# 1 Excellence

## 1.1 Quality and pertinence of the project's research and innovation objectives (and the extent to which they are ambitious, and go beyond the state of the art)

### Introduction, historical remarks and state-of-the-art.

Motivation

In a very basic and general approach, most sciences and scientific disciplines follow a standard scientific method. Very generally speaking, this method consists of a series of clearly defined steps that can be summarised as: observation, measurement, experimentation, and formulation, testing and modification of hypotheses.

The development of theoretical mathematical knowledge requires the formulation and formal proof of hypotheses. For this reason, it is almost impossible to follow the standard scientific steps mentioned above. Every day work of a mathematician usually starts with an idea that eventually becomes a hypothesis based on merely intuition and sometimes a handful of examples. More often than not the hypothesis turns to be incorrect and this only shows when failing to give a formal and complete proof. Usually the hypotheses needs to be modified and the process must start over again, without being able to report any knowledge from those previous attempts. Moreover, this need not to be the worst case scenario, it could be that a hypothesis is perfectly correct but the tools needed to develop a formal proof are not developed yet. Reference to say, Fermat's last theorem?

What is OurP ?

Clearly, the two scenarios presented above show that there is a need to improve and speed the way mathematical knowledge is developed. Unfortunately, there is not much to do for the latter situation; however we could try to improve the quotidian labor of a theoretical mathematician. This is precisely where OurP fits. It is not uncommon that examples of mathematical objects are easy to construct in a theoretical context but hard to visualize in a concrete way. This usually comes from the very nature of the mathematical objects. However, discrete and combinatorial objects often possess a finite and natural way of save them as computational objects. Of course, this natural setting is usually limited by computational power or by the lack of efficient representation of such objects. OurP 's main objective is to develop data sets and computational tools of highly symmetric maps, abstract polytopes and maniplexes.

The nature of OurP involves a two-way flow of knowledge: With the aim of generating new datasets of highly symmetric polytopes and related objects, we will develop new theoretical results and methods, enabling us to devise new algorithms and combinatorial representation for constructing highly symmetric abstract poltytopes. The algorithms will then be carefully implemented and executed. The obtained datasets will then be analysed with the aim of finding interesting patterns, suggesting new conjectures and proposing new directions for further research.

Datasets of mathematical objects

Enumeration and classification of mathematical objects is a natual way of conducting research. The discrete nature of combinatorial objects turn them into natural candidates to not only classify families of interesting objects but to enumerate and explicitly list the elements of such families. This research approach has resulted in the development of interesting data sets of combinatorial objects. Highly symmetric graphs is arguably the most studied family of combinatorial objects from the approach of building datasets of objects. It is believed that empirical study of symmetric graphs of small valence started in 1930s, when R.M. Foster began collecting examples of interesting graphs that could serve as models for electrical networks.[1]

Of course, this area of research has taken advantage of the development and improvement of computational power but the theoretical research goes back to Tutte and his work on classifying 3-valent arc-transitive graphs[2,3] . It is believed that empirical study of symmetric graphs of small valence started in 1930s, when R.M. Foster began collecting examples of interesting graphs that could serve as models. His work was published in a book which now carries the name Foster's census[4] . The development of the

[1] R. M. Foster. "Geometrical Circuits of Electrical Networks". In: *Transactions of the American Institute of Electrical Engineers* 51.2 (June 1932), pp. 309–317.

[2] W. T. Tutte. "A family of cubical graphs". In: *Proc. Cambridge Philos. Soc.* 43 (1947), pp. 459–474.

[3] W. T. Tutte. "On the symmetry of cubic graphs". In: *Canadian J Math* 11 (1959), pp. 621–624.

[4] R. Foster. "A census of trivalent symmetrical graphs". In: *Conference on Graph Theory and Combinatorial Analysis, University of Waterloo, Ontario.* 1966.

theory, together with more powerful computers, resulted in a breakthrough of datasets of highly symmetric graphs constructions. Using the classification of automorphism groups of 3-valent arc-transitive[5] and bitransitive[6] graphs, together with new methods for finding normal subgroups of finite index in a finitely presented group allowed a construction of complete list of all 3-valent arc-transitive graphs[7,8] of order up to 10000 vertices, and a list of all 3-valent bitransitive graph on up to 768 vertices .[9] Based on their deep theoretical result on the order of automorphism groups[10] , Spiga, Verret and Potočnik compiled a complete list[11] of all trivalent vertex-transitive graphs of order at most 1280. Very recently, using the database of vertex-transitive groups of small degree, Conder and Verret have compiled a complete list of all edge-transitive graphs (of arbitrary valence) up to order 63[12] , while Holt and Royle have extended their census of all vertex-transitive graphs up to order 48[13] .

