

## 1.4 Injective Modules

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ca. 1964

from Topologie algébrique et Théorie de faisceaux

Abstract

Here is an abstract.

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#### 1 Test

An  $A$ -module  $L$  is injective if for every exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0,$$

the corresponding sequence

$$0 \longleftarrow \text{Hom}(X', L) \longleftarrow \text{Hom}(X, L) \longleftarrow \text{Hom}(X'', L) \longleftarrow 0$$

is exact as well. Alternatively  $L$  is exact in the case that the following condition holds:

(INJ) : Let  $j: X' \rightarrow X$  be an injective morphism. For any morphism  $f': X' \rightarrow L$ , there exists a morphism  $f: X \rightarrow L$  such that  $f' = f \circ j$ .

This means that any morphism into  $L$  from a submodule of  $X$  extends to a morphism from  $X$  into  $L$ .

Theorem 1.1 (1.4.1). For a left  $A$ -module to be injective, it is necessary and sufficient that, for any left ideal  $I \triangleleft A$  and any morphism  $f: I \rightarrow L$ , there exists an  $x \in L$  such that  $f(\lambda) = \lambda \cdot x$  for any  $\lambda \in I$ .

This condition is necessary since  $f$  should extend to a morphism  $A \rightarrow L$ .

Conversely, suppose the above condition holds, and consider  $X$  a module,  $X'$  a submodule of  $X$ , and  $f$  a morphism  $X' \rightarrow L$ . Consider the pairs  $(Y, g)$  where  $Y$  is a submodule of  $X$  containing  $X'$  and  $g$  a morphism  $Y \rightarrow L$  which extends  $f$ . We see by Zorn's Lemma that  $f$  admits a maximal extension  $(Y, g)$ . If  $Y$  were different from  $X$ , we could find an  $x \in X$  outside of  $Y$ , a left ideal  $I \triangleleft A$  (the set of  $\lambda \in A$  such that  $\lambda x \in Y$ ), and a morphism  $h: I \rightarrow L$ , given by  $\lambda \mapsto g(\lambda \cdot x)$ . By the hypothesis, there exists an element  $u$  of  $L$  such that the relation  $\lambda x \in Y$  implies  $g(\lambda x) = \lambda u$ ; but then  $g$  could extend to a submodule of  $X$  generated by  $Y$  and  $x$ : a contradiction.

For example, if the base ring  $A$  is a principal ideal domain, then  $L$  is injective if and only if, for every  $\lambda \neq 0$ , the endomorphism  $x \mapsto \lambda x$  of  $L$  is surjective.

Theorem (1.2.2). Every  $A$ -module is a submodule of an injective  $A$ -module.

To see this, we consider the ring  $\mathbb{Z}$  of integers, the additive group  $\mathbb{Q}$  of rational numbers, and the  $\mathbb{Z}$ -module

$$\mathbb{T} \stackrel{\text{def}}{=} \mathbb{Q}/\mathbb{Z}.$$

## 1.4 Injective Modules

This module is injective by the preceding theorem, and it is clear that any cyclic abelian group (of finite or infinite order) maps nontrivially<sup>1</sup> into  $\mathbb{T}$ . Let  $L$  be a left  $A$ -module, and consider

$$\widehat{L} \stackrel{\text{def}}{=} \text{Hom}_{\mathbb{Z}}(L, \mathbb{T});$$

this is in an evident manner a right  $A$ -module; by the same logic,

$$\widehat{\widehat{L}} \stackrel{\text{def}}{=} \text{Hom}_{\mathbb{Z}}(\widehat{L}, \mathbb{T})$$

is a left  $A$ -module, and there is a canonical morphism

$$L \rightarrow \widehat{\widehat{L}}$$

defined in an obvious way.<sup>2</sup> We now show that this morphism is injective. To that end consider a nontrivial  $x \in L$ , we can construct a morphism  $f: L \rightarrow \mathbb{T}$  such that  $f(x) \neq 0$ : let  $G$  be the cyclic group generated by  $x$  inside  $L$ ;  $G$  maps nontrivially into  $\mathbb{T}$ , let  $f$  be a nontrivial map defined on  $G$ --since  $\mathbb{T}$  is an injective  $\mathbb{Z}$ -module,  $f$  extends to  $L$ , as desired.

Now we show that if  $L$  is projective, then  $\widehat{L}$  is injective; to that end let  $X$  be a right  $A$ -module and  $f$  a morphism from a submodule  $X'$  of  $X$  into the  $A$ -module  $\widehat{L}$ . By duality we have a morphism  $\widehat{f}: \widehat{\widehat{L}} \rightarrow \widehat{X'}$  and in particular, since  $L$  embeds into  $\widehat{\widehat{L}}$ , we have a morphism  $L \rightarrow \widehat{X}$ , which in turn implies by duality a morphism  $\widehat{X} \rightarrow \widehat{L}$  which extends  $f$ , hence a fortiori extends the original  $f$  to  $X$ .

To complete the proof, we represent  $\widehat{L}$  as a quotient of a projective module  $F$ ; hence  $\widehat{\widehat{L}}$ , and a fortiori  $L$ , embeds into  $\widehat{F}$ , which is injective by what we just observed--which completes the proof.

We leave the reader the job of proving, with the help of the preceding result, that for an  $L$  module to be injective, it is necessary and sufficient that it is a direct factor in every module which contains  $L$ .

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<sup>1</sup>Godement uses the word "se plonge" which suggests that any cyclic abelian group embeds into  $\mathbb{T}$ , which is clearly false (how would the integers embed?). However, Godement's proof only needs the existence of a nontrivial morphism.

<sup>2</sup>This is the evaluation map  $x \mapsto (\phi \mapsto \phi(x))$ .