TABULATION OF CLASS NUMBERS OF IMAGINARY QUADRATIC FIELDS OF DISCRIMINANTS $|\Delta| \equiv 7, 15 \pmod{24}$ AND $|\Delta| \equiv 23, 47, 95 \pmod{120}$

M. J. JACOBSON, JR. AND A. MOSUNOV

1. Main Theorems

For $\mathbf{a} \in \mathbb{N}^3$ and $\mathbf{x} \in \mathbb{Z}^3$, we define

$$Q_{\mathbf{a}}(\mathbf{x}) = \sum_{j=1}^{3} a_j x_j^2.$$

For a non-negative integer n, let $r_{\mathbf{a}}(n)$ denote the number of solutions to the equation

$$Q_{\mathbf{a}}(\mathbf{x}) = n.$$

Define

$$\vartheta_{a,m}(q) = \sum_{k=-\infty}^{+\infty} q^{(mk+a)^2}, \quad \nabla_{a,m}(q) = \sum_{k=-\infty}^{+\infty} q^{k(mk+a)/2},$$

$$\vartheta_3(q) = \nabla_{0,1}(q^2) = 1 + 2\sum_{k=0}^{\infty} q^{k^2} = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \cdots,$$

$$\nabla(q) = \tfrac{1}{2}\nabla_{1,1}(q) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = 1 + q + q^3 + q^6 + q^{10} + \cdots.$$

In this article, we will prove the following two theorems.

Theorem 1. *Let* $\mathbf{a} = (1\ 3\ 3)^T$. *Then*

$$\sum_{k=0}^{\infty} r_{\mathbf{a}}(24k+7)q^k = 8\nabla(q)\left[\nabla_{1,3}(q)\nabla(q) + \nabla_{1,3}(q^4)\vartheta_3(q^2) + 2q\nabla_{2,3}(q^4)\nabla(q^4)\right], \tag{1}$$

$$\sum_{k=0}^{\infty} r_{\mathbf{a}}(24k+15)q^k = 8\nabla(q)\left[\nabla(q^3)\nabla(q) + q\nabla(q^{12})\vartheta_3(q^2) + \vartheta_3(q^6)\nabla(q^4)\right]. \tag{2}$$

Theorem 2. *Let* $\mathbf{a} = (2\ 5\ 10)^T$. *Then*

$$\begin{aligned}\sum_{k=0}^{\infty} r_{\mathbf{a}}(120k+23)q^k &= \nabla_{1,3}(q)(8\nabla_{2,15}(q^2)\nabla_{1,3}(q^2) + 4q\nabla_{8,15}(q^2)\nabla_{1,3}(q^2) + 4q\nabla_{7,15}(q^2)\nabla_{2,3}(q^2)\\ &\quad + 4q^3\nabla_{13,15}(q^2)\nabla_{2,3}(q^2) + 4q^3\nabla_{12,15}(q^2)\nabla(q^6) + 4\nabla_{3,15}(q^2)\vartheta_3(q^3))\\ &\quad + \nabla(q^3)(8q\nabla_{2,15}(q^2)\nabla(q^6) + 8q^2\nabla_{8,15}(q^2)\nabla(q^6) + 4q\nabla_{7,15}(q^2)\vartheta_3(q^3) + 4q^3\nabla_{13,15}(q^2)\vartheta_3(q^3))\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{\infty} r_{\mathbf{a}}(120k+47)q^k &= \nabla_{1,3}(q)(4\nabla_{4,15}(q^2)\nabla_{1,3}(q^2) + 4q^3\nabla_{14,15}(q^2)\nabla_{1,3}(q^2) + 4\nabla_{1,15}(q^2)\nabla_{2,3}(q^2)\\ &\quad + 4q^2\nabla_{11,15}(q^2)\nabla_{2,3}(q^2) + 4q\nabla_{6,15}(q^2)\nabla(q^6) + 2q\nabla_{9,15}(q^2)\vartheta_3(q^3))\\ &\quad + \nabla(q^3)(8q\nabla_{4,15}(q^2)\nabla(q^6) + 8q^4\nabla_{14,15}(q^2)\nabla(q^6) + 4\nabla_{1,15}(q^2)\vartheta_3(q^3) + 4q^2\nabla_{11,15}(q^2)\vartheta_3(q^3)).\end{aligned}$$

2010 *Mathematics Subject Classification.* Primary 11E76.

$$\sum_{k=0}^{\infty} r_{\mathbf{a}}(120k+95)q^k = \nabla_{1,3}(q)(4q\nabla_{10,15}(q^2)\nabla_{1,3}(q^2) + 4\nabla_{5,15}(q^2)\nabla_{2,3}(q^2) + 2\vartheta_3(q^{15})\nabla(q^6) + 2q^3\nabla(q^{30})\vartheta_3(q^3)) \\ + \nabla(q^3)(4\nabla_{5,15}(q^2)\vartheta_3(q^3) + 8q^2\nabla_{10,15}(q^2)\nabla(q^6)).$$

By the result of Bringmann and Kane, if $\mathbf{a} = (1\ 3\ 3)^T$, $n \equiv 7 \pmod 8$ and $9 \nmid n$, then

$$r_{\mathbf{a}}(n) = 8\left(1 + \left(\frac{n}{3}\right)\right) H(n),$$

and if $\mathbf{a} = (2\ 5\ 10)^T$, $n \equiv 23 \pmod{24}$ and $25 \nmid n$, then

$$r_{\mathbf{a}}(n) = 2\left(1 - \left(\frac{n}{5}\right)\right) H(n).$$

Thus we obtain the following corollaries to Theorems 1 and 2.

