(M,g)

Heat eq^n: $\frac{\partial u}{\partial t} = -\Delta_x u$ initial cond^n $u(x,0) = u_0(x)$ $u : M \times \mathbb{R}^+ \to \mathbb{C}$ smooth

$u(x,t)$ - temp at x at time t

Formal sol^n (formal : "not rigorous)

Heat semigroup: $S(t) = e^{-t\Delta}$ $t>0$

sol^n: $u(x,t) = e^{-t\Delta} u_0(x)$

$$e^{-t\Delta} := \begin{bmatrix} e^{-t\lambda_0} & & \\ & e^{-t\lambda_1} & \\ & & \ddots \end{bmatrix}$$

[Spectral decomposition th^m $\{\varphi_i\}_{i=0}^{\infty}$ - orthonormal basis of $L^2(M)$

$\Delta \varphi_i = \lambda_i \varphi_i$ spec $(\Delta) = \{\lambda_i\}_{i=0}^{\infty}$

$$e^{-t\Delta} := \begin{bmatrix} e^{-t\lambda_0} & & \\ & e^{-t\lambda_1} & \\ & & \ddots \end{bmatrix}$$

$u(x,t) = \sum e^{-t\lambda_i} \varphi_i(x) \langle u_0, \varphi_i \rangle$, $u_0 = \sum \langle u_0, \varphi_i \rangle \varphi_i$

[$u(x,t) \in L^2$ but does it $\in C^\infty$?
in $\mathbb{R}^n$: yes, using Fourier theory (ex)
also for S^1 & any flat tori

Heat kernel (Fundamental sol^n)

f^n: $K : M \times M \times \mathbb{R}^{>0} \to \mathbb{R}$ is a heat kernel for (M,g) if

1. $K(x,y,t)$ is C^0 in x,y,t, C^2 in y, C^1 in t
2. $\forall x \in M$, $(\Delta_y + \partial_t) K = 0$ - sol^n of heat in y (infinite family of sol^n)
3. $\forall x \in M$, $K(x,-,t) \to \delta_x$ as $t \to 0^+$

[δ_x - Dirac delta f^n at x, $\delta_x(\varphi) = \varphi(x)$ $\forall \varphi \in C_c^\infty(M)$
(distribution)

$\Updownarrow$

$\forall \varphi \in C_c^\infty(M)$,

$$\int_M K(x,y,t)\varphi(y)\, dvol_g(y) \longrightarrow \varphi(x) \quad \text{as } t \to 0^+$$

[Interpretation: K density of heat or temp
at $t \to 0^+$, at x - ∞ temp
as t inc, heat eq studies spread of temp

Eg 1. $\mathbb{R}^n$, flat $\Delta = -\sum \partial_i^2$

$$K(x,y,t) = (4\pi t)^{-n/2} e^{\left(\frac{-|x-y|^2}{4t}\right)} \text{ - Gaussian}$$

3. $\circ$ $\int_{\mathbb{R}^n} K(x,y,t)\, dy = 1$ $\forall x,t$

$\circ$ away from x, $|K(x,y,t)| < \varepsilon$ for small enough t

(ex)

Not unique (ex)

2. $M = S^1 = \mathbb{R}/2\pi\mathbb{Z}$

($\mathbb{R} \to S^1$ - universal cover)

$$K_1(x,y,t) = \sum_{n\in\mathbb{Z}} (4\pi t)^{-1/2} \exp\left(\frac{-|x-y-2\pi n|^2}{4t}\right) \quad \text{(ex)}$$

(use eigenvalues & eigenf^n of $\Delta = -\frac{d^2}{dx^2}$: $\lambda_n = n$, $\varphi_n = e^{inx}$, $n \in \mathbb{Z}$)

$$\Rightarrow K_2(x,y,t) = \sum_n e^{-tn^2} e^{in(x-y)} \quad \text{(ex)}$$

ex: $K_1 = K_2$ (Poisson summation formula)

Constructing heat kernel for closed Riemannian mfld

1. from eigenvalues & eigenf^n

Th^m (Dodzick) $K(x,y,t) = \sum_n e^{-t\lambda_n} \varphi_n(x) \bar{\varphi}_n(y)$

series is convergent to a C^∞ f^n K, which is a heat kernel

2. from asymptotic exp^n

K for flat space, geodesic distance

$$K(x,y,t) \sim (4\pi t)^{-n/2} e^{-\frac{d(x,y)^2}{4t}} \{u_0(x,y) + u_1(x,y)t + u_2(x,y)t^2 + \dots\}$$

