *A continuous linear functional is a state, if for some approximate identity e_λ, $\|\phi\| = 1 = \lim \phi(e_\lambda)$.*

GNS-representation

Definition

Representation *Representation of A on a Hilbert space H* is a $*$-homomorphism from A to $\mathcal{B}(H)$.

Cyclic vector $\xi \in H$, for the representation $\pi : A \to \mathcal{B}(H)$ if $\{\pi(x)\xi \mid x \in A\}$ is dense in H.

Theorem (GNS construction)

For any state, ϕ on A, there exists a representation π_ϕ on a Hilbert space H_ϕ, with a cyclic vector ξ_ϕ such that $\|\phi\| = \|\xi_\phi\|^2$ and

$$\phi(x) = \langle \pi_\phi(x)\xi_\phi, \xi_\phi \rangle \ \ \forall x \in A.$$

$(H_\phi, \pi_\phi, \xi_\phi)$ is called the GNS triple for ϕ. We denote the vector $\pi_\phi(x)\xi_\phi$ as $\hat{x}$.

Weak topology

Definition (Weak topology)

$\{f_i : X \to X_i\}_{i \in I}$ is a family of maps, and τ_i is a topology on X_i with subbases S_i. We define a topology τ on X, called the *weak topology induced by* $\{f_i\}$, by defining a subbases, $S = \{f_i^{-1}(V) \mid V \in S_i\}_{i \in I}$.

Theorem

(i) *τ is the smallest topology such that the f_i's are continuous.*

(ii) *For a topological space Z and a function $g : Z \to X$, g is continuous if and only if $f_i \circ g$ is continuous for all $i \in I$.*

(iii) *A net $\{x_\lambda\}_{\lambda \in \Lambda}$ converges to x in τ if and only if the net $f_i(x_\lambda)_{\lambda \in \Lambda}$ converges to $f_i(x)$ in τ_i for all $i \in I$.*

Weak topologies

Definition (Topology induced by seminorms)

If X is a vector space, and $\{p_i \mid i \in I\}$ is a family of seminorms on X. For $x \in X,\ i \in I$, define linear forms $f_{i,x} : X \to [0, \infty)$, $f_{i,x}(y) = p_i(y - x)$. Then the topology τ on X induced by these linear forms is called the *topology induced by seminorms*.

Theorem

(i) *For each* $x \in X,\ i \in I,\ \varepsilon > 0.$ *define* $U_{(i,x,\varepsilon)} = \{y \in X \mid p_i(y - x) < \varepsilon\}.$ *The family* $\{U_{(i,x,\varepsilon)} \mid x \in X,\ i \in I,\ \varepsilon > 0\}$ *forms a subbases for* τ.

(ii) *A net* $\{x_\lambda\}_{\lambda \in \Lambda}$ *converges to* x *in* τ *if and only if the net* $\{p_i(x_\lambda - x)\}_{\lambda \in \Lambda}$ *converges to* 0 *in* $\mathbb{R}$ *for all* $i \in I$.

(iii) (X, τ) *is a topological vector space.*

(iv) τ *is the smallest topology such that* (X, τ) *is a topological vector space with* p_i *continuous for all* $i \in I$.

Topologies on $\mathcal{B}(\mathcal{H})$

Weak operator is induced by the family of semi-norms

$$x \mapsto |\langle x\xi, \eta\rangle| \text{ for } \xi, \eta \in \mathcal{H}.$$

Strong operator is induced by the family of semi-norms

$$x \mapsto \|x\xi\| \text{ for } \xi \in \mathcal{H}.$$

Ultraweak or σ-weak or w-topology is induced by the family of semi-norms

$$x \mapsto \left|\sum_{k=1}^{\infty} \langle x\xi_k, \eta_k\rangle\right| \text{ for } \{\xi_k\}, \{\eta_k\} \in \mathcal{H},$$

$$\sum_{k=1}^{\infty} \|\xi_k\|^2 < \infty, \sum_{k=1}^{\infty} \|\eta_k\|^2 < \infty.$$

Topologies on $\mathcal{B}(\mathcal{H})$

- For $\xi, \eta \in \mathcal{H}$, define the linear forms on $\mathcal{B}(\mathcal{H})$, $\omega_{\xi,\eta}(x) = \langle x\xi, \eta\rangle$.
- Let $\mathcal{B}(\mathcal{H})_\sim$ be the vector space generated by these forms in $\mathcal{B}(\mathcal{H})^*$ and let $\mathcal{B}(\mathcal{H})_*$ be the norm closure of $\mathcal{B}(\mathcal{H})_\sim$ in $\mathcal{B}(\mathcal{H})^*$.
- Then wo-topology is also $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H})_\sim)$. And we can show w-topology is given by $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H})_*)$. This follows from the density of finite-rank operators in trace-class operators.

