1 Introduction: A Proposition for a Metatheory

Categorical logic and categorical semantics are a generalization of a certain triangle of structures in propositional logic. First, there is the *propositional theory* itself. This is the propositional version of an *equational theory*. It is syntax, and like syntax in a programming language, we can study transformations between different languages: *translations*. The second structure is an algebraic model, called the *classifying category* of the theory. The algebraic model plays the role of an *equational theory* for the syntax. If we temporarily fast-forward to the simply-typed λ-calculus, we find an example of structure appearing in the algebraic model.

> *Well-typed λ-terms do not form a Cartesian-closed category. However, if we identify terms up to $\beta\eta$-equivalence, the resultant equivalence classes do form a Cartesian-closed category.*

This is the general idea of the algebraic model: find structure in the syntax by identifying terms which are proof-theoretically the same. The third structure is the *semantics* of the logic. The semantics is the "ground-truth" of the logic. We can think of this as observational equivalence. Semantic truth, i.e., "$\models$" is difficult to reason with, so we refer to the first two models to explore as much of the semantics as possible while remaining tractable. For this reason, soundness and completeness theorems are valuable.

> *Soundness:* the first two models are not more powerful than the semantics.
> *Completeness:* the first two models are at least as powerful as the semantics.

Let us begin by describing this triangle for propositional logic. This will be a high-level overview of the ideas in Chapter 3 of [Halvorson, 2019].

Syntax

We will work in the classical fragment of propositional logic. Given a set of atomic propositions $P, Q, \ldots$, we can form derived propositions $P \vee Q, \neg P, P \to Q, \ldots$ using the connectives

$$\wedge, \vee, \Rightarrow, \neg, \top, \bot.$$

The meaning of these connectives come from the usual introduction and elimination rules. These allow us to build Gentzen-style proof trees, shown below.

$$\frac{A \in \Gamma}{\Gamma \vdash A} \qquad \frac{\Delta \vdash A \quad \Delta \subseteq \Gamma}{\Gamma \vdash A} \qquad \frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A \quad \Gamma \vdash B}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \qquad \frac{\Gamma \vdash A \Rightarrow B}{\Gamma, A \vdash B} \qquad \left(\frac{\Gamma \vdash A \Rightarrow B \wedge B \Rightarrow A}{\Gamma \vdash A \Leftrightarrow B} \right)$$

$$\frac{\Gamma \vdash A \to \bot}{\Gamma \vdash \neg A} \qquad \frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \top} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash A}$$

Definition 1.1. A **propositional theory** is a pair $T = (\Sigma, \Delta)$ consisting of a set Σ of atomic propositions, called the **signature** of the theory, and a set Δ of sequents of derived propositions of Σ, called the **axioms** of the theory.

Propositional logic is the study of propositional theories. Using the axioms of a theory T, we can form a new relation $\vdash_T$, where $\Gamma \vdash_T A$ signifies the existence of a proof-tree whose leaves are axioms of T such that every internal node is an application of a classical introduction or elimination rule. One could pause here and study this relation in depth, but we will keep going. Propositional theories have their own notion of transpiler.

Definition 1.2. Let T_1 and T_2 be two propositional theories. A **translation** $F : T_1 \to T_2$ is a map sending atomic propositions in Σ_1 to derived propositions of Σ_2. By declaring that this map preserves the logical connectives, this extends to a map from derived propositions of Σ_1 to derived propositions of Σ_2. We require that F satisfy the additional constraint:

$$\Gamma \vdash_{T_1} A \implies F\Gamma \vdash_{T_2} FA.$$

We are often interested in asking whether two theories T_1 and T_2 have the same level of expressivity and proving power. Given that we have a notion of arrow between theories, we might try to study *isomorphisms*. However, the constraint that $GF = \mathsf{id} : T_1 \to T_1$ for a pair of translations $F : T_1 \to T_2$ and $G : T_2 \to T_1$ is often too restrictive. For one, atomic propositions must be sent to themselves, but the image of a derived proposition under a translation is never atomic. Therefore, F must send any atomic proposition to an atomic proposition. If we have an isomorphism, the same must be true for G. This shows that any isomorphism of propositional theories must be just a relabeling of the symbols. Not only that, theories also have axioms, and isomorphisms do not care about the axioms. We are led to a weakening of isomorphism which is still good enough to model the logic inside a propositional theory.

