Multiple View Geometry - Assignment 2

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May 18, 2021

The Fundamental Matrix

Exercise 1

$$P_1 = [I\ 0]$$

$$P_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, t = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$
$$\Rightarrow [t]_x = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

Compute the Fundamental matrix:

$$F = [t]_x\, A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & -2 & 0 \end{bmatrix}$$

$$\boldsymbol{F} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{-2} & \mathbf{-2} & \mathbf{0} \end{bmatrix}$$

Compute the epipolar line in the second image generated by $x = (1, 1)$

$$l = Fx = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}$$

$$\boldsymbol{l} = \begin{bmatrix} \mathbf{2} \\ \mathbf{0} \\ \mathbf{-4} \end{bmatrix}$$

For the points $a = (2,0)\,, b = (2,1)\,, c = (4,2)$ notice that:

$$\langle l, a\rangle = \left\langle \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\rangle = 0$$

$$\langle l, b\rangle = \left\langle \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\rangle = 0$$

$$\langle l, c\rangle = \left\langle \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \right\rangle = -4$$

Since only a, b are on the epipolar line of x on the image P_2 then a, b **each could be a projection of the point X onto P_2, but c cannot**.

Exercise 2

Compute the epipoles, by projecting the center of each camera to the other one:

$$P_1 = [I\ 0] \Rightarrow C_1 = -I^{-1}0 = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow C_2 = -\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right.$$

$$C_2 = -\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -0.5 & -1 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$e_2 \propto P_2 C_1 = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\boldsymbol{e_2} = \begin{bmatrix} \mathbf{2} \\ \mathbf{2} \\ \mathbf{0} \end{bmatrix}$$

$$e_1 \propto P_1 C_2 = [I\ 0] \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\boldsymbol{e_1} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}$$

Compute the fundamental matrix, its determinant and verify that $e_2^T F = 0$ and $Fe_1 = 0$:

$$P_1 = [I\ 0]$$

$$P_2 = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, t = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow [t]_x = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix}$$

$$F = [t]_x A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & -2 \end{bmatrix}$$

$$\boldsymbol{F} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{-2} \\ \mathbf{-2} & \mathbf{2} & \mathbf{-2} \end{bmatrix}$$

$$\begin{aligned} \det(F) &= 0 \det \begin{pmatrix} 0 & -2 \\ 2 & -2 \end{pmatrix} - 0 \det \begin{pmatrix} 0 & -2 \\ -2 & -2 \end{pmatrix} + 2 \det \begin{pmatrix} 0 & 0 \\ -2 & 2 \end{pmatrix} \\ &= 2 \det \begin{pmatrix} 0 & 0 \\ -2 & 2 \end{pmatrix} = 0 \end{aligned}$$

$$\mathbf{det}\,(\boldsymbol{F}) = \mathbf{0}$$

$$e_2^T F = \begin{bmatrix} 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} = \bar{0}$$

$$\boldsymbol{e_2^T F} = \bar{\mathbf{0}}$$

$$Fe_1 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \bar{0}$$

$$\boldsymbol{Fe_1 = \bar{0}}$$

For a general camera pair $P_1 = [I\ 0] P_2 = [A\ t]$ compute the epipoles:

$$C_1 = -I^{-1}0 = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_2 = -A^{-1}t$$

$$e_2 \propto P_2 C_1 = [A\ t] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = t \Rightarrow \boldsymbol{e_2 \propto t}$$

$$e_1 \propto P_1 C_2 = [I\ 0] \begin{bmatrix} -A^{-1}t \\ 1 \end{bmatrix} = -A^{-1}t \Rightarrow \boldsymbol{e_1 \propto -A^{-1}t}$$

