1 Preorders are Indexed Propositions

Let $(X, \leqslant)$ be a preorder. Then we can form the classifying category $\mathcal{C}\ell(X, \leqslant)$ as before. We also defined $\Omega \stackrel{\text{def}}{=} \{\mathsf{True}, \mathsf{False}\}$. This is the preorder of *propositions*. It is essentialy the preorder of booleans `bool`. However, for foundational reasons, it is useful to think of Ω has having slightly less structure than the full boolean operations. In the first warmup, we constructed a functor $\downarrow x : \mathcal{C}\ell(X, \leqslant)^{\mathsf{op}} \to \Omega$ by forming the *downset*

$$\downarrow x(y) \stackrel{\text{def}}{=} \begin{cases} \mathsf{True} & \text{if } y \leqslant x \\ \mathsf{False} & \text{otherwise.} \end{cases}$$

The downset construction $x \mapsto \downarrow x$ allows us to reason about the abstract preorder $(X, \leqslant)$ as if it were a collection of propositions, indexed by the elements of X.

Definition 1.1. Given a preorder $(X, \leqslant)$, a **downward closed set** is a subset $A \subseteq X$ satisfying the following closure property:

$$\frac{x \leqslant y}{y \in A \Rightarrow x \in A.}$$

There is another way to represent downward closed sets, in terms of functions $X \to \Omega$. Given a subset $A \subseteq X$, we can form the *membership function* $\chi_A : X \to \Omega$:

$$\chi_A(x) \stackrel{\text{def}}{=} \begin{cases} \mathsf{True} & x \in A, \\ \mathsf{False} & \text{otherwise.} \end{cases}$$

Conversely, any function $\varphi : X \to \Omega$ defines a subset $\varphi^{-1}(\mathsf{True}) \stackrel{\text{def}}{=} \{x \in X \mid \varphi(x) = \mathsf{True}\}$.

Proposition 1.2. *A subset $A \subseteq X$ is downward closed if and only if its membership function $\chi_A : X \to \Omega$ is order-reversing. Similarly, a function $\varphi : X \to \Omega$ is order-reversing if and only if its corresponding subset $\varphi^{-1}(\mathsf{True})$ is downward closed.*

Proof. Consider a subset $A \subseteq X$. Suppose that A is downward closed, i.e.,

$$\frac{x \leqslant y}{y \in A \Rightarrow x \in A.}$$

Note $y \in A$ if and only if $\chi_A(y) = \mathsf{True}$. Therefore the above judgment is equivalent to

$$\frac{x \leqslant y}{\chi_A(y) = \mathsf{True} \Rightarrow \chi_A(x) = \mathsf{True}\,.}$$

This is the statement that χ_A is order-reversing. Conversely, if χ_A is assumed to be order-reversing, then the equivalence of the above two judgments implies A is downward closed.

The result for functions $\varphi : X \to \Omega$ follows a similar argument. $\square$

The above proposition demonstrates a bijection between the following sets of structures

$$\begin{array}{ccc} \{\text{Downward-closed subsets } A \subseteq X\} & \cong & \{\text{Order-reversing functions } X \to \Omega\} \\ A & \mapsto & \chi_A \\ \varphi^{-1}(\mathsf{True}) & \leftarrow\!\shortmid & \varphi \end{array}$$

Recall that an order-preserving function $(X, \leqslant) \to (Y, \leqslant)$ is equialent to the data of a functor between classifying categories $\mathcal{C}\ell(X, \leqslant) \to \mathcal{C}\ell(Y, \leqslant)$. Furthermore, an order-reversing function $(X, \leqslant) \to (Y, \leqslant)$ is the same data as an order-*preserving* function—provided the preorder on X is replaced with the reverse preorder $\geqslant$. In other words, we have a chain of bijections

$$\begin{aligned} \{\text{Order-reversing functions } (X, \leqslant) \to (\Omega, \leqslant)\} &\cong \{\text{Order-preserving functions } (X, \geqslant) \to (\Omega, \leqslant)\} \\ &\cong \left\{\text{Functors } \mathcal{C}\ell(X, \leqslant)^{\mathsf{op}} \to \Omega\right\}. \end{aligned}$$

Combining this with the preceding proposition on downward closed sets, we have obtained the following theorem.