Definition 1.3. Two translations $F, F' : T_1 \to T_2$ are **provably equivalent** or **homotopic** in the case that, for any T_1-proposition ϕ,

$$\cdot \vdash_{T_2} F\phi \Leftrightarrow F'\phi.$$

An **equivalence** of theories is a pair of translations $F : T_1 \to T_2$ and $G : T_2 \to T_1$ such that GF is provably equivalent to id_{T_1} and FG is provably equivalent to id_{T_2}, i.e., for any T_1 proposition ϕ and T_2 proposition ψ,

$$\cdot \vdash_{T_1} \phi \Leftrightarrow GF\phi \text{ and } \cdot \vdash_{T_2} \psi \Leftrightarrow FG\psi.$$

If a translation $F : T_1 \to T_2$ is part of an equivalence, then F resembles a *fully abstract* compiler. Our metatheory of translations should explain when equivalences exist, and it should provide methods to construct them.

Semantics

I mentioned that equivalences are good enough to model the logic inside a propositional theory. Let's take a moment to formalize what is meant by "modeling the logic" inside a theory. For classical propositional logic, the only thing that matters is truth tables, essentially. Let $\Omega \stackrel{\text{def}}{=} \{\mathsf{True}, \mathsf{False}\}$ be the set of classical truth values.

Definition 1.4. Let Σ be a propositional signature. An **interpretation** (or, if you want, a **denotational semantics**) for the signature Σ is a map $[\![\cdot]\!]$ from atomic propositions in Σ to truth values Ω

We can extend the map from a denotational semantics $[\![\cdot]\!] : \Sigma \to \Omega$ to derived propositions in Σ. We do this by sending the logical connectives to the appropriate boolean operations

$$[\![A \wedge B]\!] \stackrel{\text{def}}{=} [\![A]\!] \,\mathsf{and}\, [\![B]\!] \qquad [\![A \vee B]\!] \stackrel{\text{def}}{=} [\![A]\!] \,\mathtt{or}\, [\![B]\!] \qquad [\![A \Rightarrow B]\!] \stackrel{\text{def}}{=} \mathtt{if}\, [\![A]\!] \,\mathtt{then}\, [\![B]\!] \,\mathtt{else}\, \mathsf{True}$$
$$[\![\neg A]\!] \stackrel{\text{def}}{=} \mathtt{not}\, [\![A]\!] \qquad [\![\top]\!] \stackrel{\text{def}}{=} \mathsf{True} \qquad [\![\bot]\!] \stackrel{\text{def}}{=} \mathsf{False}\,.$$

Definition 1.5. Let T be a propositional theory with signature Σ and axioms Δ. A **model** of the theory T is an interpretation $[\![\cdot]\!]$ of Σ such that, for any sequent $\phi \vdash \psi$ in Δ, $[\![\phi]\!] \leqslant [\![\psi]\!]$, where the ordering $\leqslant$ is the usual ordering on Ω, interpreted as a boolean algebra. If we give the model a name, say, $M : \Sigma \to \Omega$, then we define the notation $\phi \vDash_M \psi$ to mean $[\![\phi]\!]^M \leqslant [\![\psi]\!]^M$.

We can treat the semantics as the ground-truth of the propositional theory. We are concerned with which propositions are true, and the classical introduction and elimination rules help us explore the space of true propositions by identifying operations which preserve the ground truth. This is a soundness theorem.

Theorem 1.6 (Soundness theorem)**.** *Let $[\![\cdot]\!]^M$ be a model of a propositional theory T. Suppose $\phi \vdash_T \psi$ for some propositions ϕ and ψ. Then $\phi \vDash_M \psi$.*

Similarly, our notion of translation can also be justified by the semantics. Consider a translation $F : T_1 \to T_2$, and suppose $[\![\cdot]\!]^M$ is a model of T_2. We obtain a model of $[\![\cdot]\!]^{F^*M}$ of T_1, given by composition, essentially

$$\Sigma_1 \xrightarrow{F} \mathsf{Prop}(\Sigma_2) \xrightarrow{[\![\cdot]\!]^M} \Omega.$$

Proposition 1.7. *$[\![\cdot]\!]^{F^*M}$ is indeed a model of T_1.*

Proof. We need to show that $F^*[\![\cdot]\!]$ preserves the axioms of T_1. Let $\phi \vdash \psi$ be such an axiom. Since F is a translation, $F\phi \vdash_{T_2} F\psi$. By the soundness theorem, $F\phi \vDash_M F\psi$. Finaly, observe that

$$[\![\phi]\!]^{F^*M} \stackrel{\text{def}}{=} [\![F\phi]\!]^M.$$

Therefore $F\phi \vDash_M F\psi$ implies $\phi \vDash_{F^*M} \psi$, as desired. □

Recall that our notion of translation was too strict to have interesting inverses. Here is another reason why we need to consider "homotopic" translations.

