Multiple View Geometry - Assignment 1

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April 17, 2021

1. \

   (a) \

$$x_1 = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} = \lambda_1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \text{2D Cartesian coordinates of } x_1\text{: } \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} = \lambda_2 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \text{2D Cartesian coordinates of } x_2\text{: } \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} 4\lambda \\ -2\lambda \\ 2\lambda \end{bmatrix} = \lambda_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \text{2D Cartesian coordinates of } x_3\text{: } \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

   (b) Any line on the 2D image plane can be represented by a plane in 3D which includes that line and also the origin.
   This plane can be represented by a 3D normal vector.
   If we observe the two lines which are represented by the normal vectors of their 3D planes $l_1 : \begin{pmatrix} -2 \\ 4 \\ c_1 \end{pmatrix}, l_2 : \begin{pmatrix} -2 \\ 4 \\ c_2 \end{pmatrix}$ then we can make two observations:
   First, these lines are parallel.
   Second, these lines intersect at $l_1 \times l_2 = \begin{pmatrix} -2 \\ 4 \\ c_1 \end{pmatrix} \times \begin{pmatrix} -2 \\ 4 \\ c_2 \end{pmatrix} = \begin{pmatrix} 4c_2 - 4c_1 \\ -2c_1 + 2c_2 \\ 0 \end{pmatrix} = (c_2 - c_1) \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} \propto \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$.
   So, to sum up, the point with homogenous coordinates $x_4 = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$

represents a point at infinity where all the parallel lines on the image - that their representation is of the form $l : \alpha \begin{pmatrix} -2 \\ 4 \\ c \end{pmatrix}$ - intersect.

2. \

(a) The lines $l_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $l_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ intersect at the point $p = l_1 \times l_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-2 \\ 3-1 \\ 2-3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$.
The point of intersecrion p has homogenous coordinates and it intersects P^2 at $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

(b) The lines $l_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $l_4 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ intersect at the point $p = l_3 \times l_4 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2-6 \\ 3-1 \\ 2-2 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}$.
The point of intersection p does not intersect the projective space P^2.
This means geomentrically that the lines l_3, l_4 are parallel in P^2.

(c) The points with cartesian coordinates $x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, x_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ have the homogeneous coordinates $x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, x_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.
The line that goes trough these 2 points in P^2 can be represented by the plane that includes the two lines in R^3.
The normal to this plane is $l = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$ (using the calculations from the line intersection above).

3. \

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Points before homography:

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Points after homography:

$$y_1 \propto Hx_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow y_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$y_2 \propto Hx_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow y_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Line before homography:

$$l_1 = x_1 \times x_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Line after homography:

$$l_2 = y_1 \times y_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$H^{-1}: \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

$$H^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \Rightarrow \left(H^{-1}\right)^T = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left(H^{-1}\right)^T l_1 = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\left(H^{-1}\right)^T l_1 = l_2$$

For each line l_1 we know that it is all the points $P = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ s.t. $l_1^T P = 0$.

$$\begin{aligned} l_1^T P = 0 &\Rightarrow l_1^T \left(H^{-1} H\right) P = 0 \\ &\Rightarrow \left(l_1^T H^{-1}\right) (HP) = 0 \\ &\Rightarrow \left(\left(H^{-1}\right)^T l_1\right)^T (HP) = 0 \end{aligned}$$

Therefore if we define $l_2 = \left(H^{-1}\right)^T l_1$ we get that all points P that were on the line l_1, after transformation $y \propto (HP)$ belong to l_2: $l_2^T y = 0$.

4. \

$$P = \begin{bmatrix} M & p_4 \end{bmatrix}$$

$$x = (X, Y, Z, 1)^T$$

Let us calculate the coordinates of x in the coordinate system of the camera:

$$Px = \begin{bmatrix} M & p_4 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & p_{4x} \\ m_{21} & m_{22} & m_{23} & p_{4y} \\ m_{31} & m_{32} & m_{33} & p_{4z} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11}X + m_{12}Y + m_{13}Z + p_{4x} \\ m_{21}X + m_{22}Y + m_{23}Z + p_{4y} \\ m_{31}X + m_{32}Y + m_{33}Z + p_{4z} \end{bmatrix}$$

So, the depth of the x w.r.t the camera is $(Px)_Z = m_{31}X + m_{32}Y + m_{33}Z + p_{4z}$

5. \

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$Px_1 \propto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \Rightarrow Px_1 = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

$$Px_2 \propto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \Rightarrow Px_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

$$Px_3 \propto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow Px_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

The geometric interpretation of the projection of x_3 is that it is a point at infinity w.r.t this camera.

