OPTION PRICING WHEN UNDERLYING STOCK RETURNS ARE DISCONTINUOUS*

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The validity of the classic Black–Scholes option pricing formula depends on the capability of investors to follow a dynamic portfolio strategy in the stock that replicates the payoff structure to the option. The critical assumption required for such a strategy to be feasible, is that the underlying stock return dynamics can be described by a stochastic process with a continuous sample path. In this paper, an option pricing formula is derived for the more-general case when the underlying stock returns are generated by a mixture of both continuous and jump processes. The derived formula has most of the attractive features of the original Black–Scholes formula in that it does not depend on investor preferences or knowledge of the expected return on the underlying stock. Moreover, the same analysis applied to the options can be extended to the pricing of corporate liabilities.

1. Introduction

In their classic paper on the theory of option pricing, Black and Scholes (1973) present a mode of analysis that has revolutionized the theory of corporate liability pricing. In part, their approach was a breakthrough because it leads to pricing formulas using, for the most part, only observable variables. In particular, their formulas do not require knowledge of either investors' tastes or their beliefs about expected returns on the underlying common stock. Moreover, under specific posited conditions, their formula must hold to avoid the creation of arbitrage possibilities.[1]

To derive the option pricing formula, Black and Scholes[2] assume 'ideal conditions' in the market for the stock and option. These conditions are:

[1]For an alternative derivation of the Black–Scholes model and a discussion of option pricing models in general, see Merton (1973b). For applications of the Black–Scholes technique to other financial instruments, see Merton (1974) and Ingersoll (1975). As Samuelson (1973, p. 16) has pointed out, violation of the Black–Scholes formula implies arbitrage opportunities only if their assumptions hold with certainty.

[2]In this paper, the term 'option' refers to a call option although a corresponding analysis would apply to put options. For a list of the assumptions used to derive their formula, see Black and Scholes (1973, p. 640).

(1) 'Frictionless' markets: there are no transactions costs or differential taxes. Trading takes place continuously in time. Borrowing and short-selling are allowed without restriction and with full proceeds available. The borrowing and lending rates are equal. (2) The short-term interest rate is known and constant through time. (3) The stock pays no dividends or other distributions during the life of the option. (4) The option is 'European' in that it can only be exercised at the expiration date. (5) The stock price follows a 'geometric' Brownian motion through time which produces a log-normal distribution for stock price between any two points in time.

In a subsequent, alternative derivation of the Black–Scholes formula, Merton (1973b) demonstrated that their basic mode of analysis obtains even when the interest rate is stochastic; the stock pays dividends; and the option is exercisable prior to expiration. Moreover, it was shown that as long as the stock price dynamics can be described by a continuous-time diffusion process whose sample path is continuous with probability one,[3] then their arbitrage technique is still valid. Thorp (1973) has shown that dividends and restrictions against the use of proceeds of short-sales do not invalidate the Black–Scholes analysis. Moreover, the introduction of differential taxes for capital gains versus dividends or interest payments does not change the analysis either [see Ingersoll (1975)].

As was pointed out in Merton (1973, pp. 168-169), the critical assumptions in the Black–Scholes derivation is that trading takes place continuously in time and that the price dynamics of the stock have a continuous sample path with probability one. It would be pedantic to claim that the Black–Scholes analysis is invalid because continuous trading is not possible and because no empirical time series has a continuous sample path. In Merton and Samuelson (1974, pp. 85–92), it was shown that the continuous-trading solution will be a valid asymptotic approximation to the discrete-trading solution provided that the dynamics have continuous sample paths. Under these same discrete-trading conditions, the returns on the Black–Scholes 'no-risk' arbitrage portfolio will have some risk. However, the magnitude of this risk will be a bounded, continuous function of the trading interval length, and the risk will go to zero as the trading interval goes to its continuous limit. Thus, provided that the interval length is not 'too large', the difference between the Black–Scholes continuous-trading option price and the 'correct', discrete-trading price cannot differ by much without creating a 'virtual' arbitrage possibility.

However, the Black–Scholes solution is not valid, even in the continuous limit, when the stock price dynamics cannot be represented by a stochastic process with a continuous sample path. In essence, the validity of the B–S formula depends on whether or not stock price changes satisfy a kind of 'local' Markov property. I.e., in a short interval of time, the stock price can only change by a small amount.

The antipathetical process to this continuous stock price motion would be a 'jump' stochastic process defined in continuous time. In essence, such a process

[3]See Merton (1973b, pp. 164–165).

allows for a positive probability of a stock price change of extraordinary magnitude, no matter how small the time interval between successive observations. Indeed, since empirical studies of stock price series[4] tend to show far too many outliers for a simple, constant-variance log-normal distribution, there is a 'prima facie' case for the existence of such jumps. On a less scientific basis, we have all observed price changes in stocks (usually in response to some announcement) which, at least on the surface, appear to be 'jumps.' The balance of this paper examines option pricing when the stock price dynamics include the possibility of non-local changes. To highlight the impact of non-continuous stock price dynamics on option pricing, all the other assumptions made by Black and Scholes are maintained throughout the analysis.

2. The stock price and option price dynamics

The total change in the stock price is posited to be the composition of two types of changes: (1) The 'normal' vibrations in price, for example, due to a temporary imbalance between supply and demand, changes in capitalization rates, changes in the economic outlook, or other new information that causes marginal changes in the stock's value. In essence, the impact of such information per unit time on the stock price is to produce a marginal change in the price (almost certainly). This component is modeled by a standard geometric Brownian motion with a constant variance per unit time and it has a continuous sample path.[5] (2) The 'abnormal' vibrations in price are due to the arrival of important new information about the stock that has more than a marginal effect on price. Usually, such information will be specific to the firm or possibly its industry. It is reasonable to expect that there will be 'active' times in the stock when such information arrives and 'quiet' times when it does not although the 'active' and 'quiet' times are random. By its very nature, important information arrives only at discrete points in time. This component is modeled by a 'jump' process reflecting the non-marginal impact of the information.

