Discrete Maths Notes

August 31, 2023

Contents

1 Logic

1.1 Propositional Logic

1.1.1 Basics

A **proposition** is a statement that is either true or false.

It's not always easy to determine if they're true/false.

Prepositions will be represented mathematically with capital letters A, B, C...
These prepositions are then are connected into more complex compound prepositions using *connectives*. Connectives are statements like "and, implies, if-then" and are represented mathematically with the symbols below.

Connectives			
Symbol	Name	English Term(s)	Reading
$\wedge$	AND	And, But, Also	A and B
$\vee$	OR	-	A or B
$\implies$	IMPLICATION	If A, Then B If A, then B A implies B A, therefore B A only if B B follows from A A is a sufficient condition for B B is a necessary condition for A	A implies B
$\iff$	BICONDITIONAL	If & only if A is necessary and sufficient for B	A if and only if B
$\neg$	NEGATION	Not...	Not A

A Biconditional can also be thought of $(A \implies B) \wedge (B \implies A)$
Negation may sometimes be represented as A' *or* $\overline{A}$

1.1.2 Terminology

$A \wedge B$ - conjunction of conjuncts A and B
$A \vee B$ - disjunction of disjuncts A and B
$A \implies B$ - A is the hypothesis$_{/\text{antecedent}}$ and B is the conclusion$_{/\text{consequence}}$

1.1.3 Examples

1 Compound Proposition

If all humans are mortal$_{\text{prp } A}$ and all Greeks are human$_{\text{prp } B}$
then all Greeks are mortal$_{\text{prp } C}$ can be represented as $A \wedge B \implies C$

2 Negation

Chocolate is sweet $\rightarrow$ Chocolate is not sweet

Peter is tall and thin $\rightarrow$ Peter is short or fat

The river is shallow or polluted $\rightarrow$ The river is deep and polluted.

ℹ Short and fat would be incorrect!

ℹ Not shallow or not polluted would be incorrect!

3 Implication: hypothesis and conclusion

If the rain continues then the river will flood

A sufficient condition for a network failure is that the central switch goes down

The avocados are ripe only if they are dark and soft

A good diet is a necessary condition for a healthy cat

1.1.4 Satifiability, Tautology, Contradiction

A proposition is satisfiable if it is true for *at least one* combination of boolean values.

ℹ You don't need a whole truth table for this, just look for one!

A Boolean Satisfiability Problem (SAT) is checking for satisfiability in a propositional logic formula.

A Tautology is a proposition that is always true
$_{\text{ex}}$ $A \vee \neg A$

A Contradiction is a proposition that is always false.
$_{\text{ex}}$ $A \wedge \neg A$

1.2 Truth Tables

1.2.1 Basics

Truth Tables are used for determining all the possible outputs of a complex compound propostion.

The Columns Are for the prepositions, intermediate compound prepositions and the whole compound preposition.

ℹ The intrmt' prepositions are optional steps to make solving easier, use as needed.

The Rows Are to contain the different sets of possible truth values for each proposition. You will have 2^p rows where p is the number of propositions (then +1 for the header).

⚠ The connectives in a compound propositional logic problem follow an order of precedence (the PEMDAS of logic) in the following order;
$\neg$, $\wedge$, $\vee$, $\implies$, $\iff$

1.2.2 Connective Outputs

Negation

A	$\neg A$
T	F
F	T

And

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Or

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Implication

A	B	$A \implies B$
T	T	T
T	F	F
F	T	T
F	F	T

An implication is true when the hypothesis is false or the conclusion is true.

Biconditional

A	B	$A \iff B$
T	T	T
T	F	F
F	T	F
F	F	T

A Biconditional is true when the two prepositions have the same value.

Out of all these outputs, the most unintuitive is the 3rd implication output $(F, T \implies T)$. The easiest way to understand this output is with the proposition "If it is raining, then the ground is wet"; now say you step outside and it is not raining, but the ground is wet. The entire statement isn't false or incorrect, but the first part of it still has a false value.
The only way to make an implication false is when the hypothesis is true but the conclusion is false.