Theorem 1.3 (Grothendieck construction for preorders)**.** *There is a bijective correspondence*

$$\{\textit{Downward-closed subsets } A \subseteq X\} \cong \left\{\textit{Functors } \mathcal{C}\ell(X, \leqslant)^{\mathsf{op}} \to \Omega\right\}$$

given by sending a downward closed subset A to the functor associated to χ_A and by sending a functor $F : \mathcal{C}\ell(X, \leqslant)^{\mathsf{op}} \to \Omega$ to the set of objects $x \in X$ such that $F(x) = \mathsf{True}$.

Remark 1.1

The above theorem allows us to make our first abuse of notation! In these notes we will now identify the functor $\downarrow x : \mathcal{C}\ell(X, \leqslant)^{\mathsf{op}} \to \Omega$ with its associated downward closed set

$$\{y \in X \mid \downarrow x(y) = \mathsf{True}\}.$$

In retrospect, it would have been better to work entirely with downward-closed sets first, and introducing functors later. In the sequel, where we repeat the above with presheaves, it will be cleaner to avoid this abuse of notation.

Downward closed sets inherit a partial order given by inclusion of subsets. This allows us to talk about the **partial order of downward closed sets**

$$(\mathrm{Dwd}(X), \subseteq).$$

The downset construction $x \mapsto \downarrow x$ is an embedding $X \to \mathrm{Dwd}(X)$ with some nice properties.

Lemma 1.4 (Yoneda lemma for preorders)**.** *The operation $\downarrow : X \to \mathrm{Dwd}(X)$ is fully faithful, i.e., $x \leqslant y$ if and only if $\downarrow x \subseteq \downarrow y$. In particular, $\downarrow x = \downarrow y$ if and only if $x \leqslant y$ and $y \leqslant x$. Furthermore, for any downward closed set $A \in \mathrm{Dwd}(X)$,*

$$x \in A \textit{ if and only if } \downarrow x \subseteq A.$$

Proof. The first part, that $\downarrow$ is fully faithful, was Exercise 3 of the warmup. The second part, that $x \in A$

if and only if $\downarrow x \subseteq A$ follows a similar proof.

Assume $x \in A$. The downset associated to $\downarrow x$ is the collection of all elements $y \in X$ such that $y \leqslant x$. Thus,

$$\dfrac{x \in A \qquad \dfrac{\dfrac{y \in \downarrow x}{y \leqslant x}\text{Defn. }\downarrow x}{}\; A \in \mathrm{Dwd}(X)}{y \in A.}$$

Since y is arbitrary, this shows $\downarrow x \subseteq A$, as desired. Conversely, assume $\downarrow x \subseteq A$. By reflexivity, $x \leqslant x$, so $x \in \downarrow x$. In particular, $x \in A$, as desired. $\square$

Remark 1.5. *Exercise 3 and the above lemma are proven using the same technique. The judgment $\downarrow x \subseteq A$ relates arrows $y \to x$ to elements of A. Among all arrows $y \to x$, there is a unique "universal" choice, namely the identity $\mathsf{id}_x : x \to x$ (universal in the sense that it always exists). In similar situations, where one is quantifying over arrows in a category, picking the identity is often a decisive trick.*

The Grothendieck construction for preorders and the Yoneda lemma allow us to study a preorder $(X, \leqslant)$ through its downward closed sets $(\mathrm{Dwd}(X), \subseteq)$. In this sense, the only preorder relation is the subset inclusion relation $\subseteq$: all others are abstracted "substructures" of a preorder of subsets, if we pay the price of treating a pair of elements $x, y \in X$ as "equivalent" if $x \leqslant y$ and $y \leqslant x$. This price is often worth paying, since $\mathrm{Dwd}(X)$ has more structure than X.

Observation 1.6. $\mathrm{Dwd}(X)$ *is a complete Heyting algebra.*

Since $\mathrm{Dwd}(X)$ is a complete Heyting algebra, we can freely take unions and intersections of downward-closed sets. This will be the basis of the following section.

2 Limits and Colimits of Preorders

In the previous section, we saw that we can study a preorder $(X, \leqslant)$ using its collection of downward closed sets $\mathrm{Dwd}(X)$. This was justified by the embedding $\downarrow : X \to \mathrm{Dwd}(X)$. Furthermore, $\mathrm{Dwd}(X)$ is a complete Heyting algebra. We can use this Heyting algebra structure to define operations in X. For example, we can form intersections.

