1 Confidence (Interval) Band and Prediction (Interval) Band

I got great questions in the Lab today: **how come the confidence band is narrower at the center and wider at the end**? For those who are interested in this, below is some explanation that I think might help.

1.0.1 Confidence Interval

The confidence interval is the interval for the **MEAN** response, it tells us the range of the **MEAN** response might fall into at a given $X = x'$ with certain confidence. The formula of the confidence interval at a given $X = x'$ is

$$\widehat{y} \pm t_{1-\alpha/2,n-2} s_y \sqrt{\frac{1}{n} + \frac{(x' - \overline{x})^2}{(n-1)\, s_x^2}}$$

where $\widehat{y}$ is the fitted value, $s_y = \sqrt{\frac{\sum (y_i - \widehat{y}_i)^2}{n-2}}$ is the standard deviation of the residuals. Take a close look at this formula, $t_{1-\alpha/2,n-2}$ is a constant. Once we obtain the data, n is a constant, the center of explanatory variable $\overline{x}$ is a constant, the sample variance of explanatory variable s_x^2 is a constant. Once we fit our simple linear regression, s_y is a constant. The width of condifence interval

$$\text{width of CI} = 2 t_{1-\alpha/2,n-2} s_y \sqrt{\frac{1}{n} + \frac{(x' - \overline{x})^2}{(n-1)\, s_x^2}}$$

depends only on the value of x' and the distance between x' and the center of data $\overline{x}$. As we can see, the width of CI is minimized when $x' = \overline{x}$, i.e. the CI is narrowest at the center of data. When x' is far away from $\overline{x}$, $(x' - \overline{x})^2$ becomes bigger and therefore widen the CI. That is why the CI becomes wider as it departs away from the center.

1.0.2 Prediction Interval

The same thing happens for prediction interval. The formula of the prediction interval at a given $X = x'$ is

$$\widehat{y} \pm t_{1-\alpha/2,n-2} s_y \sqrt{1 + \frac{1}{n} + \frac{(x' - \overline{x})^2}{(n-1)\, s_x^2}}$$

As we can see here, the only difference is there is an additional 1 inside the squared root.