SPLIT-PLOT DESIGNS

Split-plot designs

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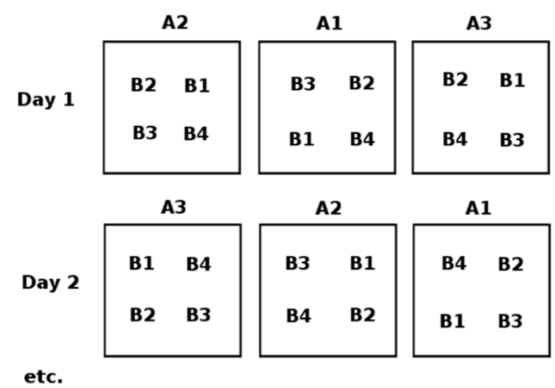
EXAMPLES

Example 1

Example (Hinkelmann & Kempthorne, p. 428). Breaking strength of plates as a function of temperature (factor A, with three levels), and chemical composition (factor B, four levels). Three furnaces are used in the following design:



We start with an example: suppose we want to study the breaking strength of dinner plates manufactured by using different chemical compounds and baking it at different temperatures.

The two treatment factors are temperature with three levels and chemical compositions with four levels.

There are three furnaces available. Each furnace can hold four dinner plates.

The temperature can only be adjusted at the furnace level. The chemical composition can vary at the plate level.

In such a case, it is reasonable to consider a split-plot design: We carry out the experiment over r days. On each day, the three temperature levels are randomly assigned to the three furnaces, and the four chemical compositions were randomly assigned to the four plates within each furnace.

The figure shows the layout of one possible implementation of such a split-plot design.

Example 1 (continued)

If we repeat this process on r days, we have r replications of each temperature, but $3r$ replications of each chemical compound. So the effects of temperature and composition will be estimated with different precision.

If we ignore factor B in the diagram above, we have factor A being tested in a randomized complete block design (r days or blocks, 3 levels of A tested in each block).

Within days, we have factor B being tested in a randomized complete block design (3 blocks or temperatures, 4 levels of B tested in each block).

Thus, this split-plot design is like one randomized complete block design superimposed on another.

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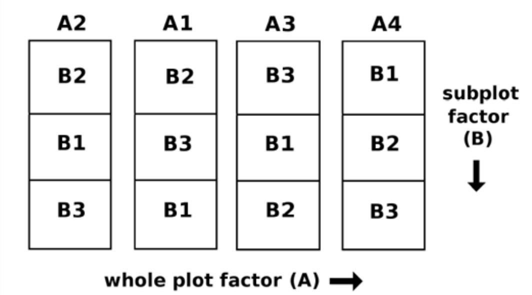
Example 2

Example (Kuehl, p. 469). Quality of turf on golf greens (chlorophyll content of grass clippings), as influenced by (A) type of nitrogen fertilizer, and (B) amount of thatch accumulation (buildup of dead grass).

Nitrogen treatment plots were arranged in two replications of a randomized complete block design. One replication looks like this:



