Multiple View Geometry - Assignment 4

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Plane Fitting

Exercise 1

Any plane can be completely described by 4 parameters (a, b, c, d) as all the points that suffice $ax + by + cz + d = 0$.
However, all the planes $(\lambda a, \lambda b, \lambda c, \lambda d)$ are the same, since $ax + by + cz + d = 0 \iff \lambda ax + \lambda by + \lambda cz + \lambda d = 0$.
This means that (a, b, c, d) determines a plane up-to-scale.
Thus, there are 3 DoF for a 3D plane.

Each point given decreases a single degree of freedom (adds a single linear constraint).
Thus, we will need a sample of $k = 3$ points to compute a plane (a model).
This coincide with the fact that we all know that 3 points define a plane in 3D (excluding special cases).
If we have 10% outliers then we have 90% inliers - $q = 0.9$.
As seen in class, if we want to achieve a success rate of $P = 98\%$, we need to draw at least $N \geq \frac{1}{q^k} \log\left(\frac{1}{1-P}\right)$ samples:

$$N \geq \frac{1}{q^k} \log\left(\frac{1}{1-P}\right) = \frac{1}{0.9^3} \log\left(\frac{1}{1-0.98}\right) \approx 5.366$$

Thus, we need to draw at least 6 sample sets (of 3 points each time) to achieve the required success rate.

Exercise 2

a.

$$f(a, b, c, d) = \sum_{i=1}^{m} (ax_i + by_i + cz_i + d)^2$$

The problem is

$$\min(f(a, b, c, d)) \;\; s.t. \; a^2 + b^2 + c^2 = 1$$

$$\frac{\partial f}{\partial a} = \sum_{i=1}^{m} 2\left(ax_i + by_i + cz_i + d\right) x_i$$

$$\frac{\partial f}{\partial b} = \sum_{i=1}^{m} 2\left(ax_i + by_i + cz_i + d\right) y_i$$

$$\frac{\partial f}{\partial c} = \sum_{i=1}^{m} 2\left(ax_i + by_i + cz_i + d\right) z_i$$

$$\frac{\partial f}{\partial d} = \sum_{i=1}^{m} 2\left(ax_i + by_i + cz_i + d\right)$$

$$\nabla f = \begin{bmatrix} \sum_{i=1}^{m} 2x_i\left(ax_i + by_i + cz_i + d\right) \\ \sum_{i=1}^{m} 2y_i\left(ax_i + by_i + cz_i + d\right) \\ \sum_{i=1}^{m} 2z_i\left(ax_i + by_i + cz_i + d\right) \\ \sum_{i=1}^{m} 2\left(ax_i + by_i + cz_i + d\right) \end{bmatrix}$$

Note that the optimal solution will suffice $\nabla f = 0$ (extreme point). This implies, in particular, that $\frac{\partial f}{\partial d} = 0$:

$$\begin{aligned}
\frac{\partial f}{\partial d} = 0 &\Rightarrow \sum_{i=1}^{m} 2\left(ax_i + by_i + cz_i + d\right) = 0 \\
&\Rightarrow \sum_{i=1}^{m} \left(ax_i + by_i + cz_i + d\right) = 0 \\
&\Rightarrow \sum_{i=1}^{m} \left(ax_i + by_i + cz_i\right) + \sum_{i=1}^{m} d = 0 \\
&\Rightarrow \sum_{i=1}^{m} \left(ax_i + by_i + cz_i\right) + md = 0 \\
&\Rightarrow d = -\frac{1}{m}\sum_{i=1}^{m} \left(ax_i + by_i + cz_i\right) \\
&\Rightarrow d = -\frac{1}{m}\sum_{i=1}^{m} ax_i - \frac{1}{m}\sum_{i=1}^{m} by_i - \frac{1}{m}\sum_{i=1}^{m} cz_i \\
&\Rightarrow d = -\left(\frac{1}{m}\sum_{i=1}^{m} ax_i + \frac{1}{m}\sum_{i=1}^{m} by_i + \frac{1}{m}\sum_{i=1}^{m} cz_i\right) \\
&\Rightarrow d = -\left(a\frac{1}{m}\sum_{i=1}^{m} x_i + b\frac{1}{m}\sum_{i=1}^{m} y_i + c\frac{1}{m}\sum_{i=1}^{m} z_i\right) \\
&\Rightarrow d = -\left(a\bar{x} + b\bar{y} + c\bar{z}\right)
\end{aligned}$$

b. For $(\tilde{x}_i, \tilde{y}_i, \tilde{z}_i) = (x_i - \bar{x}, y_i - \bar{y}, z_i - \bar{z})$ the problem is:

$$\min\left(\sum_{i=1}^{m} \left(a\tilde{x}_i + b\tilde{y}_i + c\tilde{z}_i\right)^2\right) \; s.t.\; 1 - \left(a^2 + b^2 + c^2\right) = 0$$