The classification and enumeration of groups has been also an intriguing problem since the beginning of theory. In 1854 Cayley[14] introduced the axiomatic definition of a group and enumerated the groups of order up to 6. Of course this is just the first step in what became an active research in both, theoretical mathematics[15] , as well as a motivation to develop computation tools such as the library `SmallGrp`[16] of small groups of GAP . In fact, one of the principal motivators on the study of symmetries of discrete objects the the *classification of Finite Simple Groups.* It turns out that many of the so-called sporadic simple groups can be understood as symmetry groups of discrete objects. This classification eventually derived in the construction of the Atlas of Finite Groups[17] .

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Once upon a time of polytopes

The enumerations and classification of the five Platonic Solids is one the most antique classification problems. There is archaeological evidence of stone balls representing what we now know as the symmetry groups of the five Platonic Solids. This evidence was discovered in Scotland and dates from the first half of the third millennia BCE. It is also known that the Egyptians and Babillonials were aware of the existence of such object but undoubtedly the Greeks have the credit of studying them form a purely mathematical interest. In fact, the thirteenth book of Euclid's Elements is devoted to the classification of the five Platonic Solids. A classification problem often relates to the definition of the objects that are being classified. In a paint from 1420 by Paolo Uccello and an engraving from 1568 by Wenzel Jamnitzer appear the oldest representation of what we know as *regular stellated polyhedra.* These are objects that share many properties with platonic solids but have the special characteristic of having stellated polygons as faces. These polyhedra were rediscovered first by Kepler in the late 1500's and then by Poinsot in 1809 who also discovered their duals; both authors described them with a mathematical approach. Soon later, in 1811 Cauchy show that the four objects descibed by Poinsot were the only possible regular stellated polyhedra. In the second half of the 19th century that Schäfli formally studied the symmetries of Platonic Solids and their higher dimensional analogous, those that we today know as regular convex

[5] D. Ž. Djoković and G. L. Miller. "Regular groups of automorphisms of cubic graphs". In: *J. Combin. Theory Ser. B* 29.2 (1980), pp. 195–230.

[6] D. M. Goldschmidt. "Automorphisms of trivalent graphs". In: *Ann. of Math. (2)* 111.2 (1980), pp. 377–406.

[7] M. Conder and P. Dobcsányi. "Trivalent symmetric graphs on up to 768 vertices". In: *J. Combin. Math. Combin. Comput.* 40 (2002), pp. 41–63.

[8] M. Conder. *Trivalent (cubic) symmetric graphs on up to 10000 vertices.* https://www.math.auckland.ac.nz/~conder/symmcubic10000list.txt, Accessed online January 7th 2022.

[9] M. Conder et al. "A census of semisymmetric cubic graphs on up to 768 vertices". In: *J. Algebraic Combin.* 23.3 (2006), pp. 255–294.

[10] P. Potočnik, P. Spiga, and G. Verret. "Bounding the order of the vertex-stabiliser in 3-valent vertex-transitive and 4-valent arc-transitive graphs". en. In: *J. Combin. Theory Ser. B* 111 (Mar. 2015), pp. 148–180.

[11] P. Potočnik, P. Spiga, and G. Verret. "A census of 4-valent half-arc-transitive graphs and arc-transitive digraphs of valence two". In: *Ars Math. Contemp.* 8.1 (2015), pp. 133–148.

[12] M. D. E. Conder and G. Verret. "Edge-transitive graphs of small order and the answer to a 1967 question by Folkman". In: *Algebr. Comb.* 2.6 (2019), pp. 1275–1284.

[13] D. Holt and G. Royle. "A census of small transitive groups and vertex-transitive graphs". In: *J. Symbolic Comput.* 101 (2020), pp. 51–60.

[14] A. Cayley. "On the theory of groups, as depending on the symbolic equation $\theta^n = 1$". In: *Philos Mag* 7.42 (Jan. 1854), pp. 40–47.

[15] S. R. Blackburn, P. M. Neumann, and G. Venkataraman. *Enumeration of Finite Groups.* Cambridge Tracts in Mathematics. Cambridge: Cambridge University Press, 2007.

[16] H. U. Besche, B. Eick, and E. A. O'Brien. "The groups of order at most 2000". In: *Electronic Research Announcements of the American Mathematical Society* 7 (2001), pp. 1–4.

[17] J. H. Conway. *Atlas of Finite Groups. Maximal Subgroups and Ordinary Characters for Simple Groups.* Oxford University Press, USA, 1986, p. 284.

polytopes. To this point it is important to remark that geometrical properties of convex polytopes allow us to fully classify them combinatorially by a sequence of numbers, the Schläfli symbol (explained in detail below), and hence their reconstruction from a computational viewpoint is extremely simple.

The theory of polyhedral-like structures took a complete new breath with the contributions of H.S.M. Coxeter. Those contributions are extremely numerous to list in here and spread all along the 20th century. Coxeter's monograph[18] on regular polytopes is most likely its most influential publication, but some of his remarkable contributions date as early as 1937 when together with J.F. Petrie described the *regular skew polyhedra*[19] as infinite analogues of Platonic solids. Coxeter is also attributed to classify the groups generated by hyperplane reflections, leading to what we today know as *Coxeter groups*. Coxeter groups have an influential role in several branches of mathematics. They of course, appear as the symmetry groups of regular polytopes and tessellations of the Euclidean and Hyperbolic spaces,[20] but they have made their way to Tits geometries,[21] computational Lie group theory, Hecke algebras,[22] just to mention some.