Verify that for the fundamental matrix $F = [t]_\times A$ the epipoles will always fulfill $e_2^T F = 0$ and $Fe_1 = 0$:

$$\begin{aligned} e_2^T F &= t^T [t]_\times A \\ &= \begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix} \begin{bmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{bmatrix} A \\ &= \begin{bmatrix} (t_2t_3 - t_3t_2) & (-t_1t_3 + t_3t_1) & (t_1t_2 - t_2t_1) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} = 0 \end{aligned}$$

$$\boldsymbol{e_2^T F = 0}$$

$$\begin{aligned} Fe_1 &= [t]_\times A \left(-A^{-1}t\right) \\ &= -[t]_\times AA^{-1}t \\ &= -[t]_\times t \\ &= -\begin{bmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \\ &= -\begin{bmatrix} -t_3t_2 + t_2t_3 \\ t_3t_1 - t_1t_3 \\ -t_2t_1 + t_1t_2 \end{bmatrix} = -\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \end{aligned}$$

$$\boldsymbol{Fe_1 = 0}$$

The Fundamental matrix has to have determinant 0 since its kernel is not empty $e_1 \in N_F$ for $e_1 \neq \bar{0} \Rightarrow F$ is not invertible $\Rightarrow \det(F) = 0$.

Exercise 3

$$\tilde{x_1} \sim N_1 x_1 \Rightarrow \tilde{x_1} = \lambda_1 N_1 x_1$$
$$\tilde{x_2} \sim N_2 x_2 \Rightarrow \tilde{x_2} = \lambda_2 N_2 x_2$$

$$\begin{aligned}
\tilde{x_2}^T \tilde{F} \tilde{x_1} = 0 &\Rightarrow (\lambda_2 N_2 x_2)^T \tilde{F} (\lambda_1 N_1 x_1) = 0 \\
&\Rightarrow \lambda_2 x_2^T N_2^T \tilde{F} \lambda_1 N_1 x_1 = 0 \\
&\Rightarrow x_2^T N_2^T \tilde{F} N_1 x_1 = 0 \\
&\Rightarrow x_2^T \left(N_2^T \tilde{F} N_1 \right) x_1 = 0
\end{aligned}$$

$F = N_2^T \tilde{F} N_1$ fulfills $x_2^T F x_1 = 0$.

Computer Exercise 1

F is $\begin{bmatrix} 0 & 0 & 0.0058 \\ 0 & 0 & -0.0267 \\ -0.0072 & 0.0263 & 1 \end{bmatrix}$

The mean epipolar distances with normalization is: 0.36123

The mean epipolar distances without normalization is: 0.48784

The histogram, the plot of the epipolar lines and the plot of epipolar constraints (making sure they are close to 0) can be found in the attached matlab_results.pdf

Exercise 4

$$F = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

As we know, $e_2 = \begin{bmatrix} e_{2\_x} \\ e_{2\_y} \\ 1 \end{bmatrix}$ is a point such that $e_2^T F = 0$. Let us calculate e_2:

$$e_2^T F = \begin{bmatrix} e_{2\_x} & e_{2\_y} & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} e_{2\_y} & e_{2\_x} + 1 & e_{2\_x} + 1 \end{bmatrix} = \bar{0}$$

$$\Rightarrow e_{2\_y} = 0; \; e_{2\_x} = -1$$

$$e_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$[e_2]_\times = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Calculate the camera matrices P_1, P_2:

$$P_1 = [I\ 0]$$

$$\begin{aligned}
P_2 &= \left[[e_2]_\times F\ e_2\right] \\
&= \left[\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right] \\
&= \left[\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 2 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right]
\end{aligned}$$

$$P_2 = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & 2 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

Calculate x_1, x_2 for the first point $X_1 = (1, 2, 3)$

$$x_1 \propto P_1 \begin{bmatrix} X_1 \\ 1 \end{bmatrix} = [I\ 0] \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$x_1 = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$x_2 \propto P_2 \begin{bmatrix} X_1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & 2 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 10 \\ 0 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} -2 \\ 10 \\ 0 \end{bmatrix}$$

Verify that the projections of the point X_1 in the cameras P_1, P_2 fulfill the epipolar constraint:

$$\begin{aligned}
\boldsymbol{x_2^T F x_1} &= \begin{bmatrix} -2 & 10 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \frac{1}{3} \begin{bmatrix} 10 & -2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \frac{1}{3} 0 \boldsymbol{= 0}
\end{aligned}$$

Calculate x_1, x_2 for the second point $X_2 = (3, 2, 1)$

$$x_1 \propto P_1 \begin{bmatrix} X_2 \\ 1 \end{bmatrix} = [I\ 0] \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$x_2 \propto P_2 \begin{bmatrix} X_2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & 2 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \\ -2 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

Verify that the projections of the point X_2 in the cameras P_1, P_2 fulfill the epipolar constraint:

$$\begin{aligned} \boldsymbol{x_2^T F x_1} &= \begin{bmatrix} 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \mathbf{0} \end{aligned}$$