To be consistent with the general efficient market hypothesis of Fama (1970) and Samuelson (1965b), the dynamics of the unanticipated part of the stock

[4]There have been a variety of alternative explanations for these observations. Among them, non-stationarity in Cootner (1964); finite-variance, subordinated processes in Clark (1973); non-local jump processes in Press (1967); non-stationary variance in Rosenberg (1972); stable Paretian, infinite-variance processes in Mandelbrot (1963) and Fama (1965). The latter stable Paretian hypothesis is not, in my opinion, a reasonable description of security returns because it allows for negative prices as does the corresponding finite-variance, Gaussian hypothesis. Of course, limited liability can be imposed by specifying that the logarithmic returns are stable Paretian, and therefore, the distribution of stock prices would be log-stable Paretian (the analog to log-normal for the Gaussian case). However, under this specification, the expected (arithmetic) return on such securities would be infinite, and it is not clear in this case that the equilibrium interest rate would be finite.

[5]The properties of this process in an economic context are discussed in Cootner (1964), Samuelson (1965a, 1973), Merton (1971, 1973a, 1973b), and Merton and Samuelson (1974). For a more formal analysis, see McKean (1969), Kushner (1967), and Cox and Miller (1968).

price motions should be a martingale. Just as once the dynamics are posited to be a continuous-time process, the natural prototype process for the continuous component of the stock price change is a Wiener process, so the prototype for the jump component is a 'Poisson-driven' process.[6]

The 'Poisson-driven' process is described as follows: The Poisson-distributed 'event' is the arrival of an important piece of information about the stock. It is assumed that the arrivals are independently and identically distributed. Therefore, the probability of an event occurring during a time interval of length h (where h is as small as you like) can be written as

Prob {the event does not occur in the time interval $(t, t+h)$} $= 1 - \lambda h + O(h)$,

Prob {the event occurs once in the time interval $(t, t+h)$} $= \lambda h + O(h)$,

Prob {the event occurs more than once in the time interval $(t, t+h)$} $= O(h)$,

$$\text{(1)}$$

where $O(h)$ is the asymptotic order symbol defined by $\psi(h) = O(h)$ if $\lim_{h \to 0} [\psi(h)/h] = 0$, and $\lambda = $ the mean number of arrivals per unit time.

Given that the Poisson event occurs (i.e., some important information on the stock arrives), then there is a 'drawing' from a distribution to determine the impact of this information on the stock price. I.e., if $S(t)$ is the stock price at time t and Y is the random variable description of this drawing, then, neglecting the continuous part, the stock price at time $t+h$, $S(t+h)$, will be the random variable $S(t+h) = S(t)Y$, given that one such arrival occurs between t and $(t+h)$. It is assumed throughout that Y has a probability measure with compact support and $Y \geq 0$. Moreover, the $\{Y\}$ from successive drawings are independently and identically distributed.

As discussed in Merton (1971), there is a theory of stochastic differential equations to describe the motions of continuous sample path stochastic processes. There is also a similar theory of stochastic differential equations for Poisson-driven processes.[7] The posited stock price returns are a mixture of both types and can be formally written as a stochastic differential equation [conditional on $S(t) = S$], namely, as

$$dS/S = (\alpha - \lambda k)\,dt + \sigma\,dZ + dq, \qquad (2)$$

where α is the instantaneous expected return on the stock; σ^2 is the instantaneous variance of the return, conditional on no arrivals of important new information (i.e., the Poisson event does not occur); dZ is a standard Gauss–Wiener process;

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[6]Both types of processes are infinitely-divisible in time and, appropriately scaled, have independent increments. See Kushner (1967) and Cox and Miller (1968).

[7]See Merton (1971, pp. 395–401) and Kushner (1967, pp. 18–22).

$q(t)$ is the independent Poisson process described in (1); dq and dZ are assumed to be independent; λ is the mean number of arrivals per unit time; $k \equiv \varepsilon(Y-1)$ where $(Y-1)$ is the random variable percentage change in the stock price if the Poisson event occurs; and ε is the expectation operator over the random variable Y.

The 'σdZ' part describes the instantaneous part of the unanticipated return due to the 'normal' price vibrations, and the 'dq' part describes the part due to the 'abnormal' price vibrations. If $\lambda = 0$ (and therefore, $dq \equiv 0$), then the return dynamics would be identical to those posited in the Black and Scholes (1973) and Merton (1973b) papers. (2) can be rewritten in a somewhat more cumbersome form as

$$dS/S = (\alpha - \lambda k)dt + \sigma dZ, \qquad \text{if the Poisson event does}$$
$$\text{not occur,}$$

$$= (\alpha - \lambda k)dt + \sigma dZ + (Y-1), \quad \text{if the Poisson event occurs,} \quad (2')$$

where, with probability one, no more than one Poisson event occurs in an instant, and if the event does occur, then $(Y-1)$ is an impulse function producing a finite jump in S to SY. The resulting sample path for $S(t)$ will be continuous most of the time with finite jumps of differing signs and amplitudes occurring at discrete points in time. If α, λ, k, and σ are constants, then the random variable ratio of the stock price at time t to the stock at time zero [conditional on $S(0) = S$] can be written as

$$S(t)/S = \exp[(\alpha - \sigma^2/2 - \lambda k)t + \sigma Z(t)] Y(n), \qquad (3)$$

where $Z(t)$ is a Gaussian random variable with a zero mean and variance equal to t; $Y(n) = 1$ if $n = 0$; $Y(n) = \prod_{j=1}^{n} Y_j$ for $n \geq 1$ where the Y_j are independently and identically distributed and n is Poisson distributed with parameter λt.

In the special case when the $\{Y_j\}$ are themselves log-normally distributed, then the distribution of $S(t)/S$ will be log-normal with the variance parameter a Poisson-distributed random variable. In this form, the posited dynamics are similar to those used by Press (1967).

