AN INTRODUCTION TO MODEL CATEGORIES & HOMOTOPY LIMITS

Saturday, April 10, 2021 11:26 PM

following Dwyer & Spalinski

Motivation:

In Top we have

$$\begin{tikzcd}
A & D^n \arrow[d, "a"', "\simeq"] & S^{n-1} \arrow[l, hook'] \arrow[r, hook] \arrow[d, "b"', "\simeq"] & D^n \arrow[d, "c"', "\simeq"] \\
B & * & S^{n-1} \arrow[l] \arrow[r] & *
\end{tikzcd}
\xrightarrow{\text{colim}}
\begin{tikzcd}
\operatorname{colim} A \arrow[d] \arrow[r, phantom, "\approx"] & S^n \arrow[d, "f"] \\
\operatorname{colim} B \arrow[r, phantom, "\approx"] & *
\end{tikzcd}$$

$\Rightarrow$ a, b, c are homotopy equivalences <u>but</u> f is not!!

$\Rightarrow$ colim does not behave well. wrt htpy. equiv.

This boils down to the fact that, for the purposes of colim, $S^{n-1} \hookrightarrow D^n$ is a good map

while $S^{n-1} \to *$ is not. How to fix this?

- Model categories

Def A model category $\mathcal{C}$ is a category w/

WE, F, C $\subseteq$ mor $\mathcal{C}$ st:

(0) WE, F, C closed under comp. and
$\xrightarrow{\sim}$, $\twoheadrightarrow$, $\hookrightarrow$
containing all id's;

(1) $\mathcal{C}$ is finitely bicomplete (all finite co/lims),
in particular
product

pr

$$\begin{tikzcd} A \times_C B \arrow[r, "pr_1"] \arrow[d, "pr_0"'] & B \arrow[d] \\ A \arrow[r] & C \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d] \\ 1 \end{tikzcd}$$

terminal ob.

$$\begin{tikzcd} C \arrow[r] \arrow[d] & B \arrow[d, "in_1"] \\ A \arrow[r, "in_0"'] & A +_C B \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[d] \\ X \end{tikzcd}$$

initial ob.

coproduct

② WE satisfies "2 out of 3": if two of f, g, fg are in WE then so is the third

③ If $g \in$ WE, F, or C and $\exists$ retract diagram

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend left, "id_X"] & Y \arrow[r] & X \end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[d, "f"] & {} \arrow[d, "g"] & {} \arrow[d, "f"] \\
X' \arrow[r] \arrow[rr, bend right, "id_{X'}"'] & Y' \arrow[r] & X'
\end{tikzcd}$$

then $f \in WE, F,$ or C

(4) Given

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "i"'] & Y \arrow[d, "p"] \\
X' \arrow[r] & Y'
\end{tikzcd}$$

and either $i \in C$, $p \in WE \cap F$

or

$i \in WE \cap C$, $p \in F$

then $\exists\, h: X' \to Y$ making the triangles commute

(i has LLP, p has RLP)

(5)

$$X \xrightarrow{f} Z \;=\; \begin{cases} X \overset{\sim}{\hookrightarrow} Y \twoheadrightarrow Z \\ X \hookrightarrow Y \overset{\sim}{\twoheadrightarrow} Z \end{cases} \qquad \forall f \in \operatorname{mor} \mathcal{C}$$

Note that the axioms are dualizable (i.e. $F \longleftrightarrow C$)

- Examples

(i) Ch_R : chain complexes of R-modules w/

$$C = \left\{ f : M_\bullet \longrightarrow N_\bullet \;\middle|\; \begin{array}{l} f_k : M_k \longrightarrow N_k \text{ is mono} \\ \text{and coker}\cdot f_k \text{ is projective} \end{array} \right\}$$

$$F = \left\{ f : M_\bullet \longrightarrow N_\bullet \;\middle|\; f_k : M_k \longrightarrow N_k \text{ is epi} \right\}$$

$$WE = \left\{ f : M_\bullet \longrightarrow N_\bullet \;\middle|\; f_* : H_k(M_\bullet) \longrightarrow H_k(N_\bullet) \text{ is iso} \right\}$$