1.2.3 Examples

$A \implies B \iff B \implies A$

A	B	$A \implies B$	$B \implies A$	——
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

$A \wedge \neg B \implies \neg C$

A	B	C	$A \wedge \neg B$	——
T	T	T	F	T
T	T	F	F	T
T	F	T	T	F
T	F	F	T	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

ℹ *Remember, columns like $A \implies B$ are optional in-between steps to help solve each problem.*

1.2.4 Exercise: Finding Tautologies, Satisfiable & Contradicting Props'

Indicate whether each of the following is a tautology, satisfiable but not a tautology or a contradiction;

$A \implies B$

$A \implies A$

$A \implies \neg B \vee \neg C$

$A \vee B \implies B$

$(A \wedge B) \implies (A \vee B)$

$A \vee \neg A \implies B \wedge \neg B$

(Answers and explanations on the next page...)

ℹ *Notice how none of these rely on drawing out a whole truth table! Focus on trying to find a way to get each proposition to output true and a way to get it to output false!*

$A \implies B$
Satisfiable but not a tautology
Just knowing the properties of an implication you should know there's way to get true outputs and a false output.

$A \implies A$
Tautology
Only would be $T \implies T$ or $F \implies F$, both of which result in true.

$A \implies \neg B \vee \neg C$
Satisfiable but not a tautology
Instead of making a long unpleasant truth table, it's easiest to start by simply looking for one true and one false possible output.
We can make the left side true simply by making A false, since all that remains is an or statement we now have a true output.
We can just as easily find a false output for this proposition with $A = T$, $B = T(\neg B = F)$ to make the implication false, then we can just make $\neg C$ false to make the or output false.

$A \vee B \implies B$
Satisfiable but not a tautology
If we make B true then the biconditional will always be true regardless of A.
There is only one way to make an implication false, so if we can set up A and B to result in that false output, it won't be a tautology. If we make A true and B false it will make the implication false!

$(A \wedge B) \implies (A \vee B)$
Tautology
Remember the only way to make an implication false is if the hypothesis is true and the conclusion is false. There is absolutely no way to do this because of the and/or setup!

$A \vee \neg A \implies B \wedge \neg B$
Contradiction
The left side is always true and the right side is always false. So the result of the implication is always false!

1.3 Equivalence

1.3.1 Introduction to Equivalence

Two (compound) propositions P and Q are **logically equivalent** when their truth values always match (Meaning they'll have the same truth table!).
Equivalence is denoted by $P \equiv Q$.

Equivalence relates heavily to the concept of Tautologies;

P and Q are equivalent when $P \iff Q$ is a tautology.

A proposition P is a tautology iff (if and only if) it is equivalent to T (true), i.e $P \equiv T$

Examples

1
Given the implication $A \implies B$, are the following equivalent?

The contrapositive: $\neg B \implies \neg A$

The converse: $B \implies A$

A	B	$A \implies B$	$\neg B \implies \neg A$	$B \implies A$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

Looking at the table we can see that $A \implies B$ and $\neg B \implies \neg A$ are equivalent.

Now, what about $\neg A \vee B$?

A	B	$A \implies B$	$\neg A \vee B$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Yep! $\neg A \vee B \equiv A \implies B$.

This is actually one of the equivalence laws you'll see in the next section!

Understanding equivalent boolean expressions is very important in computer science (for code) and chip design (for logic gates). Consider the code below;

```
if(x > 0 || (x <= 0 && y > 100))
```

Lets see if we can change this expression to something equivalent but simplified.
Let A be `x > 0` and let B be `y > 100`
Now we can compare the truth values of $A \vee (\neg A \wedge B)$ and $A \vee B$.

A	B	$A \vee (\neg A \wedge B)$	$A \vee B$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

They're equivalent! We can reduce the if statement's expression to simply;

```
if(x > 0 || y > 100)
```

1.3.2 Equivalence Laws

For more complex propositions it is impractical to create a set of massive truth tables to check for equivalence. So instead we utilize equivalence laws to directly prove equivalence.

