1 Excellence

1.1 Quality and pertinence of the project's research and innovation objectives (and the extent to which they are ambitious, and go beyond the state of the art)

Introduction, historical remarks and state-of-the-art.

Motivation

The standard scientific method does not work in mathematics. Generally speaking, this method consists of a series of clearly defined steps that can be summarised as: observation, measurement, experimentation, and formulation, testing and modification of hypotheses.

In classical Physics, for example, theories are formulated from observation and measurement, and they are confirmed by experiments. The development of theoretical mathematical knowledge requires the formulation and formal proof of hypotheses. The process usually starts with an idea that eventually becomes a hypothesis based on merely intuition and sometimes a handful of examples, but the observation and measurement analogous are usually overseen, often due to the nature of the objects.

What is PolyData ?

In the last few years, with the development of computers, mathematicians have found a way to run model, experiment and to some extend, visualize their job. This is precisely where PolyData fits. Discrete and combinatorial objects often possess a finite and natural way to save them as computational objects. Of course, this natural setting is usually limited by computational power or by the lack of efficient representation of such objects. PolyData 's main objective is to develop datasets and computational tools of highly symmetric maps, abstract polytopes and maniplexes.

The nature of PolyData involves a two-way flow of knowledge: With the aim of generating new datasets of highly symmetric polytopes and related objects, we will develop new theoretical results and methods, enabling us to devise new algorithms and combinatorial representation for constructing highly symmetric abstract polytopes. The algorithms will then be carefully implemented and executed. The obtained datasets will then be analysed with the aim of finding interesting patterns, suggesting new conjectures and proposing new directions for further research.

The context explained above bring us to the two main research goals for PolyData : *Build complete, accessible useful and reliable datasets of abstract polytopes*, which lead us to Objectives 1 and 2; and *Develop new theoretical constructions for polytopes*, which is explained in detail in Objective 3. With the success of the research the we propose we also have a long-term goal: that PolyData eventually becomes a standard in the area to keep and publish eventually new datasets of polytopes and related objects.

Datasets of mathematical objects

Enumeration and classification of mathematical objects is a natural way of conducting research. The discrete nature of combinatorial objects turn them into natural candidates to not only classify families of interesting objects but to enumerate and explicitly list the elements of such families. This research approach has resulted in the development of interesting data sets of combinatorial objects. Highly symmetric graphs is arguably the most studied family of combinatorial objects from the approach of building datasets of objects. It is believed that empirical study of symmetric graphs of small valence started in 1930s, when R.M. Foster began collecting examples of interesting graphs that could serve as models for electrical networks.[1]

Of course, this area of research has taken advantage of the development and improvement of computational power but the theoretical research goes back to Tutte and his work on classifying 3-valent arc-transitive graphs[2,3] . It is believed that empirical study of symmetric graphs of small valence started in 1930s, when R.M. Foster began collecting examples of interesting graphs that could serve as models. His work was published in a book which now carries the name Foster's census.[4] The development of the theory, together with more powerful computers, resulted in a breakthrough of datasets of highly symmetric graphs constructions. Using the classification of automorphism groups of 3-valent arc-transitive[5] and bitransitive[6] graphs, together with new methods for finding normal subgroups of finite index in a finitely presented group allowed a construction of complete list of all 3-valent arc-transitive graphs[7,8] of order up to 10000 vertices, and a list of all 3-valent bitransitive graph on up to 768 vertices .[9] Based on their deep theoretical result on the order of

[1] R. M. Foster. "Geometrical Circuits of Electrical Networks". In: *Transactions of the American Institute of Electrical Engineers* 51.2 (June 1932), pp. 309–317.

[2] W. T. Tutte. "A family of cubical graphs". In: *Proc. Cambridge Philos. Soc.* 43 (1947), pp. 459–474.

[3] W. T. Tutte. "On the symmetry of cubic graphs". In: *Canadian J Math* 11 (1959), pp. 621–624.

[4] R. Foster. "A census of trivalent symmetrical graphs". In: *Conference on Graph Theory and Combinatorial Analysis, University of Waterloo, Ontario.* 1966.

[5] D. Ž. Djoković and G. L. Miller. "Regular groups of automorphisms of cubic graphs". In: *J. Combin. Theory Ser. B* 29.2 (1980), pp. 195–230.

[6] D. M. Goldschmidt. "Automorphisms of trivalent graphs". In: *Ann. of Math. (2)* 111.2 (1980), pp. 377–406.

[7] M. Conder and P. Dobcsányi. "Trivalent symmetric graphs on up to 768 vertices". In: *J. Combin. Math. Combin. Comput.* 40 (2002), pp. 41–63.

[8] M. Conder. *Trivalent (cubic) symmetric graphs on up to 10000 vertices.* `https://www.math.auckland.ac.nz/~conder/symmcubic10000list.txt`, Accessed online January 7th 2022.

[9] M. Conder et al. "A census of semisymmetric cubic graphs on up to 768 vertices". In: *J. Algebraic Combin.* 23.3 (2006), pp. 255–294.

automorphism groups,[10] Spiga, Verret and Potočnik compiled a complete list[11] of all trivalent vertex-transitive graphs of order at most 1280. Very recently, using the database of vertex-transitive groups of small degree, Conder and Verret have compiled a complete list of all edge-transitive graphs (of arbitrary valence) up to order 63 ,[12] while Holt and Royle have extended their census of all vertex-transitive graphs up to order 48.[13]

The classification and enumeration of groups has been also an intriguing problem since the beginning of theory. In 1854 Cayley[14] introduced the axiomatic definition of a group and enumerated the groups of order up to 6. Of course this is just the first step in what became an active research in both, theoretical mathematics[15] , as well as a motivation to develop computation tools such as the library `SmallGrp`[16] of small groups of GAP . In fact, one of the principal motivators on the study of symmetries of discrete objects the the *classification of Finite Simple Groups*. It turns out that many of the so-called sporadic simple groups can be understood as symmetry groups of discrete objects. This classification eventually derived in the construction of the Atlas of Finite Groups[17] .