Using Lagrange multipliers, the solution to such a problem fulfills:

$$\nabla\left(\sum_{i=1}^{m}\left(a\tilde{x_i}+b\tilde{y_i}+c\tilde{z_i}\right)^2\right)+\lambda\nabla\left(1-\left(a^2+b^2+c^2\right)\right)=0$$

$$\begin{bmatrix} \sum_{i=1}^{m} 2\tilde{x_i}\left(a\tilde{x_i}+b\tilde{y_i}+c\tilde{z_i}\right) \\ \sum_{i=1}^{m} 2\tilde{y_i}\left(a\tilde{x_i}+b\tilde{y_i}+c\tilde{z_i}\right) \\ \sum_{i=1}^{m} 2\tilde{z_i}\left(a\tilde{x_i}+b\tilde{y_i}+c\tilde{z_i}\right) \end{bmatrix}+\lambda\begin{bmatrix} -2a \\ -2b \\ -2c \end{bmatrix}=0$$

$$2\sum_{i=1}^{m}\begin{bmatrix} a\tilde{x_i}^2+b\tilde{x_i}\tilde{y_i}+c\tilde{x_i}\tilde{z_i} \\ a\tilde{y_i}\tilde{x_i}+b\tilde{y_i}^2+c\tilde{y_i}\tilde{z_i} \\ a\tilde{z_i}\tilde{x_i}+b\tilde{z_i}\tilde{y_i}+c\tilde{z_i}^2 \end{bmatrix}+\lambda\begin{bmatrix} -2a \\ -2b \\ -2c \end{bmatrix}=0$$

$$2\sum_{i=1}^{m}\left(\begin{bmatrix} \tilde{x_i}^2 & \tilde{x_i}\tilde{y_i} & \tilde{x_i}\tilde{z_i} \\ \tilde{y_i}\tilde{x_i} & \tilde{y_i}^2 & \tilde{y_i}\tilde{z_i} \\ \tilde{z_i}\tilde{x_i} & \tilde{z_i}\tilde{y_i} & \tilde{z_i}^2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)-2\lambda\begin{bmatrix} a \\ b \\ c \end{bmatrix}=0$$

$$\left(\sum_{i=1}^{m}\begin{bmatrix} \tilde{x_i}^2 & \tilde{x_i}\tilde{y_i} & \tilde{x_i}\tilde{z_i} \\ \tilde{y_i}\tilde{x_i} & \tilde{y_i}^2 & \tilde{y_i}\tilde{z_i} \\ \tilde{z_i}\tilde{x_i} & \tilde{z_i}\tilde{y_i} & \tilde{z_i}^2 \end{bmatrix}\right)\begin{bmatrix} a \\ b \\ c \end{bmatrix}=\lambda\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

So, we have proven that the solution to the problem is an eigenvector of the matrix $\sum_{i=1}^{m}\begin{bmatrix} \tilde{x_i}^2 & \tilde{x_i}\tilde{y_i} & \tilde{x_i}\tilde{z_i} \\ \tilde{y_i}\tilde{x_i} & \tilde{y_i}^2 & \tilde{y_i}\tilde{z_i} \\ \tilde{z_i}\tilde{x_i} & \tilde{z_i}\tilde{y_i} & \tilde{z_i}^2 \end{bmatrix}$.

