Finding the median of three permutations under the Kendall-tau distance

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No Institute Given

1 Indroduction

Given m permutations π^1, $\pi^2 \ldots \pi^m$ of $\{1, 2, \ldots, n\}$ and a distance function d, the **median** problem is to find a permutation π^* that is the "closest" of the m given permutations. More formally, we want to find π^* such that, for all $\pi \in \mathcal{S}_n$, $\sum_{i=1}^{m} d(\pi^*, \pi^i) \leq \sum_{i=1}^{m} d(\pi, \pi^i)$. (ICI, BIBLIO DE CE QUI A ETE FAIT)
In this article, we choose to study the problem under the **Kendall-Tau distance**, denoted d_{KT}, that count the number of pairwise disagreements between two permutations. More formally we have, for permutation π and σ that

$$d_{KT}(\pi, \sigma) = |(i, j) : i < j \textit{ and } [(\pi[i] < \pi[j] \textit{ and } \sigma[i] > \sigma[j])$$

$$\textit{or } (\pi[i] > \pi[j] \textit{ and } \sigma[i] < \sigma[j])|,$$

where $\pi[i]$ denote the position of integer i in permutation π.

The median problem under the Kendall-tau distance is trivial when $m = 2$ (just take $\pi^* = \pi^1$ or π^2) and as been proved to be NP-hard for $m \geq 4$. For $m = 3$ it is not known if the problem is NP or not. For this reason, this article deals in better understanding the following problem:

> Given π^1, π^2 and π^3, we want to find π^* such that
>
> $$d_{KT}(\pi^*, \pi^1) + d_{KT}(\pi^*, \pi^2) + d_{KT}(\pi^*, \pi^3) \leq d_{KT}(\pi, \pi^1) + d_{KT}(\pi, \pi^2) + d_{KT}(\pi, \pi^3),$$
>
> for all $\pi \in \mathcal{S}_n$,

that we will call the **median of three problem under the Kendall-Tau distance**.

2 Some definitions

Let us briefly recall the standard definitions needed for our problem.

A **permutation** π is a bijection of $[n] = \{1, 2 \ldots, n\}$ onto itself. The set of all permutations of $[n]$ is denoted $\mathcal{S}_n$. As usual we denote a permutation π of $[n]$ as $\pi = \pi_1 \pi_2 \ldots \pi_n$. The **identity permutation** correspond to the identity bijection of $[n]$ and is denoted $Id = 12\ldots n$. A pair (π_i, π_j) of elements of the permutation π is called an **inversion** if $\pi_i > \pi_j$ and $i < j$. The **number of inversion** of a permutation π is denoted $inv(\pi)$.[1] We can easily computed $inv(\pi)$ as $inv(\pi) = d_{KT}(\pi, ID)$.

Definition 1 *We call the* **disagreements graph** *of the median* $\pi^* = \pi_1^*\pi_2^*\ldots\pi_n^*$ *with respect to* $\pi^1 = \pi_1^1\ldots\pi_n^1$, $\pi^2 = \pi_1^2\ldots\pi_n^2$ *and* $\pi^3 = \pi_1^3\ldots\pi_n^3$*, denoted* $\mathcal{G}(\pi^*)$*, the graph obtained from* π^* *by drawing weighted edges between each pairs* (π_i^*, π_j^*)*, with* $i < j$*. The* **weight** *of an edge* (π_i^*, π_j^*)*, denoted* $w(\pi_i^*, \pi_j^*)$*, represent the number of disagreements of this pair in* π^* *with the same pair of elements in* π^1*,* π^2 *and* π^3*, i.e the distance contribution of this pair in the total Kendall-tau distance.*

Example 1 *Given* $\pi^1 = 2134$*,* $\pi^2 = 4123$ *and* $\pi^3 = 4231$ *we can compute (since here* n *is small) the median* π^* *by choosing, in all permutation of 4 elements, the one that minimize the Kendall-Tau distance. Doing that, we know here that the median is* $\pi^* = 4213$*. The disagreements graph for this* π^* *is the following:*