Coxeter's work serve as inspiration for many mathematicians, one of them being Branko Grünbaum who in the 70's introduced[23] the notion of *polystroma*, which is an ancestor of what we today call *abstract polytopes*. Grünbaum is also responsible of first treating symmetric polyhedra from a combinatorial viewpoint. By relaxing the definition of a regular polyhedron he presented[24] a list of 47 regular polyhedra which included the Platonic Solids, Stellated polyhedra as well as Petrie-Coxeter skew polyhedra. Soon after A. Dress describes[25] another polyhedron and proves[26] that the list of 48 regular polyhedra is complete.

Almost at the same time as Grünbaum was describing his families of regular polyhedra on the Euclidean space, G. Jones and D. Singerman published his classical manuscript[27] which settle the necessary theory to identify maps on orientable surfaces with what in modern terminology we called its *monodromy group*. The ideas behind this paper show important equivalences between topological maps (embedding of graphs on orientable surfaces), certain quotients triangular groups (Coxeter groups of rank 3), maps on Riemmann surfaces and certain permutations on the darts of the map. These equivalences are a combinatorial/discrete version of the classical Uniformization theorem[28] for Riemann surfaces, since in particular show that every map can be obtained as a quotient of a regular tessellation of the sphere $\mathbb{S}^2$, the Euclidean plane $\mathbb{E}^2$ or the hyperbolic plane $\mathbb{H}^2$. The work of Jones and Singerman was an important contribution on the theory of discrete group actions on Riemann surfaces and it was eventually connected the theory Grothendieck's *Dessins d'enfant*.[29]

Some combinatorial equivalences of maps on surfaces were also explored by Tutte, who gave an axiomatic combinatorial definition of a map.[30] Vince extended[31] these equivalences and introduced a very general definition of combinatorial maps. A slightly more restrictive definition is that of a *maniplex* in-

[18] H. S. M. Coxeter. *Regular polytopes*. Third. Dover Publications, Inc., New York, 1973, pp. xiv+321.
[19] H. S. M. Coxeter. "Regular Skew Polyhedra in Three and Four Dimension, and their Topological Analogues". In: *Proc. London Math. Soc.* S2-43.1 (1937), p. 33.
[20] J. E. Humphreys. *Reflection Groups and Coxeter Groups*. Vol. 29. Cambridge Studies in Advanced Mathematics. Cambridge University Press, June 1990, pp. xii+204.
[21] J. Tits. *Buildings of spherical type and finite BN-pairs*. Lecture Notes in Mathematics, Vol. 386. Springer-Verlag, Berlin-New York, 1974, pp. x+299.
[22] A. M. Cohen. "Coxeter Groups and three Related Topics". en. In: ed. by A. Barlotti et al. NATO ASI Series. Dordrecht: Springer Netherlands, 1991, pp. 235–278.
[23] B. Grünbaum. "Regularity of graphs, complexes and designs". In: *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*. Vol. 260. Colloq. Internat. CNRS. CNRS, Paris, 1978, pp. 191–197.
[24] B. Grünbaum. "Regular polyhedra—old and new". In: *Aequationes Math.* 16.1-2 (1977), pp. 1–20.
[25] A. W. M. Dress. "A combinatorial theory of Grünbaum's new regular polyhedra. I. Grünbaum's new regular polyhedra and their automorphism group". In: *Aequationes Math.* 23.2-3 (1981), pp. 252–265.
[26] A. W. M. Dress. "A combinatorial theory of Grünbaum's new regular polyhedra. II. Complete enumeration". In: *Aequationes Math.* 29.2-3 (1985), pp. 222–243.
[27] G. A. Jones and D. Singerman. "Theory of maps on orientable surfaces". In: *Proc. London Math. Soc. (3)* 37.2 (1978), pp. 273–307.
[28] W. Abikoff. "The Uniformization Theorem". In: *The American Mathematical Monthly* 88.8 (Oct. 1981), pp. 574–592.
[29] G. A. Jones and J. Wolfart. *Dessins d'Enfants on Riemann Surfaces*. Springer London, Limited, 2016.
[30] W. T. Tutte. "What is a map?" In: *New directions in the theory of graphs (Proc. Third Ann Arbor Conf., Univ. Michigan, Ann Arbor, Mich., 1971)*. 1973, pp. 309–325.
[31] A. Vince. "Combinatorial maps". In: *Journal of Combinatorial Theory. Series B* 34.1 (1983), pp. 1–21.