Having established the stock price dynamics, I now turn to the dynamics of the option price. Suppose that the option price, W, can be written as a twice-continuously differentiable function of the stock price and time: namely, $W(t) = F(S, t)$. If the stock price follows the dynamics described in (2), then the option return dynamics can be written in a similar form as

$$dW/W = (\alpha_W - \lambda k_W)dt + \sigma_W dZ + dq_W, \qquad (4)$$

where σ_W is the instantaneous expected return on the option; σ_W^2 is the instantaneous variance of the return, conditional on the Poisson event not occurring. $q_W(t)$ is an independent Poisson process with parameter λ. $k_W \equiv \varepsilon(Y_W - 1)$

where $(Y_W - 1)$ is the random variable percentage change in the option price if the Poisson event occurs and ε is the expectation operator over the random variable Y_W.

Using Itô's Lemma for the continuous part and an analogous lemma for the jump part,[8] we have the following important relationships:

$$\alpha_W \equiv [\tfrac{1}{2}\sigma^2 S^2 F_{SS}(S, t) + (\alpha - \lambda k)SF_S(S, t) + F_t + \lambda\varepsilon\{F(SY, t)$$
$$- F(S, t)\}]/F(S, t), \tag{5a}$$

$$\sigma_W \equiv F_S(S, t)\sigma S/F(S, t), \tag{5b}$$

where subscripts on $F(S, t)$ denote partial derivatives.

Further, the Poisson process for the option price, $q_W(t)$, is perfectly functionally dependent on the Poisson process for the stock price, $q(t)$. Namely, the Poisson event for the option price occurs if and only if the Poisson event for the stock price occurs. Moreover, if the Poisson event for the stock occurs and the random variable Y takes on the value $Y = y$, then the Poisson event for the option occurs and the random variable Y_W takes on the value, $F(Sy, t)/F(S, t)$. I.e., $Y_W \equiv F(SY, t)/F(S, t)$. Warning: even though the two processes are perfectly dependent, they are *not* linearly dependent because F is a non-linear function of S.

Consider a portfolio strategy which holds the stock, the option, and the riskless asset with return r per unit time in proportions w_1, w_2, and w_3 where $\sum_{j=1}^{3} w_j = 1$. If P is the value of the portfolio, then the return dynamics on the portfolio can be written as

$$dP/P = (\alpha_p - \lambda k_p)dt + \sigma_p dZ + dq_p, \tag{6}$$

where α_p is the instantaneous expected return on the portfolio; σ_p^2 is the instantaneous variance of the return, conditional on the Poisson event not occurring. $q_p(t)$ is an independent Poisson process with parameter λ. $k_p \equiv \varepsilon(Y_p - 1)$ where $(Y_p - 1)$ is the random variable percentage change in the portfolio's value if the Poisson event occurs and ε is the expectation operator over the random variable Y_p.

From (2) and (4), we have that

$$\alpha_p \equiv w_1(\alpha - r) + w_2(\alpha_w - r) + r, \tag{7a}$$

$$\sigma_p \equiv w_1\sigma + w_2\sigma_W, \tag{7b}$$

$$Y_p - 1 \equiv w_1(Y - 1) + w_2[F(SY, t) - F(S, t)]/F(S, t), \tag{7c}$$

where $w_3 = 1 - w_1 - w_2$ has been substituted out.

[8]See Merton (1971, p. 375) for a statement for Itô's Lemma. Its proof can be found in McKean (1969, pp. 32–35). For a description of the corresponding lemma for Poisson processes, see Kushner (1967, p. 20) and Merton (1971, p. 396).

In the Black–Scholes analysis where $\lambda = 0$ (and therefore, $dq = dq_w = dq_p \equiv 0$), the portfolio return could be made riskless by picking $w_1 = w_1^*$ and $w_2 = w_2^*$ so that $w_1^*\sigma + w_2^*\sigma_W = 0$. This done, it must be that to avoid arbitrage, the expected (and realized) return on the portfolio with weights w_1^* and w_2^*, is equal to the riskless rate, r. From (7a) and (7b), this condition implies that

$$(\alpha - r)/\sigma = (\alpha_W - r)/\sigma_W. \tag{8}$$

From (5a) (with $\lambda = 0$), (5b) and (8), they arrive at their famous partial differential equation for the option price. Namely,

$$\tfrac{1}{2}\sigma^2 S^2 F_{SS} + rSF_S - rF + F_t = 0. \tag{9}$$

Unfortunately, in the presence of the jump process, dq, the return on the portfolio with weights w_1^* and w_2^* will not be riskless. Moreover, inspection of (7c) shows that there does not exist a set of portfolio weights (w_1, w_2) that will eliminate the 'jump' risk (i.e., make $Y_p \equiv 1$). The reason is that portfolio mixing is a *linear* operation and the option price is a non-linear function of the stock price. Therefore, if Y has positive dispersion,[9] then for any w_1 and w_2, $(Y_p - 1)$ will take on non-zero values for some possible values of Y. Since the analysis is already in continuous time, the Black–Scholes 'hedge' will not be riskless even in the continuous limit.

However, one can still work out the return characteristics on the portfolio where the Black–Scholes hedge is followed. Let P^* denote the value of the portfolio. Then from (6), we have that

$$dP^*/P^* = (\alpha_p^* - \lambda k_p^*)dt + dq_p^*. \tag{10}$$

Note: the return on the portfolio is a 'pure' jump process because the continuous parts of the stock and option price movements have been 'hedged' out. (10) can be rewritten in an analogous form to (2') as

$$
\begin{aligned}
dP^*/P^* &= (\alpha_p^* - \lambda k_p^*)dt, && \text{if the Poisson event does not occur,}\\
&= (\alpha_p^* - \lambda k_p^*)dt + (Y_p^* - 1), && \text{if the Poisson event occurs.}
\end{aligned}
\tag{10'}
$$

From (10') it is easy to see that 'most of the time', the return on the portfolio will be predictable and yield $(\alpha_p^* - \lambda k_p^*)$. However, on average, once every $(1/\lambda)$

[9]In the case where σ^2 equals zero and Y is not a random variable (i.e., a pure Poisson process), then a riskless hedge is possible. These twin assumptions are used by Cox and Ross (1975) to deduce by a different route this special case of the formula derived here.

units of time, the portfolio's value will take an unexpected jump. Further, we can work out further qualitative characteristics of the return. Namely, from (7c) and (5b),

$$Y_p^* - 1 = w_2^*[F(SY, t) - F(S, t) - F_S(S, t)(SY - S)]/F(S, t). \qquad (11)$$

By the strict convexity of the option price in the stock price, $[F(SY, t) - F(S, t) - F_S(S, t)(SY - S)]$ is positive for every value of Y. Hence, if w_2^* is positive, then $(Y_p^* - 1)$ will be positive, and the unanticipated return on the hedge portfolio will always be positive. If $w_2^* < 0$, then the unanticipated return will be negative. Moreover, the sign of k_p^* will be the same as the sign of w_2^*.

