A few notes about the tensor product.

a. If $v \otimes w = 0$ in $V \otimes_R W$, this means that $B(v, w) = 0$ for *every* bilinear map $B : V \times W \to P$ (otherwise, the universal property would fail!). So if you can find a bilinear map such that $B(v, w) \neq 0$, then $v \otimes w \neq 0$.

b. If $\sum_i a_i v_i \otimes w_i = 0$, then this means that $\sum_i a_i B(v_i, w_i) = 0$ for *every* bilinear maps $B : V \times W \to P$ (otherwise, the universal property would fail!). So if you can find a bilinear map such that $\sum_i a_i B(v_i, w_i) \neq 0$, then the sum of the tensors cannot be zero.

c. If $V \otimes W = 0$, then this means that there *are no bilinear maps* $B : V \times W \to P$ for any R module P.

1. Let R be a commutative ring and I, J be ideals in R. Prove that $R/I \otimes_R R/J \cong R/(I + J)$. [1] Consider the function $B : R/I \times R/J \to R/(I + J)$ defined by $B(r_1 + I, r_2 + J) = r_1 r_2 + (I + J)$.

   (a) B is well-defined: let $r_1' + I = r_1 + I$. Then $r_1' = r_1 + i$ for some $i \in I$. Thus, $B(r_1' + I, r_2 + J) = r_1' r_2 + (I + J) = (r_1 + i)r_2 + (I + J) = r_1 r_2 + (ir_2 + I + J)$ but $r_1 r_2 + ir_2 - r_1 r_2 = ir_2 \in I \subset I + J$, so $B(r_1' + I, r_2 + J) = B(r_1 + I, r_2 + J)$. The other side is similar.

   (b) B is bilinear: $B((r_1 + r_1') + I, r_2 + J) = (r_1 + r_1')r_2 + (I + J) = r_1 r_2 + r_1' r_2 + (I + J) = B(r_1 + I, r_2 + J) + B(r_1' + I, r_2 + J)$. Also, $B(\lambda r_1 + I, r_2 + J) = \lambda r_1 r_2 + (I + J) = \lambda(r_1 r_2 + (I + J)) = \lambda B(r_1 + I, r_2 + J)$.

   (c) Thus, we have a homomorphism $\Phi : R/I \otimes_R R/J \to R/(I + J)$ with $\Phi((r_1 + I) \otimes (r_2 + J)) = r_1 r_2 + (I + J)$..

   (d) To show that Φ is onto, it's enough to show that $1 + (I + J)$ is in the image of Φ. This is true, for example, because $\Phi(1 + I, 1 + J) = 1 + (I + J)$.

   (e) Now consider $x = \sum_i c_i (r_i + I) \otimes (r_i' + J) \in R/I \otimes R/J$. Using linearity in the first component, we can factor out r_i and similarly, we factor out r_i' from the second. Hence, $x = \sum_i c_i r_i r_i' (1 + I) \otimes (1 + J)$. Then we'll reabsorb the sum and the coefficients into the second component, say, so $x = (1 + I) \otimes (\sum_i c_i r_i r_i') + J$. Assume that $\Phi(x) = 0$. Then $\sum_i c_i r_i r_i' \in I + J$, so $\sum_i c_i r_i r_i' = z + z'$ where $z \in I$ and $z' \in J$. So $x = (1 + I) \otimes (z + z' + J) = (1 + I) \otimes (z + J) = z(1 + I) \otimes (1 + J) = (z + I) \otimes (1 + J) = 0 \otimes (1 + J) = 0$. Hence, $\ker \Phi = 0$, so Φ is injective.

2. Let $V = F^n, W = F^m$ be two finite-dimensional vector spaces over a field F. Let A be a matrix in $\mathrm{Mat}_{n \times m}(F)$. Prove that the function $B : V \times W \to F$ defined by $B(v, w) = v^T A w$ is a bilinear map. [2] $B(v + v', w) = (v + v')^T A w = v^T A w + v'^T A w$, which holds on the other side as well. Further,

[1] Hint: Use multiplication as your bilinear map, but you must show it is well-defined.

[2] If $n = m$ and A is the identity matrix, $B(v, w)$ is simply the dot product. So this is a generalization of that idea.

$B(\lambda v, w) = (\lambda v)^T Aw = \lambda v^T Aw = \lambda B(v, w)$, and similarly on the other side. Hence, B is bilinear. (This means that every inner product on $V \times W$ gives a homomorphism $V \otimes W \to F$, and vise-versa.)

