Turbulence Model

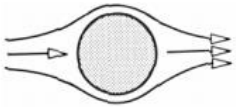
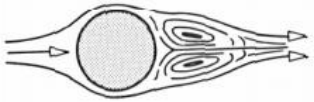
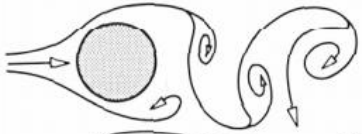
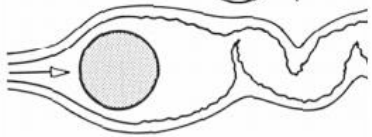
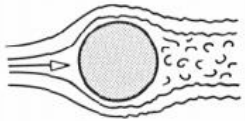
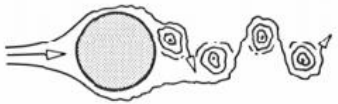
Theory and Implementation

Turbulence

- In fluid dynamics, **turbulence** is a flow regime characterized by chaotic property changes.
- Inertial forces versus viscous forces
- Use Reynolds number to quantify

$$Re = \frac{uL}{\nu}$$

- Laminar flow: low Reynolds number, viscous forces dominates, smooth, constant motion
- Turbulence: High Reynolds number, inertial forces dominates, eddies, instabilities.

$Re < 5$		**Creeping flow (no separation)**
$5\text{-}15 < Re < 40$		**A pair of stable vortices in the wake**
$40 < Re < 150$		**Laminar vortex street**
$150 < Re < 3\times10^5$		**Laminar boundary layer up to the separation point, turbulent wake**
$3\times10^5 < Re < 3.5\times10^6$		**Boundary layer transition to turbulent**
$Re > 3.5\times10^6$		**Turbulent vortex street, but the separation is narrower than the laminar case**

Turbulence models

- A turbulence model is a computational procedure to complete system of mean flow equations.
- There is no single turbulence model, which can predict reliably all kinds of turbulent flows
- Computational effort versus the accuracy required by the particular application.

Reynolds Average

- **Time average**: only for steady flow
- **Space average**: only for homogeneous flow
- **Ensemble average**: for any flow

  Perform the same experiment many times, then take average.

- They are equivalent when flow is steady and homogeneous.
- Usually use $\bar{u}, U, \langle u \rangle$ to indicate mean velocity

Reynolds Decomposition

- Any velocity field can be decomposed to mean flow and turbulent fluctuations

$$u = \bar{u} + u'$$

  - u: velocity field
  - $\bar{u}$: mean velocity
  - u': turbulent fluctuation
  - $\overline{u'} = \overline{u - \bar{u}} = \bar{u} - \bar{u} = 0$
  - $\overline{\bar{u}v} = \bar{u}\bar{v}$
  - $\overline{u + v} = \bar{u} + \bar{v}$

Reynolds Averaged NS equaiton

- Continuity equation:

$$\frac{\partial u_i}{\partial x_i} = 0$$

- Take average:

$$\frac{\partial \overline{u_i}}{\partial x_i} = 0$$

- Mean flow satisfies divergence free condition

Reynolds Averaged NS equaiton

- Momentum equation:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + f_i$$

- Take average:

$$\frac{\partial \bar{u}_i}{\partial t} + \overline{u_j \frac{\partial u_i}{\partial x_j}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} + \bar{f}_i$$

- Substitute $u = \bar{u} + u'$:

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial(\overline{u'_i u'_j})}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} + \bar{f}_i$$

- The equations are formally identical to the Navier-Stokes equations with the exception of the additional term

Reynolds Stress

- Reynolds stress tensor consists in 3D of 9 components:

$$\begin{bmatrix} \overline{(u')^2} & \overline{u'v'} & \overline{u'w'} \\ \overline{v'u'} & \overline{(v')^2} & \overline{v'w'} \\ \overline{w'u'} & \overline{w'v'} & \overline{(w')^2} \end{bmatrix}$$