Notice that we have proven two specific cases of the general case:

$$\begin{aligned} x_2^T F x_1 &= \left(P_2 \begin{bmatrix} X \\ 1 \end{bmatrix} \right)^T F P_1 \begin{bmatrix} X \\ 1 \end{bmatrix} \\ &= \left(\begin{bmatrix} [e_2]_\times F & e_2 \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix} \right)^T F \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix} \\ &= \left([e_2]_\times FX + e_2 \right)^T FX \\ &= X^T F^T [e_2]_\times^T FX + e_2^T FX \\ &= (FX)^T [e_2]_\times^T FX + e_2^T FX \\ &= (FX)^T (e_2 \times (FX)) + \left(e_2^T F \right) X = 0 \end{aligned}$$

The last equation is true since $a^T (b \times a) = 0$ and $e_2^T F = 0$.

Regarding the camera center we are looking for a point C that it's projection onto P_2 is $\bar{0}$.

If we assume towards contradiction that C is not at infinity then:

$$P_2 \begin{bmatrix} C \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & 2 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_x \\ C_y \\ C_z \\ 1 \end{bmatrix} = \begin{bmatrix} -C_x - 1 \\ 2C_y + 2C_z \\ -C_x + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get that $C_x = -1$ and $C_x = 1$. This is a contradiction. Thus, we know that C is at infinity:

$$P_2 \begin{bmatrix} C \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & 2 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_x \\ C_y \\ C_z \\ 0 \end{bmatrix} = \begin{bmatrix} -C_x \\ 2C_y + 2C_z \\ -C_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \lambda \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Arbitrarily choosing $\lambda = 1$ we get that the camera center of P_2 is at $\boldsymbol{C_2} = \begin{bmatrix} \mathbf{0} \\ \mathbf{-1} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}$.

Computer Exercise 2

The appearance of the 3D plot is VERY BAD.
This is because the problem of extracting cameras from F has infinitly many possible solutions:

$$P_2 = \begin{bmatrix} [e_2]_\times F + e_2 v & \lambda e_2 \end{bmatrix} \; \forall v \in R^3 \; \lambda \neq 0$$

and we chose a specific one, $v = 0, \lambda = 1$, without any good reason.
Thus we obtained a possible solution that is probably not the correct one (by viewing the 3D plot - it doesn't look like I expected).
The two images with the image points and the projected points, and the 3D plot can be found in the attached matlab_results.pdf

The Essential Matrix

Exercise 5

1. .

$$[t]_\times = USV^T$$
$$[t]_\times^T = VS^TU^T$$

$$
\begin{aligned}
[t]_\times^T [t]_\times &= \left(VS^TU^T\right)\left(USV^T\right) \\
&= VS^TU^TUSV^T \\
&= VS^TSV^T \\
&= V\left(S^2\right)V^T \\
&= V\left(S^2\right)V^{-1}
\end{aligned}
$$

Since $\left([t]_\times^T [t]_\times\right)$ is diagonalizeable by $V\left(S^2\right)V^{-1}$ we know that s_1^2, s_2^2, s_3^2 (the entries on the diagonal matrix S^2) are eigenvalues of $\left([t]_\times^T [t]_\times\right)$.

2. Notice that $[t]_\times = \begin{bmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{bmatrix}$ is skew-symmetric, i.e. $[t]_\times^T = -[t]_\times$.
   For an eigenvector w corresponding to the eigenvalue λ of $[t]_\times$ it holds that:

$$
\begin{aligned}
\left([t]_\times^T [t]_\times\right) w = \lambda w &\iff [t]_\times^T \left([t]_\times w\right) = \lambda w \\
&\iff [t]_\times^T (t \times w) = \lambda w \\
&\iff -[t]_\times (t \times w) = \lambda w \\
&\iff -t \times (t \times w) = \lambda w
\end{aligned}
$$

3. .

$$
u \times (v \times w) = (u \cdot w)\, v - (u \cdot v)\, w
$$

For $w = t$ we can show that:

$$
\begin{aligned}
-t \times (t \times w) &= -t \times (t \times t) \\
&= -\left((t \cdot t)\, t - (t \cdot t)\, t\right) \\
&= 0t = 0w
\end{aligned}
$$

As proven in exercise 5.2 we know that $-t \times (t \times w) = \lambda w$ iff w is an eigenvector of $[t]_\times^T [t]_\times$ with the corresponding eigenvalue λ.
Thus, in this case, we have shown that t is an eigenvector of $[t]_\times^T [t]_\times$ with the corresponding eigenvalue 0.

