Ergodic theory and Multiple recurrence theorem

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Recurrence

Definition (Measure-preserving system (MPS))

A quadruple $(X, \mathcal{B}, \mu, T)$, where $(X, \mathcal{B}, \mu)$ is a probability space and $T : X \to X$ is a *measure-preserving transformation*.

Theorem (Poincare recurrence theorem)

Let A be a measurable set with positive measure, then almost every point of A returns to A, i.e., there is a set $E \subset A$ with 0 measure such that if $x \in A \setminus E$, then there exist $n \in \mathbb{N}$, $T^n x \in A$. Furthermore, the points return infinitely often, i.e., there are infinitely many n such that $T^n x \in A$.

Ergodicity

Definition (Invariant set)

A measurable set A is an *invariant set* if $T^{-1}A = A$.

Definition (Ergodic system)

A measurable-preserving system $(X, \mathcal{B}, \mu, T)$ is *ergodic* if there are no non-trivial invariant sets, i.e., if A is an invariant set, then $\mu(A) = 0$ or 1.

Theorem

$(X, \mathcal{B}, \mu, T)$ is a MPS, then the following are equivalent,

(i) *T is ergodic.*

(ii) *if $f : X \to \mathbb{R}$ is T-invariant measurable, then f is constant a.e.*

(iii) *if $A \in \mathcal{B}, \mu(A) > 0$, then $\bigcup_{n=m}^{\infty} T^{-n}A = X$ mod-μ $\forall m$.*

(iv) *if $A, B \in \mathcal{B}$, $\mu(A), \mu(B) > 0$, then $\mu(T^{-n}A \cap B) > 0$ for infinitely many n.*

Measure Disintegration

Definition (Standard measurable space)

A measurable space $(X, \mathcal{B})$ is *standard* if there exists a complete and separable metric on X for which $\mathcal{B}$ is the Borel σ-algebra.

Definition (Probability kernel)

Let $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ be two measurable spaces and $\{\theta_x\}_{x \in X}$ be a family of probability measures on Y, then $\{\theta_x\}_{x \in X}$ is called a *probability kernel from* $(X, \mathcal{B})$ *to* $(Y, \mathcal{C})$ if for each $E \in \mathcal{C}$, the map $x \mapsto \theta_x(E)$ is $\mathcal{B}$-measurable.

Measure Disintegration

Theorem (Measure disintegration)

Let $(X, \mathcal{B}, \mu)$ be a measure space and $\mathcal{F} \subset \mathcal{B}$ a sub-σ-algebra, then there exists a unique kernel from $(X, \mathcal{F})$ to $(X, \mathcal{B})$, $\{\theta_x\}_{x \in X}$, called the disintegration of μ over $\mathcal{F}$ if

1. $\mu(E) = \int \theta_x(E) d\mu(x), \ \forall E \in \mathcal{B}$.
2. *if $f : X \to \mathbb{C}$ is a bounded $\mathcal{B}$-measurable function, then*

$$\mathbb{E}(f|\mathcal{F})(x) = \int f(t) d\theta_x(t) \ \mu\text{-a.e.} \tag{1}$$

Theorem (Ergodic decomposition)

Let $(X, \mathcal{B}, \mu, T)$ be a standard measure preserving system, and $\mathcal{F} \subset \mathcal{B}$ the sub-σ-algebra of the T-invariant sets. If $\{\theta_x\}_{x \in X}$ is the disintegration of μ over $\mathcal{F}$, then θ_x is T-invariant and ergodic for μ-a.e.

Mean Ergodic Theorem

Theorem (von Neumann's mean ergodic theorem for Hilbert spaces)

$\mathcal{H}$ is a Hilbert space and T is a contraction [i.e. T is a bounded operator and $\|T\| \leq 1$]. Let $\mathcal{M} = \{v \in \mathcal{H} \mid Tv = v\}$ and $\pi : \mathcal{H} \to \mathcal{M}$ be the orthogonal projection. Then

$$S_n(v) := \frac{1}{n} \sum_{k=0}^{n-1} T^k(v) \to \pi(v) \quad \forall v \in \mathcal{H}.$$

Sketch of proof.

