Presheaves and Indexed Set Theory *NeuPPL Category Seminar*

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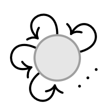
The first half of these notes developed a categorical perspective on the theory of preorders. One key step in this theory is the observation that a preorder $(X, \leqslant)$ may be treated as a family of indexed propositions $\mathcal{C}\ell(X, \leqslant)^{\mathsf{op}} \to \Omega$. This allowed us to lift operations on truth values Ω to universal constructions in a preorder. In a category C, we can no longer treat objects like indexed *propositions*, but there is a way to generalize this philosophy. Instead, we need to study indexed sets, i.e., *presheaves*. Similar to the preorder situation, operations between sets can be lifted to universal constructions in a category.

1 A brief account of categories

A category is a collection of objects and arrows between them which can be composed. There are myriad situations where such structure arises. Perhaps two archetypes are interfaces and theories. Qua interface, a category is a collection of states, called *objects*, and families of transitions between states, called *arrows*. Functors $F : C \to \mathsf{Set}$ can be seen as concrete implementations of this interface. This philosophy is explained well in [Fong and Spivak, 2019]. This approach also forms the semantics for algebraic effects. For example, algebras of the `Writer m` monad, for some monoid `m` are described by the category consisting of a single object and an arrow for every element of the monoid `m`. Functors



from this category to, e.g., Set identifies a set X (the image of the single object) and a family of functions $X \to X$ for every element of `m`. This is a `Writer m` algebra.

Qua theory, a category C is the classifying category of *some theory*. Functors out of this category are the same as models of this theory. Given a propositional theory T, we saw that functors from $\mathcal{C}\ell(T)$ to a preorder X were the same as models of T in X. This perspective is not fundamentally different to the interface perspective.

We will be using categories to describe the logical relations of a programming language with state, e.g., name allocation or step-indexing. We will identify a category C whose objects will represent states of a heap. Arrows of C will identify how a heap state may be extended with fresh names. We will write our logical relations as sets indexed by heap states, i.e., functors out of C into Set.

For the sake of completeness, here is a definition of a (small) category.

Definition 1.1. A **small category** C is a pair of sets $\left(C^{(0)}, C^{(1)}\right)$, called its set of **objects** and its set of **arrows**, respectively, and a family of functions.

- A pair $s, t : C^{(1)} \to C^{(0)}$ of **source** and **target** functions. For any arrow $f \in C^{(1)}$, $s(f)$ is its source object, and $t(f)$ is its target object. We usually write this as $f : s(f) \to t(f)$.
- A function $1 : C^{(0)} \to C^{(1)}$ sending every object $X \in C^{(0)}$ to its **identity arrow** $1_X : X \to X$.
- A partial function $C^{(1)} \times C^{(1)} \to C^{(1)}$ called **composition**. $m(g, f)$ is defined when $s(g) = t(f)$.

These functions are required to satisfy two axioms. First, m must be an associative operation:

$$m(h, m(g, f)) = m(m(h, g), f),$$

wherever defined. Second, 1 must be a left and right identity for m:

$$m(f, 1_{s(f)}) = m(1_{t(f)}, f) = f,$$

for any $f \in C^{(1)}$.

Usually composition $m(g, f)$ in a category is written as $g \circ f$ or by juxtaposition gf.

Definition 1.2. Given small categories C and D, a **functor** $F : C \to D$ is a family of functions $F^{(0)} : C^{(0)} \to D^{(0)}$ and $F^{(1)} : C^{(1)} \to D^{(1)}$ which preserve the functions $s, t, 1$, and m.

Given a category C, swapping the functions s and t yields a new category, C^{op}, called its **opposite category**. This is a generalization of the relation between $\leqslant$ and $\geqslant$ for preordered sets.