Let us notice that $f=\lambda$:

$$\begin{aligned}
f&=\sum_{i=1}^{m}\left(a\tilde{x_i}+b\tilde{y_i}+c\tilde{z_i}\right)^2\\
&=\sum_{i=1}^{m}\left(a^2\tilde{x_i}^2+ab\tilde{x_i}\tilde{y_i}+ac\tilde{x_i}\tilde{z_i}+ab\tilde{x_i}\tilde{y_i}+b^2\tilde{y_i}^2+bc\tilde{y_i}\tilde{z_i}+ac\tilde{x_i}\tilde{z_i}+bc\tilde{y_i}\tilde{z_i}+c^2\tilde{z_i}^2\right)\\
&=a^2\sum_{i=1}^{m}\tilde{x_i}^2+b^2\sum_{i=1}^{m}\tilde{y_i}^2+c^2\sum_{i=1}^{m}\tilde{z_i}^2+2ab\sum_{i=1}^{m}\tilde{x_i}\tilde{y_i}+2ac\sum_{i=1}^{m}\tilde{x_i}\tilde{z_i}+2bc\sum_{i=1}^{m}\tilde{y_i}\tilde{z_i}\\
&=a\left(a\sum_{i=1}^{m}\tilde{x_i}^2+b\sum_{i=1}^{m}\tilde{x_i}\tilde{y_i}+c\sum_{i=1}^{m}\tilde{x_i}\tilde{z_i}\right)+b\left(a\sum_{i=1}^{m}\tilde{x_i}\tilde{y_i}+b\sum_{i=1}^{m}\tilde{y_i}^2+c\sum_{i=1}^{m}\tilde{y_i}\tilde{z_i}\right)+c\left(a\sum_{i=1}^{m}\tilde{x_i}\tilde{z_i}+b\sum_{i=1}^{m}\tilde{y_i}\right.\\
&=a\left(\lambda a\right)+b\left(\lambda b\right)+c\left(\lambda c\right)=\lambda\left(a^2+b^2+c^2\right)=\lambda
\end{aligned}$$

Since the solution, by definition of it being a solution, minimizes f, then the eigenvalue of $\sum_{i=1}^{m}\begin{bmatrix} \tilde{x_i}^2 & \tilde{x_i}\tilde{y_i} & \tilde{x_i}\tilde{z_i} \\ \tilde{y_i}\tilde{x_i} & \tilde{y_i}^2 & \tilde{y_i}\tilde{z_i} \\ \tilde{z_i}\tilde{x_i} & \tilde{z_i}\tilde{y_i} & \tilde{z_i}^2 \end{bmatrix}$ corresponding to the eigenvector solution - is the smallest eigenvalue.

Computer Exercise 1

a.
RMS distance between all 3D points and least squares plane = 0.516774

b.
RANSAC found a model with ~740 inliers (out of a total of 797 points).
RMS distance between inlier points and RANSAC plane = 0.028330
This is a great improvement to the original RMS.
NOTE: I plotted the absolute distance between the plane and ALL the 3D points (both inliers and outliers).

c.
RMS distance between inlier points and least squares inliers plane = 0.025421
NOTE: I plotted the absolute distance between the plane and ALL the 3D points (both inliers and outliers).
This is ever better than the ransac plane - this estimation produced the best result.
The projection of the inliers onto the image is located on one of the walls of the house (a 3D plane).

d.
The points that were originally on the wall seem to be transformed correctly, as opposed to points that were elsewhere.
This is because the homography transforms points on a specific plane between two images (2 cameras).
2D points that correspond to 3D points that are not on that plane - will not be transformed correctly using this homography.

All the figures can be found in the attached matlab_results.pdf file.

Robust Homography Estimation and Stiching

Exercise 3

$$P_1 = \begin{bmatrix} A_1 & t_1 \end{bmatrix}, \; P_2 = \begin{bmatrix} A_2 & t_2 \end{bmatrix}$$
$$C_1 = -A_1^{-1} t_1, \; C_2 = -A_2^{-1} t_2$$

$$\begin{aligned} C_1 = C_2 &\Rightarrow -A_1^{-1} t_1 = -A_2^{-1} t_2 \\ &\Rightarrow A_1^{-1} t_1 = A_2^{-1} t_2 \\ &\Rightarrow t_2 = A_2 A_1^{-1} t_1 \end{aligned}$$

$$x_1 \propto P_1 \begin{bmatrix} X \\ 1 \end{bmatrix} \Rightarrow x_1 = \lambda_1 P_1 \begin{bmatrix} X \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} A_1 & t_1 \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix} = \lambda_1 (A_1 X + t_1)$$

