1. Dummit & Foote, problem 10.3 #2 Suppose that $R^m \cong R^n$. Let I be a maximal ideal of R. Then $F^m = R^m/I^m \cong R^n/I^n = F^n$ where $F = R/I$ is a field. For finite dimensional vector spaces, the dimension is unique, so $m = n$. On the other hand, suppose that $m = n$. Then $R^m = R^n$, so the isomorphism is the identity.

2. Let $R = \mathrm{Mat}_{n\times n}(\mathbb{C})$, and $M = \mathbb{C}^n$ be the R module with $A \cdot \vec{v} = A\vec{v}$ (given by left matrix multiplication). Prove that M is a cyclic R-module. Consider the vector $e_1 \in M$, the first standard basis vector. Then Ae_1 is the first column of A. Hence, for all $v \in M$, $v = [v * **]e_1$ where $[v * **]$ is the matrix with v as first column and anything else in the other columns. Thus, $Re_1 = M$.

3. Dummit & Foote, problem 10.2 #4 Let A be a $\mathbb{Z}$ module.

   (a) Suppose that $a \in A$. Then the map $\varphi_a : \mathbb{Z}/n\mathbb{Z} \to A$ defined by $\varphi_a([k]_n) = ka$ is well-defined if and only if $na = 0$. Indeed, if $\varphi_a([k]_n) = ka$ is well-defined, then $na = \varphi_a([n]_n) = \varphi_a([0]_n) = 0 \cdot a = 0$. On the other hand, suppose that $na = 0$. if $k - k' = cn$ for some integers k, k', c, then $ak - ak' = acn = 0$, so $ak = ak'$, i.e., $\varphi([k]_n) = \varphi([k']_n)$.

   (b) Now consider $M = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A)$. If $\varphi \in M$, then let $a = \varphi([1])$. Then $\varphi([k]) = ak = \varphi_a([k])$. Hence, $\varphi = \varphi_a$, so all elements of M are of the form φ_a for some $a \in A$. Now define the function $G : M \to A_n$ vial $G(\varphi_a) = \varphi_a(1) = a$. By the above description, G is onto. Furthermore, if $G(\varphi_a) = G(\varphi_b)$, then $a = b$, so G is one-to-one. Furthemore, $G(\varphi + \phi) = (\varphi + \phi)(1) = \varphi(1) + \phi(1) = G(\varphi) + G(\phi)$, so G is a homomorphism, and by the other work above, it is an isomorphism.

4. Consider the ring $R = \mathbb{C}[t]/(t^2)$. Classify all *irreducible* R-modules and all *indecomposable* R-modules and prove your result. Recall that an R-module M is a $\mathbb{C}$ vector space with the action of a linear map $t : M \to M$ and we need to insist that $t^2 = 0$ since this is the case in R. By Jordan Canonical Form, this means that there is a basis for M for which t take the matrix form consisting of $1x1$ and $2x2$ Jordan blocks. But if $\dim M > 1$, we know by Cayley's Theorem that t has at least one Eigenvector, and since t is nilpotent, the Eigenvalue of this is 0. Hence, $\mathrm{span}(v)$ is a submodule since $t.v = 0v \in \mathrm{span}(v)$. Hence, if M is irreducible, $\dim M = 1$ or $\dim M = 0$. If $\dim M = 1$, then t is a nilpotent linear operator, so $t.v = 0$ for all v. Thus, the only non-zero irreducibles are $(\mathbb{C}, [0])$. By the above discussion, for any R-module M with $\dim M = d$ we can find a basis for M such that the matrix of the action of t is given by $\begin{array}{|c|c|c|c|}\hline J_{n_1}(0) & 0 & \dots & 0 \\ \hline 0 & J_{n_2}(0) & \dots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \dots & J_{n_k}(0) \\ \hline \end{array}$ where $J_{n_i}(0)$ are Jordan blocks of Eigenvalue 0 of size at most 2. We showed in linear algebra that this indicates $M \cong \bigoplus_{i=1}^{k} (\mathbb{C}^{n_i}, J_{n_i}(0))$ is the direct sum decomposition of M. Therefore, the

only indecomposable modules are those of the form $(\mathbb{C}^2, J_2(0))$ and $(\mathbb{C}, J_1(0))$.

5. Dummit & Foote, problem 10.3 #7 Suppose that N and M/N are finitely generated, with N generated by $\{n_1, \dots, n_l\}$ and M/N generated by $\{m_1 + N, \dots, m_k + N\}$. Then I claim that $\{n_1, \dots, n_l, m_1, \dots, m_k\}$ generate M. Indeed, let $x \in M$. Then

$$\begin{aligned} x + N &= \sum_i r_i(m_i + N) \\ &= \sum_i r_i m_i + N \end{aligned}$$

since the cosets of m_i generate M/N. Thus, $x - \sum_i r_i m_i \in N$. But N is generated by the n_j, so $x - \sum_i r_i m_i = \sum_j s_j n_j$, and thus $x = \sum_i r_i m_i + \sum_j s_j n_j$. This shows that the original set generates M.

6. Recall that if Q is the quiver $1 \xrightarrow{a} 2$, then FQ is the F-vector space with basis $\{x_{e_1}, x_{e_2}, x_a\}$ and multiplication defined by concatenation of paths. We showed in class that a module is defined by three pieces of data: vector spaces V_1 and V_2, and a linear map $V_a : V_1 \to V_2$. Classify the *irreducible* FQ modules. Suppose that $\mathbb{V} = (V_1, V_2, V_a)$ is a module. A submodule is a collection of subspaces (W_1, W_2) with $W_i \subset V_i$ such that $V_a(W_1) \subset W_2$. Notice that $\ker V_a \subset V_1$, and $V_a(\ker V_a) = V_a(0) = 0$, so $(\ker V_a, 0)$ is a submodule of $\mathbb{V}$. Thus, if $V_1 \neq 0$, $V_2 = 0$. Further, $(\mathbb{C}^n, 0, 0)$ has $(\operatorname{span}(e_1), 0, 0)$ as a submodule, so it is irreducible if and only if $n = 1$. **Solution 1: $(\mathbb{C}^1, 0, 0)$ is irreducible. Secondly, suppose that $V_2 \neq 0$. Then for any subspace $W_2 \subset V_2$, we have $(0, W_2, 0)$ a submodule (since $V_a(0) = 0 \subset W_2$). So if $\dim V_2 \neq 0$ for some irreducible, then $\dim V_1 = 0$. Further, if $\dim V_2 > 1$, we can take $(0, \operatorname{span}(e_1), 0)$ which is also a submodule, so the only irreducible with $\dim V_2 \neq 0$ is $(0, \mathbb{C}, 0)$. These are, therefore, the only irreducible modules over $\mathbb{C}[t]$.**

7. Show that if M_1 and M_2 are irreducible R-modules and $f : M_1 \to M_2$ is a homomorphism, then f is either invertible of the zero map. We did this on another homework: $\ker f$ is a submodule of M_1, so if f is not zero, $\ker f = 0$. Similarly, $\operatorname{im}(f)$ is a submodule of M_2, so if f is non-zero, $\operatorname{im}(f) = M_2$. Thus, f is an isomorphism if it is non-zero.