1 Excellence

1.1 Quality and pertinence of the project's research and innovation objectives (and the extent to which they are ambitious, and go beyond the state of the art)

Motivation

There are fundamental differences between the standard scientific method and the method of mathematical research. Generally speaking, the scientific method is based on observations of natural phenomena and measurement of physical quantities. These are then used to pose hypotheses, which are then tested against cleverly designed experiments. In this way, many false hypotheses can be quickly rejected, while those that survive these tests are eventually recognised as laws of nature.

On the other hand, in theoretical mathematics we are often faced with a lack of initial data upon which we can formulate hypotheses. The process of mathematical research thus usually starts with an abstract idea, sometimes based on intuition and hopefully a handful of examples. These ideas eventually evolve to conjectures. However, unlike in natural sciences, in many branches of mathematics the notion of experiment is non existent making it very difficult to test these conjectures against experimental data. As we all know too well, this often results in hours spent on attempts of proving false conjectures, hours that could be spent on proving theorems instead.

Mathematicians are so used to this way of conducting research that the possibility of using experimental data is often overlooked even when available or not too difficult to obtain. In the last few years, with the development of computers, running experiments and computing large datasets of mathematical objects has opened up new opportunities in mathematical research. Very broadly speaking, the aim of the proposed project PolyData is to use this opportunity in an area of mathematics where such data-driven approach is feasible.

The main objective of PolyData is to develop datasets and computational tools for highly symmetrical discrete objects, in particular polytopes, maps and maniplexes. More precisely, the main research goals for PolyData are *building complete, accessible, useful and reliable datasets of abstract polytopes* (see RO1 and RO2) and *developing new theoretical tools for constructing abstract polytopes, maps and maniplexes* (see RO3). Finally, we shall encourage the mathematical community to adopt PolyData as a standard for creating, storing and presenting datasets of combinatorial objects.

The nature of PolyData involves a two-way flow of knowledge: With the aim of generating new datasets of highly symmetric polytopes and related objects, we will develop new theoretical results and methods, enabling us to devise new algorithms and combinatorial representation for constructing highly symmetric abstract polytopes. The algorithms will then be carefully implemented and executed. The obtained datasets will then be analysed with the aim of finding interesting patterns, suggesting new conjectures and proposing new directions for further research.

As a result of the sucess of our research we will develop a family of repositories, datasets and software packages that allow both researchers and students experiment, identify patterns and formulate conjectures on abstract polytopes. To achieve this succes, the project will bring together a reseracher with expertice on construction of abstract polytopes with prescribed combinatorial conditions with a supervisor with large experience on datases of discrete objects, in particular of highly symmetric graphs and maps.

Research objectives

Abstract polytopes are combinatorial objects that generalise (the face lattice) of convex polytopes. Enumeration and classification of highly symmetric convex polytopes goes all the way to the Greeks and the classical problem of classifying what today we know as *Platonic Solids* to the beginning of last century when the classification of higher dimensional convex polytopes was achieved. By considering combinatorial objects and hence removing the geometric constraints, often imposed by the ambiance space, opens the possibilities for abstract polytopes and turns the complete classification problem into a series of enumerating families with particular characteristic.

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first, 2-orbit, then chiral as the most prominent case

The degree of symmetry of a polytope can be measured by the number of orbits of certain substructures called *flags*. *Regular polytopes* have 1 flag-orbit (1-orbit polytopes). This notion of regularity coincides with the classical geometrical one. Informally speaking, *chiral polytopes* are those that admit maximal degree of combinatorial rotations but that do not admit mirror reflections. As I will described in detail below, the classes of regular and chiral polytopes are by far the most studied symmetry classes of abstract polytopes. This has led to the constructions of some datasets of highly symmetric polytopes, which shall be described in detail below.

Chiral polytopes are just one of $2^n - 1$ possible symmetry type class of 2-orbit n-polytopes. Classical examples of some of the other classes of the 2-orbit polytopes are known. However, the general problem of determining if

Dataset	Rank 3	Rank 4	Rank $\geqslant 5$
Hartley - Regular	64.55%	31.61%	3.84%
Hartley - Chiral	85.71%	14.29%	0.00%
Conder - Regular	61.51%	34.70%	3.79%
Conder - Chiral	87.01%	12.87%	0.12%
Leemans - Regular	95.35%	4.37%	0.30%
Leemans - Chiral	87.82%	11.97%	0.21%

Table 1: Percentages of examples according to rank

for every pair (n, T) with $n > 3$ and T a 2-orbit symmetry type exists an n-polytope of symmetry type T remains open. Some examples of maniplexes (a slightly more general class of objects) were build very recently,[1] but whether or not they are polytopal remains unknown. Constructing and classifying n-polytopes with k-orbits and given symmetry type is still a widely open problem[2] and it is one of the main motivations for the development of PolyData .

These existing datasets of polytopes suffer of the following restrictions

(i) They are mainly focused on regular or chiral polytopes.
(ii) The size of the examples is very restrictive.
(iii) They often exhibit numerous examples of rank 3 but the amount of examples of rank higher than 4 drops dramatically.
(iv) They are not very user-friendly, either because they exist only as raw data or because the are specific-programming language oriented.

In Table 1 we show the proportion of examples according to ranks on the existing datasets of polytopes with examples of rank higher than 3.

There are two obvious gaps that need to be pushed forward, not necessarily in an independent way, on the process of building new datasets of abstract polytopes: finding examples of higher ranks and building sets that consider different types of symmetries (besides chiral or regular). With the emerging development of theoretical results for less symmetric polytopes and the need to identify patterns and to find new constructions to attack the numerous open problems related to the existence of polytopes, it is clear that building new datasets of polytopes that overturn the restrictions mentioned above would not only be beneficial but it is almost necessary.

The problematic expressed above outlines our first objective.

RO1 *Extend the existing and build new datasets of abstract polytopes and related structures with particular focus on*

(i) Building examples on ranks higher than 3
(ii) Exploring different symmetry types.

RO1 is of course very general but also very ambitious and should be interpreted as the general research line of PolyData . Any contribution to this objective is a good way of measuring the global success of PolyData .