Reference (?)

troduced by Wilson[32] more than twenty years after Vince's manuscript. We shall describe this notion slightly more detail in Section **??**

The central class of objects in OurP is that of *highly symmetric abstract polytopes*. Abstract polytopes were introduced by Schulte in his PhD thesis[33] and he also established most of the early results. Abstract polytopes are a particular class of partially ordered sets that combinatorially generalise the (face-lattices) of convex polytopes but alson include the incidence structure of many other geometrical objects such as tilings of $\mathbb{E}^n$ and $\mathbb{H}^n$ as well as most maps on surfaces. Early research focused on *regular polytopes*, that is, those with the higest degree of symemtries, being one the most remakable results the correspondence between regular polytopes and *string C-groups*, that is smooth quotients of Coxeter groups that satisfy the Intersection Property[34,35] .

The correspondence between regular polytopes and its automorphism group made possible turn the combinatorial problem of building regular polytopes into a group theoretical problem. This approach has been the standard techinque to build regular polytopes and it would be impossible to list them all. Just to show the different approaches we should mention that there exist universal constructions[36,37] , constructions prescribing local combinatorics[38,39,40] and constructions fixing interesting families of groups as automorphism groups[41,42,43,44] .

The second most studied symmetry class of polytopes is that of *chiral polytopes*. Informally speaking, a chiral polytope is a polytope having full degree of (combinatorial) rotational symmetry without having (combinatorial) reflections. They were introduced by Schulte and Weiss in 1990[45] where an analogous result to the one for the automorphism group of regular polytopes was established. Chiral polytopes were introduced as a natural generalization of *chiral maps*, which have been part of the classical theory of maps from it beggiging and numerous examples exist[46,47,48] . However, the problem of constructing chiral polytopes of higher ranks has proved to be much harder to that of constructing regular polytopes. Some rank 4 examples were constructed as quotients of hyperbolic tilings[49,50,51,52] . A universal construction[53] was used to produce the first (infinite) example of a rank-5 chiral polytope. However, the first finite

[32] S. Wilson. "Maniplexes: Part 1: maps, polytopes, symmetry and operators". In: *Symmetry* 4.2 (2012), pp. 265–275.
[33] E. Schulte. "Reguläre Inzidenzkomplexe". PhD thesis. University of Dortmund, 1980.
[34] L. Danzer and E. Schulte. "Reguläre Inzidenzkomplexe. I". in: *Geom. Dedicata* 13.3 (1982), pp. 295–308.
[35] E. Schulte. "Reguläre Inzidenzkomplexe. II, III". in: *Geom. Dedicata* 14.1 (1983), pp. 33–56, 57–79.
[36] E. Schulte. "On arranging regular incidence-complexes as faces of higher-dimensional ones". In: *European J. Combin.* 4.4 (1983), pp. 375–384.
[37] E. Schulte. "Extensions of regular complexes". In: *Finite geometries (Winnipeg, Man., 1984)*. Vol. 103. Lecture Notes in Pure and Appl. Math. Dekker, New York, 1985, pp. 289–305.
[38] L. Danzer. "Regular incidence-complexes and dimensionally unbounded sequences of such. I". in: *Convexity and graph theory (Jerusalem, 1981)*. Vol. 87. North-Holland Math. Stud. North-Holland, Amsterdam, 1984, pp. 115–127.
[39] D. Pellicer. "Extensions of regular polytopes with preassigned Schläfli symbol". In: *J. Combin. Theory Ser. A* 116.2 (2009), pp. 303–313.
[40] D. Pellicer. "Extensions of dually bipartite regular polytopes". In: *Discrete Math.* 310.12 (2010), pp. 1702–1707.
[41] P. J. Cameron et al. "Highest rank of a polytope for $A_n$". In: *Proc. Lond. Math. Soc. (3)* 115.1 (2017), pp. 135–176.
[42] M. E. Fernandes and D. Leemans. "C-groups of high rank for the symmetric groups". In: *Journal of Algebra* 508 (2018), pp. 196–218.
[43] D. Leemans, J. Moerenhout, and E. O'Reilly-Regueiro. "Projective linear groups as automorphism groups of chiral polytopes". In: *Journal of Geometry* 108.2 (2017), pp. 675–702.
[44] D. Pellicer. "CPR graphs and regular polytopes". In: *European J. Combin.* 29.1 (2008), pp. 59–71.
[45] E. Schulte and A. I. Weiss. "Chiral polytopes". In: *Applied geometry and discrete mathematics*. Vol. 4. DIMACS Ser. Discrete Math. Theoret. Comput. Sci. Amer. Math. Soc., Providence, RI, 1991, pp. 493–516.
[46] H. S. M. Coxeter and W. O. J. Moser. *Generators and relations for discrete groups*. Third. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 14. Springer-Verlag, New York-Heidelberg, 1972, pp. ix+161.
[47] M. Conder and P. Dobcsányi. "Determination of all regular maps of small genus". In: *J. Combin. Theory Ser. B* 81.2 (2001), pp. 224–242.
[48] F. A. Sherk. "A family of regular maps of type $\{6, 6\}$". In: *Canad. Math. Bull.* 5 (1962), pp. 13–20.
[49] B. Nostrand, E. Schulte, and A. I. Weiss. "Constructions of chiral polytopes". In: *Proceedings of the Twenty-fourth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1993)*. Vol. 97. 1993, pp. 165–170.
[50] E. Schulte and A. I. Weiss. "Chirality and projective linear groups". In: *Discrete Math.* 131.1-3 (1994), pp. 221–261.
[51] B. Nostrand. "Ring extensions and chiral polytopes". In: *Proceedings of the Twenty-fifth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1994)*. Vol. 102. 1994, pp. 147–153.
[52] B. Nostrand and E. Schulte. "Chiral polytopes from hyperbolic honeycombs". In: *Discrete Comput. Geom.* 13.1 (1995), pp. 17–39.
[53] E. Schulte and A. I. Weiss. "Free extensions of chiral polytopes". In: *Canad. J. Math.* 47.3 (1995), pp. 641–654.