Thus, if an investor follows a Black–Scholes hedge where he is long the stock and short the option (i.e., $w_2^* < 0$), then most of the time, he will earn more than the expected return, α_p^*, on the hedge because $k_p^* < 0$. However, in those 'rare' occasions when the stock price 'jumps', he will suffer a comparatively large loss. Of course, these large losses occur just frequently enough so as to, on average, offset the almost-steady 'excess' return, $-\lambda k_p^*$. Conversely, if an investor follows a (reverse) Black–Scholes hedge where he is short the stock and long the option (i.e., $w_2^* > 0$), then most of the time, he will earn less than the expected return. But if the stock price 'jumps', then he will make large positive returns.

Thus, in 'quiet' periods when little company-specific information is arriving, writers of options will tend to make what appear to be positive excess returns, and buyers will 'lose'. However, in the relatively infrequent, 'active' periods, the writers will suffer large losses and the buyers will 'win'. Of course, if arrival of an 'active' period is random, then there is no systematic way to exploit these findings. It should be emphasized that the large losses suffered by writers during 'active' periods are not the result of an 'underestimated' variance rate. In general, there is no finite variance rate that could have been used in the formula to 'protect' the writer against the losses from a jump.

3. An option pricing formula

As was demonstrated in the previous section, there is no way to construct a riskless portfolio of stock and options, and hence, the Black–Scholes 'no arbitrage' technique cannot be employed. Of course, along the lines of Samuelson (1965a), if one knew the required expected return on the option (as a function of the stock price and time to expiration), then an option pricing formula could be derived. Let $g(S, \tau)$ be the equilibrium, instantaneous expected rate of return on the option when the current stock price is S and the option expires at time τ in the future. Then, from (5a), we have that F (written as a function of time until expiration instead of time) must satisfy

$$0 = \tfrac{1}{2}\sigma^2 S^2 F_{SS} + (\alpha - \lambda k)SF_S - F_\tau - g(S, \tau)F$$
$$\qquad + \lambda \varepsilon\{F(SY, \tau) - F(S, \tau)\}, \qquad (12)$$

subject to the boundary conditions

$$F(0, \tau) = 0, \tag{12a}$$

$$F(S, 0) = \text{Max } [0, S - E], \tag{12b}$$

where 'E' is the exercise price of the option.

Eq. (12) is a 'mixed' partial differential-difference equation, and although it is linear, such equations are difficult to solve. Moreover, the power and beauty of the original Black–Scholes derivation stems from not having to know either α or $g(S, \tau)$ to compute the option's value, and both are required to solve (12).

A second approach to the pricing problem follows along the lines of the original Black–Scholes derivation which assumed that the Capital Asset Pricing model[10] was a valid description of equilibrium security returns. In section 2, the stock price dynamics were described as the resultant of two components: the continuous part which is a reflection of new information which has a marginal impact on the stock's price and the jump part which is a reflection of important new information that has an instantaneous, non-marginal impact on the stock. If the latter type information is usually firm (or even industry) specific, then it may have little impact on stocks in general (i.e., the 'market'). Examples would be the discovery of an important new oil well or the loss of a court suit.

If the source of the jumps is such information, then the jump component of the stock's return will represent 'non-systematic' risk. I.e., the jump component will be uncorrelated with the market. Suppose that this is generally true for stocks. Return now to the P^* hedge portfolio of the previous section. Inspection of the return dynamics in eq. (10) shows that the only source of uncertainty in the return is the jump component of the stock. But by hypothesis, such components represent only non-systematic risk, and therefore the 'beta' of this portfolio is zero. If the Capital Asset Pricing model holds, then the expected return on all zero-beta securities must equal the riskless rate. Therefore, $\alpha_p^* = r$. But, from (7a), this condition implies that $w_1^*(\alpha - r) + w_2^*(\alpha_W - r) = 0$, or substituting for w_1^* and w_2^*, we have that

$$(\alpha - r)/\sigma = (\alpha_W - r)/\sigma_W. \tag{13}$$

But, (13) together with (5a) and (5b) imply that F must satisfy

$$0 = \tfrac{1}{2}\sigma^2 S^2 F_{SS} + (r - \lambda k)SF_S - F_\tau - rF + \lambda \varepsilon \{F(SY, \tau) - F(S, \tau)\}, \tag{14}$$

subject to the boundary conditions (12a) and (12b). While (14) is formally the same type of equation as (12), note that (14) does not depend on either α or $g(S, \tau)$. Instead, as in the standard Black–Scholes case, only the interest rate, r, appears. Moreover, (14) reduces to the Black–Scholes equation (9) if $\lambda = 0$ i.e.,

[10]See Black and Scholes (1973, pp. 645–646). The Capital Asset Pricing Model is derived in Sharpe (1964), Lintner (1965), and Mossin (1966). An intertemporal version is derived in Merton (1973c). Jensen (1972) provides an excellent survey article on the model.

if there are no jumps). It is important to note that even though the jumps represent 'pure' non-systematic risk, the jump component does affect the equilibrium option price. I.e., one cannot 'act as if' the jump component was not there and compute the correct option price.

While a complete closed-form solution to (14) cannot be written down without a further specification of the distribution for Y, a partial solution which is in a reasonable form for computation can be.

