MVG Assignment 3
Epipolar Geometry

Instructions

- Due date: May 18 2021.
- Submission link: `https://www.dropbox.com/request/Bt0KbpNMIQaMqKnYZ9Y5`
- You should submit a zip file containing: A pdf file with all complete solutions and the code with a main script which calls the required scripts/functions for each computer exercise.
- The report should be written individually, however you are encouraged to work together.
- We highly recommend typing your solutions in Latex.
- Notice that in the end of this exercise you are given a list of useful matlab commands for all the computer exercises.

The Fundamental Matrix

Exercise 1

If $P_1 = \begin{bmatrix} I & 0 \end{bmatrix}$ and:

$$P_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{1}$$

Compute the fundamental matrix.
Given the point $x = (1, 1)$ which is the projection of a 3D-point X onto P_1, **compute** the epipolar line in the second image generated from x.
Which of the points $\mathbf{a} = (2, 0), \mathbf{b} = (2, 1), \mathbf{c} = (4, 2)$ could be a projection of the same point X onto P_2?

Exercise 2

If $P_1 = \begin{bmatrix} I & 0 \end{bmatrix}$ and:

$$P_2 = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{2}$$

Compute the epipoles, by projecting the center of each camera to the other one.
Compute the fundamental matrix, its determinant and verify that $e_2^T F = 0$ and $F e_1 = 0$.
For a general camera pair $P_1 = \begin{bmatrix} I & 0 \end{bmatrix}$ and $P_2 = \begin{bmatrix} A & \mathbf{t} \end{bmatrix}$ **compute** the epipoles, by projecting the camera centers. (You may assume that A is invertible.)
Verify that for the fundamental matrix $F = [\mathbf{t}]_\times A$ the epipoles will always fulfill $e_2^T F = 0$ and $F e_1 = 0$.
Given the above result **explain** why the fundamental matrix has to have determinant 0.

Exercise 3

When computing the fundamental matrix F using the 8-point algorithm it is recommended to use normalization. Suppose the image points have been normalized using

$$\tilde{\mathbf{x}}_1 \sim N_1 \mathbf{x}_1 \text{ and } \tilde{\mathbf{x}}_2 \sim N_2 \mathbf{x}_2 \tag{3}$$

If $\tilde{F}$ fulfills $\tilde{\mathbf{x}}_2^T \tilde{F} \tilde{\mathbf{x}}_1 = 0$ **what** is the fundamental matrix F that fulfills $x_2^T F x_1 = 0$ for the original (unnormalized) points?

(a) kronan1.JPG

(b) kronan2.JPG

Figure 1

Computer Exercise 1

In this exercise you will compute the fundamental matrix for two images of the fort Kronan in Gothenburg, seen in figure 1.
The file `compEx1data.mat` contains a cell `x` with matched points for the two images.

First **compute** normalization matrices N_1 and N_2. These matrices should subtract the mean and re-scale using the standard deviation, as in assignment 2. **Normalize** the image points of the two images with N_1 and N_2 respectively.

Set up the matrix M in the eight point algorithm (use all the points), and **solve** the homogeneous least squares system using SVD. Check that the minimum singular value and $\|M\mathbf{v}\|$ are both small.

Construct the normalized fundamental matrix from the solution $\mathbf{v}$. Make sure that $\det(\tilde{F}) = 0$ by setting the third singular value of F to zero. Check that the epipolar constraints $\tilde{\mathbf{x}}_2^T \tilde{F} \tilde{\mathbf{x}}_1 = 0$ are roughly fulfilled.

Compute the un-normalized fundamental matrix F (using the formula from exercise 3) and the epipolar lines $l = F\mathbf{x}_1$. Pick 20 points in the second image at random and **plot** these in the same figure as the image. Now **plot** the corresponding epipolar lines in the same image using the function `rital.m`. Are they close to each other?
Compute the distance between all the points and their corresponding epipolar lines and plot these in a histogram with 100 bins. What is the mean distance?
Repeat the question without normalization (that is, set $N_1 = N_2 = I$ and run the code again). **What** is the mean distance in this case?

For the report: The fundamental matrix for the original (un-normalized) points (make sure that F(3;3) =

1 by dividing by this coordinate), the histogram, the plot of the epipolar lines, the mean epipolar distances obtained with and without normalization.

Exercise 4

Consider the fundamental matrix

$$F = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \tag{4}$$

Verify that the projection of the scene points $(1, 2, 3)$ and $(3, 2, 1)$ in the cameras $P_1 = \begin{bmatrix} I & 0 \end{bmatrix}$ and $P_2 = \begin{bmatrix} [e_2]_\times F & e_2 \end{bmatrix}$ fulfill the epipolar constraint ($x_2^T F x_1 = 0$).
What is the camera center of P_2?

Computer Exercise 2

Use the fundamental matrix F that you obtained in Computer Exercise 1 to **compute** the camera matrices in Exercise 4.
Use triangulation (with DLT) to **compute** the 3D-points.
For the two images, **Plot** both the image, the image points, and the projected 3D points in the same figure.

Note that you need to normalize when triangulating. Since the point sets are the same as in Computer Exercise 1 the normalization matrices will also be the same. (Alternatively one could compute cameras and 3D points from the fundamental matrix $\tilde{F}$ obtained with the normalized points and transform the cameras afterwards. This also gives a valid solution, but it is a different one.)
Plot the 3D-points in a 3D plot. **Does** it look like you expected?
For the report: The two images with the image points and the projected points, a 3D plot. A short answer about the appearance of the 3D plot.

