On the simplicial construction for the classifying space of a group

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Fix a group G. Recall that we have a construction (due to Milnor) for EG, the classifying space of G, given by iterating the *join operation*: namely,

$$EG = \varinjlim_k G^{*(k+1)}$$

where

$$X * Y := \frac{X \times [0,1] \times Y}{\substack{(g,0,h) \sim (g,0,h') \\ (g,1,h) \sim (g',1,h)}}.$$

The intuitive picture of a join is a tetrahedron with one edge identified with X and the opposite edge identified with Y, with the 3-dimensional space in between filled by shrinking copies of X and Y placed at every point of the unit interval. This picture naively suggests that there might be a way to construct EG using *simplicial sets*, which we now introduce. In fact, not only is this construction possible, but it surprisingly coincides with Milnor's join construction given above.

1 Simplicial sets and the geometric realization

Definition. Let Δ be the category of finite linear orders up to isomorphism with morphisms all non-decreasing maps. We can take as representatives for the objects the sets $[n] := \{0, 1, \dots, n\}$. The hom sets are generated by the following *face* and *degeneracy* maps:

$$[n-1] \xrightarrow{d_i} [n] \qquad\qquad [n] \xrightarrow{s_i} [n-1]$$

$$k \mapsto \begin{cases} k & k < i, \\ k+1 & k \geq i; \end{cases} \qquad\qquad k \mapsto \begin{cases} k & k \leq i, \\ k-1 & k > i. \end{cases}$$

A *simplicial set* (or, more generally, a simplicial *thing*) is a functor $F : \Delta^{\mathrm{op}} \to \mathsf{Set}$ (or, more generally, to a category C with "things" as objects). Simplicial sets and natural transformations form a category denoted SSet, which is simply the category of presheaves over Delta.

More concretely, a simplicial set X is a collection of sets $X_n := X([n])$ labelled by non-negative integers that we think of as containing the *n-simplices of X*, with maps $d^i : X_n \to X_{n-1}$ taking the *ith face* of a simplex and $s^i : X_{n-1} \to X_n$ including an $(n-1)$-simplex as a *degenerate* n-simplex; the maps are required to satisfy various relations induced by the relations between d_i and s_i in Δ.

The advantage of simplicial sets is that they can easily be thought of geometrically – they are, after all, a generalization of triangulated spaces. There are three intuitive criteria that this geometric representation should satisfy:

1. each n-simplex should have the same shape, a generalized triangle in n dimensions;
2. we should be able to glue n-simplices along their faces much like in a triangulation;
3. this representation should be functorial, since we're talking about functors in the first place.

To satisfy the first requirement we define the *standard (topological) n-simplex*:

$$\Delta^n_T := \left\{ (t_0, \dots, t_n) \middle| t_i \geq 0, \sum_i t_i = 1 \right\} \subseteq [0,1]^{n+1} \subseteq \mathbb{R}^{n+1}.$$

Then Δ^0_T is the point with coordinate 1 in $\mathbb{R}$, Δ^1_T is the straight line segment connecting the points $(0,1)$ and $(1,0)$ in $\mathbb{R}^2$, and so on. In general, it is easy to see that Δ^n_T is the convex hull of the standard basis $\{e_1, \dots, e_{n+1}\}$ in $\mathbb{R}^{n+1}$.

There are maps $\delta_i : \Delta^{n-1}_T \to \Delta^n_T$, given by inserting a 0 in the ith slot, and $\sigma_i : \Delta^n_T \to \Delta^{n-1}_T$, given by summing the ith and the $(i+1)$th coordinate, that respectively include an $(n-1)$-simplex as the ith face of an n-simplex and project an n-simplex linearly onto its ith face.