Theorem

Let $\mathcal{H}$ be a Hilbert space. Then

(i) *$\mathcal{B}(\mathcal{H})_\sim$ is the set of all wo-continuous linear forms on $\mathcal{B}(\mathcal{H})$.*

(ii) *$\mathcal{B}(\mathcal{H})_*$ is the set of all w-continuous linear forms on $\mathcal{B}(\mathcal{H})$.*

(iii) *wo-topology and w-topology coincide in $\mathcal{B}(\mathcal{H})_1$.*

(iv) *A linear form ϕ on $\mathcal{B}(\mathcal{H})$ is w-continuous $\iff$ its restriction to $\mathcal{B}(\mathcal{H})_1$ is wo-continuous.*

von Neumann algebra

Definition (Commutant)

Let $\mathcal{X} \subset \mathcal{B}(\mathcal{H})$, then commutant of $\mathcal{X}$,
$\mathcal{X}' = \{x' \in \mathcal{B}(\mathcal{H}) \mid x'x = xx' \text{ for all } x \in \mathcal{X}\}$

Theorem (von Neumann density theorem)

*Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a unital *-subalgebra, then*

$$\mathcal{A}'' = \overline{\mathcal{A}}^{wo} = \overline{\mathcal{A}}^{so} = \overline{\mathcal{A}}^{\sigma-wo}$$

Definition (von Neumann algebra)

A subalgebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is a *von Neumann algebra* if it is unital, self-adjoint and equals one of the above.

Operators on Hilbert space

On the set of all self-adjoint operators, $\mathcal{B}(\mathcal{H})_{\mathrm{sa}}$, we have an order relation:

$$x, y \in \mathcal{B}(\mathcal{H})_{\mathrm{sa}}, \; x \geq y \iff x - y \geq 0.$$

Theorem

Let $\{x_i\} \subset \mathcal{B}(\mathcal{H})_{sa}$ be a bounded increasing net. Then, there is an $x \in \mathcal{B}(\mathcal{H})_{sa}$ such that $x = \sup_i x_i$. Moreover, $x = so\text{-}\lim_i x_i$.

Theorem (Borel functional calculus)

*$x \in \mathcal{B}(\mathcal{H})_{sa}$, then we have a *-homomorphism $\mathcal{B}(\sigma(x)) \ni f \mapsto f(x) \in \mathcal{R}(\{x\})$, the von Neumann algebra generated by x.*

Corollary

A von Neumann algebra equals the norm-closed linear span of its projections.

Lattice of projections

Denote the set of all projections on $\mathcal{B}(\mathcal{H})$ by $\mathcal{P}_{\mathcal{B}(\mathcal{H})}$.

Definition

Let $\{e_i\}_{i\in I} \subset \mathcal{P}_{\mathcal{B}(\mathcal{H})}$. Define

- $\bigvee_{i\in I} e_i =$ projection onto $\overline{\sum_{i\in I} e_i\mathcal{H}}$.
  It is the least upper bound of the family $\{e_i\}$
- $\bigwedge_{i\in I} e_i =$ projection onto $\bigcap_{i\in I} e_i\mathcal{H}$.
  It is the greatest lower bound of the family $\{e_i\}$

Theorem

$\mathcal{P}_{\mathcal{B}(\mathcal{H})}$ *forms a complete lattice.*

Theorem

Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{N}$ be a left ideal, then there exists a unique projection $e \in \mathcal{M}$ such that $\overline{\mathcal{N}}^w = \mathcal{M}e$.