The Essential Matrix

Exercise 5

The goal of this exercise is to show that an essential matrix has two nonzero identical singular values.
Suppose the 3×3 skew symmetric matrix $[t]_\times$ has a singular value decomposition

$$[t]_\times = USV^T \tag{5}$$

where U, V are orthogonal and S diagonal with non-negative elements.

a.

Show that the eigenvalues of $[t]_\times^T[t]_\times$ are the squared singular values. (Hint: Show that $S^TS = S^2$ is the diagonalization of $[t]_\times^T[t]_\times$)

b.

Verify that the eigenvalues of $[t]_\times^T[t]_\times$ fulfill

$$-t \times (t \times w) = \lambda w \tag{6}$$

c.

Given the formula:

$$u \times (v \times w) = (u \cdot w)v - (u \cdot v)w \tag{7}$$

Show that $w = t$ is an eigenvector to $[t]_\times^T[t]_\times$ with eigenvalue 0 and that any w that is perpendicular to t is an eigenvector with eigenvalue $\|t\|^2$. Are these all of the eigenvectors?

d.

Show that the singular values of $[t]_\times$ are 0, $\|t\|$ and $\|t\|$.
If $E = [t]_\times R$ has the SVD in equation 4, state an SVD of E. What are the singular values of E?

Computer Exercise 3

The file `compEx3data.mat` contains the calibration matrix K for the two images in Computer Exercise 1.
Normalize the image points using the inverse of K.
Set up the matrix M in the eight point algorithm, and solve the homogeneous least squares system using SVD. Check that the minimum singular value and $M\mathbf{v}$ are both small.
Construct the Essential matrix from the solution $\mathbf{v}$. Don't forget to make sure that E has two equal singular values and the third one zero. Check that the epipolar constraints $\tilde{\mathbf{x}}_2^T E\tilde{\mathbf{x}}_1 = 0$ are roughly fulfilled.
Compute the fundamental matrix for the un-normalized coordinate system from the essential matrix and **compute** the epipolar lines $l = Fx_1$. **Pick** 20 of the detected points in the second image at random and **plot** these in the same figure as the image. Also plot the corresponding epipolar lines in the same figure using the function `rital.m`.
Compute the distance between the points and their corresponding epipolar lines and plot these in a histogram with 100 bins. How does this result compare to the corresponding result Computer Exercise 1?

For the report: the Essential matrix where E(3,3) = 1, the histogram and the plot of the epipolar lines.

Exercise 6

An essential matrix has the singular value decomposition

$$E = U\text{diag}([1, 1, 0])V^T \tag{8}$$

Where:

$$U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{9}$$

Verify that $\det(UV^T) = 1$.

a.

Compute the essential matrix and **verify** that $x_1 = (0, 0)$ (in camera 1) and $x_2 = (1, 1)$ (in camera 2) is a plausible correspondence.

b.

If x_1 is the projection of X in $P_1 = \begin{bmatrix} I & 0 \end{bmatrix}$, show that X must be one of the points

$$X(s) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ s \end{bmatrix} \tag{10}$$

c.

For each of the solutions

$$P_2 = \begin{bmatrix} UWV^T & u_3 \end{bmatrix} \text{ or } \begin{bmatrix} UWV^T & -u_3 \end{bmatrix} \text{ or } \begin{bmatrix} UW^TV^T & u_3 \end{bmatrix} \text{ or } \begin{bmatrix} UW^TV^T & -u_3 \end{bmatrix} \tag{11}$$

Where

$$W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12}$$

and u_3 is the third column of U, **compute** s such that $X(s)$ projects to x_2.

d.

For which of the camera pairs is the 3D point $X(s)$ in front of both cameras?

Computer Exercise 4

For the essential matrix obtained in Computer Exercise 3 compute four camera solutions in equation 11.
Triangulate the points using DLT for each of the four camera solutions, and determine for which of the solutions the points are in front fo the cameras. (Since there is noise involved it might not be possible to find a solution with all points in front of the cameras. In that case select the one with the highest number of points in front of the cameras.)
Compute the corresponding camera matrices for the original (un-normalized) coordinate system and plot the image the points and the projected 3D-points in the same figure. Do the errors look small?
Plot the 3D points and camera centers and principal axes in a 3D plot. Does it look like you expected it to?

For the report: The figures.

Usefull Matlab commands

1

- xx = x2n(:,i)*x1n(:,i)';

% Computes a 3x3 matrix containing all multiplications of coordinates from x1n(:,i) and x2n(:,i).

- M(i ,:) = xx(:)'; % Reshapes the matrix above and adds to the M matrix
- Fn = reshape(v ,[3 3]); % Forms an F- matrix from the solution v of the leat squares problem
- plot(diag(x2n'* Fn*x1n)); % Computes and plots all the epipolar constraints (should be roughly 0)
- l = F*x1; % Computes the epipolar lines
- l = l./ sqrt(repmat(l(1 ,:).^2 + l(2 ,:).^2 ,[3 1]));

% Makes sure that the line has a unit normal (makes the distance formula easier)

- hist(abs(sum(l.*x2)) ,100);

% Computes all the the distances between the points and there corresponding lines , and plots in a histogram

2

- e2 = null(F'); % Computes the null space of F
- e2x = [0 -e2(3) e2(2); e2(3) 0 -e2(1); -e2(2) e2(1) 0]; % Constructs the cross product matrix

3

```
- [U,S,V] = svd(Eapprox);
if det(U*V') >0
E = U*diag ([1 1 0])*V';
else
V = -V;
E = U* diag([1 1 0])*V';
end
% Creates a valid essential matrix from an approximate solution.
```

% Note: Computing svd on E may still give U and V that does not fulfill det(U*V') = 1 since the svd is not unique. So don't recompute the svd after this step.