There is one more perspective on categories which ties this back to the study on preorders. Recall, for a preordered set $(X, \leqslant)$, the classifying category $\mathcal{C}\ell(X, \leqslant)$ is *thin*—between any two objects x, y in $\mathcal{C}\ell(X, \leqslant)$, there is at most one arrow $x \to y$. A category C is an intensional form of a preorder. For a category, it is not enough to know that there exists an arrow between objects X and Y, the specific choice of arrow $f : X \to Y$ is now relevant. Similarly, Set is the intesional analogue of truth values: it is not enough to know that a set X is nonempty; rather, the specific choice of element $x \in X$ matters. This perspective tells us that universal constructions ofr categories are intensional/proof-relevant variants of the universal constructions for preorders. We saw some of these generalizations in the notes on Galois connections.

2 Indexed sets

We begin by describing a presheaf as an indexed set. By indexed set, we mean indexed in the following manner. Given a family of sets $\{X_\alpha \mid \alpha \in I\}$, we can describe this family with a function

$$\pi : X \to I,$$

where $X \stackrel{\text{def}}{=} \coprod_{\alpha \in I} X_\alpha$. Given a function $x \in X$, x is contained in some X_α for a unique α. $\pi(x)$ is defined to be this index α. In the case where I is replaced with a category C, we follow the same idea, but we introduce some extra structure to make the set X compatible with the arrows of C.

Definition 2.1. Let C be a small category. A C**-indexed set** is a set X equipped with a function $\pi : X \to C^{(0)}$ and a partial function $\mu : X \times C^{(1)} \to X$ satisfying the following constraints.

- $\mu(x, f)$ is defined whenever $t(f) = \pi(x)$.
- $\pi(\mu(x, f)) = s(f)$.
- $\mu(x, 1_{\pi(x)}) = x$.
- If $g \circ f$ is defined, then $\mu(x, g \circ f) = \mu(\mu(x, g), f)$, wherever defined.

Anticipating an equivalence between C-indexed sets and presheaves on C, we could write $x\big|_f$ in lieu of $\mu(x, f)$.

Calling this a C-indexed set is admiteddly uncommon. Another name for a C-indexed set is a *discrete opfibration*. This is more precise, and it alludes to a more general notion, called a *fibration of categories*. On the other hand, there is a hint of the notion of `Writer m` algebra in a C-indexed set. A category with one object, i.e., a category C where $C^{(0)} \cong \{*\}$, is the same structure as a monoid on the set $C^{(1)}$. In this sense categories are generalized monoids, where the multiplication is a partial operation. Given that categories are generalized monoids, C-indexed sets are generalized modules.

Proposition 2.2. *Let C be a category with one object. Then a C-indexed set (X, π, μ) is the same data as a right-action of the monoid $C^{(1)}$ on the set X.*

Proof. Consider a C-indexed set (X, π, μ). Since $C^{(0)} \cong \{*\}$, the projection $\pi : X \to C^{(0)}$ can only be the map sending any element of X to the single element of $C^{(0)}$. In this case, $\mu : X \times C^{(1)} \to X$ is total. We define the right action on X by $C^{(1)}$ by the rule

$$x \cdot f \stackrel{\text{def}}{=} \mu(x, f).$$

Conversely, given a right action of $C^{(1)}$ on X, we give X the structure of a C-indexed set in the following manner. The projection π is unique, since $C^{(0)}$ is a singleton set. This leaves the map $\mu : X \times C^{(1)} \to X$, which we define using the same rule as before.

$$\mu(x, f) \stackrel{\text{def}}{=} x \cdot f.$$

$\square$

Due to the above proposition, presheaves are occaisionally called *right C-modules*. Section V.7 of [Mac Lane and Moerdijk, 1994] describes the theory of right C-modules. Furthermore, to feed mathematicians' preference for infix notation, we will now write $x \cdot f$ instead of $\mu(x, f)$ for any C-indexed set.

Given a C-indexed set $(X, \pi, \cdot)$, we can lift the categorical structure on C to X. In order to give this category a name, we will anticipate presheaves and abuse notation slightly to call this category $\mathrm{el}(X)$, the *category of elements* of X.