The main step is to prove if $\mathcal{N} = \{v - Tv \mid v \in \mathcal{H}\}$, then $\mathcal{M}^\perp = \overline{\mathcal{N}}$. □

Mean ergodic theorem

Definitions

- Koopman operator, $U_T : L^p(\mu) \to L^p(\mu)$, $(1 \le p \le \infty)$,

$$U_T f = f \circ T$$

  U_T is an isometry.
- The average operator on $L^1(\mu)$,

$$S_n f = \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k.$$

- $\mathcal{F}$ be the σ-algebra of T-invariant subsets of $\mathcal{B}$, i.e. $M_T = \sigma\{E \in \mathcal{B} \mid T^{-1}E = E\}$.

Mean Ergodic Theorem

Corollary (for dynamical systems)

Let $(X, \mathcal{B}, \mu, T)$ be a MPS, then for $f \in L^2(X, \mathcal{B}, \mu)$,

$$S_n(f) \to \mathbb{E}(f|M_T) \ in \ \|\cdot\|_2 .$$

If the system is ergodic, then

$$S_n(U_T)(f) \to \int f d\mu \ in \ \|\cdot\|_2 .$$

Mean Ergodic Theorem

Theorem (for Banach spaces)

Let X be a reflexive Banach space, and $T \in \mathcal{B}(X)$ such that $\sup_{n \in \mathbb{N}} \|T^n\| < \infty$, *then*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k(x) \text{ exists for all } x \in X.$$

Pointwise Ergodic Theorem

Theorem (Birkhoff's pointwise ergodic theorem)

For $f \in L^1(\mu)$, the following holds,

$$S_n f(x) \to \mathbb{E}(f|\mathcal{F})(x) \text{ a.e.} \tag{2}$$

If the system is ergodic, then $S_n f(x) \to \int f d\mu$ a.e.

Proof.

Steps of proof

i. We first find a dense set $S \subset L^1$, where the statement holds. $S_1 = \{f \in L^2 \mid fT = f\}$ and $S_2 = \{g - gT \mid g \in L^\infty\}$.

$\square$

Pointwise Ergodic Theorem

Steps of proof.

ii To extend to all of L^1, we require the maximal inequality,

Theorem (Maximal inequality)

$f \in L^1, \ f \geq 0$, then $\forall t > 0$,

$$\mu\{x \in X \mid \sup_n S_n f(x) > t\} \leq \frac{1}{t} \int f d\mu.$$

iii Then, we can show $\limsup |S_n f - \mathbb{E}(f|\mathcal{F})(x)| = 0$.

$\square$

Weak-mixing

Theorem

$(X, \mathcal{B}, \mu, T)$ is ergodic if and only if $A, B \in \mathcal{B}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(A \cap T^{-k}B) = \mu(A)\mu(B).$$

Definition (Weakly mixing)

$(X, \mathcal{B}, \mu, T)$ is *weakly mixing* if for all $A, B \in \mathcal{B}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\mu(A \cap T^{-k}B) - \mu(A)\mu(B)| = 0.$$

Weak-mixing

Theorem

The following are equivalent definitions of weak-mixing for $(X, \mathcal{B}, \mu, T)$.

1. $(X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu, T \times T)$ *is ergodic.*
2. $X \times Y$ *is ergodic for every ergodic system* $(Y, \mathcal{C}, \nu, S)$.
3. *If* T *is ergodic, then it is weak mixing if and only if the point-spectrum* $\sigma_p = \{1\}$.

Theorem

The following are some properties of a weak-mixing system, $(X, \mathcal{B}, \mu, T)$.

