Feynman's Operational Calculus and the Evolution Equation

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- In the 1951 paper, Feynman introduced three heuristic rules to compute such functions and determine the order in which the operators would act.
- Abstract approach used here was developed later by B. Jefferies, M. L. Lapidus, G. W. Johnson, and L. Nielsen, starting from the eary 80's.

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- In a product, the operators act in succession from right to left: $(AB)(x) = A(B(x))$. If we have $(AB)(x) = (BA)(x)$ for all x, then A and B are said to commute. Composition is, in general, non-commutative (think of matrices).

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- Let A and B are operators: what can we say about $f(A, B)$?
- If A and B commute, then there's no ambiguity in writing $f(A, B) = AB^2$. If however they do not commute, then their order in the expression matters and one has to chose how to write that product: AB^2, BAB, B^2A, $\frac{1}{3}AB^2 + \frac{1}{3}BAB + \frac{1}{3}B^2A$ or any expression that would give back AB^2 if A and B commuted.

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  3. "Disentangle" the expression by restoring the conventional order of the operators.
- Of the disentangling process, Feynman states "The process is not always easy to perform and, in fact, is the central problem of this operator calculus." One should note that Feynman did not attempt to supply rigorous proofs of his results and, in fact, it is not always clear how Feynman's rules are to be applied, even heuristically.

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- This process is better understood with an example.

Rule 1: Attaching time indexes

- If A and B are operators, then we define

$$A(t_1)B(t_2) = \begin{cases} AB & \text{if } t_2 < t_1 \\ BA & \text{if } t_1 < t_2 \\ \text{undefined} & \text{if } t_1 = t_2 \end{cases}$$

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- The most common way to attach a time index to A is through an integral:

$$A = \frac{1}{t}\int_0^t A(s)ds,$$

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  where $A(s) = A$ for all $s \in [0, t]$.
- Note: the integral in this example is the usual integral over the real line (i.e. with respect to Lebesgue measure), but a variety of different measures (and hence integrals) can be used, as we will see.

Rule 2: Computing the function

- Let $f(x, y) = xy$. Associate Lebesgue measure ℓ to both operators A and B, i.e. set

$$A = \frac{1}{t}\int_0^t A(s_1)ds_1 \quad \text{and} \quad B = \frac{1}{t}\int_0^t B(s_2)ds_2,$$

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- Then we have

$$f_{\ell,\ell}(A, B) = \left(\frac{1}{t}\int_0^t A(s_1)ds_1\right)\left(\frac{1}{t}\int_0^t B(s_2)ds_2\right)$$

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- Then we have

$$\begin{aligned} f_{\ell,\ell}(A,B) &= \left(\frac{1}{t}\int_0^t A(s_1)ds_1\right)\left(\frac{1}{t}\int_0^t B(s_2)ds_2\right) \\ &= \frac{1}{t^2}\int_0^t \int_0^t A(s_1)B(s_2)ds_1 ds_2, \end{aligned}$$

  where the subscripts serve to emphasize the measures assigned to the two operators.

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f_{\ell,\ell}(A, B) = & \frac{1}{t^2} \int_{\{(s_1,s_2):s_1<s_2\}} A(s_1)B(s_2)ds_1 ds_2 \\
& + \frac{1}{t^2} \int_{\{(s_1,s_2):s_2<s_1\}} A(s_1)B(s_2)ds_1 ds_2
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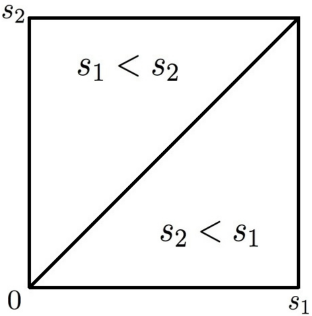

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- Finally, we compute the integrals:

$$f_{\ell,\ell}(A,B) = \frac{1}{t^2}\int_{\{(s_1,s_2):s_1<s_2\}} B(s_2)A(s_1)ds_1 ds_2 + \frac{1}{t^2}\int_{\{(s_1,s_2):s_2<s_1\}} A(s_1)B(s_2)ds_1 ds_2$$

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f_{\ell,\ell}(A,B) &= \frac{1}{t^2}\int_{\{(s_1,s_2):s_1<s_2\}} B(s_2)A(s_1)\,ds_1\,ds_2 \\
&\quad + \frac{1}{t^2}\int_{\{(s_1,s_2):s_2<s_1\}} A(s_1)B(s_2)\,ds_1\,ds_2 \\
&= \frac{1}{t^2}\left(\frac{t^2}{2}BA\right) + \frac{1}{t^2}\left(\frac{t^2}{2}AB\right)
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- It is worth noting that if A and B commute then $f_{\ell,\ell}(A,B) = AB$, like we expected.

