This time we return to [Fong and Spivak, 2019] to discuss a special kind of relation between preorders: *Galois connections*. These are the preorder-specific forms of an adjunction between categories.

1 Definition of a Galois connection

The setting of a Galois connection is a pair of preorders $(X, \leqslant)$ and $(Y, \leqslant)$. In what follows, I will write X for the pair $(X, \leqslant)$, etc., using the same symbol for the order-relation in any preorder.

Definition 1.1. A **Galois connection** between the preordered sets X and Y is a pair of monotone maps $f : X \to Y$ and $g : Y \to X$ satisfying the following property. For any pair of elements $x \in X$ and $y \in Y$,

$$\frac{f(x) \quad \leqslant y}{\overline{\quad x \quad \leqslant g(y).}}$$

In this case f is called the **left adjoint**, and g is called the **right adjoint**. We express the relationship between f and g by writing $f \dashv g$.

One powerful property of Galois connections (and adjunctions, more generally) is there preservation of limits and colimits.

Proposition 1.2. *Let $f : X \to Y$ and $g : Y \to X$ form a Galois connection $f \dashv g$ between preordered sets X and Y. Then f preserves all colimits (i.e., suprema), and g preserves all limits (i.e., infima).*

Remark 1.3. *A functor which preserves colimits is sometimes called a* right-exact functor*, and a functor which preserves limits is sometimes called a* left-exact functor. *Therefore the above proposition can be restated more mnemonically as:* if a functor has a right adjoint, then it is right exact, and if it has a left adjoint, then it is left exact.

Proof of Proposition 1.2. We will show that f preserves suprema. The proof that g preserves infima is formally dual. Consider a diagram $J : D \to X$. Assuming the colimit along J exists, we need to show

$$f\left(\bigvee_{i \in D} J(i)\right) \cong \bigvee_{i \in D} f(J(i)).$$

To that end, let x be an element in X representing $\bigvee_{i \in D} J(i)$. We will show that $f(x)$ is the least upper bound of the set $\{f(J(i)) \mid i \in D\}$. Since $J(i) \leqslant x$ for any $i \in D$, $f(J(i)) \leqslant f(x)$ since f is order-preserving. Therefore $f(x)$ is an upper bound. On the other hand, suppose $y \in Y$ is an element satisfying $f(J(i)) \leqslant y$ for any $i \in D$. Since f is left adjoint to g, this implies $J(i) \leqslant g(y)$ for any $i \in D$. Since x is a least upper bound of $\{J(i) \mid i \in D\}$, $x \leqslant g(y)$. We use the Galois connection again to deduce from this that $f(x) \leqslant y$. Therefore $f(x)$ is a least upper bound, as desired. $\square$

We now apply the fundamental trick in category theory: use `refl`. Every preorder is reflexive, so $f(x) \leqslant f(x)$, for any $x \in X$. By the above property of Galois connections

$$\frac{f(x) \quad \leqslant f(x)}{x \quad \leqslant g(f(x)).}$$

Similarly, $g(y) \leqslant g(y)$, so the Galois connection implies

$$\frac{g(y) \quad \leqslant g(y)}{f(g(y)) \quad \leqslant y.}$$

In fact, these properties characterize a Galois connection.

Theorem 1.4. *Let $f : X \to Y$ and $g : Y \to X$ be a pair of order-preserving maps. f and g form a Galois connection $f \dashv g$ if and only if, for any $x \in X$ and $y \in Y$,*

$$x \leqslant g(f(x)) \text{ and } f(g(y)) \leqslant y.$$

> The conditions $x \leqslant g(f(x))$ and $f(g(y)) \leqslant y$ for any $x \in X$ and $y \in Y$ show that $g \circ f$ is a *closure operator* and $f \circ g$ is an *interior operator* (this is the terminology used in the warm-up). This is the preorder-variant of a more general property of adjunctions. Given any adjunction $F \dashv G$ between categories, GF is a monad, and FG is a comonad.