Proposition 1.8. *Let $F, G : T_1 \to T_2$ be a pair of translations. If F and G are homotopic, then, for any model $[\![\cdot]\!]^M$ of T_2, we have, for any derived proposition ϕ of T_1,*

$$[\![\phi]\!]^{F^*M} = [\![\phi]\!]^{G^*M} : \Omega.$$

Proof. Since F and G are homotopic, the sequent $\cdot \vdash_{T_2} F\phi \leftrightarrow G\phi$ is provable in T_2. Since M is a model, this means

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 if True
then ((if ⟦Fφ⟧^M then ⟦Gφ⟧^M else True)
       and
      (if ⟦Gφ⟧^M then ⟦Fφ⟧^M else True))
else
   True
```

evaluates to True. Furthermore, the laws of the boolean operations imply that the above expression is True if and only if $[\![F\phi]\!]^M \leqslant [\![G\phi]\!]^M$ and vice versa. Antisymmetry now implies $[\![\phi]\!]^{F^*M} = [\![\phi]\!]^{G^*M}$, as desired. □

In fact, we could have defined two translations to be homotopic in the case that they are indistinguishable in any model. This was the original approach, when homotopy of translations was defined in [Ahlbrandt and Ziegler, 1986]. Proving that this is equivalent to the syntax-focused approach mentioned above requires a completeness theorem for propositional theories.

Remark 1.9. *To make sense of an interpretation and model, we only need a notion of truth value and boolean operations on truth values. Therefore, we could replace Ω with any boolean algebra B and define a notion of B-**valued model** $\Sigma \to B$. All the results in this passage transfer to B-valued models.*

Algebra

We now present the third side of the triangle. In the case of propositional logic, the equational theory is trivial. However the change in perspective from introducing category theory introduces the key ideas in categorical logic, and it shows that a propositional theory is fully subsumed by its classifying category.

Definition 1.10. Let T be a propositional theory. Notice that the set of derived propositions $\mathrm{Prop}(T)$ has a preorder, given by $\vdash_T$:

$$\phi \leqslant \psi \text{ if and only if } \phi \vdash_T \psi.$$

In the warm-up we saw that we can define the classifying category of a preorder. The **classifying category** of the theory T is the category

$$\mathcal{C}\ell(T) \stackrel{\text{def}}{=} \mathcal{C}\ell(\mathrm{Prop}(T), \vdash_T).$$

Explicitly, the objects of $\mathcal{C}\ell(T)$ are derived propositions in $\mathrm{Prop}(T)$, and there exists a unique arrow $\phi \to \psi$ in $\mathcal{C}\ell(T)$ in the case that $\phi \vdash_T \psi$.

I am choosing to call this category the classifying category of T, following [Jacobs, 1999]. There are many other names for this category; another common name is the *syntactic category* of the theory T. The rest of this section will describe how a propositional theory can be described in terms of classifying categories.

For example, a common thing to ask when given a category is *what are its isomorphisms?* Since there is at most one arrow between two objects in $\mathcal{C}\ell(T)$, the only arrow $\phi \to \phi$ must be the identity arrow, for any object ϕ in $\mathcal{C}\ell(T)$. This means two objects ϕ and ψ are isomorphic in $\mathcal{C}\ell(T)$ if and only if there exist arrows $\phi \to \psi$ and $\psi \to \phi$. Unpacking the meaning of an arrow in $\mathcal{C}\ell(T)$, we obtain the following proposition.

Proposition 1.11. *Let T be a propositional theory, and let ϕ and ψ be two objects of $\mathcal{C}\ell(T)$, i.e., derived propositions of T. ϕ and ψ are isomorphic in $\mathcal{C}\ell(T)$ if and only if $\phi \vdash_T \psi$ and $\psi \vdash_T \phi$, equivalently,*