The enumerations and classification of the five Platonic Solids is one the most antique classification problems. In fact, the thirteenth book of Euclid's Elements is devoted to the classification of the five Platonic Solids. A classification problem often relates to the definition of the objects that are being classified. By relaxing geometrical conditions on the definition of polyhedra, new objects emerged. In a paint from 1420 by Paolo Uccello and an engraving from 1568 by Wenzel Jamnitzer appear the oldest representation of what we know as *regular stellated polyhedra*. These polyhedra were rediscovered by Kepler in the late 1500's and then by Poinsot in 1809. Soon later, in 1811 Cauchy show that the four objects described by Poinsot were the only possible *regular stellated polyhedra*. In the second half of the 19th century that Schäfli formally studied the symmetries of Platonic Solids and their higher dimensional analogous.

The theory of polyhedral-like structures took a complete new breath with the contributions of H.S.M. Coxeter. Those contributions are extremely numerous to list in here and spread all along the 20th century. Coxeter's monograph[18] on regular polytopes is most likely its most influential publication, but some of his remarkable contributions date as early as 1937 when together with J.F. Petrie described the *regular skew polyhedra*[19] as infinite analogues of Platonic solids. Coxeter is also attributed to classify the groups generated by hyperplane reflections, leading to what we today know as *Coxeter groups*. Coxeter groups have an influential role in several branches of mathematics. They of course, appear as the symmetry groups of regular polytopes and tessellations of the Euclidean and Hyperbolic spaces,[20] but they have made their way to Tits geometries,[21] computational Lie group theory, Hecke algebras,[22] just to mention some.

Coxeter's work serve as inspiration for many mathematicians, one of them being Branko Grünbaum who in the 70's introduced[23] the notion of *polystroma*, which is an ancestor of what we today call *abstract polytopes*. Grünbaum is also responsible of first treating symmetric polyhedra from a combinatorial viewpoint. By relaxing the definition of a regular polyhedron he presented[24] a list of 47 regular polyhedra which included the Platonic Solids, Stellated polyhedra as well as Petrie-Coxeter skew polyhedra. Soon after A. Dress describes[25] another polyhedron and proves[26] that the list of 48 regular polyhedra is complete.

[10] P. Potočnik, P. Spiga, and G. Verret. "Bounding the order of the vertex-stabiliser in 3-valent vertex-transitive and 4-valent arc-transitive graphs". en. In: *J. Combin. Theory Ser. B* 111 (Mar. 2015), pp. 148–180.

[11] P. Potočnik, P. Spiga, and G. Verret. "A census of 4-valent half-arc-transitive graphs and arc-transitive digraphs of valence two". In: *Ars Math. Contemp.* 8.1 (2015), pp. 133–148.

[12] M. D. E. Conder and G. Verret. "Edge-transitive graphs of small order and the answer to a 1967 question by Folkman". In: *Algebr. Comb.* 2.6 (2019), pp. 1275–1284.

[13] D. Holt and G. Royle. "A census of small transitive groups and vertex-transitive graphs". In: *J. Symbolic Comput.* 101 (2020), pp. 51–60.

[14] A. Cayley. "On the theory of groups, as depending on the symbolic equation $\theta^n = 1$". In: *Philos Mag* 7.42 (Jan. 1854), pp. 40–47.

[15] S. R. Blackburn, P. M. Neumann, and G. Venkataraman. *Enumeration of Finite Groups*. Cambridge Tracts in Mathematics. Cambridge: Cambridge University Press, 2007.

[16] H. U. Besche, B. Eick, and E. A. O'Brien. "The groups of order at most 2000". In: *Electronic Research Announcements of the American Mathematical Society* 7 (2001), pp. 1–4.

[17] J. H. Conway. *Atlas of Finite Groups. Maximal Subgroups and Ordinary Characters for Simple Groups.* Oxford University Press, USA, 1986, p. 284.

[18] H. S. M. Coxeter. *Regular polytopes*. Third. Dover Publications, Inc., New York, 1973, pp. xiv+321.

[19] H. S. M. Coxeter. "Regular Skew Polyhedra in Three and Four Dimension, and their Topological Analogues". In: *Proc. London Math. Soc.* S2-43.1 (1937), p. 33.

[20] J. E. Humphreys. *Reflection Groups and Coxeter Groups*. Vol. 29. Cambridge Studies in Advanced Mathematics. Cambridge University Press, June 1990, pp. xii+204.

[21] J. Tits. *Buildings of spherical type and finite BN-pairs*. Lecture Notes in Mathematics, Vol. 386. Springer-Verlag, Berlin-New York, 1974, pp. x+299.

[22] A. M. Cohen. "Coxeter Groups and three Related Topics". en. In: ed. by A. Barlotti et al. NATO ASI Series. Dordrecht: Springer Netherlands, 1991, pp. 235–278.

[23] B. Grünbaum. "Regularity of graphs, complexes and designs". In: *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*. Vol. 260. Colloq. Internat. CNRS. CNRS, Paris, 1978, pp. 191–197.

[24] B. Grünbaum. "Regular polyhedra—old and new". In: *Aequationes Math.* 16.1-2 (1977), pp. 1–20.

[25] A. W. M. Dress. "A combinatorial theory of Grünbaum's new regular polyhedra. I. Grünbaum's new regular polyhedra and their automorphism group". In: *Aequationes Math.* 23.2-3 (1981), pp. 252–265.