Another pressing issue to address is that the existent datasets of highly symmetric maps and polytopes are not only limited in the sense discussed above, they have not been exploded to its full capacity. One of the reasons behind it is that the information in most of these data sets is not very user-friendly. Even the small amount of existing data sets have been developed by several people, mostly in an independent way, using different notation and different computer algebra systems. Moreover, many of these datasets exist only as raw text which is not always easy to consult.

RO2 *Collect, unify and make the existing data sets user-friendly and in a FAIR-ly way so that PolyData eventually becomes the standard to-go when consulting or publishing new data on highly symmetric polytopes by*

[1] D. Pellicer, P. Potočnik, and M. Toledo. "An existence result on two-orbit maniplexes". In: *J. Combin. Theory Ser. A* 166 (Aug. 2019), pp. 226–253.
[2] G. Cunningham and D. Pellicer. "Open problems on k-orbit polytopes". In: *Discrete Math.* 341.6 (2018), pp. 1645–1661.

(i) *Creating and developing a unique data set from the existent ones that unifies notation and identification of the data and make it available so that other researches could experiment and eventually contribute to* PolyData .
(ii) *Building a web-based interface to our data set for easy and quick consultation.*
(iii) *Developing this data set in a way that can be easily implemented in some of the standards computer algebra systems such as* MAGMA , *GAP and* `SageMath` .
(iv) *Writing appropriate documentation so that* PolyData *eventually becomes available for others to contribute.*

RO2 is very concrete and very easy to verify. It very easily show progress on PolyData while at the same time sets a good start point to our ambitious general objective. The current state-of-the-art allows us to start with from the very beginning of PolyData

Many of the first data sets of abstract polytopes are based on the (small) size of the objects. Either by taking advantage of previously computed objects (such as the library `SmallGrp`) or by using computational routines that, because of their own nature are limited by the size of the input (such as the `LowIndexNormalSubgroups`). However, it has been shown that the size of the smallest regular polytope of rank n grows exponentially with n[3] , while the size of the smallest chiral n-polytope is at least of factorial growth with respect to n.[4] This explains why the amount of examples of higher rank polytopes drops dramatically on the current available datasets.

The lack of not only datasets but also theoretical constructions of certain symmetry types for polytopes motivates the following objective.

RO3 *Develop new constructions of highly symmetric polytopes. In particular, focus on the constructions of abstract regular polytopes with given symmetry type to eventually build the corresponding data sets.*

RO3 by itself represents an extremely ambitious and it goes beyond any two year project. Any theoretical contribution is already of great interest to the community. However, we strongly believe that in order to really cause an impact on the workflow of research on abstract polytopes all theoretical contributions should be accompanied by its computational analogue. As mentioned before, this RO should go parallel to RO1, meaning that those theoretical construction should allow us to build new datasets and by exploring those datasets we should be able to identify pattern, formulate conjectures and develop new theoretical constructions.

Of course, PolyData is aimed to become a long term and eventually permanent resource for the community doing research on symmetries of maps and abstract polytopes. This will not be achievable without the involvement of such community.

RO4 *Encourage and motivate both well-established and young researchers to use and contribute to* PolyData *so that it eventually becomes the standard way to explore, experiment and publish data sets, routines and computational tools for the development of the research of abstract polytopes.*

RO4 is very ambitious and we acknowledge that it depends on the community more that on ourselves, but we strongly believe that our approach and the the current status of the could fit together to fill a gap that has been present for many years now. We shall achieve this objective by actively consulting an *external committee* of leading experts on the community.

State of the art

Enumeration and classification of mathematical objects is a natural way of conducting research. The discrete nature of combinatorial objects turn them into natural candidates to not only classify families of interesting objects but to enumerate and explicitly list the elements of such families. This research approach has resulted in the development of interesting data sets of combinatorial objects. Highly symmetric graphs is arguably the most studied family of combinatorial objects from the approach of building datasets of objects. It is believed that empirical study of symmetric graphs of small valence started in 1930s, when R.M. Foster began collecting examples of interesting graphs that could serve as models for electrical networks.[5]

Of course, this area of research has taken advantage of the development and improvement of computational power but the theoretical research goes back to Tutte and his work on classifying 3-valent arc-transitive graphs[6,7]

[3] M. Conder. "The smallest regular polytopes of given rank". en. In: *Adv Math* 236 (Mar. 2013), pp. 92–110.
[4] G. Cunningham. "Non-flat regular polytopes and restrictions on chiral polytopes". In: *Electron. J. Combin.* 24.3 (2017), Paper 3.59, 14.
[5] R. M. Foster. "Geometrical Circuits of Electrical Networks". In: *Transactions of the American Institute of Electrical Engineers* 51.2 (June 1932), pp. 309–317.
[6] W. T. Tutte. "A family of cubical graphs". In: *Proc. Cambridge Philos. Soc.* 43 (1947), pp. 459–474.
[7] W. T. Tutte. "On the symmetry of cubic graphs". In: *Canadian J Math* 11 (1959), pp. 621–624.

. The development of the theory, together with more powerful computers, resulted in a breakthrough of datasets of highly symmetric graphs constructions. Using the classification of automorphism groups of 3-valent arc-transitive[8] and bitransitive[9] graphs, together with new methods for finding normal subgroups of finite index in a finitely presented group allowed a construction of complete list of all 3-valent arc-transitive graphs[10,11] of order up to 10000 vertices, and a list of all 3-valent bitransitive graph on up to 768 vertices .[12] Based on their deep theoretical result on the order of automorphism groups,[13] Spiga, Verret and Potočnik compiled a complete list[14] of all trivalent vertex-transitive graphs of order at most 1280. Very recently, using the database of vertex-transitive groups of small degree, Conder and Verret have compiled a complete list of all edge-transitive graphs (of arbitrary valence) up to order 63 ,[15] while Holt and Royle have extended their census of all vertex-transitive graphs up to order 48.[16]