rank 5 polytopes were constructed by Conder et al. in 2008.[54] It was until 2010 that Pellicer showed[55] the existance of chiral polytopes of rank $n$ for ever $n \geqslant 4$; However the result by Pellicer, although constructive, is not very practical. The size of his examples grow as a tower of exponential functions with length depending on $n$. Later on examples of new chiral polytopes have been constructed from previously known ones[56,57,58,59] .

The degree of symmetry of a polytope can be measured by the number of orbits of *flags*. Regular polytopes have 1 flag-orbit (1-orbit poltyopes). Chiral polytopes are just one of $2^n - 1$ possible symmetry type class of 2-orbit polytopes. Classical examples of some of the other classes of the 2-orbit polytopes are known. However, the general problem of determining if for every pair $(n, T)$ with $n > 3$ and $T$ a 2-orbit symmetry type exists an $n$-polytope of symmetry type $T$ remains open. Some examples of maniplexes were build very recently,[60] but whether or not they are polytopal remains unknown. Constructing and classifying $n$-polytopes with $k$-orbits and given symmetry type is still a widely open problem[61] and it is one of the main motivations for the development of OurP .

Datasets of polytopes

As shown above, classification and enmeration of highly symmetric polytopes has been part of the theory from the beggining. Both, the classification of the 5 Platonic Solids to the ennumeration of the 48 Grünbaum-Dress polyhedra in $\mathbb{E}^3$ depend on strong geometric restrictions. However, the combinatorial nature of abstact poltypes open the possibilities to, in principle, have numerous examples of abstract polytopes. These has lead to the construction of some datasets of highly symmetric polytopes, however as we shall see below, these datasets suffer of thee main restrictions

*(i)* They are mainly focused on regular or chiral polytopes.
*(ii)* The size of the examples is very restrictive.
*(iii)* They often exhibit noumerous examples of rank 3 but the amount of examples of rank higher than 4 drops dramatically.
*(iv)* They are not very user-friendly, either because they exist only as raw data or because the are specific-programming language oriented.

With the emerging development of theoretical results for less symmetric poltyopes and the need to identify patterns and to find new constructions to attack the numerous open problems related to the existence of polytopes, it is clear that building new datasets of polyopes that overturn the restrictions mentioned above would not only be beneficial but it is almost necessary.

We review below the existing datasets of highly symmetric polytopes.

**Conder - Regular orientable maps by genus**[62] Computed by M. Conder it originally contained all (3378) regular maps on orientable surfaces of genus 2 to 101 up to isomorphism an duallity. It was later[63] extended to include genus up to 301 for a total of 15824 maps. Computed with the help of `LowIndexNormalSubgroups` routine of MAGMA and published as raw text.

**Conder - Regular non-orientable maps by genus**[64] Every map on a non-orientable surface admits an orientable double cover. Conder used this fact to originally compute all (862) non-orientable regular maps of genus 2 to 202 and then[65] extended to genus up to 602 for a total of 3260 maps. Computed with

[54] M. Conder, I. Hubard, and T. Pisanski. "Constructions for chiral polytopes". In: *J. Lond. Math. Soc. (2)* 77.1 (2008), pp. 115–129.
[55] D. Pellicer. "A construction of higher rank chiral polytopes". In: *Discrete Math.* 310.6-7 (2010), pp. 1222–1237.
[56] G. Cunningham and D. Pellicer. "Chiral extensions of chiral polytopes". In: *Discrete Math.* 330 (2014), pp. 51–60.
[57] M. D. E. Conder and W.-J. Zhang. "Abelian covers of chiral polytopes". In: *J. Algebra* 478 (2017), pp. 437–457.
[58] A. Montero. "Chiral extensions of toroids". PhD thesis. National University of Mexico, 2019.
[59] A. Montero. "On the Schläfli symbol of chiral extensions of polytopes". en. In: *Discrete Math* 344.11 (Nov. 2021), p. 112507.
[60] D. Pellicer, P. Potočnik, and M. Toledo. "An existence result on two-orbit maniplexes". In: *J. Combin. Theory Ser. A* 166 (Aug. 2019), pp. 226–253.
[61] G. Cunningham and D. Pellicer. "Open problems on $k$-orbit polytopes". In: *Discrete Math.* 341.6 (2018), pp. 1645–1661.
[62] M. Conder. *Regular orientable maps of genus 2 to 101.* 2006. URL: https://www.math.auckland.ac.nz/~conder/RegularOrientableMaps101.txt.
[63] M. Conder. *Regular orientable maps of genus 2 to 301.* Total number of maps in list below: 15824. 2011. URL: https://www.math.auckland.ac.nz/~conder/RegularOrientableMaps301.txt.
[64] M. Conder. *Regular non-orientable maps of genus 2 to 202.* Total number of maps in list below: 862. 2006. URL: https://www.math.auckland.ac.nz/~conder/RegularNonorientableMaps202.txt.
[65] M. Conder. *Regular non-orientable maps of genus 2 to 602.* Total number of maps in list below: 3260. 2012. URL: https://www.math.auckland.ac.nz/~conder/RegularNonorientableMaps602.txt.