[26] A. W. M. Dress. "A combinatorial theory of Grünbaum's new regular polyhedra. II. Complete enumeration". In: *Aequationes Math.* 29.2-3 (1985), pp. 222–243.

Almost at the same time as Grünbaum was describing his families of regular polyhedra on the Euclidean space, G. Jones and D. Singerman published his classical manuscript[27] which settle the necessary theory to identify maps on orientable surfaces with what in modern terminology we called its *monodromy group*. The ideas behind this paper show important equivalences between topological maps (embedding of graphs on orientable surfaces), certain quotients triangular groups (Coxeter groups of rank 3), maps on Riemann surfaces and certain permutations on the darts of the map.

These equivalences are a combinatorial/discrete version of the classical Uniformization theorem[28] for Riemann surfaces. The work of Jones and Singerman was an important contribution on the theory of discrete group actions on Riemann surfaces and it was eventually connected the theory Grothendieck's *Dessins d'enfant*.[29] Some other combinatorial equivalences of maps on surfaces were also explored by Tutte,[30] Vince[31] and Wilson.[32]

The central class of objects in PolyData is that of *highly symmetric abstract polytopes*. Abstract polytopes were introduced by Schulte in his PhD thesis[33] and he also established most of the early results. Abstract polytopes are a particular class of partially ordered sets that combinatorially generalise the (face-lattices) of convex polytopes but also include the incidence structure of many other geometrical objects such as tilings of $\mathbb{E}^n$ and $\mathbb{H}^n$ as well as most maps on surfaces. Early research focused on *regular polytopes*, that is, those with the highest degree of symmetries.

Most of this early theory can be found in the very dense and comprehensive monograph written by Schulte a McMullen.[34] Of our particular interest is the problem of building regular polytopes, for which numerous publications exists. We should mention that there exist universal constructions,[35,36] constructions prescribing local combinatorics[37,38,39] and constructions fixing interesting families of groups as automorphism groups[40,41,42,43] .

The second most studied symmetry class of polytopes is that of *chiral polytopes*. Informally speaking, a chiral polytope is a polytope having full degree of (combinatorial) rotational symmetry without having (combinatorial) reflections. They were introduced by Schulte and Weiss in 1990[44] Chiral polytopes were introduced as a natural generalization of *chiral maps*, which have been part of the classical theory of maps from it begging and numerous examples exist[45,46,47] . However, the problem of constructing chiral polytopes of higher ranks has proved to be much harder to that of constructing regular polytopes. Some rank 4 examples were constructed as quotients of hyperbolic tilings[48,49,50,51] . A universal construction[52] was used to produce the first (infinite) example of a rank-5 chiral polytope. However, the first

[27] G. A. Jones and D. Singerman. "Theory of maps on orientable surfaces". In: *Proc. London Math. Soc. (3)* 37.2 (1978), pp. 273–307.
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[35] E. Schulte. "On arranging regular incidence-complexes as faces of higher-dimensional ones". In: *European J. Combin.* 4.4 (1983), pp. 375–384.
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[41] M. E. Fernandes and D. Leemans. "C-groups of high rank for the symmetric groups". In: *Journal of Algebra* 508 (2018), pp. 196–218.
[42] D. Leemans, J. Moerenhout, and E. O'Reilly-Regueiro. "Projective linear groups as automorphism groups of chiral polytopes". In: *Journal of Geometry* 108.2 (2017), pp. 675–702.
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[51] B. Nostrand and E. Schulte. "Chiral polytopes from hyperbolic honeycombs". In: *Discrete Comput. Geom.* 13.1 (1995), pp. 17–39.
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finite rank 5 polytopes were constructed by Conder et al. in 2008.[53] It was until 2010 that Pellicer showed[54] the existence of chiral polytopes of rank n for ever $n \geqslant 4$; The result by Pellicer, although constructive, is not very practical. The size of his examples grow as a tower of exponential functions with length depending on n. Later on examples of new chiral polytopes have been constructed from previously known ones[55,56,57,58] .

The degree of symmetry of a polytope can be measured by the number of orbits of *flags*. Regular polytopes have 1 flag-orbit (1-orbit polytopes). Chiral polytopes are just one of $2^n - 1$ possible symmetry type class of 2-orbit polytopes. Classical examples of some of the other classes of the 2-orbit polytopes are known. However, the general problem of determining if for every pair (n, T) with $n > 3$ and T a 2-orbit symmetry type exists an n-polytope of symmetry type T remains open. Some examples of maniplexes were build very recently,[59] but whether or not they are polytopal remains unknown. Constructing and classifying n-polytopes with k-orbits and given symmetry type is still a widely open problem[60] and it is one of the main motivations for the development of PolyData .

Datasets of polytopes

As shown above, classification and enumeration of highly symmetric polytopes has been part of the theory from the beginning. Both, the classification of the 5 Platonic Solids to the enumeration of the 48 Grünbaum-Dress polyhedra in $\mathbb{E}^3$ depend on strong geometric restrictions. However, the combinatorial nature of abstract poltypes open the possibilities to, in principle, have numerous examples of abstract polytopes. These has lead to the construction of some datasets of highly symmetric polytopes, which we review below.

Conder - Regular orientable maps by genus[61] Computed by M. Conder it originally contained all (3378) regular maps on orientable surfaces of genus 2 to 101 up to isomorphism an duality. It was later[62] extended to include genus up to 301 for a total of 15824 maps. Computed with the help of `LowIndexNormalSubgroups` routine of MAGMA and published as raw text.

Conder - Regular non-orientable maps by genus[63] Every map on a non-orientable surface admits an orientable double cover. Conder used this fact to originally compute all (862) non-orientable regular maps of genus 2 to 202 and then[64] extended to genus up to 602 for a total of 3260 maps. Computed with the help of `LowIndexNormalSubgroups` routine of MAGMA and published as raw text.