$$
\begin{aligned}
x_2 \propto P_2 \begin{bmatrix} X \\ 1 \end{bmatrix} \Rightarrow x_2 = \lambda_2 P_2 \begin{bmatrix} X \\ 1 \end{bmatrix} = \lambda_2 \begin{bmatrix} A_2 & t_2 \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix} &= \lambda_2 \left(A_2 X + t_2 \right) \\
&= \lambda_2 \left(A_2 X + A_2 A_1^{-1} t_1 \right) \\
&= \lambda_2 A_2 \left(X + A_1^{-1} t_1 \right) \\
&= \lambda_2 A_2 \frac{A_1^{-1}}{\lambda_1} \lambda_1 A_1 \left(X + A_1^{-1} t_1 \right) \\
&= \frac{\lambda_2}{\lambda_1} A_2 A_1^{-1} \left(\lambda_1 \left(A_1 X + t_1 \right) \right) \\
&= \frac{\lambda_2}{\lambda_1} A_2 A_1^{-1} x_1
\end{aligned}
$$

$$x_2 \propto \left(A_2 A_1^{-1} \right) x_1$$

Therefore, there is a homography $H = A_2 A_1^{-1}$ that transforms the first image into the second one: $x_2 \propto H x_1$.
Notice that during the proof I never assumed that P_1 or P_2 are calibrated. Thus, this applies even if they are not (we only assumed A_1, A_2 are invertible).

Exercise 4

A general homography is a matrix $H \in R^{3\times3}$ with 9 entries.
However, all the homographies $\{\lambda H\}$ are the same, meaning that H determines a homography up-to-scale.
Thus, a homography has 9 DOF up-to-scale, which leaves it with 8 DOF overall.

Each point correspondence (two corresponding points) - $\{(x_1, y_1), (x_2, y_2)\}$ emposes two linear constraints:

$$X_1 \propto H X_2$$

$$
\begin{aligned}
\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} &\propto \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} h_{11}x_2 + h_{12}y_2 + h_{13} \\ h_{21}x_2 + h_{22}y_2 + h_{23} \\ h_{31}x_2 + h_{32}y_2 + h_{33} \end{bmatrix}
\end{aligned}
$$

$$\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{h_{11}x_2 + h_{12}y_2 + h_{13}}{h_{31}x_2 + h_{32}y_2 + h_{33}} \\ \frac{h_{21}x_2 + h_{22}y_2 + h_{23}}{h_{31}x_2 + h_{32}y_2 + h_{33}} \\ 1 \end{bmatrix}$$

$$x_1 = \frac{h_{11}x_2 + h_{12}y_2 + h_{13}}{h_{31}x_2 + h_{32}y_2 + h_{33}}, \; y_1 = \frac{h_{21}x_2 + h_{22}y_2 + h_{23}}{h_{31}x_2 + h_{32}y_2 + h_{33}}$$

The linear constraints:

$$h_{11}x_2 + h_{12}y_2 + h_{13} - h_{31}x_1x_2 - h_{32}x_1y_2 - h_{33}x_1 = 0$$

$$h_{21}x_2 + h_{22}y_2 + h_{23} - h_{31}y_1x_2 - h_{32}y_1y_2 - h_{33}y_1 = 0$$

Thus, the minimal number of point correspondences required to determine a homography is $k = 4$.

If we have 10% outliers then we have 90% inliers - $q = 0.9$.
As seen in class, if we want to achieve a success rate of $P = 98\%$, we need to draw at least $N \geq \frac{1}{q^k} \log\left(\frac{1}{1-P}\right)$ samples:

$$N \geq \frac{1}{q^k} \log\left(\frac{1}{1-P}\right) = \frac{1}{0.9^4} \log\left(\frac{1}{1-0.98}\right) \approx 5.963$$

Thus, we need to draw at least 6 samples (of 2 points each time).

Computer Exercise 2

RANSAC found a model with ~150 inliers (out of a total of 204 point correspondences).

NOTE: although the RANSAC calculations showed that we need to draw 6 sample sets to achieve the required success rate, in practice 6 was not enough.
This might be due to the fact that we have more outliers in practice than what we calculated with.
In my optimal solution I found 150 inliers out of 204 (26% outliers). New calculation will show that we need at least 14 sample sets to achieve the required success rate (but, again, in practice that wasn't enough).

The transformed image in the attached matlab_results.pdf file.