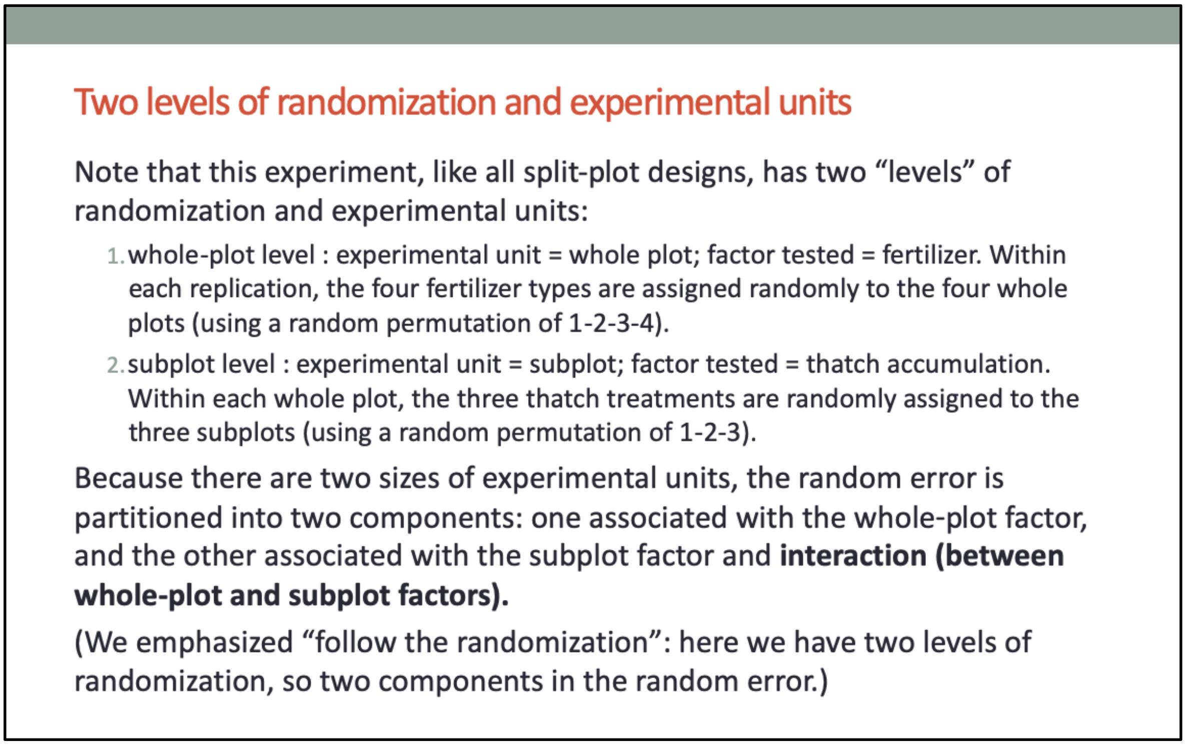
There are 4 whole plots and 12 subplots per replication, for a total of 8 whole plots and 24 subplots.

Now let's see another example where researchers want to study the effect of fertilizer and thatch accumulation on the quality of turf on golf greens.

Four types of fertilizers were randomly assigned in a randomized complete design with 2 replications: so there are two blocks and four whole plots within each block.

Within each whole plot, three levels of turf thatch accumulation are randomly assigned to the three subplots.

The picture here shows the treatment assignment in one of the blocks.



Note that this experiment, like all split-plot designs, has two “levels” of randomization and experimental units:

At the whole-plot level, the experimental unit is the whole plot; the factor tested is fertilizer.

Within each replication, the four fertilizer types are assigned randomly to the four whole plots (using a random permutation of 1-2-3-4).

At the subplot level, the experimental unit is the subplot; the factor tested is the thatch accumulation.

Within each whole plot, the three thatch treatments are randomly assigned to the three subplots (using a random permutation of 1-2-3).

Because there are two sizes of experimental units, the random error is partitioned into two components: one associated with the whole-plot factor, and the other associated with the subplot factor and interaction (between whole-plot and subplot).

We have emphasized “follow the randomization”: here we have two levels of randomization, so correspondingly two components in the random error.

Other examples

Whole plot (factor)	Subplot (factor)
Environmental chamber (temperature)	Flask (culture medium)
Classroom (teaching method)	Subgroups of students (reference materials)
Batch of material (material type)	Subbatch (curing time)

Split-plot designs are useful when one treatment is easier to apply for larger plots, and another treatment is easier to apply to smaller plots.

Here we list a few other examples where one may consider using a split-lot design:

In plant experiments, researcher often use environmental chambers. Factors such as temperature can be adjusted at the chamber level. Other factors, such a chemical contents, can be applied to smaller units (such as flasks).

