The beta-binomial value for k is:

$$\int_0^1 L_{n,k}(p)\pi_{\alpha,\beta}(p)dp$$

Let $\Pi_{\alpha,\beta}$ be the integral of $\pi_{\alpha,\beta}$. (with respect to R, we have $\pi =$ `dbeta()`, $\Pi =$ `pbeta()`, $\Pi^{-1} =$ `qbeta()`)

By change of variables:

$$\begin{aligned}
\int_0^1 L_{n,k}(\Pi^{-1}_{\alpha,\beta}(u))du &= \int_{\Pi_{\alpha,\beta}(0)}^{\Pi_{\alpha,\beta}(1)} L_{n,k}(\Pi^{-1}_{\alpha,\beta}(u))du \\
&= \int_0^1 L_{n,k}(\Pi^{-1}_{\alpha,\beta}(\Pi_{\alpha,\beta}(p)))\pi_{\alpha,\beta}(p)dp \\
&= \int_0^1 L_{n,k}(p)\pi_{\alpha,\beta}(p)dp
\end{aligned}$$

I would like to integrate:

$$\int_0^1 L_{n_1,k_1}(\Pi^{-1}_{\alpha_1,\beta_1}(u))L_{n_2,k_2}(\Pi^{-1}_{\alpha_2,\beta_2}(u))du$$

This is:

$$\binom{n_1}{k_1}\binom{n_2}{k_2}\int_0^1 \Pi_1^{-1}(u)^{k_1}(1-\Pi_1^{-1}(u))^{n_1-k_1}\Pi_2^{-1}(u)^{k_2}(1-\Pi_2^{-1}(u))^{n_2-k_2}du$$

Where:

$$\Pi_i^{-1}(u) = \Pi^{-1}_{\alpha_i,\beta_i}(u)$$

We can probably solve this with adaptive quadrature in R, i.e. `integrate()`.