(i) X_1, X_2 *is weak mixing* $\implies$ $X_1 \times X_2$ *is weak mixing.*

(ii) T *is weak mixing* $\implies$ T^n *is weak mixing for all* $n \in \mathbb{N}$

(iii) *if* T *is invertible,* T *is weak mixing* $\iff$ T^{-1} *is weak mixing*

Multiple Recurrence

Theorem (Multiple Recurrence Theorem)

For a measure-preserving system $(X, \mathcal{B}, \mu, T)$*, and a set* $A \in \mathcal{B}$*, with* $\mu(A) > 0$*, and for any* $k \in \mathbb{N}$*, there is some* $n \geq 1$ *such that*

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \ldots T^{-kn}A) > 0$$

Multiple Recurrence

Theorem (Multiple Recurrence Theorem)

For a measure-preserving system $(X, \mathcal{B}, \mu, T)$, and a set $A \in \mathcal{B}$, with $\mu(A) > 0$, and for any $k \in \mathbb{N}$, there is some $n \geq 1$ such that

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \dots T^{-kn}A) > 0$$

We will prove a stronger statement:

Theorem (Uniform Multiple Recurrence Theorem (UMR))

For a measure-preserving system $(X, \mathcal{B}, \mu, T)$, and a set $A \in \mathcal{B}$, with $\mu(A) > 0$, and for any $k \in \mathbb{N}$, we have

$$\liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n}A \cap T^{-2n}A \cap \dots T^{-kn}A) > 0$$

Outline of Proof of UMR Theorem

We first show that UMR property is satisfied by some systems, such as, weak-mixing systems and Kronecker systems. The main ingredient is understanding how the property lifts by weak-mixing and compact extensions, and the understanding the relationship between the two extensions.

Outline of Proof of UMR Theorem

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Definition (Extensions and factors)

Let $(X, \mathcal{B}_X, \mu, T)$ and $(Y, \mathcal{B}_Y, \nu, S)$ be two MPS. Y is a *factor* of X if there are sets $X' \in \mathcal{B}_X$, $Y' \in \mathcal{B}_Y$ with $\mu(X') = 1$, $\nu(Y') = 1$, $TX' \subset X'$, $SY' \subset Y'$, and a measure-preserving map $\phi : X' \to Y'$, such that $\phi \circ T = S \circ \phi$.
And X is called an extension of Y.

Theorem

Factors of a system are in 1-1 correspondence with invariant sub-σ-algebras.

Outline of Proof of UMR Theorem

Step 1

Reduction to standard measurable spaces

i. Every MPS has an invertible extension and UMR property is preserved under these extensions.
ii. Every invertible system has a standard factor that is a standard measurable space.

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Reduction to standard measurable spaces

i. Every MPS has an invertible extension and UMR property is preserved under these extensions.
ii. Every invertible system has a standard factor that is a standard measurable space.

Now, it is sufficient to prove that any standard measure-preserving system $(X, \mathcal{B}, \mu, T)$ satisfies UMR property.

Outline of Proof of UMR Theorem

Step 2

i. Weak-mixing and Kronecker systems satisfy UMR property.
ii. A system is not weak-mixing if and only if it has a non-trivial Kronecker factor.
iii. If $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots$ is an increasing chain of factors of X that satisfy UMR property, then $\sigma(\cup_{n\geq 1}\mathcal{A}_n)$ also satisfies it.

Outline of Proof of UMR Theorem

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So, if X is weak-mixing, we are done. If it is not weak-mixing, then consider the family of factors, $\mathcal{A} \subset \mathcal{B}$ such that $(X, \mathcal{A}, \mu, T)$ satisfy UMR property. We know, it is non-empty, since there is a Kronecker factor that satisfies it. Let $\mathcal{B}_\infty$ be the maximal sub-σ-algebra of this family. (Existence is shown using Zorn's lemma and iii.)

Outline of Proof of UMR Theorem

Step 3

i. If X is a weak-mixing extension of Y, which satisfies UMR property, then X also satisfies UMR property.

ii. If X is a compact extension of Y, which satisfies UMR property, then X also satisfies UMR property.

iii. If $X \to Y$ is not a weak-mixing extension, then there exists an intermediate factor of $X \to Z$, such that $Z \to Y$ is a compact extension.

Outline of Proof of UMR Theorem

Step 3

Hence, if $(X, \mathcal{B}, \mu, T) \to (X, \mathcal{B}_\infty, \mu, T)$ is a weak-mixing extension, we are done. And if it is not, there is a non-trivial compact extension $(X, \mathcal{C}) \to (X, \mathcal{B}_\infty)$. So $(X, \mathcal{C})$ satisfies UMR property, which contradicts the maximality of $\mathcal{B}_\infty$. Thus, the extension must be weak-mixing, and this completes the proof.