Definition 2.3. Given the C-indexed set $(X, \pi, \cdot)$, its **category of elements** is the following category $\mathrm{el}(X)$. The objects of $\mathrm{el}(X)$ are the elements of X. Furthermore, there is an arrow $\widetilde{f} : x \to y$ in $\mathrm{el}(X)$ for every arrow $f \in C^{(1)}$ such that $x = y \cdot f$. Composition of arrows in $\mathrm{el}(X)$ is the same as composition in C.

It can be helpful to have a picture for how the arrows in $\mathrm{el}(X)$ are related to arrows in C.

$$\begin{tikzcd}
x \arrow[r, dashed, "\widetilde{f}"] \arrow[d, rightsquigarrow] & y \arrow[d, rightsquigarrow] \\
\pi(x) \arrow[r, "f"] & \pi(y)
\end{tikzcd}$$

Here a squiggly arrow $x \rightsquigarrow A$ signifies the equation $\pi(x) = A$. The arrow notation is suggestive: the projection $\pi : X \to C$ defines a functor $\widetilde{\pi} : \mathrm{el}(X) \to C$. We prove this now.

Proposition 2.4. *Let $\widetilde{\pi} : \mathrm{el}(X) \to C$ be the map*

$$x \mapsto \pi(x), \quad \widetilde{f} \mapsto f.$$

Then $\widetilde{\pi}$ is a functor.

Proof. We need to show that $\widetilde{\pi}$ preserves identities and composition. Given an object x in $\mathrm{el}(X)$ (i.e., an element $x \in X$), its identity arrow 1_x is presented by the identity $\widetilde{1_{\pi(x)}}$, so

$$\pi(1_x) = \pi(\widetilde{1_{\pi(x)}}) = 1_{\pi(x)},$$

so $\widetilde{\pi}$ preserves identities. A similar argument works for composition. □

The functor $\widetilde{\pi}$ is what gives a C-indexed set the structure of a discrete opfibration. A fibration of categories is a categorification of $\widetilde{\pi}$, adding more arrows in $\mathrm{el}(X)$ besides the ones in C.

We now use C-indexed sets to study the category C. While the objects of C may not be faithfully represented by sets, it is true that they are faithfully represented by C-indexed sets. This property is a generalization of the embedding of a preorder into its downward closed sets.

Definition 2.5. Let $X \in C^{(0)}$ be an object of C. Its **indexed downset** $\downarrow X$ is the C-indexed set defined in the following manner.

- The elements of $\downarrow X$ are arrows $f : Y \to X$ in C, for any object Y in C.
- The projection $\pi : \downarrow X \to C$ sends an arrow $f : Y \to X$ to the object Y.
- The multiplication $\_ \cdot \_ : X \times C^{(1)} \to X$ sends an arrow $f : Y \to X$ and an arrow $g : Z \to Y$ to the composition $f \circ g$:
$$f \cdot g \overset{\text{def}}{=} f \circ g.$$

Remark 2.6. *The category of elements for the indexed downset $\downarrow X$ has a familiar name. It is the **slice category** $C_{/X}$. This category is sometimes called $C \downarrow X$, an auspicious hint of the slice category's connection to the downsets of a preordered set.*

In order to make sense of how C is embedded into its C-indexed sets, we need to make a notion of C-indexed function.

Definition 2.7. Let X and Y be a pair of C-indexed sets. A C**-indexed function** $\phi : X \to Y$ is a function ϕ_* between the underlying sets of X and Y which preserves π and $\_ \cdot \_$, i.e.,

$$\pi_Y(\phi_* x) = \pi_X(x), \quad (\phi_* x) \cdot_Y f = \phi_*(x \cdot_X f).$$

C-indexed functions may be composed, by composing the underlying functions between sets. Furthermore, there is an obvious identity function. This gives a category of C-indexed sets, which we call C-Set.

