NeuPPL Categories Seminar
Categorical Semantics of Programming Languages

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These notes are lecture notes from a summer seminar at Northeastern University to study category theory and its applications to the semantics of programming languages. The first half of these notes were written during the seminar, whereas the latter half was written after the last meeting. The contents of the latter half are, more or less, what was covered in these meetings.

Our goal with this seminar was to develop the category theory necessary to understand the relationship between step-indexed logical relations and the Kripke-Joyal semantics of the topos of trees, i.e., the topos of presheaves over the first countable ordinal. The seminar concluded with an overview of Birkedal et al's article *First Steps in Synthetic Guarded Domain Theory: Step-Indexing in the Topos of Trees* [Birkedal et al., 2011].

In between meetings, we read Fong and Spivak's *An Invitation to Applied Category Theory* [Fong and Spivak, 2019]. As such, these notes assume that the reader has an idea for what is a category and a functor—though perhaps not familiar with category *theory*!—approximately at the level of someone who has seen the Curry-Howard correspondence between certain categories and the simply typed λ calculus. On the other hand, these notes, and our seminar, were designed as a review and crash-course of the aspects of category theory needed to describe the Kripke-Joyal semantics of a topos. This aim motivates the order in which things are presented. First, the connection to propositional logic and preordered sets is established. We spend an extended time with preordered sets in order to develop the key results of category theory for this special case. We then introduce categories as an intensional/proof-relevant generalization of a preorder and presheaves as the corresponding generalization of a downward-closed set.

This preorder-focused approach gives these notes a unique perspective on category theory. It is my hope that they emphasize the role of a topos as a context-aware universe of sets. These context-aware "sets" appear in the logical relations of virtually any programming language with more features than the simply-typed λ calculus, e.g., languages with recursive types or with references. These logical references can be complex: any quantifiers involved in their definition are guarded with conditions described their dependence on contextual data. In the case of recursive types, these extra conditions involve a *step-index*. In the case of references, these conditions involve *heap extensions*. In any case, these conditions are precisely those which appear from the Kripke-Joyal semantics. This gives the PL theorist a new tool: by working internal to a topos of contexts, one can replace the indexed logical relations and their extra conditions with the simpler relations used for the simply typed λ calculus.

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1 Introduction: A Proposition for a Metatheory

Categorical logic and categorical semantics are a generalization of a certain triangle of structures in propositional logic. First, there is the *propositional theory* itself. This is the propositional version of an *equational theory*. It is syntax, and like syntax in a programming language, we can study transformations between different languages: *translations*. The second structure is an algebraic model, called the *classifying category* of the theory. The algebraic model plays the role of an *equational theory* for the syntax. If we temporarily fast-forward to the simply-typed λ-calculus, we find an example of structure appearing in the algebraic model.

> *Well-typed λ-terms do not form a Cartesian-closed category. However, if we identify terms up to $\beta\eta$-equivalence, the resultant equivalence classes do form a Cartesian-closed category.*

This is the general idea of the algebraic model: find structure in the syntax by identifying terms which are proof-theoretically the same. The third structure is the *semantics* of the logic. The semantics is the "ground-truth" of the logic. We can think of this as observational equivalence. Semantic truth, i.e., "$\vDash$" is difficult to reason with, so we refer to the first two models to explore as much of the semantics as possible while remaining tractable. For this reason, soundness and completeness theorems are valuable.

> *Soundness:* the first two models are not more powerful than the semantics.
> *Completeness:* the first two models are at least as powerful as the semantics.

Let us begin by describing this triangle for propositional logic. This will be a high-level overview of the ideas in Chapter 3 of [Halvorson, 2019].

Syntax

We will work in the classical fragment of propositional logic. Given a set of atomic propositions $P, Q, \ldots$, we can form derived propositions $P \vee Q, \neg P, P \to Q, \ldots$ using the connectives

$$\wedge, \vee, \Rightarrow, \neg, \top, \bot.$$

The meaning of these connectives come from the usual introduction and elimination rules. These allow us to build Gentzen-style proof trees, shown below.

$$\frac{A \in \Gamma}{\Gamma \vdash A} \qquad \frac{\Delta \vdash A \quad \Delta \subseteq \Gamma}{\Gamma \vdash A} \qquad \frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A \quad \Gamma \vdash B}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \qquad \frac{\Gamma \vdash A \Rightarrow B}{\Gamma, A \vdash B} \qquad \left(\frac{\Gamma \vdash A \Rightarrow B \wedge B \Rightarrow A}{\Gamma \vdash A \Leftrightarrow B}\right)$$

$$\frac{\Gamma \vdash A \to \bot}{\Gamma \vdash \neg A} \qquad \frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \top} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash A}$$

Definition 1.1. A **propositional theory** is a pair $T = (\Sigma, \Delta)$ consisting of a set Σ of atomic propositions, called the **signature** of the theory, and a set Δ of sequents of derived propositions of Σ, called the **axioms** of the theory.

Propositional logic is the study of propositional theories. Using the axioms of a theory T, we can form a new relation $\vdash_T$, where $\Gamma \vdash_T A$ signifies the existence of a proof-tree whose leaves are axioms of T such that every internal node is an application of a classical introduction or elimination rule. One could pause here and study this relation in depth, but we will keep going. Propositional theories have their own notion of transpiler.

Definition 1.2. Let T_1 and T_2 be two propositional theories. A **translation** $F : T_1 \to T_2$ is a map sending atomic propositions in Σ_1 to derived propositions of Σ_2. By declaring that this map preserves the logical connectives, this extends to a map from derived propositions of Σ_1 to derived propositions of Σ_2. We require that F satisfy the additional constraint:

$$\Gamma \vdash_{T_1} A \implies F\Gamma \vdash_{T_2} FA.$$

We are often interested in asking whether two theories T_1 and T_2 have the same level of expressivity and proving power. Given that we have a notion of arrow between theories, we might try to study *isomorphisms*. However, the constraint that $GF = \mathsf{id} : T_1 \to T_1$ for a pair of translations $F : T_1 \to T_2$ and $G : T_2 \to T_1$ is often too restrictive. For one, atomic propositions must be sent to themselves, but the image of a derived proposition under a translation is never atomic. Therefore, F must send any atomic proposition to an atomic proposition. If we have an isomorphism, the same must be true for G. This shows that any isomorphism of propositional theories must be just a relabeling of the symbols. Not only that, theories also have axioms, and isomorphisms do not care about the axioms. We are led to a weakening of isomorphism which is still good enough to model the logic inside a propositional theory.

Definition 1.3. Two translations $F, F' : T_1 \to T_2$ are **provably equivalent** or **homotopic** in the case that, for any T_1-proposition ϕ,

$$\cdot \vdash_{T_2} F\phi \Leftrightarrow F'\phi.$$

An **equivalence** of theories is a pair of translations $F : T_1 \to T_2$ and $G : T_2 \to T_1$ such that GF is provably equivalent to id_{T_1} and FG is provably equivalent to id_{T_2}, i.e., for any T_1 proposition ϕ and T_2 proposition ψ,

$$\cdot \vdash_{T_1} \phi \Leftrightarrow GF\phi \text{ and } \cdot \vdash_{T_2} \psi \Leftrightarrow FG\psi.$$

If a translation $F : T_1 \to T_2$ is part of an equivalence, then F resembles a *fully abstract* compiler. Our metatheory of translations should explain when equivalences exist, and it should provide methods to construct them.

Semantics

I mentioned that equivalences are good enough to model the logic inside a propositional theory. Let's take a moment to formalize what is meant by "modeling the logic" inside a theory. For classical propositional logic, the only thing that matters is truth tables, essentially. Let $\Omega \stackrel{\text{def}}{=} \{\mathsf{True}, \mathsf{False}\}$ be the set of classical truth values.

Definition 1.4. Let Σ be a propositional signature. An **interpretation** (or, if you want, a **denotational semantics**) for the signature Σ is a map $[\![\cdot]\!]$ from atomic propositions in Σ to truth values Ω

We can extend the map from a denotational semantics $[\![\cdot]\!] : \Sigma \to \Omega$ to derived propositions in Σ. We do this by sending the logical connectives to the appropriate boolean operations

$$[\![A \wedge B]\!] \stackrel{\text{def}}{=} [\![A]\!] \;\mathsf{and}\; [\![B]\!] \qquad [\![A \vee B]\!] \stackrel{\text{def}}{=} [\![A]\!] \;\mathtt{or}\; [\![B]\!] \qquad [\![A \Rightarrow B]\!] \stackrel{\text{def}}{=} \mathtt{if}\; [\![A]\!] \;\mathtt{then}\; [\![B]\!] \;\mathtt{else}\; \mathsf{True}$$
$$[\![\neg A]\!] \stackrel{\text{def}}{=} \mathsf{not}\; [\![A]\!] \qquad [\![\top]\!] \stackrel{\text{def}}{=} \mathsf{True} \qquad [\![\bot]\!] \stackrel{\text{def}}{=} \mathsf{False}\,.$$

Definition 1.5. Let T be a propositional theory with signature Σ and axioms Δ. A **model** of the theory T is an interpretation $[\![\cdot]\!]$ of Σ such that, for any sequent $\phi \vdash \psi$ in Δ, $[\![\phi]\!] \leqslant [\![\psi]\!]$, where the ordering $\leqslant$ is the usual ordering on Ω, interpreted as a boolean algebra. If we give the model a name, say, $M : \Sigma \to \Omega$, then we define the notation $\phi \vDash_M \psi$ to mean $[\![\phi]\!]^M \leqslant [\![\psi]\!]^M$.

We can treat the semantics as the ground-truth of the propositional theory. We are concerned with which propositions are true, and the classical introduction and elimination rules help us explore the space of true propositions by identifying operations which preserve the ground truth. This is a soundness theorem.

Theorem 1.6 (Soundness theorem)**.** *Let $[\![\cdot]\!]^M$ be a model of a propositional theory T. Suppose $\phi \vdash_T \psi$ for some propositions ϕ and ψ. Then $\phi \vDash_M \psi$.*

Similarly, our notion of translation can also be justified by the semantics. Consider a translation $F : T_1 \to T_2$, and suppose $[\![\cdot]\!]^M$ is a model of T_2. We obtain a model of $[\![\cdot]\!]^{F^*M}$ of T_1, given by composition, essentially

$$\Sigma_1 \xrightarrow{F} \mathsf{Prop}(\Sigma_2) \xrightarrow{[\![\cdot]\!]^M} \Omega.$$

Proposition 1.7. *$[\![\cdot]\!]^{F^*M}$ is indeed a model of T_1.*

Proof. We need to show that $F^*[\![\cdot]\!]$ preserves the axioms of T_1. Let $\phi \vdash \psi$ be such an axiom. Since F is a translation, $F\phi \vdash_{T_2} F\psi$. By the soundness theorem, $F\phi \vDash_M F\psi$. Finaly, observe that

$$[\![\phi]\!]^{F^*M} \stackrel{\text{def}}{=} [\![F\phi]\!]^M.$$

Therefore $F\phi \vDash_M F\psi$ implies $\phi \vDash_{F^*M} \psi$, as desired. □

Recall that our notion of translation was too strict to have interesting inverses. Here is another reason why we need to consider "homotopic" translations.

Proposition 1.8. *Let $F, G : T_1 \to T_2$ be a pair of translations. If F and G are homotopic, then, for any model $[\![\cdot]\!]^M$ of T_2, we have, for any derived proposition ϕ of T_1,*

$$[\![\phi]\!]^{F^*M} = [\![\phi]\!]^{G^*M} : \Omega.$$

Proof. Since F and G are homotopic, the sequent $\cdot \vdash_{T_2} F\phi \leftrightarrow G\phi$ is provable in T_2. Since M is a model, this means

$$
\begin{array}{l}
\textbf{if}\ \ \mathrm{True} \\
\textbf{then}\ \ ((\textbf{if}\ \ [\![F\phi]\!]^M\ \ \textbf{then}\ \ [\![G\phi]\!]^M\ \ \textbf{else}\ \ \mathrm{True}) \\
\qquad\quad \textbf{and} \\
\qquad\quad (\textbf{if}\ \ [\![G\phi]\!]^M\ \ \textbf{then}\ \ [\![F\phi]\!]^M\ \ \textbf{else}\ \ \mathrm{True})) \\
\textbf{else} \\
\quad \mathrm{True}
\end{array}
$$

evaluates to True. Furthermore, the laws of the boolean operations imply that the above expression is True if and only if $[\![F\phi]\!]^M \leqslant [\![G\phi]\!]^M$ and vice versa. Antisymmetry now implies $[\![\phi]\!]^{F^*M} = [\![\phi]\!]^{G^*M}$, as desired. □

In fact, we could have defined two translations to be homotopic in the case that they are indistinguishable in any model. This was the original approach, when homotopy of translations was defined in [Ahlbrandt and Ziegler, 1986]. Proving that this is equivalent to the syntax-focused approach mentioned above requires a completeness theorem for propositional theories.

