Problem 1

$\boldsymbol{F}_1 = \lambda_1 + \alpha_1$

$\boldsymbol{F}_2 = \lambda_2$

$\boldsymbol{F}_3 = \lambda_3$

$$[\boldsymbol{F}] = \begin{bmatrix} \lambda_1 & 0 & \alpha \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

a. $\boldsymbol{J} = det(\boldsymbol{F}) = det(\frac{\delta \mathbf{x}}{\delta \mathbf{X}}) = \mathbf{1}$ for this deformation to be isochoric.

$det(\boldsymbol{F}) = \lambda_1(\lambda_2\lambda_3 - 0) - 0(0\lambda_3 - 0) + \alpha(0 - 0\lambda_2) = \lambda_1\lambda_2\lambda_3$

$\therefore \lambda_1, \lambda_2, \lambda_3 = 1$ for $det(\boldsymbol{F}) = \lambda_1\lambda_2\lambda_3 = 1$ and this deformation to be isochoric.

b. $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})$

$$\boldsymbol{E} = \frac{1}{2}\left(\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ \alpha & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \alpha \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} - \boldsymbol{I} \right)$$

$$\boldsymbol{E} = \frac{1}{2}\left(\begin{bmatrix} \lambda_1^2 & 0 & \alpha\lambda_1 \\ 0 & \lambda_2^2 & 0 \\ \alpha\lambda_1 & 0 & \alpha^2 + \lambda_3^2 \end{bmatrix} - \boldsymbol{I} \right)$$

$$\boldsymbol{E} = \frac{1}{2}\left(\begin{bmatrix} \lambda_1^2 - 1 & 0 & \alpha\lambda_1 \\ 0 & \lambda_2^2 - 1 & 0 \\ \alpha\lambda_1 & 0 & \alpha^2 + \lambda_3^2 - 1 \end{bmatrix} \right)$$

$$\boldsymbol{E} = \begin{bmatrix} \frac{1}{2}(\lambda_1^2 - 1) & 0 & \frac{1}{2}(\alpha\lambda_1) \\ 0 & \frac{1}{2}(\lambda_2^2 - 1) & 0 \\ \frac{1}{2}(\alpha\lambda_1) & 0 & \frac{1}{2}(\alpha^2 + \lambda_3^2 - 1) \end{bmatrix}$$

```
%% Problem 1
% initialize symbolic variables
syms lambda1 lambda2 lambda3 alpha E

% define the deformation gradient
F = [lambda1 0 alpha;
    0 lambda2 0;
    0 0 lambda3];

% problem 1a
% solve for the determinant of the deformation gradient
det(F);

% problem 1b
% define the indentity matrix
I = [1 0 0; 0 1 0; 0 0 1];

% calculate the Green-Lagrange strain tensor
E = (1/2) * (transpose(F) * F - I);
```

Problem 2

$$\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F}$$

$$\boldsymbol{C} = \begin{bmatrix} 1.2287 & -0.8604 & 0 \\ 0.8604 & 1.2287 & 0 \\ 0 & 0 & 0.4444 \end{bmatrix} \begin{bmatrix} 1.2287 & 0.8604 & 0 \\ -0.8604 & 1.2287 & 0 \\ 0 & 0 & 0.4444 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 2.2500 & 0 & 0 \\ 0 & 2.2500 & 0 \\ 0 & 0 & 0.1975 \end{bmatrix}$$

$$\boldsymbol{U} = \boldsymbol{C}^{\frac{1}{2}}$$

$$\boldsymbol{U} = \begin{bmatrix} 2.2500 & 0 & 0 \\ 0 & 2.2500 & 0 \\ 0 & 0 & 0.1975 \end{bmatrix}^{\frac{1}{2}}$$

$$\boldsymbol{U} = \begin{bmatrix} 1.5000 & 0 & 0 \\ 0 & 1.5000 & 0 \\ 0 & 0 & 0.4444 \end{bmatrix}$$

$$\boldsymbol{R} = \boldsymbol{F}\boldsymbol{U}^{-1}$$

$$\boldsymbol{R} = \begin{bmatrix} 1.2287 & -0.8604 & 0 \\ 0.8604 & 1.2287 & 0 \\ 0 & 0 & 0.4444 \end{bmatrix} \begin{bmatrix} 1.5000 & 0 & 0 \\ 0 & 1.5000 & 0 \\ 0 & 0 & 0.4444 \end{bmatrix}^{-1}$$

$$\boldsymbol{R} = \begin{bmatrix} 0.8191 & 0.5736 & 0 \\ -0.5736 & 0.8191 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boldsymbol{U}$ represents stretching by 1.5 times the reference configuration in the x and y axes and stretching by 0.4444 times the reference configuration in the z axis.