Conder - Chiral maps by genus[65] Census containing all (594) chiral maps on orientable surfaces of genus 2 to 101. This census was later[66] extended to genus up to 301 for a total of 3870. Computed with the help of `LowIndexNormalSubgroups` routine of MAGMA and published as raw text.

Conder - Rotary maps by size[67] This census contains all rotary (that is regular or chiral) maps whose rotation group has less than 2000 elements (equivalently, such that the map has less than 1000 edges). Computed with the help of `LowIndexNormalSubgroups` routine of MAGMA and published as raw text. There exist versions of this census containing only regular and only chiral maps.

Potocnik - Regular maps by size[68] An improvement on Conder's census containing all (255, 980) regular maps whose automorphism group is of order less than 6, 000 for orientable maps and 3, 000 for non-orientable maps. Published as MAGMA files available to download with a CVS-file of precomputed information.

Potocnik - Chiral maps by size[69] An analogous to the one above but for chiral maps. It contains a total of 122, 092 chiral maps whose automorphism group has order less than 6, 000.

[53] M. Conder, I. Hubard, and T. Pisanski. "Constructions for chiral polytopes". In: *J. Lond. Math. Soc. (2)* 77.1 (2008), pp. 115–129.
[54] D. Pellicer. "A construction of higher rank chiral polytopes". In: *Discrete Math.* 310.6-7 (2010), pp. 1222–1237.
[55] G. Cunningham and D. Pellicer. "Chiral extensions of chiral polytopes". In: *Discrete Math.* 330 (2014), pp. 51–60.
[56] M. D. E. Conder and W.-J. Zhang. "Abelian covers of chiral polytopes". In: *J. Algebra* 478 (2017), pp. 437–457.
[57] A. Montero. "Chiral extensions of toroids". PhD thesis. National University of Mexico, 2019.
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[59] D. Pellicer, P. Potočnik, and M. Toledo. "An existence result on two-orbit maniplexes". In: *J. Combin. Theory Ser. A* 166 (Aug. 2019), pp. 226–253.
[60] G. Cunningham and D. Pellicer. "Open problems on k-orbit polytopes". In: *Discrete Math.* 341.6 (2018), pp. 1645–1661.
[61] M. Conder. *Regular orientable maps of genus 2 to 101.* 2006. URL: https://www.math.auckland.ac.nz/~conder/RegularOrientableMaps101.txt.
[62] M. Conder. *Regular orientable maps of genus 2 to 301.* Total number of maps in list below: 15824. 2011. URL: https://www.math.auckland.ac.nz/~conder/RegularOrientableMaps301.txt.
[63] M. Conder. *Regular non-orientable maps of genus 2 to 202.* Total number of maps in list below: 862. 2006. URL: https://www.math.auckland.ac.nz/~conder/RegularNonorientableMaps202.txt.
[64] M. Conder. *Regular non-orientable maps of genus 2 to 602.* Total number of maps in list below: 3260. 2012. URL: https://www.math.auckland.ac.nz/~conder/RegularNonorientableMaps602.txt.
[65] M. Conder. *Chiral orientably-regular maps of genus 2 to 101.* Total number of maps in list below: 594. 2006. URL: https://www.math.auckland.ac.nz/~conder/ChiralMaps101.txt.
[66] M. Conder. *Chiral rotary maps of genus 2 to 301.* Total number of maps in list below: 3870. 2013. URL: https://www.math.auckland.ac.nz/~conder/ChiralMaps301.txt.
[67] M. Conder. *Rotary maps (on orientable or non-orientable surfaces) with up to 1000 edges.* 2012. URL: https://www.math.auckland.ac.nz/~conder/RotaryMapsWithUpTo1000Edges.txt.
[68] P. Potočnik. *Census of regular maps.* 2014. URL: https://www.fmf.uni-lj.si/~potocnik/work.htm.
[69] P. Potočnik. *Census of chiral maps.* 2014. URL: https://www.fmf.uni-lj.si/~potocnik/work.htm.

Dataset	Rank 3	Rank 4	Rank ⩾ 5
Hartley - Regular	64.55%	31.61%	3.84%
Hartley - Chiral	85.71%	14.29%	0.00%
Conder - Regular	61.51%	34.70%	3.79%
Conder - Chiral	87.01%	12.87%	0.12%
Leemans - Regular	95.35%	4.37%	0.30%
Leemans - Chiral	87.82%	11.97%	0.21%

Table 1: Percentages of examples according to rank

Hartley - The Atlas of Small Regular Polytopes.[70] It was build using `SmallGrp` routine of GAP and contains all regular polytopes with at most 2000 flags, except those of size 1024 and 1536. It contains 9212 examples. They are presented in a nice web interface and the code is available to download.

Hartley - The Atlas of Small Chiral Polytopes[71] Every chiral polytope admits a minimal regular cover. Hartley used this fact to compute the first atlas of chiral polytopes. This dataset consists of all chiral polytopes whose minimal regular cover belongs to the Atlas of Small Regular Polytopes. This gave a total of 48 chiral polytopes of rank 3 and 8 polytopes of rank 4.

Conder - Regular polytopes up to 2,000 flags[72] A dataset containing, up to duality, all (5809) regular polytopes with at most 2000 flags (which is the same as the order of the automorphism group).

Conder - Chiral polytopes up to 2,000 flags[73] A dataset containing, up to duality, all (839) chiral polytopes with at most 2000 flags (which is the twice the order of the automorphism group).

Leemans et al. - An Atlas of polytopes for small simple groups[74] This is an ongoing atlas that contains regular polytopes whose automorphism group is an almost simple group. It currently contains $55,575$ regular polytopes. The atlas is presented on a website with downloadable data.

