Throughout, R is a ring with $1 \neq 0$. When a question says "describe" or "find" or something like that, I'm still asking you for a proof!

1. Prove the following equivalences for commutative rings R:
   (a) An ideal I is prime if and only if R/P is an integral domain.
   (b) An ideal I is maximal if and only if R/P is a field.

2. Suppose that $I_1, I_2, \ldots, I_n$ are two-sided ideals in a ring R.
   (a) Prove that $\bigcap_{j=1}^{n} I_j$ is a two-sided ideal.
   (b) Prove that $\prod_{j=1}^{n} I_j$ is a two-sided ideal.[1]
   (c) Determine the containment dependence between (a) and (b) (that is, which one must be contained in the other).

3. Suppose that $\varphi : R \to S$ is a homomorphism of rings (assume for this problem that $\varphi(1_R) = 1_S$.
   (a) Prove that if P is a prime ideal in S, then $\varphi^{-1}(P)$, its preimage in R, is a prime ideal in R.
   (b) Suppose that M is a maximal ideal in S and that φ is surjective. Prove that $\phi^{-1}(M)$ is a maximal ideal in R.
   (c) Give an example of a homomorphism of rings $\varphi : R \to S$ and a maximal ideal $M \subset S$ such that $\varphi^{-1}(M)$ is *not* a maximal ideal in R.

4. Suppose that R is an integral domain and that $D \subset R$ is a non-empty multiplicative set not containing 0. Prove that $D^{-1}R$ is also an integral domain. (Be very careful about what it means to equal 0 in $D^{-1}R$.)

5. Consider the set $\mathbb{R}[[t]] = \{\sum_{i \geq 0} a_i t^i \mid a_i \in \mathbb{R}\}$, known as the ring of formal power series in the variable t, with addition and multiplication as in the case of the polynomial ring. [2]
   (a) Describe all the units in $\mathbb{R}[[t]]$.
   (b) Prove that $\mathbb{R}[[t]]$ is a local ring by finding its unique maximal ideal.

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[1] $I_1 I_2 := \{\sum_{k=1}^{L} r_k^{(1)} r_k^{(2)} \mid r_k^{(1)} \in I_1, r_k^{(2)} \in I_2$ is the product of ideals, and the n-fold product is defined recursively.

[2]
$$\left(\sum_{i \geq 0} a_i t^i\right) \cdot \left(\sum_{i \geq 0} b_i t^i\right) = \sum_{i \geq 0} c_i t^i$$

and $c_i = \sum_{j=0}^{i} a_j b_{i-j}$.