The classification and enumeration of groups has been also an intriguing problem since the beginning of theory. In 1854 Cayley[17] introduced the axiomatic definition of a group and enumerated the groups of order up to 6. Of course this is just the first step in what became an active research in both, theoretical mathematics[18] , as well as a motivation to develop computation tools such as the library `SmallGrp`[19] of small groups of GAP . In fact, one of the principal motivators on the study of symmetries of discrete objects the the *classification of Finite Simple Groups*. It turns out that many of the so-called sporadic simple groups can be understood as symmetry groups of discrete objects. This classification eventually derived in the construction of the Atlas of Finite Groups .[20]

The problem of enumerating and classifying regular polyhedra in the Euclidean space is as old as formal mathematics themselves. The enumeration and classification of the five Platonic Solids is one the most antique classification problems. For many years it was considered a complete classification problem (and it was) but later it was shown that by relaxing geometrical constraints we could generalise platonic solids to stellated polyhedra,[21,22,23] *infinite skew polyhedra*[24] and finally to *Grünbaum-Dress polyhedra*,[25,26,27] which is what today is accepted as the complete classification of regular polyhedra in the Euclidean space.[28]

The problem of classifying regular polyhedra shows how by relaxing geometrical conditions one can open the door to new objects. In fact, B. Grünbaum was one of the firsts that formally treated geometrical polyhedra-

[8] D. Ž. Djoković and G. L. Miller. "Regular groups of automorphisms of cubic graphs". In: *J. Combin. Theory Ser. B* 29.2 (1980), pp. 195–230.

[9] D. M. Goldschmidt. "Automorphisms of trivalent graphs". In: *Ann. of Math. (2)* 111.2 (1980), pp. 377–406.

[10] M. Conder and P. Dobcsányi. "Trivalent symmetric graphs on up to 768 vertices". In: *J. Combin. Math. Combin. Comput.* 40 (2002), pp. 41–63.

[11] M. Conder. *Trivalent (cubic) symmetric graphs on up to 10000 vertices*. https://www.math.auckland.ac.nz/~conder/symmcubic10000list.txt, Accessed online January 7th 2022.

[12] M. Conder et al. "A census of semisymmetric cubic graphs on up to 768 vertices". In: *J. Algebraic Combin.* 23.3 (2006), pp. 255–294.

[13] P. Potočnik, P. Spiga, and G. Verret. "Bounding the order of the vertex-stabiliser in 3-valent vertex-transitive and 4-valent arc-transitive graphs". en. In: *J. Combin. Theory Ser. B* 111 (Mar. 2015), pp. 148–180.

[14] P. Potočnik, P. Spiga, and G. Verret. "A census of 4-valent half-arc-transitive graphs and arc-transitive digraphs of valence two". In: *Ars Math. Contemp.* 8.1 (2015), pp. 133–148.

[15] M. D. E. Conder and G. Verret. "Edge-transitive graphs of small order and the answer to a 1967 question by Folkman". In: *Algebr. Comb.* 2.6 (2019), pp. 1275–1284.

[16] D. Holt and G. Royle. "A census of small transitive groups and vertex-transitive graphs". In: *J. Symbolic Comput.* 101 (2020), pp. 51–60.

[17] A. Cayley. "On the theory of groups, as depending on the symbolic equation $\theta^n = 1$". In: *Philos Mag* 7.42 (Jan. 1854), pp. 40–47.

[18] S. R. Blackburn, P. M. Neumann, and G. Venkataraman. *Enumeration of Finite Groups*. Cambridge Tracts in Mathematics. Cambridge: Cambridge University Press, 2007.

[19] H. U. Besche, B. Eick, and E. A. O'Brien. "The groups of order at most 2000". In: *Electronic Research Announcements of the American Mathematical Society* 7 (2001), pp. 1–4.

[20] J. H. Conway. *Atlas of Finite Groups. Maximal Subgroups and Ordinary Characters for Simple Groups*. Oxford University Press, USA, 1986, p. 284.

[21] J. Kepler. *Harmonia Mundi in Opera Omnia, Editor: Frisch CH, Heyder & Zimmer, Frankfurt, Vol.* 1864.

[22] L. Poinsot. *Mémoire sur les polygones et sur les polyèdres*. 1810.

[23] A. Cauchy. "AL Cauchy, Recherche sur les polydres-premier mémoire". In: *Journal de l'Ecole Polytechnique* 9 (1813), p. 6686.

[24] H. S. M. Coxeter. "Regular Skew Polyhedra in Three and Four Dimension, and their Topological Analogues". In: *Proc. London Math. Soc.* S2-43.1 (1937), p. 33.

[25] B. Grünbaum. "Regular polyhedra—old and new". In: *Aequationes Math.* 16.1-2 (1977), pp. 1–20.

[26] A. W. M. Dress. "A combinatorial theory of Grünbaum's new regular polyhedra. I. Grünbaum's new regular polyhedra and their automorphism group". In: *Aequationes Math.* 23.2-3 (1981), pp. 252–265.

[27] A. W. M. Dress. "A combinatorial theory of Grünbaum's new regular polyhedra. II. Complete enumeration". In: *Aequationes Math.* 29.2-3 (1985), pp. 222–243.

[28] P. McMullen and E. Schulte. "Regular polytopes in ordinary space". In: *Discrete Comput. Geom.* 17.4 (1997). Dedicated to Jörg M. Wills, pp. 449–478.

like objects as purely combinatorial objects by introducing the notion of the notion *polystroma*.[29] This notion eventually evolved to what we know today as *abstract polytopes*, introduced in the early 80's[30,31,32] .

The theory of highly symmetric abstract polytopes nourishes from several branches of mathematics, including group theory, topology and geometry. Coxeter is also attributed to classify the groups generated by hyperplane reflections, leading to what we today know as *Coxeter groups*. They of course, appear as the symmetry groups of regular polytopes and tessellations of the Euclidean and Hyperbolic spaces,[33] but they have made their way to Tits geometries,[34] computational Lie group theory, Hecke algebras,[35] just to mention some.