- To complete the RANS equations, we need turbulence model

One-Equation Model	Two-Equation Models	Reynolds Stress Model
Spalart-Allmaras	Standard $k-\epsilon$ RNG $k-\epsilon$ Realizable $k-\epsilon$	SST Transition Model

Boussinesq Hypothesis

- $-\overline{u'_i u'_j} = 2\nu_T \left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i} \right) - \frac{2}{3} K \delta_{ij}$
- $\mu_T = \rho \nu_T$ stands for the eddy viscosity
- Unlike the molecular viscosity, the eddy viscosity represents no physical characteristic of the fluid, but it is a function of the local flow condition.
- μ_T is not homogeneous in space but assumed to be isotropic
- If the turbulent-viscosity hypothesis is accepted, all that remain is to determine ν_T

Mixing-Length Hypothesis

- Prandtl(1925) visualized a simplified model for turbulent motion in which fluid particles coalesce into lumps and move as a unit.
- Simply apply molecular theory to turbulence
- The task of specifying μ_T is generally approached through specifications of u^* and l^*

Mixing-length theory	Molecular theory
$\mu_T = \rho u^* l^*$ u^*: turbulent velocity scale l^* : mixing length	$\mu = \rho \bar{u} \lambda$ $\bar{u}$: average molecular speed λ: mean free path

$k-\epsilon$ Model

- k: turbulent kinetic energy
- ϵ: energy dissipation rate
- $u^* = C_\mu k^{1/2},\ l^* = k^{3/2}/\epsilon, C_\mu = 0.09$
- $\mu_T = \rho u^* l^* = 0.09\rho k^2/\epsilon$
- μ_T only depends on k and ϵ, no need to specify mixing length

Standard $k-\epsilon$ Model

- $\frac{\partial k}{\partial t}+\nabla\cdot\left[k\bar{u}-\left(\frac{\nu_T}{\sigma_K}+\nu\right)\nabla k\right]=P_k-\epsilon$
- $\frac{\partial \epsilon}{\partial t}+\nabla\cdot\left[\epsilon\,\bar{u}-\left(\frac{\nu_T}{\sigma_\epsilon}+\nu\right)\nabla \epsilon\right]=\frac{\epsilon}{k}(C_1P_k-C_2\epsilon)$
- $P_k=0.5\nu_T\left(\frac{\partial \overline{u_i}}{\partial x_j}+\frac{\partial \overline{u_j}}{\partial x_i}\right)^2$ is the production of turbulent kinetic energy

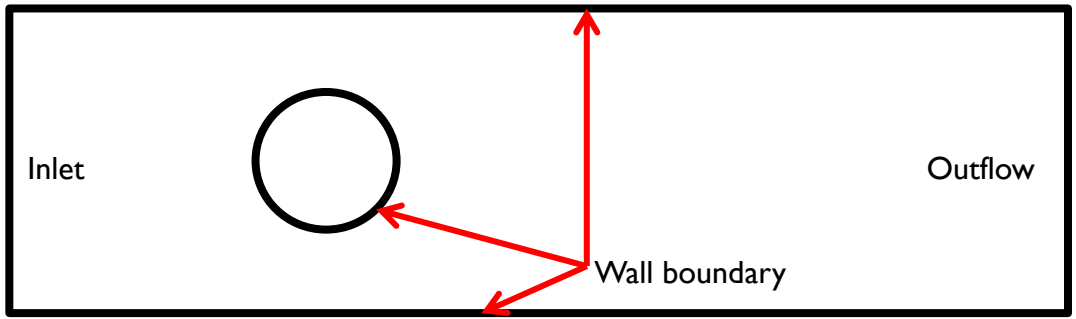
Boundary Condition

- Inlet

$$u = u_{in}, \quad k = C_{bc}|\boldsymbol{u}|^2, \quad \epsilon = C_{\mu}\frac{k^{3/2}}{l_0}$$