(ii) Top : topological spaces w/

$$F = \left\{ f : X \to Y \;\middle|\; \begin{array}{l} \forall \text{ CW cpx } A,\ f \text{ has the} \\ \text{RLP wrt } A \times \{0\} \dashrightarrow A \times [0,1] \end{array} \right\}$$

$$WE = \{ f : X \to Y \mid f_* : \pi_k(X, x) \to \pi_k(Y, f(x)) \text{ is iso}, \ \forall x \in X, \forall k \}$$

$$C = \{ f : X \to Y \mid f \text{ has the LLP wrt } F \cap WE \}$$

<u>Terminology</u>

(a) $WE \cap C$ = <u>acyclic</u> cofibrations

$WE \cap F$ = " fibrations

(b) $0 \hookrightarrow A \implies A$ <u>cofibrant</u>

Ex. Π_0 w/ projective Π_k for all k in Ch_R or retracts of CW complexes in Top

$A \twoheadrightarrow 1 \implies A$ <u>fibrant</u>

Ex. all objects in Top & Ch_R

Converse to 4? If $f : X \to Z$ has the LLP wrt $WE \cap F$ then

5

$$\begin{tikzcd} X \arrow[d, hook, "i"] \arrow[dd, bend right=60, "f"'] \\ Y \arrow[d, "p", "\sim"'] \\ Z \end{tikzcd} \implies \begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "f"'] & Y \arrow[d, "p", "\sim"'] \\ Z \arrow[r, "="'] & Z \end{tikzcd} \overset{4}{\implies} \begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[d, "p"] \\ Z \arrow[r, "="'] \arrow[ur, dashed, "\exists h"] & Z \end{tikzcd}$$

$$\implies \begin{tikzcd} X \arrow[r, "="] \arrow[d, "f"'] & X \arrow[r, "="] \arrow[d, "i"] & X \arrow[d, "f"] \\ Z \arrow[r, "h"'] \arrow[rr, bend right, "="'] & Y \arrow[r, "p"'] & Z \end{tikzcd} \overset{3}{\implies} f \in C \quad \checkmark$$

f is in the same class as i., so if $p \in F$ and $i \in WE \cap C$ then $f \in WE \cap C$

Conclusion:

$$C = \{ f \in \mathrm{mor}\,\mathcal{C} \mid f \text{ has the LLP wrt } WE \cap F \}$$

$$WE \cap C = \{ \;" \;\mid\; " \quad " \quad " \quad F \}$$

and duals: replace C w/ F and LLP w/ RLP

- **Homotopy**

Idea: in Top, $f \simeq g : X \to Y$ if $\exists\, H : X \times I \to Y$ st $H(x,0) = f(x)$ and $H(x,1) = g(x)$.

$$\begin{tikzcd}
X + X \arrow[r, "f+g"] \arrow[d, hook, "i"'] & Y \\
X \times I \arrow[ur, dashed, "H"'] &
\end{tikzcd}$$

$$i(x) = \begin{cases} (x,0) & \text{if } x \in X_{(1)} \\ (x,1) & \text{if } x \in X_{(2)} \end{cases}$$

Also, $X \times I \simeq X$?

!

Def A cylinder object for A is $A \wedge I$ w/

$$\begin{tikzcd} A + A \arrow[r, "i"'] \arrow[rr, bend right, "\mathrm{id}_A + \mathrm{id}_A"'] & A \wedge I \arrow[r, "\sim", "p"'] & A \end{tikzcd}$$

(i) $A \wedge I$ is good if $i \in C$

(ii) $A \wedge I$ is very good if (i) and $p \in F$

Note: (ii) satisfied by some $A \wedge I$ by ⑤. Also $(j = 0, 1)$

$$\begin{tikzcd} A \arrow[r, "in_j"] \arrow[rr, bend right, "i_j"'] \arrow[rrr, bend right=50, "="'] & A + A \arrow[r] & A \wedge I \arrow[r, "\sim"] & A \end{tikzcd}$$

$\rightsquigarrow$ $i_j \in WE$

Def $f, g : A \to X$ are **left homotopic** ($f \overset{l}{\simeq} g$) if

$\exists\, H : A \wedge I \to X$ st:

$$\begin{tikzcd} A + A \arrow[r, "f+g"] \arrow[d] & X \\ A \wedge I \arrow[ur, dashed, "H"'] & \end{tikzcd}$$

H: left homotopy

In Top, $\overset{l}{\sim}$ is an equivalence relation on $\mathrm{Top}(A, X)$

Take an arbitrary model category $\mathcal{C}$.