In 1978 G. Jones and D. Singerman published his classical manuscript[36] which settle the necessary theory to identify maps on orientable surfaces with what in modern terminology we called its *monodromy group*. The ideas behind this paper show important equivalences between topological maps (embedding of graphs on orientable surfaces), certain quotients triangular groups (Coxeter groups of rank 3), maps on Riemann surfaces and certain permutations on the darts of the map. These equivalences are a combinatorial/discrete version of the classical Uniformization theorem[37] for Riemann surfaces. The work of Jones and Singerman was an important contribution on the theory of discrete group actions on Riemann surfaces and it was eventually connected the theory Grothendieck's *Dessins d'enfant*.[38] Some other combinatorial equivalences of maps on surfaces were also explored by Tutte,[39] Vince[40] and Wilson.[41]

Abstract polytopes include many of the object mentioned above. Regular abstract polytopes, those with larger degree of symmetry are by far the most studied class of abstract polytopes. Most of this early theory can be found in the very dense and comprehensive monograph written by Schulte a McMullen.[42] Of our particular interest is the problem of building regular polytopes, for which numerous publications exists. We should mention that there exist universal constructions,[43,44] constructions prescribing local combinatorics[45,46,47] and constructions fixing interesting families of groups as automorphism groups[48,49,50,51] .

The second most studied symmetry class of polytopes is that of *chiral polytopes*. Informally speaking, a chiral polytope is a polytope having full degree of (combinatorial) rotational symmetry without having (combinatorial) reflections. They were introduced by Schulte and Weiss in 1990[52] Chiral polytopes were introduced as a natural

[29]B. Grünbaum. "Regularity of graphs, complexes and designs". In: *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*. Vol. 260. Colloq. Internat. CNRS. CNRS, Paris, 1978, pp. 191–197.
[30]E. Schulte. "Reguläre Inzidenzkomplexe". PhD thesis. University of Dortmund, 1980.
[31]L. Danzer and E. Schulte. "Reguläre Inzidenzkomplexe. I". in: *Geom. Dedicata* 13.3 (1982), pp. 295–308.
[32]E. Schulte. "Reguläre Inzidenzkomplexe. II, III". in: *Geom. Dedicata* 14.1 (1983), pp. 33–56, 57–79.
[33]J. E. Humphreys. *Reflection Groups and Coxeter Groups*. Vol. 29. Cambridge Studies in Advanced Mathematics. Cambridge University Press, June 1990, pp. xii+204.
[34]J. Tits. *Buildings of spherical type and finite BN-pairs*. Lecture Notes in Mathematics, Vol. 386. Springer-Verlag, Berlin-New York, 1974, pp. x+299.
[35]A. M. Cohen. "Coxeter Groups and three Related Topics". en. In: ed. by A. Barlotti et al. NATO ASI Series. Dordrecht: Springer Netherlands, 1991, pp. 235–278.
[36]G. A. Jones and D. Singerman. "Theory of maps on orientable surfaces". In: *Proc. London Math. Soc. (3)* 37.2 (1978), pp. 273–307.
[37]W. Abikoff. "The Uniformization Theorem". In: *The American Mathematical Monthly* 88.8 (Oct. 1981), pp. 574–592.
[38]G. A. Jones and J. Wolfart. *Dessins d'Enfants on Riemann Surfaces*. Springer London, Limited, 2016.
[39]W. T. Tutte. "What is a map?" In: *New directions in the theory of graphs (Proc. Third Ann Arbor Conf., Univ. Michigan, Ann Arbor, Mich., 1971)*. 1973, pp. 309–325.
[40]A. Vince. "Combinatorial maps". In: *Journal of Combinatorial Theory. Series B* 34.1 (1983), pp. 1–21.
[41]S. Wilson. "Maniplexes: Part 1: maps, polytopes, symmetry and operators". In: *Symmetry* 4.2 (2012), pp. 265–275.
[42]P. McMullen and E. Schulte. *Abstract regular polytopes*. Vol. 92. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2002, pp. xiv+551.
[43]E. Schulte. "On arranging regular incidence-complexes as faces of higher-dimensional ones". In: *European J. Combin.* 4.4 (1983), pp. 375–384.
[44]E. Schulte. "Extensions of regular complexes". In: *Finite geometries (Winnipeg, Man., 1984)*. Vol. 103. Lecture Notes in Pure and Appl. Math. Dekker, New York, 1985, pp. 289–305.
[45]L. Danzer. "Regular incidence-complexes and dimensionally unbounded sequences of such. I". in: *Convexity and graph theory (Jerusalem, 1981)*. Vol. 87. North-Holland Math. Stud. North-Holland, Amsterdam, 1984, pp. 115–127.
[46]D. Pellicer. "Extensions of regular polytopes with preassigned Schläfli symbol". In: *J. Combin. Theory Ser. A* 116.2 (2009), pp. 303–313.
[47]D. Pellicer. "Extensions of dually bipartite regular polytopes". In: *Discrete Math.* 310.12 (2010), pp. 1702–1707.
[48]P. J. Cameron et al. "Highest rank of a polytope for A_n". In: *Proc. Lond. Math. Soc. (3)* 115.1 (2017), pp. 135–176.
[49]M. E. Fernandes and D. Leemans. "C-groups of high rank for the symmetric groups". In: *Journal of Algebra* 508 (2018), pp. 196–218.
[50]D. Leemans, J. Moerenhout, and E. O'Reilly-Regueiro. "Projective linear groups as automorphism groups of chiral polytopes". In: *Journal of Geometry* 108.2 (2017), pp. 675–702.
[51]D. Pellicer. "CPR graphs and regular polytopes". In: *European J. Combin.* 29.1 (2008), pp. 59–71.
[52]E. Schulte and A. I. Weiss. "Chiral polytopes". In: *Applied geometry and discrete mathematics*. Vol. 4. DIMACS Ser. Discrete Math. Theoret. Comput. Sci. Amer. Math. Soc., Providence, RI, 1991, pp. 493–516.

generalization of *chiral maps*, which have been part of the classical theory of maps from it begging and numerous examples exist[53,54,55] . However, the problem of constructing chiral polytopes of higher ranks has proved to be much harder to that of constructing regular polytopes. Some rank 4 examples were constructed as quotients of hyperbolic tilings[56,57,58,59] . A universal construction[60] was used to produce the first (infinite) example of a rank-5 chiral polytope. However, the first finite rank 5 polytopes were constructed by Conder et al. in 2008.[61] It was until 2010 that Pellicer showed[62] the existence of chiral polytopes of rank n for ever $n \geqslant 4$; The result by Pellicer, although constructive, is not very practical. The size of his examples grow as a tower of exponential functions with length depending on n. Later on examples of new chiral polytopes have been constructed from previously known ones[63,64,65,66] .