$\boldsymbol{R}$ represents 35.0017° of rotation in the clockwise direction about the z axis.

```matlab
%% Problem 2
% define the deformation gradient
F = [1.2287 0.8604 0;
    -0.8604 1.2287 0;
    0 0 0.4444];

% right deformation tensor
C = transpose(F) * F; % multiply transpose F by F

% right stretch tensor
U = sqrtm(C); % calculate stretch tensor by taking the square root of C

% right polar dicomposition, F = RU
R = F * inv(U); % multiply F by inverse of stretch tensor

% calculate the degrees of rotation
theta = -asind(R(1, 2)); % solve -sin(theta) = 0.5736
theta2 = asind(R(2, 1)); % solve sin(theta) = -0.5736, confirms clockwise rotation
```

Problem 3

$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{U}^T \boldsymbol{R}^T \boldsymbol{R} \boldsymbol{U} - \boldsymbol{I})$

Since $\boldsymbol{U}$ is symmetric and $\boldsymbol{R}$ is a proper orthogonal rotation matrix ($\boldsymbol{R}^T \boldsymbol{R} = \mathbf{1}$),

$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{U}\boldsymbol{U} - \boldsymbol{I}) = \frac{1}{2}(\boldsymbol{U}^2 - \boldsymbol{I})$

$$\boldsymbol{E} = \frac{1}{2}\left(\begin{bmatrix}1 & 0 & 0\\0 & \lambda & 0\\0 & 0 & \lambda\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & \lambda & 0\\0 & 0 & \lambda\end{bmatrix} - \boldsymbol{I}\right)$$

$$\boldsymbol{E} = \frac{1}{2}\left(\begin{bmatrix}1 & 0 & 0\\0 & \lambda^2 & 0\\0 & 0 & \lambda^2\end{bmatrix} - \boldsymbol{I}\right)$$

$$\boldsymbol{E} = \begin{bmatrix}0 & 0 & 0\\0 & \frac{1}{2}(\lambda^2 - 1) & 0\\0 & 0 & \frac{1}{2}(\lambda^2 - 1)\end{bmatrix}$$

```
%% Problem 3
% initialize symbolic variables
syms gamma lambda E

% define the rotation matrix, R
R = [cos(gamma) 0 sin(gamma);
    0 1 0;
    -sin(gamma) 0 cos(gamma)];

% define the stretch matrix, U
U = [1 0 0;
    0 lambda 0;
    0 0 lambda];

% define the indentity matrix
I = [1 0 0; 0 1 0; 0 0 1];

% Green-Lagrange strain tensor
% E = (1/2) * ((transpose(U) * transpose(R) * R * U) - 1);

% since R is orthogonal and U is symmetric
% Green-Lagrange strain tensor does not depend on R
E = (1/2) * ((U * U) - I);
```

Problem 4

a) For each marker pair, $\boldsymbol{dX}$ and $\boldsymbol{dx}$ can be defined as:

$$\boldsymbol{dX} = \begin{bmatrix}0 & 1 & 0\\2 & 0 & 0\\0 & 1 & 0\\2 & 0 & 0\\2 & 1 & 0\\-2 & 1 & 0\end{bmatrix}$$

$$\boldsymbol{dx} = \begin{bmatrix} 0.3 & 1.3 & 0 \\ 1.5 & 0.1 & 0 \\ 0.1 & 1.2 & 0 \\ 1.7 & 0.2 & 0 \\ 1.8 & 1.4 & 0 \\ -1.4 & 1.1 & 0 \end{bmatrix}$$

Then, the reference and deformed lengths of $\boldsymbol{dX}$, $\boldsymbol{dS}$ and $\boldsymbol{ds}$, can be found by taking the vector-wise norm of each row in $\boldsymbol{dX}$ and $\boldsymbol{dx}$ (or you can calculate the squared lengths by taking the dot products of $\boldsymbol{dX}$ and $\boldsymbol{dx}$, respectively, to use directly in the system of equations below):

$$\boldsymbol{dS} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 2.2361 \\ 2.2361 \end{bmatrix}$$

$$\boldsymbol{ds} = \begin{bmatrix} 1.3342 \\ 1.5033 \\ 1.2042 \\ 1.7117 \\ 2.2804 \\ 1.7804 \end{bmatrix}$$

