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Section a. State-of-the-art and objectives

Motivation and background

There are fundamental differences between the standard scientific method and the method of mathematical research. In essence, the scientific method is founded upon the observation of natural phenomena and the measurement of physical quantities. These are then used to pose hypotheses, which are then tested against cleverly designed experiments. In this way, many false hypotheses can be quickly rejected, while those that survive these tests are eventually recognised as laws of nature. Conversely, in theoretical mathematics, there is frequently a paucity of initial data upon which hypotheses can be formulated. The process of mathematical research thus typically commences with an abstract concept, which may be based on intuition and a limited number of illustrative examples. These ideas eventually evolve to conjectures. However, in contrast to the natural sciences, experimental methodology is absent in many mathematical disciplines, making it challenging to validate conjectures through empirical evidence. As we all know too well, this frequently results in significant time investment in attempts to disprove erroneous conjectures, hours that could be spent on proving theorems instead.

Mathematicians are so used to this way of conducting research that the possibility of using experimental data is often overlooked even when available or not too difficult to obtain. This is why we consider the construction of explicit and manageable interesting examples to be one of the most important lines of research in mathematics. In recent years, with the development of computers, the ability to run experiments and compute large datasets of mathematical objects has opened up new possibilities in mathematical research.

Of our particular interest is to build objects with a large degree of mathematical symmetry. The notion of symmetry is naturally present in almost any scientific discipline. Noether's First Theorem [51] links conservation laws in physical systems with the presence of certain degree of symmetries. This result is nowadays regarded as a breakthrough contribution to modern theoretical physics (see [32]). The notion of *chirality*, first introduced by Lord Kelvin [31], describes the property of a molecule that cannot be superposed by rigid movements to its mirror image [41]. In this case the (lack of) symmetry properties impose different chemical behaviour (see [30], for example).

While each science focus on the consequences of the presence or absence of symmetries, in mathematics we study the phenomenon in arguably its purest essence. This is usually controlled by the action of a certain group of transformations that respect the global structure, often called *automorphisms*. An object is often considering symmetric if it has a *rich* automorphism group; this notion of *richness* might be present in the form of having a large automorphism group or admitting such a group acting in a specific way (for example, transitively) on the atomic parts of the mathematical object.

Sometimes the presence of symmetry impose conditions on the structure of the mathematical object. There are many examples of this phenomenon: from the classification of Platonic solids, arguably the first mathematical result involving symmetries; the Uniformization Theorem [1] and Thurston's geometrization conjecture [64], where local symmetric properties impose global uniformity structure; or the polycirculant conjecture in graph theory [38]. On the other hand, sometimes structural properties of the objects impose restrictions on the presence of symmetry. Classical results in this direction are those obtained by Hurwitz [29] to bound the order of the automorphism group of a compact Riemann surface or those by Tutte [65] where the order of the automorphism group of an arc-transitive cubic graphs is bounded.

In general, when studying symmetries from a mathematical perspective we are interested in the following questions

Problem 1. How symmetrical a given mathematical object can be?; what are the structural properties imposed by high degree of symmetry? what are the possible *symmetry types* of a family of objects? are all those symmetry types achievable?

The core ideas of this proposal pose the questions presented in Problem 1 in the context of highly symmetric abstract polytopes. Our research approach is inspired by the first paragraphs of this introduction and relies on the construction of datasets of such objects (explained in detail below).

Highly symmetric polytopes and related objects

Abstract polytopes are combinatorial objects that generalise (the face lattice) of convex polytopes. Enumeration and classification of highly symmetric convex polytopes goes all the way to the Greeks and the classical problem of classifying what today we know as *Platonic Solids*, to the beginning of last century when the classification of higher dimensional convex polytopes was achieved. By considering combinatorial objects and hence removing the geometric constraints, often imposed by the ambient space, the possibilities for constructions of highly symmetric abstract polytopes are now wide open, and a complete classification problem becomes unattainable and rather turns into a series of construction and enumeration problems of families with particular characteristics.