- Outlet

$$n \cdot \nabla u = 0\,, \quad n \cdot \nabla k = 0, \qquad \mathrm{n} \cdot \nabla \epsilon = 0$$

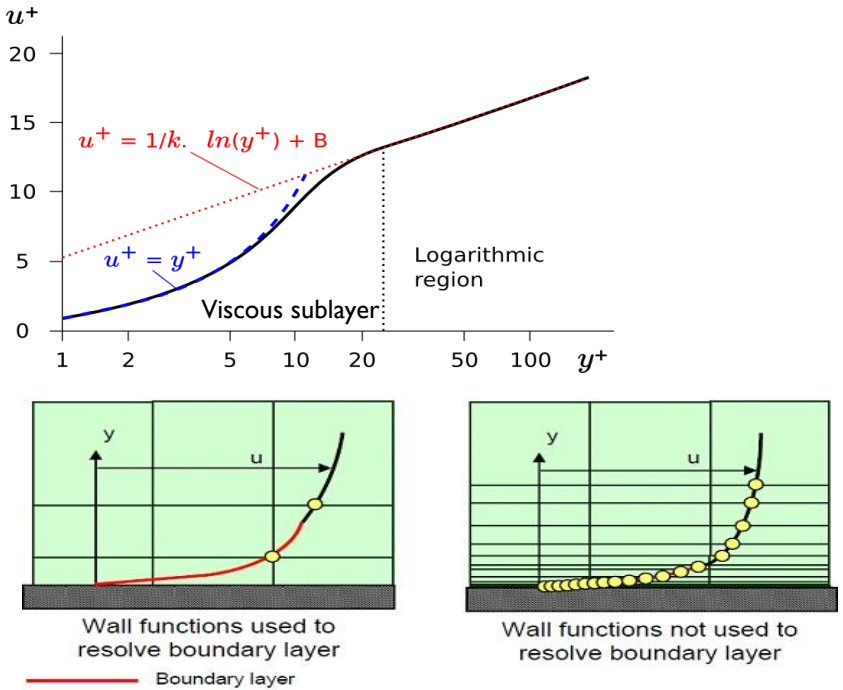


Wall Boundary

- Slip Boundary: the normal component of the velocity is set equal to zero, while tangential slip is permitted
- Non-slip Boundary: both normal component and tangential component set to be zero
- For viscous flow non-slip is realistic, however slip boundary is often used if the resolution near the wall is not high enough and the wall effects is modeled by wall function.

Law of the Wall

- Boundary Layer is divided to: viscous sublayer and logarithmic sublayer
- In viscous sublayer, viscous effects dominate and flow becomes laminar
- High resolution is needed to resolve the steep velocity profile
- Alternatively, one can use "wall function" to bridge the first node and the wall
- $u^+ = u/u_\tau, y^+ = yu_\tau/\nu$



Wall Functions

- To use wall functions, the first node should locate as $30 < y^+ < 100$, which is in logarithmic region and out of viscous sublayer
- Therefore, the log-law can be applied to the first node.
- This could either be made through an added source term simulating the wall friction or a modified viscosity ensuring the correct friction.
- This is more robust than specify the velocity

Wall Functions

- $n \cdot \nabla k_p = 0$: k at first interior node p
- $u_\tau = {C_\mu}^{1/4}\sqrt{k_p}$: friction velocity at node p
- $\mu_e = \frac{\rho u_\tau y_p \kappa}{\ln(E {y_p}^+)}$: effective viscosity at node p
- $\epsilon = \frac{{u_\tau}^3}{\kappa y_p}$: dissipation rate at node p
- y_p: distance between node p and wall
- $y^+ = \frac{y u_\tau}{\nu}$: distance in wall coordinates