Corollary 3.

$$2\sum_{k=0}^{\infty} H(24k+7)q^k = \nabla(q)\left[\nabla_{1,3}(q)\nabla(q) + \nabla_{1,3}(q^4)\vartheta_3(q^2) + 2q\nabla_{2,3}(q^4)\nabla(q^4)\right],$$

$$2\sum_{k=0}^{\infty} H(24k+15)q^k = \nabla(q)\left[\nabla(q^3)\nabla(q) + q\nabla(q^{12})\vartheta_3(q^2) + \vartheta_3(q^6)\nabla(q^4)\right].$$

Corollary 4.

$$\sum_{k=0}^{\infty} H(120k+23)q^k = \nabla_{1,3}(q)(2\nabla_{2,15}(q^2)\nabla_{1,3}(q^2) + q\nabla_{8,15}(q^2)\nabla_{1,3}(q^2) + q\nabla_{7,15}(q^2)\nabla_{2,3}(q^2) \\ + q^3\nabla_{13,15}(q^2)\nabla_{2,3}(q^2) + q^3\nabla_{12,15}(q^2)\nabla(q^6) + \nabla_{3,15}(q^2)\vartheta_3(q^3)) \\ + \nabla(q^3)(2q\nabla_{2,15}(q^2)\nabla(q^6) + 2q^2\nabla_{8,15}(q^2)\nabla(q^6) + q\nabla_{7,15}(q^2)\vartheta_3(q^3) + q^3\nabla_{13,15}(q^2)\vartheta_3(q^3))$$

$$2\sum_{k=0}^{\infty} H(120k+47)q^k = \nabla_{1,3}(q)(2\nabla_{4,15}(q^2)\nabla_{1,3}(q^2) + 2q^3\nabla_{14,15}(q^2)\nabla_{1,3}(q^2) + 2\nabla_{1,15}(q^2)\nabla_{2,3}(q^2) \\ + 2q^2\nabla_{11,15}(q^2)\nabla_{2,3}(q^2) + 2q\nabla_{6,15}(q^2)\nabla(q^6) + q\nabla_{9,15}(q^2)\vartheta_3(q^3)) \\ + \nabla(q^3)(4q\nabla_{4,15}(q^2)\nabla(q^6) + 4q^4\nabla_{14,15}(q^2)\nabla(q^6) + 2\nabla_{1,15}(q^2)\vartheta_3(q^3) + 2q^2\nabla_{11,15}(q^2)\vartheta_3(q^3)).$$

$$2\sum_{k=0}^{\infty} H(120k+95)q^k = \nabla_{1,3}(q)(2q\nabla_{10,15}(q^2)\nabla_{1,3}(q^2) + 2\nabla_{5,15}(q^2)\nabla_{2,3}(q^2) + \vartheta_3(q^{15})\nabla(q^6) + q^3\nabla(q^{30})\vartheta_3(q^3)) \\ + \nabla(q^3)(2\nabla_{5,15}(q^2)\vartheta_3(q^3) + 4q^2\nabla_{10,15}(q^2)\nabla(q^6)).$$

Before we outline the proof of Theorems 1 and 2, let us introduce some notation and prove four auxiliary lemmas. For a positive integer m and an integer a, we denote the set of all numbers congruent to a modulo m by $[a]_m$. If $a_1, \ldots, a_k$ and $m_1, \ldots, m_k$ are fixed, we refer to $([\pm a_1]_{m_1}, \ldots, [\pm a_k]_{m_k})$ as a *k-tuple of congruence classes*. If $2a \equiv 0 \pmod m$, we write $[a]_m$ instead of $[\pm a]_m$ for brevity. Given a k-tuple of congruence classes $\mathcal{A} = ([\pm a_1]_{m_1}, \ldots, [\pm a_k]_{m_k})$, for $x_1, \ldots, x_k \in \mathbb{Z}$ we write $(x_1, \ldots, x_k) \in \mathcal{A}$ if and only if $x_i \equiv \pm a_i \pmod{m_i}$ for all $i = 1, \ldots, k$. If $\mathcal{A}_1, \ldots, \mathcal{A}_m$ are k-tuples of congruence classes, we write $(x_1, \ldots, x_k) \in \bigcup_{i=1}^m \mathcal{A}_i$ if and only if there exists $i \in \{1, \ldots, m\}$ such that $(x_1, \ldots, x_k) \in \mathcal{A}_i$.

The first two lemmas follows from basic number theoretical observations.

