

Discrete Maths Notes

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1 Logic

1.1 Propositional Logic

1.1.1 Basics

A **proposition** is a statement that is either true or false.

Prepositions will be represented mathematically with capital letters A, B, C...

These prepositions are then are connected into more complex compound prepositions using *connectives*. Connectives are statements like "and, implies, if-then" and are represented mathematically with the symbols below.

❗ It's not always easy to determine if they're true/false.

Connectives			
Symbol	Name	English Term(s)	Reading
\wedge	AND	And, But, Also	A and B
\vee	OR	-	A or B
\implies	IMPLICATION	If A, Then B If A, then B A implies B A, therefore B A only if B B follows from A A is a sufficient condition for B B is a necessary condition for A	A implies B
\iff	BICONDITIONAL	If & only if A is necessary and sufficient for B	A if and only if B
\neg	NEGATION	Not...	Not A

❗ A Biconditional can also be thought of $(A \implies B) \wedge (B \implies A)$

Negation may sometimes be represented as A' or \overline{A}

1.1.2 Terminology

$A \wedge B$ - conjunction of conjuncts A and B

$A \vee B$ - disjunction of disjuncts A and B

$A \implies B$ - A is the hypothesis/antecedent and B is the conclusion/consequence

1.1.3 Examples

1 Compound Proposition

If all humans are mortal_{prp A} and all Greeks are human_{prp B}
then all Greeks are mortal_{prp C} can be represented as $A \wedge B \implies C$

2 Negation

Chocolate is sweet \rightarrow Chocolate is not sweet

Peter is tall and thin \rightarrow Peter is short or fat

The river is shallow or polluted \rightarrow The river is deep and polluted.

❗ Short and
fat would be
incorrect!

❗ Not shallow
or not pol-
luted would
be incorrect!

3 Implication: hypothesis and conclusion

If the rain continues then the river will flood

A sufficient condition for a network failure is that the central switch goes down

The avocados are ripe only if they are dark and soft

A good diet is a necessary condition for a healthy cat

1.1.4 Satisfiability, Tautology, Contradiction

A proposition is satisfiable if it is true for *at least one* combination of boolean values.

A Boolean Satisfiability Problem (SAT) is checking for satisfiability in a propositional logic formula.

❗ You don't
need a whole
truth table for
this, just look
for one!

A Tautology is a proposition that is always true

ex $A \vee \neg A$

A Contradiction is a proposition that is always false.

ex $A \wedge \neg A$

1.2 Truth Tables

1.2.1 Basics

Truth Tables are used for determining all the possible outputs of a complex compound proposition.

The Columns Are for the prepositions, intermediate compound prepositions and the whole compound proposition.

The Rows Are to contain the different sets of possible truth values for each proposition. You will have 2^p rows where p is the number of propositions (then +1 for the header).

ⓘ The intrmt³ prepositions are optional steps to make solving easier, use as needed.

▲ The connectives in a compound propositional logic problem follow an order of precedence (the PEMDAS of logic) in the following order;

\neg , \wedge , \vee , \implies , \iff

1.2.2 Connective Outputs

Negation	
A	$\neg A$
T	F
F	T

And		
A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Or		
A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Implication		
A	B	$A \implies B$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional		
A	B	$A \iff B$
T	T	T
T	F	F
F	T	F
F	F	T

An implication is true when the hypothesis is false or the conclusion is true.

A Biconditional is true when the two propositions have the same value.

Out of all these outputs, the most unintuitive is the 3rd implication output ($F, T \implies T$). The easiest way to understand this output is with the proposition “If it is raining, then the ground is wet”; now say you step outside and it is not raining, but the ground is wet. The entire statement isn’t false or incorrect, but the first part of it still has a false value. The only way to make an implication false is when the hypothesis is true but the conclusion is false.

1.2.3 Examples

$$A \implies B \iff B \implies A$$

A	B	$A \implies B$	$B \implies A$	—
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

$$A \wedge \neg B \implies \neg C$$

A	B	C	$A \wedge \neg B$	—
T	T	T	F	T
T	T	F	F	T
T	F	T	T	F
T	F	F	T	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

i Remember, columns like $A \implies B$ are optional in-between steps to help solve each problem.

1.2.4 Exercise: Finding Tautologies, Satisfiable & Contradicting Props'

Indicate whether each of the following is a tautology, satisfiable but not a tautology or a contradiction;

$$A \implies B$$

$$A \implies A$$

$$A \implies \neg B \vee \neg C$$

$$A \vee B \implies B$$

$$(A \wedge B) \implies (A \vee B)$$

$$A \vee \neg A \implies B \wedge \neg B$$

(Answers and explanations on the next page...)

❶ Notice how none of these rely on drawing out a whole truth table! Focus on trying to find a way to get each proposition to output true and a way to get it to output false!

$$A \implies B$$

Satisfiable but not a tautology

Just knowing the properties of an implication you should know there's way to get true outputs and a false output.

$$A \implies A$$

Tautology

Only would be $T \implies T$ or $F \implies F$, both of which result in true.

$$A \implies \neg B \vee \neg C$$

Satisfiable but not a tautology

Instead of making a long unpleasant truth table, it's easiest to start by simply looking for one true and one false possible output.

We can make the left side true simply by making A false, since all that remains is an or statement we now have a true output.

