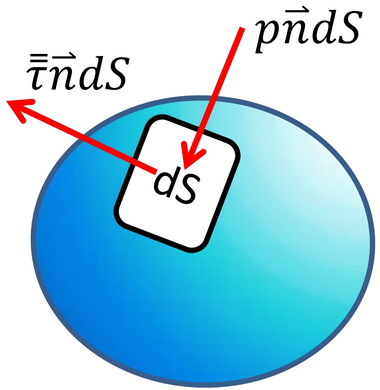
Viscous stresses

Complete system of Navier-Stokes equation

Conservation laws



- $\frac{\partial}{\partial t}\int_{\Omega} \rho \, d\Omega + \int_{\partial\Omega} \rho(v \cdot \vec{n}) \, dS = 0$
- $\frac{\partial}{\partial t}\int_{\Omega} \rho\vec{v} \, d\Omega + \oint_{\partial\Omega} \rho\vec{v}(\vec{v} \cdot \vec{n}) dS$

$= \int_{\Omega} \rho\vec{f}_e d\Omega - \oint_{\partial\Omega} p \cdot \vec{n} dS + \oint_{\partial\Omega} (\overline{\overline{\tau}} \cdot \vec{n}) dS$

(Moment equation)

Conservation laws

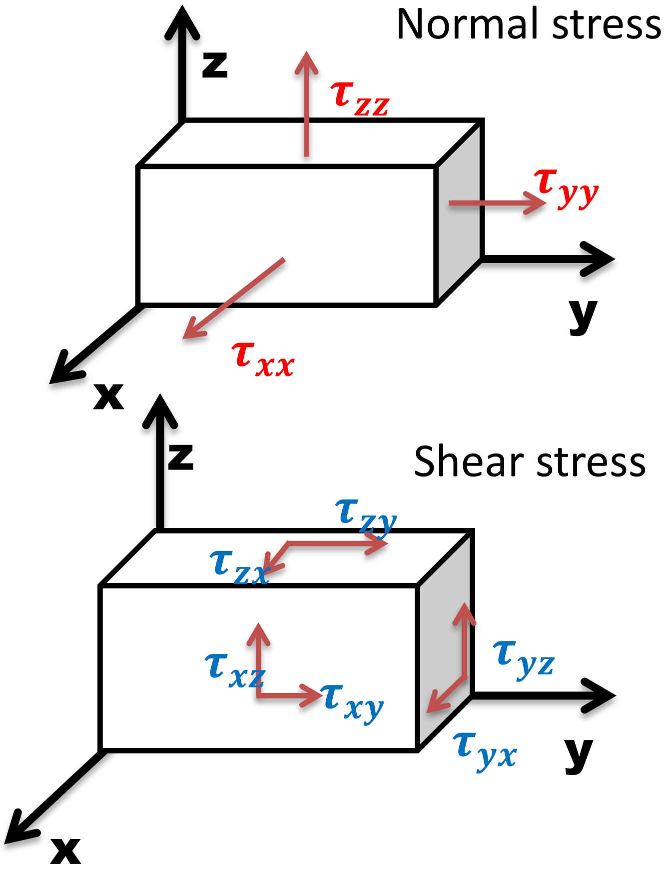
- $\frac{\partial}{\partial t}\int_{\Omega} \rho E \, d\Omega + \oint_{\partial\Omega} \rho e(\vec{v} \cdot \vec{n}) dS = \oint_{\partial\Omega} k(\nabla T \cdot \vec{n}) dS + \int_{\Omega} (\rho \vec{f}_e \cdot \vec{v}) d\Omega + \oint_{\partial\Omega} (\bar{\bar{\tau}} \cdot \vec{v}) \cdot \vec{n} \, dS$ (Energy equation)

Viscous stress

- Stress Tensor
- $\bar{\bar{\tau}} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$
- τ_{xy}

Along y direction

On the face ⊥ x



Viscous stress

- Proportional to the velocity gradient:

  Newtonian fluid (water, air, thin oil)

- Otherwise:

  Non-Newtonian fluid (butter, ketchup)

