1. The following question gives a criterion for determining if M is a direct sum. Let M be an R-module and L a submodule of M. Let $\pi : M \to M/L$ be the projection homomorphisms ($\pi : m \mapsto m + L$). Suppose that there exists a homomorphism $p : M/L \to M$ such that $\pi \circ p = \mathrm{id}_{M/L}$ (such a function is called a *section* of π). Prove that $M \cong L \oplus \mathrm{image}(p)$, and conclude that $M \cong L \oplus M/L$. Suppose that $\pi \circ p = \mathrm{id}_{M/L}$. Note that $\mathrm{im}(p)$ and L are both submodules of M. To prove that $M \cong \mathrm{im}(p) \oplus L$, we need only show that (i) $\mathrm{im}(p) \cap L = \{0\}$ and (ii) $\mathrm{im} + L = M$.

   i. Suppose that $l \in L$ and $l \in \mathrm{im}(p)$. Hence, there exists an element $n + L \in M/L$ with $p(n + L) = l$.

   $$\begin{aligned} \pi(l) &= \pi(p(n + L)) \\ 0 &= \mathrm{id}_{M/L}(n + L) \\ 0 &= n + L \end{aligned}$$

   [here, the first line is simply expressing l as $p(n+L)$, the second is recognizing that $\pi(L) = 0$ and that $\pi \circ p$ is the identity.] Hence, $n \in L$, and so $p(n + L) = 0$. Therefore, $l = 0$.

   ii. Let $m \in M$. Denote by m' the composition $p \circ \pi$ applied to m. I.e., $m' = p(\pi(m))$. Note that

   $$\begin{aligned} \pi(m') &= \pi(p(\pi(m))) \\ \pi(m') &= \mathrm{id}_{M/L}\,\pi(m) \\ \pi(m') &= \pi(m).m' + L & &= m + L \end{aligned}$$

   In particular, $m - m' \in L$. Therefore, $m = m' + l$ for some $l \in L$. Thus, $L + \mathrm{im}(p) = M$.

   By these two statement, $M \cong \mathrm{im}(p) \oplus L$. Finally, since $\pi \circ p$ is injective (it's equal to the identity), we must have p is injective, so $\mathrm{im}(p)$ is isomorphic to M/L. Thus, $M \cong M/L \oplus L$.

2. Recall that if M and N are R-modules, then

   $$\mathrm{Hom}_R(M, N) = \{\varphi : M \to N \mid \varphi \text{ is a homomorphisms of } R\text{-modules}\}.$$

   Prove the following:

   - If R is commutative, $\mathrm{Hom}_R(R, M) \cong M$ (in particular, you have to show that $\mathrm{Hom}_R(R, M)$ is an R module in this case). Let $G : \mathrm{Hom}_R(R, M) \to M$ be the function with $G(\varphi) = \varphi(1)$.

     (a) $G(\varphi + r\varphi') = \varphi(1) + r\varphi'(1) = G(\varphi) + rG(\varphi')$, so G is indeed a homomorphism.

     (b) We showed in a previous homework that $\mathrm{Hom}_R(R, M)$ is an R-module (when R is commutative).

(c) Now assume that $G(\varphi) = 0$. Then $\varphi(1) = 0$, so $\varphi(r) = r\varphi(1) = 0$ for all $r \in R$, so $\varphi = 0$.

(d) Finally, let $m \in M$. Define the function $\varphi_m(r) = rm$. We've shown previously that this function is a homomorphism, and clearly $G(\varphi_m) = m$, so G is onto.

(e) Thus, G is an isomorphism.

- $\mathrm{Hom}_R(N \oplus L, M) \cong \mathrm{Hom}_R(N, M) \oplus \mathrm{Hom}_R(L, M)$ Define $\Psi : \mathrm{Hom}_R(N, M) \oplus \mathrm{Hom}_R(L, M) \to \mathrm{Hom}_R(N \oplus L, M)$ via $\Psi(f, g)(n, l) = f(n) + g(l)$. We claim this is an isomorphism (of abelian groups, since these aren't necessarily modules).

  (a) $\Psi((f, g) + (f', g'))(n, l) = \Psi((f + f', g + g'))(n, l)$ and $(f + f')(n) + (g + g')(l) = f(n) + g(l) + f'(n) + g'(l) = \Psi(f, g) + \Psi(f', g')$.

  (b) $\ker \Psi = \{(f, g) \mid f(n) + g(l) = 0 \forall n \in N, l \in L\}$, but since $l = 0 \in L$, $f(n) + 0 = 0$ for all $n \in N$, so $f = 0$, and similarly, since $n = 0 \in N$, $0 + g(l) = 0$ for all $l \in L$, so $f = 0$ and $g = 0$.

