Low Regularity Non-$L^2(\mathbb{R}^n)$ Local Solutions to the gMHD-α system

Existence of Solutions with Poorly Behaved Initial Conditions: An Exposition

Lorenzo Riva

Department of Mathematics
Creighton University

Mathfest 2019

Outline

What are the Magneto-Hydrodynamic equations?

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - (B \cdot \nabla)B - \nu_1 \Delta u = -\nabla p - \frac{1}{2}\nabla |B|^2,$$
$$\frac{\partial B}{\partial t} + (v \cdot \nabla)B - (B \cdot \nabla)u - \nu_2 \Delta B = 0,$$
$$\operatorname{div} u = \operatorname{div} B = 0,$$
$$u(0,x) = u_0(x), \qquad B(0,x) = B_0(x).$$

- In the above,
  - $u : [0,T) \times \mathbb{R}^n \to \mathbb{R}^n$ is the fluid velocity,
  - $B : [0,T) \times \mathbb{R}^n \to \mathbb{R}^n$ is the magnetic field,
  - $p : [0,T) \times \mathbb{R}^n \to \mathbb{R}$ is the pressure, and
  - $0 < \nu_1, \nu_2 \in \mathbb{R}$ are the fluid viscosity and magnetic diffusivity.
- This is hard! We study a modified version:

The generalized MHD-α (gMHD-α) equations

$$
\begin{aligned}
&\frac{\partial v}{\partial t} + (u \cdot \nabla)\, v + \sum_{i=1}^{n} v_i \nabla u_i - (B \cdot \nabla) B - \nu_1 \mathcal{L}_1 v = -\nabla p - \frac{1}{2} \nabla |B|^2, \\
&\frac{\partial B}{\partial t} + (u \cdot \nabla) B - (B \cdot \nabla) u - \nu_2 \mathcal{L}_2 B = 0, \\
&v = (1 - \alpha^2 \mathcal{L}_3) u, \\
&\operatorname{div} u = \operatorname{div} B = 0, \\
&u(0,x) = u_0(x), \qquad B(0,x) = B_0(x), \quad x \in \mathbb{R}^n.
\end{aligned}
$$

The generalized MHD-α (gMHD-α) equations

$$
\begin{aligned}
&\frac{\partial v}{\partial t} + (u\cdot\nabla)v + \sum_{i=1}^{n} v_i \nabla u_i - (B\cdot\nabla)B - \mathcal{L}_1 v = -\nabla p - \frac{1}{2}\nabla |B|^2, \\
&\frac{\partial B}{\partial t} + (u\cdot\nabla)B - (B\cdot\nabla)u - \mathcal{L}_2 B = 0, \\
&v = (1-\mathcal{L}_3)u, \\
&\operatorname{div} u = \operatorname{div} B = 0, \\
&u(0,x) = u_0(x), \qquad B(0,x) = B_0(x), \quad x\in\mathbb{R}^n.
\end{aligned}
$$

