You must complete problems 1-5. Each is worth 10 points. For a possible 5 point bonus, you can complete an extra problem. Make sure you indicate which one is to be your bonus.

1. Define the following terms:
   (a) A *two-sided ideal* I in a ring R
   (b) A *prime element* p in a commutative ring R
   (c) An *irreducible element* q in a commutative ring R
   (d) A *principal ideal* I in a commutative ring R

2. Complete **one** of the following:
   (a) Suppose that R is a Euclidean domain with norm $N : R \to \mathbb{Z}_{\geq 0}$. Let I be an ideal in R, and let $d \in I \setminus \{0\}$ be an element of minimal norm (i.e., $N(x) \geq N(d)$ for all $x \in I$). Prove that $I = (d)$.
   (b) Let F be a field, and consider the ring $R = F[t]$.
      i. Describe the one-dimensional R-modules (that is, the modules which are one-dimensional when considered as vector spaces over F).
      ii. Let $V = F^n$ be the n-dimensional F-vector space with basis $\{e_1, \ldots, e_n\}$, and define $T : V \to V$ to be the linear transformation such that $T(e_i) = e_{i-1}$ for $2 \leq i \leq n$ and $T(e_1) = 0$. Prove that $U_k := \operatorname{span}(e_1, \ldots, e_k)$ is an $F[t]$-submodule of V for each $k = 1, \ldots, n$.

3. Prove **one** of the following.
   (a) Let M be an R module and $N \subset M$ be a submodule. Prove that if N and M/N are finitely-generated over R, then M is also finitely generated.
   (b) Suppose that $0 \to N \xrightarrow{i} M \xrightarrow{\pi} L \to 0$ is a short exact sequence of R modules and that there exists a homomorphism $p : L \to M$ such that $\pi \circ p = \mathrm{id}_L$. Prove that $M \cong N \oplus L$.

4. Fun with isomorphism theorems. Prove **one** of the following:
   (a) Let R be a commutative ring and J an ideal in R. Prove that there is a bijection between ideals I in R containing J and ideals in R/J, call it Ψ. Then show that $\Psi(I)$ is a prime ideal in R/J if and only if I is a prime ideal in R (which contains J).
   (b) Prove the first isomorphism theorem for modules: If M and N are R-modules and $\varphi : M \to N$ is an R-module homomorphism, then $M/\ker\varphi \cong \operatorname{image}(\varphi)$.

5. Complete **one** of the following problems:
   (a) Show that if R is a ring and M is an R-module, then $R \otimes_R M \cong M$. Construct this homomorphism via the universal property.

(b) Let $S \subset R$ be a multiplicatively closed subset of a commutative ring R, and $S^{-1}R$ be the ring of fractions of S in R.

i. Prove that the function $\Psi : \{\text{ideals } I \subset R \mid I \cap S = \emptyset\} \to \{\text{ideals } S^{-1}I \subset S^{-1}R\}$ defined by

$$\Psi(I) = \{(x, s) \mid x \in I, s \in S\}$$

has the property that $\Psi(P)$ is prime whenever P is.

ii. Let $S = \{2^k \mid k \in \mathbb{Z}_{\geq 0}\}$ and $R = \mathbb{Z}$. List the prime ideals in $S^{-1}\mathbb{Z}$.

(c) Let R be a commutative ring and M an R module. Prove that

$$\mathrm{Hom}_R(R^n, M) \cong \underbrace{M \oplus M \oplus \ldots \oplus M}_{n}.$$

Note: you must define the isomorphism and prove your result.