Lemma 5. *Let n be a non-negative integer, and suppose that $x^2 + 3y^2 + 3z^2 = n$ for some integers x, y, z.*

- *If* $n \equiv 7 \pmod{24}$, *then*

$$(x_1, x_2, x_3) \in ([\pm 1]_6, [1]_2, [1]_2) \cup ([\pm 2]_{12}, [0]_4, [1]_2) \cup ([\pm 2]_{12}, [1]_2, [0]_4) \cup ([\pm 4]_{12}, [2]_4, [1]_2) \cup ([\pm 4]_{12}, [1]_2, [2]_4)\,.$$

- *If* $n \equiv 15 \pmod{24}$, *then*

$$(x_1, x_2, x_3) \in ([3]_6, [1]_2, [1]_2) \cup ([6]_{12}, [0]_4, [1]_2) \cup ([6]_{12}, [1]_2, [0]_4) \cup ([0]_{12}, [2]_4, [1]_2) \cup ([0]_{12}, [1]_2, [2]_4)\,.$$

Lemma 6. *Let n be a non-negative integer, and suppose that $2x^2 + 5y^2 + 10z^2 = n$ for some integers x, y, z.*

- *If* $n \equiv 23 \pmod{120}$, *then*

$$(x_1, x_2, x_3) \in ([\pm 2]_{30}, [\pm 1]_6, [\pm 1]_6) \cup ([\pm 2]_{30}, [3]_6, [3]_6) \cup ([\pm 3]_{30}, [\pm 1]_6, [0]_6) \cup ([\pm 7]_{30}, [\pm 1]_6, [\pm 2]_6) \cup ([\pm 7]_{30}, [3]_6, [0]_6) \cup ([\pm 8]_{30}, [\pm 1]_6, [\pm 1]_6) \cup ([\pm 8]_{30}, [3]_6, [3]_6) \cup ([\pm 12]_{30}, [\pm 1]_6, [3]_6) \cup ([\pm 13]_{30}, [\pm 1]_6, [\pm 2]_6) \cup ([\pm 13]_{30}, [3]_6, [0]_6)\,.$$

- *If* $n \equiv 47 \pmod{120}$, *then*

$$(x_1, x_2, x_3) \in ([\pm 1]_{30}, [\pm 1]_6, [\pm 2]_6) \cup ([\pm 1]_{30}, [3]_6, [0]_6) \cup ([\pm 4]_{30}, [\pm 1]_6, [\pm 1]_6) \cup ([\pm 4]_{30}, [3]_6, [3]_6) \cup ([\pm 6]_{30}, [\pm 1]_6, [3]_6) \cup ([\pm 9]_{30}, [\pm 1]_6, [0]_6) \cup ([\pm 11]_{30}, [\pm 1]_6, [\pm 2]_6) \cup ([\pm 11]_{30}, [3]_6, [0]_6) \cup ([\pm 14]_{30}, [\pm 1]_6, [\pm 1]_6) \cup ([\pm 14]_{30}, [3]_6, [3]_6)\,.$$

- *If* $n \equiv 95 \pmod{120}$, *then*

$$(x_1, x_2, x_3) \in ([0]_{30}, [\pm 1]_6, [3]_6) \cup ([\pm 5]_{30}, [\pm 1]_6, [\pm 2]_6) \cup ([\pm 5]_{30}, [3]_6, [0]_6) \cup ([\pm 10]_{30}, [\pm 1]_6, [\pm 1]_6) \cup ([\pm 10]_{30}, [3]_6, [3]_6) \cup ([15]_{30}, [\pm 1]_6, [0]_6)\,.$$

For a non-negative integer n and a 3-tuple of congruence classes $\mathcal{A}$, let $r_{\mathbf{a},\mathcal{A}}(n)$ denote the number of solutions to the equation $Q_{\mathbf{a}}(\mathbf{x}) = n$ such that $(x_1, x_2, x_3) \in \mathcal{A}$.

Lemma 7. *Let* $\mathbf{a} = (1\ 3\ 3)^T$.

1. *If* $\mathcal{A} = ([\pm 1]_6, [1]_2, [1]_2)$, *then*

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(24k+7) q^k = 8\nabla_{1,3}(q)\nabla(q)^2.$$

2. *If* $\mathcal{A} = ([\pm 2]_{12}, [0]_4, [1]_2)$, *then*

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(24k+7) q^k = 4\nabla_{1,3}(q^4)\vartheta_3(q^2)\nabla(q).$$

3. *If* $\mathcal{A} = ([\pm 4]_{12}, [2]_4, [1]_2)$, *then*

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(24k+7) q^k = 8q\nabla_{2,3}(q^4)\nabla(q^4)\nabla(q).$$

4. *If* $\mathcal{A} = ([3]_6, [1]_2, [1]_2)$, *then*

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(24k+15) q^k = 8\nabla(q^3)\nabla(q)^2.$$

5. If $\mathcal{A} = ([6]_{12}, [0]_4, [1]_2)$, then

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(24k+15)q^k = 4q\nabla(q^{12})\vartheta_3(q^2)\nabla(q).$$

6. If $\mathcal{A} = ([0]_{12}, [2]_4, [1]_2)$, then

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(24k+15)q^k = 4\vartheta_3(q^6)\nabla(q^4)\nabla(q).$$