For studying teaching methods, maybe the teaching method can only be changed at the classroom level; but within each classroom, we can divide the students in to subgroups and ask each subgroup to use different reference materials. Under such a situation, we can use split-plot design to study the effects of teaching methods and reference materials on learning outcome.

In material science, one may want to study the effect of material type and curing time on strength of bonding. Different batches of materials can be the whole-plot factor and the curing time can be the subplot factor and applied to subbatches.

MODEL AND ANALYSIS

Model for the split-plot design

The split-plot design usually involves a mixed-model formulation. For the design in the golf-green example (randomized complete block design superimposed on a RCBD):

$$y_{ijk} = \mu + \alpha_i + \rho_k + d_{ik} + \beta_j + (\alpha\beta)_{ij} + e_{ijk},$$

where

- μ = overall mean
- α_i = fixed effect of i th level of factor A ($i = 1, \ldots, a$)
- ρ_k = random effect of block (replication) ($k = 1, \ldots, r$)
- d_{ik} = whole-plot random error $N(0, \sigma_d^2)$
- $\beta_j j$ = fixed effect of jth level of factor B ($j = 1, \ldots, b$)
- $(\alpha\beta)_{ij}$ = fixed effect of interaction between A and B
- e_{ijk} = subplot random error $\sim N(0, \sigma_e^2)$.

The split-plot design usually involves a mixed-model formulation: since there are two levels of randomization for the whole-plot and subplot factors, there should be two random effect terms corresponding to the whole-plot randomization and subplot randomization respectively.

For the design in the golf-green example where a randomized complete block design was superimposed on a RCBD, we can consider a model equation as follows:

Note that the (d i k) term represent the whole plot error—which corresponds to the whole-plot randomization (or random sampling in an observational study).

Note also that the fixed-effect terms for the two factors, the block effect term (which is random here) are similar to those in other two-factor designs.

Sum of squares and ANOVA table

Sums of squares are computed as for a three-way analysis of variance without replication, and are then re-arranged as shown in Table 14.1 (next page).

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ANOVA table

Table 14.1 Expected mean squares for the split-plot analysis of variance

Source of Variation	*Degrees of Freedom*	*Mean Square*	*Expected Mean Square*
Blocks	$r-1$	*MS* Blocks	
A	$a-1$	*MS**A***	$\sigma_e^2 + b\sigma_d^2 + rb\theta_a^2$
Error(1)	$(a-1)(r-1)$	*MSE*(1)	$\sigma_e^2 + b\sigma_d^2$
B	$b-1$	*MS**B***	$\sigma_e^2 + ra\theta_b^2$
AB	$(a-1)(b-1)$	*MS*(**AB**)	$\sigma_e^2 + r\theta_{ab}^2$
Error(2)	$a(r-1)(b-1)$	*MSE*(2)	σ_e^2

'block' refers to the RCB design that the whole plots are placed in; in the golf-green example, it is one replication of the 12 subplots shown in the earlier diagram.

Here we show the ANOVA table for a split-plot design. Note that there are two error terms: one corresponds to the whole-plot randomization, and the other corresponds to the subplot randomization.

Hypothesis tests

The expected mean squares from the ANOVA table lead to the following hypothesis tests:

- To test main effects of factor A, compare $F_0 = MSA/MSE(1)$ to the $F_{a-1,(a-1)(r-1)}$ distribution.
- To test main effects of factor B, compare $F_0 = MSB/MSE(2)$ to the $F_{b-1,a(r-1)(b-1)}$ distribution.
- To test the interaction of A and B, compare $F_0 = MSAB/MSE(2)$ to the $F_{(a-1)(b-1),a(b-1)(r-1)}$ distribution.

In the example (see Kuehl, p. 475-476), nitrogen and thatch accumulation have statistically significant effects on chlorophyll content, and there is marginal evidence of an interaction between these two factors.