the help of `LowIndexNormalSubgroups` routine of MAGMA and published as raw text.

**Conder - Chiral maps by genus**[66] Census containing all (594) chiral maps on orientable surfaces of genus 2 to 101. This census was later[67] extended to genus up to 301 for a total of 3870. Computed with the help of `LowIndexNormalSubgroups` routine of MAGMA and published as raw text.

**Conder - Rotary maps by size**[68] This census contains all rotary (that is regular or chiral) maps whose rotation group has less than 2000 elements (equivalently, such that the map has less than 1000 edges). Computed with the help of `LowIndexNormalSubgroups` routine of MAGMA and published as raw text. There exist versions of this census containing only regular and only chiral maps.

**Potocnik - Regular maps by size**[69] An improvement on Conder's census containing all (255, 980) regular maps whose automorphism group is of order less than 6, 000 for orientable maps and 3, 000 for non-orientable maps. Published as MAGMA files available to download with a CVS-file of precomputed information.

**Potocnik - Chiral maps by size**[70] An analogous to the one above but for chiral maps. It contains a total of 122, 092 chiral maps whose automorphism group has order less than 6, 000.

**Hartley - The Atlas of Small Regular Polytopes.**[71] It was build using `SmallGrp` routine of GAP and contains all regular polytopes with at most 2000 flags, except those of size 1024 and 1536. It contains 9212 examples. They are presented in a nice web interface and the code is available to download.

**Hartley - The Atlas of Small Chiral Polytopes**[72] Every chiral polytope admits a minimal regular cover. Hartley used this fact to compute the first atlas of chiral poltyopes. This dataset consists of all chiral polytopes whose minimal regular cover belongs to the Atlas of Small Regular Polytopes. This gave a total of 48 chiral poltyopes of rank 3 and 8 poltyopes of rank 4.

**Conder - Regular polytopes up to 2,000 flags**[73] A dataset containing, up to duality, all (5809) regular polytopes with at most 2000 flags (which is the same as the order of the automorphism group).

**Conder - Chiral polytopes up to 2,000 flags**[74] A dataset containing, up to duality, all (839) chiral polytopes with at most 2000 flags (which is the twice the order of the automorphism group).

**Leemans et al. - An atrlas of polytopes for small simple groups**[75] This is an ongoing atlas that contains regular poltyopes whose automorphism group is an almost simple group. It currently contains 55, 575 regular poltyopes. The atlas is presented on a website with downloadable data.

**Leemans et al. - An atrlas of chiral polytopes for small simple groups**[76] It is the analogous to the one above but for chiral poltyopes. It currently contains a total of 19, 964 polytopes.

As explained before, most of the existent datasets of polytopes are either completely focused on 3-poltyopes (maps) or have very little examples of higher ranks. In Table 1 we show the proportion of examples according to ranks. The current datasets are not optimal if examples of higher ranks need to be tested, and new datasets need to be developed.

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[66] M. Conder. *Chiral orientably-regular maps of genus 2 to 101.* Total number of maps in list below: 594. 2006. URL: https://www.math.auckland.ac.nz/~conder/ChiralMaps101.txt.

[67] M. Conder. *Chiral rotary maps of genus 2 to 301.* Total number of maps in list below: 3870. 2013. URL: https://www.math.auckland.ac.nz/~conder/ChiralMaps301.txt.

[68] M. Conder. *Rotary maps (on orientable or non-orientable surfaces) with up to 1000 edges.* 2012. URL: https://www.math.auckland.ac.nz/~conder/RotaryMapsWithUpTo1000Edges.txt.

[69] P. Potočnik. *Census of regular maps.* 2014. URL: https://www.fmf.uni-lj.si/~potocnik/work.htm.

[70] P. Potočnik. *Census of chiral maps.* 2014. URL: https://www.fmf.uni-lj.si/~potocnik/work.htm.

[71] M. Hartley. *The Atlas of Small Regular Polytopes.* 2006. URL: https://www.abstract-polytopes.com/atlas/.

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[75] D. Leemans et al. *An Atlas of Polytopes for Small Almost Simple Groups.* URL: https://leemans.dimitri.web.ulb.be/polytopes/index.html.