$$\cdot \vdash_T \phi \Leftrightarrow \psi.$$

Translations

We can describe translations as functors.

Proposition 1.12. *Let $F : T_1 \to T_2$ be a translation between propositional theories. Then F defines a mapping $\mathrm{Prop}(T_1) \to \mathrm{Prop}(T_2)$ which is monotonic. Therefore F defines a functor*

$$\mathcal{C}\ell(F) : \mathcal{C}\ell(T_1) \to \mathcal{C}\ell(T_2).$$

Furthermore, two translations $F_1, F_2 : T_1 \to T_2$ are provably equivalent (i.e., homotopic) if and only if $\mathcal{C}\ell(F_1)$ and $\mathcal{C}\ell(F_2)$ are naturally isomorphic.

Proof. The statement that F is a monotonic map means, for any pair of derived propositions $\phi, \psi \in \mathrm{Prop}(T_1)$,

$$\phi \vdash_{T_1} \psi \text{ implies } F\phi \vdash_{T_2} F\psi.$$

This is part of the definition of a translation. In the warm-up, we showed that a monotonic map between preordered sets defines a functor between their classifying categories. An object of $\mathcal{C}\ell(T_1)$ is a derived proposition $\phi \in \mathrm{Prop}(T_1)$; we define

$$\mathcal{C}\ell(F)\phi \stackrel{\text{def}}{=} F\phi.$$

We move to the final part of the proposition. Assume F_1 and F_2 are provably equivalent translations from T_1 to T_2. The statement that $\mathcal{C}\ell(F_1)$ and $\mathcal{C}\ell(F_2)$ are naturally isomorphic unpacks to the following. For any arrow $\phi \to \psi$ in $\mathcal{C}\ell(T_1)$, there exists arrows $F_1\phi \to F_2\phi$, $F_1\psi \to F_2\psi$, $F_2\phi \to F_1\phi$, and $F_2\psi \to F_1\psi$ fitting into a pair of commuting squares in $\mathcal{C}\ell(T_2)$.

$$\begin{tikzcd} F_1\phi \arrow[r] \arrow[d] & F_1\psi \arrow[d] & F_2\phi \arrow[r] \arrow[d] & F_2\psi \arrow[d] \\ F_2\phi \arrow[r] & F_2\psi & F_1\phi \arrow[r] & F_1\psi. \end{tikzcd}$$

We now need to prove that these squares exist. Let $\phi \to \psi$ be an arrow in $\mathcal{C}\ell(T_1)$, so $\phi \vdash_T \psi$. Since F_1 and F_2 are translations, the following sequents are provable in T_2:

$$F_1\phi \vdash_{T_2} F_1\psi \quad F_2\phi \vdash_{T_2} F_2\psi.$$

Since F_1 and F_2 are provably equivalent, the following sequents are provable in T_2:

$$\begin{gathered} F_1\phi \vdash_{T_2} F_2\phi \quad F_2\phi \vdash_{T_2} F_1\phi \\ F_1\psi \vdash_{T_2} F_2\psi \quad F_2\psi \vdash_{T_2} F_1\psi. \end{gathered}$$

These six sequents define six arrows in $\mathcal{C}\ell(T_2)$. These six arrows can be arranged to make the sides in two commuting squares above. Thus, $\mathcal{C}\ell(F_1)$ and $\mathcal{C}\ell(F_2)$ are naturally isomorphic.

Conversely, assume that $\mathcal{C}\ell(F_1)$ and $\mathcal{C}\ell(F_2)$ are naturally isomorphic. Then for any arrow $\phi \to \psi$ in $\mathcal{C}\ell(T_1)$ we can find two commuting squares of the form shown above. Let ϕ be an arbitrary derived proposition in $\mathrm{Prop}(T_1)$, i.e., an arbitrary object of $\mathcal{C}\ell(T_1)$. If we apply this to the identity arrow $\phi \to \phi$, then we obtain arrows $F_1\phi \to F_2\phi$ and $F_2\phi \to F_1\phi$ (these are the vertical sides of the two squares). Unpacking the meaning of an arrow in $\mathcal{C}\ell(T_2)$, this means

$$F_1\phi \vdash_{T_2} F_2\phi \quad F_2\phi \vdash_{T_2} F_1\phi.$$

Since ϕ is arbitrary, this means F_1 and F_2 are provably equivalent. □

Recall that two translations $F : T_1 \to T_2$ and $G : T_2 \to T_1$ define an equivalence of theories in the case that GF and FG are provably equivalent to id_{T_1} and id_{T_2}. The above proposition determines when two translations are equivalent in terms of their associated functors. This gives us the following corollary.