Remark 1.9. *To make sense of an interpretation and model, we only need a notion of truth value and boolean operations on truth values. Therefore, we could replace Ω with any boolean algebra B and define a notion of B-**valued model** $\Sigma \to B$. All the results in this passage transfer to B-valued models.*

Algebra

We now present the third side of the triangle. In the case of propositional logic, the equational theory is trivial. However the change in perspective from introducing category theory introduces the key ideas in categorical logic, and it shows that a propositional theory is fully subsumed by its classifying category.

Definition 1.10. Let T be a propositional theory. Notice that the set of derived propositions $\text{Prop}(T)$ has a preorder, given by $\vdash_T$:

$$\phi \leqslant \psi \text{ if and only if } \phi \vdash_T \psi.$$

In the warm-up we saw that we can define the classifying category of a preorder. The **classifying category** of the theory T is the category

$$\mathcal{C}\ell(T) \stackrel{\text{def}}{=} \mathcal{C}\ell(\text{Prop}(T), \vdash_T).$$

Explicitly, the objects of $\mathcal{C}\ell(T)$ are derived propositions in $\text{Prop}(T)$, and there exists a unique arrow $\phi \to \psi$ in $\mathcal{C}\ell(T)$ in the case that $\phi \vdash_T \psi$.

I am choosing to call this category the classifying category of T, following [Jacobs, 1999]. There are many other names for this category; another common name is the *syntactic category* of the theory T. The rest of this section will describe how a propositional theory can be described in terms of classifying categories.

For example, a common thing to ask when given a category is *what are its isomorphisms?* Since there is at most one arrow between two objects in $\mathcal{C}\ell(T)$, the only arrow $\phi \to \phi$ must be the identity arrow, for any object ϕ in $\mathcal{C}\ell(T)$. This means two objects ϕ and ψ are isomorphic in $\mathcal{C}\ell(T)$ if and only if there exist arrows $\phi \to \psi$ and $\psi \to \phi$. Unpacking the meaning of an arrow in $\mathcal{C}\ell(T)$, we obtain the following proposition.

Proposition 1.11. *Let T be a propositional theory, and let ϕ and ψ be two objects of $\mathcal{C}\ell(T)$, i.e., derived propositions of T. ϕ and ψ are isomorphic in $\mathcal{C}\ell(T)$ if and only if $\phi \vdash_T \psi$ and $\psi \vdash_T \phi$, equivalently,*

$$\cdot \vdash_T \phi \Leftrightarrow \psi.$$

Translations

We can describe translations as functors.

Proposition 1.12. *Let $F : T_1 \to T_2$ be a translation between propositional theories. Then F defines a mapping* $\text{Prop}(T_1) \to \text{Prop}(T_2)$ *which is monotonic. Therefore F defines a functor*

$$\mathcal{C}\ell(F) : \mathcal{C}\ell(T_1) \to \mathcal{C}\ell(T_2).$$

Furthermore, two translations $F_1, F_2 : T_1 \to T_2$ are provably equivalent (i.e., homotopic) if and only if $\mathcal{C}\ell(F_1)$ and $\mathcal{C}\ell(F_2)$ are naturally isomorphic.

Proof. The statement that F is a monotonic map means, for any pair of derived propositions $\phi, \psi \in \text{Prop}(T_1)$,

$$\phi \vdash_{T_1} \psi \text{ implies } F\phi \vdash_{T_2} F\psi.$$

This is part of the definition of a translation. In the warm-up, we showed that a monotonic map between preordered sets defines a functor between their classifying categories. An object of $\mathcal{C}\ell(T_1)$ is a derived proposition $\phi \in \text{Prop}(T_1)$; we define

$$\mathcal{C}\ell(F)\phi \stackrel{\text{def}}{=} F\phi.$$

We move to the final part of the proposition. Assume F_1 and F_2 are provably equivalent translations from T_1 to T_2. The statement that $\mathcal{C\ell}(F_1)$ and $\mathcal{C\ell}(F_2)$ are naturally isomorphic unpacks to the following. For any arrow $\phi \to \psi$ in $\mathcal{C\ell}(T_1)$, there exists arrows $F_1\phi \to F_2\phi$, $F_1\psi \to F_2\psi$, $F_2\phi \to F_1\phi$, and $F_2\psi \to F_1\psi$ fitting into a pair of commuting squares in $\mathcal{C\ell}(T_2)$.

$$\begin{array}{ccc} F_1\phi & \longrightarrow & F_1\psi \\ \downarrow & & \downarrow \\ F_2\phi & \longrightarrow & F_2\psi \end{array} \qquad \begin{array}{ccc} F_2\phi & \longrightarrow & F_2\psi \\ \downarrow & & \downarrow \\ F_1\phi & \longrightarrow & F_1\psi. \end{array}$$

We now need to prove that these squares exist. Let $\phi \to \psi$ be an arrow in $\mathcal{C\ell}(T_1)$, so $\phi \vdash_T \psi$. Since F_1 and F_2 are translations, the following sequents are provable in T_2:

$$F_1\phi \vdash_{T_2} F_1\psi \quad F_2\phi \vdash_{T_2} F_2\psi.$$

Since F_1 and F_2 are provably equivalent, the following sequents are provable in T_2:

$$F_1\phi \vdash_{T_2} F_2\phi \quad F_2\phi \vdash_{T_2} F_1\phi$$
$$F_1\psi \vdash_{T_2} F_2\psi \quad F_2\psi \vdash_{T_2} F_1\psi.$$

These six sequents define six arrows in $\mathcal{C\ell}(T_2)$. These six arrows can be arranged to make the sides in two commuting squares above. Thus, $\mathcal{C\ell}(F_1)$ and $\mathcal{C\ell}(F_2)$ are naturally isomorphic.

Conversely, assume that $\mathcal{C\ell}(F_1)$ and $\mathcal{C\ell}(F_2)$ are naturally isomorphic. Then for any arrow $\phi \to \psi$ in $\mathcal{C\ell}(T_1)$ we can find two commuting squares of the form shown above. Let ϕ be an arbitrary derived proposition in Prop(T_1), i.e., an arbitrary object of $\mathcal{C\ell}(T_1)$. If we apply this to the identity arrow $\phi \to \phi$, then we obtain arrows $F_1\phi \to F_2\phi$ and $F_2\phi \to F_1\phi$ (these are the vertical sides of the two squares). Unpacking the meaning of an arrow in $\mathcal{C\ell}(T_2)$, this means

$$F_1\phi \vdash_{T_2} F_2\phi \quad F_2\phi \vdash_{T_2} F_1\phi.$$

Since ϕ is arbitrary, this means F_1 and F_2 are provably equivalent. □

Recall that two translations $F : T_1 \to T_2$ and $G : T_2 \to T_1$ define an equivalence of theories in the case that GF and FG are provably equivalent to id_{T_1} and id_{T_2}. The above proposition determines when two translations are equivalent in terms of their associated functors. This gives us the following corollary.

Corollary 1.13. *Let $F : T_1 \to T_2$ and $G : T_2 \to T_1$ be a pair of translations. F and G define an equivalence of theories if and only if $\mathcal{C\ell}(F)$ and $\mathcal{C\ell}(G)$ define an equivalence of categories*

$$\mathcal{C\ell}(F) : \mathcal{C\ell}(T_1) \to \mathcal{C\ell}(T_2)$$
$$\mathcal{C\ell}(G) : \mathcal{C\ell}(T_2) \to \mathcal{C\ell}(T_1).$$

We can actually strengthen the above result on translations and fully identify which functors $\mathcal{C\ell}(T_1) \to \mathcal{C\ell}(T_2)$ come from translations $T_1 \to T_2$. This requires a finer description of the structure of $\mathcal{C\ell}(T)$, which we will get to later. Let's just state the result now.

Proposition 1.14. *Let $F : T_1 \to T_2$ be a translation. Then $\mathcal{C}\ell(F) : \mathcal{C}\ell(T_1) \to \mathcal{C}\ell(T_2)$ satisfies the following properties.*

- *$\mathcal{C}\ell(F)$ preserves finite limits (i.e., infima);*
- *$\mathcal{C}\ell(F)$ preserves finite colimits (i.e., suprema);*
- *$\mathcal{C}\ell(F)$ preserves exponentials (i.e., $\Rightarrow$).*

Furthermore, any functor $\mathcal{C}\ell(T_1) \to \mathcal{C}\ell(T_2)$ satisfying these properties is described by a translation $T_1 \to T_2$.

Interpretations

Perhaps what is more surprising is that interpretations of a propositional theory T into a boolean algebra B can also be described as a functor. A boolean algebra has a canonical partial order associated to it: for any elements a, b in a boolean algebra B, we write $a \leqslant b$ in the case that $a \wedge b = a$. A partial order is a special kind of preorder, so we can refer to the classifying category of a boolean algebra.

Definition 1.15. Given a boolean algebra B, the **classifying category** of B is the classifying category of the canonical partial order on B:

$$\mathcal{C}\ell(B) \stackrel{\text{def}}{=} \mathcal{C}\ell(B, \leqslant).$$

Similar to the story with translations, any boolean algebra homomorphism $F : B_1 \to B_2$ defines a functor $\mathcal{C}\ell(F) : \mathcal{C}\ell(B_1) \to \mathcal{C}\ell(B_2)$, since the set of objects of $\mathcal{C}\ell(B)$ are the elements of B.

On the other hand, models are also functors. Given a propositional theory T and a B-valued model $[\![\cdot]\!]^M : T \to B$, we obtain a functor $M : \mathcal{C}\ell(T) \to \mathcal{C}\ell(B)$ by setting

$$M(\phi : \mathcal{C}\ell(T)) \stackrel{\text{def}}{=} [\![\phi : \mathrm{Prop}(T)]\!]^M.$$

Proposition 1.16. *M is indeed a functor from $\mathcal{C}\ell(T)$ to $\mathcal{C}\ell(B)$.*

Proof. We need to verify that M sends arrows of $\mathcal{C}\ell(T)$ to arrows of $\mathcal{C}\ell(B)$. Let $\phi \to \psi$ be an arrow in $\mathcal{C}\ell(T)$. Since $[\![\cdot]\!]^M$ is a model, this implies

$$[\![\phi]\!]^M \leqslant [\![\psi]\!]^M.$$

Therefore, there exists an arrow $[\![\phi]\!]^M \to [\![\psi]\!]^M$ in $\mathcal{C}\ell(B)$. Since $M(\phi) = [\![\phi]\!]^M$, this is an arrow $M(\phi) \to M(\psi)$, as desired. $\square$

Given a translation $F : T_1 \to T_2$, recall we obtained a "pull-back" operation

$$F^* : \mathrm{Mod}(T_2, B) \to \mathrm{Mod}(T_1, B),$$

where $\mathrm{Mod}(T, B)$ denotes the set of B-valued models of the theory T. From the perspective of the classifying categories, this operation is just composition.

Theorem 1.17. *Let $F : T_1 \to T_2$ be a translation between propositional theories. Let $[\![\cdot]\!]^M : T_2 \to B$ be a B-valued model of T_2. Then $[\![\cdot]\!]^{F^*M} : T_1 \to B$ is the pulled-back model of T_1 under F. From these we obtain* *functors*

$$\mathcal{C}\ell(F) : \mathcal{C}\ell(T_1) \to \mathcal{C}\ell(T_2)$$
$$M : \mathcal{C}\ell(T_2) \to \mathcal{C}\ell(B)$$
$$F^*M : \mathcal{C}\ell(T_1) \to \mathcal{C}\ell(B).$$

These functors fit into a commuting diagram

$$\begin{tikzcd}
\mathcal{C}\ell(T_2) \arrow[r, "M"] & \mathcal{C}\ell(B) \\
\mathcal{C}\ell(T_1). \arrow[u, "\mathcal{C}\ell(F)"] \arrow[ur, dashed, "F^*M"'] &
\end{tikzcd}$$

In other words, $F^*M = M \circ \mathcal{C}\ell(F)$.