Viscosity

In the case of a Newtonian fluid

$$\tau_{xx} = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial u}{\partial x}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

λ is second viscosity coefficient

μ is dynamic viscosity

$\nu = \mu/\rho$ kinematic viscosity

Stokes hypothesis:
$\lambda = -2/3\mu$

$$\bar{\bar{\tau}} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

Complete system of the NS equation

- In the case of a Newtonian fluid,

  conservation law gives Navier-Stokes equations
- Take the momentum equation as an example

Complete system of the NS equation

- $\frac{\partial}{\partial t}\int_{\Omega} \rho\vec{v}\, d\Omega + \int_{\partial\Omega} \rho\vec{v}(\vec{v}\cdot\vec{n}) + p\vec{n} dS$

$= \int_{\partial\Omega} \bar{\bar{\tau}}\cdot\vec{n}\, dS + \int_{\Omega} \rho\overrightarrow{f_e}\, d\Omega$

Using divergence theorem:

- $\frac{\partial}{\partial t}\int_{\Omega} \rho\vec{v}\, d\Omega + \int_{\Omega} \rho\vec{v}(\nabla\cdot\vec{v}) + \nabla p d\Omega$

$= \int_{\Omega} \nabla\cdot\bar{\bar{\tau}}\, d\Omega + \int_{\Omega} \rho\overrightarrow{f_e}\, d\Omega$

Complete system of the NS equation

- $\frac{\partial}{\partial t}(\rho\vec{v}) + \rho\vec{v}(\nabla\cdot\vec{v}) + \nabla p = \nabla\cdot\bar{\bar{\tau}} + \rho\overrightarrow{\mathrm{f_e}}$
- $\nabla\cdot\bar{\bar{\tau}} = \nabla\cdot\begin{bmatrix}\tau_{xx} & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \tau_{yy} & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \tau_{zz}\end{bmatrix} = \begin{bmatrix}\frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial x}(\tau_{xy}) + \frac{\partial}{\partial z}(\tau_{xz})\\ \frac{\partial}{\partial x}(\tau_{yx}) + \frac{\partial}{\partial x}(\tau_{yy}) + \frac{\partial}{\partial z}(\tau_{yz})\\ \frac{\partial}{\partial x}(\tau_{zx}) + \frac{\partial}{\partial x}(\tau_{zy}) + \frac{\partial}{\partial z}(\tau_{zz})\end{bmatrix}$
- For incompressible fluid, we can further simplify
- Take the first row as an example

Complete system of the NS equation

$$\nabla \cdot \bar{\bar{\tau}} = \nabla \cdot \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

- Take the first row as an example
- $\frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{xy}) + \frac{\partial}{\partial z}(\tau_{xz})$

$$= \frac{\partial}{\partial x}\left(2\mu \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right]$$

$$= 2\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + \mu \frac{\partial^2 v}{\partial xy} + \mu \frac{\partial^2 u}{\partial z^2} + \mu \frac{\partial^2 w}{\partial xz}$$

$$= \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \mu \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

Complete system of the NS equation

- $\frac{\partial}{\partial t}(\rho\vec{v}) + \rho\vec{v}(\nabla\cdot\vec{v}) + \nabla p$
$= \nabla\cdot\bar{\bar{\tau}} + \rho\overrightarrow{\mathrm{f_e}}$
$= \mu\Delta\vec{v} + \rho\overrightarrow{\mathrm{f_e}}$

Assumption:

1. Viscosity is constant
2. Newtonian fluid
3. Simplified mass continuity equation $\nabla\cdot\vec{v} = 0$

Reference

- J. Blazek. Computational Fluid Dynamics: Principles and Applications: 2001
- X. Li. Partial Differential Equations and Applications in Computational Fluid Dynamics: 2014
- Wikipidia. Derivation of the Navier–Stokes equations

Complete system of the NS equation

- Five equations for five conservative variables

$\rho, \rho u, \rho v, \rho w, \rho E$

- But they contain seven unknown variables:

ρ, u, v, w, p, T, E

We have to supply two additional equations

We have to provide the viscosity coefficient μ and the thermal conductivity coefficient κ