Fraud and Monitoring in Noncompetitive Elections: Supplemental Information

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This document contains proofs of the results in the main text and formal statements and proofs for the extensions.

Formal Definition of Equilibrium

Write the utility functions for the actors as:

$$U_{I}(x,m;s(m,e,r)) = q(x,m) \int b(s(m,\omega+x,x))f(\omega)d\omega$$

$$+ (1-q(x,m)) \int b(s(m,\omega+x,0))f(\omega)d\omega - c(x)$$

$$U_{A}(s(m,e,r);x,m) = -q(x,m) \int (s(m,\omega+x,x)-\omega)^{2}f(\omega)d\omega$$

$$- (1-q(x,m)) \int (s(m,\omega+x,0)-\omega)^{2}f(\omega)$$
(2)

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An equilibrium must satisfy:

$$s^*(m, e, r) \in \arg\max_{s} U_A(s, m, x^*(m)), \forall m, e, r$$
(3)

$$x^*(m) \in \arg\max_{x} U_I(x, m, s^*(m, e, r)), \forall m$$
(4)

$$m^* \in \arg\max_{m \in [m,\overline{m}]} U_I(x^*(m), m, s^*(m, e, r))$$
 (5)

We do not need to separately specify the Audience beliefs about ω because 1) every contingency where r > 0 is at a singleton information set, and 2) for contingencies where r = 0 all election results are on the equilibrium path and the support level must be a best response to $x^*(m)$. The only aspect of the conditions stronger than Subgame Perfect Nash Equilibrium is that the actors choose mutual best responses for all m, even those not on the equilibrium path. This is loosely analogous to the common sequential rationality requirement where the audience makes no inference about Nature's move if an uninformed actor (i.e., the Incumbent) takes an observed off the path action (i.e., the monitoring level).

Alternatively, if Nature moves after the Incumbent makes their monitoring and fraud decision, then the fraud decision is at a singleton information set and hence subgame perfection would be equivalent to the condition above.

Proofs of Main Results

Proof of proposition 1 and discussion of functional form assumptions: The derivative of the incumbent utility with respect to x when the audience plays the best response to

¹That is, the conditions of best responding given the proper belief about ω and best responding to x are identical. This would not be the case if, for example, we considered a mixed strategy equilibrium or a different information structure.

 $x^*(m)$ is:

$$\frac{\partial u_I}{\partial x} = \frac{\partial q}{\partial x} (\mathbb{E}[b(\omega)] - \mathbb{E}[b(\omega + x - x^*(m))]) + (1 - q(x, m))\mathbb{E}[b'(\omega + x - x^*(m))] - c'(x)$$
 (6)

Plugging in $x = x^*(m)$ and setting the equation equal to zero gives the first order condition in the main text: $c'(x^*(m)) = (1 - q(x^*(m), m))\mathbb{E}[b'(\omega)]$. By the convexity and range of c and the fact that q is decreasing in x we know that there exists a unique $x^*(m)$ solving this equation, and hence a unique potential pure strategy equilibrium for a fixed m.

To check that this is a global maximum, first note that for any $x < x^*(m)$, $c'(x) < (1 - q(x, m))\mathbb{E}[b'(\omega)]$, and $\mathbb{E}[b'(\omega + x - x^*(m))] > \mathbb{E}[b'(\omega)]$. Plugging these inequalities into equation 6 implies that $\frac{\partial u_I}{\partial x} > 0$ for $x < x^*(m)$. An identical argument implies that $\frac{\partial u_I}{\partial x} < 0$ for $x > x^*(m)$, so $x = x^*(m)$ is a global maximizer when the audience expects $x^*(m)$, hence $x^*(m)$ is the unique equilibrium level of fraud for monitoring level m.

Though the objective function is not necessarily globally concave, there are multiple forces that make it "concave enough" to ensure a unique global maximum. Roughly speaking, these are the facts that 1) c is convex, 2) b is concave, and 3) $\frac{\partial q}{\partial x} \geq 0$, which induces additional concavity to the benefit for committing more fraud when this inequality is strict. All three of these assumptions seem individually reasonable, but if any fails than the other two may be sufficient to ensure a unique pure strategy equilibrium.