Kaplansky density theorem

Theorem

*Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If $\mathcal{A} \subset \mathcal{M}$ be a so-dense *-subalgebra of $\mathcal{M}$, then the unit ball of $\mathcal{A}$ is so-dense in the unit ball of $\mathcal{M}$.*

Corollary

*Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a unital *-algebra. Then $\mathcal{M}$ is a von Neumann algebra if and only if $\mathcal{M}_1$ is w-compact.*

Corollary

*Let $\phi : \mathcal{M}_1 \to \mathcal{M}_2$ be a unital w-continuous *-homomorphism between von Neumann algebra, then $\phi(\mathcal{M}_1)$ is a von Neumann algebra.*

W*-algebra

Definition (W*-algebra)

A C*-algebra $\mathcal{M}$ if it admits a predual $\mathcal{M}_*$, that is, $\mathcal{M}$ is isometrically isomorphic to the dual space of some Banach space, which we call the predual of $\mathcal{M}$ and denote by $\mathcal{M}_*$.

Theorem

Every von Neumann algebra is a W-algebra.*

(The converse is also true)
Hence, the double dual of a C*-algebra is a von Neumann algebra and by Goldstine theorem, it is weak*-dense. Using this we can reduce C*-algebras problems to von Neumann algebra.

Conditional expectation

Definition

Let $\mathcal{A}$ be a C*-algebra, $\mathcal{B} \subset \mathcal{A}$ be a C*-subalgebra, then a linear map $\Phi : \mathcal{A} \to \mathcal{B}$ is a

Projection if $\Phi(b) = b$ for every $b \in \mathcal{B}$. Then $\Phi \circ \Phi = \Phi$.

$\mathcal{B}$-linear if $\Phi(ab) = \Phi(a)b$ and $\Phi(ba) = b\Phi(a)$ for $a \in \mathcal{A}$, $b \in \mathcal{B}$.

Conditional expectation if it is $\mathcal{B}$-linear and a positive map.

Theorem

Every projection of norm 1, $\Phi : \mathcal{A} \to \mathcal{B}$ is a conditional expectation.

Envelopping von Neumann algebra

Definition (Enveloping von Neumann algebra of a C*-algebra)

Let $\mathcal{A}$ be a C*-algebra and consider the universal representation,

$$\pi_{\mathcal{A}} = \bigoplus_{\phi \in S(\mathcal{A})} \pi_{\phi} : \mathcal{A} \to \mathcal{B}(\mathcal{H}_{\mathcal{A}}).$$

The *enveloping von Neumann algebra of* $\mathcal{A}$, $N_{\mathcal{A}}$ is defined to be w-closure of $\pi_{\mathcal{A}}(\mathcal{A})$.

Theorem

There is a map $N_{\mathcal{A}} \to \mathcal{A}^{**}$ *which is a surjective linear isometry and a* $(w, \sigma(\mathcal{A}^{**}, \mathcal{A}^{*}))$*-homeomorphism.*

Theorem

Let $\mathcal{M}$ *be a von Neumann algebra with predual* $\mathcal{M}_{*}$*, there exists a unique central projection* $p \in N_{\mathcal{M}}$ *such that the map* $\mathcal{M} \ni x \mapsto \pi_{\mathcal{M}}(x)p \in (N_{\mathcal{M}})p$ *is a surjective *-isomorphism.*

Poincare recurrence in von Neumann algebra

Theorem

*Let $\mathcal{M}$ be a von Neumann algebra, ϕ a faithful normal state on $\mathcal{M}$, and $\alpha : \mathcal{M} \to \mathcal{M}$ be a *-homomorphism such that $\phi \circ \alpha = \phi$. Then, for every $p \in \mathcal{M}$ and every $n \in \mathbb{N}$,*

$$\bigvee_{k=n}^{\infty} \alpha^k(p) = \bigvee_{k=0}^{\infty} \alpha^k(p) \geq p.$$

This implies Poincare recurrence: for every projection $p \in \mathcal{M}$,

$$p \wedge \bigwedge_{n=0}^{\infty} \bigvee_{k=n}^{\infty} \alpha^k(p) = p.$$

Poincare recurrence in von Neumann algebra

Proof.

