

Throughout we assume that R is a commutative ring with identity $1 \neq 0$.

1. Recall that $S^{-1}R$ is defined to be the set of equivalence classes in $R \times S$ with the equivalence relation $(r, s) \sim (r', s')$ if there exists an element $t \in S$ such that $t(rs' - r's) = 0$. Prove that the multiplication $(r_1, s_1) \cdot (r_2, s_2) = (r_1 r_2, s_1 s_2)$ is a well-defined operation on $S^{-1}R$.
2. Define Φ to be the function which takes a subset of R to a subset of $S^{-1}R$ in the following way:

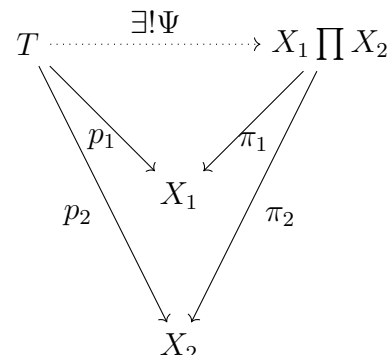
$$\Phi(I) = \{(x, s) \mid x \in I, s \in S\}.$$

- Prove that if I is an ideal in R , then $\Phi(I)$ is an ideal in $S^{-1}R$.
 - Prove that if P is a prime ideal that does not intersect S , and S has no zero divisors, then $\Phi(I)$ is a prime ideal in $S^{-1}R$.
3. Suppose that $S \subset R$ is a multiplicative set in R . Prove that the homomorphism $\phi : R \rightarrow S^{-1}R$ is injective if and only if S contains no zero-divisors.
 4. Let p be a prime integer, and consider the ring $R = \mathbb{Z}$. Let $S = \mathbb{Z} \setminus p\mathbb{Z}$. The ring $S^{-1}R$ is called the localization of \mathbb{Z} at p . Find all of the ideals of this ring, and describe the maximal ideal (there is only one).
 5. This problem is to help us look ahead. Let $\varphi(n)$ denote the number of units in $\mathbb{Z}/n\mathbb{Z}$. (From previous homework, we know this is the same as computing the number of integers $1 \leq k < n$ that are relatively prime to n .)
 - (a) Compute $\varphi(p^k)$ when p is prime and k is a non-negative integer.
 - (b) Compute $\varphi(n)$ for $n \in \{6, 10, 12, 18, 24, 36\}$ and make a conjecture about the relationship between $\varphi(n)$ and the prime power decomposition of n .

Here I'll recall the universal property that defines a product, and introduce a new one that defines the coproduct.

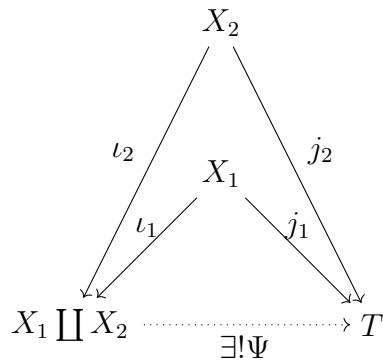
Definition: Let X_1, X_2 be objects in a category \mathcal{C} . Define $X_1 \amalg X_2$ to be the object with the following universal property: There are morphisms $\pi_i : X_1 \amalg X_2 \rightarrow X_i$ such that if there is any other object T with morphisms $\varphi_i : T \rightarrow X_i$ for $i \in \{1, 2\}$, then there is a unique morphism $\Psi : T \rightarrow X_1 \amalg X_2$ such that $\pi_i \circ \Psi = \varphi_i$.

This is conveniently captured in the diagram to the right, and is read to mean that you start with the object $X_1 \amalg X_2$ and its morphisms to X_1 and X_2 set in stone. Any time there is an object T with morphisms as in the diagram, there is a morphism Ψ that can fill in that dotted arrow so that the diagram commutes, meaning any two ways to get to the same place produce the same morphism.



6. Prove that if R_1 and R_2 are rings, then the product ring $\{(r_1, r_2) \mid r_1 \in R_1, r_2 \in R_2\}$ is actually the object $R_1 \amalg R_2$. I.e., verify that it satisfies the universal property.
7. Consider now the category \mathcal{C} whose objects are positive integers and so that
$$\text{Mor}_{\mathcal{C}}(n, m) = \begin{cases} \{m/n\} & \text{if } m \mid n \\ \emptyset & \text{otherwise} \end{cases}.$$
 Given two integers n, m , compute $m \amalg n$ in this context (i.e., what integer has the desired universal property).

If you really like the last problem, awesome. You can try your hand at a new one. This is just for the superfans. Try to write what you think the universal property of the coproduct should be given just the diagram:



Then, compute $m \amalg n$ for objects m, n in \mathcal{C} .