Contents

1 Test

An A-module L is *injective* if for every exact sequence

$$0 \to X' \to X \to X'' \to 0,$$

the corresponding sequence

$$0 \leftarrow \mathrm{Hom}\,(X', L) \leftarrow \mathrm{Hom}\,(X, L) \leftarrow \mathrm{Hom}\,(X'', L) \leftarrow 0$$

is exact as well. Alternatively L is exact in the case that the following condition holds:
(INJ) : *Let $j : X' \to X$ be an injective morphism. For any morphism $f' : X' \to L$, there exists a morphism $f : X \to L$ such that $f' = f \circ j$.*

This means that any morphism into L from a submodule of X extends to a morphism from X into L.

Theorem 1.1 (1.4.1)**.** *For a left A-module to be injective, it is necessary and sufficient that, for any left ideal $I \triangleleft A$ and any morphism $f : I \to L$, there exists an $x \in L$ such that $f(\lambda) = \lambda \cdot x$ for any $\lambda \in I$.*

This condition is necessary since f should extend to a morphism $A \to L$.

Conversely, suppose the above condition holds, and consider X a module, X' a submodule of X, and f a morphism $X' \to L$. Consider the pairs (Y, g) where Y is a submodule of X containing X' and g a morphism $Y \to L$ which extends f. We see by Zorn's Lemma that f admits a maximal extension (Y, g). If Y were different from X, we could find an $x \in X$ outside of Y, a left ideal $I \triangleleft A$ (the set of $\lambda \in A$ such that $\lambda x \in Y$), and a morphism $h : I \to L$, given by $\lambda \mapsto g(\lambda \cdot x)$. By the hypothesis, there exists an element u of L such that the relation $\lambda x \in Y$ implies $g(\lambda x) = \lambda u$; but then g could extend to a submodule of X generated by Y and x: a contradiction.

For example, if the base ring A is a principal ideal domain, then L is injective if and only if, for every $\lambda \neq 0$, the endomorphism $x \mapsto \lambda x$ of L is surjective.

Theorem (1.2.2)**.** *Every A-module is a submodule of an injective A-module.*

To see this, we consider the ring Z of integers, the additive group $\mathbb{Q}$ of rational numbers, and the $\mathbb{Z}$-module

$$\mathbb{T} \stackrel{\text{def}}{=} \mathbb{Q}/\mathbb{Z}.$$

This module is injective by the preceding theorem, and it is clear that any cyclic abelian group (of finite or infinite order) maps nontrivially[1] into $\mathbb{T}$.

Let L be a left A-module, and consider

$$\widehat{L} \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathbb{Z}}\,(L, \mathbb{T});$$

this is in an evident manner a right A-module; by the same logic,

$$\widehat{\widehat{L}} \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathbb{Z}}\left(\widehat{L}, \mathbb{T}\right)$$

[1]Godement uses the word "se plonge" which suggests that any cyclic abelian group *embeds* into $\mathbb{T}$, which is clearly false (how would the integers embed?). However, Godement's proof only needs the existence of a nontrivial morphism.

is a left A-module, and there is a canonical morphism

$$L \to \widehat{\widehat{L}}$$

defined in an obvious way.[2] We now show that this morphism is injective. To that end consider a nontrivial $x \in L$, we can construct a morphism $f : L \to \mathbb{T}$ such that $f(x) \neq 0$: let G be the cyclic group generated by x inside L; G maps nontrivially into $\mathbb{T}$, let f be a nontrivial map defined on G—since $\mathbb{T}$ is an injective $\mathbb{Z}$-module, f extends to L, as desired.

Now we show that *if L is projective, then $\widehat{L}$ is injective*; to that end let X be a right A-module and f a morphism from a submodule X' of X into the A-module $\widehat{L}$. By duality we have a morphism $\widehat{f} : \widehat{\widehat{L}} \to \widehat{X'}$ and in particular, since L embeds into $\widehat{\widehat{L}}$, we have a morphism $L \to \widehat{X}$, which in turn implies by duality a morphism $\widehat{\widehat{X}} \to \widehat{L}$ which extends f, hence *a fortiori* extends the original f to X.

To complete the proof, we represent $\widehat{L}$ as a quotient of a projective module F; hence $\widehat{\widehat{L}}$, and *a fortiori* L, embeds into $\widehat{F}$, which is injective by what we just observed—which completes the proof.

We leave the reader the job of proving, with the help of the preceding result, that *for an L module to be injective, it is necessary and sufficient that it is a direct factor in every module which contains L.*

[2] This is the *evaluation* map $x \mapsto (\phi \mapsto \phi(x))$.