Exponential of a bounded operator

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$$\frac{\partial u}{\partial t} = Au \qquad u(0) = u_0,$$

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- Given its similarity to the differential equation that defines the exponential function, we write its solution as

$$u = u_0 e^{tA} := u_0 \sum_{k=0}^{\infty} \frac{(tA)^k}{k!},$$

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- Note that, since A is a bounded operator, the series will always converge, and it is thus well defined. But what would happen if A was unbounded?

Exponential of an unbounded operator

- Take, for example the heat equation

$$\frac{\partial u}{\partial t} = \Delta u,$$

or the Schrödinger equation (in natural units)

$$\frac{\partial \Psi}{\partial t} = -i\left(-\frac{1}{2}\Delta + V\right)\Psi,$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ is the Laplacian, V is a potential function, and $i^2 = -1$.

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- The operator $A = \frac{i}{2}\Delta$ is unbounded and (the operator of multiplication by $-iV$) $B = -iV$ may be unbounded, and so e^{tA} and e^{tB} cannot defined using a power series.
- The operational calculus will allow us to evaluate expressions like $e^{t\Delta}e^{tV}$ in a rigorous fashion.

Disentangling the exponential function: setup

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- A will be the infinitesimal generator of a strongly continuous (C_0) semigroup of operators: that is, a family $\{T(t)\}_{t\geq 0}$ of linear operators for which $T(t)T(s) = T(t+s)$ for all $t, s \geq 0$ and $T(0) = I$, and

$$Ax = \lim_{t\searrow 0} \frac{T(t)x - x}{t}.$$

Furthermore, we usually write $T(t) = e^{tA}$

Disentangling the exponential function: setup

- In particular, for $\zeta \in \mathbb{C}_+$, we set $A = \frac{H_0}{\zeta} = -\frac{1}{2\zeta}\Delta$ and we associate Lebesgue measure ℓ on $[0, T]$ to A. We know that the operator A is the generator of a strongly continuous semigroup with action

$$\left(e^{-tA}\varphi\right)(\xi) = \left(\frac{\zeta}{2\pi t}\right)^{d/2} \int_{\mathbb{R}^d} \varphi(u) \exp\left(-\frac{\zeta \left\|\xi - u\right\|^2}{2t}\right) du$$

for $\varphi \in L^2(\mathbb{R}^d)$ and $\xi \in \mathbb{R}^d$. By the Wiener integration formula, the above is equal to a function-space (or path) integral over
$C_0^t = \{y : [0, t] \to \mathbb{R}^d | y(0) = 0 \text{ and } y \text{ is continuous}\}$ with respect to Wiener measure $\mathfrak{m}$:

$$\left(e^{-tA}\varphi\right)(\xi) = \int_{C_0^t} \varphi\left(\zeta^{-1/2} y(t) + \xi\right) \mathfrak{m}(dy).$$

Disentangling the exponential function: setup

- We also set $V(s) \in \mathcal{L}(L^2(\mathbb{R}^d))$ to be the operator of multiplication by a bounded function $V : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$: for $f \in L^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, $(V(s)f)(x) = V(s,x)f(x)$. We associate the measure $\lambda = \mu + \omega\delta_\tau$ to V : μ is a continuous measure on $[0, t]$ and $\omega\delta_\tau$ is the Dirac point mass at $\tau \in [0, t]$ and weight ω.

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- Finally, we set $f(z_0, z_1) = e^{z_0}e^{z_1}$ and use the notation

$$f^t_{\ell;\lambda}(-H_0/\zeta; V) = \exp^t_{\ell,\lambda}\left(-t\frac{H_0}{\zeta} + \int_{[0,t]} V(s)\lambda(ds)\right);$$

the subscripts serve to emphasize the measure associated to each operator.

