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Preorders as Categories

Let X be a set. A **preorder** on X is a reflexive and transitive relation $\_ \leqslant \_ \subseteq X \times X$. For example, any partial order is a preorder (but not all preorders are partial orders!). We will show that preordered sets form a special class of categories.

Exercise 1 : Let $(X, \leqslant)$ be a preordered set. Define a category $\mathcal{C}\ell(X, \leqslant)$ in the following manner.

- The objects of $\mathcal{C}\ell(X, \leqslant)$ are the elements of X.
- There is precisely one arrow $x \to y$ in $\mathcal{C}\ell(X, \leqslant)$ in the case that $x \leqslant y$.

Verify that $\mathcal{C}\ell(X, \leqslant)$ is a category. Can any of the hypotheses on the relation $\leqslant$ be relaxed while still ensuring that $\mathcal{C}\ell(X, \leqslant)$ is a category?

The category $\mathcal{C}\ell(X, \leqslant)$ is called the **classifying category** of the preorder $(X, \leqslant)$.

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Exercise 2 :

- Unpack what it means for two objects x, y of $\mathcal{C}\ell(X, \leqslant)$ to be isomorphic, in terms of the preordered set $(X, \leqslant)$.
- Unpack what is a functor $\mathcal{C}\ell(X, \leqslant) \to \mathcal{C}\ell(Y, \leqslant)$, in terms of a function between preordered sets $(X, \leqslant) \to (Y, \leqslant)$.
- The opposite category $\mathcal{C}\ell(X, \leqslant)^{\mathsf{op}}$ is the classifying category of another preordered set $(X', \leqslant)$. What is this preordered set?

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Exercise 3 : Let $\Omega \stackrel{\text{def}}{=} \{\mathsf{False}, \mathsf{True}\}$ with the usual preordering $\mathsf{False} \leqslant \mathsf{True}$. To keep things clean, let's also write Ω for $\mathcal{C}\ell(\Omega, \leqslant)$. Given an object x of $\mathcal{C}\ell(X, \leqslant)$, we can write a functor $\downarrow x : \mathcal{C}\ell(X, \leqslant)^{\mathsf{op}} \to \Omega)$:

$$\downarrow x(y) \stackrel{\text{def}}{=} \begin{cases} \mathsf{True} & \text{if } y \leqslant x \\ \mathsf{False} & \text{otherwise.} \end{cases}$$

Write $\downarrow x \stackrel{\text{nat}}{\leqslant} \downarrow y$ in the case that, for any object z of $\mathcal{C}\ell(X, \leqslant)$, $\downarrow x(z) \leqslant \downarrow y(z)$. Show for any pair of elements $x, y \in X$,

$$x \leqslant y \text{ if and only if } \downarrow x \stackrel{\text{nat}}{\leqslant} \downarrow y.$$

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Monads for Preorders (Extra)

Given two categories C, D and a functor $F : C \to D$, a **right adjoint** to F is a functor $G : D \to C$ satisfying the following universal property.

$$\frac{f : FX \to Y}{\overline{f^\dagger : X \to GY.}}$$

We read this as: *for any construction of an arrow from FX to Y in D, there exists a unique dual construction of an arrow form X to GY in C.* In the case where C and D are the classifying categories of preordered sets, $C = \mathcal{C}\ell(X, \leqslant)$ and $D = \mathcal{C}\ell(Y, \leqslant)$, then there is only one way to make an arrow from FX to Y and from X to GY. Then the above universal property reduces to

$$\frac{FX \leqslant Y}{\overline{X \leqslant GY}}$$

This pair $F \dashv G$ is called a **Galois connection**.

Exercise 4 : Given a Galois connection $F \dashv G$, where $F : \mathcal{C}\ell(X, \leqslant) \to \mathcal{C}\ell(Y, \leqslant)$ and $G : \mathcal{C}\ell(Y, \leqslant) \to \mathcal{C}\ell(X, \leqslant)$, we can apply G after F to obtain a "loop":

$$GF : \mathcal{C}\ell(X, \leqslant) \to \mathcal{C}\ell(X, \leqslant).$$

Show that GF is a **closure operator**, i.e., for any object x of $\mathcal{C}\ell(X, \leqslant)$, $x \leqslant GFx$ and $GF(GFx) \leqslant GFx$.

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Exercise 5 : We can reverse the previous exercise. Apply F after G to obtain another "loop":

$$FG : \mathcal{C}\ell(Y, \leqslant) \to \mathcal{C}\ell(Y, \leqslant).$$

Show that FG is an **interior operator**, i.e., for any object y of $\mathcal{C}\ell(Y, \leqslant)$, $FGy \leqslant y$ and $FGy \leqslant FG(FGy)$.

Here is an example of a closure operator which will appear in different forms later. Let $(X, \leqslant)$ be a preorder, as before. We saw in Exercise 3 that we can "embed" the elements of X into maps $\mathcal{C}\ell(X, \leqslant)^{\mathsf{op}} \to \Omega$. In fact, we can make the set of all such maps into a preorder using $\overset{\text{nat}}{\leqslant}$. Given a pair of maps $F, F' : \mathcal{C}\ell(X, \leqslant)^{\mathsf{op}} \to \Omega$, write $F \overset{\text{nat}}{\leqslant} F'$ in the case that, for any element $x \in X$, we have

$$Fx \leqslant Fx'.$$

A **Lawvere-Tierney topology** on $(X, \leqslant)$ is a closure operator J on the set of maps $\mathcal{C}\ell(X, \leqslant)^{\mathsf{op}} \to \Omega$ satisfying:

- J is **idempotent**: $J(JF) \overset{\text{nat}}{\leqslant} J(F)$ (and $J(F) \overset{\text{nat}}{\leqslant} J(J(F))$.
- J preserves infima.