- $f + f = f(id_A + id_A)$

$$\begin{tikzcd} A + A \arrow[r, "f+f"] \arrow[d, "id_A + id_A"'] & X \\ A \arrow[ur, "f"'] & \end{tikzcd}$$

  so $f \overset{l}{\simeq} f$

- If $f \overset{l}{\simeq} g$ then

$$\begin{tikzcd} A + A \arrow[r, "g+f"] \arrow[d, "s"'] & X \\ A + A \arrow[ur, "f+g"'] & \end{tikzcd}$$

  $f + g = H i$ and so

$$g+f = (f+g)s = His$$

$$\begin{tikzcd} A+A \arrow[d, "i"'] & \\ A\wedge I \arrow[ur, "H"'] & \end{tikzcd}$$

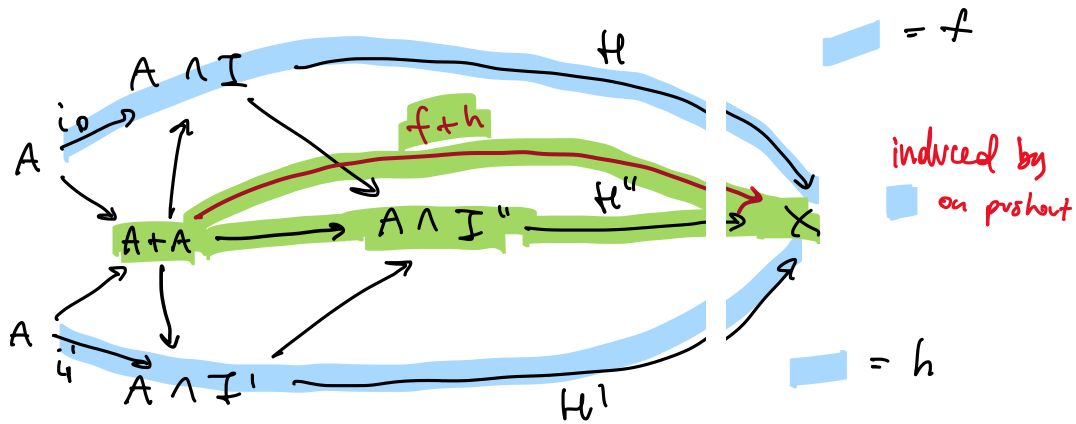
- If $f \overset{l}{\sim} g$ and $g \overset{l}{\sim} h$, then

$$\begin{tikzcd} A+A \arrow[r, "f+g"] \arrow[d, "i"'] & X \\ A\wedge I \arrow[ur, "H"'] & \end{tikzcd} \qquad \begin{tikzcd} A+A \arrow[r, "g+h"] \arrow[d, "i'"'] & X \\ A\wedge I' \arrow[ur, "H'"'] & \end{tikzcd}$$

$$\begin{tikzcd} & A\wedge I \arrow[dr, "a"] \arrow[drr, bend left, "\sim"] & & \\ A \arrow[ur, "i_1", "\sim"'] \arrow[dr, "i'_0"', "\sim"] & A+A \arrow[u, "i"'] \arrow[d, "i'"] \arrow[r, tail] & A\wedge I'' \arrow[r, "c"] & X \\ & A\wedge I' \arrow[ur, "b"'] \arrow[urr, bend right, "\sim"'] & & \end{tikzcd}$$

If $a, b \in WE \Rightarrow c \in WE \Rightarrow A \wedge I''$ is cylinder object.

Then



Turns out a, b can't "just" be WE :((something to do w/ WE not being nice wrt LLP / RRP). But

A cofibrant $\overset{(*)}{\Longrightarrow}$ $i_1 \in C \cap WE$, $i_0' \in C \cap WE$ $\Rightarrow$ $b \in C \cap WE$, $a \in C \cap WE$

$\Downarrow$

$\stackrel{\ell}{\sim}$ is equiv. rel. on $\mathrm{Hom}_{\mathcal{C}}(A,X)$ $\Longleftarrow$ $\underline{a, b \in WE}$

$\Downarrow$

$\circledast$ need to pick good cyl. obj.

Def $\underline{\pi^{\ell}(A,X)} = \mathrm{Hom}_{\mathcal{C}}(A,X)/\stackrel{\ell}{\sim}$

$\uparrow$

or the e.r. generated by $\stackrel{\ell}{\sim}$

<u>Properties</u> of $\pi^{\ell}(A,X)$

- covariant in X : every $p: X \to Y$ induces (when A cofibrant)

$$\begin{aligned} p_* : \pi^{\ell}(A,X) &\longrightarrow \pi^{\ell}(A,Y) \\ [\ell] &\longmapsto [p\ell] \end{aligned}$$

p_* bijection when $p \in F \cap WE$

- If X fibrant and $f \stackrel{\ell}{\sim} g$, then $fh \stackrel{\ell}{\sim} gh$ (for $h: A' \to A$)

• " " " , $\pi^l(A,B) \times \pi^l(B,X) \longrightarrow \pi^l(A,X)$ well-defined
$\qquad [f] \quad [g] \longmapsto [gf]$

We have <u>dual notions</u> (invert all arrows and properties):

