1. Let R be a ring. Prove the following basics:

   (a) $0 \cdot a = 0$ for all $a \in R$.

   (b) If $a, b \in R$, then $a \cdot (-b) = -(a \cdot b)$. (Be careful as this isn't just "moving around a minus sign"; it says that a times the additive inverse of b is suppose to be the additive inverse of ab, and that's what you must prove.)

2. Let n be a positive integer. Prove that the set of zero divisors of $\mathbb{Z}/n\mathbb{Z}$ is precisely the set of elements in $\mathbb{Z}/n\mathbb{Z}$ that are not relatively prime to n, and that the set of units in $\mathbb{Z}/n\mathbb{Z}$ is the set of elements that *are* relatively prime to n.

3. Prove that if R is an integral domain and the cardinality of R is finite, then R is a field.

4. Let R be an integral domain, and $a, b \in R$ be elements with $a \neq 0$. Prove that the equation $ax^2 = b$ has at most two solutions in R. Then find an example of an integral domain R and an equation $ax^2 = b$ that has no solutions, one that has exactly one solution, and one that has exactly two solutions.

5. Let $\varphi : R \to R'$ be a ring homomorphism. Prove that $\ker \varphi$ is a subring with the additional *absorbtion* property. That is, if $x \in \ker \varphi$ and $r \in R$, then $rx \in \ker \varphi$ and $xr \in \ker \varphi$. Such a subring is called a *two-sided ideal* in R. Find all two-sided ideals in $\mathbb{Z}/60\mathbb{Z}$.

6. Consider the ring $\mathbb{Z}$ and a two-sided ideal $I \subset \mathbb{Z}$. Prove that $x, y \in I$ if and only if $\gcd(x, y) \in I$.

7. Let F be a field and $a \in F$ be an arbitrary element. Define the function $\mathrm{ev}_a : F[x] \to F$ via $\mathrm{ev}_a(f) = f(a)$ (i.e., just replace x with a and evaluate).

   (a) Prove that ev_a is a ring homomorphism.

   (b) Compute the kernel of ev_a and prove your result.

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