The approach described above aligns with our research approach presented above and motivates our deepest mathematical interests: developing combinatorial constructions of abstract polytopes with prescribed combinatorial properties.

A *flag* of an abstract polytope of rank n is a n-tuple of mutually incident faces, one of each rank (dimension). The degree of symmetry of a polytope can be measured by the number of flag-orbits of its automorphism group. This combinatorial way of measuring symmetry usually agrees with the classical geometrical notion (see [13, 14, 21], for example). *Regular polytopes* are those that are flag-transitive, meaning they have exactly 1 flag-orbit. This *symmetry type* of polytopes has been traditionally the most studied one and includes classical examples as Platonic solids and regular tilings of Euclidean and Hyperbolic spaces. A slightly less symmetric class of polytopes is that of 2-*orbit polytopes* and among those, *chiral polytopes* are the most studied. The notion of chirality is a classical one in other natural sciences, notably in Chemistry. However, in the context of abstract polytopes, chirality has a very specific meaning. Chiral polytopes were formally introduced in [62] by Schulte and Weiss as a combinatorial generalisation of Coxeter's twisted honeycombs [12]. An abstract polytope is chiral if it has 2 orbits on flags such that adjacent flags belong to different orbits. Informally speaking, a chiral polytope is one that admits maximal degree of combinatorial rotations but that do not admit mirror reflections.

Chiral polytopes are, without a doubt, the second most studied symmetry type of abstract polytopes. Even so, the amount of theory developed around this symmetry type is very limited, in particular when compared with the regular case. One potential reason behind this is the building manageable examples of chiral polytopes has proved to be a extremely difficult problem. Moreover the mere existence of regular polytopes in *higher ranks* (every rank larger than 3) was established fairly recently [53]. However, the examples provided by Pellicer are very quickly impractical due to their size. The existence of chiral polytopes in higher ranks can be obtained also a consequence of recent results by Conder, Hubard and O'Reilly [10]. In a joint work with Toledo we used a similar general approach to obtain some results that also prove the existence of chiral polytopes in higher ranks (see [49]). We discuss these results in slightly deeper detail below.

Chiral polytopes are just one of $2^n - 1$ possible symmetry types of 2-orbit n-polytopes. Classical examples of some of the other classes of the 2-orbit polytopes are known, particularly in small ranks. However, the general problem of determining if for every pair (n, T) with $n > 3$ and T a 2-orbit symmetry type exists an n-polytope of symmetry type T remains open.

Maniplexes, premaniplexes, symmetry type graphs

When we study abstract regular polytopes with less symmetry, that is, with several flag orbits, the number of such orbits gives us only partial information. Here is where the notion of the *symmetry type graph (STG)* comes in handy. The *flag-graph* of a polytopes the (edge-coloured) graph whose nodes are the flags of the polytope in such a way that two of them are connected by an edge of colour i ($i \in \{0, \dots, n-1\}$) if the corresponding flags differ exactly on the face of rank i. It can be proved that the automorphism group of a polytope acts as colour-preserving graph automorphisms of the flag-graph. Moreover this action is semiregular. The *symmetry type graph (STG)* of a polytope is the quotient of its flag-graph by its automorphism group [16]. This graph has one vertex for each flag-orbit and its connections show the local arrangement of the flag orbits.

Maniplexes arise naturally as graph-theoretical generalisations of abstract polytopes. A *maniplex of rank n* (or n-maniplex) is a properly-edge-colour n-valent graph that satisfies some properties (that can be made precise) that resemble those of the flag-graph of a polytope. *Premaniplexes* (see [26, 27]) are properly edge coloured graphs that satisfy certain (natural) conditions so that, when connected, become the most natural candidate to be the symmetry graph of a polytope, in the sense that every STG of a k-orbit polytope is a connected premaniplex with k vertices.

Determining weather or not there exist polytopes or maniplexes with a given symmetry type is one of the most relevant and long lasting problems on the theory; it that has shown to be extremely difficult even for the simplest cases. However, we pose the following conjecture as theoretical driving goal of our research.