$k - \epsilon$ Model Discussion

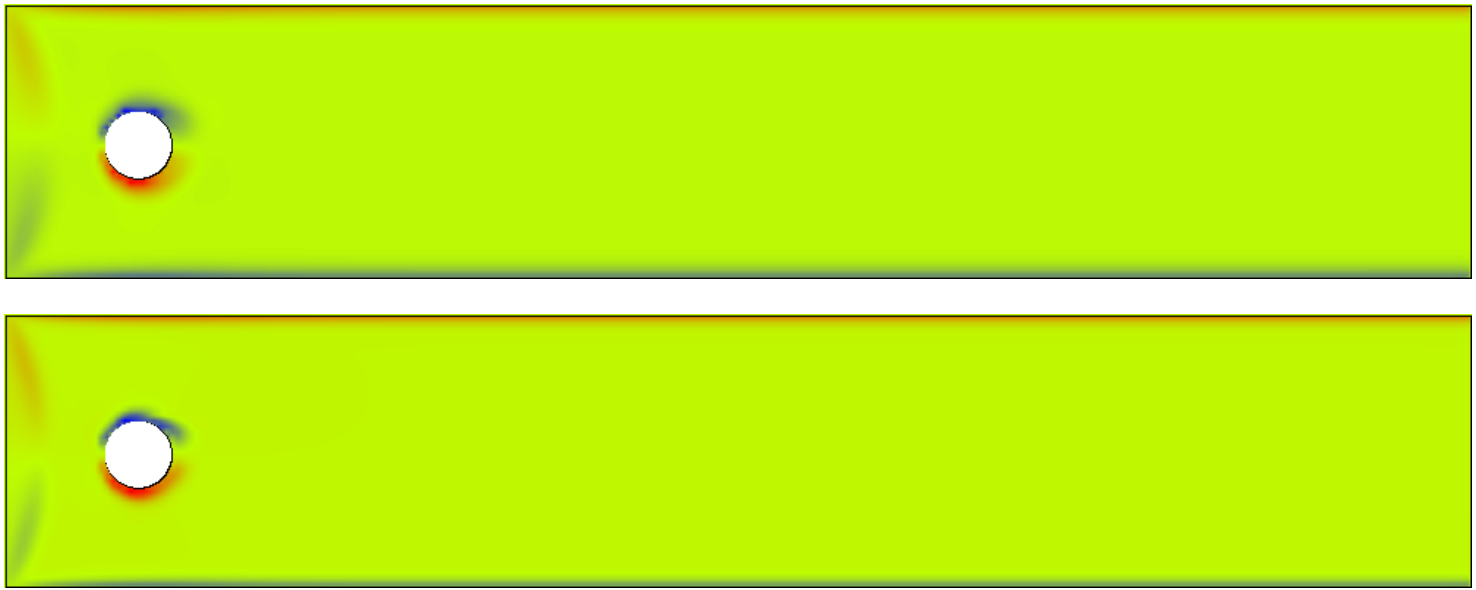
- Advantages:
  - Relatively simple to implement.
  - Leads to stable calculations that converge relatively easily.
  - Reasonable predictions for many flows.
- Disadvantages:
  - Poor predictions for:
    - swirling and rotating flows,
    - flows with strong separation,
    - axisymmetric jets,
    - certain unconfined flows
    - fully developed flows in non-circular ducts.
  - Valid only for fully turbulent flows, no transition

Summary

1. Calculate characteristic Reynolds number and determine whether flow is turbulent.
2. If the flow is transition range, consider use of the turbulence transition models
3. Estimate wall-adjacent cell y+ before generate mesh
4. Prepare wall functions except for low-Re model
5. Begin with RKE and change to the other models if needed
6. Use RSM for highly swirling, 3-D, rotating flows

Numerical Results

- $k-\epsilon$ model fail to generate vortex, because this is a transition process Re = 100