Proof. We will prove only Part 1, as the other parts can be established analogously. Let $\mathcal{A} = ([\pm 1]_6, [1]_2, [1]_2)$. For a positive integer m and an integer a, define

$$\vartheta_{a,m}(q) = \sum_{k=-\infty}^{+\infty} q^{(mk+a)^2}.$$

We claim that

$$\sum_{n=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(n)q^n = 2\vartheta_{1,6}(q)\vartheta_{1,2}(q^3)^2. \tag{3}$$

To see that this is the case, first note that the n-th coefficient of the series

$$2\vartheta_{1,6}(q) = \sum_{x \,\equiv\, \pm 1 \pmod 6} q^{x^2} = 2q + 2q^{25} + 2q^{49} + 2q^{121} + 2q^{169} + \cdots$$

is equal to the number of solutions to the equation $x^2 = n$ with $x \equiv \pm 1 \pmod 6$. Second, observe that the n-th coefficient of the series

$$\vartheta_{1,2}(q^3) = \sum_{y \text{ odd}} q^{3y^2} = 2q^3 + 2q^{27} + 2q^{75} + 2q^{147} + \cdots$$

is equal to the number of solutions y to the equation $3y^2 = n$ with y odd. Consequently, the number of solutions to the equation $3(y^2+z^2) = n$ with y and z odd is equal to the n-th coefficient of the series $\vartheta_{1,2}(q^3)^2$, since

$$\vartheta_{1,2}(q^3)^2 = \left(\sum_{y \text{ odd}} q^{3y^2}\right)^2 = \sum_{y \text{ odd}} q^{3y^2} \times \sum_{z \text{ odd}} q^{3z^2} = \sum_{y,\, z \text{ odd}} q^{3(y^2+z^2)}.$$

Finally, we conclude that

$$\begin{aligned} 2\vartheta_{1,6}(q)\vartheta_{1,2}(q^3)^2 &= \sum_{x \,\equiv\, \pm 1 \pmod 6} q^{x^2} \times \sum_{y,\, z \text{ odd}} q^{3(y^2+z^2)} \\ &= \sum_{(x_1,x_2,x_3)\in\mathcal{A}} q^{x^2+3(y^2+z^2)} \\ &= \sum_{n=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(n)q^n. \end{aligned}$$

Next, observe that $n \not\equiv 7 \pmod{24}$ implies $r_{\mathbf{a},\mathcal{A}}(n) = 0$. Hence we can rewrite (3) as

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(24k+7)q^{24k+7} = 2\vartheta_{1,6}(q)\vartheta_{1,2}(q^3)^2.$$

Substituting q with $q^{1/24}$ and then multiplying both sides by $q^{-7/24}$, we obtain

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(24k+7)q^k = 2\left[q^{-1/24}\vartheta_{1,6}(q^{1/24})\right]\cdot\left[q^{-1/8}\vartheta_{1,2}(q^{1/8})\right]^2.$$

The result follows once we note that

$$q^{-1/24}\vartheta_{1,6}(q^{1/24}) = q^{-1/24}\cdot\sum_{k=-\infty}^{+\infty} q^{(6k+1)^2/24} = \sum_{k=-\infty}^{+\infty} q^{k(3k+1)/2} = \nabla_{1,3}(q)$$

and

$$q^{-1/8}\vartheta_{1,2}(q^{1/8}) = q^{-1/8}\sum_{k=-\infty}^{+\infty} q^{(2k+1)^2/8} = \sum_{k=-\infty}^{+\infty} q^{k(k+1)/2} = 2\nabla(q).$$

□

Lemma 8. *Let* $\mathbf{a} = (2\ 5\ 10)^T$.

1. *If* $\mathcal{A} = ([\pm 2]_{30}, [\pm 1]_6, [\pm 1]_6)$, *then*
$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+23)q^k = 8\nabla_{2,15}(q^2)\nabla_{1,3}(q)\nabla_{1,3}(q^2).$$
2. *If* $\mathcal{A} = ([\pm 2]_{30}, [3]_6, [3]_6)$, *then*
$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+23)q^k = 8q\nabla_{2,15}(q^2)\nabla(q^3)\nabla(q^6).$$
3. *If* $\mathcal{A} = ([\pm 3]_{30}, [\pm 1]_6, [0]_6)$, *then*
$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+23)q^k = 4\nabla_{3,15}(q^2)\nabla_{1,3}(q)\vartheta_3(q^3).$$
4. *If* $\mathcal{A} = ([\pm 7]_{30}, [\pm 1]_6, [\pm 2]_6)$, *then*
$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+23)q^k = 4q\nabla_{7,15}(q^2)\nabla_{1,3}(q)\nabla_{2,3}(q^2).$$
5. *If* $\mathcal{A} = ([\pm 7]_{30}, [3]_6, [0]_6)$, *then*
$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+23)q^k = 4q\nabla_{7,15}(q^2)\nabla(q^3)\vartheta_3(q^3).$$
6. *If* $\mathcal{A} = ([\pm 8]_{30}, [\pm 1]_6, [\pm 1]_6)$, *then*
$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+23)q^k = 4q\nabla_{8,15}(q^2)\nabla_{1,3}(q)\nabla_{1,3}(q^2).$$
7. *If* $\mathcal{A} = ([\pm 8]_{30}, [3]_6, [3]_6)$, *then*
$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+23)q^k = 8q^2\nabla_{8,15}(q^2)\nabla(q^3)\nabla(q^6).$$
8. *If* $\mathcal{A} = ([\pm 12]_{30}, [\pm 1]_6, [3]_6)$, *then*
$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+23)q^k = 4q^3\nabla_{12,15}(q^2)\nabla_{1,3}(q)\nabla(q^6).$$