Define $W(S, \tau; E, r, \sigma^2)$ to be the Black–Scholes option pricing formula for the no-jump case. Then W will satisfy eq. (9) subject to the boundary conditions (12a) and (12b). From the Black and Scholes paper (1973, p. 644, eq. 13), W can be written as

$$W(S, \tau; E, r\,\sigma^2) = S\Phi(d_1) - Ee^{-r\tau}\Phi(d_2),\qquad(15)$$

where

$$\Phi(y) \equiv \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{y} e^{-s^2/2}\,\mathrm{d}s,$$

the cumulative normal distribution function,

$$d_1 \equiv [\log(S/E) + (r + \sigma^2/2)\tau]/\sigma\sqrt{\tau},$$

and

$$d_2 \equiv d_1 - \sigma\sqrt{\tau}. \quad\cdot$$

Define the random variable, X_n, to have the same distribution as the product of n independently and identically distributed random variables, each identically distributed to the random variable Y defined in (2), where it is understood that $X_0 \equiv 1$. Define 'ε_n' to be the expectation operator over the distribution of X_n.

The solution to eq. (14) for the option price when the current stock price is S can be written as [11]

$$F(S, \tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} [\varepsilon_n\{W(SX_n e^{-\lambda k\tau}, \tau; E, \sigma^2, r\}].\qquad(16)$$

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[11]A verification that (16) is indeed a solution to (14) is provided in the appendix. The method of obtaining this solution is as follows: In Merton (1973a, p. 38), it was pointed out that the mathematical form of the Black–Scholes equation was formally equivalent to that of Samuelson's (1965a) 'first moment' analysis where the expected return on the stock and the option in his analysis are set equal to the interest rate. I.e., to obtain a solution to Black–Scholes, one can 'pretend' that the required expected return on both the stock and option must equal the riskless rate. While at first a bit counter-intuitive, this result follows because the Black–Scholes solution does not depend on risk preferences. So, in particular, it must be consistent with risk-neutral preferences which require that expected returns on all securities must equal the interest rate. Cox and Ross (1975) provide an explicit demonstration of this point. Warning: while this method is valid for obtaining solutions, it does not imply that the *actual* expected return on the option is equal to the interest rate. Indeed, from (5b) and (13), we have that $\alpha_W = r + F_S S(\alpha - r)/F$, and therefore, $\alpha_W \neq r$ unless $\alpha = r$.

While (16) is not a closed-form solution, it does admit to reasonable computational approximation provided that the density functions for the $\{X_n\}$ are not too complicated.

There are two special cases where (16) can be vastly simplified. The first is the one described by Samuelson (1973, p. 16, fn. 6) where there is a positive probability of immediate ruin. I.e., if the Poisson event occurs, then the stock price goes to zero. In our notation, this case corresponds to $Y \equiv 0$ with probability one. Clearly, $X_n = 0$ for $n \neq 0$, and $k = -1$. So, in this case, eq. (16) can be written as

$$F(S, \tau) = e^{-\lambda \tau} W(Se^{\lambda \tau}; E, \sigma^2, r)$$

$$= W(S, \tau; E, \sigma^2, r+\lambda). \tag{17}$$

Formula (17) is identical to the standard Black–Scholes solution but with a larger 'interest rate', $r' \equiv r+\lambda$, substituted in the formula. As was shown in Merton (1973b), the option price is an increasing function of the interest rate, and therefore an option on a stock that has a positive probability of complete ruin is more valuable than an option on a stock that does not. This result verifies a conjecture of Samuelson.

The second special case of no little interest occurs when the random variable Y has a log-normal distribution. Let δ^2 denote the variance of the logarithm of Y and let $\gamma \equiv \log(1+k)$. In this case, X_n will have a log-normal distribution with the variance of the logarithm of X_n equal to $\delta^2 n$ and $\varepsilon_n(X_n) = \exp[n\gamma]$. Moreover, define $f_n(S, \tau)$ by

$$f_n(S, \tau) \equiv W(S, \tau; E, v_n^2, r_n), \tag{18}$$

where $v_n^2 \equiv [\sigma^2 + n\delta^2/\tau]$ and $r_n \equiv r - \lambda k + n\gamma/\tau$. $f_n(S, \tau)$ is the value of a standard Black–Scholes option where the 'formal' variance per unit time on the stock is v_n^2 and the 'formal', instantaneous rate of interest is r_n.[12] If Y has a log-normal distribution, then (16) can be written as

$$F(S, \tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda' \tau}(\lambda' \tau)^n}{n!} f_n(S, \tau), \tag{19}$$

where $\lambda' \equiv \lambda(1+k)$. Clearly, $f_n(S, \tau)$ is the value of the option, conditional on knowing that exactly n Poisson jumps will occur during the life of the option. The actual value of the option, $F(S, \tau)$, is just the weighted sum of each of these prices where each weight equals the probability that a Poisson random variable

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[12]The term 'formal' is used because if the variance per unit time were really v_n^2, then the variance over the life of the option would be the limit as $m \to 0$ of $(\sigma^2 \tau + n\delta^2[\log(\tau) - \log(m)])$, and not $(\sigma^2 \tau + n\delta^2)$. So, actually v_n^2 is the average variance per unit time, and a similar interpretation holds true for r_n.

with characteristic parameter $\lambda'\tau$, will take on the value n.[13] From (19), it is clear that k does not net out of the option price formula although the total expected return on the stock, α, does.

Formula (16) was deduced from the twin assumptions that securities are priced so as to satisfy the Sharp–Lintner–Mossin Capital Asset Pricing model and that the jump component of a security's return is uncorrelated with the market. While the CAPM has been extensively tested, its validity as a descripter of equilibrium returns is still an open question.[14] To my knowledge, there have been no empirical studies of the correlation between the jump component of stocks' returns and the market return. So one can hardly claim strong empirical evidence to support these assumptions.