The generalized MHD-α (gMHD-α) equations

$$
\begin{aligned}
&\frac{\partial v}{\partial t} + (u \cdot \nabla) v + \sum_{i=1}^{n} v_i \nabla u_i - (B \cdot \nabla) B - \mathcal{L}_1 v = -\nabla p - \frac{1}{2} \nabla |B|^2, \\
&\frac{\partial B}{\partial t} + (u \cdot \nabla) B - (B \cdot \nabla) u - \mathcal{L}_2 B = 0, \\
&v = (1 - \mathcal{L}_3) u, \\
&\operatorname{div} u = \operatorname{div} B = 0, \\
&u(0,x) = u_0(x), \qquad B(0,x) = B_0(x), \quad x \in \mathbb{R}^n.
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial v}{\partial t} + P\left((u\cdot\nabla)v + \sum_{i=1}^{n} v_i \nabla u_i - (B\cdot\nabla)B\right) - \mathcal{L}_1 v = 0,\\
&\frac{\partial B}{\partial t} + P\left((u\cdot\nabla)B - (B\cdot\nabla)u\right) - \mathcal{L}_2 B = 0,\\
&v = (1-\mathcal{L}_3)u,\\
&\operatorname{div} u = \operatorname{div} B = 0,\\
&u(0,x) = u_0(x), \qquad B(0,x) = B_0(x), \quad x\in\mathbb{R}^n.
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + \underbrace{P(1-\mathcal{L}_3)^{-1}\left((u\cdot\nabla)v + \sum_{i=1}^{n} v_i \nabla u_i - (B\cdot\nabla)B\right)}_{G_1(u,B)} - \mathcal{L}_1 u = 0,\\
&\frac{\partial B}{\partial t} + \underbrace{P\left((u\cdot\nabla)B - (B\cdot\nabla)u\right)}_{G_2(u,B)} - \mathcal{L}_2 B = 0,\\
&v = (1-\mathcal{L}_3)u,\\
&\operatorname{div} u = \operatorname{div} B = 0,\\
&u(0,x) = u_0(x), \qquad B(0,x) = B_0(x), \quad x \in \mathbb{R}^n.
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = \mathcal{L}_1 u - G_1(u, B),\\
&\frac{\partial B}{\partial t} = \mathcal{L}_2 B - G_2(u, B),\\
&v = (1 - \mathcal{L}_3)u,\\
&\operatorname{div} u = \operatorname{div} B = 0,\\
&u(0, x) = u_0(x), \qquad B(0, x) = B_0(x), \quad x \in \mathbb{R}^n.
\end{aligned}
$$

Goal

- Guarantee the **existence** of a **local** solution with **rough** initial conditions and arbitrary **integrability**.
  - Will not be written explicitly
  - Will exist on some finite interval $[0, T]$
  - Initial conditions will have very few derivatives
  - Integrability:
$$\int_{\mathbb{R}^n} |f(x)|^p \, dx < \infty.$$

Duhamel's principle

- Here are our equations:

$$\frac{\partial u}{\partial t} = \mathcal{L}_1 u - G_1(u,B),$$
$$\frac{\partial B}{\partial t} = \mathcal{L}_2 B - G_2(u,B).$$

- Apply Duhamel's principle to find formal solutions in terms of integrals:

$$u = e^{t\mathcal{L}_1} u_0 - \int_0^t e^{(t-s)\mathcal{L}_1} G_1(u,B)\,ds,$$
$$B = e^{t\mathcal{L}_2} B_0 - \int_0^t e^{(t-s)\mathcal{L}_2} G_2(u,B)\,ds.$$

Contraction mapping theorem

- Are there any (u, B) that satisfy the integral equations? Maybe. Use the contraction mapping theorem:

Theorem

Let $F : X \to X$ be a map on a complete normed space. If there is a $0 \leq k < 1$ such that

$$\|F(x_1) - F(x_2)\| \leq k \|x_1 - x_2\|$$

for all $x_1, x_2 \in X$, then F is called a contraction and it has a unique fixed point ($F(y) = y$).

- Examples: $F(x) = \frac{1}{3}x$ or a geographic map.

- Define $F = (F_1, F_2)$ by

$$F_1(u, B) := e^{t\mathcal{L}_1} u_0 - \int_0^t e^{(t-s)\mathcal{L}_1} G_1(u, B) ds,$$
$$F_2(u, B) := e^{t\mathcal{L}_2} B_0 - \int_0^t e^{(t-s)\mathcal{L}_2} G_2(u, B) ds.$$

- We want (F_1, F_2) to have a unique fixed point (u, B).

Sobolev spaces

$$L^p(\mathbb{R}^n) := \left\{ f : \mathbb{R}^n \to \mathbb{R} : \|f\|_p := \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p} < \infty \right\}$$

$$\Downarrow$$

$$H^{r,p}(\mathbb{R}^n) := \{ f \in L^p(\mathbb{R}^n) : D^r f \in L^p(\mathbb{R}^n) \}$$

- $D^r f$ indicates r derivatives of f in any combination of directions.
- Now $F : Z \to Z$, where Z is an interval (for the time variable) and a Sobolev space (for the space variable).

Summary

- To recap:
  - Found an integral formulation of the solutions using Duhamel's principle.
  - Used those solutions to define an operator $F : Z \to Z$.
  - Have to show that F is a contraction on Z.
- Previous work:
  - $\mathcal{L}_3 = \Delta$,
  - integrability: $p = 2$,
  - regularity: high.
- My project:
  - arbitrary $\mathcal{L}_3$,
  - integrability: $p \geq n$,
  - regularity: low.