<u>cylinder</u> $\begin{tikzcd} A+A \arrow[r, "i"] \arrow[rr, bend right, "\mathrm{id}_A+\mathrm{id}_A"'] & A\wedge I \arrow[r, "p", "\sim"'] & A \end{tikzcd}$	$\begin{tikzcd} X \arrow[r, "i", "\sim"'] \arrow[rr, bend right, "(\mathrm{id}_X,\mathrm{id}_X)"'] & X^I \arrow[r, "p"] & X\times X \end{tikzcd}$ <u>path</u>
good $\quad i \in C$ very good $\quad p \in F$	$p \in F \quad$ good $i \in C \quad$ very good
A cofibrant, $A\wedge I$ good $\Rightarrow i_0, i_1 \in C \cap WE$	X fibrant, X^I good $\Rightarrow p_0, p_1 \in F \cap WE$

$f \overset{l}{\sim} g : A \to X$ when

$$\begin{tikzcd} A + A \arrow[r, "f+g"] \arrow[d] & X \\ A \wedge I \arrow[ur, dashed, "H"'] & \end{tikzcd}$$

$f \overset{r}{\sim} g : A \to X$ when

$$\begin{tikzcd} & A \arrow[d, "{(f,g)}"] \arrow[dl, dashed, "H"'] \\ X^I \arrow[r] & X \times X \end{tikzcd}$$

A cofibrant $\Rightarrow$ $\overset{l}{\sim}$ is e.r. on $\mathrm{Hom}_{\mathcal{C}}(A,X)$ and $\pi^l(A,X)$ is covariant in X (f_* bijective when $f \in F \cap WE$)	X is fibrant $\Rightarrow$ $\overset{r}{\sim}$ is e.r. on $\mathrm{Hom}_{\mathcal{C}}(A,X)$ and $\pi^r(A,X)$ is contravariant in A. (f^* bijective when $f \in C \cap WE$)

de k

- <u>Homotopy and WE</u>

$\boxed{\text{Nice result}}$ If A cofibrant, X fibrant, then $\overset{l}{\sim} = \overset{r}{\sim}$ on $\mathrm{Hom}_{\mathcal{C}}(A,X)$. use $\sim$

Pf Sufficient to show that A cofibrant and $f \overset{l}{\sim} g \Rightarrow f \overset{r}{\sim} g$

Choose good objects and left homotopy

$$\underbrace{\begin{tikzcd} A+A \arrow[r, "i_0+i_1"] \arrow[d, "f+g"'] & A\wedge I \arrow[r, "j"] \arrow[dl, "H"] & A \\ X & & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "q"] & X^I \arrow[r, two heads, "(p_0,p_1)"] & X\times X \end{tikzcd}}$$

$$\begin{tikzcd} A \arrow[r, "qf"] \arrow[d, "i_0"', "\wr"] & X^I \arrow[d, "(p_0,p_1)"] \\ A\wedge I \arrow[r, "(fj,H)"'] \arrow[ur, dashed, "K"] & X\times X \end{tikzcd}$$

w/ $i_0 \in C\cap WE$

$(p_0,p_1)\in F$

$\Downarrow$ ④

$\exists$ lift K

$\Downarrow$

right homotopy

$$\begin{tikzcd} & A \arrow[dl, "ki_1"'] \arrow[d, "(f,g)"] \\ X^I \arrow[r, "(p_0,p_1)"'] & X\times X \end{tikzcd}$$

holds since

$$\begin{aligned}(p_0,p_1)ki_1 &= (p_0ki_1, p_1ki_1)\\ &= (fji_1, Hi_1)\\ &= (f, g)\end{aligned}$$

as $ji_1 + ji_0 = id_A + id_A$ $\blacksquare$

$\boxed{\text{Other nice result}}$ Assume A, X fibrant & cofibrant, $f: A \to X$. Then

$$f \in WE \iff \exists g: X \to A \text{ w/ } \begin{array}{l} fg \sim id_X \\ gf \sim id_A \end{array}$$

"homotopy inverse"

Pf Say $f \in WE$ and factor it as $A \overset{q}{\underset{\sim}{\hookrightarrow}} C \overset{p}{\underset{\sim}{\twoheadrightarrow}} X$.

A ,

$$
\begin{tikzcd}
A \arrow[r, "="] \arrow[d, "q"', "\sim"] & A \arrow[d] \\
C \arrow[r] & 1
\end{tikzcd}
\quad \& \quad
\begin{tikzcd}
0 \arrow[r] \arrow[d] & C \arrow[d, "p", "\sim"'] \\
X \arrow[r, "="'] & X
\end{tikzcd}
\overset{(4)}{\Longrightarrow} \exists\, q^{-1}: C \to \cdot,\ p^{-1}: X \to C \text{ st}
$$

$$q^{-1}q = id_A$$
$$p p^{-1} = id_X$$

But:

q^* bijective and $q^*([qq^{-1}]) = [qq^{-1}q] = [q] \Longrightarrow qq^{-1} \sim id_A$

p_* bijective and $p_*([p^{-1}p]) = [pp^{-1}p] = [p] \Longrightarrow p^{-1}p \sim id_X$

$\Longrightarrow f^{-1} = q^{-1}p^{-1}$ is left/right homotopy inverse for f

Say f has homotopy inverse $g: X \to A$. Factor f as

$A \overset{q}{\underset{\sim}{\hookrightarrow}} C \overset{p}{\twoheadrightarrow} X$. Enough to show $p \in WE$.