Conjecture 1 (Symmetry-type conjecture)**.** *For every n-premaniplex $\mathcal{T}$ there exist an abstract polytope $\mathcal{P}$ such that the symmetry type graph of $\mathcal{P}$ is isomorphic to $\mathcal{T}$.*

Research Objective 1. Prove the Symmetry-type conjecture (Conjecture 1).

Of course Research objective 1 is very ambitious and represents a challenging research goal. This conjecture has been presented before in the form of a problem (see [19, Problem 12]). In [55], Pellicer, Potočnik and Toledo build examples of 2-orbit n-maniplexes for $n \geqslant 4$ and every symmetry type with 2 vertices. Very recently Mochán [42] proved that several of them are actually polytopes. However, it is still unknown if Conjecture 1 holds for most of the 2-orbit symmetry types. Conjecture 1 is complete solved when $\mathcal{T}$ has 3 vertices, that is, for $n \geqslant 3$ there exist n-polytopes with each of the possible connected premaniplexes with 3 vertices as their symmetry type graph (see [19, Theorem 4.3])

In [27] we propose the approach of operations as a way of constructing k-orbit (for $k \geqslant 4$) maniplexes. That is, the use of *voltage operations*. Voltage operations are a graph theoretical (hence combinatorial) generalisation of many of the classical geometrical operations to polytopes and related objects. In our second publication on the topic [28] we explore the symmetry properties of the resulting objects. These two publications settle a new research approach towards solving Conjecture 1.

In [27, Thm 5.1] we proved that if $\mathcal{O}$ is the operation associated to the voltage operator $(\mathcal{Y}, \eta)$ and $\mathcal{P}$ is a regular polytope, then the symmetry type graph of $\mathcal{O}(\mathcal{P})$ is a quotient of $\mathcal{Y}$. However, this quotient can be trivial, in other words, it is possible that $\mathcal{O}(\mathcal{P})$ has extra symmetry. The problem of understanding this additional symmetry is the main topic of the manuscript [28] and it should be a step towards proving Conjecture 1.

Extensions of abstract polytopes

A slightly different approach to building abstract polytopes is that of extensions. An $(n+1)$-polytope $\mathcal{P}$ is an *extension* of an n-polytope $\mathcal{K}$ if all the facets (maximal faces) of $\mathcal{P}$ are isomorphic to $\mathcal{K}$.

Every polytope $\mathcal{K}$ admits a *trivial extension* $\{\mathcal{K}, 2\}$ with exactly 2 facets. The problem of determining the existence of extensions for polytopes becomes more interesting when symmetry conditions are prescribed over the potential extension $\mathcal{P}$, which in turn impose symmetry conditions of over $\mathcal{K}$. If $\mathcal{P}$ is regular, then $\mathcal{K}$ must also be regular. This situation has been explored deeply and several papers have appeared on the topic (see [20, 52, 54, 59–61], for example) If $\mathcal{P}$ is chiral, then $\mathcal{K}$ must be either regular or chiral with regular facets. In the latter situation universal [63] and finite [18, 48] constructions are known. However, if $\mathcal{K}$ is regular, little is known on the problem of determining whether or not $\mathcal{K}$ admits a chiral extension. In [15] Cunningham has proved that if $\mathcal{K}$ is $(1, n-1)$-flat then $\mathcal{K}$ does not admit a chiral extension. Polytopes satisfying this condition are considered somehow degenerate. On the other extreme, in a recent paper [10] Conder, Hubard and O'Reilly proved that for every $n \geqslant 4$, the n-simplex admits a chiral extension. In my dissertation [47] it is proved that if n es even, all but finitely many regular tessellations of the n-dimensional torus admit a chiral extension. In a recently finished paper with Toledo [49], we extend this result to all possible values of n. However, the problem of finding chiral extensions of regular polytopes is mostly open and due to its importance we pose it as one of our driving questions:

Problem 2. Determine which regular polytopes admit a chiral extension.

