Throughout, unless otherwise stated, R is a ring with identity and M is a (left) R-module.

1. Suppose that I is an ideal in R, and define by
$$IM = \left\{ \sum_{finite} a_i m_i \mid a_i \in I,\ m_i \in M \right\}.$$
Prove that IM is a submodule of M. Suppose that $A = \sum_{k\in K} a_k m_k$ and $B = \sum_J b_j m_j$ are elements of IM where $a_k, b_j \in I$ and K and J are finite index sets. Then
$$\begin{aligned} A + rB &= \sum_{k\in K} a_k m_k + r \sum_{j\in J} b_j m_j \\ &= \sum_{l\in K\cup J} (a_l + r b_l) m_l \end{aligned}$$
which is in IM since $al + rbl \in I$ and $K \cup J$ is finite.

2. Let M and M' be modules over R. Denote by
$$\mathrm{Hom}_R(M, M') = \{ f : M \to M' \mid f(rm) = r \cdot f(m) \forall r \in R,\ m \in M \}.$$
Prove that if R is a commutative ring, then $\mathrm{Hom}_R(M, M')$ is an R-module. First, we have seen in group theory that $\mathrm{Hom}_R(M, M')$ is an abelian group, so we only need to demonstrate the action of R on this set. Let $\varphi \in \mathrm{Hom}_R(M, M')$ and $r \in R$. Define $[r\varphi] : M \to M'$ by $[r\varphi](m) = r(\varphi(m))$. Then
   (a) $[r\varphi](m + n) = r(\varphi(m + n)) = r\varphi(m) + r\varphi(n) = [r\varphi](m) + [r\varphi](n)$
   (b) $[r\varphi](sm) = r\varphi(sm) = rs\varphi(m) = sr\varphi(m) = s[r\varphi](m)$

   Hence, $[r\varphi]$ is a module homomorphism (you can check the other conditions).

3. Suppose that $R = \mathbb{R}[t]$, and consider the space $M = \mathbb{R}^2$. Recall that we can make M into an R-module by providing a linear transformation T on M. To this end, let T be the transformation given by the matrix
$$\begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}.$$
Find all R-submodules of this module M. Suppose that $N \subset M$ is a submodule. This means that $rn \in N$ for all $n \in N$ and $r \in R$, and that $n+n' \in N$ whenever $n, n' \in N$. In particular, we consider the constant polynomials $c \in R$. This menas that $cn \in N$ for all $n \in N$. The rest of the module axioms show that N must be a vector subspace of M. Since M is two-dimensional over $\mathbb{R}$, its only subspaces are $0, M$ and $\mathrm{span}(v)$ where $v \in M$. Next, we consider the polynomial t. We require that $tv \in \mathrm{span}(v)$, so $tv = \lambda v$ for some $\lambda \in \mathbb{R}$. This holds if and only if v is an Eigenvector of T. The Eigenvectors of this matrix are $(1, 0)$ and $(-2, 1)$. Hence, the submodules are $\mathrm{span}((1, 0))$ and $\mathrm{span}((-2, 1))$. t acts on the first via multiplication by 2, and t acts on the second via multiplication by -1.

4. Suppose that M and M' are R-modules and $f : M \to M'$ is a module homomorphism.

   (a) Prove that $\ker f$ is a submodule of M. We know that the kernel of a group homomorphism, so $\ker f$ is an abelian group. Furthermore, if $r \in R$ and $x \in \ker f$, then $f(rx) = rf(x) = r \cdot 0 = 0$. So $\ker f$ is a submodule of M.

   (b) Prove that $\operatorname{im} f$ is a submodule of M'. Similarly, from group theory, the image of a homomorphism is a subgroup of the codomain. So suppose that $x \in \operatorname{im} f$, so $x = f(y)$ for some $y \in M$. Then $rx = rf(y) = f(ry)$, so $rx \in \operatorname{im} f$.

   (c) Prove that $M/\ker f \cong \operatorname{im} f$ (you'll have to construct the isomorphism, of course, but you can assume that M/N is a module whenever N is a submodule of M). Consider the function $F : M/\ker f \to M'$ defined via $F(m + \ker f) = f(m)$. We already know that F is a well-defined group isomorphism from the first isomorphism theorem from group theory. We need only show, then, that F is also a module homomorphism. To this end, $F(r(m+kerf)) = F(rm+\ker f) = f(rm) = rf(m) = rF(m+\ker f)$, so indeed, F is an R-module isomorphism.

5. M is said to be *irreducible* if its only submodules are 0 and M itself.

   (a) Prove that a non-trivial module M is irreducible if and only if M can be generated by any of its non-zero elements.[1] Suppose that M is irreducible and $x \neq 0$ in M. Then Rx is a submodule. Since $x \neq 0$, then $x \in Rx$ so $Rx \neq 0$. Hence, $Rx = M$. On the other hand, suppose that M can be generated by any of its non-zero elements, and let $N \subset M$ be a non-zero submodule. Then there is an element $x \in N$, so by assumption, $M = Rx$. But $Rx \subset N$, so $N = M$. Hence, M is irreducible.

   (b) Prove that if M, N are irreducible modules, and $f : M \to N$ is a non-zero homomorphism, then f is an isomorphism. If M and N are irreducible and $f : M \to N$ is a module homomorphism. Then $\ker f$ and $\operatorname{im} f$ are submodules of M and N as shown above. Since f is non-zero, $\ker f \neq M$, so, since M is irreducible, $\ker f = 0$ (i.e., f is injective). Similarly, $\operatorname{im} f \neq 0$, so $\operatorname{im} f = N$.