9. If $\mathcal{A} = ([\pm 13]_{30}, [\pm 1]_6, [\pm 2]_6)$, *then*

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+23)q^k = 4q^3\nabla_{13,15}(q^2)\nabla_{1,3}(q)\nabla_{2,3}(q^2).$$

10. If $\mathcal{A} = ([\pm 13]_{30}, [3]_6, [0]_6)$, *then*

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+23)q^k = 4q^3\nabla_{13,15}(q^2)\nabla(q^3)\vartheta_3(q^3).$$

11. If $\mathcal{A} = ([\pm 1]_{30}, [\pm 1]_6, [\pm 2]_6)$, *then*

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+47)q^k = 4\nabla_{1,15}(q^2)\nabla_{1,3}(q)\nabla_{2,3}(q^2).$$

12. If $\mathcal{A} = ([\pm 1]_{30}, [3]_6, [0]_6)$, *then*

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+47)q^k = 4\nabla_{1,15}(q^2)\nabla(q^3)\vartheta_3(q^3).$$

13. If $\mathcal{A} = ([\pm 4]_{30}, [\pm 1]_6, [\pm 1]_6)$, *then*

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+47)q^k = 4\nabla_{4,15}(q^2)\nabla_{1,3}(q)\nabla_{1,3}(q^2).$$

14. If $\mathcal{A} = ([\pm 4]_{30}, [3]_6, [3]_6)$, *then*

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+47)q^k = 8q\nabla_{4,15}(q^2)\nabla(q^3)\nabla(q^6).$$

15. If $\mathcal{A} = ([\pm 6]_{30}, [\pm 1]_6, [3]_6)$, *then*

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+47)q^k = 4q\nabla_{6,15}(q^2)\nabla_{1,3}(q)\nabla(q^6).$$

16. If $\mathcal{A} = ([\pm 9]_{30}, [\pm 1]_6, [0]_6)$, *then*

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+47)q^k = 2q\nabla_{9,15}(q^2)\nabla_{1,3}(q)\vartheta_3(q^3).$$

17. If $\mathcal{A} = ([\pm 11]_{30}, [\pm 1]_6, [\pm 2]_6)$, *then*

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+47)q^k = 4q^2\nabla_{11,15}(q^2)\nabla_{1,3}(q)\nabla_{2,3}(q^2).$$

18. If $\mathcal{A} = ([\pm 11]_{30}, [3]_6, [0]_6)$, *then*

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+47)q^k = 4q^2\nabla_{11,15}(q^2)\nabla(q^3)\vartheta_3(q^3).$$

19. If $\mathcal{A} = ([\pm 14]_{30}, [\pm 1]_6, [\pm 1]_6)$, *then*

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+47)q^k = 4q^3\nabla_{14,15}(q^2)\nabla_{1,3}(q)\nabla_{1,3}(q^2).$$

20. If $\mathcal{A} = ([\pm 14]_{30}, [3]_6, [3]_6)$*, then*

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+47)q^k = 8q^4 \nabla_{14,15}(q^2)\nabla(q^3)\nabla(q^6).$$

21. If $\mathcal{A} = ([0]_{30}, [\pm 1]_6, [3]_6)$*, then*

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+95)q^k = 2\vartheta_3(q^{15})\nabla_{1,3}(q)\nabla(q^6).$$

22. If $\mathcal{A} = ([\pm 5]_{30}, [\pm 1]_6, [\pm 2]_6)$*, then*

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+95)q^k = 4\nabla_{5,15}(q^2)\nabla_{1,3}(q)\nabla_{2,3}(q^2).$$

23. If $\mathcal{A} = ([\pm 5]_{30}, [3]_6, [0]_6)$*, then*

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+95)q^k = 4\nabla_{5,15}(q^2)\nabla(q^3)\vartheta_3(q^3).$$

24. If $\mathcal{A} = ([\pm 10]_{30}, [\pm 1]_6, [\pm 1]_6)$*, then*

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+95)q^k = 4q\nabla_{10,15}(q^2)\nabla_{1,3}(q)\nabla_{1,3}(q^2).$$

25. If $\mathcal{A} = ([\pm 10]_{30}, [3]_6, [3]_6)$*, then*

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+95)q^k = 8q^2\nabla_{10,15}(q^2)\nabla(q^3)\nabla(q^6).$$