We show α is unital and normal. This gives,
$\ker(\alpha) = \mathcal{M}(1_{\mathcal{M}} - q)$, which implies $\mathcal{M}q \cong \alpha(\mathcal{M})$. Then,

$$\begin{aligned}\alpha\left(\bigvee_{k=n}^{\infty} \alpha^k(p)\right) &= \alpha\left(\left(\bigvee_{k=n}^{\infty} \alpha^k(p)\right) q\right) \\ &= \alpha\left(\left(\bigvee_{k=n}^{\infty} \alpha^k(p) q\right)\right) \\ &= \bigvee_{k=n}^{\infty} \alpha(\alpha^k(p)q) \\ &= \bigvee_{k=n}^{\infty} \alpha^{k+1}(p)\end{aligned}$$

$\square$

C*-dynamical systems

Definition (C*-dynamical system)

A *C*-dynamical system* is a triplet $(\mathfrak{A}, \phi, \alpha)$, where $\mathfrak{A}$ is a C*-algebra, ϕ is a state and $\alpha : \mathfrak{A} \to \mathfrak{A}$ is a C*-algebra homomorphism.

Definition (State-preserving C*-dynamical system)

A C*-dynamical system $(\mathfrak{A}, \phi, \alpha)$ is *state-preserving* if $\phi \circ \alpha = \phi$.

Example

Given a MPS $(X, \mathcal{B}, \mu, T)$, we have a C*-dynamical system, $(\mathfrak{A}, \phi, \alpha)$, where $\mathfrak{A} = L^\infty_\mu(X)$, $\phi(f) = \int f d\mu$, and $\alpha = U_T$.

Noncommutative Poincare Recurrence

Definition (Relatively dense set)

A subset $N \subset \mathbb{N}$ is *relatively dense* if there is an $L > 0$ such that every interval in $\mathbb{N}$ of length L has an element of N.

Note, relatively dense set will have positive density.

Theorem (Noncommutative Khintchine recurrence theorem)

In state-preserving C-dynamical system, $(\mathfrak{A}, \phi, \alpha)$, for every $x \in \mathfrak{A}$ and every $\varepsilon > 0$, there is a relatively dense set $N \subset \mathbb{N}$ such that, for all $n \in n$,*

$$\Re\phi(\alpha^n(x^*)x) \geq |\phi(x)|^2 - \varepsilon.$$

Noncommutative Poincare Recurrence

Theorem (Noncommutative Poincare recurrence theorem)

Let $(\mathfrak{A}, \phi, \alpha)$ *be a state-preserving C*-dynamical system. Then*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\phi(\alpha^k(a^*)a)| > 0,$$

for every $a \in \mathfrak{A}$ *with* $\phi(a) > 0$*.*

Recall,

Lemma

For bounded sequences a_n*,*
$d\text{-}\lim_{n\to\infty} a_n = a \iff \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |a_k - a| = 0.$

Noncommutative Khintchine Recurrence

Definition

Let $\mathfrak{A}$ be a C*-algebra, ϕ a state and $\alpha : \mathfrak{A} \to \mathfrak{A}$ a positive linear map such that $\phi \circ \alpha = \phi$. Consider the GNS triple $(H_\phi, \pi_\phi, \xi_\phi)$ for state ϕ. The linear map $\alpha : \mathfrak{A} \to \mathfrak{A}$ induces a linear map $U_\alpha : H_\phi \to H_\phi,\ U_\alpha(\hat{x}) = \widehat{\alpha(x)}$.

Lemma

1. *If $\phi(\alpha(x)^*\alpha(x)) \leq \phi(x^*x)$ for all $x \in \mathfrak{A}$, then*
   - (i) *U_α is a contraction.*
   - (ii) *$U_\alpha \xi_\phi = \xi_\phi$.*
   - (iii) *If P is the orthogonal projection onto $\{\xi \in H_\phi \mid U_\alpha \xi = \xi\}$, then $U_\alpha P = PU_\alpha = U$, and $\frac{1}{n}\sum_{k=0}^{n-1} U_\alpha^k \xrightarrow{so} P$.*
2. *If α is multiplicative, then*
   - (i) *U_α is an isometry.*
   - (ii) *$U_\alpha U_\alpha^*$ is the orthogonal projection onto $\overline{\pi_\phi(\alpha(\mathfrak{A}))\xi_\phi}$, and thus, belongs to the commutant of $\pi_\phi(\alpha(\mathfrak{A}))$.*
   - (iii) *$U_\alpha \circ \pi_\phi(a) = \pi_\phi(\alpha(a)) \circ T_\alpha$ for all $a \in \mathfrak{A}$.*

Noncommutative Khintchine Recurrence

We consider a unital C*-dynamical system for simplicity.