The key observation behind all of the proofs so far is that the classifying category of a theory is almost a copy of the theory. We will see later a construction, called the *internal logic* of a category, which will recover the original theory from its classifying category.

2 The Lindenbaum Algebra

The Lindenbaum algebra is another algebraic model of a propositional theory. We saw that, for the purposes of understanding the semantics of a theory, we only need equivalence classes of translations up to homotopy/provable equivalence. This was another point in a pattern of observations that the syntax of propositional logic hides certain symmetries which appear from the introduction and elimination rules. The Lindenbaum algebra of a propositional theory is a quotient

$$\text{Propositions}\big/_{\text{Proof theory}}.$$

This is the simplest form of an *equational theory*. Traditionally, the algebraic semantics was given by a boolean algebra of propositions modulo the double-sided implication $\leftrightarrow$.

Definition 2.1. Let T be a classical propositional theory. The **Lindenbaum algebra** is the boolean algebra $(L(T), \leqslant)$ of equivalence classes of propositions $[\phi]$, where two propositions ϕ and ϕ' are in the same class in the case

$$\phi \vdash_T \phi' \text{ and } \phi' \vdash_T \phi.$$

The ordering $\leqslant$ is given by

$$[\phi] \leqslant [\psi] \overset{\text{def}}{\equiv} \phi \vdash_T \psi.$$

There are a few nice properties of the Lindenbaum algebra, which highlights the framework in [Halvorson, 2019].

Proposition 2.2. *Let T be a propositional theory. The mapping $\phi \mapsto [\phi]$ defines a $L(T)$-valued model of T. This mapping is called the **canonical interpretation** of T in $L(T)$.*

Proposition 2.3. *There is a bijective correspondence between provable-equivalence classes of translations $F : T_1 \to T_2$ and boolean algebra homomorphisms $L(T_1) \to L(T_2)$, fitting into a square, where the vertical arrows* *are the canonical interpretations.*

$$\begin{tikzcd} T_1 \arrow[r, "F"] \arrow[d] & T_2 \arrow[d] \\ L(T_1) \arrow[r, dashed] & L(T_2). \end{tikzcd}$$

Proposition 2.4. *Let T be a propositional theory and B a boolean algebra. There is a bijective correspondence between B-valued models of T and boolean algebra homomorphisms $L(T) \to B$, related by the canonical interpretation*

$$\begin{tikzcd} L(T) \arrow[r, dashed] & B \\ T. \arrow[u] \arrow[ur] & \end{tikzcd}$$

For classical propositional logic, the Lindenbaum algebra is sufficient. However, our goal is to apply categorical logic to programming languages, where the proof theory is not so simple. I mentioned that the Lindenbaum algebra $L(T)$ is the quotient of T by $\leftrightarrow$. Under the Curry-Howard interpretation, this shows that the Lindenbaum algebra $L(T)$ identifies isomorphic types!

For this reason, it will be helpful to take a step backwards and permit ourselves to provide our own equational theory. This will require more complicatd algebra, but it will build a system which is amenable to the Curry-Howard correspondence.

Preorders as Categories

Let X be a set. A **preorder** on X is a reflexive and transitive relation $\_ \leqslant \_ \subseteq X \times X$. For example, any partial order is a preorder (but not all preorders are partial orders!). We will show that preordered sets form a special class of categories.

Exercise 1 : Let $(X, \leqslant)$ be a preordered set. Define a category $\mathcal{C}\ell(X, \leqslant)$ in the following manner.

- The objects of $\mathcal{C}\ell(X, \leqslant)$ are the elements of X.
- There is precisely one arrow $x \to y$ in $\mathcal{C}\ell(X, \leqslant)$ in the case that $x \leqslant y$.

Verify that $\mathcal{C}\ell(X, \leqslant)$ is a category. Can any of the hypotheses on the relation $\leqslant$ be relaxed while still ensuring that $\mathcal{C}\ell(X, \leqslant)$ is a category?

The category $\mathcal{C}\ell(X, \leqslant)$ is called the **classifying category** of the preorder $(X, \leqslant)$.

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Exercise 2 :

- Unpack what it means for two objects x, y of $\mathcal{C}\ell(X, \leqslant)$ to be isomorphic, in terms of the preordered set $(X, \leqslant)$.
- Unpack what is a functor $\mathcal{C}\ell(X, \leqslant) \to \mathcal{C}\ell(Y, \leqslant)$, in terms of a function between preordered sets $(X, \leqslant) \to (Y, \leqslant)$.
- The opposite category $\mathcal{C}\ell(X, \leqslant)^{\mathsf{op}}$ is the classifying category of another preordered set $(X', \leqslant)$. What is this preordered set?

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Exercise 3 : Let $\Omega \stackrel{\text{def}}{=} \{\mathsf{False}, \mathsf{True}\}$ with the usual preordering $\mathsf{False} \leqslant \mathsf{True}$. To keep things clean, let's also write Ω for $\mathcal{C}\ell(\Omega, \leqslant)$. Given an object x of $\mathcal{C}\ell(X, \leqslant)$, we can write a functor $\downarrow x : \mathcal{C}\ell(X, \leqslant)^{\mathsf{op}} \to \Omega$):

$$\downarrow x(y) \stackrel{\text{def}}{=} \begin{cases} \mathsf{True} & \text{if } y \leqslant x \\ \mathsf{False} & \text{otherwise.} \end{cases}$$

Write $\downarrow x \stackrel{\text{nat}}{\leqslant} \downarrow y$ in the case that, for any object z of $\mathcal{C}\ell(X, \leqslant)$, $\downarrow x(z) \leqslant \downarrow y(z)$. Show for any pair of elements $x, y \in X$,

$$x \leqslant y \text{ if and only if } \downarrow x \stackrel{\text{nat}}{\leqslant} \downarrow y.$$

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Monads for Preorders (Extra)

Given two categories C, D and a functor $F : C \to D$, a **right adjoint** to F is a functor $G : D \to C$ satisfying the following universal property.

$$\frac{f : FX \to Y}{\overline{f^\dagger : X \to GY.}}$$

We read this as: *for any construction of an arrow from FX to Y in D, there exists a unique dual construction of an arrow form X to GY in C.* In the case where C and D are the classifying categories of preordered sets, $C = \mathcal{C}\ell(X, \leqslant)$ and $D = \mathcal{C}\ell(Y, \leqslant)$, then there is only one way to make an arrow from FX to Y and from X to GY. Then the above universal property reduces to

$$\frac{FX \leqslant Y}{\overline{X \leqslant GY}}$$

This pair $F \dashv G$ is called a **Galois connection**.

Exercise 4 : Given a Galois connection $F \dashv G$, where $F : \mathcal{C}\ell(X, \leqslant) \to \mathcal{C}\ell(Y, \leqslant)$ and $G : \mathcal{C}\ell(Y, \leqslant) \to \mathcal{C}\ell(X, \leqslant)$, we can apply G after F to obtain a "loop":

$$GF : \mathcal{C}\ell(X, \leqslant) \to \mathcal{C}\ell(X, \leqslant).$$

Show that GF is a **closure operator**, i.e., for any object x of $\mathcal{C}\ell(X, \leqslant)$, $x \leqslant GFx$ and $GF(GFx) \leqslant GFx$.

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Exercise 5 : We can reverse the previous exercise. Apply F after G to obtain another "loop":

$$FG : \mathcal{C}\ell(Y, \leqslant) \to \mathcal{C}\ell(Y, \leqslant).$$

Show that FG is an **interior operator**, i.e., for any object y of $\mathcal{C}\ell(Y, \leqslant)$, $FGy \leqslant y$ and $FGy \leqslant FG(FGy)$.

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Here is an example of a closure operator which will appear in different forms later. Let $(X, \leqslant)$ be a preorder, as before. We saw in Exercise 3 that we can "embed" the elements of X into maps $\mathcal{C}\ell(X, \leqslant)^{\mathsf{op}} \to \Omega$. In fact, we can make the set of all such maps into a preorder using $\overset{\text{nat}}{\leqslant}$. Given a pair of maps $F, F' : \mathcal{C}\ell(X, \leqslant)^{\mathsf{op}} \to \Omega$, write $F \overset{\text{nat}}{\leqslant} F'$ in the case that, for any element $x \in X$, we have

$$Fx \leqslant Fx'.$$

A **Lawvere-Tierney topology** on $(X, \leqslant)$ is a closure operator J on the set of maps $\mathcal{C}\ell(X, \leqslant)^{\mathsf{op}} \to \Omega$ satisfying:

- J is **idempotent**: $J(JF) \overset{\text{nat}}{\leqslant} J(F)$ (and $J(F) \overset{\text{nat}}{\leqslant} J(J(F))$.
- J preserves infima.

3 Preorders are Indexed Propositions

Let $(X, \leqslant)$ be a preorder. Then we can form the classifying category $\mathcal{C}\ell(X, \leqslant)$ as before. We also defined $\Omega \stackrel{\text{def}}{=} \{\mathsf{True}, \mathsf{False}\}$. This is the preorder of *propositions*. It is essentialy the preorder of booleans `bool`. However, for foundational reasons, it is useful to think of Ω has having slightly less structure than the full boolean operations. In the first warmup, we constructed a functor $\downarrow x : \mathcal{C}\ell(X, \leqslant)^{\mathsf{op}} \to \Omega$ by forming the *downset*

$$\downarrow x(y) \stackrel{\text{def}}{=} \begin{cases} \mathsf{True} & \text{if } y \leqslant x \\ \mathsf{False} & \text{otherwise.} \end{cases}$$

The downset construction $x \mapsto \downarrow x$ allows us to reason about the abstract preorder $(X, \leqslant)$ as if it were a collection of propositions, indexed by the elements of X.

Definition 3.1. Given a preorder $(X, \leqslant)$, a **downward closed set** is a subset $A \subseteq X$ satisfying the following closure property:

$$\frac{x \leqslant y}{y \in A \Rightarrow x \in A.}$$

There is another way to represent downward closed sets, in terms of functions $X \to \Omega$. Given a subset $A \subseteq X$, we can form the *membership function* $\chi_A : X \to \Omega$:

$$\chi_A(x) \stackrel{\text{def}}{=} \begin{cases} \mathsf{True} & x \in A, \\ \mathsf{False} & \text{otherwise.} \end{cases}$$

Conversely, any function $\varphi : X \to \Omega$ defines a subset $\varphi^{-1}(\mathsf{True}) \stackrel{\text{def}}{=} \{x \in X \mid \varphi(x) = \mathsf{True}\}$.

Proposition 3.2. *A subset $A \subseteq X$ is downward closed if and only if its membership function $\chi_A : X \to \Omega$ is order-reversing. Similarly, a function $\varphi : X \to \Omega$ is order-reversing if and only if its corresponding subset $\varphi^{-1}(\mathsf{True})$ is downward closed.*

Proof. Consider a subset $A \subseteq X$. Suppose that A is downward closed, i.e.,

$$\frac{x \leqslant y}{y \in A \Rightarrow x \in A.}$$

Note $y \in A$ if and only if $\chi_A(y) = \mathsf{True}$. Therefore the above judgment is equivalent to

$$\frac{x \leqslant y}{\chi_A(y) = \mathsf{True} \Rightarrow \chi_A(x) = \mathsf{True}\,.}$$

This is the statement that χ_A is order-reversing. Conversely, if χ_A is assumed to be order-reversing, then the equivalence of the above two judgments implies A is downward closed.

The result for functions $\varphi : X \to \Omega$ follows a similar argument. □

The above proposition demonstrates a bijection between the following sets of structures

$$\begin{array}{ccc} \{\text{Downward-closed subsets } A \subseteq X\} & \cong & \{\text{Order-reversing functions } X \to \Omega\} \\ A & \mapsto & \chi_A \\ \varphi^{-1}(\mathsf{True}) & \leftarrow\!\shortmid & \varphi \end{array}$$

Recall that an order-preserving function $(X, \leqslant) \to (Y, \leqslant)$ is equialent to the data of a functor between classifying categories $\mathcal{C}\ell(X, \leqslant) \to \mathcal{C}\ell(Y, \leqslant)$. Furthermore, an order-reversing function $(X, \leqslant) \to (Y, \leqslant)$ is the same data as an order-*preserving* function—provided the preorder on X is replaced with the reverse preorder $\geqslant$. In other words, we have a chain of bijections

$$\begin{aligned} \{\text{Order-reversing functions } (X, \leqslant) \to (\Omega, \leqslant)\} &\cong \{\text{Order-preserving functions } (X, \geqslant) \to (\Omega, \leqslant)\} \\ &\cong \left\{\text{Functors } \mathcal{C}\ell(X, \leqslant)^{\mathsf{op}} \to \Omega\right\}. \end{aligned}$$

Combining this with the preceding proposition on downward closed sets, we have obtained the following theorem.