The problem of classifying and enumerating highly symmetric polytopes has been part of the theory from the beginning. Even before the emergence of computers appeared the first Both, the classification of the 5 Platonic Solids to the enumeration of the 48 Grünbaum-Dress polyhedra in $\mathbb{E}^3$ depend on strong geometric restrictions. However, the combinatorial nature of abstract poltypes open the possibilities to, in principle, have numerous examples of abstract polytopes. These has lead to the construction of some datasets of highly symmetric polytopes, which we review below.

Conder - Regular orientable maps by genus Computed by M. Conder it originally contained all (3378) regular maps on orientable surfaces of genus 2 to 101 up to isomorphism an duality. It was later extended to include genus up to 301 for a total of 15824 maps. Computed with the help of `LowIndexNormalSubgroups`routine of MAGMA and published as raw text.

Conder - Regular non-orientable maps by genus Every map on a non-orientable surface admits an orientable double cover. Conder used this fact to originally compute all (862) non-orientable regular maps of genus 2 to 202 and then extended to genus up to 602 for a total of 3260 maps. Computed with the help of `LowIndexNormalSubgroups`routine of MAGMA and published as raw text.

Conder - Chiral maps by genus Census containing all (594) chiral maps on orientable surfaces of genus 2 to 101. This census was later extended to genus up to 301 for a total of 3870. Computed with the help of `LowIndexNormalSubgroups`routine of MAGMA and published as raw text.

Conder - Rotary maps by size This census contains all rotary (that is regular or chiral) maps whose rotation group has less than 2000 elements (equivalently, such that the map has less than 1000 edges). Computed with the help of `LowIndexNormalSubgroups`routine of MAGMA and published as raw text. There exist versions of this census containing only regular and only chiral maps.

Potocnik - Regular maps by size An improvement on Conder's census containing all $(255,980)$ regular maps whose automorphism group is of order less than $6,000$ for orientable maps and $3,000$ for non-orientable maps. Published as MAGMA files available to download with a CVS-file of precomputed information.

Potocnik - Chiral maps by size An analogous to the one above but for chiral maps. It contains a total of $122,092$ chiral maps whose automorphism group has order less than $6,000$.

Hartley - The Atlas of Small Regular Polytopes It was build using `SmallGrp` routine of GAP and contains all regular polytopes with at most 2000 flags, except those of size 1024 and 1536. It contains 9212 examples. They are presented in a nice web interface and the code is available to download.

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[53] H. S. M. Coxeter and W. O. J. Moser. *Generators and relations for discrete groups*. Third. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 14. Springer-Verlag, New York-Heidelberg, 1972, pp. ix+161.

[54] M. Conder and P. Dobcsányi. "Determination of all regular maps of small genus". In: *J. Combin. Theory Ser. B* 81.2 (2001), pp. 224–242.

[55] F. A. Sherk. "A family of regular maps of type $\{6,6\}$". In: *Canad. Math. Bull.* 5 (1962), pp. 13–20.

[56] B. Nostrand, E. Schulte, and A. I. Weiss. "Constructions of chiral polytopes". In: *Proceedings of the Twenty-fourth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1993)*. Vol. 97. 1993, pp. 165–170.

[57] E. Schulte and A. I. Weiss. "Chirality and projective linear groups". In: *Discrete Math.* 131.1-3 (1994), pp. 221–261.

[58] B. Nostrand. "Ring extensions and chiral polytopes". In: *Proceedings of the Twenty-fifth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1994)*. Vol. 102. 1994, pp. 147–153.

[59] B. Nostrand and E. Schulte. "Chiral polytopes from hyperbolic honeycombs". In: *Discrete Comput. Geom.* 13.1 (1995), pp. 17–39.

[60] E. Schulte and A. I. Weiss. "Free extensions of chiral polytopes". In: *Canad. J. Math.* 47.3 (1995), pp. 641–654.

[61] M. Conder, I. Hubard, and T. Pisanski. "Constructions for chiral polytopes". In: *J. Lond. Math. Soc. (2)* 77.1 (2008), pp. 115–129.

[62] D. Pellicer. "A construction of higher rank chiral polytopes". In: *Discrete Math.* 310.6-7 (2010), pp. 1222–1237.

[63] G. Cunningham and D. Pellicer. "Chiral extensions of chiral polytopes". In: *Discrete Math.* 330 (2014), pp. 51–60.

[64] M. D. E. Conder and W.-J. Zhang. "Abelian covers of chiral polytopes". In: *J. Algebra* 478 (2017), pp. 437–457.

[65] A. Montero. "Chiral extensions of toroids". PhD thesis. National University of Mexico, 2019.

[66] A. Montero. "On the Schläfli symbol of chiral extensions of polytopes". en. In: *Discrete Math* 344.11 (Nov. 2021), p. 112507.

Hartley - The Atlas of Small Chiral Polytopes Every chiral polytope admits a minimal regular cover. Hartley used this fact to compute the first atlas of chiral polytopes. This dataset consists of all chiral polytopes whose minimal regular cover belongs to the Atlas of Small Regular Polytopes. This gave a total of 48 chiral polytopes of rank 3 and 8 polytopes of rank 4.

Conder - Regular polytopes up to 2,000 flags A dataset containing, up to duality, all (5809) regular polytopes with at most 2000 flags (which is the same as the order of the automorphism group).