This is a standard topological simplex. But what is a standard *simplicial* simplex? That is, if we were to force all simplicial sets to be glued out of basic pieces (which is the content of the second requirement), what would those basic pieces be? The Yoneda lemma answers this question for us. We know that every presheaf $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ of sets is exactly isomorphic (in the category of presheaves over C) to a colimit of representable presheaves:

$$F \cong \operatorname*{colim}_{c \in \mathsf{C}, f \in F(c)} \mathrm{Hom}_{\mathsf{C}}(-, c).$$

It thus makes sense to consider $\Delta^n := \mathrm{Hom}_{\mathsf{Delta}}(-, [n])$ as the basic n-simplex from which all simplicial sets can be glued (via a colimit). Yoneda also gives us an extra treat: since

$$X_n = X([n]) \cong \mathrm{Hom}_{\mathsf{SSet}}(\mathrm{Hom}_\Delta(-, [n]), X) = \mathrm{Hom}_{\mathsf{SSet}}(\Delta^n, X),$$

we can consider an n-simplex in X (an element of X_n) as one way to embed the standard n-simplex Δ^n into X (an element of $\mathrm{Hom}_{\mathsf{SSet}}(\Delta^n, X)$), which further justifies our choice of notation.

Now the third requirement for the geometric representation of X forces us to postulate the following reasonable definition.

Definition. The *geometric realization functor* $|-| : \mathsf{SSet} \to \mathsf{Top}$ is the only colimit-preserving functor satisfying $|\Delta^n| = \Delta^n_T$.

Exploiting the concrete construction of colimits in Set as quotients of disjoint unions, we get that

$$|X| = \operatorname*{colim}_{n\in\mathbb{N},\sigma\in X_n} \cong \coprod_{n\in\mathbb{N}} X_n \times \Delta^n_T \Big/ \begin{matrix}(d^i(x), \bar{t}) \sim (x, \delta_i(\bar{t})) \\ (s^i(x), \bar{t}) \sim (x, \sigma_i(\bar{t}))\end{matrix}$$

where we consider each X_n to have the discrete topology. Namely, we take one standard topological n-simplex for each n-simplex of X and 1. if it's a face of some simplex y, we glue it to the face of the standard topological $(n+1)$-simplex associated to y, and 2. if it's a degenerate simplex coming from some y, we glue it to the degenerate standard topological $(n-1)$-simplex associated to y.

2 The construction we want

Our goal now is to define a simplicial set $\mathcal{E}G$ whose realization is an acyclic principal G-bundle. This will give us our desired universal bundle.

Definition. Let $\mathcal{E}G$ be the simplicial set with G^{n+1} as its set of n-simplices, face maps $d^i : G^{n+1} \to G^n$ given by

$$(g_0, \dots, g_n) \mapsto (g_0, \dots, \widehat{g_i}, \dots, g_n),$$

and degeneracy maps $s^i : G^n \to G^{n+1}$ given by

$$(g_0, \dots, g_{n-1}) \mapsto (g_0, \dots, g_i, g_i, \dots, g_{n-1}).$$

Proposition. $|\mathcal{E}G|$ is aspherical.

Proof. Let

$$|\mathcal{E}G|^{(n)} := \coprod_{k=0}^{n} G^{k+1} \times \Delta^k_T \Big/ \sim$$

be the n-skeleton of $|\mathcal{E}G|$, with $\sim$ the identification given by the realization functor. It will be enough to show that the inclusion $|\mathcal{E}G|^{(n)} \hookrightarrow |\mathcal{E}G|^{(n+1)}$ is null-homotopic for each n: in that case, any map $S^k \to |\mathcal{E}G|$ will factor through some $|\mathcal{E}G|^{(n)}$ for a high enough n (by compactness of S^k) and thus S^k will be null-homotopic in the larger space $|\mathcal{E}G|^{(n+1)}$, thus showing that $\pi_k(|\mathcal{E}G|)$ is trivial.

We construct an explicit homotopy j_r from $j : |\mathcal{E}G|^{(n)} \hookrightarrow |\mathcal{E}G|^{(n+1)}$ to a constant map. Notice that the definition of the face maps in $\mathcal{E}G$ forces every simplex in G^{k+1} for $k < n$ to be a face of some simplex in G^{k+2} since we can adjoin an arbitrary element g to a $(k+1)$-tuple $(g_0, \ldots, g_k)$ in any of its $k+2$ available slots. Therefore every element of $|\mathcal{E}G|^{(n-1)}$ in $|\mathcal{E}G|^{(n)}$ is identified via $\sim$ to some element in $G^{n+1} \times \Delta^n_T$, which allows us to focus only on those elements when we consider the map $|\mathcal{E}G|^{(n)} \hookrightarrow |\mathcal{E}G|^{(n+1)}$. Recall also that $\sim$ is defined on faces by $(d^i(x), \bar{t}) \sim (x, \delta_i(\bar{t}))$. In our case, that corresponds to

$$((g_0, \ldots, \widehat{g_i}, \ldots, g_k), \bar{t}) \sim ((g_0, \ldots, g, \ldots, g_k), \delta_i(\bar{t}))$$

for any k-tuple $(g_0, \ldots, \widehat{g_i}, \ldots, g_k) \in G^k$ and any $g \in G$.