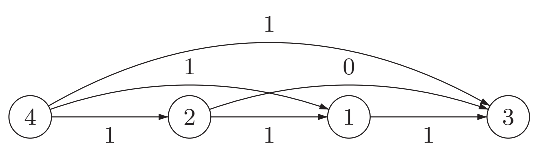


Fig. 1. Disagreements graph of $\pi^* = 4213$ with $\pi^1 = 2134$, $\pi^2 = 4123$ and $\pi^3 = 4231$.

3 Reducing the search space

When dealing with permutations, searching the whole set of permutations quickly becomes a problem since there is $n!$ permutations of $[n]$. To be able to compare our heuristic with the brute force algorithm for permutations of $[n]$ where $n > 12$ we need to reduce the search space so that the computation will take place in a reasonable time. Here, given three permutation π^1, π^2 and π^3, we derived some combinatorial properties of their median π^* which will considerably reduce the search space for π^*.

3.1 Combinatorial properties of the median

Theorem 1 *Let* $\pi^* = \pi_1^*\ldots\pi_n^*$ *be the median of* π^1*,* π^2 *and* π^3*, three permutations of* $[n]$*, with respect to the Kendall-tau distance. Then, for all pairs* (i, j) *such that* $i < j$ *and* $\pi^k[i] < \pi^k[j]$ *for all* $1 \leq k \leq 3$*, (respectively* $\pi^k[i] > \pi^k[j]$ *for all* $1 \leq k \leq 3$*), we have* $\pi^*[i] < \pi^*[j]$ *(respectively* $\pi^*[i] > \pi^*[j]$*).*

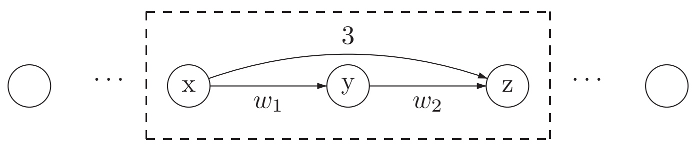
[1] Since the inversions are generators of $\mathcal{S}_n$, we can view $\mathcal{S}_n$ with these generators as a Coxeter group. In this context, the number of inversions of a permutation π is called the **length** of π and is denoted by $\ell(\pi)$. See Chapter 5 of [1] for more details.

Proof. What we want to show is that there is no edge of weight three in $\mathcal{G}(\pi^*)$ (see Definition 1). We will do the proof by induction on m, the number of elements between pairs π_i^* and π_j^*, $i < j$ in π^*.

- if $m = 0$

  In this case, π_i^* and π_j^* are consecutive elements in π^*, i.e that $j = i + 1$. It is easy to see that the weight of the edges between consecutive elements of π^* in $\mathcal{G}(\pi^*)$ need to be 0 or 1, since, if not, permuting these two consecutive elements in π^* will only affect the weight of the edge between this pair giving a permutation π^{**} that is closer to π^1, π^2 and π^3, with respect to the Kendall-tau distance. This contradicts our choice of π^* as the median of π^1, π^2 and π^3. This prove that in π^*, we have $w(\pi_i^*, \pi_{i+1}^*) = 0$ or 1, for all $1 \le i \le n - 1$.
- if $m = 1$

  Let x, y and z be three consecutive elements in π^*. We want to show that in $\mathcal{G}(\pi^*)$, we can not have the following subgraph:



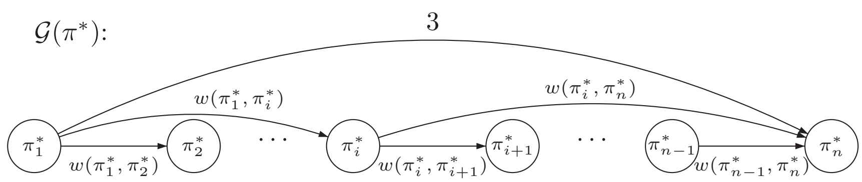
$\mathcal{G}_{xyz}(\pi^*)$

  where w_1 and w_2 are either 0 or 1 since x, y and y, z are consecutive elements of π^*. We can prove this by contradiction:
  Suppose that we do have $\mathcal{G}_{xyz}(\pi^*)$ as a subgraph of $\mathcal{G}(\pi^*)$ and suppose that $w_1 = w(x, y) = 0$ (resp. $w_2 = 0$). Since $w(x, z) = 3$ we have that z appears before x in all three permutations π^1, π^2 and π^3. Now, since $w(x, y) = 0$ (resp. $w(y, z) = 0$), we have that x appears before y (resp. y appears before z) in all three permutations π^1, π^2 and π^3. That implies that z appears before y (resp. y appears before x) in all three permutations π^1, π^2 and π^3. This means that $w(y, z) = 3$ (resp. $w(x, y) = 3$), which contradicts the fact that in π^* edges between consecutive elements should not be greater than 1.
  Now suppose that $w_1 = w_2 = 1$. Again, since $w(x, z) = 3$ we have that z appears before x in all three permutations π^1, π^2 and π^3. The fact that $w_1 = 1$ implies that in two out of the three permutations we have that x appears before y and the fact that $w_2 = 1$ implies that in two out of the three permutations we have that y appears before z. That means that in at least one permutation we need to have z before x before y before z, which is impossible.

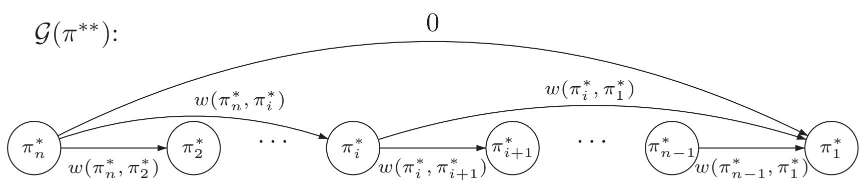
Induction hypothesis: Suppose that there is no edge of weight three between pairs π_i^* and π_j^*, $i < j$, in π^*, where π_i^* and π_j^* are separated by m elements, $0 \le m \le n - 3$.

- We need to show that the hypothesis still hold for $m = n - 2$:

What this means is that the edge between π_1^* and π_n^* in $\mathcal{G}(\pi^*)$ can not be weighted by 3. Again, we will prove this by contraction by supposing that $w(\pi_1^*, \pi_n^*) = 3$. If so, $\mathcal{G}(\pi^*)$ look like this (where not all the edges are draw to facilitate the comprehension):



Now, considerer the permutation π^{**} obtained from π^* by interchanging the positions of π_1^* and π_n^*. We have for π^{**} the following disagreements graph:



The Kendall-tau distance of π^* (resp. π^{**}) from π^1, π^2 and π^3 is the sum of the weights of all the edges in $\mathcal{G}(\pi^*)$ (resp. $\mathcal{G}(\pi^{**})$). Only the weights of the edges that touch either π_1^* or π_n^* are different between the two graphs. To derive a contradiction, we need to show that the sum of the weights of these edges is smaller in $\mathcal{G}(\pi^{**})$ then it is in $\mathcal{G}(\pi^*)$. Let us prove this fact.
By induction hypothesis, we have that in $\mathcal{G}(\pi^*)$, $w(\pi_1^*, \pi_i^*)$, for all $3 \leq i \leq n-2$, is either 1 or 2. Let us take a look at these two cases:

- $w_{\mathcal{G}(\pi^*)}(\pi_1^*, \pi_i^*) = 1$: If this is true then in two out of the three permutation π^1, π^2 and π^3 we need to have that π_1^* appears before π_i^*. Also, since $w_{\mathcal{G}(\pi^*)}(\pi_1^*, \pi_n^*) = 3$, we have in all three permutations π^1, π^2 and π^3 that π_n^* appears before π_1^*. These two facts implies that $w_{\mathcal{G}(\pi^*)}(\pi_i^*, \pi_n^*) = 2$ (in fact it could equals 2 or 3, but 3 is impossible by induction hypothesis). So, in this case, for all $3 \leq i \leq n-2$, we have that $w_{\mathcal{G}(\pi^*)}(\pi_1^*, \pi_i^*) + w_{\mathcal{G}(\pi^*)}(\pi_i^*, \pi_n^*) = 1 + 2 = w_{\mathcal{G}(\pi^{**})}(\pi_n^*, \pi_i^*) + w_{\mathcal{G}(\pi^{**})}(\pi_i^*, \pi_1^*)$.

- $w_{\mathcal{G}(\pi^*)}(\pi_1^*, \pi_i^*) = 2$: If this is true then in one out of the three permutation π^1, π^2 and π^3 we need to have that π_1^* appears before π_i^*. Also, since $w_{\mathcal{G}(\pi^*)}(\pi_1^*, \pi_n^*) = 3$, we have in all three permutations π^1, π^2 and π^3 that π_n^* appears before π_1^*. These two facts implies that $w_{\mathcal{G}(\pi^*)}(\pi_i^*, \pi_n^*) = 1$ or 2 (in fact it could equals 1, 2 or 3, but 3 is impossible by induction hypothesis). So, in this case, for all $3 \leq i \leq n-2$, we have that $w_{\mathcal{G}(\pi^*)}(\pi_1^*, \pi_i^*) + w_{\mathcal{G}(\pi^*)}(\pi_i^*, \pi_n^*) = 2 + (1 \text{ or } 2) = 3 \text{ or } 4$ and $w_{\mathcal{G}(\pi^{**})}(\pi_n^*, \pi_i^*) + w_{\mathcal{G}(\pi^{**})}(\pi_i^*, \pi_1^*) = (2 \text{ or } 1) + 1 = 2 \text{ or } 3$. So, here $w_{\mathcal{G}(\pi^*)}(\pi_1^*, \pi_i^*) + w_{\mathcal{G}(\pi^*)}(\pi_i^*, \pi_n^*) \leq w_{\mathcal{G}(\pi^{**})}(\pi_n^*, \pi_i^*) + w_{\mathcal{G}(\pi^{**})}(\pi_i^*, \pi_1^*)$.

We have already consider all the edges that touch either π_1^* or π_n^* in $\mathcal{G}(\pi^*)$ except the following edges: $w(\pi_1^*, \pi_2^*)$, $w(\pi_1^*, \pi_{n-1}^*)$, $w(\pi_1^*, \pi_n^*)$, $w(\pi_2^*, \pi_n^*)$ and $w(\pi_{n-1}^*, \pi_n^*)$. First, since $w(\pi_1^*, \pi_2^*)$ and $w(\pi_{n-1}^*, \pi_n^*)$ are edges between consecutive elements in the median π^*, their weight can only be 0 or 1. Now, since we have $w(\pi_1^*, \pi_n^*) = 3$, $w(\pi_1^*, \pi_2^*) = 0$ (resp. $w(\pi_{n-1}^*, \pi_n^*) = 0$) implies that $w(\pi_2^*, \pi_n^*) = 3$ (resp. $w(\pi_1^*, \pi_{n-1}^*) = 3$) which is impossible by induction hypothesis. So, we need to have in $\mathcal{G}(\pi^*)$, $w(\pi_1^*, \pi_2^*) = w(\pi_{n-1}^*, \pi_n^*) = 1$. Finally, $w(\pi_1^*, \pi_n^*) = 3$ and $w(\pi_1^*, \pi_2^*) = 1$ (resp. $w(\pi_{n-1}^*, \pi_n^*) = 1$) implies that $w(\pi_2^*, \pi_n^*) = 2$ (resp. $(\pi_1^*, \pi_{n-1}^*) = 2$). So, in $\mathcal{G}(\pi^*)$ we have $w(\pi_1^*, \pi_2^*) + w(\pi_1^*, \pi_{n-1}^*) + w(\pi_1^*, \pi_n^*) + w(\pi_2^*, \pi_n^*) + w(\pi_{n-1}^*, \pi_n^*) = 1+2+3+2+1 = 9$, where in $\mathcal{G}(\pi^{**})$ we have $w(\pi_2^*, \pi_1^*) + w(\pi_{n-1}^*, \pi_1^*) + w(\pi_n^*, \pi_1^*) + w(\pi_n^*, \pi_2^*) + w(\pi_n^*, \pi_{n-1}^*) = 2+1+0+1+2 = 6$. This conclude the proof that the Kendall-tau distance of $\mathcal{G}(\pi^{**})$ from π^1, π^2 and π^3 is less than the Kendall-tau distance of $\mathcal{G}(\pi^*)$ from π^1, π^2 and π^3, which contradicts our choice of median. ■