If $H : fg \sim id_X$ then

$$\begin{tikzcd}
X + X \arrow[r, "fg + id_X"] \arrow[d] & X \\
X \wedge I \arrow[ur, "H"'] &
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
X \arrow[r, "qg"] \arrow[d, hook, "i_0"', "\sim"] & C \arrow[d, two heads, "p"] \\
X \wedge I \arrow[r, "H"'] & X
\end{tikzcd}
\overset{4}{\Rightarrow}$$

$\exists$ lift $H' : X \wedge I \to C$

is left htpy, not going to prove it

Let $s = H'i_1$, so $ps = pH'i_1 = Hi_1 = id_X$. Then

$$\begin{tikzcd}
C \arrow[r, "="] \arrow[d, "p"'] & C \arrow[r, "="] \arrow[d, "sp"] & C \arrow[d, "p"] \\
X \arrow[r, "s"'] & C \arrow[r, "p"'] & X
\end{tikzcd}
\overset{5}{\Rightarrow}$$

if $sp \in WE$ then $p \in WE$

But $H'(i_0 + i_1) = H'i_0 + H'i_1 = qg + s$, so $s \sim qg$ and thus

$$sp \sim (qg)(f\overset{WE}{q}{}^{-1}) \sim q\, id_A\, q^{-1} \sim id_C \Rightarrow sp \in WE$$
$$\Rightarrow p \in WE$$
$$\Rightarrow f \in WE \quad \blacksquare$$

- **The homotopy category**

Idea: We want a better $\mathcal{C}$ where $Hom_{\mathcal{C}}(A,X) = \pi(A,X)$ so that WE has inverses

Problem: π ??? where do we put the "bad" (non-fibrant, non-cofibrant) objects ??

Solution: ignore them!!

$$0 \longrightarrow A \quad = \quad 0 \hookrightarrow \underbrace{QA}_{\text{Q for Qofibrant}} \overset{\sim}{\underset{p_A}{\twoheadrightarrow}} A$$

By ⑤ :

$X \longrightarrow 1 \quad : \quad X \overset{}{\underset{i_X}{\hookrightarrow}} \underbrace{RX}_{R \text{ for fibrant}} \twoheadrightarrow 1$

requires introducing corresponding categories

Q and R are "functorial": map $f : A \longrightarrow X$ induces

$$\begin{tikzcd} QA \arrow[r, "\tilde{f}"] \arrow[d, "p_A"'] & QX \arrow[d, "p_X"] \\ A \arrow[r, "f"'] & X \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i_A"'] & X \arrow[d, "i_X"] \\ RA \arrow[r, dashed, "\bar{f}"'] & RX \end{tikzcd}$$

w/ nice properties (uniqueness up to homotopy, depends only on f)
Now RQX is fibrant and cofibrant, so $\pi|_{\text{im } RQ}$ is e.r.

<u>Def</u> The **homotopy category** $Ho(\mathcal{C})$ of a model category $\mathcal{C}$ has $\text{ob } Ho(\mathcal{C}) = \text{ob } \mathcal{C}$ and

$\forall\, X, Y \in \mathrm{ob}\,\mathcal{C}$: $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(RQX, RQY)$

$$= \pi(RQX, RQY)$$

We have a functor $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$

objects: $X \mapsto X$

morphisms: $f: X \to Y \mapsto RQf : RQX \to RQY$

such that $\gamma(WE) \subseteq \{\text{isomorphisms}\}$ (since w.e.s have homotopy inverses). In a sense, $\mathrm{Ho}(\mathcal{C})$ is the "best" category with this property:

if $F : \mathcal{C} \to \mathcal{D}$ has $F(WE) \subseteq \{\text{isomorphisms}\}$ then

$\exists\, G : \mathrm{Ho}(\mathcal{C}) \to \mathcal{D}$ functor st. $G \circ \gamma = F$,

i.e. $\mathrm{Ho}(\mathcal{C}) = (WE)^{-1}\mathcal{C}$ is the <u>localization</u> of $\mathcal{C}$ wrt WE.

What can be said about general functors?