[76] M. Hartley, I. Hubard, and D. Leemans. *An Atlas of Chiral Polytopes for Small Almost Simple Groups.* URL: https://leemans.dimitri.web.ulb.be/CHIRAL/index.html.

| Dataset | Rank 3 | Rank 4 | Rank $\geqslant 5$ |
|---|---|---|---|
| Hartley - Regular | 64.55% | 31.61% | 3.84% |
| Hartley - Chiral | 85.71% | 14.29% | 0.00% |
| Conder - Regular | 61.51% | 34.70% | 3.79% |
| Conder - Chiral | 87.01% | 12.87% | 0.12% |
| Leemans - Regular | 95.35% | 4.37% | 0.30% |
| Leemans - Chiral | 87.82% | 11.97% | 0.21% |

Table 1: Percentages of examples according to rank

### Problem identification and Research and Innovation objectives.

The general objective of OurP is to build a environment of data sets and computational tools for maps, abstract polytopes and maniplexes. We expect this environment to be not only of the interest but more importantly, useful to the community doing research on this area. In the following paragraphs, we explain how we split this general objective into several particular and very concrete objectives. The existent datasets of highly symmetric maps and polytopes are not only limited but they have not been exploded to its full capacity. One of the reasons behind it is that the information in most of these data sets is not very user-friendly. Even the small amount of existing data sets have been developed by several people, mostly in an independent way, using different notation and different computer algebra systems. This usually stops a researcher to use a given data set just because he or she might not be familiar with the notation or programming language. Moreover, many of these datasets exist only as raw text which is not always easy to consult. However, we could use the fact the the amount of data is not huge in our advantage. All the facts mentioned above outline our first objective:

**Objective 1** Collect, unify and make the existing data sets user-friendly and in a FAIR-ly way so that OurP eventually becomes the standard to-go when consulting or publishing new data on highly symmetric polytopes by

*(i)* Creating and developing a unique data set from the existent ones that unifies notation and identification of the data and make it available so that other researches could experiment and eventually contribute to OurP .

*(ii)* Building a web-based interface to our data set for easy and quick consultation.

*(iii)* Developing this data set in a way that can be easily implemented in some of the standards computer algebra systems such as `MAGMA` , GAP and `SageMath` .

*(iv)* Writing appropriate documentation so that OurP eventually becomes available for others to contribute.

Objective 1 is very concrete and very easy to verify. It very easily show progress on OurP while at the same time sets a good start point to our ambitious general objective. The current state-of-theart allows us to start with Objective 1 from the very beginning of OurP

Many of the first data sets of abstract polytopes are based on the (small) size of the objects. Either by taking advantage of previously computed objects (such as the library `SmallGrp` ) or by using computational routines that, because of their own nature are limited by the size of the input (such as the `LowIndexNormalSubgroups` ). However, it has been shown that the size of the smallest regular polytope of rank $n$ grows exponentially with $n$[77] While the size of the smallest chiral $n$-polytope is at least of factorial growth with respect to $n$.[78] This explains why the amount of examples of higher rank polytopes drops dramatically on the current available datasets.

On the other hand, there are interesting infinite families (eg, the Toroids) or constructions (eg. generalized cubes) that of course will not come as a result of an exhaustive exploration but that are not too demanding from a computational approach. Motivated from the previous discussion we propose our second objective.

[77] M. Conder. "The smallest regular polytopes of given rank". en. In: *Adv Math* 236 (Mar. 2013), pp. 92–110.

[78] G. Cunningham. "Non-flat regular polytopes and restrictions on chiral polytopes". In: *Electron. J. Combin.* 24.3 (2017), Paper 3.59, 14.

Objective 2. Explore the literature and implement appropriate routines to construct new examples of polytopes from previously known ones.

Of course the previous very practical objective comes alongside with the next one which, just by its theoretical relevance, presents a very ambitious part of OURPROJECT.

Objective 3. Develop new constructions of highly symmetric polytopes. In particular, focus on the constructions of abstract regular polytopes with given symmetry type to eventually build data sets of given symmetry type (besides regular and chiral).

Objective 3 by itself represents an extremely ambitious and it goes beyond any two year project. Any theoretical contribution is already of great interest to the community. However, we strongly believe that in order to really cause an impact on the workflow of research on abstract polytopes all theoretical contributions should be accompanied by its computational analogue.

Of course, OURPROJECT is aimed to become a long term and eventually permanent resource for the community doing research on symmetries of maps and abstract polytopes. This will not be achievable without the involvement of such community.

Objective 4. Encourage and motivate both well-established and young reseachers to use and contribute to OURPROJECT so that it eventually becomes the standard way to explore, experiment and publish data sets, routines and computational tools for the development on the research of abstract polytopes.

Of course, Objective 4 is very ambitious and we acknowledge that it depends on the community more that on ourselves, but we strongly believe that our approach and the the current status of the could fit together to fill a gap that has been present for many year now. It is important to remark that the community has faced a similar scenario before. Many of the early research on abstract polytopes was collected on the comprehensive manuscript [REF! ARP] and nowadays it serves as the natural and standard theoretical reference. Our expectations is that OURPROJECT eventually becomes the computational analogue for our community.