26. If $\mathcal{A} = ([15]_{30}, [\pm 1]_6, [0]_6)$*, then*

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+95)q^k = 2q^3\nabla(q^{30})\nabla_{1,3}(q)\vartheta_3(q^3).$$

Proof. We will prove only Part 1, as the other parts can be established analogously. Let $\mathcal{A} = ([\pm 2]_{30}, [\pm 1]_6, [\pm 1]_6)$. We claim that

$$\sum_{n=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(n)q^n = 8\nabla_{2,15}(q^2)\nabla_{1,3}(q)\nabla_{1,3}(q^2). \tag{4}$$

To see that this is the case, note that

- the coefficient of q^m of the series
$$2\vartheta_{2,30}(q^2) = \sum_{x \equiv \pm 2 \pmod{30}} q^{2x^2} = 2q^8 + 2q^{1568} + 2q^{2048} + 2q^{6728} \cdots$$
is equal to the number of solutions to the equation $2x^2 = m$ with $x \equiv \pm 2 \pmod{30}$.
- the coefficient of q^m of the series
$$2\vartheta_{1,6}(q^5) = \sum_{y \equiv \pm 1 \pmod 6} q^{5y^2} = 2q^5 + 2q^{125} + 2q^{245} + 2q^{605} + \cdots$$
is equal to the number of solutions to the equation $5y^2 = m$ with $x \equiv \pm 1 \pmod 6$;

- the coefficient of q^m of the series
$$2\vartheta_{1,6}(q^{10}) = \sum_{z \equiv \pm 1 \pmod 6} q^{10z^2} = 2q^{10} + 2q^{250} + 2q^{490} + 2q^{1210} + \cdots$$
is equal to the number of solutions to the equation $10z^2 = m$ with $x \equiv \pm 1 \pmod 6$.

Combining the above observations, we find that

$$\begin{aligned} 8\vartheta_{2,30}(q^2)\vartheta_{1,6}(q^5)\vartheta_{1,6}(q^{10}) &= \sum_{x \equiv \pm 2 \pmod{30}} q^{2x^2} \times \sum_{y \equiv \pm 1 \pmod 6} q^{5y^2} \times \sum_{z \equiv \pm 1 \pmod 6} q^{10z^2} \\ &= \sum_{(x,y,z)\in\mathcal{A}} q^{2x^2+5y^2+10z^2} \\ &= \sum_{n=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(n) q^n. \end{aligned}$$

Next, observe that $n \not\equiv 23 \pmod{120}$ implies $r_{\mathbf{a},\mathcal{A}}(n) = 0$. Hence we can rewrite (4) as

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+23) q^{120k+23} = 8\vartheta_{2,30}(q^2)\vartheta_{1,6}(q^5)\vartheta_{1,6}(q^{10}).$$

Substituting q with $q^{1/120}$ and then multiplying both sides by $q^{-23/120}$, we obtain

$$\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}}(120k+23) q^k = 8\left[q^{-1/15}\vartheta_{2,30}(q^{1/60})\right]\cdot\left[q^{-1/24}\vartheta_{1,6}(q^{1/24})\right]\cdot\left[q^{-1/12}\vartheta_{1,6}(q^{1/12})\right].$$

The result follows once we note that

$$q^{-1/15}\vartheta_{2,30}(q^{1/60}) = \sum_{k=-\infty}^{+\infty} q^{k(15k+2)} = \nabla_{2,15}(q^2),$$

$$q^{-1/24}\vartheta_{1,6}(q^{1/24}) = \sum_{k=-\infty}^{+\infty} q^{k(3k+1)/2} = \nabla_{1,3}(q),$$

and

$$q^{-1/12}\vartheta_{1,6}(q^{1/12}) = \sum_{k=-\infty}^{+\infty} q^{3k(k+1)} = \nabla_{1,3}(q^2).$$

□

We can now prove Theorems 1 and 2.

Proof of Theorem 1. We will demonstrate only the identity (1), as (2) can be established analogously. Let n be an integer congruent to 7 modulo 24. By Lemma 5, the equality $x^2 + 3y^2 + 3z^2 = n$ implies that $(x_1, x_2, x_3) \in \mathcal{A}_i$, where $\mathcal{A}_i$ is one of the following five 3-tuples of congruence classes:

$$\begin{aligned} \mathcal{A}_1 = ([\pm 1]_6, [1]_2, [1]_2), \mathcal{A}_2 = ([\pm 2]_{12}, [0]_4, [1]_2), \mathcal{A}_3 = ([\pm 2]_{12}, [1]_2, [0]_4), \\ \mathcal{A}_4 = ([\pm 4]_{12}, [2]_4, [1]_2), \mathcal{A}_5 = ([\pm 4]_{12}, [1]_2, [2]_4). \end{aligned}$$

Hence

$$r_{\mathbf{a}}(n) = \sum_{i=1}^{5} r_{\mathbf{a},\mathcal{A}_i}(n).$$

Since $r_{\mathbf{a},\mathcal{A}_2}(n) = r_{\mathbf{a},\mathcal{A}_3}(n)$ and $r_{\mathbf{a},\mathcal{A}_4}(n) = r_{\mathbf{a},\mathcal{A}_5}(n)$, the identity (1) follows from Lemma 7. □