$u_i : M \times M \to \mathbb{R}$ smooth, $u_i(x,x)$ are uniquely determined

$u_0(x,x) = 1$, $u_1(x,x) = \frac{1}{6} s(x)$, $s(x)$ = scalar curvature of (M,g) at x (to show)

$u_i(x,x)$ can be computed (in principle) (not even well defined)

- asymptotic:

(?) $N+1-N/2$

$\forall N \geq 0$ $|K(x,y,t) - F(x,y,t)(u_0 + u_1 t^1 + \dots u_N t^N)| \leq c t^{N+1}$ $\forall x,y \in M$ & t small enough

- let $x=y$, $K(x,x,t) \sim (4\pi t)^{-n/2}(u_0(x,x) + u_1(x,x)t + \dots)$

$u_i(x,x)$ are constructed from curvature tensor R_{ijkl} & its covariant derivatives

- $Z_M(t) := Tr\, e^{-t\Delta}$

(to show) $= \int K(x,x,t)\, dvol_g(x) \sim (4\pi t)^{n/2}(a_0 + a_1 t + a_2 t^2 + \dots)$

$a_i = \int u_i(x,x)\, dvol_g(x)$ $a_0 = vol(M)$

integral of asymptote $a_1 = \frac{1}{6}\int s(x)\, dvol_g(x)$ - total scalar curvature

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(M,g) - closed, connected Riemannian manifold

Th^m $K(x,y,t) = \sum_n e^{-t\lambda_n} \varphi_n(x)\varphi_n(y)$ is a heat kernel

Assuming 1. is satisfied

2. is easy

3. $\int_M K(x,y,t)\varphi(y)\, dvol_g(y)$

$$= \sum_i e^{-t\lambda_i} \varphi_i(x) \int_M \varphi_i(y)\varphi(y)\, dvol_g(y) \longrightarrow \sum_i \varphi_i(x)\,(\,) = \varphi(x)$$

$\hookrightarrow$ fourier coefficient $\quad$ $\hookrightarrow$ fourier tranform

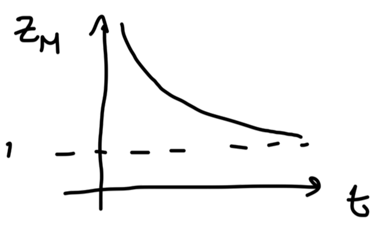
For 1. restrict to $x=y$, $K(x,x,t) = \sum e^{-\lambda_i t} \varphi_i(x)^2$

$\int_M K(x,x,t)\, dvol_g(y) = \sum e^{-\lambda_i t}$ (orthonormal basis)

$= Z_M(t)$ - **Heat trace** / Partition f^n

$Z_M(t)$ is a spectral invariant: $Z_M(t) = Z_{M'}(t) \Leftrightarrow (M,g), (M',g')$ isospectral

Cor $\sum e^{-\lambda_i t} \sim (4\pi t)^{-n/2}(a_0 + a_1 t + \dots)$ $t \to 0$



Tauberium Th^m (Karamata)

Let $d\mu(\lambda)$ be a +ve measure on $\mathbb{R}^{>0}$ st $\int_0^\infty e^{-t\lambda} d\mu(\lambda) < \infty$ $\forall t>0$ & $\lim_{t\to 0^+} t^\alpha \int_0^\infty e^{-t\lambda} d\mu(\lambda) = c$ $(c>0, \alpha>0)$

(laplace transform)

If $f(x)$ is cont^n on $[0,1]$

then $\lim_{t\to 0^+} t^\alpha \int_0^\infty f(e^{-t\lambda}) e^{t\lambda} d\mu(\lambda) = \frac{c}{\Gamma(\alpha)} \int_0^\infty f(e^{-t}) t^{\alpha-1} e^{-t}\, dt$

[Gamma f^n $\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} ds$
$\cdot$ $\Gamma(s)$ is abs conv, for $Re(s)>0$
$\cdot$ $\Gamma(s+1) = s\Gamma(s)$ $\quad$ $\cdot$ $\Gamma(n) = (n-1)!$

Pf f - cont $\Rightarrow$ by stone - weierstrass, enough to show for $f(x) = x^n$

$$\lim_{t\to 0} t^\alpha \int_0^\infty e^{-(n+1)t\lambda} d\mu(\lambda) = c(n+1)^{-\alpha}$$

$$\frac{c}{\Gamma(\alpha)} \int_0^\infty e^{-(n+1)t} t^{\alpha-1}\, dt = c(n+1)^{-\alpha}$$

Weyl's Law

Pf Take μ - counting measure and let $f(x) = \begin{cases} 1/x & 1/e \leq x \leq 1 \\ 0 & 0 < x < 1/e \end{cases}$