The expected mean squares from the ANOVA table suggests the correct denominator to use in the following hypothesis tests:

To test main effects of the factor A—the whole-plot factor, we divide MSA by MSE(1) and compare the resulting F-statistic value to a F distribution with (a-1) numerator degrees of freedom and (a-1)*(r-1) denominator degrees of freedom.

To test main effects of factor B—the subplot factor, we divide MSB by MSE (2), and compare the resulting F-statistic value to a F distribution with (b-1) numerator degrees of freedom and a(b-1)(r-1) denominator degrees of freedom.

To test the interaction of A and B, we divide MSAB by MSE(2), and compare the resulting F-statistic value to a F distribution with (a-1)(b-1) numerator degrees of freedom and a(b-1)(r-1) denominator degrees of freedom.

Note that once we know the form of the F-test, we can get the numbers of degrees freedom from the ANOVA table.

Example

Table 14.3 Analysis of variance of chlorophyll content of Penncross creeping bent grass clippings

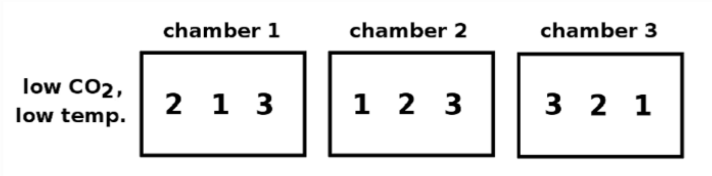
Source of Variation	*Degrees of Freedom*	*Sum of Squares*	*Mean Square*	*F*	*Pr > F*
Total	23	48.78			
Block	1	0.51	0.51		
Nitrogen (N)	3	37.32	12.44	29.62	0.010
Error (1)	3	1.26	0.42		
Thatch (T)	2	3.82	1.91	9.10	0.009
$N \times T$	6	4.15	0.69	3.29	0.065
Error (2)	8	1.72	0.21		

In Example 2, a split-plot experiment was used to study the effects of nitrogen fertilizer and thatch accumulation on chlorophyll content of grass clippings.

Here is the ANOVA table for that example (see also Kuehl, p. 475-476). The F-tests show that nitrogen and thatch accumulation have statistically significant effects on chlorophyll content, and there is marginal evidence of an interaction between these two factors.

Example

Example: Effects of CO_2 and soil temperature on growth of three species of plants (data set `plant`). Twelve chambers were randomly assigned to 4 treatment combinations ($CO_2 \times$ temperature); each chamber had 12 pots of each of three species. Plant weight was summarized as the average over 12 plants of a particular species in a particular chamber. The layout for one of the four combinations of temperature and CO_2 level looks like this:



The *whole plot* is chamber, testing $CO_2 \times$ temperature in a factorial design. The *subplot* is a third of a chamber, testing species in a randomized complete block design.

Now let's look at another example. In this study, researchers want to study the effects of CO2 and soil temperature on growth of three species of plants.