Solve the system of equations to find the unknown strains, E_{11}, E_{22}, E_{12} as:

$$2 \begin{bmatrix} dX_1^2 & dY_1^2 & 2dX_1dY_1 \\ dX_2^2 & dY_2^2 & 2dX_2dY_2 \\ dX_3^2 & dY_3^2 & 2dX_3dY_3 \\ dX_4^2 & dY_4^2 & 2dX_4dY_4 \\ dX_5^2 & dY_5^2 & 2dX_5dY_5 \\ dX_6^2 & dY_6^2 & 2dX_6dY_6 \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{12} \end{bmatrix} = \begin{bmatrix} (ds_1)^2 - (dS_1)^2 \\ (ds_2)^2 - (dS_2)^2 \\ (ds_3)^2 - (dS_3)^2 \\ (ds_4)^2 - (dS_4)^2 \\ (ds_5)^2 - (dS_5)^2 \\ (ds_6)^2 - (dS_6)^2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 0 \\ 8 & 0 & 0 \\ 0 & 2 & 0 \\ 8 & 0 & 0 \\ 8 & 2 & 8 \\ 8 & 2 & -8 \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{12} \end{bmatrix} = \begin{bmatrix} 0.78 \\ -1.74 \\ 0.45 \\ -1.07 \\ 0.20 \\ -1.83 \end{bmatrix}$$

where the system above can be represented as:

$[\boldsymbol{A}][\boldsymbol{x}] = [\boldsymbol{b}]$

$[\boldsymbol{x}] = [\boldsymbol{A}]^{-1}[\boldsymbol{b}]$

$E_{11} = -0.1761$

$E_{22} = 0.3033$

$E_{12} = 0.1269$

```
%% Problem 4a
% define the four markers of the element, El_0
P1_0 = [0 0 0];
P2_0 = [0 1 0];
```

```
P3_0 = [2 1 0];
P4_0 = [2 0 0];

% calculate the vectors between nodes
P12_0 = P2_0 - P1_0;
P23_0 = P3_0 - P2_0;
P34_0 = P3_0 - P4_0;
P14_0 = P4_0 - P1_0;
P13_0 = P3_0 - P1_0;
P24_0 = P2_0 - P4_0;

% define the four markers of element, El_x
P1_x = [0 0 0];
P2_x = [0.3 1.3 0];
P3_x = [1.8 1.4 0];
P4_x = [1.7 0.2 0];

% calculate the vectors between nodes
P12_x = P2_x - P1_x;
P23_x = P3_x - P2_x;
P34_x = P3_x - P4_x;
P14_x = P4_x - P1_x;
P13_x = P3_x - P1_x;
P24_x = P2_x - P4_x;

% define reference vectors of El, 6x3 matrix
dX = [P12_0; P23_0; P34_0; P14_0; P13_0; P24_0];

% define deformed vectors of El, 6x3 matrix
dx = [P12_x; P23_x; P34_x; P14_x; P13_x; P24_x];

% calculate the reference lengths of dX
dS = vecnorm(dX, 2, 2); % 6x1 vector

% or calculate the squared lengths of dX
dS2 = dot(dX, dX, 2); % 6x1 vector

% calculate the deformed lengths of dx
ds = vecnorm(dx, 2, 2); % 6x1 vector

% or calculate the squared lengths of dx
ds2 = dot(dx, dx, 2); % 6x1 vector

% calculate b, 6x1 vector
% b = ds.^2 - dS.^2;
b = ds2 - dS2;

% calculate A, 6x3 matrix
A = 2 .* [dX(:, 1).^2 dX(:, 2).^2 2.*dX(:, 1).*dX(:, 2)];

% solve the system of equations to find the Green-Lagrange strain elements
% of El_x, E_11, E_22, and E_12, as the least-squares solution
E = A \ b; % 3x1 vector
```

b) The left ventricle is contracting, where the walls are pinching/shrinking and stretching/thickening, to increase pressure in the ventricle and force blood out through the aortic valve to be distributed throughout the body.