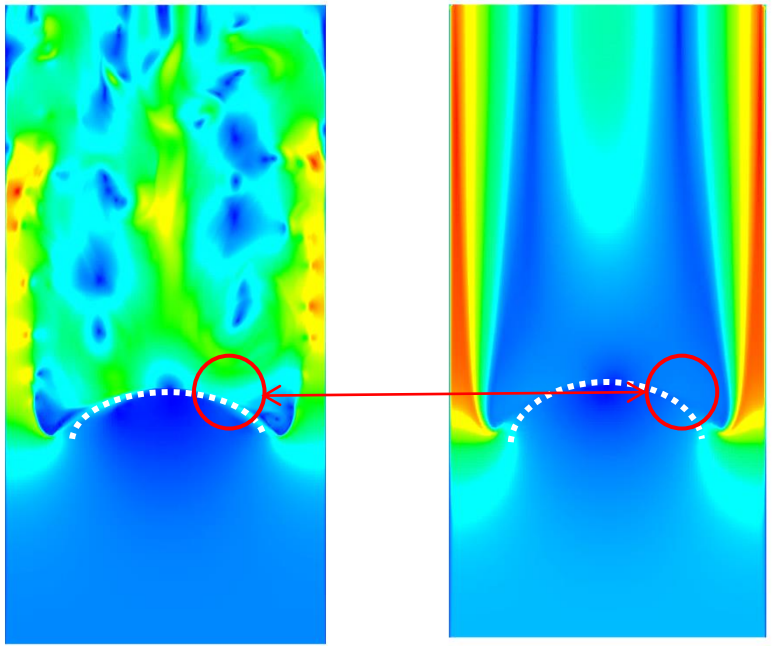


Use eddy viscosity

Without eddy viscosity

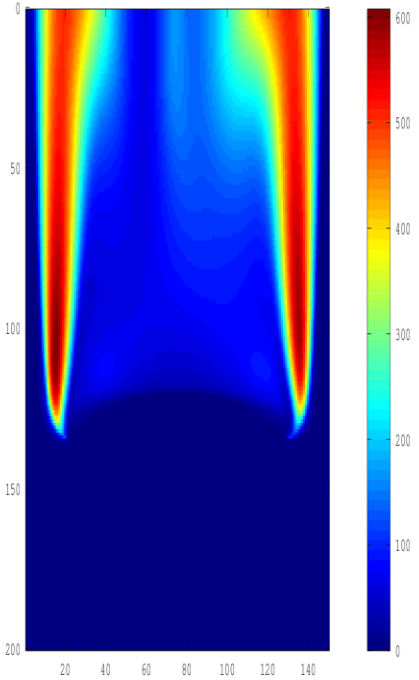
Numerical Results

$k - \epsilon$ fails to predict separation and reattachment

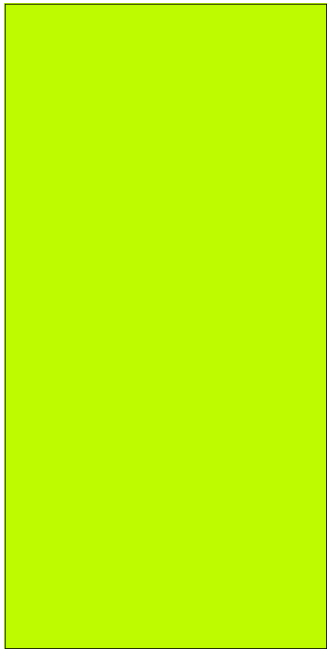


Velocity magnitude (DNS)

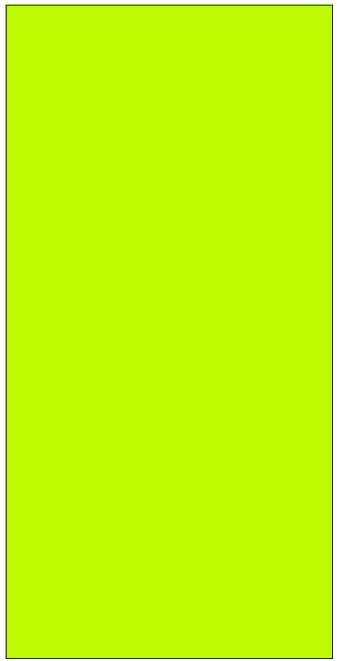
Velocity magnitude (RANS)