Conder - Chiral polytopes up to 2,000 flags A dataset containing, up to duality, all (839) chiral polytopes with at most 2000 flags (which is the twice the order of the automorphism group).

Leemans et al. - An Atlas of polytopes for small simple groups This is an ongoing atlas that contains regular polytopes whose automorphism group is an almost simple group. It currently contains 55, 575 regular polytopes. The atlas is presented on a website with downloadable data.

Leemans et al. - An Atlas of chiral polytopes for small simple groups It is the analogous to the one above but for chiral polytopes. It currently contains a total of 19, 964 polytopes.

include a paragraph about what is missing

1.2 Soundness of the proposed methodology

Overall methodology

The nature of the research on symmetries of abstract polytopes sits in the interplay between combinatorics, group theory, geometry and topology. In order to build new datasets of abstract polytopes we need computational efficient ways to compute and storage such objects and the other way around: from analysing new datasets we can identify patterns, formulate new conjectures and eventually develop new theoretical constructions.

In the following paragraphs we present some techniques and methodologies to build highly symmetric abstract polytopes and hence, datasets of such.

Abstract polytopes from their automorphism group. Automorphism groups of regular and chiral polytopes have been characterised .[67,68] Such characterisations have been the main tool used to build the existing datasets of polytopes. Very recently Mochán[69] fully the automorphism groups of an abstract polytope of arbitrary symmetry type and described how to build the polytope form the group. Using these novel results I shall explore known libraries of groups (such as `SmallGrp` of GAP) to build abstract polytopes from their automorphism group with different symemtry types (besides regular and chiral). This should be a first approach towards RO1.

Moreover, whenever we develop or group theoretical construction these results can be used to determine whether or not the given construction is an abstract poltype.

Schreier coset graphs and permutation groups. Schreier coset graphs are a classical tool to represent a permutation group. Large groups can be represented with relative small graphs; for example, the symmetric group S_n with $n!$ elements admits a representation on a graph with n vertices. This tool shall be used with two main purposes: as a way to overcome the restriction of size on the computational representation of objects during the development of RO1, but also as a theoretical tool to build new abstract polytopes while working on RO3.

It is important to remark that these graphs already have shown potential to improve known computational methods. In a joint manuscript[70] with A. I. Weiss, we build infinite families of regular hypertopes (a generalisation of abstract polytopes) using Schreier coset graphs, solving an open question from a previous manuscript[71] where the authors were not able to find a single example using traditional computational tools (in particular `LowIndexNormalSubgroups`form MAGMA).

Maniplexes. Maniplexes are a graph theoretical generalisation of an abstract polytopes. They were introduced by Wilson[72] in 2012 but just recently have proved to be useful on solving classical problems on abstract polytopes. This graph representation of polytopes is a natrual candidate to be our main way fo storing polytopes. By representing polytopes as graphs we are able to use graph-theoretical techniques such as *covers*, *voltage assignments* and *extensions* to build new polytopes and hence new datasets of such, which aligns with both RO1 and RO3.

[67] Danzer and Schulte, "Reguläre Inzidenzkomplexe. I".
[68] Schulte and Weiss, "Chiral polytopes".
[69] E. Mochán. "Abstract polytopes from their symmetry type graph". PhD thesis. National University of Mexico, 2021.
[70] A. Montero and A. I. Weiss. "Proper locally spherical hypertopes of hyperbolic type". en. In: *J Algebr Comb* (Oct. 2021).
[71] M. E. Fernandes, D. Leemans, and A. I. Weiss. "An Exploration of Locally Spherical Regular Hypertopes". In: *Discrete & Computational Geometry* (June 2020).
[72] Wilson, "Maniplexes: Part 1: maps, polytopes, symmetry and operators".

Moreover, by using maniplexes we will explore computational techniques of data management such as variants of the canonical labelling algorithm for graphs. This shall allow us to represent our data not only in an efficient way but also following the FAIR principles, which aligns with RO1, RO2 and RO4.

We shall emphasise that this approach has been barely used before and sits in a great place for our research. The supervisor is an expert on methods of building datasets of graphs and maniplexes are a stron bridge between graph-theoretical techniques and problems on abstract polytopes.

Extensions. A polytope $\mathcal{P}$ is an extension of a polytope $\mathcal{K}$ if all the facets of $\mathcal{P}$ are isomorphic to $\mathcal{K}$. Symmetry conditions on $\mathcal{K}$ imply symmetry restriction on $\mathcal{P}$. Most of my career I have worked with problems related to extension of polytopes and such techniques have proved to be useful to build new polytopes from previously known ones. The specific technique varies from graph theoretical (using maniplexes and coset graphs) to purely group theoretical (such as free products with amalgamations). These techniques naturally sit on our research and will be used to build new abstract polytopes from previously existing ones hence expanding and creating new datasets from the existing ones. Observe that the use of extensions is a natural way to overcome the need of natural examples and datasets of higher rank polytopes.

Operations. Very informally, an operation is a mapping O that assigns a polytope $O(\mathcal{K})$ to each polytope $\mathcal{K}$. Usually the symmetries of $O(\mathcal{K})$ are related to the symmetries of $\mathcal{K}$. Very recently in a joint work with Hubard and Mochán ,[73] we have developed a theoretical techniqueto build polytopes with prescribed symmetry type. We shall use this technique to build new abstract polytopes from previously known ones by using known families of operations. Implementing these operations in a computational way should be one of our first approaches to attack both RO1 and RO3 and should allow us to build the first known dataset of non regular or chiral abstract polytopes.

Open science practices and FAIR principles

The nature of our research does not forces us to keep it closed at any step of the process. Therefore, our research will be as open as necessary. We shall keep preliminary versions of our research manuscripts on ArXiv and submit the final versions to high quality open access journals. We shall use open-source software and keep the development of our own packages and datasets in a public git repository. Final version of our datasets will be publicly available on a website and in a FAIR-repository (such as Zenodo, MathDataHub).