Problem 2 has different variants. One of particular interest is the one presented below. A way of understanding Problem 3 is as a stronger version of Problem 2, where not only symmetry conditions are prescribed on the extension but also constraints on its local combinatorics. The Schläfli symbol of an abstract n-polytope $\mathcal{K}$ is a sequence $\{p_1, \dots, p_{n-1}\}$ that describes the local combinatorics of a $\mathcal{K}$. In particular, if $\mathcal{K}$ has Schläfli symbol $\{p_1, \dots, p_{n-1}\}$, then the number of facets of $\mathcal{K}$ around each $(n-3)$-face is p_{n-1}. Not every abstract polytope has a well-defined Schläfli symbol, however, regular an chiral ones do. If $\mathcal{P}$ is a regular or chiral extension of $\mathcal{K}$ and $\mathcal{K}$ has Schläfli symbol $\{p_1, \dots, p_{n-1}\}$, then $\mathcal{P}$ must have Schläfli symbol $\{p_1, \dots, p_{n-1}, q\}$ for some (possibly infinity) q.

Problem 3. Let $\mathcal{K}$ be a regular n-polytope or a chiral polytope with regular facets and assume that $\mathcal{K}$ has Schläfli symbol $\{p_1, \dots, p_{n-1}\}$. Determine de possible integers q so that $\mathcal{K}$ admits a chiral extension of type $\{p_1, \dots, p_{n-1}, q\}$.

The analogous problem for regular extensions was solved by Pellicer in [52, 54] proving that every regular polytope admits a regular extension with an arbitrary even number as the last entry of its Schläfli symbol; while Hartley proved that the n-hemicube, for $n \geqslant 4$, does not admit a regular extension with an odd number as the last entry of its Schläfli symbol.

Clearly there is a need of building examples of chiral polytopes and the extensions approach naturally attacks this problem, hence we pose it as our next research objective.

Research Objective 2. Develop constructions of chiral extensions of regular polytopes. More precisely, solve Problem 2 and Problem 3.

As mentioned above, chiral polytopes are just one of $2^n - 1$ possible symmetry types of 2-orbit n-polytopes. Each symmetry type are in correspondence with a proper subset I of $\{0, \dots, n-1\}$ defined by the property that $i \in I$ if and only if any two i-adjacent flags are on the same orbit. The symmetry type is known in the literature as *class* 2_I (see [25, 39, 55]), being chiral the class $2_\emptyset$. If $I \neq \{0, \dots, n-2\}$, then a polytope in class 2_I is facet-transitive hence it makes explore the extension problem: given $\mathcal{K}$ an n-polytope, does $\mathcal{K}$ admit an extension in class 2_I for $I \neq \{0, \dots, n-1\}$ (note the shift on the rank). An obvious necessary condition is that $\mathcal{K}$ must be either regular of in class $2_{I \setminus \{n-1\}}$. Many of the questions explored chiral extensions have their analogues for extensions in class 2_I, but other than the results in [55], little is known about this topic.

The case $I = \{0, \dots, n-1\}$ is in correspondence with the problem of constructing *semiregular alternating polytopes* (see [43–46]). These polytopes have two different types of regular facets occurring in an alternating way around each $(n-3)$-face. Several open problems on constructions are known, many of those are analogous to the extension problem and therefore natural candidates to apply similar techniques.

Problem 4. Classify the triplets $(\mathcal{P}, \mathcal{Q}, k)$ with $\mathcal{P}$ and $\mathcal{Q}$ regular n-polytopes and $k \geqslant 2$ such that there exists a semiregular alternating polytope with facets isomorphic to $\mathcal{P}$ and $\mathcal{Q}$ and interlacing number k.

Given the existing knowledge gap on the theory of extensions of non-regular polytopes and the need of constructing highly symmetric examples of such towards solving Conjecture 1 and complete our Research objective 1, we propose the following research objective:

Research Objective 3. Develop the theory of 2-orbit non chiral extensions of abstract polytopes and, as a consequence, push forward the results about alternating semiregular polytopes.

