Problem 1: [20 points]

For the three-link cylindrical manipulator (RPP) shown below, solve the inverse kinematics problem. You can choose either the algebraic or the geometric approach–but you must show complete details of your solution. Please note the given link dimensions.

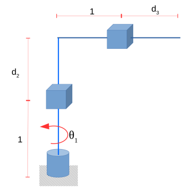


Figure 1: A Cylindrical (RRP) robot arm.

Consider the following alternate perspective of Figure 1:

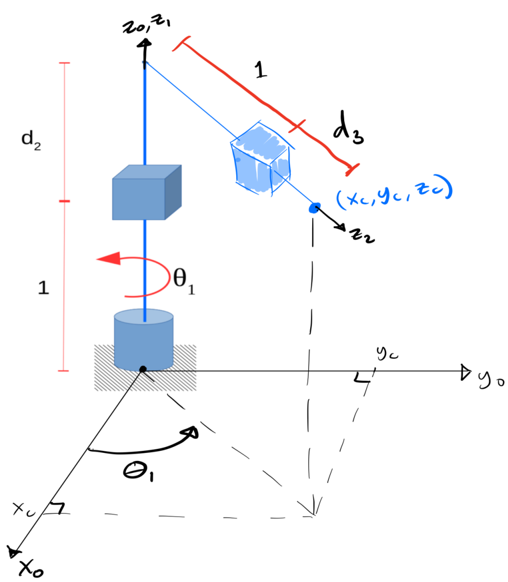


Figure 2: Alternate perspective of robot seen in Figure 1 with base frame shown.

Using Figure 2 we will now solve the inverse kinematics problem given a target end effector position of (x_c, y_c, y_c) as seen from the base frame. Beginning with joint variable θ_1, note the projection of (x_c, y_c, y_c) onto the $x_0 - y_0$ plane. From the projected construction lines, it is easy to see, geometrically, that $\theta_1 = Atan2(x_c, y_c)$ where $Atan2(x, y)$ is the two-argument arctangent function which returns a unique angle in the range $[-\pi, \pi]$ based upon the signs of the x and y arguments. Continuing, joint variable d_2 can be found by solving $z_c = d_2 + 1$ for d_2. It follows that $d_2 = z_c - 1$. Finally, joint variable d_3 can be found by once again observing the projection of (x_c, y_c, z_c) onto the $x_0 - y_0$ plane. Notice that the hypotenuse of the same right triangle used to find θ_1 is equal to the length $1 + d_3$. Using the pythagorean theorem, then, it follows that $1 + d_3 = \sqrt{(x_c)^2 + (y_c)^2}$. Hence, $d_3 = \sqrt{(x_c)^2 + (y_c)^2} - 1$.

Therefore, the inverse kinematic solution for the cylindrical robot arm depicted in Figures 1 and 2, given an end effector position of (x_c, y_c, y_c) relative to the base frame, is:

$$\theta_1 = Atan2(x_c, y_c)$$
$$d_2 = z_c - 1$$
$$d_3 = \sqrt{(x_c)^2 + (y_c)^2} - 1$$

Problem 2: [20 + 10 = 30 points]

A kinematic model of the human arm (Figure 3) co-locates joints (1-3) at the shoulder creating spherical joint; joint (4), a pin joint axis–shown coming out of the page–at the elbow; joint (5) along the forearm (axial roll), and then joints (6,7) at the wrist for yaw and then pitch. Starting with the z_0 axis coming out at you from the page, and the x_0 axis to your left, as shown (i.e., θ_1 rotation lowers/raises the arm from the side), neatly draw and label all the z and x axes and make a DH table for all 7 inter-joint transformations, including the end-effector. Show and label all non-zero displacements on the diagram and include them in your table. The "end-effector", that is the hand frame is as shown. Points will be given for clarity.

Note that in Figure 3, axes z_0, x_2, z_3, z_5, x_7, and n lie perpendicular to the page. Furthermore, if the axis label is red (i.e. x_7 and n), then the positive direction of that axis points into the page. Conversely, if the axis label is black–for one of the axes which lie perpendicular to the page–then the positive direction of that axis comes out of the page.

An 8th "fixed" joint was included to account for the fact that the origin of the end-effector frame does not lie at the intersection of the z_6 and a axes as is dicated by the DH convention given that z_6 and a (where a is the end-effector frame's z-axis or approach) are not parallel but still intersect.