As explained in Section 1.1, data management is still a young discipline in mathematics. Data production and storage is has not been a key part of the traditional development of theoretical mathematics. Moreover, although most theoretical mathematicians embrace the notion of *open science* the nature of the objects has restrict the use of FAIR principle for mathematical data. This problem has been previously addressed and the notion of *Deep FAIR* was introduced by Berčič et al.[74] as a particularization of FAIR principles to mathematical objects. We shall follow their line and at the same time use our project an its data-oriented nature to impulse *Deep FAIR* principles.

We shall make our datasets *findable* by publishing datasets into an open FAIR-repository (Zenodo, Math-DadaHub). We shall write appropriate metadata and documentation so that our datasets are not only accesible from the corresponding research papers (as it is usually the standard on mathematics) but from both a web-based interfaced and a downloadable source. This shall include corresponding software packages to manipulate and experiment with the datasets. Our data and metadata will be presented in web-based format as well as in platform independent formats (PDF, CVS). Full *interoperability* of software and mathematical objects is almost impossible but we shall at least make our datasets compatible with most popular computer algebra systems (GAP, `SageMath` , MAGMA). We shall follow community standards in order to create datasets and software packages as *reusable* as possible.

We shall finish this section by pointing out that there the nature of our research, being purely abstract, *does not involve any gender dimension or other diversity aspects.*

1.3 Quality of supervision, training, and knowledge transfer

Add some words about Primož

During the MCS fellowship in Ljubljana, I will be supervised by prof. Primož Potočnik. He is a leading expert in the area of algebraic graph theory. His work includes purely graph theoretical results, applications of group

[73] I. Hubard, E. Mochán, and A. Montero. “Voltage operations on maniplexes”. In preparation.

[74] K. Berčič, M. Kohlhase, and F. Rabe. “(Deep) FAIR mathematics”. en. In: *it - Information Technology* 62.1 (Feb. 2020), pp. 7–17.

theory in algebraic graph theory, as well as computational aspects in group theory and discrete mathematics. He has proved a number of deep theoretical results as well as developed many new computational approaches that enabled him (and his coworkers Pablo Spiga, Gabriel Verret) to significantly increase the scope of existing datasets of highly symmetric graphs. His achievements include the census of all cubic vertex-transitive graphs of order up to 1280, the census of all tetravalent arc-transitive graphs of order up to 640, the census of all rotary maps on at most 1.500 edges etc. These datasets are one of the most used and cited resources in algebraic graph theory.

**** Maybe the following paragraph could be moved to Section 3.2 ***** Prof. Potočnik has successfully supervised three PhD students (Gabriel Verret, Katja Berčič, Micael Toledo), currently supervises a postdoctoral student Alejandra Ramos and has informally advised several junior members of the department. He is the leader of of a long-term research programme funded by the Slovenian Research Agency (ARRS) has been the head of the PhD programme in mathematics at University of Ljubljana. He has strong research ties with a number of leading research groups in discrete mathematics (such as the groups at University of Auckland, University of Ottawa, Comenius University in Bratislava, Univeristá dagli Studi Milano etc) and regularly hosts top researchers in the area.

Working under his supervision as a MCS fellow, I hope to draw from his vast experience on theoretical and computational issues in algebraic graph theory. In particular, I hope acquire several new skills, including:

- ability of using advanced group theoretical methods relevant for the topic of the proposed research;
- proficiency in development of software packages (GAP, `SageMath` , MAGMA) for discrete mathematics;
- internet programming skills, such as construction of user-friendly internet platforms;
- understanding mathematical knowledge management (presenting and storing mathematical data under FAIR principles).

The training and joint research with the supervisor will consists of regular research meetings with the supervisor (at least 4 times a week), attending the meetings of the departments discrete mathematics group (once a week) and intensive research retreats (at least once per 2 months). Regular weekly meetings will consist of discussing the weekly plan (on Mondays) at least one long joint research session per week and a recap session at the end of the week.

**

While developing the proposed research I will receive training on two different *theoretical skills* that have been previously used to build datasets of graphs. We shall explore if and how they could be adapted to developing datasets of polytopes. On the other hand, to develop the computational part of the project I will acquire new skills on developing and managing datasets. This skills includes *development of software packages* (GAP, `SageMath` , MAGMA), *database management* (SQL) and *data publishing* (FAIR repositories, website building, CVS-files manipulation).

On the other hand I am a young researcher on the subject of abstract polytopes with expertise on problems related to building abstract polytopes and similar objects with prescribed combinatorial constraints or imposed symmetry conditions. FMF-UL is already a leading institution on discrete mathematics and some successful researchers on abstract polytopes and related objects have been formed in this institution in the past. However, there is no currently an active researcher working on this topic. My presence in FMF-UL will serve as fists steps to revive the area in an already strong environment on discrete mathematics.

The Faculty of Mathematics and Physics of the University of Ljubljana (FMF-UL) has a strong group on computer science who . In particular, Dr. Katja Berčič is an expert on *mathematical knowledge management* and will serve as a consultant for the computational part of the development of the project.

Tero: Say something about how they will interact, drop some names?

Mention external committee here?

2 Impact

2.1 Credibility of the measures to enhance the career perspectives and employability of the researcher and contribution to his/her skills development

The candidate's long-term goal is to obtain a permanent academic position. The interdisciplinary nature of the proposed research will train the candidate in two main sets of skills. On the one hand, with the guidance of the supervisor, the candidate will consolidate his young career as a theoretical mathematician working with highly symmetric objects. In particular, the candidate will gain training on *permutation groups* and *symmetries of graphs*. This will increase the candidate's presence on the community, and hence improve his chances to obtain a permanent academic job. On the other hand, the computational skills needed to build the proposed datasets will make the candidate a desirable researcher not only for pure math departments but also for computer science departments.

Moreover, the time that the candidate will spend in the *Faculty of Mathematics and Physics* of the University of Ljubljana will give the candidate opportunities to increase his teaching experience.

Finally, the programming and data management training obtained as a result of the proposed research will not only improve the candidate's chances to get not only an academic position but also a non academic job.

2.2 Suitability and quality of the measures to maximise expected outcomes and impacts, as set out in the dissemination and exploitation plan, including communication activities

As described in previous section, our project has two main branches: a purely theoretical one and a computational one. The activities planned to disseminate and communicate our activities can be naturally divided in those two branches as well.