We can just as easily find a false output for this proposition with $A = T$, $B = T$ ($\neg B = F$) to make the implication false, then we can just make $\neg C$ false to make the or output false.

$$A \vee B \implies B$$

Satisfiable but not a tautology

If we make B true then the biconditional will always be true regardless of A.

There is only one way to make an implication false, so if we can set up A and B to result in that false output, it won't be a tautology. If we make A true and B false it will make the implication false!

$$(A \wedge B) \implies (A \vee B)$$

Tautology

Remember the only way to make an implication false is if the hypothesis is true and the conclusion is false. There is absolutely no way to do this because of the and/or setup!


$$A \vee \neg A \implies B \wedge \neg B$$

Contradiction

The left side is always true and the right side is always false. So the result of the implication is always false!

1.3 Equivalence

1.3.1 Introduction to Equivalence

 Two (compound) propositions P and Q are **logically equivalent** when their truth values always match (Meaning they'll have the same truth table!). Equivalence is denoted by $P \equiv Q$.

Equivalence relates heavily to the concept of Tautologies;

P and Q are equivalent when $P \iff Q$ is a tautology.

A proposition P is a tautology iff (if and only if) it is equivalent to T (true), i.e $P \equiv T$

Examples

¹

Given the implication $A \implies B$, are the following equivalent?

The contrapositive: $\neg B \implies \neg A$

The converse: $B \implies A$


A	B	$A \implies B$	$\neg B \implies \neg A$	$B \implies A$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

Looking at the table we can see that $A \implies B$ and $\neg B \implies \neg A$ are equivalent.

Now, what about $\neg A \vee B$?

A	B	$A \implies B$	$\neg A \vee B$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Yep! $\neg A \vee B \equiv A \implies B$.

 This is actually one of the equivalence laws you'll see in the next section!

Understanding equivalent boolean expressions is very important in computer science (for code) and chip design (for logic gates). Consider the code below;

```
if (x > 0 || (x <= 0 && y > 100))
```

Lets see if we can change this expression to something equivalent but simplified.

Let A be $x > 0$ and let B be $y > 100$

Now we can compare the truth values of $A \vee (\neg A \wedge B)$ and $A \vee B$.

A	B	$A \vee (\neg A \wedge B)$	$A \vee B$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

They're equivalent! We can reduce the if statement's expression to simply;

```
if (x > 0 || y > 100)
```


1.3.2 Equivalence Laws

For more complex propositions it is impractical to create a set of massive truth tables to check for equivalence. So instead we utilize equivalence laws to directly prove equivalence.

Nine Equivalence Laws;

Many of these are pretty self-explanatory

Double Negation Law: $\neg(\neg A) \equiv A$

Identity Laws: $A \wedge T \equiv A$ $A \vee F \equiv A$

Domination Laws: $A \vee T \equiv T$ $A \wedge F \equiv F$

Commutative Laws: $A \wedge B \equiv B \wedge A$ $A \vee B \equiv B \vee A$

Associative Laws: $(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$ $(A \vee B) \vee C \equiv A \vee (B \vee C)$

Idempotent Laws: $A \wedge A \equiv A$ $A \vee A \equiv A$

Distributive Laws: $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$

DeMorgan's Laws: $\neg(A \wedge B) \equiv \neg A \vee \neg B$ $\neg(A \vee B) \equiv \neg A \wedge \neg B$

Implication Laws: $A \implies B \equiv \neg B \implies \neg A \equiv \neg A \vee B$

ⓘ Very similar to the algebraic distributive law

TODO: Maybe reformat this as a table so its a bit easier to quickly reference?

Examples

₁ Prove $A \vee (\neg A \wedge B) \equiv A \vee B$

$$\begin{aligned} A \vee (\neg A \wedge B) &\equiv (A \vee \neg A) \wedge (A \vee B) && \text{(Distributive)} \\ &\equiv T \wedge (A \vee B) \\ &\equiv A \vee B && \text{(Identity)} \end{aligned}$$

i In solving these, the goal should be to reduce the # of letters in the propositions. Focus on the side of an equivalence with more going on and try to reduce it down since the more complex proposition will have more opportunities to utilize the different equivalence laws.

₂ Simplify $A \wedge \neg(A \wedge B)$

$$\begin{aligned} A \wedge \neg(A \wedge B) &\equiv A \wedge (\neg A \vee \neg B) && \text{(DeMorgan's)} \\ &\equiv (A \wedge \neg A) \vee (A \wedge \neg B) && \text{(Distributive)} \\ &\equiv F \vee (A \wedge \neg B) \\ &\equiv A \wedge \neg B && \text{(Identity)} \end{aligned}$$

i You don't have to name the laws you're using in the homework, the simple \equiv down the middle format for each step is fine.

₃ Show that $(A \wedge B) \implies (A \vee B)$ is a tautology.

$$\begin{aligned} (A \wedge B) \implies (A \vee B) &\equiv \neg(A \wedge B) \vee (A \vee B) && \text{(Implication)} \\ &\equiv (\neg A \vee \neg B) \vee (A \vee B) && \text{(DeMorgan's)} \\ &\equiv \neg A \vee \neg B \vee A \vee B && \text{(Associative)} \\ &\equiv \neg A \vee A \vee \neg B \vee B && \text{(Commutative)} \\ &\equiv (\neg A \vee A) \vee (\neg B \vee B) && \text{(Associative)} \\ &\equiv T \vee T \\ &\equiv T && \text{(Idempotent)} \end{aligned}$$