Now let $x = ((g_0, \ldots, g_n), (t_0, \ldots, t_n)) \in G^{n+1} \times \Delta^n_T \subseteq |\mathcal{E}G|^{(n)}$ be arbitrary. Its image $j(x) \in |\mathcal{E}G|^{(n+1)}$ is also denoted by the same element $x = ((g_0, \ldots, g_n), (t_0, \ldots, t_n))$, but now we can make the identification

$$((g_0, \ldots, g_n), (t_0, \ldots, t_n)) \sim ((g_0, \ldots, g_n, e), (t_0, \ldots, t_n, 0)) \in G^{n+2} \times \Delta^{n+1}_T$$

via the map d^{n+1}, where $e \in G$ is the identity. Now we homotope in time $1/(n+1)$ from $((g_0, \ldots, g_n, e), (t_0, \ldots, t_n, 0))$ to

$$j_{1/(n+1)}(x) = ((g_0, \ldots, g_n, e), (t_0, \ldots, t_{n-1}, 0, 0)),$$

which we can do via a straight line homotopy in Δ^{n+1}_T. The $\bar{t}$ coordinate is in the nth face of some Δ^{n+1}_T, so we can make the identification

$$\begin{aligned} ((g_0, \ldots, g_n, e), (t_0, \ldots, t_{n-1}, 0, 0)) &\sim (d^n(g_0, \ldots, g_n, e), (t_0, \ldots, t_{n-1}, 0)) \\ &= ((g_0, \ldots, \widehat{g_n}, e), (t_0, \ldots, t_{n-1}, 0)). \end{aligned}$$

Thanks to another identification (via d^n) we can insert e in the nth slot in $j_{1/(n+1)}(x)$ to get

$$\begin{aligned} j_{1/(n+1)}(x) &= ((g_0, \ldots, \widehat{g_n}, e), (t_0, \ldots, 0)) \\ &\sim ((g_0, \ldots, g_{n-1}, e, e), (t_0, \ldots, t_{n-1}, 0, 0)). \end{aligned}$$

Proceed by homotoping in time $1/(n+1)$ to

$$j_{2/(n+1)}(x) = ((g_0, \ldots, g_{n-1}, e, e), (t_0, \ldots, t_{n-2}, 0, 0, 0)).$$

Two more identification allow us to replace g_{n-1} by e to obtain

$$j_{2/(n+1)}(x) = ((g_0, \ldots, g_{n-2}, e, e, e), (t_0, \ldots, t_{n-2}, 0, 0, 0)).$$

We can continue homotoping and identifying as done above so that at time $r/(n+1)$ we have

$$j_{r/(n+1)}(x) = ((g_0, \ldots, g_{n-r}, e, \ldots, e), (t_0, \ldots, t_{n-r}, 0, \ldots, 0)).$$

When $r = n+1$ we will get $j_{n+1}(x) = ((e, \dots, e), (0, \dots, 0)) \in G^{n+2} \times \Delta^{n+1}$, the desired constant map. $\square$

The only other ingredient required for $|\mathcal{E}G|$ to be a universal bundle is that it carries a right action of G. This is given by the map $|\mathcal{E}G| \times G \to |\mathcal{E}G|$ induced by right multiplication on G^n:

$$g \cdot ((g_0, \dots, g_p), \bar{t}) \mapsto ((g_0 g, \dots, g_p g), \bar{t})$$

for any $g \in G$ and $((g_0, \dots, g_p), \bar{t}) \in |\mathcal{E}G|$. Thus we can define $EG := |\mathcal{E}G|$ and $BG := EG/G$, making $EG \to BG$ a universal G-bundle. BG is then the *classifying space of G*.

Notice that all of these constructions and calculations relied solely on the monoidal structure of G, i.e. we did not use the assumption that each element of G is invertible. Therefore these arguments can be extended to monoids, obtaining the classifying space BA of a monoid A, and in general to small categories, obtaining the classifying space $B\mathsf{C}$ of a category C.