Lemma

Let H be a Hilbert space and let $T : H \to H$ be an operator such that $\|Tx\| = \|x\|$, for all $x \in H$ and $Tv = v$ for some $v \in H$ with norm 1. Then, for every $x \in H$ and $\varepsilon > 0$, there is a relatively dense set $N \subset \mathbb{N}$ such that, for all $n \in n$,

$$\Re \langle T^n x, x \rangle \geq |\langle x, v \rangle|^2 - \varepsilon.$$

Proof of Khintchine recurrence.

Note, U_α is an isometry and $U_\alpha(\pi_\phi(1)) = \hat{1}$. We apply the lemma on $H = H_\phi$, $U = U_\alpha$ and $v = \hat{1}$. Then

$$\Re \langle U_\alpha^n \hat{x}, \hat{x} \rangle = \Re\phi(\alpha^n(x^*)x) \geq |\langle \pi_\phi(x), \pi_\phi(1) \rangle|^2 - \varepsilon \geq |\phi(x)|^2 - \varepsilon.$$

$\square$

Multiple recurrence for compact systems

Definition (Compact systems)

A C*-dynamical system $(\mathfrak{A}, \phi, \alpha)$ if $\{U^n(\hat{x}) \mid n \in \mathbb{N}\}$ is precompact in H_ϕ for all $x \in \mathfrak{A}$.

Theorem

Let $(\mathfrak{A}, \phi, \alpha)$ be a compact state preserving C-dynamical system, with the support projection of ϕ, $s(\phi)$ in the double dual $\mathfrak{A}^{**}$ being central, then for every $p \in \mathbb{N}$, $m_0, m_1, \ldots, m_p \in \mathbb{N}$, $x_0, x_1, \ldots, x_p \in \mathfrak{A}$ and $\varepsilon > 0$, there is a relatively dense set $N \subset \mathbb{N}$ such that*

$$|\phi(\alpha^{m_0 n}(x_0)\alpha^{m_1 n}(x_1)\ldots\alpha^{m_p n}(n_k)) - \phi(x_0 x_1 \ldots x_p)| \leq \varepsilon \textit{ for all } n \in N,$$

Note, $s(\phi)$ is central if and only if ξ_ϕ is cyclic for $\pi_\phi(\mathfrak{A})$.

Multiple recurrence for compact systems

It follows that

$\phi(\alpha^{m_0 n}(x_0)\alpha^{m_1 n}(x_1)\ldots\alpha^{m_p n}(n_p)) \geq \phi(x_0 x_1 \ldots x_p) - \varepsilon$ for all $\varepsilon > 0$.

As in the proof of noncommutative Poincare recurrence, we have the multiple recurrence property: for $0 < a \in \mathfrak{A}$, if $\phi(a) > 0$, then

$$\liminf_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} |\phi(a\alpha^{m_1 k}(a)\ldots\alpha^{m_p n}(a))| > 0.$$

Multiple recurrence for compact systems

Lemma

Let (X, d) is a totally bounded metric space and $\epsilon > 0$, then the set $\{n \in \mathbb{N} \mid \text{ there are } x_1, \ldots, x_n \in X \text{ such that } d(x_j, x_k) > \varepsilon \text{ for } j \neq k\}$ is bounded.

Theorem

Let (Ω, d) be a metric space, $T : \Omega \to \Omega$ an isometry, and $\omega \in \Omega$, then the following statements are equivalent:

(i) *the orbit of ω, $\{T^n\omega \mid n \in \mathbb{N}\}$ is totally bounded.*

(ii) *for all $\varepsilon > 0$ there exists a relatively dense set $N \subset \mathbb{N}$ such that $d(T^n\omega, \omega) \leq \varepsilon$, for all $n \in N$.*

Multiple recurrence for compact systems

Consider a contraction U on a Hilbert space H. Define the set of almost periodic vectors,

$$H_{AP}^{U} = \{\xi \in H \mid \{U^n(\xi) \mid n \in \mathbb{N}\} \text{ is relatively norm-compact}\}.$$

It is a U-invariant, linear subspace of H and it is easy to show that it closed.