The theory of k-orbit extensions of polytopes (for $k > 2$) is even less explored. In our manuscript [17] we take a slightly different approach to extensions and use use graph covering and voltage assignments to build extensions of polytopes and maniplexes. These extensions admit a group of automorphisms acting regularly on the facets and the are, thus, a generalization of Cayley maps. Since a priori we did not imposed restrictions on the symmetry conditions of the extensions, we were able to extend polytopes with diverse symmetry types, obtaining thus interesting examples of higher rank polytopes.

Datasets of abstract polytopes

The research objectives presented above are classical and natural in the theory of abstract polytopes. Our approach would not be innovative in any way if it was not connected with and experimental and data-oriented complement. We describe the corresponding research objectives of this complement below.

Theoretical constructions of abstract polytopes have lead to the constructions of some datasets of highly symmetric polytopes and related objects (see [3–8, 22–24, 33, 58]). However, the existing datasets of polytopes suffer of the following restrictions

(i) They are mainly focused on regular or chiral polytopes.
(ii) The size of the examples is very restrictive.
(iii) They often exhibit numerous examples of rank 3 but the amount of examples of rank higher than 4 drops dramatically.
(iv) They are not very user-friendly, either because they exist only as raw data or because the are specific-programming language oriented.

In Table 1 we show the proportion of examples according to ranks on the existing datasets of polytopes with examples of rank higher than 3.

Dataset	Rank 3	Rank 4	Rank $\geqslant 5$
Hartley - Regular	64.55%	31.61%	3.84%
Hartley - Chiral	85.71%	14.29%	0.00%
Conder - Regular	61.51%	34.70%	3.79%
Conder - Chiral	87.01%	12.87%	0.12%
Leemans - Regular	95.35%	4.37%	0.30%
Leemans - Chiral	87.82%	11.97%	0.21%

Table 1: Percentages of examples according to rank

There are two obvious gaps that need to be pushed forward, not necessarily in an independent way, on the process of building new datasets of abstract polytopes: finding examples of higher ranks and building datasets that consider different symmetry types (besides chiral or regular). With the emerging development of theoretical results for less symmetric polytopes and the need to identify patterns and to find new constructions to attack the numerous open problems related to the existence of polytopes (such as Problems 2 to 4), it is clear that building new datasets of polytopes that overturn the restrictions mentioned above would not only be beneficial but it is almost necessary. The problematic expressed above outlines the following research objective.

Research Objective 4. Extend the existing and build new datasets of abstract polytopes and related structures with particular focus on

(i) Building examples on ranks higher than 3;
(ii) Exploring different symmetry types.

Many of the existing datasets of abstract polytopes are based on the (small) size of the objects. Either by taking advantage of previously computed objects (such as the library `SmallGroups`) or by using computational routines that, because of their own nature are limited by the size of the input (such as `LowIndexNormalSubgroup`). However, it has been shown that the size of the smallest regular polytope of rank n grows exponentially with n [9], while the size of the smallest chiral n-polytope is at least of factorial growth with respect to n [15]. This explains why the amount of examples of higher rank polytopes drops dramatically in the current available datasets.

Another pressing issue to address is that the existent datasets of highly symmetric maps and polytopes are not only limited on size, rank and symmetry type, but they are practically not used. One of the reasons behind it is that the information in most of these data sets is not very user-friendly. Even the small amount of existing datasets have been developed by several people, mostly in an independent way, using different notation and different computer algebra systems. Moreover, many of these datasets exist only as raw text which is not always easy to consult. This motivates our next research objective.

Research Objective 5. Develop standards and a platform for storing and presenting the datasets of abstract polytopes (both new and existing) in a unified and user-friendly way, complying with the FAIR principles (([2, 66])). In particular:

(i) Survey the existing datasets of abstract polytopes and related objects, with emphasis on the ways in which they are stored, documented and presented.

(ii) Identify the strengths and weaknesses of the existing datasets and propose unified standards for presentation, management and stewardship of the datasets of abstract polytopes, following FAIR guideline principles.