If the functional form assumptions made here are not met, the only equilibrium may be in mixed strategies or with non-interior solutions. For example, in the context of choosing a level of fraud is it not entirely clear that the cost function should be convex. This assumption may be reasonable for similar reasons that cost functions are generally assumed to be convex: if we consider x to be the number of falsified votes and then order each vote by the cost of changing it, the incumbent will falsify the cheaper ones before the expensive ones. As a result, the marginal cost of falsifying votes goes up as the amount of fraud increases, generating a

convex cost function. However, the cost of falsifying vote i may not be independent of the cost of falsifying vote j, particularly when using technology such as ballot stuffing or simply strong-arming the relevant electoral body to make up the results, which may have decreasing marginal costs.

As formalized above, a non-convex cost function would not necessarily prevent a pure strategy equilibrium, but will under some circumstances result in the incumbent using a mixed fraud strategy in equilibrium. Mixed strategy equilibria are far more difficult to characterize; to my knowledge there is no work addressing mixed strategies in similar games. However, one can easily show that even even if there is a mixed strategy equilibrium for all possible monitoring levels, if \overline{m} is sufficiently high the incumbent always chooses $m^* = \overline{m}$ Further, as long as monitoring decreases the marginal benefit to committing more fraud than expected it seems likely that analogous results would hold with mixed strategies.

Competitive Elections

To illustrate how the model could incorporate competitive elections a straightforward fashion, suppose the incumbent payoff is:

$$u_I(x, m, s, e) = b(s) - c(x) + vPr(e > e^*)$$

for some $e^* \in \mathbb{R}$. That is, the incumbent wins the election if $e > e^*$, and the value of winning the election relative to losing is v. Of course the effect of the audience support may be different or irrelevant if the incumbent loses the election; this parameterization simply eases the presentation.

The incumbent utility for choosing x when the audience expects $x^*(m)$ becomes:

$$u_I(x, x^*(m)) = q(x, m)\mathbb{E}[b(\omega)] + (1 - q(x, m))\mathbb{E}[b(\omega + x - x^*(m))] - c(x) + vPr(x + \omega > e^*)$$

Giving first order condition:

$$\frac{\partial u_I}{\partial x}\bigg|_{x=x^*(m)} = (1 - q(x^*(m)), m)\mathbb{E}[b'(\omega)] - c'(x^*(m)) + vf(e^* - x^*(m)) = 0$$
$$c'(x^*(m)) = (1 - q(x^*(m)), m)\mathbb{E}[b'(\omega)] + vf(e^* - x^*(m))$$

Without adding more assumptions about f, this condition could be met for multiple $x^*(m)$, and if two of these intersections correspond to peaks of the objective function that have the exact same value, there can be multiple optimal fraud choices. This possibility can be ruled out if, for example, f' is always negative, v is sufficiently small, c'' is sufficiently positive everywhere or $\mathbb{E}[b'']$ is sufficiently negative everywhere, any of which would imply that the LHS of the equilibrium condition is always increasing faster than the LHS.

If the fraud choice is unique, implicitly differentiating gives $\frac{\partial x^*}{\partial m} < 0$. Now consider the change in the equilibrium utility with respect to m:

$$\frac{\partial u_I(x^*(m), x^*(m))}{\partial m} = \frac{\partial x^*}{\partial m} (-c'(x^*(m)) + vf(e^* - x^*(m))) > 0$$

Where the inequality comes from substituting in the first order condition. So the incumbent still invites full monitoring.

However, this is assuming that fraud detected by monitors still counts towards winning the election. If the falsified votes are nullified by the monitoring report, or incumbents caught cheating (or cheating above a certain threshold) automatically lose the election, then monitoring carries an additional cost. It will be particularly costly when v is high and $f(e^* - x^*)$ is high, i.e., the incremental gains to fraud via increasing the chance of winning the election are high. Using standard functional forms, this will tend to be the case when the election is close. So, loosely speaking, the desire to invite monitoring for the mechanism described here should hold for somewhat competitive elections.