First approach

- Try setting $Z = X_{T,M} \times Y_{T,M}$:

$$X_{T,M} = \left\{ f : [0,T) \times \mathbb{R}^n \to \mathbb{R}^n : \sup_{(0,T)} \left\| f(t,\cdot) - e^{t\mathcal{L}_1} u_0 \right\|_{r_0,p_0} < M \right\},$$

$$Y_{T,M} = \left\{ f : [0,T) \times \mathbb{R}^n \to \mathbb{R}^n : \sup_{(0,T)} \left\| f(t,\cdot) - e^{t\mathcal{L}_2} B_0 \right\|_{r_2,p_2} < M \right\}.$$

- We run into a problem.

$$\left\| P(1-\mathcal{L}_3)^{-1} \sum_{i=1}^{n} v_i \nabla u_i \right\|_{r_0,\pi_2} \leq \cdots \leq C \left\| u \right\|_{\gamma_3,p'} \left\| u \right\|_{1,p''},$$

where π_2, p', p'' are intermediate integrability indices that can be ignored.

- We could use a Sobolev embedding to obtain

$$\left\| u \right\|_{\gamma_3,p'} \leq \left\| u \right\|_{r_0,p_0} \text{ and } \left\| u \right\|_{1,p''} \leq \left\| u \right\|_{r_0,p_0},$$

but that would require $r_0 \geq \gamma_3$ and $r_0 \geq 1$.

- We want r_0 to be as small as possible, so this does not work.

Auxiliary space and algorithm

- We let u be in two spaces: one of regularity r_0, and one of regularity $r_1 \geq r_0$ (which we care less about) that satisfies the following conditions, for a particular $a_1 > 0$:

$$\|u\|_{a_1;r_1,p_1} := \sup_{(0,T)} t^{a_1} \|u(t)\|_{r_1,p_1} < \infty \text{ and } \lim_{t \to 0^+} t^{a_1} u(t) = 0.$$

Now

$$X_{T,M} = \left\{ f : \sup_{(0,T)} \left\| f(t,\cdot) - e^{t\mathcal{L}_1} u_0 \right\|_{r_0,p_0} + \|f\|_{a_1;r_1,p_1} < M \right\}$$

$$
\begin{aligned}
I_1 &= \sup_{(0,T)} t^{a_1} \left\| e^{t\mathcal{L}_1} u_0 \right\|_{r_1,p_1} < M/4, \\
I_2 &= \sup_{(0,T)} \left\| \int_0^t e^{(t-s)\mathcal{L}_1} G_1(u,B) ds \right\|_{r_0,p_0} < M/4, \\
I_3 &= \sup_{(0,T)} t^{a_1} \left\| \int_0^t e^{(t-s)\mathcal{L}_1} G_1(u,B) ds \right\|_{r_1,p_1} < M/4, \\
I_4 &= \sup_{(0,T)} \left\| \int_0^t e^{(t-s)\mathcal{L}_2} G_2(u,B) ds \right\|_{r_2,p_2} < M/4.
\end{aligned}
$$

- Each integral can be further split up into its individual terms, e.g.

$$I_2 \leq \sup_{(0,T)} \int_0^t \left(\left\| e^{(t-s)\mathcal{L}_1} P(1-\mathcal{L}_3)^{-1}(u \cdot \nabla)v \right\|_{r_0,p_0} + \left\| e^{(t-s)\mathcal{L}_1} P(1-\mathcal{L}_3)^{-1} \left(\sum_{i=1}^{n} v_i \nabla u_i \right) \right\|_{r_0,p_0} + \left\| e^{(t-s)\mathcal{L}_1} P(1-\mathcal{L}_3)^{-1}(B \cdot \nabla)B \right\|_{r_0,p_0} \right) ds.$$