Theorem 3.3 (Grothendieck construction for preorders)**.** *There is a bijective correspondence*

$$\{\textit{Downward-closed subsets } A \subseteq X\} \cong \left\{\textit{Functors } \mathcal{C}\ell(X, \leqslant)^{\mathsf{op}} \to \Omega\right\}$$

given by sending a downward closed subset A to the functor associated to χ_A and by sending a functor $F : \mathcal{C}\ell(X, \leqslant)^{\mathsf{op}} \to \Omega$ to the set of objects $x \in X$ such that $F(x) = \mathsf{True}$.

Remark 3.1

The above theorem allows us to make our first abuse of notation! In these notes we will now identify the functor $\downarrow x : \mathcal{C}\ell(X, \leqslant)^{\mathsf{op}} \to \Omega$ with its associated downward closed set

$$\{y \in X \mid \downarrow x(y) = \mathsf{True}\}.$$

In retrospect, it would have been better to work entirely with downward-closed sets first, and introducing functors later. In the sequel, where we repeat the above with presheaves, it will be cleaner to avoid this abuse of notation.

Downward closed sets inherit a partial order given by inclusion of subsets. This allows us to talk about the **partial order of downward closed sets**

$$(\mathrm{Dwd}(X), \subseteq).$$

The downset construction $x \mapsto \downarrow x$ is an embedding $X \to \mathrm{Dwd}(X)$ with some nice properties.

Lemma 3.4 (Yoneda lemma for preorders)**.** *The operation $\downarrow : X \to \mathrm{Dwd}(X)$ is fully faithful, i.e., $x \leqslant y$ if and only if $\downarrow x \subseteq \downarrow y$. In particular, $\downarrow x = \downarrow y$ if and only if $x \leqslant y$ and $y \leqslant x$. Furthermore, for any downward closed set $A \in \mathrm{Dwd}(X)$,*

$$x \in A \textit{ if and only if } \downarrow x \subseteq A.$$

Proof. The first part, that $\downarrow$ is fully faithful, was Exercise 3 of the warmup. The second part, that $x \in A$

if and only if $\downarrow x \subseteq A$ follows a similar proof.

Assume $x \in A$. The downset associated to $\downarrow x$ is the collection of all elements $y \in X$ such that $y \leqslant x$. Thus,

$$\dfrac{x \in A \qquad \dfrac{y \in \downarrow x}{y \leqslant x}\ \text{Defn. } \downarrow x}{y \in A.}\ A \in \mathrm{Dwd}(X)$$

Since y is arbitrary, this shows $\downarrow x \subseteq A$, as desired. Conversely, assume $\downarrow x \subseteq A$. By reflexivity, $x \leqslant x$, so $x \in \downarrow x$. In particular, $x \in A$, as desired. □

Remark 3.5. *Exercise 3 and the above lemma are proven using the same technique. The judgment $\downarrow x \subseteq A$ relates arrows $y \to x$ to elements of A. Among all arrows $y \to x$, there is a unique "universal" choice, namely the identity $\mathsf{id}_x : x \to x$ (universal in the sense that it always exists). In similar situations, where one is quantifying over arrows in a category, picking the identity is often a decisive trick.*

The Grothendieck construction for preorders and the Yoneda lemma allow us to study a preorder $(X, \leqslant)$ through its downward closed sets $(\mathrm{Dwd}(X), \subseteq)$. In this sense, the only preorder relation is the subset inclusion relation $\subseteq$: all others are abstracted "substructures" of a preorder of subsets, if we pay the price of treating a pair of elements $x, y \in X$ as "equivalent" if $x \leqslant y$ and $y \leqslant x$. This price is often worth paying, since $\mathrm{Dwd}(X)$ has more structure than X.

Observation 3.6. $\mathrm{Dwd}(X)$ *is a complete Heyting algebra.*

Since $\mathrm{Dwd}(X)$ is a complete Heyting algebra, we can freely take unions and intersections of downward-closed sets. This will be the basis of the following section.

4 Limits and Colimits of Preorders

In the previous section, we saw that we can study a preorder $(X, \leqslant)$ using its collection of downward closed sets $\mathrm{Dwd}(X)$. This was justified by the embedding $\downarrow : X \to \mathrm{Dwd}(X)$. Furthermore, $\mathrm{Dwd}(X)$ is a complete Heyting algebra. We can use this Heyting algebra structure to define operations in X. For example, we can form intersections.

Definition 4.1. Let $x, y \in X$ be a pair of elements. A **conjunction** of x, y is an element $z \in X$ which satisfies the equation of downward-closed subsets.

$$\downarrow z = \downarrow x \cap \downarrow y.$$

In other words, a conjunction in X of x and y is an element which behaves like the intersection of x and y. Since the embedding $\downarrow : X \to \mathrm{Dwd}(X)$ is fully faithful, this can be formalized in the manner above. Another way to describe a conjunction is via a *universal property*.

Proposition 4.2. *Let $x, y \in X$ be a pair of elements. An element $z \in X$ is a conjunction of x and y if and only if, for any element $w \in X$,*

$$w \leqslant z \text{ if and only if } w \leqslant x \text{ and } w \leqslant y.$$

In category theory, we usual write universal properties like the one in the above proposition as a diagram. In this case, the diagram depicts arrows in the classifying category $\mathcal{Cl}(X, \leqslant)$. Here is the diagram for a binary conjunction.

$$\begin{tikzcd}
w \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& z \arrow[r] \arrow[d] & x \\
& y &
\end{tikzcd}$$

Every arrow in this diagram, being an arrow in $\mathcal{Cl}(X, \leqslant)$ represents a comparison using $\leqslant$. The solid arrows represent the hypotheses of the universal property, and the dashed line represents the conclusion. In this case, the diagram translates to

Given $z \leqslant x$ and $z \leqslant y$. If $w \leqslant x$ and $w \leqslant y$, then $w \leqslant z$.

I have also added the symbol $\exists!$ to the diagram. This is not relevant for preorders, because there is always at most one arrow between any two objects of $\mathcal{Cl}(X, \leqslant)$. However, in general category theory, universal properties almost always require the dashed line to be unique.

Proof of Proposition 4.2. Assume z is a conjunction of x and y. We need to show, for any element $w \in X$, $w \leqslant z$ if and only if $w \leqslant x$ and $w \leqslant y$. We now chase a chain of iffs

$$\begin{aligned}
w \leqslant z &\text{ iff } w \in {\downarrow} z \\
&\text{ iff } w \in {\downarrow} x \cap {\downarrow} y \\
&\text{ iff } w \in {\downarrow} x \text{ and } w \in {\downarrow} y \\
&\text{ iff } w \leqslant x \text{ and } w \leqslant y.
\end{aligned}$$

Here the second iff holds because z is a conjunction of x and y. The converse direction is a similar argument. □

Remark 4.3. *In the above, z is called* a *conjunction of x and y rather than* the *conjunction of x and y. This is because conjunctions are technically not unique—but they are unique* up to isomorphism*. Suppose z is a conjunction of x and y, and suppose that there exists an element $z' \in X$ such that $z' \leqslant z$ and $z \leqslant z'$. (These conditions are equivalent to the statement that z and z' are isomorphic in $\mathcal{Cl}(X, \leqslant)$.) Then ${\downarrow} z' = {\downarrow} z$, so z' is also a conjunction of x and y. In practice, identifying isomorphic objects is manageable, so we usually give the isomorphism class of conjunctions of x and y a name*

$$z \cong x \wedge y.$$

Furthermore, it is common in category theory to write $x \wedge y$ instead of a explicit representative of the conjunction.

We now consider two examples.

Example 4.4 (A preorder of subsets)**.** Suppose $(X, \leqslant)$ is a preorder of subsets $(X, \leqslant) = (\mathcal{P}(V), \subseteq)$. Consider a pair of subsets $A, B \in \mathcal{P}(V)$. A conjunction $A \wedge B$ of A and B satisfies the universal property:

$$\forall W \subseteq V, W \subseteq A \wedge B \Leftrightarrow W \subseteq A \text{ and } W \subseteq B.$$

If we pick singleton subsets $\{v\}$ for W (so v is some element of V), then the above universal property implies

$$\forall v \in V, v \in A \wedge B \Leftrightarrow v \in A \text{ and } v \in B.$$

This sentence uniquely characterizes the intersection $A \cap B$. Therefore

$$A \wedge B \cong A \cap B.$$

(Furthermore, since $\subseteq$ is an antisymmetric relation, this immediately implies $A \wedge B = A \cap B$.) Power sets are the nicest type of preorder: they have a generating set of singleton sets, also called *atoms*, which can distinguish any pair of objects in the power set. This property is called *well-pointedness*. For this reason, we could be silly and say that a power set $\mathcal{P}(V)$ is an example of a *atomic 0-topos with enough points*. This probably doesn't help to understand preorders, but it can be useful to intuit how an atomic topos generalizes the power-set situation!

Example 4.5 (A propositional theory)**.** Let $T = (\Sigma, \Delta)$ be a propositional theory. We constructed the classifying category of T, $\mathcal{C}\ell(T)$, whose objects are propositions ϕ in T and whose arrows $\phi \to \psi$ correspond to sequents $\phi \vdash_T \psi$. Given two propositions ϕ and ψ of T, we can ask for their conjunction. This conjunction is a proposition, P such that, for any other proposition Q,

$$Q \vdash_T P \Leftrightarrow Q \vdash_T \phi \text{ and } Q \vdash_T \psi.$$

The latter condition, $Q \vdash_T \phi$ and $Q \vdash_T \psi$ is equivalent to $Q \vdash_T \phi \wedge \psi$. Thus, a conjunction P satisfies

$$Q \vdash_T P \Leftrightarrow Q \vdash_T \phi \wedge \psi.$$

This implies $P \vdash_T \phi \wedge \psi$ and $\phi \wedge \psi \vdash_T P$. Thus $P \cong \phi \wedge \psi$. In other words, the conjunctions in $\mathcal{C}\ell(T)$ are provably equivalent to the usual conjunctions from propositional logic. This observation justifies our use of the symbol $\wedge$ to denote conjunction. It also an example of the motivation of categorical logic: *features in a language correspond to universal properties of the classifying category.*

We defined conjunctions by an equation of downsets. Nothing is stopping us from picking other equations in order to study other universal properties. For example, we could ask for larger conjunctions. Given a family x_α of elements of X, a conjunction $\bigwedge_\alpha x_\alpha$ is an element $z \in X$ satisfying the equation

$$\downarrow z = \bigcap_\alpha \downarrow x_\alpha.$$

From this we can derive the universal property like before. Instead we move ahead to the general picture.

Definition 4.6. Let $(X, \leqslant)$ be a preordered set. A **diagram** is an order-preserving map $J : (D, \leqslant) \to (X, \leqslant)$. In other words, a diagram is a functor $J : \mathcal{C}\ell(D, \leqslant) \to \mathcal{C}\ell(X, \leqslant)$.

To every diagram $J : (D, \leqslant) \to (X, \leqslant)$, we can create a downward closed set $\downarrow J$ in $\mathrm{Dwd}(X)$:

$$\downarrow J \overset{\text{def}}{=} \bigcap_{i \in D} \downarrow (J(i)).$$

Thus, $x \in \downarrow J$ if and only if $x \leqslant J(i)$ for every $i \in D$.

