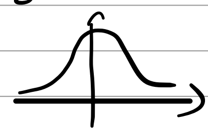
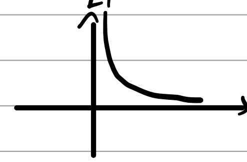
1. χ^2 Chi-Square Distribution

$\boxed{\text{pdf}}$

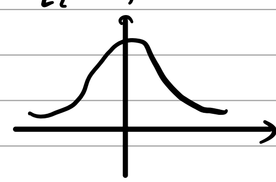
$Z_1 \sim N(0,1)$



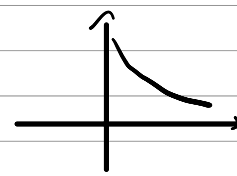
Z_1^2



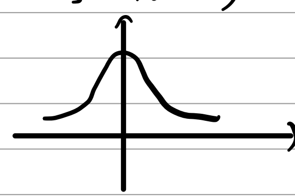
$Z_2 \sim N(0,1)$



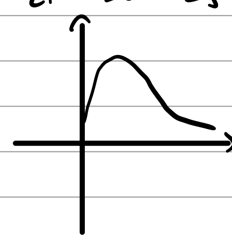
$Z_1^2 + Z_2^2$



$Z_3 \sim N(0,1)$



$Z_1^2 + Z_2^2 + Z_3^2$



$Y = \chi^2(k) = \sum_{i=1}^{k} Z_i^2$

$E[Y] = k$ why? $E[Y] = \sum E[Z_i^2] = k$

$Var(Y) = 2k$. why? $Var(Y) = \sum Var(Z_i^2)$

$= \sum \left[\underbrace{E[Z_i^4]}_{\substack{\text{kurtosis} \\ = 3}} - \underbrace{E[Z_i^2]^2}_{1} \right]$

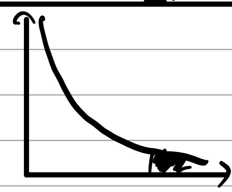
2. Goodness of Fit

	theory	sample
Left hand	9	11
Right hand	66	64

Test statistic: $\chi^2 = \sum_{i=1}^{k} \frac{(O_i - e_i)^2}{e_i}$, $df = 1$

$$\chi^2(1) = \frac{(11-9)^2}{9} + \frac{(64-66)^2}{66} = 0.505$$

差的越多，值越大

Critical Value	p-value
 5% level of significance	5% level of significance $\Rightarrow p = 0.05$ or 5%
$g(a) = 95\%$	$g(0.505) = 47.7\%$
$a = g^{-1}(95\%) = 3.841$	$\therefore$ Not reject.
$\because 0.505 < 3.841$,	
$\therefore$ Not reject.	

3. Test for independence

GoF

	obs	expected
A	22	33.3
B	19	33.3
C	59	33.3

$$\chi^2(2) = \frac{(22-33.3)^2}{33.3} + \cdots$$

	X	Y	total
A	10	12	22
B	9	10	19
C	25	34	59
	44	56	100

Q: (A, B, C) indep of (X, Y) ?

A: $P(E1 \cap E2) = P(E1) \cdot P(E2)$

If indep, what to expect ?

A	22%
B	19%
C	59%

X	Y
44%	56%

G.F

	obs	expected
A	22	33.3
B	19	33.3
C	59	33.3

Indep

	obs (X)	expected
A	10	22% × 44% ÷ 100 = 9.68
B	9	19% × 44% ÷ 100 = 8.36
C	25	59% × 56% ÷ 100 = 33.04

$$\chi^2(2) = \frac{(10-9.68)^2}{9.68} + \frac{(9-8.36)^2}{8.36} + \frac{(25-33.04)^2}{33.04}$$

In general,

	X	Y	Z
A			
B			
C			
D			

$df = (c-1)(r-1)$

$= 2 \times 3$

$= 6$

4. Where is Normal distribution?

$$\chi^2(1) = Z_1^2$$

$\uparrow$

?

CLT !

Left / Right } $X = Bern(p)$ $\quad \mu = p \quad \sigma^2 = p(1-p)$

$$\frac{\Sigma X_i}{n} \overset{CLT}{\sim} N(\mu, \frac{\sigma^2}{n})$$

$$\Sigma X_i \sim N(np, np(1-p))$$

$$Y = \Sigma X_i \sim Bin(n, p)$$

$$Y \sim N(np, np(1-p))$$

$$\frac{Y - np}{\sqrt{np(1-p)}} \sim Z$$

$$\frac{(Y-np)^2}{np(1-p)} \sim Z^2 = \chi^2(1)$$

recall:

	theory		sample	
Left hand	9	np	11	Y
Right hand	66	$n(1-p)$	64	$n-Y$

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - e_i)^2}{e_i}$$

$$= \frac{(Y - np)^2}{np} + \frac{(n-Y - n(1-p))^2}{n(1-p)}$$

Given $\dfrac{(Y-np)^2}{np(1-p)} \sim Z^2 = \chi^2(1)$

$$= \frac{(Y-np)^2 (p+(1-p))}{np(1-p)}$$

$$= \frac{(Y-np)^2}{n(1-p)} + \frac{(Y-np)^2}{np}$$

$$= \frac{(Y-np+n-n)^2}{n(1-p)} + \frac{(Y-np)^2}{np}$$

$$= \frac{(Y-n+n(1-p))^2}{n(1-p)} + \frac{(Y-np)^2}{np}$$

$$= \frac{(n-Y-n(1-p))^2}{n(1-p)} + \frac{(Y-np)^2}{np} \qquad (\checkmark)$$