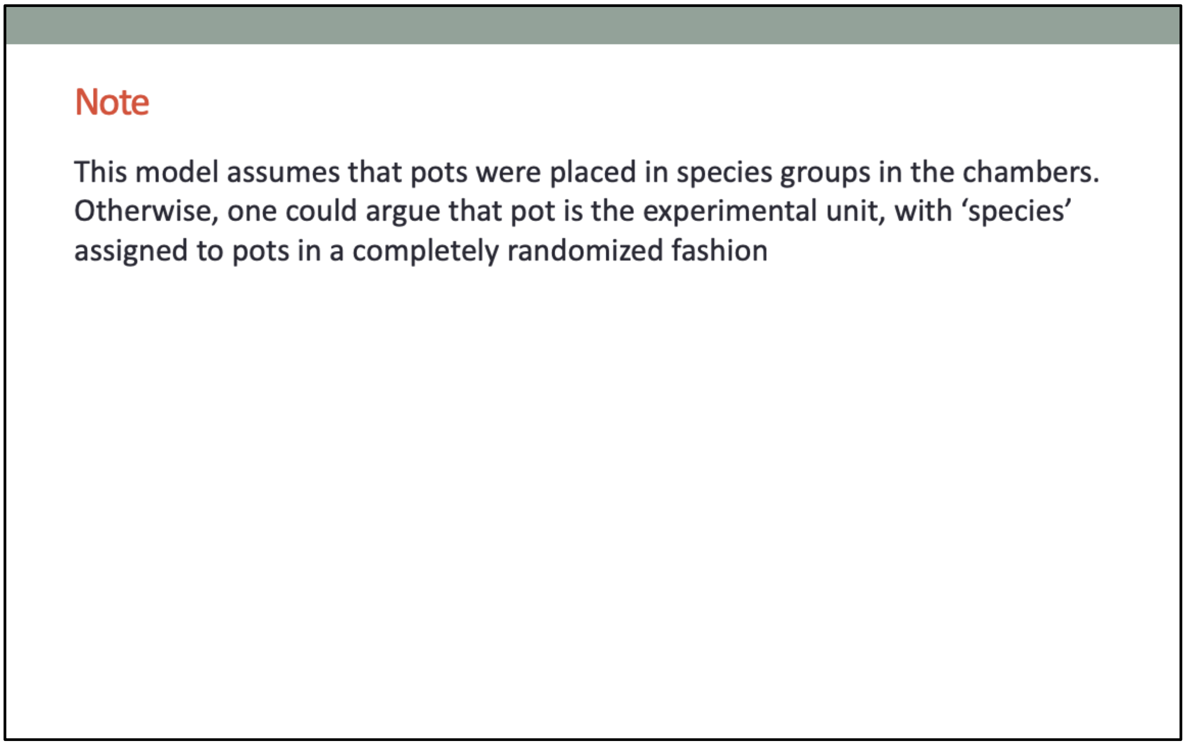
The layout for one of the four combinations of temperature and CO2 level is shown in the picture.

The 4 treatment combinations of CO2 and temperature were randomly assigned to twelve chambers.

Each chamber had 12 pots of each of three species.

Plant weight was summarized as the average over 12 plants of a particular species in a particular chamber.

This is a split-plot design: the whole-plot factor is the four combinations of temperature and CO2 level; the subplot factor is the species (the experimental units for species are groups of 12 plants).



This model assumes that pots were placed in species groups in the chambers. Otherwise, one could argue that pot is the experimental unit, with ‘species’ assigned to pots in a completely randomized fashion

Data analysis in R

In script4.html, you can see the R code for analyzing this data set. In particular, for analyzing a split-lot design, we used the R function “aov”, which allows us to specify two error terms.

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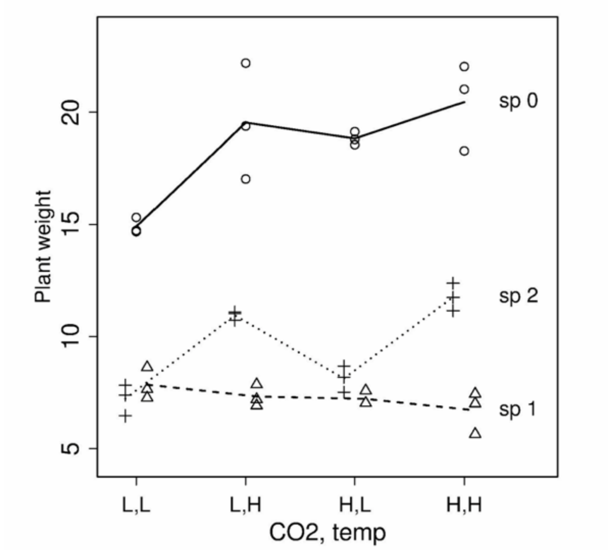
ANOVA Table

```
Error: chamber.f
             Df Sum Sq Mean Sq F value    Pr(>F)
co2.f         1  7.004   7.004  5.8646 0.0417396
temp.f        1 39.545  39.545 33.1123 0.0004269
co2.f:temp.f  1  2.378   2.378  1.9914 0.1958801
Residuals     8  9.554   1.194

(this error term includes co2*chamber, temp*chamber, and
co2*temp*chamber)

Error: chamber:species
                       Df Sum Sq Mean Sq  F value    Pr(>F)
species.f               2 837.82  418.91 380.1755 3.255e-14
co2.f:species.f         2  13.89    6.95   6.3041 0.0095726
temp.f:species.f        2  31.23   15.62  14.1719 0.0002873
co2.f:temp.f:species.f  2   4.40    2.20   1.9974 0.1681181
Residuals              16  17.63    1.10

(this error term includes species*chamber, co2*species*chamber,
temp*species*chamber, and co2*temp*species*chamber)
```