Definition 2.1. Let $x, y \in X$ be a pair of elements. A **conjunction** of x, y is an element $z \in X$ which satisfies the equation of downward-closed subsets.

$$\downarrow z = \downarrow x \cap \downarrow y.$$

In other words, a conjunction in X of x and y is an element which behaves like the intersection of x and y. Since the embedding $\downarrow : X \to \mathrm{Dwd}(X)$ is fully faithful, this can be formalized in the manner above. Another way to describe a conjunction is via a *universal property*.

Proposition 2.2. *Let $x, y \in X$ be a pair of elements. An element $z \in X$ is a conjunction of x and y if and only if, for any element $w \in X$,*

$$w \leqslant z \textit{ if and only if } w \leqslant x \textit{ and } w \leqslant y.$$

In category theory, we usual write universal properties like the one in the above proposition as a diagram. In this case, the diagram depicts arrows in the classifying category $\mathcal{C}\ell(X, \leqslant)$. Here is the diagram for a binary conjunction.

$$\begin{tikzcd}
w \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& z \arrow[r] \arrow[d] & x \\
& y &
\end{tikzcd}$$

Every arrow in this diagram, being an arrow in $\mathcal{C}\ell(X, \leqslant)$ represents a comparison using $\leqslant$. The solid arrows represent the hypotheses of the universal property, and the dashed line represents the conclusion. In this case, the diagram translates to

Given $z \leqslant x$ and $z \leqslant y$. If $w \leqslant x$ and $w \leqslant y$, then $w \leqslant z$.

I have also added the symbol $\exists!$ to the diagram. This is not relevant for preorders, because there is always at most one arrow between any two objects of $\mathcal{C}\ell(X, \leqslant)$. However, in general category theory, universal properties almost always require the dashed line to be unique.

Proof of Proposition 2.2. Assume z is a conjunction of x and y. We need to show, for any element $w \in X$, $w \leqslant z$ if and only if $w \leqslant x$ and $w \leqslant y$. We now chase a chain of iffs

$$\begin{aligned} w \leqslant z & \text{ iff } w \in {\downarrow} z \\ & \text{ iff } w \in {\downarrow} x \cap {\downarrow} y \\ & \text{ iff } w \in {\downarrow} x \text{ and } w \in {\downarrow} y \\ & \text{ iff } w \leqslant x \text{ and } w \leqslant y. \end{aligned}$$

Here the second iff holds because z is a conjunction of x and y. The converse direction is a similar argument. $\square$

Remark 2.3. *In the above, z is called* a *conjunction of x and y rather than* the *conjunction of x and y. This is because conjunctions are technically not unique—but they are unique* up to isomorphism. *Suppose z is a conjunction of x and y, and suppose that there exists an element $z' \in X$ such that $z' \leqslant z$ and $z \leqslant z'$. (These conditions are equivalent to the statement that z and z' are isomorphic in $\mathcal{C}\ell(X, \leqslant)$.) Then ${\downarrow} z' = {\downarrow} z$, so z' is also a conjunction of x and y. In practice, identifying isomorphic objects is manageable, so we usually give the isomorphism class of conjunctions of x and y a name*

$$z \cong x \wedge y.$$

Furthermore, it is common in category theory to write $x \wedge y$ instead of a explicit representative of the conjunction.

We now consider two examples.

Example 2.4 (A preorder of subsets)**.** Suppose $(X, \leqslant)$ is a preorder of subsets $(X, \leqslant) = (\mathcal{P}(V), \subseteq)$. Consider a pair of subsets $A, B \in \mathcal{P}(V)$. A conjunction $A \wedge B$ of A and B satisfies the universal property:

$$\forall W \subseteq V, W \subseteq A \wedge B \Leftrightarrow W \subseteq A \text{ and } W \subseteq B.$$

If we pick singleton subsets $\{v\}$ for W (so v is some element of V), then the above universal property implies

$$\forall v \in V, v \in A \wedge B \Leftrightarrow v \in A \text{ and } v \in B.$$

This sentence uniquely characterizes the intersection $A \cap B$. Therefore

$$A \wedge B \cong A \cap B.$$

(Furthermore, since $\subseteq$ is an antisymmetric relation, this immediately implies $A \wedge B = A \cap B$.) Power sets are the nicest type of preorder: they have a generating set of singleton sets, also called *atoms*, which can distinguish any pair of objects in the power set. This property is called *well-pointedness*. For this reason, we could be silly and say that a power set $\mathcal{P}(V)$ is an example of a *atomic 0-topos with enough points*. This probably doesn't help to understand preorders, but it can be useful to intuit how an atomic topos generalizes the power-set situation!