Costly Monitoring

Suppose that in addition to the payoffs in the baseline model, the incumbent gets a partial payoff -k(m) for choosing monitoring level m. That is, their entire payoff is:

$$u_I(x, m; s) = b(s) - c(x) - k(m)$$

Assume that k is continuous and differentiable, with $k'(\underline{m}) = 0$, $k' \geq 0$, and $\lim_{m \to \overline{m}} = k'(m) = \infty$. That is, there is no marginal cost for the lowest level of monitoring, the cost of monitoring is increasing in the level of monitoring, and monitoring becomes arbitrarily costly as the monitoring level approaches the highest level possible. This leads to a straightforward result:

Lemma 2. The interior level of monitoring is an interior $m^* \in (\underline{m}, \overline{m})$.

Proof The equilibrium condition for the monitoring level is:

$$m^* \in \arg\max_{m \in [m,\overline{m}]} \mathbb{E}[b(\omega)] - c(x^*(m)) - k(m)$$

Giving first order condition:

$$k'(m) = -c'(x^*(m))\frac{\partial x^*}{\partial m} \tag{7}$$

By the continuity of the individual functions, an interior maximum m^* must meet this condition. The right hand side is positive (since $\frac{\partial x^*}{\partial m} < 0$) and finite for all m. The left hand side has range \mathbb{R}_+ , ensuring at least one m^* that satisfies equation 7. The boundary conditions ensure that $k'(\underline{m}) = 0 < -c'(x^*(\underline{m})) \frac{\partial x^*}{\partial m}|_{m=\underline{m}}$ and $\lim_{m\to\overline{m}} k'(m) = \infty < \lim(m\to \overline{m}) - c'(x^*(\overline{m})) \frac{\partial x^*}{\partial m}|_{m=\overline{m}}$, hence there can be no boundary solution.

Since $[\underline{m}, \overline{m}]$ is compact and the functions are continuous, a maximum must exist, and by the above argument it must be interior and meet equation 7.

As with the competitive elections extension, this does not guarantee there is a unique optimal monitoring level: the LHS of equation 7 may be increasing for some m if $\frac{\partial^2 x^*}{\partial^2 m} > 0$, which means there could be multiple m's that satisfy the equilibrium condition. Even if this is true, there will only be multiple equilibrium monitoring levels if the incumbent utility function has multiple peaks at exactly the same height. To simplify the comparative static analysis I assume the optimal monitoring level is always unique.

To generate comparative statics on the general costliness of fraud and monitoring write the incumbent utility function as:

$$u_I(x, m; s) = b(s) - \beta_c c(x) - \beta_k k(m)$$

Where $\beta_c > 0$ represents the costliness of committing fraud and $\beta_k > 0$ the costliness of monitoring. To simplify some of the math, assume the probability of detection is q(x, m) = m.

The equilibrium level of fraud for a fixed level of monitoring m is now characterized by:

$$x^*(m) = (c')^{-1} \left(\frac{(1-m)}{\beta_c} \mathbb{E}[b'(\omega)] \right)$$
 (8)

Since $(c')^{-1}$ is increasing, $x^*(m)$ is decreasing in β_c . Differentiating gives:

$$\frac{\partial x^*}{\partial m} = -((c')^{-1})' \left(\frac{(1-m)}{\beta_c} \mathbb{E}[b'(\omega)] \right)$$
(9)

And the first order condition for the monitoring level, rearranged for implicit differentiation is:

$$G(x,m) \equiv \beta_k k'(m) - \beta_c c'(x^*(m)) \frac{\partial x^*}{\partial m} = 0$$
 (10)

Proposition 3. The equilibrium level of monitoring is

i Decreasing in the cost of monitoring

ii Increasing in the cost of fraud.

iii Decreasing in the popularity of the incumbent

iv Provided b''' > 0, increasing in the uncertainty about the incumbent popularity.

Proof Parts 1 and 2 follow from implicitly differentiating equation 10 and, where appropriate, substituting in equation 9.

$$\frac{\partial m^*}{\partial \beta_k} = -\frac{\frac{\partial G}{\partial \beta_k}}{\frac{\partial G}{\partial m}} = -\frac{k'(m)}{\frac{\partial G}{\partial m}} < 0$$

$$\frac{\partial m^*}{\partial \beta_c} = -\frac{\frac{\partial G}{\partial \beta_c}}{\frac{\partial G}{\partial m}} = -\frac{(1-m)\mathbb{E}[b'(\omega)]\frac{\partial^2 x^*}{\partial m\partial \beta_c}}{\frac{\partial G}{\partial m}} > 0$$

The second part relies on a substitution with the first order condition on x^* and the fact that $\frac{\partial^2 x^*}{\partial m \partial \beta_c} < 0$ from equation 9.