Definition 4.7. Given a diagram $J : (D, \leqslant) \to (X, \leqslant)$, a **limit** of J is an element $z \in X$ such that

$$\downarrow z = \downarrow J$$

as elements of $\mathrm{Dwd}(X)$. Similar to how conjunctions were written using $\wedge$, e.g., "$x \wedge y$", we write for limits

$$z \cong \varprojlim_{i \in D} J(i) \text{ or } \cong \varprojlim J.$$

There are two equivalent ways to describe limits in a preorder. First, we can describe a limit by its membership function

Observation 4.8. *Given a diagram $J : (D, \leqslant) \to (X, \leqslant)$ and an element $z \in X$, z is a limit for J if and only if, for any element $x \in X$,*

$$\downarrow z(x) = \bigwedge_{i \in D} \downarrow J(i)(x).$$

Proposition 4.9. *In a preorder $(X, \leqslant)$, limits are infima. Formally, given a diagram $J : (D, \leqslant) \to (X, \leqslant)$,*

$$\varprojlim J \cong \inf \{J(i) \in X \mid i \in D\}.$$

Proof. The universal property of the infimum on the right is: for any element $x \in X$,

$$x \leqslant \inf \{J(i) \in X \mid i \in D\} \text{ iff } \bigwedge_{i \in D} x \leqslant J(i).$$

By the preceding observation this is the same universal property as the limit $\varprojlim J$. □

> The reason that limits are the same as infima for preorders is because the classifying category $\mathcal{C}\ell(X, \leqslant)$ of a preorder is *thin*. That is, between any two objects x, y of $\mathcal{C}\ell(X, \leqslant)$, there is at most one arrow $x \to y$. More generally, we need to impose certain commutativity conditions which we will explore later.

There is one special diagram to explore. The empty set $\emptyset$ has a (unique) structure of a preordered set. Therefore, we may consider the empty map $J_\emptyset : \emptyset \to (X, \leqslant)$ as a diagram, vacuously. We can ask if X has any limits for this diagram. A limit for $J_\emptyset$ is an element $z \in X$ such that

$$\downarrow z = \bigcap_{\_ \in \emptyset} \downarrow J(\_) = X,$$

since empty intersections yield the entire preorder. This can also be seen by comparing membership functions: $x \in \downarrow z$ if and only if, for any $\_$ in $\emptyset$, $x \in \downarrow J(\_)$. Since the latter condition is always vacuously satisfied, $x \in \downarrow z$ is always satisfied, so $\downarrow z = X$.

Definition 4.10. A **terminal object** is a limit of the empty diagram. For a preorder X, a terminal object is the same as a maximum for X.

The reverse situation

We have related limits to infima. We now describe the category theory for suprema. These are called *colimits*. Their definition is more indirect—at least from the downset perspective. We turn to Observation 4.8, which describes the downset of a limit as a large conjunction. We can swap the role of z and x on the right side of that equation. This yields a colimit.

Definition 4.11. Let $J : (D, \leqslant) \to (X, \leqslant)$ be a diagram. A **colimit** of J is an object $z \in X$ such that, for any element $x \in X$,

$$\downarrow x(z) = \bigwedge_{i \in D} \downarrow J(i)(x).$$

We write a colimit using similar notation as the limit:

$$z \cong \varinjlim_{i \in D} J(i) \cong \varinjlim J.$$

For emphasis, below is the equation characterizing a limit of the diagram J, where the difference to the colimit equation highlighted.

$$\downarrow z(x) = \bigwedge_{i \in D} \downarrow J(i)(x).$$

Proposition 4.12. *Given a diagram $J : (D, \leqslant) \to (X, \leqslant)$, z is a colimit for J if and only if, for any $x \in X$,*

$$z \leqslant x \text{ iff } \forall i \in D, J(i) \leqslant x.$$

In other words, $z \cong \sup \{J(i) \mid i \in D\}$, or in terms of downsets

$$\downarrow z = \bigvee_{i \in D} \downarrow J(i).$$

Similar to the conjunctions from earlier, we can form disjunctions using a colimit.

Definition 4.13. Let $(X, \leqslant)$ be a preorded set, and let x, y be a pair of elements of X. Let $\{a, b\}$ denote the preordered set with no relation imposed on a and b. The function $J : (a, b) \to (X, \leqslant)$ sending a to x and b to y is order-preserving, so it is a diagram. A **disjunction** for x and y is a colimit for this diagram. That is, an element $x \vee y$ is a disjunction for x and y if and only if, for any element $v \in X$,

$$x \vee y \leqslant v \text{ iff } x \leqslant v \text{ and } y \leqslant v.$$

A disjunction's universal property can be expressed diagramtically, like for conjunctions (or any limit or colimit, for that matter). Here is the diagram for the disjunction of x and y.

$$\begin{tikzcd}
& x \arrow[d] \arrow[ddr, bend left] & \\
y \arrow[r] \arrow[drr, bend right] & x \vee y \arrow[dr, dashed, "\exists!"] & \\
& & v
\end{tikzcd}$$

From the diagram we can see an important relationship between limits and colimits. If we replace the order $\leqslant$ on X with its reverse $\geqslant$, then the ensuing classifying category $\mathcal{C}\ell(X, \geqslant)$ is the opposite category of $\mathcal{C}\ell(X, \leqslant)$, also called $\mathcal{C}\ell(X, \leqslant)^{\mathsf{op}}$. All the arrows in this category have flipped direction. If we flip the arrows in the above diagram, we get the same picture for a conjunction, just rotated 180 degrees (which does not change the universal property!). In other words:

Observation 4.14. *A disjunction in $(X, \leqslant)$ is the same as a conjunction in $(X, \geqslant)$.*

More generally, a colimit of $\mathcal{C}\ell(X, \leqslant)$ is the same as a limit of $\mathcal{C}\ell(X, \leqslant)^{\mathsf{op}}$. This arrow-reversing relationship is the reason x and z are swapped in Definition 4.11.

We wrap things up by describing the colimit analogue of a terminal object.

Definition 4.15. Let $(X, \leqslant)$ be a preordered set. An **initial object** of X is a colimit for the empty diagram $J_\emptyset : \emptyset \to (X, \leqslant)$.

Equivalently, an initial object is an element $z \in X$ such that, for any $x \in X$,

$$z \leqslant x \text{ iff } \forall \_ \in \emptyset, J_\emptyset(\_) \leqslant x.$$

The latter condition is always true, vacuously. Therefore, $z \leqslant x$ for every $x \in X$, so an initial object of a preorder is the same as a minimum.

Indexed propositions

The key takeaway from these constructions of limits and colimits is that they are described by operations on the downsets of elements of $(X, \leqslant)$. While X may not be a Heyting algebra, we can inherit the language of Heyting algebras by working in $\mathrm{Dwd}(X)$: the downset of a limit is an intersection of downsets, and the downset of a colimit is a union of downsets.

We may take an alternative but equivalent approach. Recall (from the Grothendieck construction for preorders) that a downward closed set is the same as a functor $\mathcal{C}\ell(X, \leqslant)^{\mathsf{op}} \to \Omega$, where Ω is the preorder of truth values. Under this correspondence, intersection of downward closed sets is identified with the operation

$$\mathsf{and} : \Omega \times \Omega \to \Omega,$$

and union of downward closed sets is identified with the dual

$$\mathtt{or} : \Omega \times \Omega \to \Omega.$$

For example, for any $v \in X$

$$\downarrow(x \wedge y)(v) = \downarrow x(v) \text{ and } \downarrow y(v).$$

Indeed, for any operation on propositions $\Omega \times \ldots \times \Omega \to \Omega$, we can create a universal property on downsets/indexed propositions to explore in a preorder.

Definition 4.16. Given an operation $\delta : \Omega^n \to \Omega$, this operation is **representable** in a preorder $(X, \leqslant)$ if, for any family of objects $x_1, \ldots, x_n$, there exists an object $z \in X$ such that for any $v \in X$

$$\downarrow z(v) = \delta(\downarrow x_1(v), \ldots, \downarrow x_n(v)).$$

Dually, δ is **co-representable** in $(X, \leqslant)$ if it is representable in $(X, \geqslant)$. Equivalently, δ is corepresentable if, for any family of objects $x_1, \ldots, x_n$, there exists an object $z \in X$ such that for any $v \in X$

$$\downarrow v(z) = \delta(\downarrow x_1(v), \ldots, \downarrow x_n(v)).$$

In this sense, a preorder has conjunctions if and is representable. This notion of representability will later allow us to describe the relation between a topos and a universe of sets. In this sense, a topos will be a category where many of the operations from set theory are either representable or co-representable.

5 Yoneda and Generalized Objects

Given a preorder $(X, \leqslant)$ the elements $x \in X$ are subsumed by their downsets $\downarrow x \in \mathrm{Dwd}(X)$. We saw this in the Yoneda lemma, where the preorder $\leqslant$ on X is modeled by the subset inclusion $\subseteq$ on $\mathrm{Dwd}(X)$. We also saw this in the previous section, where universal properties, such as infima and suprema, can be described in terms of unions and intersections of downward closed sets. Regarding the other downward-closed sets $A \in \mathrm{Dwd}(X)$, since they are downward-closed, they behave like elements in X, as far as the preordering relation is concerned. For this reason, downward-closed sets (like their categorified cousins, presheaves) are *generalized objects* of X. One way to witness this is through the *co-Yoneda lemma*.

Lemma 5.1 (Co-Yoneda Lemma for preorders)**.** *Let $A \in \mathrm{Dwd}(X)$ be a downward closed set. The membership function $\chi_A : \mathcal{C}\ell(X, \leqslant)^{\mathsf{op}} \to \Omega$ satisfies the equation*

$$\chi_A(y) = \bigvee_{x \in X} (\chi_A(x) \wedge \downarrow x(y)) = \bigvee_{x \in A} \downarrow x(y).$$

The co-Yoneda lemmma shows that any downward-closed set $A \in \mathrm{Dwd}(X)$ is a union of sets of the form $\downarrow x$, for objects $x \in X$. These equations come from the co-Yoneda lemma for presheaves.

$\bigvee_{x \in X} (\chi_A(x) \wedge \downarrow x)$	is a special case of	$\int^{x:X} \chi_A(x) \cdot よ(x).$
$\bigvee_{x \in A} \downarrow x$	is a special case of	$\varinjlim \left(\mathrm{el}(A) \to X \xrightarrow{よ} \mathrm{Psh}(X) \right).$

We will present two proofs. The first is a direct proof. The second follows the structure of the proof of the general Co-Yoneda lemma. Before the proofs, we discuss how the co-Yoneda lemma allows us to treat downward closed sets like generalized objects.

First proof. Let y be an arbitrary element of X. If $\chi_A(y) = \mathsf{True}$, then $\chi_A(y) \wedge \downarrow y(y)$ is also True. Therefore the right side of the equation is True. Conversely, suppose $\bigvee_{x \in X} (\chi_A(x) \wedge \downarrow x(y)) = \mathsf{True}$. Then there exists some $x_0 \in X$ such that $\chi_A(x_0) \wedge \downarrow x_0(y) = \mathsf{True}$. In particular, $x_0 \in A$ and $y \leqslant x_0$. Since A is downward closed, this implies $y \in A$, so $\chi_A(y) = \mathsf{True}$. Thus, the left side of the equation evaluates to True if and only if the right side of the equation evaluates to True, as desired. $\square$

Second proof. We begin by making a few observations. First, we can impose a preorder on order-reversing functions $f, g : (X, \leqslant) \to \Omega$ by defining

$$f \leqslant g \text{ if and only if } \forall x \in X. f(x) \leqslant g(x).$$

Secondly, given two order-reversing functions $f_1, f_2 : (X, \leqslant) \to \Omega$, $f_1 = f_2$ if and only if, for any other order-reversing function $g : (X, \leqslant) \to \Omega$, $f_1 \leqslant g$ if and only if $f_2 \leqslant g$.[1]Furthermore, given two order-reversing functions $f, g : (X, \leqslant) \to \Omega$, we may define a new function $f \Rightarrow g : (X, \leqslant) \to \Omega$ using the usual implication

$$(f \Rightarrow g)(x \in X) \stackrel{\text{def}}{=} f(x) \Rightarrow g(x) = \begin{cases} \mathsf{True} & f(x) \leqslant g(x) \\ \mathsf{False} & \text{otherwise.} \end{cases}$$

Lastly, we can combine the Yoneda lemma for preorders and the last observation to deduce $(\downarrow x \Rightarrow g)(y) = g(x)$ for any input $y \in X$. Based on these observations, in order to prove the desired equation

$$\chi_A(y) = \bigvee_{x \in X} (\chi_A(x) \wedge \downarrow x(y)),$$

it suffices to prove the following. For any order-reversing $g : X \to \Omega$,

$$\chi_A \leqslant g \text{ if and only if } \bigvee_{x \in X} (\chi_A(x) \wedge \downarrow x) \leqslant g.$$

We can now chase a chain of implications

$$\begin{aligned} & \bigvee_{x \in X} (\chi_A(x) \wedge \downarrow x) \leqslant g \\ \text{iff } & \bigwedge_{x \in X} ((\chi_A(x) \wedge \downarrow x) \leqslant g) \\ \text{iff } & \bigwedge_{x \in X} (\chi_A(x) \leqslant \downarrow x \Rightarrow g) \\ \text{iff } & \bigwedge_{x \in X} (\chi_A(x) \leqslant g(x)) \\ \text{iff } & \chi_A \leqslant g. \end{aligned}$$

□

[1]The second observation is the Yoneda lemma. We can use the same trick, supplying f_1 and f_2 for g, to prove it. Indeed, one could simply invoke the Yoneda lemma for preorders, because the class of order-preserving functions between two preordered sets is a set. When generalizing to presheaves, this is no longer the case, so one encounters the usual set-theoretic obstacles.