(iii) Build a web-based and user-friendly interface to datasets of abstract polytopes stored in accordance with the proposed standards.

The nature of Research objective 4 and Research objective 5 involves a two-way flow of knowledge connecting them with the Research objectives 1 to 3. With the aim of generating new datasets of highly symmetric polytopes and related objects, we will develop new theoretical results and methods, enabling us to devise new algorithms and combinatorial representation for constructing highly symmetric abstract polytopes. The algorithms will then be carefully implemented and executed. The obtained datasets will then be analysed with the aim of finding interesting patterns, suggesting new conjectures and proposing new directions for further research.

Section b. Methodology

Highly symmetric abstract polytopes from their automorphism group

A standard step in our research is the way abstract polytopes, maniplexes and related objects are represented. Quite often, when the objects possess a high degree of symmetry, its automorphism group contains enough information to fully recover the combinatorial object. Highly symmetric abstract polytopes are not the exception and a natural way of working with construction of such is via their automorphism group.

If $\mathcal{K}$ is a regular n-polytope, $\mathrm{Aut}(\mathcal{K})$ is a smooth quotient of the *Universal String Coxeter Group (of rank n)*

$$\mathcal{C}_n = \left\langle \rho_0, \dots, \rho_{n-1} : \rho_i^2 = (\rho_i\rho_j)^2 = 1 \text{ for } |i-j| > 1 \right\rangle .$$

If $\mathcal{K}$ is *orientably regular* or *chiral* (also known in the literature as *rotary*), then $\mathrm{Aut}(\mathcal{K})$ is a smooth quotient of the *Universal Rotational String Coxeter Group (of rank n)*

$$\mathcal{C}_n^+ = \left\langle \sigma_1, \dots, \sigma_{n-1} : (\sigma_i\sigma_{i+1}\cdots\sigma_j)^2 = 1 \text{ for } 1 \leqslant i < j \leqslant n-1 \right\rangle .$$

The group $\mathcal{C}^+$ is an index-2 subgroup of $\mathcal{C}$ (via the mapping $\sigma_i \mapsto \rho_i\rho_{i+1}$).

In both cases (regular and rotary) the abstract polytope can be fully recovered from a group satisfying the defining relations of $\mathcal{C}$ and $\mathcal{C}^+$, respectively, and a group theoretical condition often called the *intersection property* (see [40, Theorem] for the regular case and [62, Theorem] for the rotary one).

Most of the results involving constructions of highly symmetric polytopes use, in one way or another, this representation, turning the problem of constructing a combinatorial object into a group theoretical problem.

The approach described above has been used to build extensions of highly symmetric polytopes. If $\mathcal{P}$ is a highly symmetric extension of an n-polytope $\mathcal{K}$, quite often a (usually large) group of automorphism of $\mathcal{K}$ appears as a subgroup of $\mathcal{P}$. Equivalently, given $\mathcal{K}$, in order to build a highly symmetric extension of $\mathcal{K}$, one needs to embed a large group of automorphisms of $\mathcal{K}$ into a larger group (that will define $\mathcal{P}$). For example, in order to build a chiral extension of a rotary polytope $\mathcal{K}$, it is necessary to embed $\mathrm{Aut}(\mathcal{K})$ into a quotient Γ of $\mathcal{C}_{n+1}$.

This technique of building extensions of polytopes and related objects results particularly useful when $\mathrm{Aut}(\mathcal{K})$ is represented as a permutation group (via a natural faithful action, for example). We can then represent $\mathrm{Aut}(\mathcal{K})$ as a *permutation representation graph* (often called Schreier diagrams) and then use some clever technique to build a permutation representation graph of the corresponding goup Γ (see [10, 11, 49, 50, 52–56], just to mention some examples).

The technique explained before has offered the only results known regarding chiral extensions of regular polytopes [10, 49]. This should be our first approach towards solving Problem 2 and Problem 3.