Parts iii and iv follow from the first and second order stochastic dominance argument for the comparative statics on the level of fraud (proposition 3 in the main text).

Alternative Information Structure

Suppose that the incumbent still chooses a level of fraud x, but there is some noise that translates this into a "true" level of fraud $x + \nu_x$. So, the election result is now $e = \omega + x + \nu_x$. Let the monitoring report be a noisy signal of the true level of fraud: $r = x + \nu_x + \nu_r$.

To greatly simplify the analysis, suppose ω is normally distributed with mean μ_0 and precision τ_0 (that is, standard deviation $\tau_0^{-1/2}$), and ν_x and ν_r are independent (from each other and ω) and normally distributed with mean 0 and precisions τ_x and m, respectively. That is, more precise monitoring now more literally corresponds to a less noisy monitoring

report, where $0 < \underline{m} < \overline{m} < \infty$. Using standard properties of the normal distribution,² if the audience believes that the incumbent chooses amount of fraud x^* , then their expected belief about ω given e and m becomes:

$$\mu(x^*, e, r) = \frac{\tau_0 \mu_0 + (\tau_x + m)(e - x^*) - \tau_m(r - x^*)}{\tau_0 + \tau_x + m}$$
(11)

The incumbent payoff now must integrate over the noise terms as well:

$$u_I(x, m; x^*) = \iiint b(s^*(m, \omega + x + \nu_x, x + \nu_x + \nu_r)) f(\omega) f(\nu_x) f(\nu_r) d\omega d\nu_x d\nu_r$$
(12)

So, in a pure strategy equilibrium, it must be the case that for a fixed m:

$$s^*(m, e, r) = \frac{\tau_0 \mu_0 + (\tau_x + m)(e - x^*) - m(r - x^*)}{\tau_0 + \tau_x + m}$$
(13)

The incumbent payoff given this best response is:

$$u_{I}(x, m; x^{*}) = \iiint b \left(\frac{\tau_{0}\mu_{0} + (\tau_{x} + m)(\omega + x + \nu_{x} - x^{*}) - m(x + \nu_{x} + \nu_{m} - x^{*})}{\tau_{0} + \tau_{x} + m} \right)$$
$$f(\omega)f(\nu_{x})f(\nu_{r})d\omega d\nu_{x}d\nu_{r} - c(x)$$

We can think of the posterior belief inside the b function as a sum of a constant and three independent normal random variables:

$$\frac{\tau_0 \mu_0 + (\tau_x)(x - x^*)}{\tau_0 + \tau_x + m} + \frac{(\tau_x + m)}{\tau_0 + \tau_x + \tau_m} \omega + \frac{\tau_x}{\tau_0 + \tau_x + \tau_m} \nu_x - \frac{m}{\tau_0 + \tau_x + \tau_m} \nu_m$$

²The join distribution of ω , e, and r is a multivariate normal, so the conditional distribution of ω given e and r is multivariate normal. See Greene (2008, pg. 1014).

Which is itself a normally distributed random variable with mean:

$$\mu_{\mu}(x, x^*) = \frac{\tau_0 \mu_0 + (\tau_x)(x - x^*)}{\tau_0 + \tau_x + m} + \frac{(\tau_x + m)}{\tau_0 + \tau_x + \tau_m} \mu_0 = \mu_0 + \frac{\tau_x(x - x^*)}{\tau_0 + \tau_x + m}$$

And precision:

$$\tau_{\mu} = \left(\tau_0^{-1} \left(\frac{\tau_x + m}{\tau_0 + \tau_x + m}\right)^2 + \tau_x^{-1} \left(\frac{\tau_x}{\tau_0 + \tau_x + m}\right)^2 + m^{-1} \left(\frac{m}{\tau_0 + \tau_x + m}\right)^2\right)^{-1}$$

All that matters moving forward is that 1) the mean of the posterior distribution is independent of m in equilibrium (i.e., when $x = x^*$), and 2) the precision this distribution is decreasing in m. So, for any $m_1 < m_2$, the distribution of μ under m_1 second order stochastic dominates the distribution of μ under m_2 . That is, the more precise the monitoring signal, the less precision the incumbent places on their belief about what the audience posterior expected belief about ω , and hence the less precision in the incumbent belief about how much support they will get.