### 1.2 Soundness of the proposed methodology (including interdisciplinary approaches, consideration of the gender dimension and other diversity aspects if relevant for the research project, and the quality of open science practices, including sharing and management of research outputs and engagement of citizens, civil society and end users, where appropriate)

The nature of our project involves a two way flow of knowledge between from discrete mathematics and group theory to the development of computational tools and data management. In order to build new data sets of abstract polytopes we need computational-efitient ways to compute and storage property of such objects and the other way around, if we want to OURPROJECT to eventually becomes a useful tool on theoretical research we need to develop ways to access and present the computed information in a user friendly way. We describe our proposed metodology from this two complementary approaches.

Mathematical representation of highly symmetric abstract polytopes.

Abrstract polytopes as posets. The original definition of an abstract polytope is on the form of a poset [REF]. It makes sense from an hystorical view point. They are intendet to be combinatorial generalisation of the (geometric) convex polytopes. This generalisation was obtained by taking some of the properties of the face lattice of a convex polytope and use them as defining properties of an abstract polytope. Unfortunatelly, the computational cost and the combinatorial problem of storaging a poset seems to be very inneficient. Moreover, modern computer algebra systems do not have particullarly efficient tools to deal with posets. We should try to avoid representing abstract polytopes as parially ordered sets.

Abstract polytopes from their automorphism group. Most likely this is the most exploided representation of an abstract polytopes. When an abstract polytope has a high degree of symmetry its automorphism group contains many combinatorial information of the poltyope. In particular, regular polytopes are incorrespondence with string C-groups [REF!], which are smoot quotient of Coxeter groups satisfying certain interection property. This fact has been strongly used to build the existing datasets mentioned in Section 1.1. The census [CONDER] was built by computing all possible normal subgroups of index at most 2000 of the universal string Coxeter group. This approach is computational expensive but it has the advantage that the computations have to be done only once. It might be worthy to try an push

### 1.3 Quality of supervision, training, and knowledge transfer

## 2 Impact

### 2.1 Credibility of the measures to enhance the career perspectives and employability of the researcher and contribution to his/her skills development

### 2.2 Suitability and quality of the measures to maximise expected outcomes and impacts, as set out in the dissemination and exploitation plan, including communication activities

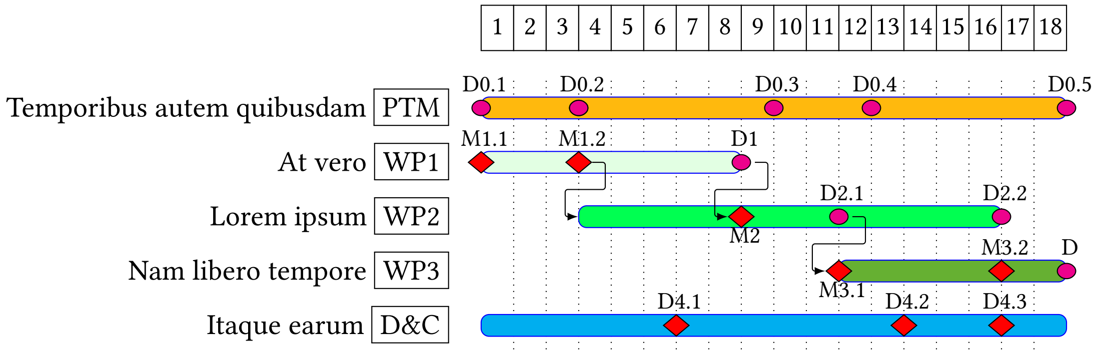
At a minimum, address the following aspects: Plan for the dissemination and exploitation activities, including communication activities:1 Describe the planned measures to maximize the impact of your project by providing a first version of your 'plan for the dissemination and exploitation including communication activities'. Describe the dissemination, exploitation measures that are planned, and the target group(s) addressed (e.g. scientific community, end users, financial actors, public at large). Regarding communication measures and public engagement strategy, the aim is to inform and reach out to society and show the activities performed, and the use and the benefits the project will have for citizens. Activities must be strategically planned, with clear objectives, start at the outset and continue through the lifetime of the project. The description of the communication activities needs to state the main messages as well as the tools and channels that will be used to reach out to each of the chosen target groups. • Strategy for the management of intellectual property, foreseen protection measures: if relevant, discuss the strategy for the management of intellectual property, foreseen protection measures, such as patents, design rights, copyright, trade secrets, etc., and how these would be used to support exploitation.

• All measures should be proportionate to the scale of the project, and should contain concrete actions to be implemented both during and after the end of the project.

### 2.3 The magnitude and importance of the project's contribution to the expected scientific, societal and economic impacts

## 3 Quality and Efficiency of the Implementation

### 3.1 Quality and effectiveness of the work plan, assessment of risks and appropriateness of the effort assigned to work packages

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Risk management

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## 3.2 Quality and capacity of the host institutions and participating organisations, including hosting arrangements