Proof of Theorem 1.4. The forward direction was proven in the above discussion. For the reverse direction, assume, for any $x \in X$ and $y \in Y$, $x \leqslant g(f(x))$ and $f(g(y)) \leqslant y$. We need to show that f and g form a Galois connection $f \dashv g$. Consider a pair of elements $x \in X$ and $y \in Y$. We have a chain of implications.

$$\dfrac{x \leqslant g(f(x)) \qquad \dfrac{f(x) \leqslant y}{g(f(x)) \leqslant g(y)}\ g \text{ order-pres.}}{x \leqslant g(y)}\ g \circ f \text{ closure operator.}$$

Therefore $f(x) \leqslant y$ implies $x \leqslant g(y)$. Conversely, we have a similar chain of implications.

$$f \circ g \text{ interior operator. } \dfrac{f \text{ order-pres. } \dfrac{x \leqslant g(y)}{f(x) \leqslant f(g(y))} \qquad f(g(y)) \leqslant y}{f(x) \leqslant y}$$

Thus $x \leqslant g(y)$ implies $f(x) \leqslant y$. Since x and y are arbitrary, this shows that f and g form a Galois connection $f \dashv g$, as desired. □

Example 1.5. During the seminar on Galois connections, Steven brought up a nice example for finding a Galois connection. Here is the example. It will motivate a theorem characterizing the existence of Galois connections. Consider the following preorders

$$X \overset{\text{def}}{=} \qquad \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

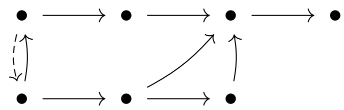
$$Y \overset{\text{def}}{=} \qquad \bullet \longrightarrow \bullet \longrightarrow \bullet$$

Given a functor $g : Y \to X$, e.g.,

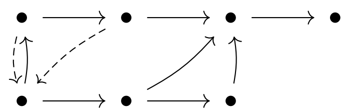
$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[ur] & \bullet \arrow[u] &
\end{tikzcd}$$

we can ask whether this functor has a left adjoint $f : X \to Y$ (forming a Galois connection $f \dashv g$. In this example, no left adjoint exists. We can see this by looking at what f must do to the right-most point of X. Recall, for any Galois connection $f \dashv g$ and any $x \in X$, $x \leqslant g(f(x))$. There is no choice for $f(x)$ to satisfy this constraint, since the image of g is strictly less than the right-most point.

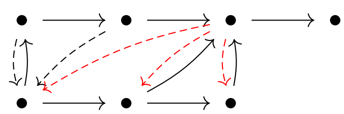
On the other hand, we can ask if $g : Y \to X$ has a *right* adjoint $h : X \to Y$ (forming a Galois connection $g \dashv h$. We can construct h by looking at the behavior of $g \circ h$ starting from the left. Recall, for a Galois connection $g \dashv h$ and any element $x \in X$, $g(h(x)) \leqslant x$. For the left most point of X, there is only one choice of $h(x)$ which satisfies this constraint, shown below.



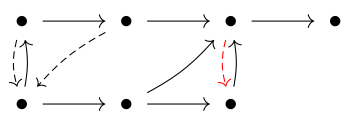
Similarly, the image of the second point under h is fixed by this constraint.



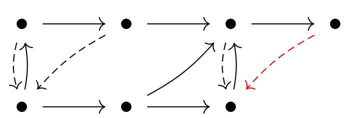
For the third point, any choice of image satisfies this constraint.



To determine which choice is the correct choice, we need to use another constraint. In the reverse direction, for any $y \in Y$, $y \leqslant h(g(y))$. When y is the right-most point of Y, this fixes the choice of red arrow.



This leaves the right-most point of X. $h : X \to Y$ must be order-preserving. This fixes the choice for the image of the right-most point.



Thus, we have found the right adjoint h using the constraints $g(h(x)) \leqslant x$ and $y \leqslant h(g(y))$.

The previous example suggests the question of uniqueness of an adjoint. Adjoints are always unique, up to isomorphism. We will prove this by writing right adjoints as colimits (and left adjoints as a limit).

Theorem 1.6. *Let $f : X \to Y$ be an order-preserving map between preorders. If f has a right adjoint $g : Y \to$*

X, then g satisfies

$$g(y) \cong \bigvee_{x \in f^{-1}(\downarrow y)} x.$$

This supremum is the preorder variant of the following colimit, which characterizes a right adjoint of a functor $F : C \to D$.

$$G(y) \cong \varinjlim (F \downarrow y \to C),$$

where $F \downarrow y$ is the *comma category* of F and y. Its objects are pairs $(x, a : F(x) \to y)$ of an object x of C and an arrow $a : F(x) \to y$ in D. An arrow $(x, a) \to (x', a')$ is an arrow $\theta : x \to x'$ in C fitting into a commuting triangle

$$\begin{tikzcd} F(x) \arrow[rr, "F(\theta)"] \arrow[dr, "a"'] & & F(x') \arrow[dl, "a'"] \\ & y. & \end{tikzcd}$$

The functor $F \downarrow y \to C$ sends a commuting triangle likke the one above to the arrow $\theta : x \to x'$.