We can now write the incumbent payoff using a single integral, giving equilibrium condition for fraud:

$$\frac{\partial u_I}{\partial x}\Big|_{x=x^*(m)} = \frac{\tau_x}{\tau_0 + \tau_x + m} \int b'(\mu) f(\mu) d\mu - c'(x^*(m)) = 0$$

$$c'(x^*(m)) = \frac{\tau_x}{\tau_0 + \tau_x + m} \int b'(\mu) f(\mu) d\mu$$

As in the baseline model, the convexity of c and concavity of b imply there exists a unique $x^*(m)$ meeting this equation for all m. The incumbent equilibrium payoff as a function of m is then:

$$u_I(m; x^*(m)) = \mathbb{E}[b(\mu)] - c(x^*(m))$$

and the derivative with respect to m is:

$$\frac{\partial u_I(m; x^*(m))}{\partial m} = \frac{\partial \mathbb{E}[b(\mu)]}{\partial m} - c'(x^*(m)) \frac{\partial x^*}{\partial m}$$

The second term is positive as in the main text. However, since b is concave and for $m_1 > m_2$, the distribution of μ under m_2 second order stochastic dominates the distribution of μ under m_1 , $\frac{\partial \mathbb{E}[b(\mu)]}{\partial m} < 0$. So, it is possible that the incumbent payoff is decreasing in m, depending on how concave the incumbent benefit function is.

Multiple, Potentially Biased Monitors

Having multiple potential monitors with a known bias can be easily modeled in either the baseline or alternative information structure. Suppose there are $N \in \mathbb{N}_+$ reports. In the baseline information structure, index the monitoring reports r_i , where:

$$r_i = \gamma_i \beta_i x$$

Where $\gamma_i = 1$ with probability $q_n(x, m)$ and 0 otherwise. As above, assume the q_n are nondecreasing in x and strictly increasing in m. That is, γ_i is the indicator for whether the incumbent fraud is caught. So, increasing m should be more literally interpreted as allowing more access to the monitors if they are invited, and n as the decision for how many monitors to invite.

The $\beta_i \geq$ term represents the bias of the nth monitor. If $\beta_i = 1$, then when $\gamma_i = 1$ monitor i reports the exact amount of fraud x. If $\beta_i < 1$, the monitor is biased "for" the incumbent, in the sense that they report less fraud then there really is. If $\beta_i = 0$, the monitor is so biased that they never report any fraud. If $\beta_i > 1$, the monitor reports more fraud then there truly was when $\gamma_i = 1$.

If the bias parameters are known by the audience, then the analysis is largely unchanged. The audience strategy is now a function mapping the monitoring level, election result, and monitoring reports $r = (r_1, ..., r_i)$ to an action s(m, e, r).

With probability $Q(x,m) \equiv 1 - \prod (1 - q_i(x,m))$ at least one monitor will report $\beta_i x$, and since the β_i are known the audience can directly infer the level of fraud. So, the equilibrium analysis for a fixed m and n is identical to the baseline with Q(x,m) replacing q(x,m).

Proposition 5. If the bias of the monitors is known, the incumbent chooses the maximal level of monitoring. The equilibrium level of fraud is given by:

$$c'(x^*(m)) = (1 - Q(x^*(\overline{m}), \overline{m}))\mathbb{E}[b'(\omega)]$$

Proof Q(x, m) is increasing in m. The incumbent payoff is increasing in m by an identical argument to lemma 1.

It is easy to see that if which monitors to invite is part of the incumbent strategy space (i.e., if the incumbent does not invite monitor $i \gamma_i = 0$ wi probability 1), they will invite all monitors with $\beta_i > 0$. Monitors such that $\beta_i = 0$ may or may not be invited in equilibrium.