Example 2.5 (A propositional theory)**.** Let $T = (\Sigma, \Delta)$ be a propositional theory. We constructed the classifying category of T, $\mathcal{C}\ell(T)$, whose objects are propositions ϕ in T and whose arrows $\phi \to \psi$ correspond to sequents $\phi \vdash_T \psi$. Given two propositions ϕ and ψ of T, we can ask for their conjunction. This conjunction is a proposition, P such that, for any other proposition Q,

$$Q \vdash_T P \Leftrightarrow Q \vdash_T \phi \text{ and } Q \vdash_T \psi.$$

The latter condition, $Q \vdash_T \phi$ and $Q \vdash_T \psi$ is equivalent to $Q \vdash_T \phi \wedge \psi$. Thus, a conjunction P satisfies

$$Q \vdash_T P \Leftrightarrow Q \vdash_T \phi \wedge \psi.$$

This implies $P \vdash_T \phi \wedge \psi$ and $\phi \wedge \psi \vdash_T P$. Thus $P \cong \phi \wedge \psi$. In other words, the conjunctions in $\mathcal{C}\ell(T)$ are provably equivalent to the usual conjunctions from propositional logic. This observation justifies our use of the symbol $\wedge$ to denote conjunction. It also an example of the motivation of categorical logic: *features in a language correspond to universal properties of the classifying category*.

We defined conjunctions by an equation of downsets. Nothing is stopping us from picking other equations in order to study other universal properties. For example, we could ask for larger conjunctions. Given a family x_α of elements of X, a conjunction $\bigwedge_\alpha x_\alpha$ is an element $z \in X$ satisfying the equation

$$\downarrow z = \bigcap_\alpha \downarrow x_\alpha.$$

From this we can derive the universal property like before. Instead we move ahead to the general picture.

Definition 2.6. Let $(X, \leqslant)$ be a preordered set. A **diagram** is an order-preserving map $J : (D, \leqslant) \to (X, \leqslant)$. In other words, a diagram is a functor $J : \mathcal{C}\ell(D, \leqslant) \to \mathcal{C}\ell(X, \leqslant)$.

To every diagram $J : (D, \leqslant) \to (X, \leqslant)$, we can create a downward closed set $\downarrow J$ in $\mathrm{Dwd}(X)$:

$$\downarrow J \stackrel{\text{def}}{=} \bigcap_{i \in D} \downarrow (J(i)).$$

Thus, $x \in \downarrow J$ if and only if $x \leqslant J(i)$ for every $i \in D$.

Definition 2.7. Given a diagram $J : (D, \leqslant) \to (X, \leqslant)$, a **limit** of J is an element $z \in X$ such that

$$\downarrow z = \downarrow J$$

as elements of $\mathrm{Dwd}(X)$. Similar to how conjunctions were written using $\wedge$, e.g., "$x \wedge y$", we write for limits

$$z \cong \varprojlim_{i \in D} J(i) \text{ or } \cong \varprojlim J.$$

There are two equivalent ways to describe limits in a preorder. First, we can describe a limit by its membership function

Observation 2.8. *Given a diagram $J : (D, \leqslant) \to (X, \leqslant)$ and an element $z \in X$, z is a limit for J if and only if, for any element $x \in X$,*

$$\downarrow z(x) = \bigwedge_{i \in D} \downarrow J(i)(x).$$

Proposition 2.9. *In a preorder $(X, \leqslant)$, limits are infima. Formally, given a diagram $J : (D, \leqslant) \to (X, \leqslant)$,*

$$\varprojlim J \cong \inf \{J(i) \in X \mid i \in D\}.$$

Proof. The universal property of the infimum on the right is: for any element $x \in X$,

$$x \leqslant \inf \{J(i) \in X \mid i \in D\} \text{ iff } \bigwedge_{i \in D} x \leqslant J(i).$$

By the preceding observation this is the same universal property as the limit $\varprojlim J$. □

> The reason that limits are the same as infima for preorders is because the classifying category $\mathcal{C}\ell(X, \leqslant)$ of a preorder is *thin*. That is, between any two objects x, y of $\mathcal{C}\ell(X, \leqslant)$, there is at most one arrow $x \to y$. More generally, we need to impose certain commutativity conditions which we will explore later.