Joint	θ_i	d_i	a_i	α_i
1	θ_1^*	0	0	$\frac{\pi}{2}$
2	θ_2^*	0	0	$\frac{\pi}{2}$
3	θ_3^*	L_1	0	$\frac{\pi}{2}$
4	θ_4^*	0	0	$\frac{\pi}{2}$
5	θ_5^*	L_2	0	$\frac{-\pi}{2}$
6	θ_6^*	0	0	$\frac{-\pi}{2}$
7	θ_7^*	0	0	$\frac{\pi}{2}$
8	0	L_h	0	0

Table 1: DH Table for the human arm model shown in Figure 3

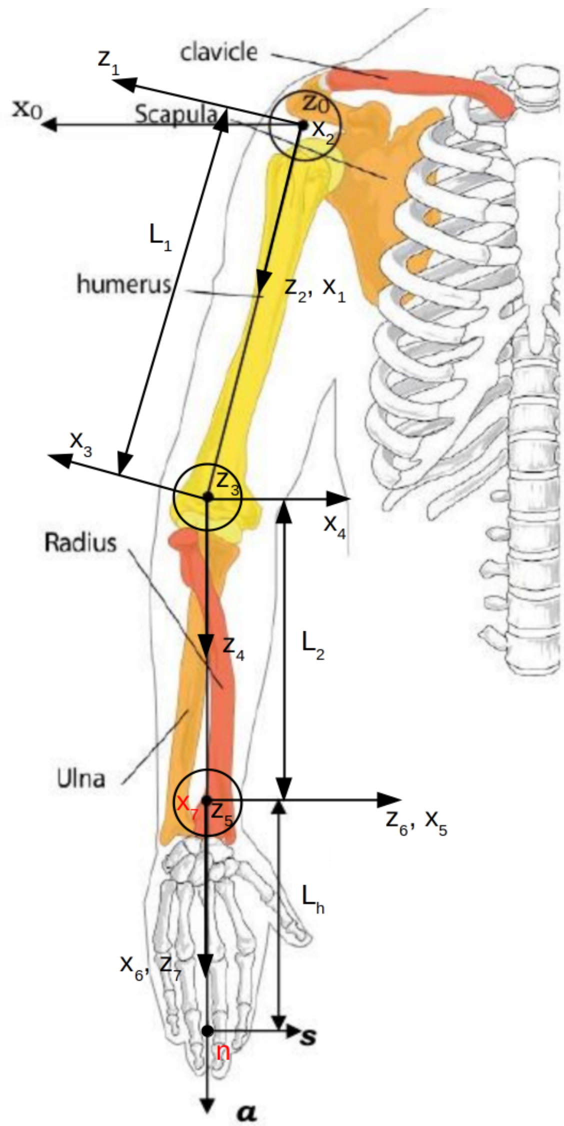


Figure 3: The kinematic model of the human arm.

Problem 3: [25 points]

Using either the algebraic or geometric method, along with the Euler solution for a spherical wrist, find the closed-form inverse kinematics solution for the MOM manipulator shown in Figure 4.

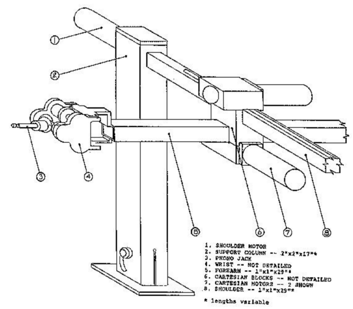


Figure 4: The MOM robot.

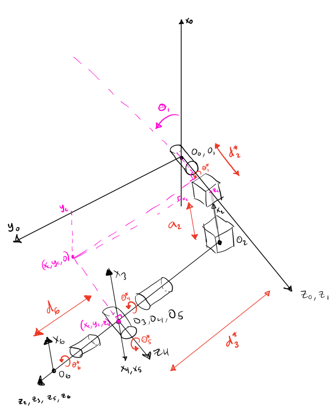
Given a homogenous transformation

$$\begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & o_x \\ r_{21} & r_{22} & r_{23} & o_y \\ r_{31} & r_{32} & r_{33} & o_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