Leemans et al. - An Atlas of chiral polytopes for small simple groups[75] It is the analogous to the one above but for chiral polytopes. It currently contains a total of $19,964$ polytopes.

Problem identification and Research and Innovation objectives.

These existing datasets of polytopes suffer of the following restrictions

(i) They are mainly focused on regular or chiral polytopes.
(ii) The size of the examples is very restrictive.
(iii) They often exhibit numerous examples of rank 3 but the amount of examples of rank higher than 4 drops dramatically.
(iv) They are not very user-friendly, either because they exist only as raw data or because the are specific-programming language oriented.

As explained before, most of the existent datasets of polytopes are either completely focused on 3-polytopes (maps) or have very little examples of higher ranks. In Table 1 we show the proportion of examples according to ranks.

There are two obvious gaps that need to be pushed forward, not necessarily in an independent way, on the process of building new datasets of abstract polytopes: finding examples of higher ranks and building sets that consider different types of symmetries (besides chiral or regular). With the emerging development of theoretical results for less symmetric polytopes and the need to identify patterns and to find new constructions to attack the numerous open problems related to the existence of polytopes, it is clear that building new datasets of polytopes that overturn the restrictions mentioned above would not only be beneficial but it is almost necessary.

The problematic expressed above outlines our first objective.

Objective 1 *Extend the existing and build new datasets of abstract polytopes and related structures with particular focus on*

(i) Building examples on ranks higher than 3

[70] M. Hartley. *The Atlas of Small Regular Polytopes.* 2006. URL: https://www.abstract-polytopes.com/atlas/.
[71] M. Hartley. *The Atlas of Small Chiral Polytopes.* 2006. URL: https://www.abstract-polytopes.com/chiral/.
[72] M. Conder. *Regular polytopes with up to 2000 flags.* 2012. URL: https://www.math.auckland.ac.nz/~conder/RegularPolytopesWithFewFlags-ByOrder.txt.
[73] M. Conder. *Chiral polytopes with up to 2000 flags.* 2012. URL: https://www.math.auckland.ac.nz/~conder/ChiralPolytopesWithFewFlags-ByOrder.txt.
[74] D. Leemans et al. *An Atlas of Polytopes for Small Almost Simple Groups.* URL: https://leemans.dimitri.web.ulb.be/polytopes/index.html.
[75] M. Hartley, I. Hubard, and D. Leemans. *An Atlas of Chiral Polytopes for Small Almost Simple Groups.* URL: https://leemans.dimitri.web.ulb.be/CHIRAL/index.html.

(ii) Exploring different symmetry types.

Objective 1 is of course very general but also very ambitious and should be interpreted as the general research line of PolyData . Any contribution to this objective is a good way of measuring the global success of PolyData .

Another pressing issue to address is that the existent datasets of highly symmetric maps and polytopes are not only limited in the sense discussed above, they have not been exploded to its full capacity. One of the reasons behind it is that the information in most of these data sets is not very user-friendly. Even the small amount of existing data sets have been developed by several people, mostly in an independent way, using different notation and different computer algebra systems. Moreover, many of these datasets exist only as raw text which is not always easy to consult.

Objective 2 *Collect, unify and make the existing data sets user-friendly and in a FAIR-ly way so that PolyData eventually becomes the standard to-go when consulting or publishing new data on highly symmetric polytopes by*

(i) Creating and developing a unique data set from the existent ones that unifies notation and identification of the data and make it available so that other researches could experiment and eventually contribute to PolyData .
(ii) Building a web-based interface to our data set for easy and quick consultation.
(iii) Developing this data set in a way that can be easily implemented in some of the standards computer algebra systems such as MAGMA , GAP and `SageMath` .
(iv) Writing appropriate documentation so that PolyData eventually becomes available for others to contribute.

Objective 2 is very concrete and very easy to verify. It very easily show progress on PolyData while at the same time sets a good start point to our ambitious general objective. The current state-of-the-art allows us to start with from the very beginning of PolyData

Many of the first data sets of abstract polytopes are based on the (small) size of the objects. Either by taking advantage of previously computed objects (such as the library `SmallGrp`) or by using computational routines that, because of their own nature are limited by the size of the input (such as the `LowIndexNormalSubgroups`). However, it has been shown that the size of the smallest regular polytope of rank n grows exponentially with n[76] , while the size of the smallest chiral n-polytope is at least of factorial growth with respect to n.[77] This explains why the amount of examples of higher rank polytopes drops dramatically on the current available datasets.

The lack of not only datasets but also theoretical constructions of certain symmetry types for polytopes motivates the following objective.

Objective 3 *Develop new constructions of highly symmetric polytopes. In particular, focus on the constructions of abstract regular polytopes with given symmetry type to eventually build the corresponding data sets.*

Objective 3 by itself represents an extremely ambitious and it goes beyond any two year project. Any theoretical contribution is already of great interest to the community. However, we strongly believe that in order to really cause an impact on the workflow of research on abstract polytopes all theoretical contributions should be accompanied by its computational analogue.

Of course, PolyData is aimed to become a long term and eventually permanent resource for the community doing research on symmetries of maps and abstract polytopes. This will not be achievable without the involvement of such community.

Objective 4 *Encourage and motivate both well-established and young researchers to use and contribute to PolyData so that it eventually becomes the standard way to explore, experiment and publish data sets, routines and computational tools for the development of the research of abstract polytopes.*

Objective 4 is very ambitious and we acknowledge that it depends on the community more that on ourselves, but we strongly believe that our approach and the the current status of the could fit together to fill a gap that has been present for many years now. It is important to remark that the community has faced a similar scenario before. Many of the early research on abstract polytopes was collected on the comprehensive manuscript[78] and nowadays it serves as a natural and standard theoretical reference. Our expectations is that PolyData eventually becomes the computational analogue for our community.