2.1 Simplicial bar construction for BG

It turns out that we can obtain a simplicial construction for BG as well.

Definition. Let $\mathcal{B}G$ be the simplicial set with G^n its set of n-simplices, face maps $d^i : G^n \to G^{n-1}$ given by

$$d^i(g_1, \dots, g_n) = \begin{cases} (g_2, \dots, g_n) & i = 0, \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n) & 1 \le i \le n-1, \\ (g_1, \dots, g_{n-1}) & i = n, \end{cases}$$

and degeneracy maps $s^i : G^{n-1} \to G^n$ given by

$$s^i(g_1, \dots, g_{n-1}) = \begin{cases} (e, g_1, \dots, g_{n-1}) & i = 0, \\ (g_1, \dots, g_i, e, g_{i+1}, \dots, g_n) & 1 \le i \le n-1. \end{cases}$$

Then we let $BG := |\mathcal{B}G|$. This is called the *simplicial bar construction* for BG.

Then we have a projection $p : \mathcal{E}G \to \mathcal{B}G$ given for n-simplices by

$$p_n(g_0, \dots, g_n) = (g_0 g_1^{-1}, g_1 g_2^{-1}, \dots, g_{n-1} g_n^{-1}).$$

It's an easy exercise to see that the p_n's commute with face and degeneracies, making p a well-defined natural transformation. The induced map $p_* : EG \to BG$ on realizations has the following property: if π is the projection map $EG \to EG/G$ then there is a homeomorphism $h : EG/G \to BG = |\mathcal{B}G|$ such that the diagram

$$\begin{tikzcd} & EG \arrow[dl, "\pi"'] \arrow[dr, "p_*"] & \\ EG/G \arrow[rr, "h"'] & & BG \end{tikzcd}$$

commutes.

3 Simplicial homology

Let's go back to arbitrary simplicial complexes. One might be interested in knowing what $H_n(|X|)$ is. It turns out that this question can be answered combinatorially.

Definition. Let $C_n(X) := F(X_n)$ be the free abelian group on X_n, the set of n-simplices of X. We can put a boundary map ∂ on $C_\bullet(X)$ defined on simplices by

$$\partial_n x := \sum_{i=0}^{n} (-1)^i d^i(x)$$

and extended linearly to the whole group. The complex $(C_\bullet(X), \partial)$ is called the *simplicial chain complex of* X. Notice the striking similarity with the simplicial chains of a topological space endowed with a Δ-complex (or simplicial complex) structure.

We now can conclude that

Proposition. There is an isomorphism $H_n(|X|) \cong H_n(C_\bullet(X))$, where the first homology is the singular homology with $\mathbb{Z}$-coefficients and the second homology is that of the singular chain complex.

Proof. The realization $|X|$ has a natural CW structure given by letting each $\{x\} \times \Delta^n_T$ be an n-cell for any $x \in X_n$ and $n \in \mathbb{N}$. Then $C_\bullet(X)$ is *almost* exactly the complex of CW chains on $|X|$, the only difference being that each $C_n(X)$ contains all the degenerate simplices as well and is thus, a priori, a much larger group. It turns out that degenerate simplices get killed in homology (see for example Corollary 3.2 and Theorem 3.3. in this nLab page), so indeed we have

$$H_n(C_\bullet(X)) \cong H_n(C_\bullet^{\mathrm{CW}}(|X|)) \cong H_n(|X|). \qquad \square$$

Since every CW complex has the homotopy type of the realization of some simplicial set (that set being the *singular simplicial set* with n simplices the continuous maps $\sigma : \Delta^n_T \to X$), this proposition gives us an explicit (though unwieldy) combinatorial technique to calculate its homology.

Specializing to $X = \mathcal{B}G$, we have that $C_n(\mathcal{B}G) \cong \mathbb{Z}[G^n] \cong \mathbb{Z}[G]^{\otimes n}$ and

$$\begin{aligned} \partial_n(a_1 \otimes \cdots \otimes a_n) = a_2 \otimes \cdots \otimes a_n \\ + \sum_{i=1}^{n-1} (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ + (-1)^n a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

This is the *bar complex* for computing the group homology of G.