An alternative derivation of formula (16) follows along the lines of the Ross (forthcoming) model for security pricing. Namely, suppose that the jump components of stocks' returns are contemporaneously independent.[15] Suppose that there are m stocks outstanding and one forms a stock-option hedge portfolio of the type described in the previous section for each of the m stocks. If P_j^* denotes the value of the hedge portfolio for stock j, then from eq. (10) we can write the return dynamics for this portfolio as

$$\mathrm{d}P_j^*/P_j^* = (\alpha_j^* - \lambda_j k_j^*)\mathrm{d}t + \mathrm{d}q_j^*, \qquad j = 1, 2, \ldots, m. \tag{20}$$

Consider forming a portfolio of these hedge portfolios and the riskless asset where x_j is the fraction of the portfolio invested in the jth hedge portfolio, $j = 1, 2, \ldots, m$, and $(1 - \sum_{j=1}^{m} x_j)$ equals the fraction allocated to the riskless asset. If the value of this portfolio of hedge portfolios is H, then the return dynamics of the portfolio can be written as

$$\mathrm{d}H/H = (\alpha_H - \lambda_H k_H)\mathrm{d}t + \mathrm{d}q_H, \tag{21}$$

where

$$\alpha_H \equiv \sum_{j=1}^{m} x_j(\alpha_j^* - r) + r, \tag{21a}$$

$$\lambda_H k_H \equiv \sum_{j=1}^{m} x_j \lambda_j k_j^*, \tag{21b}$$

$$\mathrm{d}q_H \equiv \sum_{j=1}^{m} x_j \mathrm{d}q_j^*. \tag{21c}$$

[13]In the particular case when the expected change in the stock price is zero, given that the Poisson event occurs (i.e., $k = 0$), then $r_n = r$ and $\lambda' = \lambda$. And, with the exception of $\tau = 0$, $f_n(S, \tau)$ in (18) is equivalent in value to an option on a stock with no jumps, but a non-proportional-in-time variance that approaches a non-zero limit as the option approaches expiration. In this case from (19), each weight is the probability that exactly n jumps occur, and therefore, $F(S, \tau)$ is equal to the expected value of $f_n(S, \tau)$ over the random variable n.

[14]See Black, Jensen and Scholes (1972) for an empirical test of the model and a discussion of the discrepancies. Also, see Jensen (1972) and Merton (1973c) for a theoretical discussion of why such discrepancies may occur.

[15]Actually, the assumption of strict independence can be weakened to allow for some dependence among stocks within groups (e.g., an industry), without affecting the results. See Ross (forthcoming) for a discussion of this point.

Suppose the unconstrained portfolio weights in the hedge portfolios, $\{x_j\}$, are restricted so that they can be written as $x_j \equiv \mu_j/m$ where the μ_j are finite constants, independent of the number of stocks, m. As m becomes large, Ross calls such portfolios 'well-diversified' portfolios. If $ds_j \equiv \mu_j dq_j^*$, then, ds_j has an instantaneous expected value per unit time of $\mu_j \lambda_j k_j^*$ and an instantaneous variance per unit time of $\lambda_j \mu_j^2 \text{Var}(Y_j - 1)$, where $(Y_j - 1)$ is the random variable percentage change in the jth hedge portfolio if a jump occurs in the jth stock price. By the assumption on μ_j, the instantaneous mean and variance per unit time of ds_j are bounded and independent of m.

From (21c), we have the $dq_H = (\sum_{j=1}^{m} ds_j)/m$ where the ds_j are independent because the dq_j^* are independent. Therefore, by the Law of Large Numbers, $dq_H \to \lambda_H k_H dt$ with probability one as $m \to \infty$. I.e., as the number of hedge portfolios contained in a well-diversified portfolio becomes large, the risk of that portfolio tends to zero, and it becomes virtually riskless. Thus, the realized return, dH/H, will be its expected return, $\alpha_H dt$, with probability one, and to rule out 'virtual' arbitrage $\alpha_H = r$. Substituting this condition into (21a), we have that, for large m,

$$\frac{1}{m} \sum_{j=1}^{m} \mu_j(\alpha_j^* - r) = 0. \tag{22}$$

Since the $\{\mu_j\}$ are arbitrary and (22) must hold for almost all choices for the $\{\mu_j\}$, we have that, almost certainly, $\alpha_j^* = r$, for $j = 1, 2, \ldots, m$. But, in the first derivation, it was shown that $\alpha_j^* = r$ implies that $(\alpha - r)/\sigma = (\alpha_w - r)/\sigma_w$ [eq. (13)]. But, eq. (13) was the condition required to obtain formula (16) as a valid equilibrium price for the option.

While the two derivations leading to formula (16) used different assumptions, they had in common the same basic message: Namely, if the jump component of a stock's risk can be diversified away, then the equilibrium option price must satisfy formula (16). While I am not aware of any empirical tests of this proposition, the essential test would be whether the returns on well-diversified portfolios can reasonably be described as stochastic processes with continuous sample paths or do these returns contain identifiable jump components as well.

In the 'no-jump' case, Black–Scholes (1973, p. 645, eq. 14) derive the number of shares of stock to be bought for each option sold, that will create a riskless hedge. Namely,

$$N = \partial W/\partial S$$
$$= \Phi(d_1), \tag{23}$$

where W and d_1 are defined in (15). In the jump case, there is no such riskless mix. However, there is a mix which eliminates all systematic risk, and in that sense, is a hedge. The number of shares required for this hedge, N^*, is equal to $\partial F/\partial S$

which can be obtained by differentiating formula (16). Note: while in both cases, the appropriate number of shares is equal to the derivative of the option pricing function with respect to the stock price, the formulas for the number of shares are different. So, for example, in the special case leading to formula (19), the number of shares is given by

$$N^* = \sum_{n=0}^{m} \frac{e^{-\lambda'\tau}(\lambda'\tau)^n}{n!} \, \Phi[d(n)], \qquad (24)$$

where

$$d(n) \equiv [\log (S/E) + (r_n + \sigma^2/2)\tau + n\delta^2/2] \, \sqrt{(\sigma^2\tau + n\delta^2)}.$$

Of course, when $\lambda = 0$, (24) reduces to (23).

