DL for CV - Assignment 3

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Given x, $\tilde{k_2}$ we will prove that $\left(x * \tilde{k_4}\right) \downarrow_4 = \left(x * \tilde{k_2} \downarrow_2\right) * \tilde{k_2} \downarrow_2$ for $\tilde{k_4} = \left(\tilde{k_2} * \tilde{k}_{2\_dilated}\right)$.

Disclaimer: My project with Michal is related to kernelGAN.
While reading the paper, I came across the proof in the project's website:
http://www.wisdom.weizmann.ac.il/~vision/kernelgan/resources/k_4_proof.pdf
I did my best to explain every step of the proof in order to show that I fully understand it.

Proof

I will prove the equivalence in the inifinite case - assuming $\tilde{k_2}$ is infinite.
For any finite kernel $\tilde{k_2}$, we can simply observe it as an infinite kernel padded with infinite 0-s at each size.
The resulting$\tilde{k_4}$ will also be infinite.
Since both $\tilde{k_2}$ and $\tilde{k}_{2\_dilated}$ are padded with infinite 0-s at each size, the resulting $\tilde{k_4}$ will also be padded with infinite 0-s at each size (stems from the convolution definition).
Therefore, we could later observe only the relevant part of the resulting $\tilde{k_4}$ for the finite case (the non-zero part with the relevant padding).

Let us define $y = x * \tilde{k_2}$.

$$\left(\left(\left(x * \tilde{k_2}\right)\downarrow_2\right) * \tilde{k_2}\right)[n] \overbrace{=}^{\text{definition of } y} \left((y\downarrow_2) * \tilde{k_2}\right)[n]$$

$$\overbrace{=}^{\text{convolution is commutative}} \left(\tilde{k_2} * (y\downarrow_2)\right)[n]$$

$$\overbrace{=}^{\text{definition of convolution}} \sum_{l=-\infty}^{\infty} \tilde{k_2}[n-l]\,(y\downarrow_2)[l]$$

$$\overbrace{=}^{\text{definition of subsampling}} \sum_{l=-\infty}^{\infty} \tilde{k_2}[n-l]\,y[2l]$$

$$\overbrace{=}^{\text{change of variables } m=2l *^1} \sum_{l=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} \delta[2l-m]\,\tilde{k_2}\left[n-\frac{m}{2}\right] y[m]\right)$$

$$\overbrace{=}^{\text{changing summation order}} \sum_{m=-\infty}^{\infty} \left(\sum_{l=-\infty}^{\infty} \delta[2l-m]\,\tilde{k_2}\left[\frac{1}{2}(2n-m)\right] y[m]\right)$$

$$\overbrace{=}^{l \text{ appears only at } \delta} \sum_{m=-\infty}^{\infty} \left(\left(\sum_{l=-\infty}^{\infty} \delta[2l-m]\right)\left(\tilde{k_2}\left[\frac{1}{2}(2n-m)\right] y[m]\right)\right)$$

$$\overbrace{=}^{\text{add } 2n \text{ to the Kronecker comb } *^2} \sum_{m=-\infty}^{\infty} \left(\left(\sum_{l=-\infty}^{\infty} \delta[2l-m+2n]\right)\left(\tilde{k_2}\left[\frac{1}{2}(2n-m)\right] * y[m]\right)\right)$$

$$= \sum_{m=-\infty}^{\infty} \left(\left(\sum_{l=-\infty}^{\infty} \delta[2l+(2n-m)]\,\tilde{k_2}\left[\frac{1}{2}(2n-m)\right]\right) y[m]\right)$$

$$\overbrace{=}^{\text{definition of convolution}} \left(\left(\sum_{l=-\infty}^{\infty} \delta[2l+\cdot]\,\tilde{k_2}\left[\frac{\cdot}{2}\right]\right) * y\right)[2n]$$

$$\overbrace{=}^{\text{Kronecker comb zeros out odd inputs} *^3} \left(\tilde{k}_{2\_dilated} * y\right)[2n]$$

$$\overbrace{=}^{\text{definition of } y} \left(\tilde{k}_{2\_dilated} * x * \tilde{k_2}\right)[2n]$$

$$\overbrace{=}^{\text{convolution is commutative}} \left(x * \left(\tilde{k_2} * \tilde{k}_{2\_dilated}\right)\right)[2n]$$

$$\overbrace{=}^{\text{definition of subsampling}} \left(x * \left(\tilde{k_2} * \tilde{k}_{2\_dilated}\right)\right)\downarrow_2[n]$$

$$\left(\left(\left(x * \tilde{k_2}\right)\downarrow_2\right) * \tilde{k_2}\right) = \left(x * \left(\tilde{k_2} * \tilde{k}_{2\_dilated}\right)\right)\downarrow_2$$

$$\left(\left(\left(x * \tilde{k_2}\right)\downarrow_2\right) * \tilde{k_2}\right)\downarrow_2 = \left(x * \left(\tilde{k_2} * \tilde{k}_{2\_dilated}\right)\right)\downarrow_4$$

$*^1$ - We use Kronecker comb function in the change of variables:
Assume we had some $f(l)$ and wanted to change variables so that $m = 2l$, i.e. $f\left(\frac{m}{2}\right) = f(l)$.
The problem is that $f\left(\frac{m}{2}\right)$ is not defined for odd m (since $f(l)$ is only defined for $l \in Z$).
To overcome this fact, we multiply $f\left(\frac{m}{2}\right)$ with the Kronecker comb and by that we basically set $f\left(\frac{m}{2}\right) = 0$ for odd m.
Explanation:
Notice that for each even m the function $\delta[2l-m] f(l) = \delta[2l-m] f\left(\frac{m}{2}\right) = \begin{cases} f\left(\frac{m}{2}\right) & m = 2l \\ 0 & otherwise \end{cases}$. Thus, $\sum_{m_{even}=-\infty}^{\infty} \delta[2l-m] f(l) = f\left(\frac{m}{2}\right)$ for all even m.
Notice that for each odd m the function $\delta[2l-m] f(l) = 0$ (since $2l \neq m$). Thus, $\sum_{m_{even}=-\infty}^{\infty} \delta[2l-m] f(l) = 0$ for all odd m.
This is exactly what we wanted.

$*^2$ - $\sum_{l=-\infty}^{\infty} \delta[2l-m]$ is a function of m for which $\sum_{l=-\infty}^{\infty} \delta[2l-m] = \begin{cases} 1 & m \text{ even} \\ 0 & otherwise \end{cases}$ (chain of deltas).
Shifting this function by any even number $2n$ will result in the same function.

$*^3$ - Observe $\sum_{l=-\infty}^{\infty} \delta[2l+t]$ as a function of t.
Notice that for each even t the function $\delta[2l+t] = \begin{cases} 1 & t = -2l \\ 0 & otherwise \end{cases}$. Thus, $\sum_{l=-\infty}^{\infty} \delta[2l+t] = 1$ for all even t.
Notice that for each odd t the function $\delta[2l+t] = 0$ (since $t \neq 2l$). Thus, $\sum_{l=-\infty}^{\infty} \delta[2l+t] = 0$ for all odd t.
So, to sum up (pun intended), $\sum_{l=-\infty}^{\infty} \delta[2l+t] = \begin{cases} 1 & t \text{ even} \\ 0 & otherwise \end{cases}$.
Thus, $\sum_{l=-\infty}^{\infty} \delta[2l+t] \tilde{k_2}\left[\frac{\cdot}{2}\right] = \tilde{k}_{2\_dilated}$.