1. Recall that in order to prove that given a chain of ideals $J \subset I \subset R$, there is an isomorphism $(R/I)/(J/I) \cong R/J$, we wanted to study the function

$$\varphi : R/I \to R/J$$
$$\varphi : r + I \mapsto r + J.$$

Prove that φ is a well-defined ring homomorphism with $\ker(\varphi) = J/I = \{j+I \mid j \in J\}$.

2. Describe all ideals in $\mathbb{Z}/n\mathbb{Z}$ for any positive integer n.

3. Suppose that R is an ring. Prove that $R[x]$ is an integral domain if and only if R is an integral domain.

4. Suppose that F is a field. Prove the following:

   (a) If $f(x), g(x) \in F[x]$, then there exists polynomials $q(x)$ and $r(x)$ in $F[x]$ such that $f(x) = g(x)q(x) + r(x)$ where $\deg(r(x)) < deg(g(x))$.

   (b) Suppose that $f(x), g(x) \in F[x]$. Prove that if $f(x) = g(x)q(x) + r(x)$ where $q(x), r(x)$ are the polynomials guaranteed from above, then $\gcd(f(x), g(x)) = \gcd(g(x), r(x))$. [1]

5. Suppose that R is a commutative local ring. I.e., a commutative ring with $1 \neq 0$ which has a unique maximal ideal, which we'll call $\mathfrak{m}$. Prove that $R^* = R \setminus \mathfrak{m}$.

6. Suppose that $\mathfrak{p}$ is an ideal in a commutative ring with identity $1 \neq 0$. Prove that $\mathfrak{p}$ is prime if and only if $R/\mathfrak{p}$ is an integral domain.

7. Suppose that $n, m \in \mathbb{Z}$, and consider the function

$$\varphi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$$
$$\varphi : x \mapsto ([x]_n, [x]_m)$$

where $[x]_n$ indicates the equivalence class of x modulo n.

   (a) Prove that φ is a ring homomorphism.

   (b) Compute the kernel of φ and determine the cardinality of its image.

   (c) What does this tell you in the case that $\gcd(n, m) = 1$.

[1] This can be iterated to prove that $\gcd(f(x), g(x))$ can be written as a linear combination of $f(x)$ and $g(x)$.

Name: ______________________