Nine Equivalence Laws;

Many of these are pretty self-explanatory

Double Negation Law: $\neg(\neg A) \equiv A$

Identity Laws: $A \wedge T \equiv A \qquad A \vee F \equiv A$

Domination Laws: $A \vee T \equiv T \qquad A \wedge F \equiv F$

Commutative Laws: $A \wedge B \equiv B \wedge A \qquad A \vee B \equiv B \vee A$

Associative Laws: $(A \wedge B) \wedge C \equiv A \wedge (B \wedge C) \qquad (A \vee B) \vee C \equiv A \vee (B \vee C)$

Idempotent Laws: $A \wedge A \equiv A \qquad A \vee A \equiv A$

Distributive Laws: $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C) \qquad A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$

Very similar to the algebriac distributive law

DeMorgan's Laws: $\neg(A \wedge B) \equiv \neg A \vee \neg B \qquad \neg(A \vee B) \equiv \neg A \wedge \neg B$

Implication Laws: $A \implies B \equiv \neg B \implies \neg A \equiv \neg A \vee B$

TODO: Maybe refomat this as a table so its a bit easier to quickly reference?

Examples

1 Prove $A \vee (\neg A \wedge B) \equiv A \vee B$

$$\begin{aligned}
A \vee (\neg A \wedge B) &\equiv (A \vee \neg A) \wedge (A \vee B) && \text{(Distributive)} \\
&\equiv T \wedge (A \vee B) \\
&\equiv A \vee B && \text{(Identity)}
\end{aligned}$$

> *In solving these, the goal should be to reduce the # of letters in the propositions. Focus on the side of an equivalence with more going on and try to reduce it down since the more complex proposition will have more oppurtunities to utilize the different equivalence laws.*

2 Simplify $A \wedge \neg(A \wedge B)$

$$\begin{aligned}
A \wedge \neg(A \wedge B) &\equiv A \wedge (\neg A \vee \neg B) && \text{(DeMorgan's)} \\
&\equiv (A \wedge \neg A) \vee (A \wedge \neg B) && \text{(Distributive)} \\
&\equiv F \vee (A \wedge \neg B) \\
&\equiv A \wedge \neg B && \text{(Identity)}
\end{aligned}$$

> *You don't have to name the laws you're using in the homework, the simple $\equiv$ down the middle format for each step is fine.*

3 Show that $(A \wedge B) \implies (A \vee B)$ is a tautology.

$$\begin{aligned}
(A \wedge B) \implies (A \vee B) &\equiv \neg(A \wedge B) \vee (A \vee B) && \text{(Implication)} \\
&\equiv (\neg A \vee \neg B) \vee (A \vee B) && \text{(DeMorgan's)} \\
&\equiv \neg A \vee \neg B \vee A \vee B && \text{(Associative)} \\
&\equiv \neg A \vee A \vee \neg B \vee B && \text{(Commutative)} \\
&\equiv (\neg A \vee A) \vee (\neg B \vee B) && \text{(Associative)} \\
&\equiv T \vee T \\
&\equiv T && \text{(Idempotent)}
\end{aligned}$$

1.4 Arguments

An **argument** is a sequence of propositions in which the conjunction of the initial propositions implies the final proposition
An argument can be represented as;
$P_1 \wedge P_2 \wedge P_3 ... \wedge P_n \implies Q$

Examples

If George Washington was the first president of the United States, then John Adams was the first vice president. George Washington was the first president of the United States. Therefore John Adams was the first vice president.

→ Let A be "George Washington was the first president of the United States."

→ Let B be "John Adams was the first vice president."

→ $(A \implies B) \wedge A \implies B$

If Martina is the author of the book, then the book is fiction. But the book is nonfiction. Therefore Martina is not the author.

→ Let A be "Martina is the author of the book."

→ Let B be "The book is fiction."

→ $(A \implies B) \wedge \neg B \implies \neg A$

The dog has a shiny coat and loves to bark. Consequently, the dog loves to bark.

→ Let A be "The dog has a shiny coat."

→ Let B be "The dog loves to bark."

→ $A \wedge B \implies B$

1.4.1 Valid Arguments / Inference Rules

An argument is **valid** if and only if **its conclusion is never false while its premises are true.**

We can't use a truth table to validate an argument since it only shows the truth values for the statement as a whole, instead we need to use new ***Inference Rules***

Inference Rules

$$\begin{array}{l} \quad P \\ \quad \underline{P \longrightarrow Q} \\ \therefore Q \end{array}$$

Example: If George Washington...

$$\begin{array}{l} \quad P \longrightarrow Q \\ \quad \underline{\neg Q} \\ \therefore \neg P \end{array}$$

Example: If Martina ...

$$\begin{array}{l} \quad P \\ \quad \underline{Q} \\ \therefore P \land Q \end{array}$$