Corollary 1.13. *Let $F : T_1 \to T_2$ and $G : T_2 \to T_1$ be a pair of translations. F and G define an equivalence of theories if and only if $\mathcal{C}\ell(F)$ and $\mathcal{C}\ell(G)$ define an equivalence of categories*

$$\begin{gathered} \mathcal{C}\ell(F) : \mathcal{C}\ell(T_1) \to \mathcal{C}\ell(T_2) \\ \mathcal{C}\ell(G) : \mathcal{C}\ell(T_2) \to \mathcal{C}\ell(T_1). \end{gathered}$$

We can actually strengthen the above result on translations and fully identify which functors $\mathcal{C}\ell(T_1) \to \mathcal{C}\ell(T_2)$ come from translations $T_1 \to T_2$. This requires a finer description of the structure of $\mathcal{C}\ell(T)$, which we will get to later. Let's just state the result now.

Proposition 1.14. *Let $F : T_1 \to T_2$ be a translation. Then $\mathcal{C}\ell(F) : \mathcal{C}\ell(T_1) \to \mathcal{C}\ell(T_2)$ satisfies the following properties.*

- *$\mathcal{C}\ell(F)$ preserves finite limits (i.e., infima);*
- *$\mathcal{C}\ell(F)$ preserves finite colimits (i.e., suprema);*
- *$\mathcal{C}\ell(F)$ preserves exponentials (i.e., $\Rightarrow$).*

Furthermore, any functor $\mathcal{C}\ell(T_1) \to \mathcal{C}\ell(T_2)$ satisfying these properties is described by a translation $T_1 \to T_2$.

Interpretations

Perhaps what is more surprising is that interpretations of a propositional theory T into a boolean algebra B can also be described as a functor. A boolean algebra has a canonical partial order associated to it: for any elements a, b in a boolean algebra B, we write $a \leqslant b$ in the case that $a \wedge b = a$. A partial order is a special kind of preorder, so we can refer to the classifying category of a boolean algebra.

Definition 1.15. Given a boolean algebra B, the **classifying category** of B is the classifying category of the canonical partial order on B:

$$\mathcal{C}\ell(B) \stackrel{\text{def}}{=} \mathcal{C}\ell(B, \leqslant).$$

Similar to the story with translations, any boolean algebra homomorphism $F : B_1 \to B_2$ defines a functor $\mathcal{C}\ell(F) : \mathcal{C}\ell(B_1) \to \mathcal{C}\ell(B_2)$, since the set of objects of $\mathcal{C}\ell(B)$ are the elements of B.

On the other hand, models are also functors. Given a propositional theory T and a B-valued model $[\![\cdot]\!]^M : T \to B$, we obtain a functor $M : \mathcal{C}\ell(T) \to \mathcal{C}\ell(B)$ by setting

$$M(\phi : \mathcal{C}\ell(T)) \stackrel{\text{def}}{=} [\![\phi : \text{Prop}(T)]\!]^M.$$

Proposition 1.16. *M is indeed a functor from $\mathcal{C}\ell(T)$ to $\mathcal{C}\ell(B)$.*

Proof. We need to verify that M sends arrows of $\mathcal{C}\ell(T)$ to arrows of $\mathcal{C}\ell(B)$. Let $\phi \to \psi$ be an arrow in $\mathcal{C}\ell(T)$. Since $[\![\cdot]\!]^M$ is a model, this implies

$$[\![\phi]\!]^M \leqslant [\![\psi]\!]^M.$$

Therefore, there exists an arrow $[\![\phi]\!]^M \to [\![\psi]\!]^M$ in $\mathcal{C}\ell(B)$. Since $M(\phi) = [\![\phi]\!]^M$, this is an arrow $M(\phi) \to M(\psi)$, as desired. $\square$

Given a translation $F : T_1 \to T_2$, recall we obtained a “pull-back” operation

$$F^* : \mathrm{Mod}(T_2, B) \to \mathrm{Mod}(T_1, B),$$

where $\mathrm{Mod}(T, B)$ denotes the set of B-valued models of the theory T. From the perspective of the classifying categories, this operation is just composition.

Theorem 1.17. *Let $F : T_1 \to T_2$ be a translation between propositional theories. Let $[\![\cdot]\!]^M : T_2 \to B$ be a B-valued model of T_2. Then $[\![\cdot]\!]^{F^*M} : T_1 \to B$ is the pulled-back model of T_1 under F. From these we obtain* *functors*