In the alternative information structure, let each monitoring report be:

$$r_i = x + \nu_i$$

Where the ν_i 's are iid random variables with known mean $\beta_i \in \mathbb{R}$ and precision τ_i . So, each $r_i - \beta_i$ is an unbiased signal signal of x. By standard properties of normal distributions, this information structure is identical to observing a single monitoring report with precision $m \equiv \sum_{i=1}^{N} \tau_i$, and an identical analysis as above applies.

This also allows for a convenient way to model monitors of unknown bias. Suppose each $r_i = x + \nu_i$ as above, but now the mean of ν_i is not known. Let the audience belief

about β_i be normal with mean $\overline{\beta}_i$ and precision ν_{β_i} . Again using standard properties of normal distributions, this has the same distribution as $r_i = x + \overline{\beta}_i + \tilde{\nu}_i$ where $\tilde{\nu}_i$ is normally distributed with mean 0 and precision $\frac{1}{\tau_i^{-1} + \tau_{\beta_i}^{-1}}$. So defining $m \equiv \sum_{i=1}^N \frac{1}{\tau_i^{-1} + \tau_{\beta_i}^{-1}}$, the above analysis applies.

The proposition in the main text immediately follows:

Proposition 4. The equilibrium level of fraud is decreasing in the precision of the beliefs about the monitor bias $(\frac{\partial x^*}{\partial \tau_{\beta_i}} < 0)$

Proof From the calculations above we know that $\frac{\partial m}{\partial \tau_{\beta_i}} > 0$ for all i. Combining this with $\frac{\partial x^*}{\partial m} < 0$ gives the result.

Private information and signaling

The baseline model assumed that the incumbent and audience have the same belief about the incumbent popularity. These greatly simplifies the analysis, but is surely not true. This section describes what happens when the assumptions are loosened.

Suppose the audience observes a private signal θ , and that this signal and the incumbent popularity are drawn from a joint distribution $f_2(\omega, \theta)$ where the marginal density of ω is $f(\omega)$ as above.

Suppose the audience conditional belief about ω after observing θ admits a density with support on the real line. When examining pure strategy equilibria the audience support decision remains unchanged as they directly learn the audience popularity from the monitoring report or must best respond to the true level of fraud and hence popularity when the monitoring report is null.

The incumbent payoff must now integrate over potential private signals seen by the audience, but since the audience strategy is the same for all θ , the incumbent payoff and hence optimal strategy remains unchanged as well. If the audience perfectly learns the incumbent

popularity (e.g., $\theta = \omega$), then the game changes substantially as the audience learns nothing from the election result. However, this would be a heroically strong assumption.

Now suppose the incumbent has private information about their popularity. To formalize this, suppose the incumbent is a high popularity type with probability p and a low popularity type with probability 1-p. The high popularity type has density f_h , and the low density type f_l . To formalize the idea that the high type is more popular, suppose f_h first order stochastic dominates f_l . The incumbent learns their type before choosing the monitoring level and amount of fraud.

Suppose there is an equilibrium where the types separate on the monitoring level, i.e., $m_l \neq m_h$. Call the proposed equilibrium levels of fraud played by the two types x_l^* and x_h^* . The the audience response to on-the-path monitoring decisions must be:

$$s^*(m, e, r) = \begin{cases} e - r & r > 0 \\ e - x_l^* & m = m_l, r = 0 \\ e - x_r^* & m = m_h, r = 0 \end{cases}$$

If $x_l^* > x_h^*$, then the low type could deviate to m_h , still choose fraud level x_l^* , and get a strictly higher payoff. Similarly, if $x_l^* < x_h^*$, the high type could deviate to m_l , choose fraud level x_h^* , and get a strictly higher payoff.³

It is possible to construct equilibria where $m_l = m_h < \underline{m}$. However, these equilibria 1) are not Pareto efficient, and 2) require the audience to belief that upon observe a higher than expected monitoring level that more fraud was committed than expected.

³This leaves open the case if $x_l^* = x_h^*$. Even if the first order conditions can be met for both types using a different monitoring level and the same level of fraud, this equilibrium requires the incumbent infer the incumbent is committing a strictly higher amount of fraud or a strictly less popular when inviting high monitoring.

References

Greene, William H. 2008. $Econometric\ Analysis,\ Sixth\ Edition.$ Prentice Hall.