4. A possible answer to an empirical puzzle

Using formula (16) and the strict convexity in the stock price of the Black–Scholes option price formula (15), it is a straightforward exercise to show that ceteris paribus, an option on a stock with a jump component in its return is more valuable than an option on a stock without a jump component (i.e., $[\partial F/\partial \lambda] > 0$ at $\lambda = 0$). However, a much more interesting question can be posed as follows: suppose an investor believes that the stock price dynamics follows a continuous sample-path process with a constant variance per unit time, and therefore he uses the standard Black–Scholes formula (15) to appraise the option when the true process for the stock price is described by eq. (2). How will the investor's appraised value, call it $F_e(S, \tau)$, based on a misspecified process for the stock, compare with the $F(S, \tau)$ value based on the correct process?

To make the analysis tractable, I assume the special case in the previous section where Y is log-normally distributed with the variance of the logarithm of Y equal to δ^2 and the expected value of Y equal to one. Given the investor's incorrect belief about the stock process, it would be natural for him to estimate the variance by using the past time series of the logarithmic returns on the stock.

The distribution of the logarithmic returns on the stock around the mean over any observation period, conditional on exactly n Poisson jumps occurring during the period, is a normal distribution with variance per unit time equal to $(\sigma^2 + n\delta^2/h)$ where h is the length of time between observations. Thus, if one observation period was an (ex post) 'active' period for the stock and a second observation period was an (ex post) 'quiet' period, then the investor might conclude that the variance rate on the 'perceived' process is not stationary. Moreover, there would appear to be a 'regression' effect in the variance, which has been given by Black and Scholes (1972, pp. 405–409) as a possible explanation for certain empirical discrepancies in a test of their model.

However, I will assume that the investor has a sufficiently long time series of data so that his estimate is the true, unconditional variance per unit time of the process. Namely,

$$v^2(h) = \sigma^2 + \lambda\delta^2$$

$$= v^2, \quad \text{the same for all } h. \tag{25}$$

So the issue becomes if the investor uses v^2 as his estimate of the variance rate in the standard Black–Scholes formula, then how will his appraisal of the option's value compare with the 'true' solution in formula (19)? Define the variable, for $n = 0, 1, 2, \ldots,$

$$T_n \equiv \sigma^2\tau + n\delta^2.$$

Let N be a Poisson-distributed random variable with parameter $(\lambda\tau)$ and define T to be a random variable that takes on the value T_n when the random variable N takes on the value n. Let 'ε' denote the expectation operator over the distribution of T. Then, the expected value of T can be written as

$$\bar{T} \equiv \varepsilon(T)$$

$$= (\sigma^2 + \lambda\delta^2)\tau$$

$$= v^2\tau. \tag{26}$$

I have shown elsewhere (1973b, p. 166, eq. 38) that

$$W(S, \tau; E, r, u^2) = Ee^{-r\tau}W(X, \tau'; 1, 0, 1), \tag{27}$$

where $W(\ )$ is defined in (15); $X \equiv Se^{r\tau}/E$; $\tau' \equiv u^2\tau$. I adopt the short-hand notation $W(X, \tau') \equiv W(X, \tau'; 1, 0, 1)$.

Inspection of eq. (18) shows that from (27), f_n can be rewritten as

$$f_n(S, \tau) = Ee^{-r\tau}W(X, T_n), \tag{28}$$

and from (19), that

$$F(S, \tau) = Ee^{-r\tau}\varepsilon\{W(X, T)\}. \tag{29}$$

Moreover, from (26) and (27), the investor's incorrect appraisal can be written as

$$F_\varepsilon(S, \tau) = Ee^{-r\tau}W(X, \bar{T}). \tag{30}$$

From (29) and (30), the answer to the question as to which formula gives the larger option price estimate will depend on whether $\varepsilon\{W(X, T)\} - W(X, \bar{T}) \gtreqless 0$. If $W(X, \tau')$ were either a strictly convex or strictly concave function of τ', then the answer would be unambiguous by Jensen's Inequality. Unfortunately, while $\partial W/\partial \tau' > 0$, the second derivative satisfies

$$(\partial^2 W/\partial \tau'^2)/\partial W/\partial \tau') = [a^2 - (\tau' + (\tau')^2/4)]/2(\tau')^2$$

$$\gtreqless 0, \tag{31}$$

where $a \equiv \ln(X)$. At $a = 0$ which corresponds to $S \equiv Ee^{-r\tau}$, $W(x, \tau')$ is a concave function of τ', and therefore, $F_e(S, \tau) > F(S, \tau)$ at that stock price. I.e., the Black–Scholes estimate will be larger than the true value. For small values of $(r\tau)$ which would be the case for options, one would expect by continuity, that for stock prices sufficiently near the exercise price, this same inequality would hold. Of course, as a *percentage difference*, the difference may be small.

Similarly, for $a^2 \gg 1$, one would expect that $W(X, T)$ would be convex for most of the probable range of T, and in that case $F(S, \tau) > F_e(S, \tau)$. I.e., the Black–Scholes estimate will be smaller than the true value. But, $a^2 \gg 1$ implies either $S \gg E$ or $S \ll E$, which makes this conjecture intuitively correct. Namely, for deep-out-of-the-money options, there is relatively little probability that the stock price will exceed the exercise price prior to expiration if the underlying process is continuous. However, the possibility of a large, finite jump in price significantly increases this probability, and hence, makes the option more valuable. Similarly, for deep-in-the-money options, there is relatively little probability that the stock would decline below the exercise price prior to expiration if the underlying process is continuous, and hence, the 'insurance' value of the option would be virtually nil. However, this need not be the case with jump possibilities. Moreover, these differences will be magnified as one goes to short-maturity options.