Step 1

Invertible extension

Any MPS $(X, \mathcal{B}, \mu, T)$ has an invertible extension, $(\tilde{X}, \tilde{\mathcal{B}}, \tilde{\mu}, \tilde{T})$, where

- $\tilde{X} = \{x \in X^{\mathbb{Z}} \mid Tx_k = x_{k+1} \ \forall k \in \mathbb{Z}\}$
- $(\tilde{T}x)_k = x_{k+1}$ for all $k \in \mathbb{Z}$ and $x \in \tilde{X}$
- $\tilde{\mathcal{B}}$ is the product σ-algebra
- $\tilde{\mu}$ is the product measure

$\pi_0 : \tilde{X} \to X$ is called the invertible extension, where $\pi_0 : X^{\mathbb{Z}} \to X$ is the 0-th projection..

Step 1

Theorem

A MPS has the properties: ergodicity, weak-mixing, and UMR if and only if its invertible extension does.

Theorem

An invertible system has a factor which is a standard probability space.

Proof.

Fix $A \in \mathcal{B}$ of positive measure and define
$\phi : X \to \{0,1\}^{\mathbb{Z}},\ \phi(x) = \chi_A(T^n x)$ $\square$

Step 2 i.

Definition (Kronecker systems)

A Kronecker system is a MPS $(X, \mathcal{B}, \mu, T)$, where X is a compact metrizable group, $\mathcal{B}$ is the Borel σ-algebra, λ is the Haar measure and T is an ergodic rotation, $T(x) = ax$ for a fixed $a \in X$.

Theorem

Kronecker systems satisfy UMR property.

Proof.

(i) For fixed $f \in L_\infty$, the map $\phi : X \to \mathbb{R}$, $\phi(x) = \int f(x)f(xy)\dots f(x^k y)d\lambda(y)$ is continuous.

(ii) Since T is ergodic and X is compact metrizable, it is uniquely ergodic and,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} U_T^n \phi = \int \phi(x)d\mu(x), \text{ uniformly.}$$

Step 2 ii.

Theorem

$(X, \mathcal{B}, \mu, T)$ is not weak-mixing if and only if it has a non-trivial Kronecker factor.

$\implies$.

- If T is not weak-mixing, there is a complete metric space (Y, d), with isometry $T : Y \to Y$ and a Borel map $\phi : X \to Y$ such that $\phi T = T \phi$.
- Define psuedo-metric on $\mathcal{B}$, $d(A, B) := \mu(A \Delta B)$ $= \|\chi_A - \chi_B\|_1$. By identifying sets that differ by measure 0, we can make $\mathcal{B}$ a complete metric space.
- The measure on Y is $\nu = \mu \circ \phi^{-1}$.
- Lastly, we show $\text{supp}(\nu)$ is compact.

$\square$

Step 2 iii.

Theorem

Let $(X, \mathcal{B}, \mu, T)$ be a standard invertible MPS and $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots$ is an increasing chain of factors that satisfy UMR property, then $\sigma(\cup_{n\geq 1}\mathcal{A}_n)$ also satisfies it.

(i) Let $\mathcal{A} = \sigma(\cup_{n\geq 1}\mathcal{A}_n)$, then for $A \in \mathcal{A}$, for any $\epsilon > 0$, there exists $A_1 \in \mathcal{A}_n$ for some n, such that $\mu(A \bigtriangleup A_1) < \epsilon$.

(ii) Let $\eta = 1/2(k+1)$ and $\epsilon = \frac{1}{4}\eta\nu(A)$ and define $A_0 = \{x \in A_1 \mid \mu_x(A) \geq 1 - \eta\}$.

(iii) We show $\mu(A_0) > \frac{1}{2}\mu(A) > 0$.

(iv) From definition of A_0, $\mu_x(A \cap T^{-n}A \cap \dots \cap T^{-kn}A) \geq \frac{1}{2}$.