Problem 5

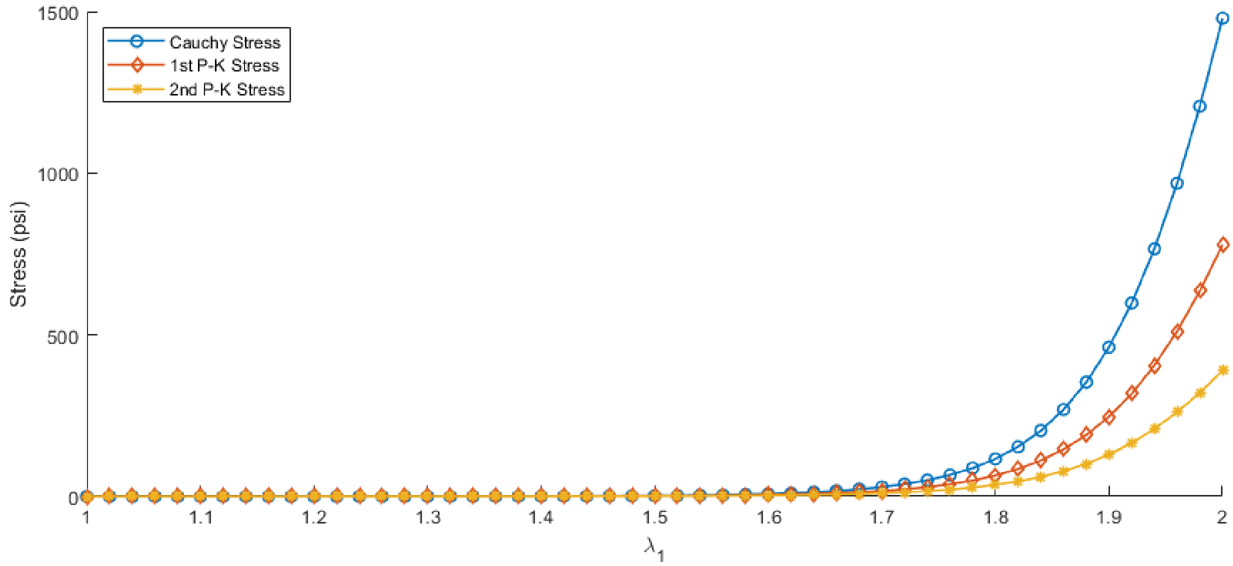

Figure 1: Cauchy stress, 1st Piola-Kirchhoff stress, and 2nd Piola-Kirchhoff stress as a function of the applied axial stretch ratio for $\lambda_1 = 1.0$ to 2.0 over 50 time steps of equal size (0.02).

```
%% Problem 5
% import data exported from FEBio
% variables: time, relative volumne (relVol), Cauchy stress (xStress),
%            stretch ratios (xInfStrain, yInfStrain, zInfStrain)
data = readtable('.\data\hw1prb5_data.xlsx'); % read as table data type

% determine the stretch ratio from data
lambda1 = 1 + data.xInfStrain;
lambda2 = 1 + data.yInfStrain;
lambda3 = 1 + data.zInfStrain;

% compose the defomration gradient
% F = [lambda1 0 0; 0 lambda2 0; 0 0 lambda3];

% extract Cauchy stress from data
T_11 = data.xStress;

% extract Jacobain (volume ratio or relative volume) from data
J = data.relVol;

% initialize vectors for 1st P-K and 2nd P-K stresses
P_11 = zeros(length(T_11), 1);
S_11 = zeros(length(T_11), 1);

for i = 1:length(T_11)
```

Problem 5 continued on next page...

```
    % compose the deformation gradient
    F = [lambda1(i, 1) 0 0; 0 lambda2(i, 1) 0; 0 0 lambda3(i, 1)];

    % Cauchy stress
    T = [T_11(i, 1) 0 0; 0 0 0; 0 0 0];

    % calculate 1st P-K stress as a function of applied axial stress ratio
    % for lambda_1 = 1.0 to 2.0
    P = J(i, 1) .* (T * transpose(inv(F)));
    P_11(i, 1) = P(1, 1);

    % calculate 2nd P-K stress as a function of applied axial stress ratio
    % for lambda_1 = 1.0 to 2.0
    S = F \ P;
    S_11(i, 1) = S(1, 1);
end

% plot applied axial stretch ratio vs.
% Cauchy stress, 1st P-K stress, and 2nd P-K stress
figure();
hold on;
plot(lambda1, T_11, '-o', 'LineWidth', 1.2);
plot(lambda1, P_11, '-d', 'LineWidth', 1.2);
plot(lambda1, S_11, '-*', 'LineWidth', 1.2);
xlabel('\lambda_{1}');
ylabel('Stress (psi)');
legend('Cauchy Stress', '1st P-K Stress', '2nd P-K Stress', 'Location', 'northwest');
hold off;
```