Of course, since both $F(S, \tau)$ and $F_e(S, \tau)$ are bounded below by $(S - E)$ and bounded above by S, the percentage difference between $F(S, \tau)$ and $F_e(S, \tau)$ cannot be large for $S \gg E$. However, in the out-of-the-money case, the percentage difference could be substantial.[16]

It is interesting to note that the qualitative discrepancies between the two formulas correspond to what practitioners often claim to observe in market prices for options. Namely, deep-in-the-money, deep-out-of-the-money, and shorter-maturity options tend to sell for more than their Black–Scholes value, and marginally-in-the-money and longer-maturity options sell for less. It would

[16]Computer analysis by J. Ingersoll and me are currently underway to determine the parameter ranges for which the Black–Scholes solution is less than or greater than the solution in this paper.

be presumptuous to claim that the model in this paper 'explains' these discrepancies from such casual empiricisms because other deviations from the original Black–Scholes assumptions might also explain them. For example, the special tax treatment of options for writers or a 'no-jump' process with a stochastic variance rate for the stock's return could cause such an effect. However, the model in this paper does suggest a direction for more, careful empirical research. Indeed, since the same analysis applied here to options can be extended to pricing corporate liabilities in general,[17] the results of such further research would be of interest to all students of Finance.

Appendix

To verify that formula (16) in the text is a solution to (14) and boundary conditions (12a) and (12b), we proceed as follows: From (16), the option price formula can be rewritten as

$$F(S, \tau) = \sum_{n=0}^{\infty} P_n(\tau) \varepsilon_n \{ W(V_n, \tau; E, \sigma^2, r) \}, \qquad (A.1)$$

where we define $P_n(\tau) \equiv \exp[-\lambda\tau](\lambda\tau)^n/n!$ and $V_n \equiv SX_n \exp[-\lambda k\tau]$.

By differentiating (A.1), we have that

$$SF_S(S, \tau) = \sum_{n=0}^{\infty} P_n(\tau) \varepsilon_n \{ V_n W_1 \}, \qquad (A.2)$$

and

$$S^2 F_{SS}(S, \tau) = \sum_{n=0}^{\infty} P_n(\tau) \varepsilon_n \{ V_n^2 W_{11} \}, \qquad (A.3)$$

where subscripts denote partial derivatives. Further, we have that

$$\begin{aligned}
F_\tau(S, \tau) &= -\lambda F - \lambda k \sum_{n=0}^{\infty} P_n(\tau) \varepsilon_n \{ V_n W_1 \} + \sum_{n=0}^{\infty} P_n(\tau) \varepsilon_n \{ W_2 \} \\
&\quad + \lambda \sum_{n=1}^{\infty} \frac{(\lambda\tau)^{n-1} e^{-\lambda\tau}}{(n-1)!} \varepsilon_n \{ W \} \\
&= -\lambda F - \lambda k S F_S + \sum_{n=0}^{\infty} P_n(\tau) \varepsilon_n \{ W_2 \} \\
&\quad + \lambda \sum_{m=0}^{\infty} P_m(\tau) \varepsilon_{m+1} \{ W(V_{m+1}, \tau; E, \sigma^2, r) \}, \qquad (A.4)
\end{aligned}$$

[17]Examples of such extensions can be found in Black and Scholes (1973), Merton (1974) and Ingersoll (1975).

where the second line follows by substituting from (A.2) and changing the summation variable in the last term by $m \equiv n-1$. Finally, we have that

$$\varepsilon_Y\{F(SY, \tau)\} = \varepsilon_Y\left[\sum_{n=0}^{\infty} P_n(\tau)\varepsilon_n\{W(V_n Y, \tau; E, \sigma^2, r)\} \right]$$

$$= \sum_{n=0}^{\infty} P_n(\tau)\varepsilon_{n+1}\{W(V_{n+1}, \tau; E, \sigma^2, r)\}, \qquad (A.5)$$

where the second line follows because by the definition of X_n, X_{n+1} and (YX_n) are identically distributed, and the operator $\varepsilon_Y \cdot \varepsilon_n$ applied to a function of (YX_n) is identical to the operator ε_{n+1} applied to the same function with X_{n+1} substituted for (YX_n).

From (A.1) – (A.5), we have that

$$\tfrac{1}{2}\sigma^2 S^2 F_{SS} + (r - \lambda k)SF_S - F_\tau - rF$$

$$= \sum_{n=0}^{\infty} P_n(\tau)\varepsilon_n\{\tfrac{1}{2}\sigma^2 V_n^2 W_{11} + rV_n W_1 - W_2 - rW\}$$

$$\quad - \lambda k SF_S + \lambda F + \lambda k SF_S - \lambda \sum_{m=0}^{\infty} P_m(\tau)\varepsilon_{m+1}\{W(V_{m+1}, \tau; E, \sigma^2, r)\}$$

$$= -\lambda[\varepsilon_Y\{F(SY, \tau) - F(S, \tau)]], \qquad (A.6)$$

because W satisfies eq. (9) in the text and therefore,

$$\tfrac{1}{2}\sigma^2 V_n^2 W_{11} + rV_n W_1 - W_2 - rW = 0,$$

for each n. It follows immediately from (A.6) that $F(S, \tau)$ satisfies eq. (14).

$S = 0$ implies that $V_n = 0$ for each n. Further, from (15), $W(0, \tau; E, \sigma^2, r) = 0$. Therefore, from (A.1), $F(0, \tau) = 0$ which satisfies boundary condition (12a).

From (15), we have

$$\varepsilon_n\{W(V_n, 0; E, \sigma^2, r)\} = \varepsilon_n\{\text{Max } [0, V_n - E]\}$$

$$\leqq \varepsilon_n\{V_n\} = S(1+k)^n. \qquad (A.7)$$

Therefore, using (A.7),

$$\lim_{\tau \to 0} \sum_{n=1}^{\infty} P_n(\tau)\varepsilon_n\{W\} \leqq \lim_{\tau \to 0} \sum_{n=1}^{\infty} \frac{Se^{-\lambda\tau}[(1+k)\lambda\tau]^n}{n!}$$

$$= \lim_{\tau \to 0} Se^{-\lambda\tau}[e^{(1+k)\lambda\tau} - 1]$$

$$= 0. \qquad (A.8)$$

And, from (A.8), it follows that

$$\lim_{\tau \to 0} F(S, \tau) = \lim_{\tau \to 0} [P_0(\tau)\varepsilon_0\{W(V_0, \tau; E, \sigma^2, r\}$$

$$= \text{Max } [0, S - E]. \tag{A.9}$$

Hence, formula (16) satisfies boundary condition (12b).

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