There is one special diagram to explore. The empty set $\emptyset$ has a (unique) structure of a preordered set. Therefore, we may consider the empty map $J_\emptyset : \emptyset \to (X, \leqslant)$ as a diagram, vacuously. We can ask if X has any limits for this diagram. A limit for $J_\emptyset$ is an element $z \in X$ such that

$$\downarrow z = \bigcap_{\_ \in \emptyset} \downarrow J(\_) = X,$$

since empty intersections yield the entire preorder. This can also be seen by comparing membership functions: $x \in \downarrow z$ if and only if, for any $\_$ in $\emptyset$, $x \in \downarrow J(\_)$. Since the latter condition is always vacuously satisfied, $x \in \downarrow z$ is always satisfied, so $\downarrow z = X$.

Definition 2.10. A **terminal object** is a limit of the empty diagram. For a preorder X, a terminal object is the same as a maximum for X.

The reverse situation

We have related limits to infima. We now describe the category theory for suprema. These are called *colimits*. Their definition is more indirect—at least from the downset perspective. We turn to Observation 2.8, which describes the downset of a limit as a large conjunction. We can swap the role of z and x on the right side of that equation. This yields a colimit.

Definition 2.11. Let $J : (D, \leqslant) \to (X, \leqslant)$ be a diagram. A **colimit** of J is an object $z \in X$ such that, for any element $x \in X$,

$$\downarrow x(z) = \bigwedge_{i \in D} \downarrow J(i)(x).$$

We write a colimit using similar notation as the limit:

$$z \cong \varinjlim_{i \in D} J(i) \cong \varinjlim J.$$

> For emphasis, below is the equation characterizing a limit of the diagram J, where the difference to the colimit equation highlighted.
>
> $$\downarrow z(x) = \bigwedge_{i \in D} \downarrow J(i)(x).$$

Proposition 2.12. *Given a diagram $J : (D, \leqslant) \to (X, \leqslant)$, z is a colimit for J if and only if, for any $x \in X$,*

$$z \leqslant x \textit{ iff } \forall i \in D, J(i) \leqslant x.$$

In other words, $z \cong \sup \{J(i) \mid i \in D\}$, or in terms of downsets

$$\downarrow z = \bigvee_{i \in D} \downarrow J(i).$$

Similar to the conjunctions from earlier, we can form disjunctions using a colimit.

Definition 2.13. Let $(X, \leqslant)$ be a preorded set, and let x, y be a pair of elements of X. Let $\{a, b\}$ denote the preordered set with no relation imposed on a and b. The function $J : (a, b) \to (X, \leqslant)$ sending a to x and b to y is order-preserving, so it is a diagram. A **disjunction** for x and y is a colimit for this diagram. That is, an element $x \vee y$ is a disjunction for x and y if and only if, for any element $v \in X$,

$$x \vee y \leqslant v \text{ iff } x \leqslant v \text{ and } y \leqslant v.$$

A disjunction's universal property can be expressed diagramtically, like for conjunctions (or any limit or colimit, for that matter). Here is the diagram for the disjunction of x and y.

$$\begin{tikzcd}
 & x \arrow[d] \arrow[ddr, bend left] & \\
y \arrow[r] \arrow[drr, bend right] & x \vee y \arrow[dr, dashed, "\exists!"] & \\
 & & v
\end{tikzcd}$$

From the diagram we can see an important relationship between limits and colimits. If we replace the order $\leqslant$ on X with its reverse $\geqslant$, then the ensuing classifying category $\mathcal{C}\ell(X, \geqslant)$ is the opposite category of $\mathcal{C}\ell(X, \leqslant)$, also called $\mathcal{C}\ell(X, \leqslant)^{\mathsf{op}}$. All the arrows in this category have flipped direction. If we flip the arrows in the above diagram, we get the same picture for a conjunction, just rotated 180 degrees (which does not change the universal property!). In other words:

Observation 2.14. *A disjunction in $(X, \leqslant)$ is the same as a conjunction in $(X, \geqslant)$.*

More generally, a colimit of $\mathcal{C}\ell(X, \leqslant)$ is the same as a limit of $\mathcal{C}\ell(X, \leqslant)^{\mathsf{op}}$. This arrow-reversing relationship is the reason x and z are swapped in Definition 2.11.

