Throughout we assume that R is a commutative ring with identity $1 \neq 0$.

1. Recall that $S^{-1}R$ is defined to be the set of equivalence classes in $R \times S$ with the equivalence relation $(r, s) \sim (r', s')$ if there exists an element $t \in S$ such that $t(rs' - r's) = 0$. Prove that the multiplication $(r_1, s_1) \cdot (r_2, s_2) = (r_1r_2, s_1s_2)$ is a well-defined operation on $S^{-1}R$.

2. Define Φ to be the function which takes a subset of R to a subset of $S^{-1}R$ in the following way:
$$\Phi(I) = \{(x, s) \mid x \in I, s \in S\}.$$
   - Prove that if I is an ideal in R, then $\Phi(I)$ is an ideal in $S^{-1}R$.
   - Prove that if P is a prime ideal that does not intersect S, and S has no zero divisors, then $\Phi(I)$ is a prime ideal in $S^{-1}R$.

3. Suppose that $S \subset R$ is a multiplicative set in R. Prove that the homomorphism $\phi : R \to S^{-1}R$ is injective if and only if S contains no zero-divisors.

4. Let p be a prime integer, and consider the ring $R = \mathbb{Z}$. Let $S = \mathbb{Z} \setminus p\mathbb{Z}$. The ring $S^{-1}R$ is called the localization of $\mathbb{Z}$ at p. Find all of the ideals of this ring, and describe the maximal ideal (there is only one).

5. This problem is to help us look ahead. Let $\varphi(n)$ denote the number of units in $\mathbb{Z}/n\mathbb{Z}$. (From previous homework, we know this is the same as computing the number of integers $1 \leq k < n$ that are relatively prime to n.)
   (a) Compute $\varphi(p^k)$ when p is prime and k is a non-negative integer.
   (b) Compute $\varphi(n)$ for $n \in \{6, 10, 12, 18, 24, 36\}$ and make a conjecture about the relationship between $\varphi(n)$ and the prime power decomposition of n.

Here I'll recall the universal property that defines a product, and introduce a new one that defines the coproduct.

Definition: Let X_1, X_2 be objects in a category $\mathcal{C}$. Define $X_1 \prod X_2$ to be the object with the following universal property: There are morphisms $\pi_i : X_1 \prod X_2 \to X_i$ such that if there is any other object T with morphisms $\varphi_i : T \to X_i$ for $i \in \{1, 2\}$, then there is a unique morphism $\Psi : T \to X_1 \times \prod X_2$ such that $\pi_i \circ \Psi = \varphi_i$.

This is conveniently captured in the diagram to the right, and is read to mean that you start with the object $X_1 \prod X_2$ and its morphisms to X_1 and X_2 set in stone. Any time there is an object T with morphisms as in the diagram, there is a morphism Ψ that can fill in that dotted arrow so that the diagram commutes, meaning any two ways to get to the same place produce the same morphism.

$$\begin{tikzcd}
T \arrow[rr, dotted, "\exists!\Psi"] \arrow[dr, "p_1"'] \arrow[ddr, "p_2"'] & & X_1 \prod X_2 \arrow[dl, "\pi_1"] \arrow[ddl, "\pi_2"] \\
& X_1 & \\
& X_2 &
\end{tikzcd}$$

6. Prove that if R_1 and R_2 are rings, then the product ring $\{(r_1, r_2) \mid r_1 \in R_1, r_2 \in R_2\}$ is actually the object $R_1 \prod R_2$. I.e., verify that it satisfies the universal property.

7. Consider now the category $\mathcal{C}$ whose objects are positive integers and so that $\mathrm{Mor}_{\mathcal{C}}(n, m) = \begin{cases} \{m/n\} & \text{if } m \mid n \\ \emptyset & \text{otherwise} \end{cases}$. Given two integers n, m, compute $m \prod n$ in this context (i.e., what integer has the desired universal property).

If you really like the last problem, awesome. You can try your hand at a new one. This is just for the superfans. Try to write what you think the universal property of the coproduct should be given just the diagram:

$$\begin{tikzcd}
& X_2 \arrow[ddl, "\iota_2"'] \arrow[ddr, "j_2"] & \\
& X_1 \arrow[dl, "\iota_1"] \arrow[dr, "j_1"'] & \\
X_1 \coprod X_2 \arrow[rr, dotted, "\exists!\Psi"'] & & T
\end{tikzcd}$$

Then, compute $m \coprod n$ for objects m, n in $\mathcal{C}$.