Theorem 2 *Let $\pi^* = \pi_1^* \ldots \pi_n^*$ be the median of π^1, π^2 and π^3, three permutations of $[n]$, with respect to the Kendall-tau distance. Without loss of generality, suppose that π^1 is the permutation that is the closest of the two others i.e that $d_{KT}(\pi^1, \pi^2) + d_{KT}(\pi^1, \pi^3) \leq d_{KT}(\pi^2, \pi^1) + d_{KT}(\pi^2, \pi^3)$ and $d_{KT}(\pi^1, \pi^2) + d_{KT}(\pi^1, \pi^3) \leq d_{KT}(\pi^3, \pi^1) + d_{KT}(\pi^3, \pi^2)$. We have that*

$$inv(\pi^*) \leq \frac{inv(\pi^1) + inv(\pi^2) + inv(\pi^3) + d_{KT}(\pi^1, \pi^2) + d_{KT}(\pi^1, \pi^3)}{3}$$

and

$$inv(\pi^*) \geq \frac{inv(\pi^1) + inv(\pi^2) + inv(\pi^3) - d_{KT}(\pi^1, \pi^2) - d_{KT}(\pi^1, \pi^3)}{3}.$$

Proof. It is easy to show that the Kendall-tau distance is indeed a distance in the mathematical sense of the term. That means, in particular, that it needs to satisfy the triangle inequality ($\forall x, y, z, d_{KT}(x, z) \leq d_{KT}(x, y) + d_{KT}(y, z)$). By the triangle inequality we have that

$$inv(\pi^*) = d_{KT}(\pi^*, Id) \leq d_{KT}(\pi^*, \pi^1) + d_{KT}(\pi^1, Id) = d_{KT}(\pi^*, \pi^1) + inv(\pi^1),$$

$$inv(\pi^*) = d_{KT}(\pi^*, Id) \leq d_{KT}(\pi^*, \pi^2) + d_{KT}(\pi^2, Id) = d_{KT}(\pi^*, \pi^2) + inv(\pi^2),$$

$$inv(\pi^*) = d_{KT}(\pi^*, Id) \leq d_{KT}(\pi^*, \pi^3) + d_{KT}(\pi^3, Id) = d_{KT}(\pi^*, \pi^3) + inv(\pi^3),$$

Summing the three inequalities we get

$$3\, inv(\pi^*) \leq d_{KT}(\pi^*, \pi^1) + d_{KT}(\pi^*, \pi^2) + d_{KT}(\pi^*, \pi^3) + inv(\pi^1) + inv(\pi^2) + inv(\pi^3).$$

Now, since π^* is the median of π^1, π^2 and π^3 with respect to the Kendall-tau distance we need to have that

$$d_{KT}(\pi^*,\pi^1)+d_{KT}(\pi^*,\pi^2)+d_{KT}(\pi^*,\pi^3) \leq d_{KT}(\pi^1,\pi^1)+d_{KT}(\pi^1,\pi^2)+d_{KT}(\pi^1,\pi^3).$$