$$\mathcal{C}\ell(F) : \mathcal{C}\ell(T_1) \to \mathcal{C}\ell(T_2)$$
$$M : \mathcal{C}\ell(T_2) \to \mathcal{C}\ell(B)$$
$$F^*M : \mathcal{C}\ell(T_1) \to \mathcal{C}\ell(B).$$

These functors fit into a commuting diagram

$$\begin{tikzcd} \mathcal{C}\ell(T_2) \arrow[r, "M"] & \mathcal{C}\ell(B) \\ \mathcal{C}\ell(T_1). \arrow[u, "\mathcal{C}\ell(F)"] \arrow[ur, dashed, "F^*M"'] & \end{tikzcd}$$

In other words, $F^*M = M \circ \mathcal{C}\ell(F)$*.*

The key observation behind all of the proofs so far is that the classifying category of a theory is almost a copy of the theory. We will see later a construction, called the *internal logic* of a category, which will recover the original theory from its classifying category.

References

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A The Lindenbaum Algebra

The Lindenbaum algebra is another algebraic model of a propositional theory. We saw that, for the purposes of understanding the semantics of a theory, we only need equivalence classes of translations up to homotopy/provable equivalence. This was another point in a pattern of observations that the syntax of propositional logic hides certain symmetries which appear from the introduction and elimination rules. The Lindenbaum algebra of a propositional theory is a quotient

$$\text{Propositions}\big/_{\text{Proof theory}}.$$

This is the simplest form of an *equational theory*. Traditionally, the algebraic semantics was given by a boolean algebra of propositions modulo the double-sided implication $\leftrightarrow$.

Definition A.1. Let T be a classical propositional theory. The **Lindenbaum algebra** is the boolean algebra $(L(T), \leqslant)$ of equivalence classes of propositions $[\phi]$, where two propositions ϕ and ϕ' are in the same class in the case

$$\phi \vdash_T \phi' \text{ and } \phi' \vdash_T \phi.$$

The ordering $\leqslant$ is given by

$$[\phi] \leqslant [\psi] \overset{\text{def}}{\equiv} \phi \vdash_T \psi.$$

There are a few nice properties of the Lindenbaum algebra, which highlights the framework in [Halvorson, 2019].

Proposition A.2. *Let T be a propositional theory. The mapping $\phi \mapsto [\phi]$ defines a $L(T)$-valued model of T. This mapping is called the* ***canonical interpretation*** *of T in $L(T)$.*

Proposition A.3. *There is a bijective correspondence between provable-equivalence classes of translations $F : T_1 \to T_2$ and boolean algebra homomorphisms $L(T_1) \to L(T_2)$, fitting into a square, where the vertical arrows are the canonical interpretations.*

$$\begin{tikzcd} T_1 \arrow[r, "F"] \arrow[d] & T_2 \arrow[d] \\ L(T_1) \arrow[r, dashed] & L(T_2). \end{tikzcd}$$

Proposition A.4. *Let T be a propositional theory and B a boolean algebra. There is a bijective correspondence between B-valued models of T and boolean algebra homomorphisms $L(T) \to B$, related by the canonical interpretation*

$$\begin{tikzcd} L(T) \arrow[r, dashed] & B \\ T. \arrow[u] \arrow[ur] & \end{tikzcd}$$

For classical propositional logic, the Lindenbaum algebra is sufficient. However, our goal is to apply categorical logic to programming languages, where the proof theory is not so simple. I mentioned that the Lindenbaum algebra $L(T)$ is the quotient of T by $\leftrightarrow$. Under the Curry-Howard interpretation, this shows that the Lindenbaum algebra $L(T)$ identifies isomorphic types!

For this reason, it will be helpful to take a step backwards and permit ourselves to provide our own equational theory. This will require more complicatd algebra, but it will build a system which is amenable to the Curry-Howard correspondence.