Proof of Theorem 2. We will demonstrate only the first identity, as the other two identities can be established analogously. Let n be a positive integer congruent to 23 modulo 120. By Lemma 6, the equality $2x^2 + 5y^2 + 10z^2 = n$ implies that $(x_1, x_2, x_3) \in \mathcal{A}_i$, where $\mathcal{A}_i$ is one of the following five 3-tuples of congruence classes:

$$\begin{array}{lll}
\mathcal{A}_1 = ([\pm 2]_{30}, [\pm 1]_6, [\pm 1]_6)\,, & \mathcal{A}_2 = ([\pm 2]_{30}, [3]_6, [3]_6) & \mathcal{A}_3 = ([\pm 3]_{30}, [\pm 1]_6, [0]_6) \\
\mathcal{A}_4 = ([\pm 7]_{30}, [\pm 1]_6, [\pm 2]_6) & \mathcal{A}_5 = ([\pm 7]_{30}, [3]_6, [0]_6) & \mathcal{A}_6 = ([\pm 8]_{30}, [\pm 1]_6, [\pm 1]_6) \\
\mathcal{A}_7 = ([\pm 8]_{30}, [3]_6, [3]_6) & \mathcal{A}_8 = ([\pm 12]_{30}, [\pm 1]_6, [3]_6) & \mathcal{A}_9 = ([\pm 13]_{30}, [\pm 1]_6, [\pm 2]_6) \\
\mathcal{A}_{10} = ([\pm 13]_{30}, [3]_6, [0]_6) & &
\end{array}$$

Hence

$$r_{\mathbf{a}}(n) = \sum_{i=1}^{10} r_{\mathbf{a},\mathcal{A}_i}(n)$$

and

$$\sum_{k=0}^{\infty} r_{\mathbf{a}}(120k+23)q^k = \sum_{i=1}^{10} \left(\sum_{k=0}^{\infty} r_{\mathbf{a},\mathcal{A}_i}(120k+23)q^k \right).$$

The result follows from Lemma 8. □

2. Tabulation Bounds

In the previous work, we derived three series for the tabulation of class numbers $h(\Delta)$ when $|\Delta| \equiv 4, 8 \pmod{16}$ and $|\Delta| \equiv 3 \pmod 8$. In this article we derived five more tabulation formulas, namely for $|\Delta| \equiv 7, 15 \pmod{24}$ and for $|\Delta| \equiv 23, 47, 95 \pmod{120}$. The cases $|\Delta| \equiv 71, 119 \pmod{120}$ have to be handled with Jacobson-Ramachandran-Williams Algorithm. Below we present the bounds for tabulation of all $h(\Delta)$ for all $|\Delta|$ up to X.

$\lvert\Delta\rvert$	$X = 2^{40}$	$X = 2^{41}$	$X = 2^{42}$	$X = 2^{43}$	$X = 2^{44}$
$4 \pmod{16}$	2^{36}	2^{37}	2^{38}	2^{39}	2^{40}
$8 \pmod{16}$	2^{36}	2^{37}	2^{38}	2^{39}	2^{40}
$3 \pmod 8$	2^{37}	2^{38}	2^{39}	2^{40}	2^{41}
$7 \pmod{24}$	2^{36}	2^{37}	2^{38}	2^{39}	2^{40}
$15 \pmod{24}$	2^{36}	2^{37}	2^{38}	2^{39}	2^{40}
$23 \pmod{120}$	2^{34}	2^{35}	2^{36}	2^{37}	2^{38}
$47 \pmod{120}$	2^{34}	2^{35}	2^{36}	2^{37}	2^{38}
$95 \pmod{120}$	2^{34}	2^{35}	2^{36}	2^{37}	2^{38}
$71, 119 \pmod{120}$	2^{35}	2^{36}	2^{37}	2^{38}	2^{39}

For example in order to compute $h(\Delta)$ for all $|\Delta|$ up to $X = 2^{42}$, one would have to

- $4 \pmod{16}$: Multiply two polynomials of degree at most 2^{38};
- $8 \pmod{16}$: Multiply two polynomials of degree at most 2^{38};
- $3 \pmod 8$: Multiply two polynomials of degree at most 2^{39};
- $7 \pmod{24}$: Multiply two polynomials of degree at most 2^{38};
- $15 \pmod{24}$: Multiply two polynomials of degree at most 2^{38};
- $23 \pmod{120}$: Multiply two polynomials of degree at most 2^{36};
- $47 \pmod{120}$: Multiply two polynomials of degree at most 2^{36};

- $95 \pmod{120}$: Multiply two polynomials of degree at most 2^{36};
- $71, 119 \pmod{120}$: Compute $h(-120k-71)$ and $h(-120k-119)$ for at most 2^{37} values of k. Not all of these values have to be computed though, because the discriminants $-120k-71$, $-120k-119$ need not be fundamental (i.e., they are not squarefree). Assuming that only $6/\pi^2 \approx 0.6079$ of these values correspond to fundamental discriminants, this number further reduces to 2^{36}.

