Multiple View Geometry - Assignment 5

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Theoretical exercises

1.

A convex optimization problem is of the form:

$$\min\left(f_0\left(x\right)\right)\ \ s.t.$$

$$f_j\left(x\right)\leq 0\ j=1\ldots m$$
$$h_i\left(x\right)=a_i^Tx-b_i=0\ i=1\ldots p$$

We shall first prove that the feasible set of a convex optimization problem is a convex set.

Let us observe a function $f_j\left(x\right)$ for some j. It is given that this function is convex.
Thus, we know that all its sublevel sets are convex sets, and in particular the sublevel set $\{x|f_j\left(x\right)\leq 0\}$ is a convex set.
Since an intersection of convex sets is a convex set, we know that the following is a convex set $\{x|f_j\left(x\right)\leq 0\ j=1\ldots m\}$.

Let us now observe a constraint $h_i\left(x\right)=0$ for some i, two points x_1,x_2 which satisfy this constraint and $\theta\in[0,1]$.

$$\begin{aligned}
a_i^T\left(\theta x_1+\left(1-\theta\right)x_2\right)-b_i &= \theta a_i^Tx_1+\left(1-\theta\right)a_i^Tx_2-b_i\\
&= \theta a_i^Tx_1+\left(1-\theta\right)a_i^Tx_2+\theta\left(-b_i\right)+\left(1-\theta\right)\left(-b_i\right)\\
&= \theta\left(a_i^Tx_1-b_i\right)+\left(1-\theta\right)\left(a_i^Tx_2-b_i\right)\\
&= \theta\left(0\right)+\left(1-\theta\right)\left(0\right)=0
\end{aligned}$$

Thus, all the points that satisfy the constraint $h_i\left(x\right)=0$ are a convex set.
As before, since an intersection of convex sets is a convex set, we know that the following is a convex set $\{x|h_i\left(x\right)=0\ i=1\ldots p\}$.

Therefore, the feasible set of a convex optimization problem is a convex set.

All that is left to prove is that while minimizing a convex function $f(x)$ over some convex domain Ω (which in our case came from the constrints), any local minimizer is a global minimizer.
Let x_1 be a local minimizer of $f(x)$. This means that $\exists \epsilon_1 \ \forall x \in (x_1 - \epsilon_1, x_1 + \epsilon_1) \ f(x_1) \leq f(x)$.
Let's assume towards constradication that our local minimizer is not a global minimizer. i.e. there exists $x_0 \in \Omega$ s.t. $f(x_0) < f(x_1)$.
Since our domain is convex, $\forall \theta \in [0,1]$ the point $(\theta x_1 + (1-\theta) x_0) \in \Omega$.
Since $f(x)$ is convex we know that $f(\theta x_1 + (1-\theta) x_0) \leq \theta f(x_1) + (1-\theta) f(x_0)$.
Putting it all together:

$$\begin{aligned} f(\theta x_1 + (1-\theta) x_0) &\leq \theta f(x_1) + (1-\theta) f(x_0) \\ &< \theta f(x_0) + (1-\theta) f(x_0) \\ &= (\theta + 1 - \theta) f(x_0) = f(x_0) \end{aligned}$$

Choosing θ small enough we would get a point $x' \in (x_1 - \epsilon_0, x_1 + \epsilon_0)$ for which $f(x') \leq f(x_0) < f(x_1)$, and this is a contradiction to the fact that $\forall x \in (x_1 - \epsilon_1, x_1 + \epsilon_1) \ f(x_1) \leq f(x)$.
Thus, we conclude that x_1 is a global minimizer.

Therefore, given a constraint convex optimization problem, any local minimizer is a global minimizer.

2.

$$f(x,y) = xy$$
$$dom(f) = R_+^2$$

The domain is obviously convex.
Let us observe a superlevel set of f: $C_\alpha = \left\{(x,y) \in R_+^2 | f(x,y) \geq \alpha\right\}$.
If we prove that it is convex, then all superlevel sets of f are convex (α is not restricted and we prove the general case) and thus f is a quasiconcave function, as required.

To prove that C_α is convex, let us observe two points in C_α: $(x_1, y_1), (x_2, y_2) \in C_\alpha$, and let us set some $\theta \in [0,1]$.

