Low Regularity Non-$L^2(\mathbb{R}^n)$ Local Solutions to the gMHD-α System

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1. Introduction

The Magneto-Hydrodynamic (MHD) system of equations governs viscous fluids subject to a magnetic field and is derived via a coupling of the Navier-Stokes equations and Maxwell's equations. It has recently become common to study generalizations of fluids-based differential equations. Here we consider the generalized Magneto-Hydrodynamic alpha (gMHD-α) system, which differs from the original MHD system by the presence of additional non-linear terms (indexed by the choice of α) and replacing the Laplace operators in the equations by more general Fourier multipliers with symbols of the form $-|\xi|^\gamma/g(|\xi|)$. In [2], Pennington considered the problem with initial data in Sobolev spaces of the form $H^{s,2}(\mathbb{R}^n)$ with $n \geq 3$. Here we consider the problem with initial data in $H^{s,p}(\mathbb{R}^n)$ with $n \geq 3$ and $p > 2$, with the goal of minimizing the regularity required to obtain unique existence results.

2. Introduction

The MHD system is a coupling of the incompressible Navier-Stokes equations, which govern fluids of constant density, and Maxwell's equations, which govern magnetic fields.

$$\begin{aligned}
&\partial_t u + (u\cdot\nabla)u - (B\cdot\nabla)B - \nu_1\Delta u = -\nabla\left(p + \frac{1}{2}|B|^2\right),\\
&\partial_t B + (u\cdot\nabla)B - (B\cdot\nabla)u - \nu_2\Delta B = 0,\\
&\operatorname{div} u = \operatorname{div} B = 0,\\
&u(0,x) = u_0(x), \quad B(0,x) = B_0(x).
\end{aligned}$$

In the above,

- $u : [0,T) \times \mathbb{R}^n \to \mathbb{R}^n$ is the fluid velocity,
- $B : [0,T) \times \mathbb{R}^n \to \mathbb{R}^n$ is the magnetic field ($B = 0$ returns Navier-Stokes),
- $p : [0,T) \times \mathbb{R}^n \to \mathbb{R}$ is the pressure,
- $0 < \nu_1, \nu_2 \in \mathbb{R}$ are the fluid viscosity and magnetic diffusivity respectively,
- div is the divergence operator.

This system is notably harder than Navier-Stokes itself, so we decided to study a generalization called the generalized MHD-α system:

$$\begin{aligned}
&\partial_t v + (v\cdot\nabla)u + \sum_{i=1}^n v_i\nabla u_i - (B\cdot\nabla)B - \nu_1\mathcal{L}_1 v = -\nabla\left(p + \frac{1}{2}|B|^2\right),\\
&\partial_t B + (u\cdot\nabla)B - (B\cdot\nabla)u - \nu_2\mathcal{L}_2 B = 0,\\
&v = (1-\mathcal{L}_3)u,\\
&\operatorname{div} u = \operatorname{div} B = 0,\\
&u(0,x) = u_0(x), \quad B(0,x) = B_0(x).
\end{aligned}$$

The new variable v, called the "filtered" fluid velocity, contains α, a parameter arising from a process called Lagrangian averaging. The other two most notable differences are the presence of the sum term and the substitution of the Laplacians with Fourier multipliers $\mathcal{L}_i$ having symbols $-|\xi|^{\gamma_i}/g_i(|\xi|)$. We want to think about these operators as generalized derivatives of order slightly less than γ_i. The g_i's will not play a significant role in the proof of our theorem, but they do require some smoothness conditions in order to behave properly.

Previous work focused on the case of $\mathcal{L}_3 = \Delta$, high regularity, and 2-integrable solutions. This project is concerned with guaranteeing the existence of a unique local non-$L^2(\mathbb{R}^n)$ solution with rough initial conditions and arbitrary integrability, with arbitrary $\mathcal{L}_3$ (and hence γ_3).

3. Setup of the proof

Plan: Use Duhamel's principle to define a map on a specific space, and, thanks to the contraction mapping theorem, showing that the map is a contraction on an appropriate space will guarantee the existence of our unique solution.

First, we simplify the system. Without loss of generality, we set $\alpha = \nu_1 = \nu_2 = 1$. Thanks to the divergence-free condition, we can change the terms of the form $(x\cdot\nabla)y$ into $\operatorname{div}(x\otimes y)$, where $x \otimes y$ is just the matrix with (i,j) entry equal to $x_i y_j$. We also get rid of the terms that are the gradient of a scalar by applying the Hodge operator P and passing onto a divergence-free subspace. Finally, we apply $(1-\mathcal{L}_3)^{-1}$ to both sides of the first equation and, noting that ∂_t, $\mathcal{L}_i$, and $(1-\mathcal{L}_3)^{-1}$ all commute because they are Fourier multipliers, we obtain