Proof of Theorem 1.6. We first show that $g(y)$ is an upper bound for $f^{-1}(\downarrow y)$. Then we will show that $g(y)$ is the least upper bound. Any element $x \in f^{-1}(\downarrow y)$ satisfies $f(x) \leqslant y$ by definition of $\downarrow y$. Since g is a right adjoint, this implies $x \leqslant g(y)$, so $g(y)$ is an upper bound. Furthermore, any Galois connection $f \dashv g$ satisfies

$$f(g(y)) \leqslant y,$$

for any $y \in Y$. Thus $g(y) \in f^{-1}(\downarrow y)$, so $g(y)$ is a least upper bound, as desired. (In fact, this shows that $g(y)$ is the maximum of $f^{-1}(\downarrow y)$ rather than just the supremum.) □

Corollary 1.7. *Let $f : X \to Y$ be an order-preserving map between preorders. Suppose $g_1 : Y \to X$ and $g_2 : Y \to X$ are right adjoints to f, i.e., there are Galois connections $f \dashv g_1$ and $f \dashv g_2$. Then g_1 and g_2 are* ***naturally isomorphic****: for any element $y \in Y$,*

$$g_1(y) \leqslant g_2(y) \text{ and } g_2(y) \leqslant g_1(y).$$

Proof. By Theorem 1.6, the two Galois connections $f \dashv g_1$ and $f \dashv g_2$ imply, for any $y \in Y$

$$g_1(y) \cong \bigvee_{x \in f^{-1}(\downarrow y)} x \cong g_2(y).$$

Thus $g_1(y) \cong g_2(y)$. Recall this means $g_1(y) \leqslant g_2(y)$ and $g_2(y) \leqslant g_1(y)$, as desired. □

We can prove the analogous things for left adjoints too.

Exercise 1 : Let $g : Y \to X$ be an order-preserving map between preorders. Suppose $f : X \to Y$ is a *left* adjoint to g. Show, for any $x \in X$,

$$f(x) \cong \bigwedge_{y \in g^{-1}(\uparrow x)} y.$$

Where $\uparrow x$ is the set of elements x' in X such that $x \leqslant x'$. This is the preorder form of the limit

$$f(x) \cong \varprojlim x \downarrow g \to Y.$$

Furthermore, show that left adjoints are unique up to natural isomorphism.
This exercise can be proven directly, using the same argument. Alternatively, use Theorem 1.6 on the opposite preorders $(X, \geqslant)$ and $(Y, \geqslant)$.

2 A very important Galois connection

Let X and Y be a pair of sets, and let $f : X \to Y$ be a function between them. From f we can define an order-preserving map $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ which sends a subset $A \in \mathcal{P}(Y)$ to its preimage

$$f^{-1}(A) \stackrel{\text{def}}{=} \{x \in X \mid f(x) \in A\} \in \mathcal{P}(X).$$

This order-preserving map participates in two Galois connections. These Galois connections are essential for interpreting first-order logic inside a topos. First, f^{-1} has a left adjoint, written $\exists_f : \mathcal{P}(X) \to \mathcal{P}(Y)$. This is the usual image map:

$$\exists_f(B \in \mathcal{P}(X)) \stackrel{\text{def}}{=} f(B) = \{y \in Y \mid \exists x \in X . f(x) = y \wedge x \in B\}.$$

Proposition 2.1. *$\exists_f$ is the left adjoint of f^{-1} (forming a Galois connection $\exists_f \dashv f^{-1}$).*