Def Let $F : \mathcal{C} \to \mathcal{D}$. A **left derived functor** for F is

a functor $LF : Ho(\mathcal{C}) \to \mathcal{D}$ w/ a natural transf.

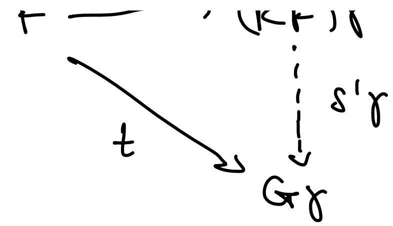
$s : (LF)\gamma \to F$ st for all $G : Ho(\mathcal{C}) \to \mathcal{D}$ and

nat. transf. $t : G\gamma \to F$, $\exists\, s' : G \to LF$ w/

$$\begin{tikzcd} (LF)\gamma \arrow[r, "s"] & F \\ G\gamma \arrow[u, dashed, "s'\gamma"] \arrow[ur, "t"'] & \end{tikzcd}$$

A **right derived functor** is the dual notion:

$$F \xrightarrow{s} (RF)\gamma$$

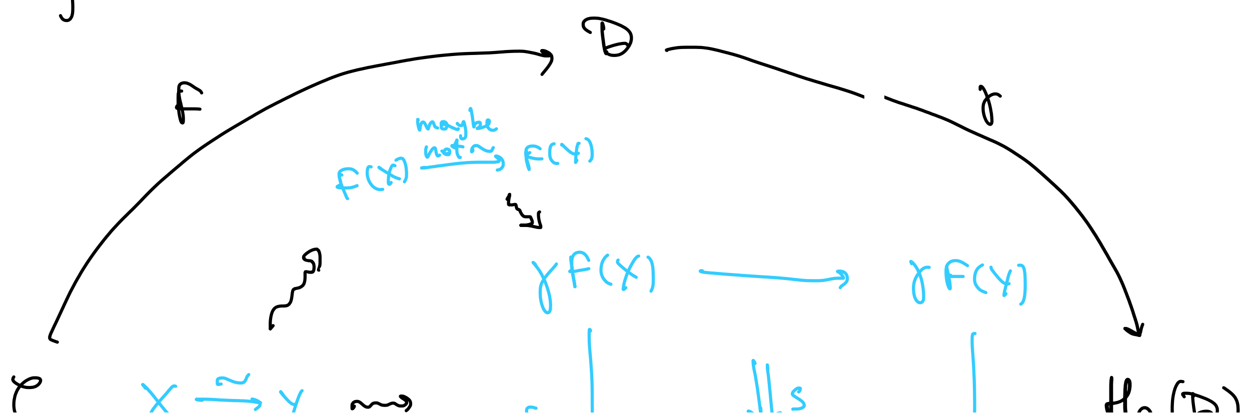


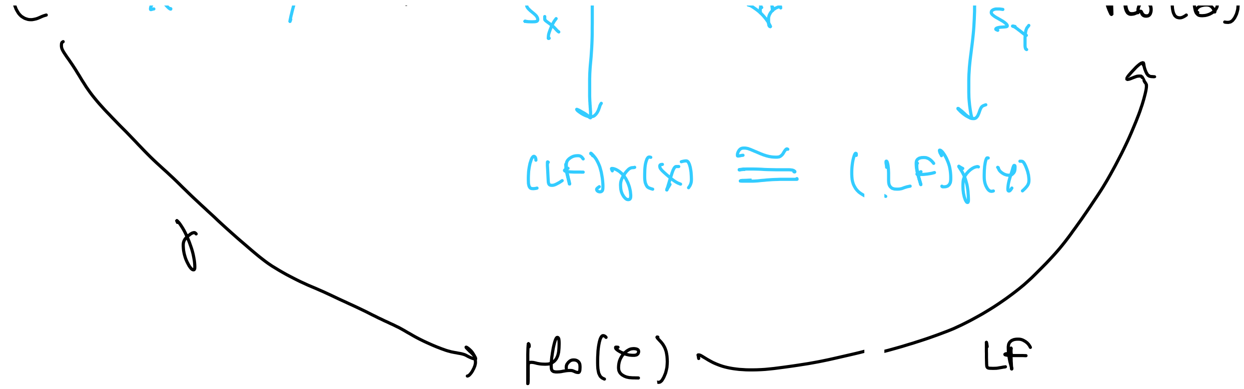
A **total left derived functor** is a functor

$$LF : Ho(\mathcal{C}) \longrightarrow Ho(\mathcal{E})$$

that is left derived for $\gamma F : \mathcal{C} \to \mathcal{D} \to Ho(\mathcal{D})$.