Here is the summary of the R “aov” output.

The comments in the parentheses were not part of the the R output: they explain the connections between the two error terms in this split-plot design and the interaction terms from fitting a four-way ANOVA model with factors species, co2, chamber and species.

Interaction plot



Here is an interaction plot of the data set: it shows all the response values and group mean for each combined level of CO2, temperature and species.

The x-axis lists all combined levels of CO2 and temperature; and each line corresponds to one species.

We see that there seems to be strong interaction between species and the effect of (CO2, temp): in other words, the effect of (C02, temp) varies across species.

An alternative approach?

An alternative (better?) approach: Fit each species separately, as a two-way ANOVA: two levels of CO_2 times two levels of temperature, with three replicates (chambers) per treatment combination.

SPECIES 0:

	Df	SS	MS	F	Pr(>F)
co2	1	17.52	17.52	6.58	0.033
temp	1	29.31	29.31	11.00	0.011
co2:temp	1	6.77	6.77	2.54	0.149
Resid.	8	21.32	2.66		

SPECIES 1:

	Df	SS	MS	F	Pr(>F)
co2	1	1.177	1.177	2.784	0.13
temp	1	0.807	0.807	1.909	0.20
co2:temp	1	0.0001	0.0001	0.0002	0.99
Resid.	8	3.38	0.423		

SPECIES 2:

	Df	SS	MS	F	Pr(>F)
co2	1	2.20	2.20	7.07	0.030
temp	1	40.66	40.66	130.75	<0.0001
co2:temp	1	0.01	0.01	0.02	0.893
Resid.	8	2.49	0.31		

Since the effect of (CO2, temp) varies across species. An alternative approach to analyzing this data set is to fit each species separately.

For each species, we fit a two-way ANOVA model to study the effects of CO2 and temp.

Here we list the three ANOVA tables for the three species.

We see that CO2 and temp have significant effects for species 0 and 2, but not for species 1.

Other forms of the split-plot design

Many other forms of the split-plot design are possible. From Hinkelmann & Kempthorne, 1994, p. 439:

Design for whole-plot treatment	Design for sub-plot treatment
RCBD	RCBD
CRD	RCBD
CRD	LSD
LSD	RCBD
CRD	IBD
GRBD	IBD
IBD	RCBD

CRD = completely randomized design; RCBD = randomized complete block design; LSD = Latin square design; IBD = incomplete block design; GRBD = generalized randomized block design.

The key feature of a split-plot design is that we have two levels of randomization and thus two levels of experimental units.

At both the whole-plot level and sub-plot level, many design choices are possible. This table list a few possible designs for a split-plot experiment.

Summary

Split-plot design - two-levels of randomization and experimental units

ANOVA table - includes whole-plot error and subplot error.

Data analysis in R

In this lecture, we discussed split-plot experiments.

The key feature of a split-plot design is that we have two levels of randomization and thus two levels of experimental units.

To correctly analyze a split-plot experiment, we have to correctly specify two levels of error terms: the whole-plot error and the subplot error.

Please study the R code and notes on how to analyze a split-plot experiment using R.