[76] M. Conder. “The smallest regular polytopes of given rank”. en. In: *Adv Math* 236 (Mar. 2013), pp. 92–110.
[77] G. Cunningham. “Non-flat regular polytopes and restrictions on chiral polytopes”. In: *Electron. J. Combin.* 24.3 (2017), Paper 3.59, 14.
[78] McMullen and Schulte, *Abstract regular polytopes*.

1.2 Soundness of the proposed methodology (including interdisciplinary approaches, consideration of the gender dimension and other diversity aspects if relevant for the research project, and the quality of open science practices, including sharing and management of research outputs and engagement of citizens, civil society and end users, where appropriate)

The nature of our project involves a two way flow of knowledge between from theoretical mathematics to the development of computational tools and data management. Moreover, symmetries of abstract polytopes sits in the interplay between combinatorics, group theory, geometry and topology. In order to build new datasets of abstract polytopes we need computational efficient ways to compute and storage such objects and the other way around: from analysing new datasets we can identify patterns, formulate new conjectures and eventually develop new theoretical constructions.

In the following paragraphs we present some techniques and methodologies to build highly symmetric abstract polytopes and hence, datasets of such.

The original[79] definition of an abstract polytope is on the form of a poset . It makes sense from an historical view point. Abstract are intended to be combinatorial generalisation of the (geometric) convex polytopes and this generalisation was obtained by taking some of the properties of the face lattice of a convex polytope and use them as defining properties of an abstract polytope. However, usually storing an abstract polytopes as a poset is redundant and inefficient. We describe below some other ways of representing a highly symmetric polytope and briefly describe how they could be useful in our objectives.

Abstract polytopes from their automorphism group. This is the most used representation of an abstract polytope. When an abstract polytope has a high degree of symmetry its automorphism group contains many combinatorial information of the polytope. In particular, regular polytopes are in correspondence with string C-groups, which are smooth quotient of Coxeter groups satisfying certain intersection property. This fact has been strongly used to build the existing datasets mentioned in Section 1.1. Conder's census of regular polytopes[80] was built by computing all possible normal subgroups of index at most 2000 of the universal string Coxeter group. This approach is computational expensive it might be worthy try to push the bound of 2000 further. Hartley's Atlas of small regular polytopes[81] and Leemans's Atlas of regular polytopes[82] were built by analysing which of the groups in the library `SmallGrp` of GAP (Hartley's) or in Conway's Atlas of finite groups[83] (Leemans's) are string C-group. The automorphism group of a chiral polytope was characterised by Schulte and Weiss[84] and similar approaches to those for regular polytopes were used to build the existing datasets of chiral polytopes.

Very recently Mochán[85] fully described how to build an abstract polytope of arbitrary symmetry type from its automorphism group. A first approximation to part *(ii)* of Objective 1 must be to explore known datasets of groups and determine which of those groups can give automorphism groups of non regular and non chiral abstract polytopes.

Schreier coset graphs and permutation groups. Schreier coset graphs are a classical tool to represent a permutation group. Large groups can be represented with relative small graphs; for example, the symmetric group S_n with $n!$ elements admits a representation on a graph with n vertices. Schreier coset graphs offer an efficient tool to represent automorphism groups of abstract polytopes as certain graph. These tools have been used on the context of polytopes before[86,87,88,89] from a theoretical interest but little has been done to implement them as a tool to building datasets.

It is important to remark that these graphs already have shown potential to improve known computational methods. In a manuscript[90] by the candidate and Weiss infinite families of regular hypertopes (a generalisation of abstract polytopes) were build using Schreier coset graphs, solving an open question from a previous manuscript[91] where the authors were not able to find a single example using traditional computational tools (in particular `LowIndexNormalSubgroups` form MAGMA).

Maniplexes. For every $0 \leqslant i \leqslant n-1$ and every flag (maximal chain) on an n-polytope $\mathcal{P}$, there exists a unique i-adjacent flag. This provides the set of flags of a polytope a structure of n-edge-coloured graph, the *flag-graph* of $\mathcal{P}$. Wilson introduced the notion of *maniplexes*[92] as a generalisation of maps. Every (flag-graph of a) polytope is a maniplex

[79] Schulte, "Reguläre Inzidenzkomplexe".
[80] Conder, *Regular non-orientable maps of genus 2 to 602*.
[81] Hartley, *The Atlas of Small Regular Polytopes*.
[82] Leemans et al., *An Atlas of Polytopes for Small Almost Simple Groups*.
[83] Conway, *Atlas of Finite Groups*.
[84] Schulte and Weiss, "Chiral polytopes".
[85] E. Mochán. "Abstract polytopes from their symmetry type graph". PhD thesis. National University of Mexico, 2021.
[86] Pellicer, "CPR graphs and regular polytopes".
[87] Pellicer, "Extensions of regular polytopes with preassigned Schläfli symbol".
[88] D. Pellicer and A. I. Weiss. "Generalized CPR-graphs and applications". In: *Contrib. Discrete Math.* 5.2 (2010), pp. 76–105.
[89] M. E. Fernandes and C. A. Piedade. "Faithful permutation representations of toroidal regular maps". In: *J Algebr Comb* (Sept. 2019).
[90] A. Montero and A. I. Weiss. "Proper locally spherical hypertopes of hyperbolic type". en. In: *J Algebr Comb* (Oct. 2021).
[91] M. E. Fernandes, D. Leemans, and A. I. Weiss. "An Exploration of Locally Spherical Regular Hypertopes". In: *Discrete & Computational Geometry* (June 2020).
[92] Wilson, "Maniplexes: Part 1: maps, polytopes, symmetry and operators".

and Hubard and Garza-Vargas characterised[93] maniplexes that are polytopes. This graph representation of polytopes improves storage and identification of isomorphic, due to the implementation of variants of the canonical labelling algorithm for graphs.