All that is left is to prove that $\theta(x_1,y_1)+(1-\theta)(x_2,y_2)\in C_\alpha$.

$$\begin{aligned}
f(\theta(x_1,y_1)+(1-\theta)(x_2,y_2)) &= f((\theta x_1,\theta y_1)+((1-\theta)x_2,(1-\theta)y_2)) \\
&= f(\theta x_1+(1-\theta)x_2,\theta y_1+(1-\theta)y_2) \\
&= (\theta x_1+(1-\theta)x_2)(\theta y_1+(1-\theta)y_2) \\
&= \theta^2 x_1y_1+(1-\theta)\theta x_2y_1+(1-\theta)\theta x_1y_2+(1-\theta)^2 x_2y_2 \\
&= \theta^2 f(x_1,y_1)+(1-\theta)^2 f(x_2,y_2)+(1-\theta)\theta x_2y_1+(1-\theta)\theta x_1y_2 \\
&\geq \theta^2\alpha+(1-\theta)^2\alpha+(1-\theta)\theta(x_2y_1+x_1y_2) \\
&= \theta^2\alpha+(1-\theta)^2\alpha+2(1-\theta)\theta\left(\frac{x_2y_1+x_1y_2}{2}\right) \\
&\geq \theta^2\alpha+(1-\theta)^2\alpha+2(1-\theta)\theta\left(\sqrt{x_2y_1x_1y_2}\right) \\
&= \theta^2\alpha+(1-\theta)^2\alpha+2(1-\theta)\theta\left(\sqrt{x_1y_1x_2y_2}\right) \\
&= \theta^2\alpha+(1-\theta)^2\alpha+2(1-\theta)\theta\left(\sqrt{f(x_1,y_1)f(x_2,y_2)}\right) \\
&= \theta^2\alpha+(1-\theta)^2\alpha+2(1-\theta)\theta\left(\sqrt{\alpha\alpha}\right) \\
&= \theta^2\alpha+(1-\theta)^2\alpha+2(1-\theta)\theta\alpha \\
&= \alpha\left(\theta^2+\theta^2-2\theta+1+2\theta-2\theta^2\right)=\alpha
\end{aligned}$$

$$f(\theta(x_1,y_1)+(1-\theta)(x_2,y_2)) \geq \alpha$$

$$\Rightarrow \theta(x_1,y_1)+(1-\theta)(x_2,y_2)\in C_\alpha$$

Thus, we have shown that C_α is convex, and therefore all sublevel sets of f are convex, and therefore f is a quasiconcave function, as required.

Computer Exercise 1 - Triangulation with DLT

The mean reprojection error for the **GT** 3D points was ~0.3209.
Triangulating the 2D points using DLT with **normalized** cameras and points gave astonishing results - mean reprojection error of ~0.3211 - very close to the GT points.
Triangulating the 2D points using DLT with **unnormalized** cameras and points gave worst results - mean reprojection error of ~10.7662 - not nearly as good.
This is of course as expected.

The histogram of the reprojection errors of the GT 3D points, as well as the exact three mean reprojection errors, are given in the attached jupyter_results.pdf

Computer Exercise 2 - Triangulation with cones

a.

$$r_i(X) = \left|\left(x_i^1 - \frac{P_i^1 X}{P_i^3 X}, x_i^2 - \frac{P_i^2 X}{P_i^3 X}\right)\right|_2$$

$$\begin{aligned}
r_i(X) \leq \gamma &\iff \left|\left(x_i^1 - \frac{P_i^1 X}{P_i^3 X}, x_i^2 - \frac{P_i^2 X}{P_i^3 X}\right)\right|_2 \leq \gamma \\
&\iff \frac{1}{P_i^3 X}\left|\left(x_i^1 P_i^3 X - P_i^1 X, x_i^2 P_i^3 X - P_i^2 X\right)\right|_2 \leq \gamma \\
&\iff \frac{1}{P_i^3 X}\left|\left(\left(x_i^1 P_i^3 - P_i^1\right) X, \left(x_i^2 P_i^3 - P_i^2\right) X\right)\right|_2 \leq \gamma \\
&\iff \left|\left(\left(x_i^1 P_i^3 - P_i^1\right) X, \left(x_i^2 P_i^3 - P_i^2\right) X\right)\right|_2 \leq \gamma P_i^3 X \\
&\iff \left|\begin{array}{c} \left(x_i^1 P_i^3 - P_i^1\right) X \\ \left(x_i^2 P_i^3 - P_i^2\right) X \end{array}\right|_2 \leq \gamma P_i^3 X \\
&\iff \left|\begin{bmatrix} x_i^1 P_i^3 - P_i^1 \\ x_i^2 P_i^3 - P_i^2 \end{bmatrix} X\right|_2 \leq \gamma P_i^3 X
\end{aligned}$$