$$
\begin{aligned}
\left\| P(1-\mathcal{L}_3)^{-1} \sum_{i=1}^{n} v_i \nabla u_i \right\|_{r_0,\pi_2} &\leq C \left\| \sum_{i=1}^{n} v_i \nabla u_i \right\|_{r_0-\gamma_3^-,\pi_2} \\
&\leq C \sum_{i=1}^{n} \left\| v_i \nabla u_i \right\|_{\pi_2} \\
&\leq C \sum_{i=1}^{n} \left\| v_i \right\|_{p'} \left\| \nabla u_i \right\|_{p''} \\
&\leq C \sum_{i=1}^{n} \left\| v \right\|_{p'} \left\| \nabla u \right\|_{p''} \\
&\leq C \left\| u \right\|_{\gamma_3^-,p'} \left\| u \right\|_{1,p''} \\
&\leq C \left\| u \right\|_{r_1,p_1}^2
\end{aligned}
$$

provided that $r_0 < \gamma_3$ and $\frac{1}{\pi_2} = \frac{1}{p'} + \frac{1}{p''} = \frac{1}{p_1} + \frac{1-r_1}{n}$.

- The rest of the proof consists in bounding appropriately each nonlinear term using various product and Sobolev estimates, always being mindful of the restrictions on r_0.
- Each estimate imposes a condition on the indices. The collection of all these conditions forms the requirements necessary for the existence of a solution and constitutes the statement of our theorem.
- This is a special case of our main result:

Theorem (Riva, 2019)

Assume $\gamma_1, \gamma_2, \gamma_3 > 0$, $\gamma_3 \leq 1$ and $p, q \geq n$. Then, for any divergence-free $u_0 \in H^{0,p}(\mathbb{R}^n)$ and $B_0 \in H^{0,q}(\mathbb{R}^n)$, there exists a unique local solution (u, B) to the generalized MHD-α system provided that

$$\gamma_1 > 6 - \gamma_3,$$
$$\gamma_2 > 1 + \frac{n}{p}.$$

Thank you! Questions?

Theorem (Riva, 2019)

Let $g_1, g_2, g_3 : [0,\infty) \to \mathbb{R}$ be non-decreasing functions bounded below by 1 satisfying condition (1). Let $\gamma_3^- - 1 \le r_0 \le \gamma_3^- \le r_1$, $r_2 - 1 + \gamma_3^- \le r_0 < n/p_0$, and let $p_0, p_1, p_2 \ge n$ with $p_0 \le p_1$. Moreover, assume that

$$2r_1 \ge \max\left\{2, 1 + \gamma_3^- - \frac{n}{p_0} + \frac{2n}{p_1}\right\}, 0 \le r_2 < \min\left\{\frac{n}{p_0}, \frac{n}{p_2}, \frac{2n}{p_2} - \frac{n}{p_0}\right\}.$$

Then, for any divergence-free $u_0 \in H^{r_0,p_0}(\mathbb{R}^n)$ and $B_0 \in H^{r_2,p_2}(\mathbb{R}^n)$, there exists a unique local solution (u, B) to the generalized MHD-α system provided that

$$\gamma_1^- > 3r_1 - 2r_0 - \gamma_3^- + \frac{3n}{p_0} - \frac{3n}{p_1},$$

$$\gamma_1^- > 1 - 2r_2 + r_1 - \gamma_3^- - \frac{n}{p_1} + \frac{2n}{p_2},$$

$$\gamma_2^- > 1 - r_2 + \frac{n}{p_0}.$$

A white lie

- At the beginning I mentioned that each $\mathcal{L}_i$ ($i = 1, 2, 3$) is a Fourier multiplier that contains γ_i derivatives. Formally, this would mean that $\mathcal{L}_i$ has a symbol

$$m_i(\xi) = -|\xi|^{\gamma_i}.$$

  E.g. the Laplacian, another Fourier multiplier, has symbol $-|\xi|^2$.
- In full generality, however, this project allowed for the $\mathcal{L}_i$'s to have a little less than γ_i derivatives: their symbols are

$$m_i(\xi) = -\frac{|\xi|^{\gamma_i}}{g_i(\xi)},$$

  where each g_i satisfies some derivative conditions that render each $\mathcal{L}_i$ a Mikhlin multiplier: $g_1, g_2, g_3 : [0, \infty) \to \mathbb{R}$ are non-decreasing functions bounded below by 1 satisfying

$$g_i^{(k)}(s) \leq Cs^{-k}. \tag{1}$$