Proof. Let $A \in \mathcal{P}(Y)$ and $B \in \mathcal{P}(X)$. Then $B \subseteq f^{-1}(A)$ if and only if every element $x \in B$ satisfies $f(x) \in A$, which is true if and only if $f(B) \subseteq A$, i.e., $\exists_f B \subseteq A$. □

f^{-1} also has a right adjoint $\forall_f : \mathcal{P}(X) \to \mathcal{P}(Y)$. This is sometimes called the *dual image* (c.f. [Makkai and Reyes, 1977]). The dual image is defined using the universal quantifier:

$$\forall_f(B \in \mathcal{P}(X)) \stackrel{\text{def}}{=} \{y \in Y \mid \forall x \in X . f(x) = y \Rightarrow x \in B\} = \big\{y \in Y \mid f^{-1}(\{y\}) \subseteq B\big\}.$$

Proposition 2.2. *$\forall_f$ is a right adjoint for f^{-1}, yielding a Galois connection $f^{-1} \dashv \forall_f$.*

Proof. Let $A \in \mathcal{P}(Y)$ and $B \in \mathcal{P}(X)$ be a pair of subsets. Suppose $f^{-1}(A) \subseteq B$. Then for any element $y \in A$, $f^{-1}(\{y\}) \subseteq B$ (since f^{-1} is order-preserving), i.e., $y \in \forall_f A$. Since y is arbitrary, this shows that $A \subseteq \forall_f B$. Conversely, suppose $A \subseteq \forall_f B$, then for any $y \in A$, $f^{-1}(\{y\}) \subseteq B$. Thus

$$f^{-1}(A) = \bigcup_{y \in A} f^{-1}(\{y\}) \subseteq B,$$

as desired. □

The previous two propositions show that we have an *adjoint triple* $\exists_f \dashv f^{-1} \dashv \forall_f$. Adjoint triples are particularly nice categorical structures. Longer adjoint *strings*, e.g., $f_1 \dashv \ldots \dashv f_n$ are rarer and have increasingly powerful properties. For example, we have a fun fact.

Proposition 2.3 ([Rosebrugh and Wood, 1994])**.** *Let C be a locally small category, and let* よ $: C \to Psh(C)$ *be its Yoneda embedding. If* よ *extends to the left to an adjoint quintuple*

$$F_1 \dashv F_2 \dashv F_3 \dashv F_4 \dashv よ,$$

then C is equivalent to the category Set*.*

The adjoint triple $\exists_f \dashv f^{-1} \dashv \forall_f$ gives an overpowered proof of the following result.

Corollary 2.4. $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ *preserves intersections and unions.*

Proof. f^{-1} has both a left and a right adjoint. Therefore, by Proposition 1.2, f^{-1} preserves infima and suprema, i.e., intersections and unions. □

Quantifiers as Adjoints

The notation in the adjoint triple $\exists_f \dashv f^{-1} \dashv \forall_f$ suggests the adjoints to f^{-1} have something to say about the quantifiers in first-order predicate logic. If we are working within a topos, there is a bijection between subsets $A \subseteq X$ and propositions $\chi_A(x : X)$ with a free variable representing an element of X. This bijection is given by

$$\phi(x : X) \mapsto \{x \in X \mid \phi(x)\} \subseteq X.$$

(This bijection exists because a topos has a *subobject classifier*.) Under this bijection, the above adjoint triple interprets the quantifiers $\exists$ and $\forall$ in the way we expect. We can also describe essentially the same adjoint triple directly in logic.

Recall, from a propositional theory T, we can form its Lindenbaum algebra $L(T)$ of propositions. To describe a first-order theory, we stitch together families of Lindenbaum algebras, indexed by a context of free variables.

Definition 2.5. A **first-order signature** Σ is a collection of the following data.

- A set of *sorts* $\sigma \in \Sigma$.
- A collection of base terms, i.e., *function symbols*

$$x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash f(x_1, \ldots, x_n) : \tau.$$

- For every context $x_1 : \sigma_1, \ldots, x_n : \sigma_n$, a collection of base propositions. e.g., $R(x_1 : \sigma_1, \ldots, x_n : \sigma_n)$.

A **first-order theory** is a first-order signature and a set of sequents, which are the axioms of the theory.