Quotients of the universal Coxeter group. The generator r_i of the *universal string Coxeter group* $\mathcal{U}$ acts on a maniplex by swapping all the i-adjacent flags. This action is transitive and the induced permutation group is called the *monodromy group*. This group keeps all the combinatorial information of a maniplex. The kernel K of the action determines the maniplex as a quotient $\mathcal{U}/N$ and several combinatorial properties arise in a group theoretical way without relying on particular symmetry properties. This approach could be potentially useful for Objective 1.

Extensions. A polytopes $\mathcal{P}$ is an extension of a polytope $\mathcal{K}$ if all the facets of $\mathcal{P}$ are isomorphic to $\mathcal{K}$. Symmetry conditions on $\mathcal{K}$ imply symmetry restriction on $\mathcal{P}$. This has been used[94,95,96]to build new polytopes from previously known ones with desired symmetry conditions.

Operations. Very informally, an operation is a mapping O that assigns a polytope $O(\mathcal{K})$ to each polytope $\mathcal{K}$. Usually the symmetries of $O(\mathcal{K})$ are related to the symmetries of $\mathcal{K}$. This approach has been used[97,98,99,100]to build polytopes with prescribe symmetry type

1.3 Quality of supervision, training, and knowledge transfer

1.4 Quality and appropriateness of the researcher's professional experience, competences and skills

The candidate is a young researcher with wide experience on abstract polytopes. He has a strong background on several topics of discrete mathematics, but in particular on abstract polytopes. He started participating on conferences and workshops more than 10 years ago and has been active on the community ever since. The early years of his academic career focused on the geometric side of abstract polytopes. He started doing research on abstract polytopes very early in his career. His undergraduate thesis[101] attacks the problem of enumerating toroidal polyhedra. Later on he did a complete classification of such polyhedra.[102] During his Ph. D. the candidate worked under the supervision of Daniel Pellicer, one of the young leading experts on abstract polytopes. The candidate moved his research interests to one of the most challenging problems on the theory: constructing chiral polytopes with prescribed regular facets. The results obtained[103,104] were only partial but trained the candidate on several approaches and the process gave him tools from geometry, combinatorics, group theory and programming.

The candidate did a Postdoctoral visit of one year in York University (Canada) under the supervision of A. Weiss, one of the most established researchers on the area were the research focus was mostly on building highly symmetric polytopal objects.[105,106] He continued his career in the National Autonomous University of Mexico, where he spent one year and a half working with I. Hubard. During this period he continued his work on building symmetrical objects. In particular, his research turn into the problem of building maniplexes and polytopes with given symmetry type. The candidate started two projects on this topic: One related to operations of maniplexes with Hubard and Mochán and another related to extensions of maniplexes in joint work with G. Cunningham and Mochán.

His academic relation with the Slovenian mathematical community began when he was a Ph. D. student in 2017 and spent a research visit of six months in the University of Ljubljana.

The candidate has experience not only doing research but also on the communications of mathematics. He has given more than 35 talks on colloquia, seminars and conferences, including some of them as invited speaker. In 2018 the candidate was awarded with the *best student talk award* in the $8th$ Ph.D. summer school in discrete mathematics. The candidate has also participated and organised outreach activities, which has given them some team management skills.

2 Impact

[93] J. Garza-Vargas and I. Hubard. "Polytopality of maniplexes". In: *Discrete Math.* 341.7 (2018), pp. 2068–2079.
[94] Pellicer, "Extensions of regular polytopes with preassigned Schläfli symbol".
[95] Cunningham and Pellicer, "Chiral extensions of chiral polytopes".
[96] Pellicer, Potočnik, and Toledo, "An existence result on two-orbit maniplexes".
[97] A. Orbanić, D. Pellicer, and A. I. Weiss. "Map operations and k-orbit maps". In: *J. Combin. Theory Ser. A* 117.4 (2010), pp. 411–429.
[98] I. Helfand. "Constructions of k-orbit Abstract Polytopes". PhD thesis. Northeastern University, 2013.
[99] G. Cunningham, D. Pellicer, and G. Williams. "Stratified operations on maniplexes". To appear.
[100] I. Hubard, E. Mochán, and A. Montero. "Voltage operations on maniplexes". In preparation.
[101] **UndergradThesis**.
[102] A. Montero. "Regular polyhedra in the 3-torus". In: *Advances in Geometry* 18.4 (2018), pp. 431–450.
[103] Montero, "Chiral extensions of toroids".
[104] A. Montero, D. Pellicer, and M. Toledo. "Chiral extensions of regular toroids". In preparation.
[105] A. Montero and A. I. Weiss. "Locally spherical hypertopes from generalised cubes". en. In: *Art Discrete Appl. Math.* 4.3 (Aug. 2021), #P3.07–#P3.07.
[106] Montero and Weiss, "Proper locally spherical hypertopes of hyperbolic type".

2.1 Credibility of the measures to enhance the career perspectives and employability of the researcher and contribution to his/her skills development

The candidate's long-term goal is to obtain a permanent academic position. The interdisciplinary nature of the proposed research will train the candidate in two main sets of skills. On the one hand, with the guidance of the supervisor, the candidate will consolidate his young career as a theoretical mathematician working with highly symmetric objects. In particular, the candidate will gain training on *permutation groups* and *symmetries of graphs*. This will increase the candidate's presence on the community, and hence improve his chances to obtain a permanent academic job. On the other hand, the computational skills needed to build the proposed datasets will make the candidate a desirable researcher not only for pure math departments but also for computer science departments.

Moreover, the time that the candidate will spend in the *Faculty of Mathematics and Physics* of the University of Ljubljana will give the candidate opportunities to increase his teaching experience.