This time we return to [Fong and Spivak, 2019] to discuss a special kind of relation between preorders: *Galois connections*. These are the preorder-specific forms of an adjunction between categories.

6 Definition of a Galois connection

The setting of a Galois connection is a pair of preorders $(X, \leqslant)$ and $(Y, \leqslant)$. In what follows, I will write X for the pair $(X, \leqslant)$, etc., using the same symbol for the order-relation in any preorder.

Definition 6.1. A **Galois connection** between the preordered sets X and Y is a pair of monotone maps $f : X \to Y$ and $g : Y \to X$ satisfying the following property. For any pair of elements $x \in X$ and $y \in Y$,

$$\frac{f(x) \quad \leqslant y}{\overline{\quad x \quad \leqslant g(y).}}$$

In this case f is called the **left adjoint**, and g is called the **right adjoint**. We express the relationship between f and g by writing $f \dashv g$.

One powerful property of Galois connections (and adjunctions, more generally) is there preservation of limits and colimits.

Proposition 6.2. *Let $f : X \to Y$ and $g : Y \to X$ form a Galois connection $f \dashv g$ between preordered sets X and Y. Then f preserves all colimits (i.e., suprema), and g preserves all limits (i.e., infima).*

Remark 6.3. *A functor which preserves colimits is sometimes called a* right-exact functor*, and a functor which preserves limits is sometimes called a* left-exact functor*. Therefore the above proposition can be restated more mnemonically as:* if a functor has a right adjoint, then it is right exact, and if it has a left adjoint, then it is left exact.

Proof of Proposition 6.2. We will show that f preserves suprema. The proof that g preserves infima is formally dual. Consider a diagram $J : D \to X$. Assuming the colimit along J exists, we need to show

$$f\left(\bigvee_{i \in D} J(i)\right) \cong \bigvee_{i \in D} f(J(i)).$$

To that end, let x be an element in X representing $\bigvee_{i \in D} J(i)$. We will show that $f(x)$ is the least upper bound of the set $\{f(J(i)) \mid i \in D\}$. Since $J(i) \leqslant x$ for any $i \in D$, $f(J(i)) \leqslant f(x)$ since f is order-preserving. Therefore $f(x)$ is an upper bound. On the other hand, suppose $y \in Y$ is an element satisfying $f(J(i)) \leqslant y$ for any $i \in D$. Since f is left adjoint to g, this implies $J(i) \leqslant g(y)$ for any $i \in D$. Since x is a least upper bound of $\{J(i) \mid i \in D\}$, $x \leqslant g(y)$. We use the Galois connection again to deduce from this that $f(x) \leqslant y$. Therefore $f(x)$ is a least upper bound, as desired. ☐

We now apply the fundamental trick in category theory: use `refl`. Every preorder is reflexive, so $f(x) \leqslant f(x)$, for any $x \in X$. By the above property of Galois connections

$$\frac{f(x) \quad \leqslant f(x)}{x \quad \leqslant g(f(x)).}$$

Similarly, $g(y) \leqslant g(y)$, so the Galois connection implies

$$\frac{g(y) \quad \leqslant g(y)}{f(g(y)) \quad \leqslant y.}$$

In fact, these properties characterize a Galois connection.

Theorem 6.4. *Let $f : X \to Y$ and $g : Y \to X$ be a pair of order-preserving maps. f and g form a Galois connection $f \dashv g$ if and only if, for any $x \in X$ and $y \in Y$,*

$$x \leqslant g(f(x)) \textit{ and } f(g(y)) \leqslant y.$$

> The conditions $x \leqslant g(f(x))$ and $f(g(y)) \leqslant y$ for any $x \in X$ and $y \in Y$ show that $g \circ f$ is a *closure operator* and $f \circ g$ is an *interior operator* (this is the terminology used in the warm-up). This is the preorder-variant of a more general property of adjunctions. Given any adjunction $F \dashv G$ between categories, GF is a monad, and FG is a comonad.

Proof of Theorem 6.4. The forward direction was proven in the above discussion. For the reverse direction, assume, for any $x \in X$ and $y \in Y$, $x \leqslant g(f(x))$ and $f(g(y)) \leqslant y$. We need to show that f and g form a Galois connection $f \dashv g$. Consider a pair of elements $x \in X$ and $y \in Y$. We have a chain of implications.

$$\dfrac{x \leqslant g(f(x)) \qquad \dfrac{f(x) \leqslant y}{g(f(x)) \leqslant g(y)}\ g \text{ order-pres.}}{x \leqslant g(y)}\ g \circ f \text{ closure operator.}$$

Therefore $f(x) \leqslant y$ implies $x \leqslant g(y)$. Conversely, we have a similar chain of implications.

$$f \circ g \text{ interior operator.}\ \dfrac{f \text{ order-pres.}\ \dfrac{x \leqslant g(y)}{f(x) \leqslant f(g(y))} \qquad f(g(y)) \leqslant y}{f(x) \leqslant y}$$

Thus $x \leqslant g(y)$ implies $f(x) \leqslant y$. Since x and y are arbitrary, this shows that f and g form a Galois connection $f \dashv g$, as desired. □

Example 6.5. During the seminar on Galois connections, Steven brought up a nice example for finding a Galois connection. Here is the example. It will motivate a theorem characterizing the existence of Galois connections. Consider the following preorders

$$X \overset{\text{def}}{=} \quad \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

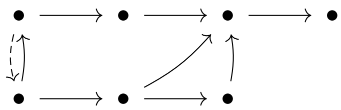
$$Y \overset{\text{def}}{=} \quad \bullet \longrightarrow \bullet \longrightarrow \bullet$$

Given a functor $g : Y \to X$, e.g.,

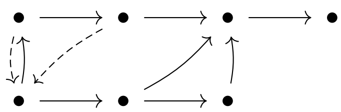
$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[ur, bend right] & \bullet \arrow[u] &
\end{tikzcd}$$

we can ask whether this functor has a left adjoint $f : X \to Y$ (forming a Galois connection $f \dashv g$. In this example, no left adjoint exists. We can see this by looking at what f must do to the right-most point of X. Recall, for any Galois connection $f \dashv g$ and any $x \in X$, $x \leqslant g(f(x))$. There is no choice for $f(x)$ to satisfy this constraint, since the image of g is strictly less than the right-most point.

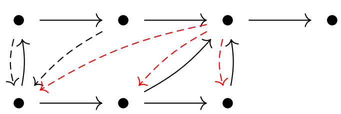
On the other hand, we can ask if $g : Y \to X$ has a *right* adjoint $h : X \to Y$ (forming a Galois connection $g \dashv h$. We can construct h by looking at the behavior of $g \circ h$ starting from the left. Recall, for a Galois connection $g \dashv h$ and any element $x \in X$, $g(h(x)) \leqslant x$. For the left most point of X, there is only one choice of $h(x)$ which satisfies this constraint, shown below.



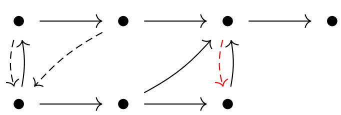
Similarly, the image of the second point under h is fixed by this constraint.



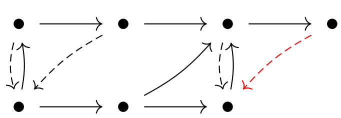
For the third point, any choice of image satisfies this constraint.



To determine which choice is the correct choice, we need to use another constraint. In the reverse direction, for any $y \in Y$, $y \leqslant h(g(y))$. When y is the right-most point of Y, this fixes the choice of red arrow.



This leaves the right-most point of X. $h : X \to Y$ must be order-preserving. This fixes the choice for the image of the right-most point.



Thus, we have found the right adjoint h using the constraints $g(h(x)) \leqslant x$ and $y \leqslant h(g(y))$.

The previous example suggests the question of uniqueness of an adjoint. Adjoints are always unique, up to isomorphism. We will prove this by writing right adjoints as colimits (and left adjoints as a limit).

Theorem 6.6. *Let $f : X \to Y$ be an order-preserving map between preorders. If f has a right adjoint $g : Y \to$*

X, then g satisfies

$$g(y) \cong \bigvee_{x \in f^{-1}(\downarrow y)} x.$$

This supremum is the preorder variant of the following colimit, which characterizes a right adjoint of a functor $F : C \to D$.

$$G(y) \cong \varinjlim (F \downarrow y \to C),$$

where $F \downarrow y$ is the *comma category* of F and y. Its objects are pairs $(x, a : F(x) \to y)$ of an object x of C and an arrow $a : F(x) \to y$ in D. An arrow $(x, a) \to (x', a')$ is an arrow $\theta : x \to x'$ in C fitting into a commuting triangle

$$\begin{tikzcd} F(x) \arrow[rr, "F(\theta)"] \arrow[dr, "a"'] & & F(x') \arrow[dl, "a'"] \\ & y. & \end{tikzcd}$$

The functor $F \downarrow y \to C$ sends a commuting triangle likke the one above to the arrow $\theta : x \to x'$.

Proof of Theorem 6.6. We first show that $g(y)$ is an upper bound for $f^{-1}(\downarrow y)$. Then we will show that $g(y)$ is the least upper bound. Any element $x \in f^{-1}(\downarrow y)$ satisfies $f(x) \leqslant y$ by definition of $\downarrow y$. Since g is a right adjoint, this implies $x \leqslant g(y)$, so $g(y)$ is an upper bound. Furthermore, any Galois connection $f \dashv g$ satisfies

$$f(g(y)) \leqslant y,$$

for any $y \in Y$. Thus $g(y) \in f^{-1}(\downarrow y)$, so $g(y)$ is a least upper bound, as desired. (In fact, this shows that $g(y)$ is the maximum of $f^{-1}(\downarrow y)$ rather than just the supremum.) ☐

Corollary 6.7. *Let $f : X \to Y$ be an order-preserving map between preorders. Suppose $g_1 : Y \to X$ and $g_2 : Y \to X$ are right adjoints to f, i.e., there are Galois connections $f \dashv g_1$ and $f \dashv g_2$. Then g_1 and g_2 are **naturally isomorphic**: for any element $y \in Y$,*

$$g_1(y) \leqslant g_2(y) \text{ and } g_2(y) \leqslant g_1(y).$$

Proof. By Theorem 6.6, the two Galois connections $f \dashv g_1$ and $f \dashv g_2$ imply, for any $y \in Y$

$$g_1(y) \cong \bigvee_{x \in f^{-1}(\downarrow y)} x \cong g_2(y).$$

Thus $g_1(y) \cong g_2(y)$. Recall this means $g_1(y) \leqslant g_2(y)$ and $g_2(y) \leqslant g_1(y)$, as desired. ☐

We can prove the analogous things for left adjoints too.

Exercise 6 : Let $g : Y \to X$ be an order-preserving map between preorders. Suppose $f : X \to Y$ is a *left* adjoint to g. Show, for any $x \in X$,

$$f(x) \cong \bigwedge_{y \in g^{-1}(\uparrow x)} y.$$

Where $\uparrow x$ is the set of elements x' in X such that $x \leqslant x'$. This is the preorder form of the limit

$$f(x) \cong \varprojlim x \downarrow g \to Y.$$

Furthermore, show that left adjoints are unique up to natural isomorphism.
This exercise can be proven directly, using the same argument. Alternatively, use Theorem 6.6 on the opposite preorders $(X, \geqslant)$ and $(Y, \geqslant)$.