We wrap things up by describing the colimit analogue of a terminal object.

Definition 2.15. Let $(X, \leqslant)$ be a preordered set. An **initial object** of X is a colimit for the empty diagram $J_\emptyset : \emptyset \to (X, \leqslant)$.

Equivalently, an initial object is an element $z \in X$ such that, for any $x \in X$,

$$z \leqslant x \text{ iff } \forall \_ \in \emptyset, J_\emptyset(\_) \leqslant x.$$

The latter condition is always true, vacuously. Therefore, $z \leqslant x$ for every $x \in X$, so an initial object of a preorder is the same as a minimum.

Indexed propositions

The key takeaway from these constructions of limits and colimits is that they are described by operations on the downsets of elements of $(X, \leqslant)$. While X may not be a Heyting algebra, we can inherit the language of Heyting algebras by working in $\mathsf{Dwd}(X)$: the downset of a limit is an intersection of downsets, and the downset of a colimit is a union of downsets.

We may take an alternative but equivalent approach. Recall (from the Grothendieck construction for preorders) that a downward closed set is the same as a functor $\mathcal{C}\ell(X, \leqslant)^{\mathsf{op}} \to \Omega$, where Ω is the preorder of truth values. Under this correspondence, intersection of downward closed sets is identified with the operation

$$\mathsf{and} : \Omega \times \Omega \to \Omega,$$

and union of downward closed sets is identified with the dual

$$\mathtt{or} : \Omega \times \Omega \to \Omega.$$

For example, for any $v \in X$

$$\downarrow(x \wedge y)(v) = \downarrow x(v) \;\mathsf{and}\; \downarrow y(v).$$

Indeed, for any operation on propositions $\Omega \times \ldots \times \Omega \to \Omega$, we can create a universal property on downsets/indexed propositions to explore in a preorder.

Definition 2.16. Given an operation $\delta : \Omega^n \to \Omega$, this operation is **representable** in a preorder $(X, \leqslant)$ if, for any family of objects $x_1, \ldots, x_n$, there exists an object $z \in X$ such that for any $v \in X$

$$\downarrow z(v) = \delta(\downarrow x_1(v), \ldots, \downarrow x_n(v)).$$

Dually, δ is **co-representable** in $(X, \leqslant)$ if it is representable in $(X, \geqslant)$. Equivalently, δ is corepresentable if, for any family of objects $x_1, \ldots, x_n$, there exists an object $z \in X$ such that for any $v \in X$

$$\downarrow v(z) = \delta(\downarrow x_1(v), \ldots, \downarrow x_n(v)).$$

In this sense, a preorder has conjunctions if and is representable. This notion of representability will later allow us to describe the relation between a topos and a universe of sets. In this sense, a topos will be a category where many of the operations from set theory are either representable or co-representable.

3 Yoneda and Generalized Objects

Given a preorder $(X, \leqslant)$ the elements $x \in X$ are subsumed by their downsets $\downarrow x \in \mathrm{Dwd}(X)$. We saw this in the Yoneda lemma, where the preorder $\leqslant$ on X is modeled by the subset inclusion $\subseteq$ on $\mathrm{Dwd}(X)$. We also saw this in the previous section, where universal properties, such as infima and suprema, can be described in terms of unions and intersections of downward closed sets. Regarding the other downward-closed sets $A \in \mathrm{Dwd}(X)$, since they are downward-closed, they behave like elements in X, as far as the preordering relation is concerned. For this reason, downward-closed sets (like their categorified cousins, presheaves) are *generalized objects* of X. One way to witness this is through the *co-Yoneda lemma*.