   (c) Describe the irreducible modules over $\mathbb{Z}$ that have finite cardinality. Suppose that M is an irreducible module over $\mathbb{Z}$ of finite cardinality. We have seen that $\mathbb{Z}$ modules are precisely the same as abelian groups, so M is a finite abelian group. By the classification theorem of finite abelian groups, we have $M \cong \bigoplus_{i=1}^{u} \mathbb{Z}/k_i\mathbb{Z}$ where $k_i \mid k_{i+1}$. In particular, if M is irreducible, it can only have one direct summand, so $M \cong \mathbb{Z}/k\mathbb{Z}$ for some integer k. But if $n \mid k$, then $n\mathbb{Z}/k\mathbb{Z}$ is a non-trivial submodule, so $\mathbb{Z}/k\mathbb{Z}$ is simple if and only if $k = p$ is a prime.

[1] We say x generates M if $M = \{rx \mid r \in R\}$.

6. Consider the quiver discussed in class: $1 \xrightarrow{a} 2$. Recall that $\mathbb{R}Q$ has a basis $\{x_{e_1}, x_a, x_{e_2}\}$.

   (a) This ring has an identity element. Find it. $(x_{e_1} + x_{e_2})()(\gamma_1 x_{e_1} + \gamma_a x_a + \gamma_2 x_{e_2}) = (\gamma_1 x_{e_1} + \gamma_a x_a + \gamma_2 x_{e_2})$, so $x_{e_1} + x_{e_2}$ is the identity.

   (b) It turns out that $\mathbb{R}Q$ is isomorphic to $T_2(\mathbb{R})$ (the ring of upper-triangular matrices with entries in $\mathbb{R}$). Find the isomorphism and prove your result. Consider the function $F : \mathbb{R}Q \to T_2(\mathbb{R})$ defined by

$$F(\gamma_1 x_{e_1} + \gamma_a x_a + \gamma_2 x_{e_2}) = \begin{bmatrix} \gamma_2 & \gamma_a \\ 0 & \gamma_1 \end{bmatrix}.$$

   The linearity is easy to verify, so you should do it. Additionally,

$$\begin{aligned} F((\gamma_1 x_{e_1} + \gamma_a x_a + \gamma_2 x_{e_2})(\alpha_1 x_{e_1} + \alpha_a x_a + \alpha_2 x_{e_2})) &= F(\gamma_1 \alpha_1 X_{e_1} \\ &\quad + (\gamma_a \alpha_1 + \alpha_a \gamma_2) x_a \\ &\quad + \gamma_2 \alpha_2 X_{e_2} \\ &= \begin{bmatrix} \gamma_2 \alpha_2 & \gamma_a \alpha_1 + \alpha_a \gamma_2 \\ 0 & \gamma_1 \alpha_1 \end{bmatrix} \\ &= \begin{bmatrix} \gamma_2 & \gamma_a \\ 0 & \gamma_1 \end{bmatrix} \cdot \begin{bmatrix} \alpha_2 & \alpha_a \\ 0 & \alpha_1 \end{bmatrix}. \end{aligned}$$

7. Consider the groups $G = \mathbb{Z}/4\mathbb{Z}$ (the cyclic group of order 4) and $H = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (the Klein 4-group).

   (a) Construct the group rings $\mathbb{R}G$ and $\mathbb{R}H$ for each. That is, describe a basis for each and the multiplicative structure on this basis. RG has basis $\{x_0, x_1, x_2, x_3\}$ in which $x_0 = 1$, $x_1^2 = x_2$, and $x_1^3 = x_3$, and $x_1^4 = 1$. So $RG = \mathbb{R}[x]/(x^4 - 1)$. $\mathbb{R}H$ has basis $\{x_{00}, x_{10}, x_{01}, x_{11}\}$. But notice that $x_{10}^2 = x_{01}^2 = 1$, and $x_{11} = x_{10}x_{01} = x_{01}x_{10}$. If we rename $x = x_{10}$ and $y = y_{01}$, then this ring is really $\mathbb{R}[x, y]/(x^2 - 1, y^2 - 1)$.

   (b) Prove that $\mathbb{R}G$ is not isomorphic to $\mathbb{R}H$. Here's one way you can go about it: We've phrased each of the rings as quotients of polynomial rings by ideals. We could ask what the maximal ideals are, since those are relatively easy to understand in the context of polynomial rings. One theorem we have is that there is a bijection between ideals in R/I and ideals in R containing J. So which ideals J contain $(x^4 - 1)$. Well we note that $x^4 - 1 = (x - 1)(x + 1)(x^2 + 1)$. If J is an ideal containing $(x^4 - 1)$, then it must be generated by $(x - 1)^n (x + 1)^m (x^2 + 1)^l$ where each of n, m, l could be either 0 or 1, but no larger. This gives us a total of 8 ideals in $\mathbb{R}G$ (one of which is $\mathbb{R}G$ itself by taking $n = m = l = 0$). On the other hand, $(x^2 - 1) = (x + 1)(x - 1)$ and $y^2 - 1 = (y + 1)(y - 1)$. Thus, the ideals that contain $(x^2 - 1, y^2 - 1)$ are the ideals I of the form $((x - 1)^n (x + 1)^m, (y - 1)^l (y + 1)^p)$ where each of n and m can be 0 or 1, but note that if $n = m = 0$ or $l = p = 0$, then $I = \mathbb{R}H$, so we're counting that one twice. This yields a total of $2^4 - 1 = 15$ ideals. Hence, $\mathbb{R}H$ and $\mathbb{R}G$ are non-isomorphic.