7 A very important Galois connection

Let X and Y be a pair of sets, and let $f : X \to Y$ be a function between them. From f we can define an order-preserving map $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ which sends a subset $A \in \mathcal{P}(Y)$ to its preimage

$$f^{-1}(A) \overset{\text{def}}{=} \{x \in X \mid f(x) \in A\} \in \mathcal{P}(X).$$

This order-preserving map participates in two Galois connections. These Galois connections are essential for interpreting first-order logic inside a topos. First, f^{-1} has a left adjoint, written $\exists_f : \mathcal{P}(X) \to \mathcal{P}(Y)$. This is the usual image map:

$$\exists_f(B \in \mathcal{P}(X)) \overset{\text{def}}{=} f(B) = \{y \in Y \mid \exists x \in X. f(x) = y \wedge x \in B\}.$$

Proposition 7.1. *$\exists_f$ is the left adjoint of f^{-1} (forming a Galois connection $\exists_f \dashv f^{-1}$).*

Proof. Let $A \in \mathcal{P}(Y)$ and $B \in \mathcal{P}(X)$. Then $B \subseteq f^{-1}(A)$ if and only if every element $x \in B$ satisfies $f(x) \in A$, which is true if and only if $f(B) \subseteq A$, i.e., $\exists_f B \subseteq A$. □

f^{-1} also has a right adjoint $\forall_f : \mathcal{P}(X) \to \mathcal{P}(Y)$. This is sometimes called the *dual image* (c.f. [Makkai and Reyes, 1977]). The dual image is defined using the universal quantifier:

$$\forall_f(B \in \mathcal{P}(X)) \overset{\text{def}}{=} \{y \in Y \mid \forall x \in X. f(x) = y \Rightarrow x \in B\} = \big\{y \in Y \mid f^{-1}(\{y\}) \subseteq B\big\}.$$

Proposition 7.2. *$\forall_f$ is a right adjoint for f^{-1}, yielding a Galois connection $f^{-1} \dashv \forall_f$.*

Proof. Let $A \in \mathcal{P}(Y)$ and $B \in \mathcal{P}(X)$ be a pair of subsets. Suppose $f^{-1}(A) \subseteq B$. Then for any element $y \in A$, $f^{-1}(\{y\}) \subseteq B$ (since f^{-1} is order-preserving), i.e., $y \in \forall_f A$. Since y is arbitrary, this shows that $A \subseteq \forall_f B$. Conversely, suppose $A \subseteq \forall_f B$, then for any $y \in A$, $f^{-1}(\{y\}) \subseteq B$. Thus

$$f^{-1}(A) = \bigcup_{y \in A} f^{-1}(\{y\}) \subseteq B,$$

as desired. □

The previous two propositions show that we have an *adjoint triple* $\exists_f \dashv f^{-1} \dashv \forall_f$. Adjoint triples are particularly nice categorical structures. Longer adjoint *strings*, e.g., $f_1 \dashv \ldots \dashv f_n$ are rarer and have increasingly powerful properties. For example, we have a fun fact.

Proposition 7.3 ([Rosebrugh and Wood, 1994])**.** *Let C be a locally small category, and let $よ : C \to Psh(C)$ be its Yoneda embedding. If $よ$ extends to the left to an adjoint quintuple*

$$F_1 \dashv F_2 \dashv F_3 \dashv F_4 \dashv よ,$$

then C is equivalent to the category Set.

The adjoint triple $\exists_f \dashv f^{-1} \dashv \forall_f$ gives an overpowered proof of the following result.

Corollary 7.4. *$f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ preserves intersections and unions.*

Proof. f^{-1} has both a left and a right adjoint. Therefore, by Proposition 6.2, f^{-1} preserves infima and suprema, i.e., intersections and unions. □

Quantifiers as Adjoints

The notation in the adjoint triple $\exists_f \dashv f^{-1} \dashv \forall_f$ suggests the adjoints to f^{-1} have something to say about the quantifiers in first-order predicate logic. If we are working within a topos, there is a bijection between subsets $A \subseteq X$ and propositions $\chi_A(x : X)$ with a free variable representing an element of X. This bijection is given by

$$\phi(x : X) \mapsto \{x \in X \mid \phi(x)\} \subseteq X.$$

(This bijection exists because a topos has a *subobject classifier*.) Under this bijection, the above adjoint triple interprets the quantifiers $\exists$ and $\forall$ in the way we expect. We can also describe essentially the same adjoint triple directly in logic.

Recall, from a propositional theory T, we can form its Lindenbaum algebra $L(T)$ of propositions. To describe a first-order theory, we stitch together families of Lindenbaum algebras, indexed by a context of free variables.

Definition 7.5. A **first-order signature** Σ is a collection of the following data.

- A set of *sorts* $\sigma \in \Sigma$.
- A collection of base terms, i.e., *function symbols*

$$x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash f(x_1, \ldots, x_n) : \tau.$$

- For every context $x_1 : \sigma_1, \ldots, x_n : \sigma_n$, a collection of base propositions. e.g., $R(x_1 : \sigma_1, \ldots, x_n : \sigma_n)$.

A **first-order theory** is a first-order signature and a set of sequents, which are the axioms of the theory.

If we fix a context, we can use the usual connectives $\wedge, \vee$, etc. to form formulae. The new expressivity comes from the terms in a first-order signature. We can freely compose terms in the usual typed manner. Given a list of terms $\Gamma \vdash t_1 : \tau_1, \ldots, \Gamma \vdash t_n : \tau_n$, which we could write as an arrow $t : \Gamma \to \Gamma'$, where Γ' is a context of fresh variables of type $\tau_1, \ldots, \tau_n$, we obtain a map

$$t^* : \mathrm{Form}(\Gamma') \to \mathrm{Form}(\Gamma)$$

given by substitution. If $\phi(\vec{x} : \Gamma')$ is a formula with context Γ', then substituting the list of terms t gives a formula $\phi(t_1, \dots, t_n)$ with context Γ. The left and right adjoints to t^* allow us to interpret quantifiers: $\exists_t \dashv t^* \dashv \forall_t$, where the preorder on $\mathrm{Form}(\Gamma)$ is given by provability $\vdash$, the same as in the Lindenbaum algebra. We bundle this data into one categorical structure.

Definition 7.6. A **first-order hyperdoctrine** is a pair (C, F) of a category C of contexts and terms and a functor

$$F : C^{\mathsf{op}} \to \text{Lindenbaum Algebras}$$

sending a context Γ to its Lindenbaum algebra of formulae with free variables in Γ. We require, for any arrow $t : \Gamma \to \Gamma'$ in C, we have an adjoint triple

$$\exists_t \dashv F(t) \dashv \forall_t.$$

Any first-order theory defines a first-order hyperdoctrine, where $F(t) = t^*$. The adjoints $\exists_t$ and $\forall_t$ are given by

$$\exists_t \phi(\vec{x'} : \Gamma') \equiv \exists \vec{x} : \Gamma.\, t(\vec{x}) = \vec{x'} \wedge \phi(\vec{x}).$$

and

$$\forall_t \phi(\vec{x'} : \Gamma') \equiv \forall \vec{x} : \Gamma.\, t(\vec{x}) = \vec{x'} \Rightarrow \phi(\vec{x}).$$

Theorem 7.7 (Quantifiers are adjoints)**.** *Given a pair of context Γ and Γ', a list of terms $t : \Gamma \to \Gamma'$, and a pair of formulae $\phi(\vec{x} : \Gamma)$ and $\psi(\vec{x'} : \Gamma')$,*

$$\phi(\vec{x}) \vdash t^*\psi(\vec{x}) \textit{ if and only if } \exists_t \phi(\vec{x'}) \vdash \psi(\vec{x'}).$$

Similarly

$$t^*\psi(\vec{x}) \vdash \phi(\vec{x}) \textit{ if and only if } \forall_t \phi(\vec{x'}) \vdash \psi(\vec{x'}).$$

In other words, we have a pair of Galois connections $\exists_t \dashv t^*$ and $t^* \dashv \forall_t$.

We now give an example of a term and its adjoint triple. Given a pair of variables $x_1 : \sigma_1, x_2 : \sigma_2$, we can form the "projection"

$$x_1 : \sigma_1, x_2 : \sigma_2 \vdash x_1 : \sigma_1,$$

representing an arrow $\pi_1 : \sigma_1, \sigma_2 \to \sigma_1$. Let $\phi(x_1, x_2$ and $\psi(x_1$ be a pair of formulae with context σ_1, σ_2 and σ_1, respectively. Note, given another pair of variables $y_1 : \sigma_1, y_2 : \sigma_2$, then

$$\pi_1^*\psi(y_1, y_2) \equiv \psi(\pi_1(y_1, y_2)) \equiv \psi(y_1),$$

Similarly,

$$\exists_{\pi_1}\phi(x_1) \equiv \exists x_1' : \sigma_1.\, \exists x_2' : \sigma_2.\, \pi_1(x_1', x_2') = x_1 \wedge \phi(x_1', x_2') \equiv \exists x_1' \exists x_2'\, x_1' = x_1 \wedge \phi(x_1', x_2').$$

The latter is provably equivalent (by $=$ elimination) to $\exists x_2'.\, \phi(x_1, x_2')$. To emphasize, we have shown:

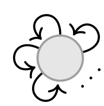
$$\exists_{\pi_1}\phi(x_1) \dashv\vdash \exists x_2'.\, \phi(x_1, x_2').$$

A similar argument shows $\forall_{\pi_1}\phi(x_1) \dashv\vdash \forall x_2'.\, \phi(x_1, x_2')$. In other words, the adjoints $\exists_\pi$ and $\forall_\pi$ recover the usual quantifiers when π is a projection.

The first half of these notes developed a categorical perspective on the theory of preorders. One key step in this theory is the observation that a preorder $(X, \leqslant)$ may be treated as a family of indexed propositions $\mathcal{C}\ell(X, \leqslant)^{\mathsf{op}} \to \Omega$. This allowed us to lift operations on truth values Ω to universal constructions in a preorder. In a category C, we can no longer treat objects like indexed *propositions*, but there is a way to generalize this philosophy. Instead, we need to study indexed sets, i.e., *presheaves*. Similar to the preorder situation, operations between sets can be lifted to universal constructions in a category.

8 A brief account of categories

A category is a collection of objects and arrows between them which can be composed. There are myriad situations where such structure arises. Perhaps two archetypes are interfaces and theories. Qua interface, a category is a collection of states, called *objects*, and families of transitions between states, called *arrows*. Functors $F : C \to \mathsf{Set}$ can be seen as concrete implementations of this interface. This philosophy is explained well in [Fong and Spivak, 2019]. This approach also forms the semantics for algebraic effects. For example, algebras of the `Writer m` monad, for some monoid `m` are described by the category consisting of a single object and an arrow for every element of the monoid `m`. Functors



from this category to, e.g., Set identifies a set X (the image of the single object) and a family of functions $X \to X$ for every element of `m`. This is a `Writer m` algebra.

Qua theory, a category C is the classifying category of *some theory*. Functors out of this category are the same as models of this theory. Given a propositional theory T, we saw that functors from $\mathcal{C}\ell(T)$ to a preorder X were the same as models of T in X. This perspective is not fundamentally different to the interface perspective.

We will be using categories to describe the logical relations of a programming language with state, e.g., name allocation or step-indexing. We will identify a category C whose objects will represent states of a heap. Arrows of C will identify how a heap state may be extended with fresh names. We will write our logical relations as sets indexed by heap states, i.e., functors out of C into Set.

For the sake of completeness, here is a definition of a (small) category.

Definition 8.1. A **small category** C is a pair of sets $\left(C^{(0)}, C^{(1)}\right)$, called its set of **objects** and its set of **arrows**, respectively, and a family of functions.

- A pair $s, t : C^{(1)} \to C^{(0)}$ of **source** and **target** functions. For any arrow $f \in C^{(1)}$, $s(f)$ is its source object, and $t(f)$ is its target object. We usually write this as $f : s(f) \to t(f)$.
- A function $1 : C^{(0)} \to C^{(1)}$ sending every object $X \in C^{(0)}$ to its **identity arrow** $1_X : X \to X$.
- A partial function $C^{(1)} \times C^{(1)} \to C^{(1)}$ called **composition**. $m(g, f)$ is defined when $s(g) = t(f)$.