For $w \perp t$ we can show that:

$$
\begin{aligned}
-t \times (t \times w) &= -\left((t \cdot w)\, t - (t \cdot t)\, w\right) \\
&= -\left(0t - (t \cdot t)\, w\right) \\
&= (t \cdot t)\, w = |t|^2\, w
\end{aligned}
$$

Thus, in this case, we have shown that $w \perp t$ is an eigenvector of $[t]_\times^T [t]_\times$ with the corresponding eigenvalue $|t|^2$.

Assume towards contradiction the there exists an eigenvector v of $[t]_\times^T [t]_\times$ which is not t and not $\perp t$.
Thus, this vector can be expressed as $v = \alpha_1 t + \alpha_2 w$ for some $w \perp t$. Notice that:

$$\begin{aligned}
\left([t]_\times^T [t]_\times\right) v &= \left([t]_\times^T [t]_\times\right)(\alpha_1 t + \alpha_2 w) \\
&= \alpha_1 \left([t]_\times^T [t]_\times\right) t + \alpha_2 \left([t]_\times^T [t]_\times\right) w \\
&= \alpha_1 0t + \alpha_2 |t|^2 w \\
&= \alpha_2 |t|^2 w \neq \lambda v
\end{aligned}$$

Thus, we know that each eigenvector is either λt or $\perp t$.

4. Observing the set $\{t, w_1, w_2\}$ for which $w_1 \perp t$ and $w_2 \perp t$ $w_2 \perp w_1$ we notice that we have three linearly-independent eigenvectors of $[t]_\times^T [t]_\times$ with the corresponding eigenvalues $0, |t|^2, |t|^2$.
Note that s_i is a singular valur of $[t]_x$ iff $s_i = \sqrt{\lambda_i}$ for λ_i an eigenvalue of $[t]_x^T [t]_x$.
Thus, the singular values of $[t]_x$ are $0, |t|, |t|$, as required.

$$E = [t]_x R = USV^T R = U \begin{bmatrix} |t| & 0 & 0 \\ 0 & |t| & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T R$$

Note that $\left(V^T R\right)^T V^T R = R^T V V^T R = R^T R = I \Rightarrow V^T R$ orthogonal $\Rightarrow$

$$E = U \begin{bmatrix} |t| & 0 & 0 \\ 0 & |t| & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(V^T R\right)$$

is the SVD decomposition of E, and therefore the singular values of E are $\{|t|, |t|, 0\}$, as required.

Computer Exercise 3

This result is worst than the corresponding result in Computer Exercise 1.
The average distance here is ~2.1 pixels instead of ~0.36 pixels.

E is: $\begin{bmatrix} 0.0032 & 0.3619 & -0.1357 \\ -0.4507 & -0.0282 & 0.8808 \\ 0.1701 & -0.9176 & -0.0004 \end{bmatrix}$

The histogram and the plot of the epipolar lines can be found in the attached matlab_results.pdf

Exercise 6

$$E = U\text{diag}\left([1,1,0]\right)V^T$$
$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned}
\det\left(UV^T\right) &= \det(U)\det\left(V^T\right) \\
&= \det\left(\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\right) \\
&= \left(\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} - 0\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} - 0\right) + 0\right)\left(1\left(0+1\right) - 0 + 0\right) \\
&= \left(\frac{1}{2} + \frac{1}{2}\right)(1) = 1
\end{aligned}$$

$$\det\left(UV^T\right) = 1$$

1. .

$$\begin{aligned}
E &= U\text{diag}\left([1,1,0]\right)V^T \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

$$E = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}
x_2^T E x_1 &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \\
&= -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 0
\end{aligned}$$

$$x_2^T E x_1 = 0$$

Thus, $x_1 = (0, 0)$ in camera 1 and $x_2 = (1, 1)$ in camera 2 is a plausible correspondence.