Given an arrow $f : X \to Y$ in C, we obtain an indexed function $f_* : \downarrow X \to \downarrow Y$, given by postcomposition:

$$f_*(g : Z \to X) \overset{\text{def}}{=} f \circ g : Z \to Y.$$

Lemma 2.8 (C-indexed Yoneda lemma)**.** *The mapping $X \mapsto {\downarrow} X$ and $f \mapsto f_*$ defines a functor*

$$\downarrow : C \to C\text{-Set}.$$

Furthermore, this functor is fully faithful. Finally, given an arbitrary C-indexed set A, there is a bijection

$$C\text{-Set}({\downarrow} X, A) \cong \{a \in A \mid \pi(a) = X\}.$$

Proof. We use the only trick in category theory: chase `refl`. We first argue that $\downarrow$ is a functor. Since f_* is defined by composition in C, this is straightforward. We now show that $\downarrow$ is faithful. Let $f, g : X \to Y$ be two arrows in C. Suppose $f_* = g_*$. We need to prove $f = g$. The identity arrow $1_X : X \to X$ is an element of ${\downarrow} X$. Then

$$f = f_*(1_X) = g_*(1_X) = g,$$

so $\downarrow$ is faithful. Next we show $\downarrow$ is full. Let $\phi : {\downarrow} X \to {\downarrow} Y$ be an indexed function. Then $\phi_*(1_X)$ is some arrow $f : X' \to Y$. Since ϕ preserves the projection π, $X' = X$. What is left is to argue that $\phi_* = f_*$. Indeed, for any element $g : Z \to X$ in ${\downarrow} X$,

$$f_* g = f \circ g = f \cdot g = \phi_*(1_X) \cdot g = \phi_*(1_X \cdot g) = \phi_*(g).$$

This shows $\phi_* = f_*$. Since ϕ is arbitrary, this shows that $\downarrow$ is full; hence $\downarrow$ is fully faithful.

Finally, we prove the bijection. Let $\phi_0 : {\downarrow} X \to A$ be an indexed function. ${\phi_0}_*(1_X)$ is an element of A; call it $a(\phi_0)$. We will show that $a : \phi_0 \mapsto a(\phi_0)$ is a bijection. We need an inverse to a. Given an element $a_0 \in A$ such that $\pi(a_0) = X$, we define an indexed function $\phi(a_0) : {\downarrow} X \to A$ in the following manner. Given an element $f : Y \to X$ in ${\downarrow} X$, define

$$\phi(a_0)_*(f) \stackrel{\text{def}}{=} a_0 \cdot f \in A.$$

Since $\pi(a_0) = X$, this function is well-defined, and it is straightforward to show that this function preserves π and $\_ \cdot \_$. Given an indexed function $\phi_0 : {\downarrow} X \to A$ and an element $f : Y \to X$ in ${\downarrow} X$,

$$\phi(a(\phi_0))_*(f) = a(\phi_0) \cdot f = {\phi_0}_*(1_X) \cdot f = {\phi_0}_*(1_X \cdot f) = {\phi_0}_*(f).$$

Thus $\phi(a(\phi_0)) = \phi_0$. Similarly given an element $a_0 \in A$ such that $\pi(a_0) = X$,

$$a(\phi(a_0)) = \phi(a_0)_*(1_X) = a_0 \cdot 1_x = a_0.$$

Therefore $a(\phi(a_0)) = a_0$. This shows that $a(\_)$ and $\phi(\_)$ are inverses, proving the desired bijection. $\square$

We have shown that $\downarrow$ is an embedding. When we did this for preorders, we used this embedding to lift operations on truth-values to universal constructions on a preorder. We do the analogous thing for indexed sets, lifting operations on sets to universal constructions on a category.

Indexed set theory

Test

3 Presheaves: Indexed sets as functors

References

[Fong and Spivak, 2019] Fong, B. and Spivak, D. I. (2019). *An Invitation to Applied Category Theory: Seven Sketches in Compositionality*. Cambridge University Press, Cambridge. https://www.cambridge.org/core/books/an-invitation-to-applied-category-theory/D4C5E5C2B019B2F9B8CE9A4E9E84D6BC.

[Mac Lane and Moerdijk, 1994] Mac Lane, S. and Moerdijk, I. (1994). *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. Universitext. Springer, New York, NY. http://link.springer.com/10.1007/978-1-4612-0927-0.