3. Suppose that V and W are vector spaces over a field F with bases $\mathcal{B} = \{e_i\}_{i\in I}$ and $\mathcal{C} = \{f_j\}_{j\in J}$. Our goal is to prove that the set $\mathcal{B} \otimes \mathcal{C} := \{e_i \otimes f_j\}_{\substack{i\in I \\ j\in J}}$ is a basis for $V \otimes W$.

   i. Prove that $\mathcal{B}\otimes\mathcal{C}$ spans $V \otimes W$. We know already that $V\otimes W$ is spanned by tensors $v \otimes w$, so if we can show that every tensor of this form is a linear combination of the elements of $\mathcal{B}\otimes\mathcal{C}$, we're done. Since the e_i form a basis for V, and similarly with f_j, write

$$v = \sum_i \lambda_i e_i$$
$$w = \sum_j \mu_j f_j$$

   Then using the multilinearity

$$\begin{aligned} v \otimes w &= (\sum_i \lambda_i e_i) \otimes (\sum_j \mu_j f_j) \\ &= \sum_{i,j} \lambda_i \mu_j (e_i \otimes f_j) \end{aligned}$$

   so indeed, $v \otimes w$ is a linear combination of the elements in $\mathcal{B}\otimes\mathcal{C}$.

   ii. For any pair of integers $k \in I$ and $l \in J$, define the function

$$\begin{aligned} B_{kl} &: V \times W \to F \\ B_{kl}(v, w) &= [v]_k[w_l] \end{aligned}$$

   the product of the k-coordinate of v and the l coordinate of w. (This can also be realized as $v^T(e_k e_l^T)w$.) Prove that $B_{kl}(v, w)$ is bilinear (see problem 2). For each k, l, $e_k e_l^T$ is a matrix, so $B_{kl}(v, w) = v^T Aw$. By problem 2, then, B is bilinear.

   iii. Utilize the map from part (ii) to prove that $\mathcal{B}\otimes\mathcal{C}$ is a linearly independent set. (Hint: use remark (b) from the notes at the top of the page!) Suppose that there is a linear dependence:

$$\sum_{i,j} a_{ij} e_i \otimes f_j = 0.$$

   By the note at the top, then,

$$\star \sum_{i,j} a_{ij} B_{kl}(e_i, f_j) = 0$$

for *any choice* of integers k, l. However, $[e_i]_k = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$ and similarly with j and l. Hence, $\star$ really says that $a_{kl}(1) = 0$ for each k, l, so $\mathcal{B} \otimes \mathcal{C}$ is linearly independent.

4. Complete the work we started in class: Prove that if V and W are finite-dimensional vector spaces over a field F, then $V^* \otimes W \cong \mathrm{Hom}_F(V, W)$ where $V^* := \mathrm{Hom}_F(V, F)$. Define $B : V^* \times W \to \mathrm{Hom}_F(V, W)$ by $B(\varphi, w) = g_{\varphi,w}$ where $g_{\varphi,w}$ is the function defined by $g_{\varphi,w}(v) = \varphi(v)w$.

   (a) B is bilinear (this is pretty easy). So we get a homomorphism $\Phi : V^* \otimes W \to \mathrm{Hom}_F(V, W)$.

   (b) Φ is onto: Let $L_{ij} : V \to W$ be the linear transformation defined by $L_{ij}(e_k) = \begin{cases} f_i & j = k \\ 0 & j \neq k \end{cases}$. It can be easily verified that these are linear transformations that form a basis for $\mathrm{Hom}_F(V, W)$ (they are the matrices with a 1 in the ij entry and zeroes everywhere else). We will show that $L_{ij} = \Phi(\varphi \otimes w)$. Indeed, consider $\varphi_j : V \to F$ defined by $\varphi_j(e_k) = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$. When extended linearly, this is a linear transformation. Consider $\Phi(\varphi_j \otimes f_i) = g_{\varphi_j, f_i}$ which has the property that $g_{\varphi_j, f_i}(e_k) = \varphi_j(e_k) f_i = \begin{cases} f_i & j = k \\ 0 & j \neq k \end{cases}$. Since L_{ij} and g agree on a basis, they are the same transformation. Hence, Φ is onto.

   (c) We showed in class that $\dim F^n \otimes F^m = n \cdot m$ when F is a field, and $\dim \mathrm{Hom}_F(V, W) = nm$ since it is identified with $\mathrm{Mat}_{m \times n}(F)$. Since Φ is an onto linear map between two spaces of equal dimension, it is bijective, so an isomorphism.

Given an R module M, we can think of $\mathrm{Hom}_R(M, -)$ as a function. It takes an R-module N as an input, and produces the abelian group $\mathrm{Hom}_R(M, N)$.

$$\mathrm{Hom}_R(M, -) : \mathrm{Mod}(R) \to \mathcal{AB}$$
$$\mathrm{Hom}_R(M, -)(N) = \mathrm{Hom}_R(M, N).$$

More can be said. If $f : N \to N'$ is a homomorphism, then we can define $\mathrm{Hom}_R(M, f)$ to be a function which takes a homomorphism from $\mathrm{Hom}_R(M, N)$ and outputs a homomorphism in $\mathrm{Hom}_R(M, N')$ by composition:

$$\mathrm{Hom}_R(M, f) : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N')$$
$$\mathrm{Hom}_R(M, f)(\varphi) = f \circ \varphi$$

A creature like $\mathrm{Hom}_R(M, -)$, which associates objects in a target category to objects in a source category in such a way that it also takes homomorphisms to homomorphisms is called a *functor*. Think of these as function whose domains and codomains are categories.