Finally, the programming and data management training obtained as a result of the proposed research will not only improve the candidate's chances to get not only an academic position but also a non academic job.

2.2 Suitability and quality of the measures to maximise expected outcomes and impacts, as set out in the dissemination and exploitation plan, including communication activities

As described in previous section, our project has two main branches: a purely theoretical one and a computational one. The activities planned to disseminate and communicate our activities can be naturally divided in those two branches as well.

For the theoretical part of our research the candidate will have day-to-day conversations not only with the supervisor but also with other researches on the math department of FMF-UL. The candidate will constantly participate on the *discrete mathematics seminar* of FMF-UL and the *combinatorics and group theory seminar* of the Faculty of Education of the University of Ljubljana (PeF-UL). The candidate will participate on the 2023 International Slovenian Conference on Graph theory, which is an international forum that occurs every 4 years and has had more than 300 participants in the last editions. These activities align perfectly with Objectives 3 and 4.

In other to develop the computational part of the project and the actual development of Objective 1, the candidate will work in close contact with Katja Berčič, who is an expert on *mathematical knowledge management* and the creation of datasets. The candidate will organise at least one *programming workshop* per year where both researcher and students will get together and discuss problems regarding datasets of discrete objects.

The research objects involved in PolyData are not only of mathematical interest but the have a natural degree of beauty. With this in mind the candidate is expected to participate in the $28th$ and $29th$ editions of the Slovenian Festival of Science, where some of the aspects of the research developed can be shared to a general audience with activities such as *Origami Polyhedra* or *Build your own kaleidoscope*.

2.3 The magnitude and importance of the project's contribution to the expected scientific, societal and economic impacts

Our research project is intended to make a heavy impact on the community doing research on highly symmetric combinatorial structures. More particularly we expect that with the success of PolyData the way and methodology used to do research on abstract polytopes improves significantly. We also expect PolyData to be the first step in a long-term project so that it eventually becomes a standard reference to look for datasets of highly symmetric polytopes that is both accessible and easy to use.

Just the theoretical implications of our project will help the community to better understand highly symmetric polytopes. However, we should emphasise that by standardising the existing and building new datasets of abstract polytopes we expect to offer a new way to do research that is closer to the way other sciences work.

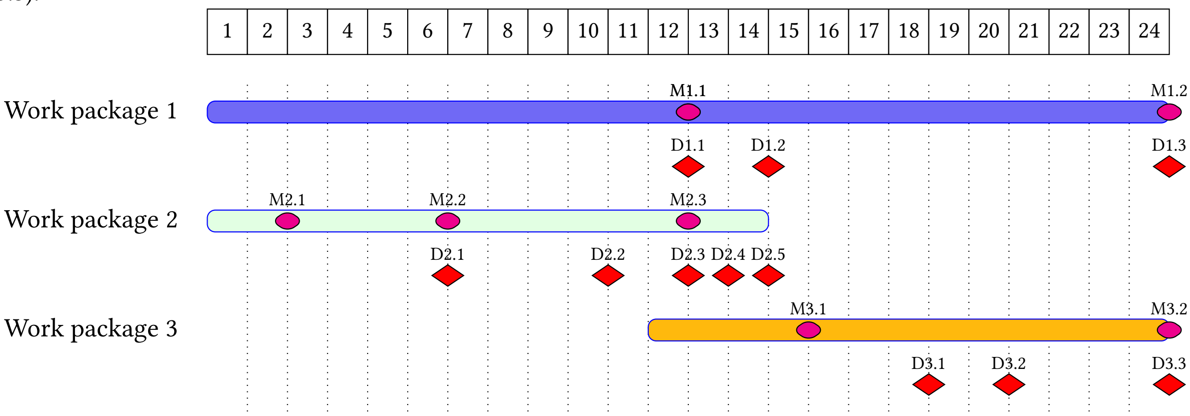
3 Quality and Efficiency of the Implementation

3.1 Quality and effectiveness of the work plan, assessment of risks and appropriateness of the effort assigned to work packages

Work package 1 *(24 moths).* **Develop theoretical constructions of non-chiral 2-orbit polytopes and k-orbit polytopes for $k \geqslant 3$.** This is our main theoretical work plan. The candidate will explore the technique of *operations* to build new polytopes from previously existing polytopes as well as the technique of *extensions* to build new polytopes with prescribed facets. *Milestone (1.1)* is set for the first 12 months and consist of having the first constructions of families of non regular o chiral polytopes. The success of this task will result on the preparation and submission of at least 2 papers to a high-quality international peer-reviewed journal (*Derivables 1.1 and 1.2*). Then the candidate will explore new theoretical constructions *(Milestone 1.2)* which eventually result on the publication of at least one manuscript more (*Derivable 1.3*).

Work package 2 *(14 months).* **Collect and unify existing datasets of highly symmetric polytopes.** First the candidate will collect the existing dataset on a common format (*Milestone 2.1*, 2 months). Then these data will be use to set a FAIR repository (*Milestone 2.2* (4 months) and *Derivable 2.1*). Finally the repository will be used to build a website (Derivable 2.2) and corresponding software packages to be used in computer algebra systems such as GAP (Derivable 2.3), MAGMA (Derivable 2.4) or `SageMath` (Derivable 2.5). This last step is Milestone 2.3 (6 months).

Work package 3 *(12 months).* **Develop new datasets of abstract polytopes.** From the success of Work package 1 and the foundations given in Work package 2, the candidate will create new datasets containing both existing (Milestone 3.1, 3 months) and newly constructed (Milestone 3.2, 9 months) highly symmetric polytopes. These new datasets will bring an update to the FAIR repository (Derivable 3.1) to the website(Derivable 3.2) and to the software packages (Derivable 3.3).



3.2 Quality and capacity of the host institutions and participating organisations, including hosting arrangements