MIXED MODELS

Mixed models (a.k.a., mixed-effects models)

In a mixed model or mixed-effect model, some of the factors are random-effect factors and some of the factors are fixed-effect factors.

In a two-factor mixed-effect model, one of the factors (A) is fixed and the other (B) is random.

In a mixed model or mixed-effect model, some of the factors are random-effect factors and some of the factors are fixed-effect factors.

In a two-factor mixed-effect model, one of the factors is fixed and the other is random.

Example

(Kuehl, p. 238) Lipid levels in serum samples run by two different chemistry methods (fixed effect) on four different days (random effect).

Why treating the two chemistry methods as fixed effects?

Suppose a scientist wants to study the lipid levels measured using two different chemistry methods on four different days.

As we discussed last time, the day effects should be treated as random effects since the days are selected to represent a population of days.

Why or when should we treat the chemistry effects as fixed?

Fixed effects versus random effects

Why treating the two chemistry methods as fixed effects?

Treating the two chemistry methods as fixed effects indicates that when we apply our conclusions from the experimental results, we will intend to apply the conclusions to the same two methods examined in the experiment.

In other words, we are only interested in these two methods. They are not a selected subset from a larger collection of methods.

In this example, treating the two chemistry methods as fixed effects indicates that when we apply our conclusions from the experimental results, we will intend to apply the conclusions to the same two methods examined in the experiment.

In other words, we are only interested in these two specific chemistry methods. They are not a selected subset from a larger collection of methods.

The model

The model for a two-factor mixed-effects experiment using a completely-randomized design:

$$y_{ijk} = \mu + \alpha_i + b_j + (ab)_{ij} + e_{ijk}$$

where

- α_i is the fixed effect of the ith method;
- b_j is the random effect of the jth day, assumed $N(0, \sigma_b^2)$; and
- $(ab)_{ij}$ is the random effect of the ijth treatment combination, assumed $N(0, \sigma_{ab}^2)$. $i = 1, \ldots, a$, $j = 1, \ldots, b$, and $k = 1, \ldots, n$.

So, $\text{Var}\, y_{ijk} = \sigma_b^2 + \sigma_{ab}^2 + \sigma^2$.

Conventions:

- We use Greek letters for fixed effects and Roman letters for random effects.
- The interaction effect is considered random if at lest one of the main effects is random.

The model equation for a two-factor mixed-effects experiment looks similar to corresponding fixed or random-effects model. But note that there is one fixed effect term (alpha i) and one random-effect term (b j) in the model equation.

Here we follow the convention of using Greek letters for fixed effects and Roman letters for random effects.

The interaction effect is considered random if at lest one of the main effects is random.

So in this model, besides the error term, there are two random-effect terms (b j) (a b i j) and two corresponding variance componets (sigma b squared) and (sigma a b squared).

ANOVA table for a two-factor mixed-effects model for a completely randomized design

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	Expected Mean Square	F-test
A	$a-1$	SSA	MSA	$\sigma^2 + r\sigma_{ab}^2 + br\sum_{i=1}^{a}\alpha_i^2$	MSA/MSAB
B	$b-1$	SSB	MSB	$\sigma^2 + r\sigma_{ab}^2 + ar\sigma_b^2$	MSB/MSAB
$A \times B$	$(a-1)(b-1)$	SSAB	MSAB	$\sigma^2 + r\sigma_{ab}^2$	MSAB/MSE
Error	$ab(r-1)$	SSE	MSE	σ^2	
Total	$abr-1$	SS Total			

Pay particular attention to the "Expected Mean Square" column: it suggests the correct dominator to use in a F-test.

The ANOVA table summarizes the degrees of freedom, sum of squares, mean squares, expected mean squares, and the F-tests for each term in the regression model.

SSA , SSB , SSAB , and SSE are calculated as the same way in the fixed-effects case.

By now, we should all understand the central role of the ANOVA table in the analysis of designed experiments:

The column of sum of squares summarizes the partitioning of the sum of squares.

The column of expected mean squares can help us determine the correct denominator to use in a F-test.

We can also construct method-of-moments estimators of the variance components based on the expected mean squares.

Example

Table 7.5 Analysis of variance for a factorial experiment with one fixed effects factor, Method, and one random effects factor, Day

Source of Variation	*Degrees of Freedom*	*Mean Square*	*Expected Mean Square*
Method	1	$MSM = 329$	$\sigma^2 + r\sigma^2_{md} + rb\theta^2_m$
Day	3	$MSD = 144$	$\sigma^2 + r\sigma^2_{md} + ra\sigma^2_d$
Interaction	3	$MS(MD) = 62$	$\sigma^2 + r\sigma^2_{md}$
Error	8	$MSE = 14$	σ^2

Source: Dr. J. Anderson, Beckman Instruments, Inc.

ANOVA Table

This is the ANOVA table for the lipid measurements data from the mixed-effects experiment with fixed-effect factor “Chemistry Method” and random-effect factor “Day”.

Given the ANOVA table on the previous page, how do you test for the interaction effect and for the two main effects?

Example (continued)

Given the ANOVA table on the previous page, how do you test for the interaction effect and for the two main effects?

Solution:

Note that the null hypothesis for testing the main effect of the random effect factor (in this case the, days) is $H_0: \sigma_d^2 = 0$.

- To test this null, use $F = MSD/MS(MD)$ as the test statistic.

The null hypothesis for testing the main effect of the fixed-effect factor (in this example, chemistry methods) is $H_0: \alpha_1 = \alpha_2 = 0$.

- To test this null, use $F = MSM/MS(MD)$ as the test statistic.

To test no interaction, use $F = MS(MD)/MSE$ as the test statistic.

Exercise: Complete the three F-tests above.

In this example, day is a random-effect factor. For a random-effect factor, no effect means that the corresponding variance component is 0. To test no day effect, we use MSD over MSDM as as the F-test statistic.

The null hypothesis for testing the main effect for the fixed-effect factor is that the corresponding regression coefficients equal 0. To test no day effect in this example, we use MSM/MS(MD) as the test statistic.

Note for testing the two main effects in a mixed effect model, we used the mean square corresponding to the interaction term as the denominator of the F-tests, same as in a random-effects model

To test no interaction, we use MS(MD)/MSE as the test statistic.

As an exercise, complete the three F-tests above.

Summary: Denominators for F-tests

Summary: Denominators for test statistics, F_0

Test for effects of	A and B fixed	A and B random	A fixed, B random
A	MSE	MSAB	MSAB
B	MSE	MSAB	MSAB
AB	MSE	MSE	MSE

In this table, we summarize the denominators to use in F-tests in fixed-effects, random-effects, or mixed-effects models.

Estimation of variance components

Similar to the random-effects model case.

Exercise: Estimate σ_{md}^2 and σ_d^2.

The estimation of the variance components is similar to the random-effects model case. We leave it as an exercise.

Data analysis using R

See example7.2.html for data analysis using R.
You can also find solutions to the exercises there.

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Summary

- Mixed-effects model
- ANOVA table for a two-factor mixed-effects model for a complete-randomized design
  - Use the correct denominator in a F-test
  - Estimate variance components

In this lecture, we discussed mixed-effects model for two-factor experiments.

Like in the case of random-effects models, the ANOVA table summarizes most of the useful information.

In particular, it tells us the correct denominator to use in an F-test and method for estimating variance components.