Disentangling the exponential function: computation

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$$\left(\exp^{t}_{\ell,\lambda}\left(-t\frac{H_0}{\zeta}+\int_{[0,t]}V(s)\lambda(ds)\right)\varphi\right)(\xi)$$

$$=\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{m}\frac{m!\omega^{m-k}}{(m-k)!}\sum_{j=0}^{k}\int_{\Delta^{t}_{k;j}}\Big(e^{-(t-s_k)H_0/\zeta}V(s_k)\cdots$$

$$\cdots V(s_{j+1})e^{-(s_{j+1}-\tau)H_0/\zeta}[V(\tau)]^{m-k}e^{-(\tau-s_j)H_0/\zeta}V(s_j)\cdots$$

$$\cdots V(s_1)e^{-s_1H_0/\zeta}\varphi\Big)(\xi)\,\mu^{k}(ds_1,\ldots,ds_k)$$

Disentangling the exponential function: computation

$$
\begin{aligned}
= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{m} \frac{m!\omega^{m-k}}{(m-k)!} \sum_{j=0}^{k} \int_{\Delta_{k;j}^{t}} \int_{C_0^t} V\left(\zeta^{-1/2} y(s_1) + \xi, s_1\right) \cdots \\
\cdots V\left(\zeta^{-1/2} y(s_{j+1}) + \xi, s_{j+1}\right) \left[V\left(\zeta^{-1/2} y(\tau) + \xi, \tau\right)\right]^{m-k} \cdot \\
V\left(\zeta^{-1/2} y(s_j) + \xi, s_j\right) \cdots V\left(\zeta^{-1/2} y(s_k) + \xi, s_k\right) \\
\varphi\left(\zeta^{-1/2} y(t) + \xi\right) \mathfrak{m}(dy) \mu^k(ds_1, \ldots, ds_k)
\end{aligned}
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\varphi\left(\zeta^{-1/2} y(t) + \xi\right) \mathfrak{m}(dy) \mu^k(ds_1, \ldots, ds_k) \\
= \int_{C_0^t} \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{m} \frac{m!\omega^{m-k}}{(m-k)!k!} \left(\int_{[0,t]} V\left(\zeta^{-1/2} y(s) + \xi, s\right) \mu(ds)\right)^k \cdot \\
\left[V\left(\zeta^{-1/2} y(\tau) + \xi, \tau\right)\right]^{m-k} \varphi(\zeta^{-1/2} y(t) + \xi) \mathfrak{m}(dy)
\end{aligned}
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Disentangling the exponential function: computation

$$= \int_{C_0^t} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\int_{[0,t]} V\left(\zeta^{-1/2} y(s) + \xi, s\right) (\mu + \omega\delta_\tau)(ds) \right)^m \cdot$$

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\begin{aligned}
&= \int_{C_0^t} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\int_{[0,t]} V\left(\zeta^{-1/2} y(s) + \xi, s\right) (\mu + \omega\delta_\tau)(ds) \right)^m \cdot \\
&\quad \varphi(\zeta^{-1/2} y(t) + \xi) \mathfrak{m}(dy) \\
&= \int_{C_0^t} \exp\left(\int_{[0,t]} V\left(\zeta^{-1/2} y(s) + \xi, s\right) \lambda(ds) \right) \varphi(\zeta^{-1/2} y(t) + \xi) \mathfrak{m}(dy)
\end{aligned}
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Disentangling the exponential function: computation

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\begin{aligned}
&= \int_{C_0^t} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\int_{[0,t]} V\left(\zeta^{-1/2} y(s) + \xi, s\right) (\mu + \omega\delta_\tau)(ds) \right)^m \cdot \\
&\qquad \varphi(\zeta^{-1/2} y(t) + \xi) \mathfrak{m}(dy) \\
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&=: K(t)
\end{aligned}
$$

Evolution equation

- The expression in the previous slide, labeled $K(t)$ for convenience, satisfies an evolution equation which represents the time evolution of this disentangled operator as a function of the time t. In fact, if we let

$$K_{\mathrm{dis}}(t) = \int_{C_0^t} \exp\left(\int_{[0,t]} V\left(\zeta^{-1/2} y(s) + \xi, s\right) \omega \delta_\tau(ds) \right) \cdot \varphi(\zeta^{-1/2} y(t) + \xi) \mathfrak{m}(dy)$$

be the discrete part of $K(t)$, then

$$K(t) = K_{\mathrm{dis}}(t) + \int_0^t e^{-(t-s)H_0/\zeta} V(s) K(s) \mu(ds).$$

Generalizations and plans

- The above evolution equation is one particular example of a much more general theorem proved by Nielsen. In his paper, he went on to prove that $K(t)$ is the solution of a class of differential equations analogous to the heat equation and the Schrödinger equation.

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- The above evolution equation is one particular example of a much more general theorem proved by Nielsen. In his paper, he went on to prove that $K(t)$ is the solution of a class of differential equations analogous to the heat equation and the Schrödinger equation.
- Right now, I'm working toward finding other differential equations by disentangling exponentials of different generators of (C_0) semigroups, like $A = -(-H_0)^{1/2}$, exploiting the Weiner integration formula when possible. There is also a generalized integral equation that follows from the evolution equation, of which I'm looking into some particular examples.

References

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