3. Next Steps

We need to implement initialization routines for the following ten series:

- $\nabla_{1,3}(q)\nabla(q) + \nabla_{1,3}(q^4)\vartheta_3(q^2) + 2q\nabla_{2,3}(q^4)\nabla(q^4)$
- $\nabla(q^3)\nabla(q) + q\nabla(q^{12})\vartheta_3(q^2) + \vartheta_3(q^6)\nabla(q^4)$
- $\nabla_{1,3}(q)$
- $\nabla(q^3)$
- $2\nabla_{2,15}(q^2)\nabla_{1,3}(q^2)+q\nabla_{8,15}(q^2)\nabla_{1,3}(q^2)+q\nabla_{7,15}(q^2)\nabla_{2,3}(q^2)+q^3\nabla_{13,15}(q^2)\nabla_{2,3}(q^2)+q^3\nabla_{12,15}(q^2)\nabla(q^6) + \nabla_{3,15}(q^2)\vartheta_3(q^3)$
- $2q\nabla_{2,15}(q^2)\nabla(q^6)+2q^2\nabla_{8,15}(q^2)\nabla(q^6)+q\nabla_{7,15}(q^2)\vartheta_3(q^3)+q^3\nabla_{13,15}(q^2)\vartheta_3(q^3)$
- $2\nabla_{4,15}(q^2)\nabla_{1,3}(q^2)+2q^3\nabla_{14,15}(q^2)\nabla_{1,3}(q^2)+2\nabla_{1,15}(q^2)\nabla_{2,3}(q^2)+2q^2\nabla_{11,15}(q^2)\nabla_{2,3}(q^2)+2q\nabla_{6,15}(q^2)\nabla(q^6) + q\nabla_{9,15}(q^2)\vartheta_3(q^3)$
- $4q\nabla_{4,15}(q^2)\nabla(q^6)+4q^4\nabla_{14,15}(q^2)\nabla(q^6)+2\nabla_{1,15}(q^2)\vartheta_3(q^3)+2q^2\nabla_{11,15}(q^2)\vartheta_3(q^3)$
- $2q\nabla_{10,15}(q^2)\nabla_{1,3}(q^2)+2\nabla_{5,15}(q^2)\nabla_{2,3}(q^2)+\vartheta_3(q^{15})\nabla(q^6)+q^3\nabla(q^{30})\vartheta_3(q^3)$
- $2\nabla_{5,15}(q^2)\vartheta_3(q^3) + 4q^2\nabla_{10,15}(q^2)\nabla(q^6)$

2021-12-30 Now that the polymult library has been updated, we can think of the practical aspects of computation. Recall that all series that we are dealing with have the form

$$cq^r\nabla_{a,m}(q^s).$$

Thus, each series can be expressed in terms of five parameters, namely (c, r, s, a, m). Polymult enables us to compute products of the form

$$(c_0q^{r_0}\nabla_{a_0,m_0}(q^{s_0})) \cdot \sum_{i=1}^{k} \left(c_{2i-1}q^{r_{2i-1}}\nabla_{a_{2i-1},m_{2i-1}}(q^{s}_{2i-1})\right) \cdot (c_{2i}q^{r_{2i}}\nabla_{a_{2i},m_{2i}}(q^{s_{2i}})) .$$

The product can be computed by calling the following in the command line:

```
./polymult [limit] [files] [bundle] [bound] [resultname] [folder]
      [c0] [r0] [s0] [m0] [a0] [c1] [r1] [s1] [m1] [a1] ...
```

Now, let us focus on a particular example. From Corollary 3 we have the identity

$$2\sum_{k=0}^{\infty} H(24k+7)q^k = \nabla(q)\left[\nabla_{1,3}(q)\nabla(q) + \nabla_{1,3}(q^4)\nabla_{0,1}(q^4) + 2q\nabla_{2,3}(q^4)\nabla(q^4)\right].$$

Recall that all series that we are dealing with have the form

$$cq^r\nabla_{a,m}(q^s).$$

Here are the parameters for the series occurring in the aforementioned identity:

Series	c	r	s	a	m
$\nabla(q)$	1	0	1	1	1
$\nabla_{1,3}(q)$	1	0	1	1	3
$\nabla(q)$	1	0	1	1	1
$\nabla_{1,3}(q^4)$	1	0	4	1	3
$\nabla_{0,1}(q^4)$	1	0	4	0	1
$2q\nabla_{2,3}(q^4)$	2	1	4	2	3
$\nabla(q^4)$	1	0	4	1	1

Thus, if we want to compute the product in the right-hand side of the above identity, we can call the following piece of code:

```
./polymult 268435456 32 256 1000000 res. / 1 0 1 1 1 1 0 1 1 3 1 0 1
            1 1 1 0 4 1 3 1 0 4 0 1 2 1 4 2 3 1 0 4 1 1
```

This

Department of Computer Science, University of Calgary, Calgary AB, Canada T2N 1N4

Email address: jacobs@ucalgary.ca

Mathematics Undergraduate Office, University of Waterloo, Waterloo ON, Canada N2L 3G1

Email address: amosunov@uwaterloo.ca