In [25] Hubard established the basis for representing non-chiral 2-orbit polytopes via its automorphism group. In [26] Hubard and Mochán fully described the necessary conditions to represent not only 2-orbit polytopes but arbitrary polytopes with its automorphism group. The results obtained in theses paper should be our starting point towards developing a theory of 2-orbit extensions of polytopes and push Research objective 3 forward.

Maniplexes and graph coverings

Maniplexes offer a direct connection between the theory of highly symmetric abstract polytopes and the theory of graph coverings.

For a graph G with the dart-set $D(G)$ and an abstract group Γ, we say that a function $\zeta : D(G) \to \Gamma$ is a *voltage assignment* on G provided that it maps inverse darts to inverse group elements. Given such a voltage assignment, one can construct the *derived covering graph* $\mathrm{Cov}(G,\zeta)$ and the *derived regular covering projection* $\wp_\zeta\colon \mathrm{Cov}(G,\zeta) \to G$ with Γ acting regularly on each fibre as a subgroup of the covering transformations of $\wp$. Conversely, every regular covering projection is isomorphic (in a sense that can be made precise) to one derived from a voltage assignment.

The automorphism group of a maniplex acts semiregularly on the flags. Hence, every maniplex may be regarded simply s a regular cover of its symmetry type graph. Equivalently, there exists a voltage assignment ζ such that if $\mathcal{T}$ is the symmetry type of a maniplex $\mathcal{M}$, then $\mathcal{M} = \mathrm{Cov}(T,\zeta)$. The symmetry type graph of a maniplex is not necessarily a maniplex but it is a *premaniplex*. We can thus reformulate Conjecture 1 as a problem in the language of coverings of graphs or rather, of premaniplexes.

Problem 5. Given an n-premaniplex $\mathcal{T}$, determine conditions on ζ so that the properly n-edge-coloured graph $\mathrm{Cov}(\mathcal{T},\zeta)$ satisfies:

a) is a maniplex with symmetry type $\mathcal{T}$; and

b) is polytopal.

A constant problem when using this approach is that quite often the graph $\mathrm{Cov}(T,\zeta)$ has more than the desired symmetry. This phenomenon can occur essentially for two reasons:

i) there might be automorphisms of $\mathcal{T}$ that lift to (undesired) automorphisms of $\mathrm{Cov}(\mathcal{T},\zeta)$ (e.g. a regular maniplex covering the symmetry type graph of chiral maniplexes);

ii) or there might be automorphisms of $\mathrm{Cov}(\mathcal{T},\zeta)$ that do not project to automorphisms of $\mathcal{T}$ (e.g. a regular maniplex covering a 3-vertex premaniplex)

Explicit examples of both situations are known (are not uncommon). The theory of coverings of graphs offers a way to understanding both phenomena via the study of the groups $\mathrm{MaxLift}(\wp)$ and $\mathrm{MaxProj}(\wp)$ that, for a graph covering $\wp : \overline{G} \to G$, describe the automorphisms G that lift to $\overline{G}$ and the automorphisms of $\overline{G}$ that project to G, respectively. (see [34–37] for details).

A graph covering $\wp : \overline{G} \to G$ is called *stable* if $\mathrm{Aut}(\overline{G}) = \mathrm{MaxProj}(\wp)$. For the purposes of solving Conjecture 1, given a premaniplex $\mathcal{T}$ we shall find a stable covering $\wp : \mathrm{Cov}(\mathcal{T},\zeta) \to \mathcal{T}$ that satisfies $\mathrm{MaxLift}(\wp) = 1$. This is not a new problem on the more general theory of coverings of graphs, in fact the following conjecture is believe to be true (see [57]):

Conjecture 2. *For every graph G and $\Gamma \leqslant \mathrm{Aut}(G)$ (satisfying some mild conditions that can be made precise), there exists a* stable *non-identity regular covering projection $\wp\colon \overline{G} \to G$ such that $\Gamma = \mathrm{MaxLift}(\wp)$.*

Clearly, proving Conjecture 2 would imply solving Problem 5 and hence proving Conjecture 1. Therefore, we propose exploring to which extend the techniques of graphs coverings can be adapted to the study of coverings of maniplexes.

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