So we have

$$3\,inv(\pi^*) \leq d_{KT}(\pi^1,\pi^2) + d_{KT}(\pi^1,\pi^3) + inv(\pi^1) + inv(\pi^2) + inv(\pi^3),$$

which gives us the first inequality. The second inequality is easily derived in the same manner from the following triangle inequalities:

$$inv(\pi^1) = d_{KT}(\pi^1, Id) \leq d_{KT}(\pi^1,\pi^*) + d_{KT}(\pi^*, Id) = d_{KT}(\pi^1,\pi^*) + inv(\pi^*),$$

$$inv(\pi^2) = d_{KT}(\pi^2, Id) \leq d_{KT}(\pi^2,\pi^*) + d_{KT}(\pi^*, Id) = d_{KT}(\pi^2,\pi^*) + inv(\pi^*),$$

$$inv(\pi^3) = d_{KT}(\pi^3, Id) \leq d_{KT}(\pi^3,\pi^*) + d_{KT}(\pi^*, Id) = d_{KT}(\pi^3,\pi^*) + inv(\pi^*).$$

■

4 Our heuristic

The idea of our algorithm is to apply a series of "good" cyclic movements on the starting permutations to make them closer to the median. We want to apply these "good" movements to all three starting permutations so to ensure that we reach the median. Formally we have the following definitions and algorithm.

Definition 2 *Given $\pi = \pi_1 \dots \pi_n$, we call a* **cyclic movement** *of a segment $\pi[i..j]$ of π, denoted $c[i,j](\pi)$, the cycling shifting of one position to the right ($c_r[i,j]$) or to the left ($c_\ell[i,j]$) of the segment inside the permutation π:*

$$c_r[i,j](\pi) = \pi_1 \dots \pi_{i-1}\pi_{\mathbf{i+1}} \dots \pi_{\mathbf{j}}\pi_{\mathbf{i}}\pi_{j+1} \dots \pi_n,$$

$$c_\ell[i,j](\pi) = \pi_1 \dots \pi_{i-1}\pi_{\mathbf{j}}\pi_{\mathbf{i}} \dots \pi_{\mathbf{j-1}}\pi_{j+1} \dots \pi_n$$

When $j = i+1$, a cyclic movement correspond to a **transposition**.

Definition 3 *We will say that a cyclic movement $c[i,j]$ is* **good** *if, when we apply it to π we obtain a permutation that is closest to the median than π i.e if*

$$\sum_{i=1}^{3} d_{KT}(c[i,j](\pi), \pi_i) < \sum_{i=1}^{3} d_{KT}(\pi, \pi_i).$$

Our algorithm:

```
Algorithm FindMedian (π, [π_1, π_2, π_3])
    n ← length(π)
    bool ← 0 (will be change to 1 if there is no more possible "good" movement)
    chang ← 0 (will tells us if some movements where made)
    WHILE bool <> 1 DO
        FOR i from 1 to n − 1 DO
            FOR j from i + 1 to n DO
                IF c_r[i, j](π) or c_ℓ[i, j](π) is a good movement THEN
                    π ← c_good[i, j](π)
                    chang ← chang +1
                END IF
            END FOR
        END FOR
        IF chang = 0 THEN
            bool ← 1
        END IF
    END WHILE
    RETOURNER π
```

If we execute algorithm FindMedian on π_1, π_2 and π_3, we will possibly get three different permutations π_1^*, π_2^* and π_3^* that can have the same d_{KT} distance from π_1, π_2 and π_3 or not. We choose π^* to be one of the permutation π_1^*, π_2^* and π_3^* that minimize the Kendall-Tau distance and **claim that it is the median**. Not, in the next section, I want to try to show why we need to apply the algorithm on each three starting permutation to ensure that we get the median.

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References

1. J.E.Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, (1990), 204 pages.