Lemma 3.1 (Co-Yoneda Lemma for preorders)**.** *Let $A \in \mathrm{Dwd}(X)$ be a downward closed set. The membership function $\chi_A : \mathcal{C}\ell(X, \leqslant)^{\mathsf{op}} \to \Omega$ satisfies the equation*

$$\chi_A(y) = \bigvee_{x \in X} (\chi_A(x) \wedge \downarrow x(y)) = \bigvee_{x \in A} \downarrow x(y).$$

The co-Yoneda lemmma shows that any downward-closed set $A \in \mathrm{Dwd}(X)$ is a union of sets of the form $\downarrow x$, for objects $x \in X$. These equations come from the co-Yoneda lemma for presheaves.

$\bigvee_{x \in X} (\chi_A(x) \wedge \downarrow x)$	is a special case of	$\int^{x:X} \chi_A(x) \cdot よ(x).$
$\bigvee_{x \in A} \downarrow x$	is a special case of	$\varinjlim \left(\mathrm{el}(A) \to X \xrightarrow{よ} \mathrm{Psh}(X) \right).$

We will present two proofs. The first is a direct proof. The second follows the structure of the proof of the general Co-Yoneda lemma. Before the proofs, we discuss how the co-Yoneda lemma allows us to treat downward closed sets like generalized objects.

First proof. Let y be an arbitrary element of X. If $\chi_A(y) = \mathsf{True}$, then $\chi_A(y) \wedge \downarrow y(y)$ is also True. Therefore the right side of the equation is True. Conversely, suppose $\bigvee_{x \in X} (\chi_A(x) \wedge \downarrow x(y)) = \mathsf{True}$. Then there exists some $x_0 \in X$ such that $\chi_A(x_0) \wedge \downarrow x_0(y) = \mathsf{True}$. In particular, $x_0 \in A$ and $y \leqslant x_0$. Since A is downward closed, this implies $y \in A$, so $\chi_A(y) = \mathsf{True}$. Thus, the left side of the equation evaluates to True if and only if the right side of the equation evaluates to True, as desired. □

Second proof. We begin by making a few observations. First, we can impose a preorder on order-reversing functions $f, g : (X, \leqslant) \to \Omega$ by defining

$$f \leqslant g \text{ if and only if } \forall x \in X. f(x) \leqslant g(x).$$

Secondly, given two order-reversing functions $f_1, f_2 : (X, \leqslant) \to \Omega$, $f_1 = f_2$ if and only if, for any other order-reversing function $g : (X, \leqslant) \to \Omega$, $f_1 \leqslant g$ if and only if $f_2 \leqslant g$.[1] Furthermore, given two order-reversing functions $f, g : (X, \leqslant) \to \Omega$, we may define a new function $f \Rightarrow g : (X, \leqslant) \to \Omega$ using the usual implication

$$(f \Rightarrow g)(x \in X) \stackrel{\text{def}}{=} f(x) \Rightarrow g(x) = \begin{cases} \mathsf{True} & f(x) \leqslant g(x) \\ \mathsf{False} & \text{otherwise.} \end{cases}$$

Lastly, we can combine the Yoneda lemma for preorders and the last observation to deduce $(\downarrow x \Rightarrow g)(y) = g(x)$ for any input $y \in X$. Based on these observations, in order to prove the desired equation

$$\chi_A(y) = \bigvee_{x \in X} (\chi_A(x) \wedge \downarrow x(y)),$$

it suffices to prove the following. For any order-reversing $g : X \to \Omega$,

$$\chi_A \leqslant g \text{ if and only if } \bigvee_{x \in X} (\chi_A(x) \wedge \downarrow x) \leqslant g.$$

We can now chase a chain of implications

$$\begin{aligned} & \bigvee_{x \in X} (\chi_A(x) \wedge \downarrow x) \leqslant g \\ \text{iff } & \bigwedge_{x \in X} ((\chi_A(x) \wedge \downarrow x) \leqslant g) \\ \text{iff } & \bigwedge_{x \in X} (\chi_A(x) \leqslant \downarrow x \Rightarrow g) \\ \text{iff } & \bigwedge_{x \in X} (\chi_A(x) \leqslant g(x)) \\ \text{iff } & \chi_A \leqslant g. \end{aligned}$$

□

[1] The second observation is the Yoneda lemma. We can use the same trick, supplying f_1 and f_2 for g, to prove it. Indeed, one could simply invoke the Yoneda lemma for preorders, because the class of order-preserving functions between two preordered sets is a set. When generalizing to presheaves, this is no longer the case, so one encounters the usual set-theoretic obstacles.