For the theoretical part of our research the candidate will have day-to-day conversations not only with the supervisor but also with other researches on the math department of FMF-UL. The candidate will constantly participate on the *discrete mathematics seminar* of FMF-UL and the *combinatorics and group theory seminar* of the Faculty of Education of the University of Ljubljana (PeF-UL). The candidate will participate on the 2023 International Slovenian Conference on Graph theory, which is an international forum that occurs every 4 years and has had more than 300 participants in the last editions. These activities align perfectly with **??** 3**??** 4.

In other to develop the computational part of the project and the actual development of RO1, the candidate will work in close contact with Katja Berčič, who is an expert on *mathematical knowledge management* and the creation of datasets. The candidate will organise at least one *programming workshop* per year where both researcher and students will get together and discuss problems regarding datasets of discrete objects.

The research objects involved in PolyData are not only of mathematical interest but the have a natural degree of beauty. With this in mind the candidate is expected to participate in the $28th$ and $29th$ editions of the Slovenian Festival of Science, where some of the aspects of the research developed can be shared to a general audience with activities such as *Origami Polyhedra* or *Build your own kaleidoscope*.

2.3 The magnitude and importance of the project's contribution to the expected scientific, societal and economic impacts

Our research project is intended to make a heavy impact on the community doing research on highly symmetric combinatorial structures. More particularly we expect that with the success of PolyData the way and methodology used to do research on abstract polytopes improves significantly. We also expect PolyData to be the first step in a long-term project so that it eventually becomes a standard reference to look for datasets of highly symmetric polytopes that is both accessible and easy to use.

Just the theoretical implications of our project will help the community to better understand highly symmetric polytopes. However, we should emphasise that by standardising the existing and building new datasets of abstract polytopes we expect to offer a new way to do research that is closer to the way other sciences work.

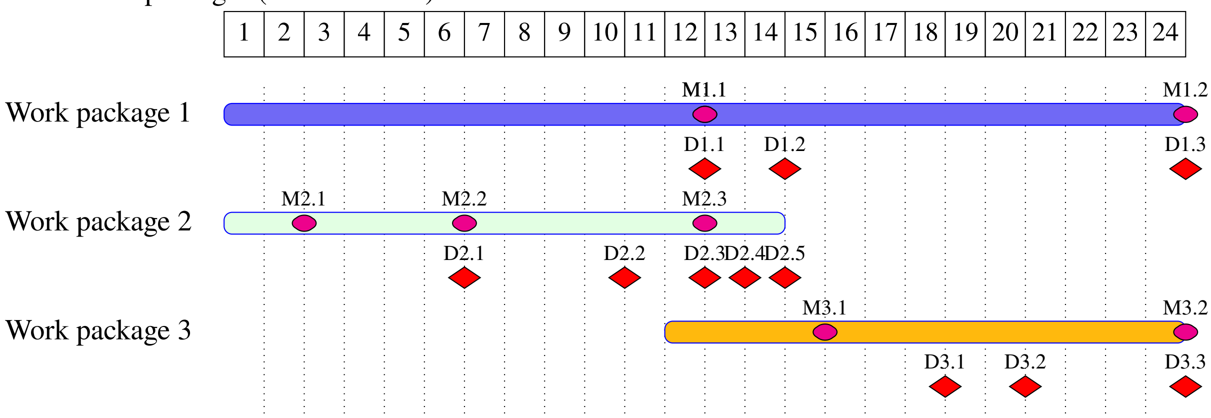
3 Quality and Efficiency of the Implementation

3.1 Quality and effectiveness of the work plan, assessment of risks and appropriateness of the effort assigned to work packages

Work package 1 *(24 moths)*. **Develop theoretical constructions of non-chiral 2-orbit polytopes and k-orbit polytopes for $k \geqslant 3$.** This is our main theoretical work plan. The candidate will explore the technique of *operations* to build new polytopes from previously existing polytopes as well as the technique of *extensions* to build new polytopes with prescribed facets. *Milestone (1.1)* is set for the first 12 months and consist of having the first constructions of families of non regular o chiral polytopes. The success of this task will result on the preparation and submission of at least 2 papers to a high-quality international peer-reviewed journal (*Derivables 1.1 and 1.2*). Then the candidate will explore new theoretical constructions *(Milestone 1.2)* which eventually result on the publication of at least one manuscript more (*Derivable 1.3*).

Work package 2 *(14 months)*. **Collect and unify existing datasets of highly symmetric polytopes.** First the candidate will collect the existing dataset on a common format (*Milestone 2.1*, 2 months). Then these data will be use to set a FAIR repository (*Milestone 2.2* (4 months) and *Derivable 2.1*). Finally the repository will be used to build a website (Derivable 2.2) and corresponding software packages to be used in computer algebra systems such as GAP(Derivable 2.3), MAGMA (Derivable 2.4) or `SageMath` (Derivable 2.5). This last step is Milestone 2.3 (6 months).

Work package 3 *(12 months)*. **Develop new datasets of abstract polytopes.** From the success of Work package 1 and the foundations given in Work package 2, the candidate will create new datasets containing both existing (Milestone 3.1, 3 months) and newly constructed (Milestone 3.2, 9 months) highly symmetric polytopes. These new datasets will bring an update to the FAIR repository (Derivable 3.1) to the website(Derivable 3.2) and to the software packages (Derivable 3.3).



3.2 Quality and capacity of the host institutions and participating organisations, including hosting arrangements

The Department of Mathematics at UL FMF hosts a very strong discrete mathematics and theoretical computer science group at the host institution, which includes high-level researchers, such as Bojan Mohar (affiliated also with Simon Fraser University, Canada), Sandi Klavžar, Tomaž Pisanski, Riste Škrekovski, Andrej Bauer, Matija Pretnar etc. Members of this group have lead several research projects and have extensive experience in supervising PhD students and postdoctoral researchers, including an MSC fellow (Daniel Ahman, under supervision of Matija Pretnar).