These functions are required to satisfy two axioms. First, m must be an associative operation:

$$m(h, m(g, f)) = m(m(h, g), f),$$

wherever defined. Second, 1 must be a left and right identity for m:

$$m(f, 1_{s(f)}) = m(1_{t(f)}, f) = f,$$

for any $f \in C^{(1)}$.

Usually composition $m(g, f)$ in a category is written as $g \circ f$ or by juxtaposition gf.

Definition 8.2. Given small categories C and D, a **functor** $F : C \to D$ is a family of functions $F^{(0)} : C^{(0)} \to D^{(0)}$ and $F^{(1)} : C^{(1)} \to D^{(1)}$ which preserve the functions $s, t, 1$, and m.

Given a category C, swapping the functions s and t yields a new category, C^{op}, called its **opposite category**. This is a generalization of the relation between $\leqslant$ and $\geqslant$ for preordered sets.

There is one more perspective on categories which ties this back to the study on preorders. Recall, for a preordered set $(X, \leqslant)$, the classifying category $\mathcal{C}\ell(X, \leqslant)$ is *thin*—between any two objects x, y in $\mathcal{C}\ell(X, \leqslant)$, there is at most one arrow $x \to y$. A category C is an intensional form of a preorder. For a category, it is not enough to know that there exists an arrow between objects X and Y, the specific choice of arrow $f : X \to Y$ is now relevant. Similarly, Set is the intesional analogue of truth values: it is not enough to know that a set X is nonempty; rather, the specific choice of element $x \in X$ matters. This perspective tells us that universal constructions ofr categories are intensional/proof-relevant variants of the universal constructions for preorders. We saw some of these generalizations in the notes on Galois connections.

9 Indexed sets

We begin by describing a presheaf as an indexed set. By indexed set, we mean indexed in the following manner. Given a family of sets $\{X_\alpha \mid \alpha \in I\}$, we can describe this family with a function

$$\pi : X \to I,$$

where $X \stackrel{\text{def}}{=} \coprod_{\alpha \in I} X_\alpha$. Given a function $x \in X$, x is contained in some X_α for a unique α. $\pi(x)$ is defined to be this index α. In the case where I is replaced with a category C, we follow the same idea, but we introduce some extra structure to make the set X compatible with the arrows of C.

Definition 9.1. Let C be a small category. A C**-indexed set** is a set X equipped with a function $\pi : X \to C^{(0)}$ and a partial function $\mu : X \times C^{(1)} \to X$ satisfying the following constraints.

- $\mu(x, f)$ is defined whenever $t(f) = \pi(x)$.
- $\pi(\mu(x, f)) = s(f)$.
- $\mu(x, 1_{\pi(x)}) = x$.
- If $g \circ f$ is defined, then $\mu(x, g \circ f) = \mu(\mu(x, g), f)$, wherever defined.

Anticipating an equivalence between C-indexed sets and presheaves on C, we could write $x\big|_f$ in lieu of $\mu(x, f)$.

Calling this a C-indexed set is admiteddly uncommon. Another name for a C-indexed set is a *discrete opfibration*. This is more precise, and it alludes to a more general notion, called a *fibration of categories*. On the other hand, there is a hint of the notion of `Writer m` algebra in a C-indexed set. A category with one object, i.e., a category C where $C^{(0)} \cong \{*\}$, is the same structure as a monoid on the set $C^{(1)}$. In this sense categories are generalized monoids, where the multiplication is a partial operation. Given that categories are generalized monoids, C-indexed sets are generalized modules.

Proposition 9.2. *Let C be a category with one object. Then a C-indexed set (X, π, μ) is the same data as a right-action of the monoid $C^{(1)}$ on the set X.*

Proof. Consider a C-indexed set (X, π, μ). Since $C^{(0)} \cong \{*\}$, the projection $\pi : X \to C^{(0)}$ can only be the map sending any element of X to the single element of $C^{(0)}$. In this case, $\mu : X \times C^{(1)} \to X$ is total. We define the right action on X by $C^{(1)}$ by the rule

$$x \cdot f \overset{\text{def}}{=} \mu(x, f).$$

Conversely, given a right action of $C^{(1)}$ on X, we give X the structure of a C-indexed set in the following manner. The projection π is unique, since $C^{(0)}$ is a singleton set. This leaves the map $\mu : X \times C^{(1)} \to X$, which we define using the same rule as before.

$$\mu(x, f) \overset{\text{def}}{=} x \cdot f.$$

□

Due to the above proposition, presheaves are occaisionally called *right C-modules*. Section V.7 of [Mac Lane and Moerdijk, 1994] describes the theory of right C-modules. Furthermore, to feed mathematicians' preference for infix notation, we will now write $x \cdot f$ instead of $\mu(x, f)$ for any C-indexed set.

Given a C-indexed set $(X, \pi, \cdot)$, we can lift the categorical structure on C to X. In order to give this category a name, we will anticipate presheaves and abuse notation slightly to call this category $\mathrm{el}(X)$, the *category of elements* of X.

Definition 9.3. Given the C-indexed set $(X, \pi, \cdot)$, its **category of elements** is the following category $\mathrm{el}(X)$. The objects of $\mathrm{el}(X)$ are the elements of X. Furthermore, there is an arrow $\widetilde{f} : x \to y$ in $\mathrm{el}(X)$ for every arrow $f \in C^{(1)}$ such that $x = y \cdot f$. Composition of arrows in $\mathrm{el}(X)$ is the same as composition in C.

It can be helpful to have a picture for how the arrows in $\mathrm{el}(X)$ are related to arrows in C.

$$\begin{tikzcd}
x \arrow[r, dashed, "\widetilde{f}"] \arrow[d, squiggly] & y \arrow[d, squiggly] \\
\pi(x) \arrow[r, "f"] & \pi(y)
\end{tikzcd}$$

Here a squiggly arrow $x \rightsquigarrow A$ signifies the equation $\pi(x) = A$. The arrow notation is suggestive: the projection $\pi : X \to C$ defines a functor $\widetilde{\pi} : \mathrm{el}(X) \to C$. We prove this now.

Proposition 9.4. *Let* $\widetilde{\pi} : \mathrm{el}(X) \to C$ *be the map*

$$x \mapsto \pi(x), \quad \widetilde{f} \mapsto f.$$

Then $\widetilde{\pi}$ *is a functor.*

Proof. We need to show that $\widetilde{\pi}$ preserves identities and composition. Given an object x in $\mathrm{el}(X)$ (i.e., an element $x \in X$), its identity arrow 1_x is presented by the identity $\widetilde{1_{\pi(x)}}$, so

$$\pi(1_x) = \pi(\widetilde{1_{\pi(x)}}) = 1_{\pi(x)},$$

so $\widetilde{\pi}$ preserves identities. A similar argument works for composition. □

The functor $\widetilde{\pi}$ is what gives a C-indexed set the structure of a discrete opfibration. A fibration of categories is a categorification of $\widetilde{\pi}$, adding more arrows in $\mathrm{el}(X)$ besides the ones in C.

We now use C-indexed sets to study the category C. While the objects of C may not be faithfully represented by sets, it is true that they are faithfully represented by C-indexed sets. This property is a generalization of the embedding of a preorder into its downward closed sets.

Definition 9.5. Let $X \in C^{(0)}$ be an object of C. Its **indexed downset** $\downarrow X$ is the C-indexed set defined in the following manner.

- The elements of $\downarrow X$ are arrows $f : Y \to X$ in C, for any object Y in C.
- The projection $\pi : \downarrow X \to C$ sends an arrow $f : Y \to X$ to the object Y.
- The multiplication $\_ \cdot \_ : X \times C^{(1)} \to X$ sends an arrow $f : Y \to X$ and an arrow $g : Z \to Y$ to the composition $f \circ g$:
$$f \cdot g \overset{\text{def}}{=} f \circ g.$$

Remark 9.6. *The category of elements for the indexed downset* $\downarrow X$ *has a familiar name. It is the* ***slice category*** $C_{/X}$. *This category is sometimes called* $C \downarrow X$, *an auspicious hint of the slice category's connection to the downsets of a preordered set.*

In order to make sense of how C is embedded into its C-indexed sets, we need to make a notion of C-indexed function.

Definition 9.7. Let X and Y be a pair of C-indexed sets. A C**-indexed function** $\phi : X \to Y$ is a function ϕ_* between the underlying sets of X and Y which preserves π and $\_ \cdot \_$, i.e.,

$$\pi_Y(\phi_* x) = \pi_X(x), \quad (\phi_* x) \cdot_Y f = \phi_*(x \cdot_X f).$$

C-indexed functions may be composed, by composing the underlying functions between sets. Furthermore, there is an obvious identity function. This gives a category of C-indexed sets, which we call C-Set.

Given an arrow $f : X \to Y$ in C, we obtain an indexed function $f_* : \downarrow X \to \downarrow Y$, given by postcomposition:

$$f_*(g : Z \to X) \overset{\text{def}}{=} f \circ g : Z \to Y.$$

Lemma 9.8 (C-indexed Yoneda lemma)**.** *The mapping $X \mapsto {\downarrow} X$ and $f \mapsto f_*$ defines a functor*

$$\downarrow : C \to C\text{-Set}.$$

Furthermore, this functor is fully faithful. Finally, given an arbitrary C-indexed set A, there is a bijection

$$C\text{-Set}({\downarrow} X, A) \cong \{a \in A \mid \pi(a) = X\}.$$

Proof. We use the only trick in category theory: chase `refl`. We first argue that $\downarrow$ is a functor. Since f_* is defined by composition in C, this is straightforward. We now show that $\downarrow$ is faithful. Let $f, g : X \to Y$ be two arrows in C. Suppose $f_* = g_*$. We need to prove $f = g$. The identity arrow $1_X : X \to X$ is an element of ${\downarrow} X$. Then

$$f = f_*(1_X) = g_*(1_X) = g,$$

so $\downarrow$ is faithful. Next we show $\downarrow$ is full. Let $\phi : {\downarrow} X \to {\downarrow} Y$ be an indexed function. Then $\phi_*(1_X)$ is some arrow $f : X' \to Y$. Since ϕ preserves the projection π, $X' = X$. What is left is to argue that $\phi_* = f_*$. Indeed, for any element $g : Z \to X$ in ${\downarrow} X$,

$$f_* g = f \circ g = f \cdot g = \phi_*(1_X) \cdot g = \phi_*(1_X \cdot g) = \phi_*(g).$$

This shows $\phi_* = f_*$. Since ϕ is arbitrary, this shows that $\downarrow$ is full; hence $\downarrow$ is fully faithful.

Finally, we prove the bijection. Let $\phi_0 : {\downarrow} X \to A$ be an indexed function. ${\phi_0}_*(1_X)$ is an element of A; call it $a(\phi_0)$. We will show that $a : \phi_0 \mapsto a(\phi_0)$ is a bijection. We need an inverse to a. Given an element $a_0 \in A$ such that $\pi(a_0) = X$, we define an indexed function $\phi(a_0) : {\downarrow} X \to A$ in the following manner. Given an element $f : Y \to X$ in ${\downarrow} X$, define

$$\phi(a_0)_*(f) \stackrel{\text{def}}{=} a_0 \cdot f \in A.$$

Since $\pi(a_0) = X$, this function is well-defined, and it is straightforward to show that this function preserves π and $\_ \cdot \_$. Given an indexed function $\phi_0 : {\downarrow} X \to A$ and an element $f : Y \to X$ in ${\downarrow} X$,

$$\phi(a(\phi_0))_*(f) = a(\phi_0) \cdot f = {\phi_0}_*(1_X) \cdot f = {\phi_0}_*(1_X \cdot f) = {\phi_0}_*(f).$$

Thus $\phi(a(\phi_0)) = \phi_0$. Similarly given an element $a_0 \in A$ such that $\pi(a_0) = X$,

$$a(\phi(a_0)) = \phi(a_0)_*(1_X) = a_0 \cdot 1_x = a_0.$$

Therefore $a(\phi(a_0)) = a_0$. This shows that $a(\_)$ and $\phi(\_)$ are inverses, proving the desired bijection. □

We have shown that $\downarrow$ is an embedding. When we did this for preorders, we used this embedding to lift operations on truth-values to universal constructions on a preorder. We do the analogous thing for indexed sets, lifting operations on sets to universal constructions on a category.

Indexed set theory

Test

10 Presheaves: Indexed sets as functors

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