(v) By integrating,

$$\mu(A \cap T^{-n}A \cap \dots \cap T^{-kn}A) \geq \frac{1}{2}\mu(A_0 \cap T^{-n}A_0 \cap \dots \cap T^{-kn}A_0)$$

Step 3 i.

Definition (Weak-mixing extension)

Let $\phi : (X, \mathcal{B}_X, \mu, T) \to (Y, \mathcal{B}_Y, \nu, S)$ be an extension. Define a measure $\tilde{\mu}$ on $(X \times X, \mathcal{B} \times \mathcal{B})$ given by disintegration (wrt $\mathcal{A} := \phi^{-1}(\mathcal{B}_Y)$),

$$\tilde{\mu}_x = \mu_x \times \mu_x.$$

Then $(X \times X, \mathcal{B} \times \mathcal{B}, \tilde{\mu}, T \times T)$ is a MPS and the extension is *weak-mixing* if this system is ergodic.

Theorem

If X is a weak-mixing extension of Y, which satisfies UMR property, then X also satisfies UMR property.

Corollary

If X is weak-mixing, then it satisfies UMR property.

Step 3 ii.

Definition (Relative almost periodic functions)

Let $(X, \mathcal{B}_X, \mu, T) \to (Y, \mathcal{B}_Y, \nu, S)$ be an extension. $f \in L^2_\mu(X)$ is *almost periodic* relative to Y if for every $\epsilon > 0$, there is an $r \in \mathbb{Z}$ and function $g_1, \ldots, g_r$ such that

$$\min_{i=1,\ldots,r} \|U_T^n f - g_i\|_{L^2_{\mu_x}} < \epsilon.$$

for all $n \in \mathbb{N}$ and a.e $x \in X$.

Definition (Compact extension)

$X \to Y$ is a *compact extension*, if the set of functions almost periodic relative to Y is dense in $L^2_\mu(X)$.

Theorem

If X is a compact extension of Y, which satisfies UMR property, then X also satisfies UMR property.

Szemeredi's Theorem

Definition (Upper density)

The *upper density* of a set $A \subset \mathbb{Z}$ is defined as

$$d(A) = \limsup_{Z \to \infty} \frac{1}{2N+1} |S \cap \{-n, -n+1, \dots, n-1, n\}|$$

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Theorem (Szemeredi's Theorem)

If $A \subset \mathbb{Z}$ has positive upper density, then A contains arithmetic progressions of arbitrary length, that is, for all $k \in \mathbb{N}$, there exists $a, b \in \mathbb{Z}, \; b \neq 0$ such that $a, a+b, a+2b, \ldots, a+kb \in A$.

Szemeredi's Theorem

Sketch of proof.

i. Define MPS: $X = 0, 1^{\mathbb{Z}}$, $\sigma : X \to X$, $(\sigma(x))_n = x_{n-1}$. Let $f = \chi_A \in X$, and define

$$\mu_n = \frac{1}{2n+1} \sum_{i=-n}^{n} \delta_{\sigma^i(f)}.$$

ii. By Riesz representation theorem, the set of probability measures, $\mathcal{P}(X)$ on X is weak-$*$ compact and we can show, if X is metrizable and compact, then $C(X)$ is separable and $\mathcal{P}(X)$ is metrizable: let $\{f_i\}$ be a countable dense subset of $C(X)$, then

$$d(\mu, \nu) = \sum_{i=1}^{\infty} \frac{1}{2^i} \left| \int f_i d\mu - \int f_i d\nu \right|$$

is a metric on $\mathcal{P}(X)$.

Szemeredi's theorem

Sketch of proof

iii. So, $\mathcal{P}(X)$ is sequentially-compact in weak-$*$ topology and let $\mu_{n_k} \to \mu$.

iv. After showing μ is σ-invariant, we have a MPS $(X, \mathcal{B}, \mu, T)$. This uses the fact that $\mu_{n_k} \to \mu \iff \int f d\mu_{n_k} \to \int f d\mu$.

v. We use multiple recurrence theorem on X, with $A = \{x \in X \mid x_0 = 1\}$. Note, $\mu(A) = d(A) > 0$.