Utility:





Thm Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be adjoint functors such that $F(C) \subseteq C$ and $G(F) \subseteq F$. Then

$$LF : Ho(\mathcal{C}) \to Ho(\mathcal{D}) \qquad \text{(total left derived)}$$

$$RG : Ho(\mathcal{D}) \to Ho(\mathcal{C}) \qquad (\text{" right "})$$

exist and are adjoints.

Moreover: if $F(f)$ is an iso. for every $X \xrightarrow{\sim} Y$ w/ X, Y cofibrant (dually for G) then

the nat. trans. $s: LF(X) \to F(X)$ is an iso.

$LF(X) \cong F(X)$ for all cofibrant X.

- **The case of pushouts**

Recall:

$$\begin{tikzcd}
A & D^n \arrow[d, "a"', "\simeq"] & S^{n-1} \arrow[l, hook'] \arrow[r, hook] \arrow[d, "b"', "\simeq"] & D^n \arrow[d, "c"', "\simeq"] \\
B & * & S^{n-1} \arrow[l] \arrow[r] & *
\end{tikzcd}
\xrightarrow{\text{colim}}
\begin{tikzcd}
\text{colim } A \simeq S^n \arrow[d, "f"] \\
\text{colim } B \simeq *
\end{tikzcd}$$

in WE   not in WE

We want a better colim : $\left\{ \begin{matrix} a \leftarrow b \rightarrow c \\ \text{in } \mathcal{C} \end{matrix} \right\} \dashrightarrow \mathcal{C}$. We need:

1. a model category structure on pushout diagrams
2. an an adjoint to colim
3. the right properties on colim and c adjoint

1. Induced model category: for $D = \{a \leftarrow b \rightarrow c\}$

$$\mathcal{C}^D = \mathrm{Fun}(D, \mathcal{C}) \cong \begin{cases} \text{obj: } X(a) \leftarrow X(b) \rightarrow X(c) \text{ in } \mathcal{C} \\ \text{mor: } \begin{tikzcd} X(a) \arrow[d, "f_a"'] & X(b) \arrow[l] \arrow[r] \arrow[d, "f_b"] & X(c) \arrow[d, "f_c"] \\ Y(a) & Y(b) \arrow[l] \arrow[r] & Y(c) \end{tikzcd} \text{ in } \mathcal{C} \end{cases}$$

Note: for $f : X \to Y$ in $\mathcal{C}^D$

$$\begin{array}{ccccc} & & P_b(f) & & \\ & & \shortparallel & & \\ X(a) & \longleftarrow & X(b) & \longrightarrow & X(c) \end{array}$$

$$\begin{tikzcd}
 & & \arrow[d, "{i_b(f) = f_b}"] & & \\
 & P_a(f) \arrow[dl, dashed, "i_a(f)"'] & Y(b) \arrow[l] \arrow[r] \arrow[dll, bend right] \arrow[drr, bend left] & P_c(f) \arrow[dr, dashed, "i_c(f)"] & \\
Y(a) & & & & Y(c)
\end{tikzcd}$$

with vertical maps f_a into $Y(a)$ and f_c into $Y(c)$.

$P_a(f)$, $P_c(f)$ pushouts,
$i_a(f)$, $i_c(f)$ induced maps

Set
$f \in WE \iff f_a, f_b, f_c \in WE$ in $\mathcal{C}$
$f \in F \iff$ " $\in F$ in $\mathcal{C}$
$f \in C \iff i_a(f), i_b(f), i_c(f) \in C$ in $\mathcal{C}$

2. adjoint to colim: take the constant functor

$$\Delta : \mathcal{C} \longrightarrow \mathcal{C}^D$$
$$x \longmapsto x \xleftarrow{=} x \xrightarrow{=} x$$

generalizes to C^D for any C and any D

$$
X \xrightarrow{f} Y \longmapsto
\begin{tikzcd}
X \arrow[d, "f"'] & X \arrow[l] \arrow[r] \arrow[d, "f"] & X \arrow[d, "f"] \\
Y & Y \arrow[l, "="'] \arrow[r, "="'] & Y
\end{tikzcd}
$$

Then

$$
\mathrm{Hom}_{\mathcal{C}^{\mathcal{D}}}(X, \Delta(Y)) = \left\{ (f_a, f_b, f_c) \;\middle|\;
\begin{tikzcd}
X(a) \arrow[dr, "f_a"'] & X(b) \arrow[l] \arrow[r] \arrow[d, "f_b"] & X(c) \arrow[dl, "f_c"] \\
& Y &
\end{tikzcd}
\right\}
$$

$$
\simeq \left\{ f \;\middle|\;
\begin{tikzcd}
X(b) \arrow[r] \arrow[d] & X(c) \arrow[d] \arrow[dr, bend left, "f_c"] & \\
X(a) \arrow[r] \arrow[rr, bend right, "f_a"'] & \operatorname{colim} X \arrow[r, "f"] \arrow[ul, phantom, "\ulcorner"] & Y
\end{tikzcd}
\right\} \quad \text{by univ. prop. of colim}
$$

$$
= \mathrm{Hom}(\operatorname{colim} X, Y)
$$

$\Rightarrow$ $\operatorname{colim} \dashv \Delta$ are adjoints!