$$\begin{aligned}
&\frac{\partial u}{\partial t} + P(1-\mathcal{L}_3)^{-1}\left(\operatorname{div}(u\otimes v) + \sum_{i=1}^n v_i\nabla u_i - \operatorname{div}(B\otimes B)\right) - \mathcal{L}_1 u = 0,\\
&\frac{\partial B}{\partial t} + P\left(\operatorname{div}(u\otimes B) - \operatorname{div}(B\otimes u)\right) - \mathcal{L}_2 B = 0,\\
&v = (1-\mathcal{L}_3)u,\\
&\operatorname{div} u = \operatorname{div} B = 0,\\
&u(0,x) = u_0(x), \qquad B(0,x) = B_0(x), \quad x \in \mathbb{R}^n.
\end{aligned}$$

We can write this more compactly as

$$\frac{\partial u}{\partial t} = \mathcal{L}_1 u - G_1(u,B), \quad \frac{\partial B}{\partial t} = \mathcal{L}_2 B - G_2(u,B),$$

where G_1 and G_2 are the collection of nonlinear terms in the first and second equation respectively.

Next, we apply Duhamel's principle to obtain an integral formulation of the simplified system:

$$\begin{aligned}
u &= e^{t\mathcal{L}_1}u_0 - \int_0^t e^{(t-s)\mathcal{L}_1}G_1(u,B)ds,\\
B &= e^{t\mathcal{L}_2}B_0 - \int_0^t e^{(t-s)\mathcal{L}_2}G_2(u,B)ds.
\end{aligned}$$

Our next step is to prove that there is at least one pair (u,B) that satisfies this integral system. We will employ the contraction mapping theorem:

Theorem. Let $F : Z \to Z$ be a map on a complete normed space Z. If F is a contraction, i.e. if there exists a $0 \leq k < 1$ such that

$$\|F(x_1) - F(x_2)\| \leq k\,\|x_1 - x_2\|$$

for all $x_1, x_2 \in Z$, then F has a fixed point ($F(y) = y$).

For our purposes, we define $F = (F_1, F_2)$ to be

$$\begin{aligned}
F_1(u,B) &:= e^{t\mathcal{L}_1}u_0 - \int_0^t e^{(t-s)\mathcal{L}_1}G_1(u,B)ds,\\
F_2(u,B) &:= e^{t\mathcal{L}_2}B_0 - \int_0^t e^{(t-s)\mathcal{L}_2}G_2(u,B)ds.
\end{aligned}$$

If F is a contraction, then its fixed point (u,B) will consist of the required solutions.

The only remaining step is to select an appropriate space Z for F map into. Remember: our goal is to guarantee solutions with rough initial conditions and arbitrary integrability, so Z will need to keep track of both. The space which will track integrability is the following:

$$L^p(\mathbb{R}^n) := \left\{ f : \mathbb{R}^n \to \mathbb{R}^n \,\middle|\, \left(\int_{\mathbb{R}^n}|f(x)|^p\,dx\right)^{1/p} < \infty \right\}.$$

Maps in $L^p(\mathbb{R}^n)$ are said to be p-integrable. Requiring that our solutions be p-integrable is essential for our functional analytical methods to work. We then define a Sobolev space:

$$H^{r,p}(\mathbb{R}^n) := \left\{ f \in L^p(\mathbb{R}^n) \,|\, D^r f \in L^p(\mathbb{R}^n) \right\},$$

where $D^r f$ denotes r weak derivatives of f in any direction.

The space Z will then be a combination of an interval, for the time variable, and a Sobolev space, for the space variable.

4. Proving that F is a contraction

We start by setting $Z = X_{T,M} \times Y_{T,M}$, where

$$\begin{aligned}
X_{T,M} &= \left\{ f : [0,T)\times\mathbb{R}^n \to \mathbb{R}^n : \sup_{(0,T)}\left\|f(t,\cdot) - e^{t\mathcal{L}_1}u_0\right\|_{r_0,p_0} < M \right\},\\
Y_{T,M} &= \left\{ f : [0,T)\times\mathbb{R}^n \to \mathbb{R}^n : \sup_{(0,T)}\left\|f(t,\cdot) - e^{t\mathcal{L}_2}B_0\right\|_{r_2,p_2} < M \right\}.
\end{aligned}$$

We subtract the $e^{t\mathcal{L}_1}u_0$ and $e^{t\mathcal{L}_2}B_0$ in the Sobolev norms to prevent the size of the initial condition to affect the size of our solutions. The rest of the proof would consist in founding appropriate bounds for the following norms:

$$\left\|\int_0^t e^{(t-s)\mathcal{L}_1}G_1(u,B)ds\right\|_{r_0,p_0} \quad\text{and}\quad \left\|\int_0^t e^{(t-s)\mathcal{L}_2}G_2(u,B)ds\right\|_{r_2,p_2}.$$