6. \

   (a) \

$$P_1 = \begin{bmatrix} I & 0 \end{bmatrix}, \ P_2 = \begin{bmatrix} R & t \end{bmatrix}$$

$$u = \begin{bmatrix} u_x \\ u_y \\ u_z \\ 1 \end{bmatrix} \text{ - 3D point in homogenous coordinates}$$

$$x \propto P_1 u = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \\ 1 \end{bmatrix} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \Rightarrow x = \begin{bmatrix} \frac{u_x}{u_z} \\ \frac{u_y}{u_z} \\ 1 \end{bmatrix}$$

We shall now show that for all $s \in R$ the point of the form $u(s)$ projects to x:

$$\forall s \in R\, u(s) = \begin{bmatrix} x \\ s \end{bmatrix} = \begin{bmatrix} \frac{u_x}{u_z} \\ \frac{u_y}{u_z} \\ 1 \\ s \end{bmatrix} \Rightarrow P_1 u(s) = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \frac{u_x}{u_z} \\ \frac{u_y}{u_z} \\ 1 \\ s \end{bmatrix} = \begin{bmatrix} \frac{u_x}{u_z} \\ \frac{u_y}{u_z} \\ 1 \end{bmatrix} = x$$

The collection of all the points $\{u(s)\}_{s\in R} = \left\{ \begin{bmatrix} \frac{u_x}{u_z} \\ \frac{u_y}{u_z} \\ 1 \\ s \end{bmatrix} \right\}_{s\in R} = \left\{ \begin{bmatrix} \frac{u_x}{u_z} \\ \frac{u_y}{u_z} \\ 1 \\ s \end{bmatrix} \right\}_{s\in R}$ should be viewed in its homogeneous form $\left\{ \begin{bmatrix} \frac{1}{s}\frac{u_x}{u_z} \\ \frac{1}{s}\frac{u_y}{u_z} \\ \frac{1}{s} \\ 1 \end{bmatrix} \right\}_{s\in R}$ which are all the 3D points $\begin{bmatrix} \frac{1}{s}x \\ 1 \end{bmatrix}$. In 3D this is the ray that goes through the origin and x. All the points on that ray are projected to the same point x on the image plane.

It is impossible to determine s using only information from P_1. As we have just shown, the projection of all of these point onto P_1 is x - regardless of the value of s.

(b) Let us assume that $u(s)$ belongs to the plane $\Pi = \begin{bmatrix} \pi \\ 1 \end{bmatrix}$:

$$\Pi^T u(s) = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & 1 \end{bmatrix} \begin{bmatrix} \frac{u_x}{u_z} \\ \frac{u_y}{u_z} \\ 1 \\ s \end{bmatrix} = 0$$

$$\Rightarrow \pi_1 \frac{u_x}{u_z} + \pi_2 \frac{u_y}{u_z} + \pi_3 + s = 0$$

$$\Rightarrow s = -\left(\pi_1 \frac{u_x}{u_z} + \pi_2 \frac{u_y}{u_z} + \pi_3\right)$$

$$\Rightarrow \boldsymbol{s = -\pi^T x}$$

$$\Rightarrow u(s) = \begin{bmatrix} \frac{u_x}{u_z} \\ \frac{u_y}{u_z} \\ 1 \\ -\pi^T x \end{bmatrix}$$

Verification:

$$x \propto P_1 u \Rightarrow x = \begin{bmatrix} \frac{u_x}{u_z} \\ \frac{u_y}{u_z} \\ 1 \end{bmatrix}$$

$$\Pi^T u = 0 \Rightarrow s = -\pi^T x \Rightarrow u = \begin{bmatrix} \frac{u_x}{u_z} \\ \frac{u_y}{u_z} \\ 1 \\ -\pi^T x \end{bmatrix}$$

$$P_2 u = \begin{bmatrix} R & t \end{bmatrix} \begin{bmatrix} \frac{u_x}{u_z} \\ \frac{u_y}{u_z} \\ 1 \\ -\pi^T x \end{bmatrix} = \begin{bmatrix} R & t \end{bmatrix} \begin{bmatrix} x \\ -\pi^T x \end{bmatrix} = Rx - t\pi^T x = \left(R - t\pi^T\right) x$$

$$y \propto P_2 u \Rightarrow y \propto \left(R - t\pi^T\right) x$$

$$y \propto Hx$$

as required.