3. colim preserves C (obviously Δ preserves F):

let $f : X \hookrightarrow Y$ in $\mathcal{C}^{D}$.

$$\begin{tikzcd}
X(b) \arrow[r] \arrow[d, hook, "f_b"'] & \operatorname{colim} X \arrow[r] \arrow[d, "\operatorname{colim} f"'] & W \arrow[d, two heads, "s"'] \\
Y(b) \arrow[r] & \operatorname{colim} Y \arrow[r] & Z
\end{tikzcd}$$

④ $\Rightarrow$ $\exists$ lift
$$\begin{tikzcd}
 & W \arrow[d] \\
Y(b) \arrow[ur, "h_b"] \arrow[r] & Z
\end{tikzcd}$$

$$\begin{tikzcd}
X(b) \arrow[r] \arrow[d] & X(a) \arrow[d] \arrow[dr, bend left] & \\
Y(b) \arrow[r] \arrow[drr, bend right, "h_b"'] & P_a(f) \arrow[ul, phantom, "\ulcorner"] & \operatorname{colim} X \arrow[d] \\
 & & W
\end{tikzcd}$$

$\Rightarrow$ $\exists$ $r_a : P_a(f) \to W$

$r_c : P_c(f) \to W$ (symmetry)

$$\begin{tikzcd}
P_{a/c}(f) \arrow[r, "{}_{a/c}"] \arrow[d, "i_{a/c}(f)"'] & W \arrow[d, "s"] \\
Y(a/c) \arrow[r] & Z
\end{tikzcd}$$

(4) $\Rightarrow$ $\exists$ lifts

$$\begin{tikzcd}
 & \arrow[d, "s"] \\
Y(a/c) \arrow[ur, "h_{a/c}"] \arrow[r] & Z
\end{tikzcd}$$

$\Rightarrow$ h_a, h_b, h_c induce lift

$$\begin{tikzcd}
\operatorname{colim} X \arrow[r] \arrow[d, "\operatorname{colim} f"'] & W \arrow[d, "s"] \\
\operatorname{colim} Y \arrow[r] \arrow[ur, dashed, "h"] & Z
\end{tikzcd}$$

$\Rightarrow$ $\boxed{\operatorname{colim} f \in C}$ $\Rightarrow$ $\operatorname{colim}(C) \subseteq C$.

Therefore by the theorem we have

$\text{ho colim} := L\operatorname{colim} : \mathcal{H}o(\mathcal{C}^{D}) \longrightarrow \mathcal{H}o(\mathcal{C})$

st $\operatorname{hocolim}(X) \cong \operatorname{colim}(X')$ for $X' \xrightarrow{\sim} X$.

cofibrant

In practice,

$$\begin{tikzcd} X(c) & & X(c)' \\ X(b) \arrow[u] \arrow[d] \arrow[r, squiggly] & & X(b)' \arrow[u, "\in C"'] \arrow[d, "\in C"] \\ X(a) & & X(a)' \end{tikzcd}$$

$\leftarrow$ cofibrant

Dually we get

$D = \{a \to b \leftarrow c\}$

$\boxed{\textbf{holim} := R\lim : Ho(\mathcal{C}^D) \longrightarrow Ho(\mathcal{C})}$

To conclude :

$$\operatorname{hocolim}\left(\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & D^n \\ D^n & \end{tikzcd}\right) \cong \operatorname{colim}\left(\begin{tikzcd} \overset{\text{cofibrant}}{S^{n-1}} \arrow[r, "\in C"] \arrow[d, "\in C"] & D^n \\ D^n & \end{tikzcd}\right)$$

$$\operatorname{hocolimit}\left(\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & * \\ * & \end{tikzcd}\right) \not\cong \operatorname{colim}\left(\begin{tikzcd} S^{n-1} \arrow[r, "\notin C"] \arrow[d, "\notin C"] & * \\ * & \end{tikzcd}\right)$$

For general homotopy limit on any diagram, the nLab says you need :

(& these)

- a symmetric monoidal category S
- $\mathcal{C}$ to be S-enriched and cofibrantly generated
- D to be small and S-enriched

(or duals

- $\mathcal{E}$ to admit copowers by hom-objects $D(x,y) \in S$ preserving trivial cofibrations

- maybe something else