2. .

$$x_1 = P_1 X$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \propto [I \ 0] \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$u \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$X_1 = X_2 = 0; \ X_3 = u$$

$$X = \begin{bmatrix} 0 \\ 0 \\ u \\ v \end{bmatrix} = u \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{v}{u} \end{bmatrix}$$

All the points in X are thus equal to $X(s) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ s \end{bmatrix}$ up to some scale.

Since all the points in a 4D ray are equivalent, we can say the X must be one of the points in $X(s) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ s \end{bmatrix}$.

3. .

$$W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} UWV^T &= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \end{bmatrix} \end{aligned}$$

$$UW^TV^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \end{bmatrix}$$

For $P_2 = \begin{bmatrix} UWV^T & u_3 \end{bmatrix}$:

$$x_2 \propto P_2X(s)$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \propto \begin{bmatrix} UWV^T & u_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ s \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ s \end{bmatrix}$$

$$s = -\frac{1}{\sqrt{2}} \Rightarrow x_2 \propto P_2X(s)$$

For $P_2 = \begin{bmatrix} UWV^T & -u_3 \end{bmatrix}$:

$$x_2 \propto P_2X(s)$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \propto \begin{bmatrix} UWV^T & -u_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ s \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ -s \end{bmatrix}$$

$$s = \frac{1}{\sqrt{2}} \Rightarrow x_2 \propto P_2X(s)$$

For $P_2 = \begin{bmatrix} UW^TV^T & u_3 \end{bmatrix}$:

$$x_2 \propto P_2X(s)$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \propto \begin{bmatrix} UW^TV^T & u_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ s \end{bmatrix}$$

$$s = \frac{1}{\sqrt{2}} \Rightarrow x_2 \propto P_2 X(s)$$

For$P_2 = \begin{bmatrix} UW^TV^T & -u_3 \end{bmatrix}$:

$$x_2 \propto P_2 X(s)$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \propto \begin{bmatrix} UW^TV^T & -u_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ -s \end{bmatrix}$$

$$s = -\frac{1}{\sqrt{2}} \Rightarrow x_2 \propto P_2 X(s)$$

4. In order for a 3D point to be in front of the cameras - the projection of the point onto the cameras must result in a positive Z-coordinate.
   As shown in the previous paragraph, $X(s)$ (that is projected onto x_2 in camera P_2) is in front of the camera P_2 in the second and third case.
   Let us check in which of these cases $X(s)$ is in front of the camera P_1:

   For$P_2 = \begin{bmatrix} UWV^T & -u_3 \end{bmatrix}$:

$$x_1 = P_1 X(s)$$

$$x_1 \propto \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \Rightarrow x_1 = \begin{bmatrix} 0 \\ 0 \\ \sqrt{2} \end{bmatrix}$$

   In this case $X(s)$ is in front of both cameras.

For$P_2 = \left[\begin{array}{cc} UWV^T & u_3 \end{array}\right]$: $x_2 \propto P_2 X(s) = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ s \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \Rightarrow$ Behind camera 2

For$P_2 = \left[\begin{array}{cc} UWV^T & -u_3 \end{array}\right]$: $x_2 \propto P_2 X(s) = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ -s \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \Rightarrow$ Behind camera 2

For$P_2 = \left[\begin{array}{cc} UW^TV^T & u_3 \end{array}\right]$: $x_2 \propto P_2 X(s) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ s \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \Rightarrow$ In front of camera 2

$x_1 \propto P_1 X(s) = \left[\begin{array}{cc} I & 0 \end{array}\right] \begin{bmatrix} 0 \\ 0 \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$ In front of camera 1

For$P_2 = \left[\begin{array}{cc} UW^TV^T & -u_3 \end{array}\right]$: $x_2 \propto P_2 X(s) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ -s \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \Rightarrow$ In front of camera 2

$x_1 \propto P_1 X(s) = \left[\begin{array}{cc} I & 0 \end{array}\right] \begin{bmatrix} 0 \\ 0 \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \Rightarrow$ Behind camera 1

Thus, only in the third case where $P_2 = \left[\begin{array}{cc} UW^TV^T & u_3 \end{array}\right]$ the 3D point $X(s)$ is in front of both cameras.

Computer Exercise 4

The errors between the 2D points and the projected 3D points indeed look small (~1.5 pixels).
The 3D reconstruction now does indeed look like I expected (as opposed to computer exercise 2).
The figures can be found in the attached matlab_results.pdf