So, by setting $A_i = \begin{bmatrix} x_i^1 P_i^3 - P_i^1 \\ x_i^2 P_i^3 - P_i^2 \end{bmatrix}$ and $c_i = \left(P_i^3\right)^T$ we get that $r_i(X) \leq \gamma \iff |A_i X|_2 \leq \gamma c_i^T X$.
Since we know that constraints of the form $|AX| \leq \gamma c^T X$ forms a second order cone, and that the following constraints are equivalent $r_i(X) \leq \gamma \iff |A_i X|_2 \leq \gamma c_i^T X$ (for the above mentioned A_i, c_i), then we have shown that **the constraint $r_i(X) \leq \gamma$ forms a second order cone**, as required.

b.

Let us observe the function $r_i(X)$ for some i.
It's domain is R^4, which is obviously convex, and each of its sublevel sets $C_\alpha = \left\{X \in R^4 | r_i(X) \leq \alpha\right\}$ are convex since we have proven in the previous exercise that the constraint $r_i(X) \leq \alpha$ forms a second order cone which is in particular a convex set.
Thus, $r_i(X)$ is quasi-convex.

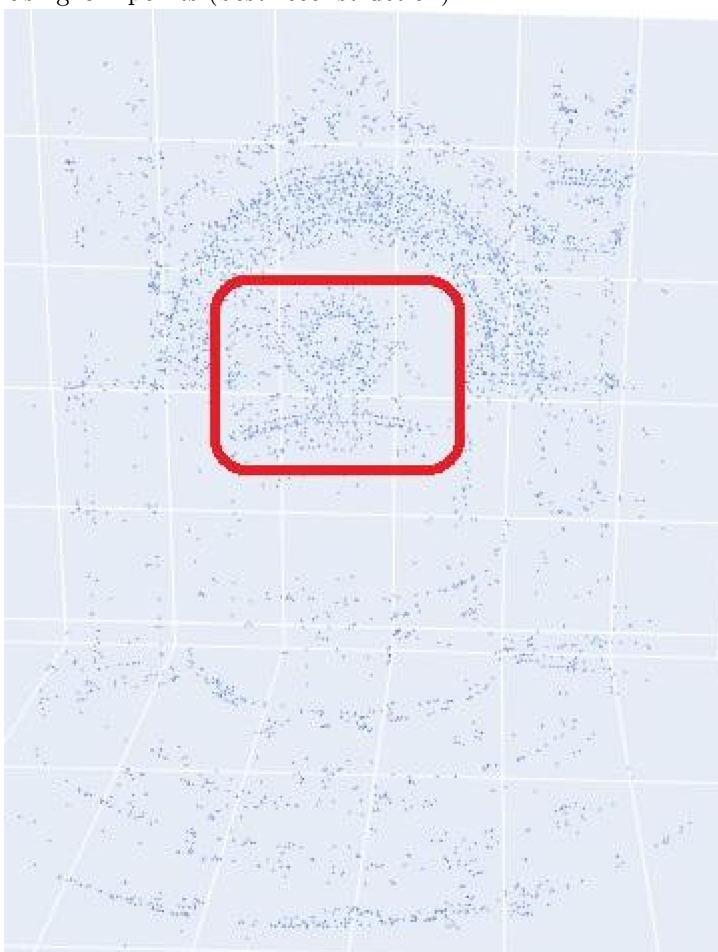
The function $f(X) = \max_i\left(r_i(X)\right)$ is simply a maximum of quasi-convex functions, and thus it is also a quasi-convex function on its own (as shown in class).

e.

As I expected, the SOCP reconstruction was very close to the GT, and in the SOCP solution the normalization is not important as it was in the DLT solution (as discussed in class).
The comparison bar chart can be found in the attached jupyter_results.pdf.

f.

The resolution between the plot with the best mean reconstruction error and worst mean reconstruction error is indeed noticeable.
Attached here are the visualizations of the reconstructions.
Using GT points (best reconstruction):



Using unnormalized DLT (worst reconstruction):