5. Describe the following abelian groups:

   (a) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ Suppose that $\phi : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ is a homomorphism. Then if $\phi([1]) = a$, $0 = \phi([2]) = \phi([1]) + \phi([1]) = a + a$, so $2a = 0$ in $\mathbb{Z}$. Thus, $a = 0$. Hence $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$

   (b) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ Since this function is defined by $f([1])$, we consider the possible values of $f([1])$. If $f([1]) = 0$, then $f = 0$ is the zero function. Otherwise, $f([1]) = [1]_2$. This is well-defined (since it is really the quotient homomorphism by the submodule $2\mathbb{Z}/4\mathbb{Z}$), and it has the property that $(f + f)([1]) = [1]_2 + [1]_2 = [2]_2 = [0]_2$, so it has order 2. Hence, this group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

   (c) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ for any integer m Similar to above, if $f([1]) \neq 0$, $f([1])$ has order 2 in $\mathbb{Z}/m\mathbb{Z}$. Hence m is divisible by 2, and $f(1) = m/2$. It's clear to show in this case that f has order 2, so $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ if m is even and 0 otherwise.

   (d) $\mathrm{Hom}_{\mathbb{C}[t]}(N, N)$ where N is the $\mathbb{C}[t]$-module which, as a vector space, is $\mathbb{C}^2$, and on which T acts via the matrix $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ Suppose that $f : N \to N$ is a homomorphism of R modules where $R = \mathbb{C}[t]$. This means that f is an abelian group homomorphism, and that $f(rn) = rf(n)$ for all $r \in R$ and $n \in N$

      i. That f is an abelian group homomorphism menas that $f(n + n') = f(n) + f(n')$ for all $n, n' \in N$.

      ii. Since the constant polynomials λ are in $\mathbb{C}[t]$, we also require that $f(\lambda n) = \lambda f(n)$ for all $n \in N$.

      iii. These two together imply that f is a linear transformation. Since N is 2 dimensional, let's say $f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

iv. The polynomial $t \in \mathbb{C}[t]$, so we need to also verify that $f(tn) = tf(n)$. But these are both linear transformations, so we need to ensure that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

This implies that $a = d$ and $c = 0$, so $f = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$.

v. Finally, we need to ensure that $f(p(t)n) = p(t)f(n)$ for all $n \in N$ and $p(t) \in \mathbb{C}[t]$. Let $p(t) = \sum_{i=0}^{m} a_i t^i$. Then $f \sum_{i=0}^{m} a_i t^i n = \sum_{i=0}^{m} a_i f t^i n = \sum_{i=0}^{m} a_i t^i f(n) = p(t)f(n)$ (since above we showed that f and t commute.

vi. Hence, $\mathrm{Hom}_{\mathbb{C}[t]}(N, N) \cong \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbb{C} \right\}$.

6. Prove that if $f : N \to N'$ is injective, then

$$\mathrm{Hom}_R(M, f) : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N')$$

is also injective. First note that $\mathrm{Hom}_R(M, f)$ is a homomorphism of abelian groups (indeed, $\mathrm{Hom}_R(M, f)(\varphi + \varphi') = f \circ (\varphi + \varphi') = f \circ \varphi + f \circ \varphi'$ since f is a homomorphism of modules). Thus, it suffices to show that $\ker(\mathrm{Hom}_R(M, f)) = 0$. So assume that $\mathrm{Hom}_R(M, f)(\varphi) = 0$. Then $f \circ \varphi(x) = 0$ for all $x \in M$. Then $f(\varphi(x)) = 0$ for all $x \in M$. But f is injective, so $\ker f = 0$, hence $\varphi(x) = 0$ for all $x \in M$. This implies $\varphi = 0$.

7. On the other hand, it is not always the case that $\mathrm{Hom}_R(M, f)$ is surjective when f is. Here, you'll work out an example. Let $f : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ be the quotient map $f([x]_4) = [x]_2$ (you probably identified it in problem 4). f is clearly surjective. Show that the map $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, f) : \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ is not surjective. Suppose that φ is the only non-trivial homomorphism in $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z})$. I.e., $\varphi([x]_2) = [2x]_4$. Then $f \circ \varphi(x) = f([2x]_4) = [2x]_2 = 0$. Hence, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, f)$ is the zero map, even though the domain and codomain are not trivial, so it is not surjective.