Example: Paul is a good swimmer. Paul is a good runner. Therefore Paul is a good swimmer and a good runner.

$$\begin{array}{l} \quad \underline{P \land Q} \\ \therefore P \end{array}$$

$$\begin{array}{l} \quad \underline{P} \\ \therefore P \lor Q \end{array}$$

Each line of these rules are basically "if this prop is true and if that prop is true then the last prop is true"

Examples (finding conclusions)

- If the car was involved in the hit-and-run, then the paint would be chipped. But the paint is not chipped.
  - $\rightarrow$ "Car was involved in a hit-and-run" $\rightarrow P$
  - $\rightarrow$ "Paint would be chipped" $\rightarrow Q$
  - $\rightarrow$ "The paint is not chipped" $\rightarrow \neg Q$
  - $\rightarrow$ Conclusion: The car was not involved in a hit-and-run. From the second rule!
- If the bill was sent today, then you will be paid tomorrow. You will be paid tomorrow.
  - $\rightarrow$ Nothing can be concluded from this. ☺
- If the program is efficient$_P$, it executes quickly$_Q$. Either the program is efficient$_P$, or it has a bug$_R$. However, the program does not execute quickly$_{\neg Q}$.
  - $\rightarrow$ "If the program is efficient" $\rightarrow P$
  - $\rightarrow$ "it executes quickly" $\rightarrow Q$
  - $\rightarrow$ "it has a bug" $\rightarrow R$
  - $\rightarrow$ "the program does not execute quickly" $\rightarrow \neg Q$
  - $\rightarrow$ We start by knowing $P \implies Q$ and $P \vee R$ and $\neg Q$...
  - $\rightarrow$ $(P \implies Q)$ and $\neg Q$ can imply $\neg P$
  - $\rightarrow$ We need to transform $P \vee R$ to use it: $P \vee R \equiv \neg(\neg P) \vee R \equiv \neg P \implies R$
  - $\rightarrow$ $\neg P \implies R$ and $\neg P$ (the first implication we isolated) now implies R by the first inference rule.

1.4.2 Proving a Valid Argument

Assuming the premises are true, apply a sequence of premises and derivation rules, which include the equivalence laws and inference.

General Steps

1. Identify all the premises (might need some transformations).
2. Think backwards. Start from what you want and then seek supporting premises, current results, and necessary equivalence laws and inference rules, until you reach the given premises.
3. Write the proof sequence, where each step is either one premise or derived from previous step(s) using equivalence laws or inference rules.

Examples

[1]
Prove $(A \implies B) \wedge (\neg C \vee A) \wedge C \implies B$
This one is already in its standard form - so we just need to identify each part of the standard $P_1 \wedge P_2 \wedge P_3 ... \wedge P_n \implies Q$ form. At the end of this we want to prove that B is true.

$(A \implies B)$	$\wedge$	$(\neg C \vee A)$	$\wedge$	C	$\implies B$
$\downarrow$	$\wedge$	Implication Law $C \implies A$	$\wedge$	C	$\downarrow$
$(A \implies B)$	$\wedge$	first inference rule law A			$\downarrow$
		B also by first inf rule			$\implies B$

This is not usually how you would format these proofs, this table was to give you an idea of the actual process. The actual proof would look like the following;

1. $A \implies B$
2. $\neg C \vee A$
3. C
4. $C \implies A$ *(2, Implication)*
5. A *(3,4)*
6. B *(1,5)*

You need to put every step in a seperate (numbered) line, starting with each component of the argument and then the transformations you do with the reason given. You dont need to name the law used but you need to mention the steps you combined to acheive the next part.

2 Prove $A \land (B \implies C) \land ((A \land B) \implies (D \lor \neg C)) \land B \implies D$

ℹ *For this one focus on step 3 ($D \lor \neg C$) as your point to figure out this argument since its the only portion that has D in it.*

1. A
2. $B \implies C$
3. $(A \land B) \implies (D \lor \neg C)$
4. B
5. $A \land B$ (1,4)
6. $D \lor \land C$ (3,5)
7. $C \implies D$ (6, Commutative, Implication) - Communicative used to swap C, D

TODO: ADD MORE NOTES TO THIS ONE, I GENIUNELY AM NOT FOLLOWING THE PROCESS HERE