  (c) If $H : N \oplus L \to M$, then we have maps $H_N : N \to M$ and $H_L : L \to M$ constituted by composing the inclusion homomorphisms with H. Then $\Psi((H_N, H_L))(n, l) = H(n, l)$.

- If F is free and R is commutative, then $\mathrm{Hom}_R(F, R) \cong F$. [Note, in the non-commutative case, this is not true! Ask about it in class.] Let F be free with basis $A = \{f_i \mid i \in I\}$. A cute approach is to use the universal property of the free modules (D&F p354): For any element x in F, write $x = \sum x_i f_i$ which is, by definition, a finite sum. This gives rise to a set function $\varphi : A \to F$ defined by $\varphi(f_i) = x_i$. By the universal property, there is a homomorphism $\Phi : F \to R$ with the property that $\Phi(\sum b_i f_i) = \sum b_i \varphi(f_i) = \sum b_i x_i$. So this is the function we'll use: For any $x \in F$, let $\Phi(x) : F \to R$ be the function defined by $\Phi(x)(\sum_i b_i f_i) = \sum_i b_i x_i$ where $x = \sum x_i f_i$.

  (a) This function is one-to-one: Assume $\Phi(x) = \Phi(y)$. Then $\Phi(x)(f_i) = x_i \cdot 1$ and $\Phi(y)(f_i) = y_i \cdot 1$. By assumption, then $x_i = y_i$ for all i, so $x = y$.

  (b) To prove that the function is onto, suppose that $\phi : F \to R$. Let $x_i = \phi(f_i)$ for each i, and $x = \sum_i x_i f_i$. Since F is of finite rank, the index set is finite. Then $\Phi(x)(y) = \sum_i x_i y_i = \sum_i y_i \phi(f_i) = \sum_i \phi(y_i f_i) = \phi(\sum_i y_i f_i) = \phi(y)$. Hence, $\Phi(x) = \phi$.

3. Suppose that M and N are R-modules which are free of finite rank, and let $\varphi : M \to N$ be an R-module homomorphism. Let $\{m_1, \ldots, m_s\}$ be a basis for M and $\{n_1, \ldots, n_t\}$ a basis for N. Prove that ϕ can be encoded as multiplication by a $t \times s$ matrix with entries in R. [1] Let $A \in \mathrm{Mat}_{t,s}(R)$ be the table of elements

[1] This is intentionally vague. Think about it for a bit, try to think about linear algebra, and ask me if you have questions after mulling it over.

defined in the following way: for each $j = 1, \dots, s$, express $\varphi(m_j)$ as a linear combination of the basis element of N, i.e., $\varphi(m_j) = \sum_i \alpha_{ij} n_i$. If $x = \sum_j x_j m_j$, then

$$\begin{aligned}
\varphi(x) &= \varphi(\sum_j x_j m_j) \\
&= \sum_j x_j \varphi(m_j) \\
&= \sum_j x_j \sum_i \alpha_{ij} n_i \\
&= \sum_i \left(\sum_j \alpha_{ij} x_j \right) n_i \\
&= \sum_i \left(A \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_s \end{bmatrix} \right)_i n_i
\end{aligned}$$

So the column vector of $\varphi(x)$ is given by $A[x]$ where $[x]$ is the column vector of x in the basis $\{m_1, \dots, m_s\}$.

4. Dummit & Foote §10.3 #12 As everyone pointed out, this was a previous exercise.

5. Dummit & Foote §10.3 #27 (This one is really wild: we showed in linear algebra that the dimension of a vector space is unique. In particular, any two bases have the same cardinality. It would certainly be weird if we could find a 2D vector space that was isomorphic to a 3D vector space. But that exact thing can happen with non-commutative rings.) Let φ_i and ψ_i be as defined in Dummit & Foote. Note that

$$\begin{aligned}
\psi_1\phi_i(a_1, a_2, \dots) + \psi_2\phi_2(a_1, a_2, \dots) &= (a_1, 0, a_3, 0, \dots) + (0, a_2, 0, a_4, \dots) \\
&= (a_1, a_2, a_3, \dots) \\
&= \mathrm{id}(a_1, \dots)
\end{aligned}$$

so $\psi_1\phi_1 + \psi_2\phi_2 = id$. In particular, if $g \in M$, then $g = (g\psi_1)\phi_1 + (g\psi_2)\phi_2$. Hence, $\{\phi_1, \phi_2\}$ span M. Furthemore, suppose that $g_1\phi_1 + g_2\phi_2 = 0$. Then

$$\begin{aligned}
(g_1\phi_1 + g_2\phi_2)(\psi_i) &= 0 \\
g_1\phi_1\psi_i + g_2\phi_2\psi_i &= 0 \\
g_i &= 0.
\end{aligned}$$

where we used the relation that $\phi_i\psi_i = 1$ and $\phi_i\psi_{\neq i} = 0$.