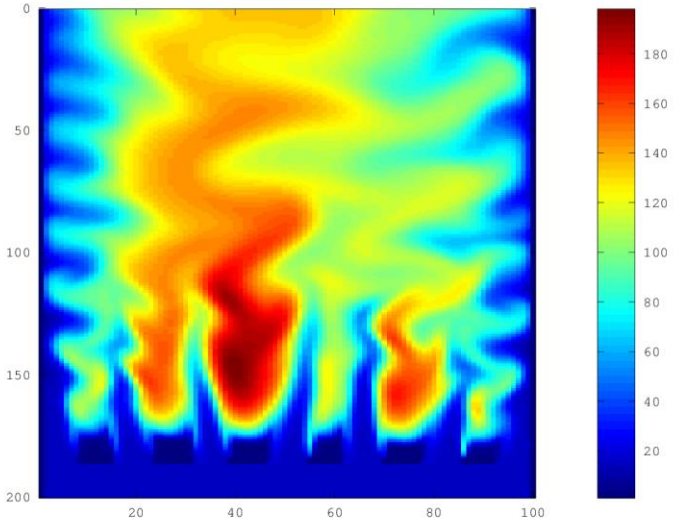
Numerical Results



Vorticity



Eddy viscosity



Eddy viscosity ratio is about 180 larger than molecular viscosity

Vorticity Equation

- Vorticity is the curl of the velocity field: $\omega = \nabla \times u$

$$\frac{d\omega}{dt} + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \nu\nabla^2\omega$$

- $(\omega \cdot \nabla)u$ is the vortex stretching term, very important in turbulent flow
- In 2-dimension, vortex stretching term vanishes

$$(\omega \cdot \nabla)u = \left[\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial z}, \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial z}, 0\right] = (0,0,0)$$

- In 2-dimension, vorticity evolves as a conserved scalar

  (qualitatively different from 3d problem)

Model	Behavior and Usage
Spalart – Allmaras	Economical for large meshes. Performs poorly for 3D flows, free shear flows, flows with strong separation. Suitable for mildly complex (quasi-2D) external/internal flows and boundary layer flows under pressure gradient (e.g. airfoils, wings, airplane fuselages, missiles, ship hulls).
Standard k–ε	Robust. Widely used despite the known limitations of the model. Performs poorly for complex flows involving severe pressure gradient, separation, strong streamline curvature. Suitable for initial iterations, initial screening of alternative designs, and parametric studies.
RNG k–ε	Suitable for complex shear flows involving rapid strain, moderate swirl, vortices, and locally transitional flows (e.g. boundary layer separation, massive separation, and vortex shedding behind bluff bodies, stall in wide-angle diffusers, room ventilation).
Realizable k–ε	Offers largely the same benefits and has similar applications as RNG. Possibly more accurate and easier to converge than RNG.
Standard k–ω	Superior performance for wall-bounded boundary layer, free shear, and low Reynolds number flows. Suitable for complex boundary layer flows under adverse pressure gradient and separation (external aerodynamics and turbomachinery). Can be used for transitional flows (though tends to predict early transition). Separation is typically predicted to be excessive and early.
SST k–ω	Offers similar benefits as standard k–ω. Dependency on wall distance makes this less suitable for free shear flows.
Reynolds Stress	Physically the most sound RANS model. Avoids isotropic eddy viscosity assumption. More CPU time and memory required. Tougher to converge due to close coupling of equations. Suitable for complex 3D flows with strong streamline curvature, strong swirl/rotation (e.g. curved duct, rotating flow passages, swirl combustors with very large inlet swirl, cyclones).

- **Discussion on Turbulence Model**