Lemma

Let H be a Hilbert space, $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ a closed linear subspace, $\xi_0 \in H$ such that $\overline{\mathcal{M}'\xi_0} = H$ and U a linear isometry on H such that $\mathcal{M}\xi_0 \subset H_{AP}^{U}$. For a linear contraction $\theta : H \to H \mid \theta(T)\xi_0 \in \overline{\{U^n T\xi_0 \mid n \in \mathbb{N}\}}$ for $T \in \mathcal{M}$. θ has the following recurrence property:
For any integer $p \in \mathbb{N},\ \theta_1, \ldots, \theta_p \in \mathcal{G},\ T_1, \ldots, T_p \in \mathcal{A},\ \xi_1,\ \ldots, \xi_p \in \mathcal{H}$ and $\varepsilon > 0$, there exists a relatively dense set $N \subset \mathbb{N}$ such that

$$\left\|\theta_j^n(T_j)\xi_j - T_j\xi_j\right\| \leq \varepsilon \text{ for } 1 \leq j \leq p,\ n \in N.$$

Multiple recurrence for compact systems

Theorem

Let $\mathfrak{A}$ be a C^*-algebra, ϕ a state on $\mathfrak{A}$ such that support $s(\phi)$ in $\mathfrak{A}^{**}$ is central, and $\alpha : \mathfrak{A} \to \mathfrak{A}$ a positive linear map such that $\phi \circ \alpha = \phi$.

(i) *If $\phi(\alpha(x)^*\alpha(x)) \leq \phi(x^*x)$, then then there is a normal positive linear map, $\Psi : \pi_\phi(\mathfrak{A})'' \to \pi_\phi(\mathfrak{A})''$ such that $\Psi(T)\xi_\phi = U_\alpha(T\xi_\phi)$ for $T \in \pi_\phi(\mathfrak{A})''$, $\Psi(1_{H_\phi}) = 1_{H_\phi}$, $\|\Psi\| \leq \|\alpha\|$, preserving $\omega_{\xi_\phi}|\pi_\phi(\mathfrak{A})''$ and $\Psi(\pi_\phi(a)) = \pi_\phi(\alpha^{**}(a))$, $a \in \mathfrak{A}^{**}$.*

(ii) *If α is multiplicative, then $\pi_\phi(\alpha(\mathfrak{A}))''$ is a von Neumann subalgebra of $\pi_\phi(\mathfrak{A})''$, the central support of the projection $U_\alpha U_\alpha^*$ in $\pi_\phi(\alpha(\mathfrak{A}))''$ is 1_{H_ϕ}, and we have $\Psi(T)U_\alpha U_\alpha^* = U_\alpha T U_\alpha^*$.*

Multiple recurrence for compact systems

Lemma

Let $\mathfrak{A}$ be a C^*-algebra, ϕ a state on $\mathfrak{A}$ such that support $s(\phi)$ in $\mathfrak{A}^{**}$ is central, and let $\mathcal{M}_{AP} = \{T \in \pi_\phi(\mathfrak{A})'' \mid T\xi_\phi \in (H_\phi)_{AP}\}$. Then,

for any $p \in \mathbb{N}$, $m_1, \dots, m_p \geq 1$, $T_1, \dots, T_p \in \mathcal{M}_{AP}$, $\xi_1, \dots, \xi_p \in H_\phi$ and $\varepsilon > 0$, there exists a relatively dense set $N \subset \mathbb{N}$ such that

$$\|\Psi^{m_j n}(T_j)\xi_j - T_j\xi_j\| \leq \varepsilon \text{ for all } 1 \leq j \leq p,\ n \in N.$$

Corollary

Let $\mathfrak{A}$ be a C^*-algebra, ϕ a state on $\mathfrak{A}$ such that support $s(\phi)$ in $\mathfrak{A}^{**}$ is central, then for any $p \in \mathbb{N}$, $m_1, \dots, m_p \geq 1$, $T_1, \dots, T_p \in \mathcal{M}_{AP}$, $S_1, \dots s_{p-1} \in \mathcal{B}(H_\phi)$, $\xi \in H_\phi$ and $\varepsilon > 0$, there exists a relatively dense set $N \subset \mathbb{N}$ such that

$$\|\Psi^{m_1 n}(T_1)S_1 \dots \Psi^{m_p n}\xi - T_1 S_1 \dots T_p\xi\| \leq \varepsilon \text{ for all } n \in N.$$

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