If we fix a context, we can use the usual connectives $\wedge, \vee$, etc. to form formulae. The new expressivity comes from the terms in a first-order signature. We can freely compose terms in the usual typed manner. Given a list of terms $\Gamma \vdash t_1 : \tau_1, \ldots, \Gamma \vdash t_n : \tau_n$, which we could write as an arrow $t : \Gamma \to \Gamma'$, where Γ' is a context of fresh variables of type $\tau_1, \ldots, \tau_n$, we obtain a map

$$t^* : \mathrm{Form}(\Gamma') \to \mathrm{Form}(\Gamma)$$

given by substitution. If $\phi(\vec{x} : \Gamma')$ is a formula with context Γ', then substituting the list of terms t gives a formula $\phi(t_1, \ldots, t_n)$ with context Γ. The left and right adjoints to t^* allow us to interpret quantifiers: $\exists_t \dashv t^* \dashv \forall_t$, where the preorder on $\mathrm{Form}(\Gamma)$ is given by provability $\vdash$, the same as in the Lindenbaum algebra. We bundle this data into one categorical structure.

Definition 2.6. A **first-order hyperdoctrine** is a pair (C, F) of a category C of contexts and terms and a functor

$$F : C^{\mathsf{op}} \to \text{Lindenbaum Algebras}$$

sending a context Γ to its Lindenbaum algebra of formulae with free variables in Γ. We require, for any arrow $t : \Gamma \to \Gamma'$ in C, we have an adjoint triple

$$\exists_t \dashv F(t) \dashv \forall_t.$$

Any first-order theory defines a first-order hyperdoctrine, where $F(t) = t^*$. The adjoints $\exists_t$ and $\forall_t$ are given by

$$\exists_t \phi(\vec{x'} : \Gamma') \equiv \exists \vec{x} : \Gamma.\, t(\vec{x}) = \vec{x'} \wedge \phi(\vec{x}).$$

and

$$\forall_t \phi(\vec{x'} : \Gamma') \equiv \forall \vec{x} : \Gamma.\, t(\vec{x}) = \vec{x'} \Rightarrow \phi(\vec{x}).$$

Theorem 2.7 (Quantifiers are adjoints)**.** *Given a pair of context Γ and Γ', a list of terms $t : \Gamma \to \Gamma'$, and a pair of formulae $\phi(\vec{x} : \Gamma)$ and $\psi(\vec{x'} : \Gamma')$,*

$$\phi(\vec{x}) \vdash t^* \psi(\vec{x}) \textit{ if and only if } \exists_t \phi(\vec{x'}) \vdash \psi(\vec{x'}).$$

Similarly

$$t^* \psi(\vec{x}) \vdash \phi(\vec{x}) \textit{ if and only if } \forall_t \phi(\vec{x'}) \vdash \psi(\vec{x'}).$$

In other words, we have a pair of Galois connections $\exists_t \dashv t^*$ and $t^* \dashv \forall_t$.

We now give an example of a term and its adjoint triple. Given a pair of variables $x_1 : \sigma_1, x_2 : \sigma_2$, we can form the "projection"

$$x_1 : \sigma_1, x_2 : \sigma_2 \vdash x_1 : \sigma_1,$$

representing an arrow $\pi_1 : \sigma_1, \sigma_2 \to \sigma_1$. Let $\phi(x_1, x_2$ and $\psi(x_1$ be a pair of formulae with context σ_1, σ_2 and σ_1, respectively. Note, given another pair of variables $y_1 : \sigma_1, y_2 : \sigma_2$, then

$$\pi_1^* \psi(y_1, y_2) \equiv \psi(\pi_1(y_1, y_2)) \equiv \psi(y_1),$$

Similarly,

$$\exists_{\pi_1} \phi(x_1) \equiv \exists x_1' : \sigma_1.\, \exists x_2' : \sigma_2.\, \pi_1(x_1', x_2') = x_1 \wedge \phi(x_1', x_2') \equiv \exists x_1' \exists x_2'\, x_1' = x_1 \wedge \phi(x_1', x_2').$$

The latter is provably equivalent (by $=$ elimination) to $\exists x_2'.\, \phi(x_1, x_2')$. To emphasize, we have shown:

$$\exists_{\pi_1} \phi(x_1) \dashv\vdash \exists x_2'.\, \phi(x_1, x_2').$$

A similar argument shows $\forall_{\pi_1} \phi(x_1) \dashv\vdash \forall x_2'.\, \phi(x_1, x_2')$. In other words, the adjoints $\exists_\pi$ and $\forall_\pi$ recover the usual quantifiers when π is a projection.

References

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