By unpacking the definitions of G_1 and G_2 and applying some functional analytic bounds on $e^{(t-s)\mathcal{L}_i}$, we obtain

$$\left\|P(1-\mathcal{L}_3)^{-1}\sum_{i=1}^n v_i\nabla u_i\right\|_{r_0,\pi_2} \leq \cdots \leq C\,\|u\|_{\gamma_3,p'}\,\|u\|_{1,p''},$$

where π_2, p', p'' are intermediate integrability indices that can be ignored. Ideally, we would want to continue with

$$\|u\|_{\gamma_3^-,p'} \leq \|u\|_{r_0,p_0} \text{ and } \|u\|_{1,p''} \leq \|u\|_{r_0,p_0};$$

unfortunately, these two inequalities require $r_0 \geq \gamma_3^-$ and $r_0 \geq 1$, both of which are detrimental to our goal of lowering r_0 as much as possible.

To solve this issue, we employ a technique introduced in [1] and construct an auxiliary space for the fluid velocity: the function u will belong in two spaces, one of regularity r_0 (which we want to minimize), and one of regularity $r_1 \geq r_0$ (which we care less about) that satisfies the following conditions, for a particular $a_1 > 0$:

$$\|u\|_{a_1;r_1,p_1} := \sup_{(0,T)} t^{a_1}\,\|u(t)\|_{r_1,p_1} < \infty \text{ and } \lim_{t\to 0^+} t^{a_1}u(t) = 0.$$

Effectively we have

$$X_{T,M} = \left\{ f : \sup_{(0,T)}\left\|f(t,\cdot) - e^{t\mathcal{L}_1}u_0\right\|_{r_0,p_0} + \|f\|_{a_1;r_1,p_1} < M \right\}.$$

With this new setup, the rest of the proof then consists in proving that

$$\begin{aligned}
I_1 &= \sup_{(0,T)} t^{a_1}\left\|e^{t\mathcal{L}_1}u_0\right\|_{r_1,p_1} < M/4,\\
I_2 &= \sup_{(0,T)}\left\|\int_0^t e^{(t-s)\mathcal{L}_1}G_1(u,B)ds\right\|_{r_0,p_0} < M/4,\\
I_3 &= \sup_{(0,T)} t^{a_1}\left\|\int_0^t e^{(t-s)\mathcal{L}_1}G_1(u,B)ds\right\|_{r_1,p_1} < M/4,\\
I_4 &= \sup_{(0,T)}\left\|\int_0^t e^{(t-s)\mathcal{L}_2}G_2(u,B)ds\right\|_{r_2,p_2} < M/4.
\end{aligned}$$

The nonlinear terms inside each integral can be expanded and dealt with individually. Each bound will require a condition on the regularity and integrability parameters, the collection of which forms our general result.

Theorem (Riva, 2019). Let $g_1, g_2, g_3 : [0,\infty) \to \mathbb{R}$ be non-decreasing functions bounded below by 1 satisfying

$$g_i^{(k)}(s) \leq Cs^{-k}$$

for $i = 1,2,3$ and $0 \leq k \leq n/2+1$. Let $\gamma_3^- - 1 \leq r_0 \leq \gamma_3^- \leq r_1$, $r_2 - 1 + \gamma_3^- \leq r_0 < n/p_0$, and let $p_0, p_1, p_2 \geq n$ with $p_0 \leq p_1$. Moreover, assume that

$$2r_1 \geq \max\left\{2, 1+\gamma_3^- - \frac{n}{p_0} + \frac{2n}{p_1}\right\}, 0 \leq r_2 < \min\left\{\frac{n}{p_0}, \frac{n}{p_2}, \frac{2n}{p_2} - \frac{n}{p_0}\right\}.$$

Then, for any divergence-free $u_0 \in H^{r_0,p_0}(\mathbb{R}^n)$ and $B_0 \in H^{r_2,p_2}(\mathbb{R}^n)$, there exists a unique local solution (u,B) to the generalized MHD-α system provided that

$$\begin{aligned}
\gamma_1^- &> 3r_1 - 2r_0 - \gamma_3^- + \frac{3n}{p_0} - \frac{3n}{p_1},\\
\gamma_1^- &> 1 - 2r_2 + r_1 - \gamma_3^- - \frac{n}{p_1} + \frac{2n}{p_2},\\
\gamma_2^- &> 1 - r_2 + \frac{n}{p_0}.
\end{aligned}$$

References

[1] Tosio Kato and Gustavo Ponce. The Navier-Stokes equation with weak initial data. Internat. Math. Res. Notices, (10):435 ff., approx. 10 pp., 1994.

[2] Nathan Pennington. Low regularity global solutions for a generalized MHD-α system. Nonlinear Anal. Real World Appl., 38:171–183, 2017.