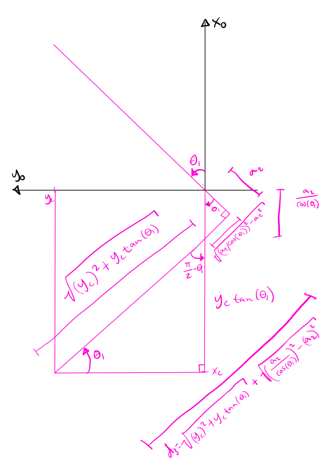
we aim to find closed form expressions for each of the six joint variables of the MOM manipulator shown in Figure 4. To assist in kinemeatic decoupling, let the point (x_c, y_c, z_c) represent the spherical wrist center as seen from the base frame. Specifically, let

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$

Observe the following figures which will be used in solving the inverse position problem:



(a) Reference frames assigned to MOM manipulator shown in Figure 4.



(b) View of MOM robot manipular projected onto the x_0, y_0 plane.

Figure 5: Two perspectives of the MOM manipulator shown in Figure 4.

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To solve this inverse kinematics problem, we will first solve the inverse position problem before solving the inverse orientation problem. Looking at Figure 5b above, we begin by finding θ_1:

$$\begin{aligned}
& x_c = y_c \tan(\theta_1) + \frac{a_2}{\cos(\theta_1)} \\
\iff\quad & x_c \cos(\theta_1) = y_c \sin(\theta_1) + a_2 \\
\iff\quad & x_c \cos(\theta_1) - y_c \sin(\theta_1) = a_2 \\
\iff\quad & \frac{x_c}{\sqrt{(x_c)^2 + (y_c)^2}} \cos(\theta_1) - \frac{y_c}{\sqrt{(x_c)^2 + (y_c)^2}} \sin(\theta_1) = \frac{a_2}{\sqrt{(x_c)^2 + (y_c)^2}}
\end{aligned}$$

Let there be some α such that:

$$\sin(\alpha) = \frac{y_c}{\sqrt{(x_c)^2 + (y_c)^2}}, \text{ and } \cos(\alpha) = \frac{x_c}{\sqrt{(x_c)^2 + (y_c)^2}}$$

It follows that,

$$\tan(\alpha) = \frac{y_c}{x_c} \iff \alpha = Atan2(x_c, y_c) \tag{1}$$

Continuing:

$$\begin{aligned}
& \frac{x_c}{\sqrt{(x_c)^2 + (y_c)^2}} \cos(\theta_1) - \frac{y_c}{\sqrt{(x_c)^2 + (y_c)^2}} \sin(\theta_1) = \frac{a_2}{\sqrt{(x_c)^2 + (y_c)^2}} \\
\iff\quad & \cos(\alpha)\cos(\theta_1) - \sin(\alpha)\sin(\theta_1) = \frac{a_2}{\sqrt{(x_c)^2 + (y_c)^2}} \\
\iff\quad & \cos(\alpha + \theta_1) = \frac{a_2}{\sqrt{(x_c)^2 + (y_c)^2}} \qquad \text{// Angle-sum identity} \\
\iff\quad & \alpha + \theta_1 = \cos^{-1}\left(\frac{a_2}{\sqrt{(x_c)^2 + (y_c)^2}}\right) \\
\iff\quad & \theta_1 = \cos^{-1}\left(\frac{a_2}{\sqrt{(x_c)^2 + (y_c)^2}}\right) - \alpha \\
\iff\quad & \theta_1 = \cos^{-1}\left(\frac{a_2}{\sqrt{(x_c)^2 + (y_c)^2}}\right) - Atan2(x_c, y_c) \qquad \text{// by Eq. (1)}
\end{aligned}$$

Therefore,

$$\theta_1 = \cos^{-1}\left(\frac{a_2}{\sqrt{(x_c)^2 + (y_c)^2}}\right) - Atan2(x_c, y_c)$$

Next we find joint variable d_2. As seen in Figure 5a, $d_2 = z_c$ since the origin of the x_1, y_1, z_1 frame is coincident with the origin of the base frame.

To complete the inverse position problem, we now find d_3. Using joint variable θ_1, we can now find d_3 using the expression found in Figure 5b since all quantities are known:

$$d_3 = \sqrt{(y_c)^2 + y_c \tan(\theta_1)} + \sqrt{\left(\frac{a_2}{\cos(\theta_1)}\right)^2 - (a_2)^2}$$

For the sake of clarity, the expression for θ_1 will not be expanded in the expression for d_3.

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With expressions for joint variables θ_1, d_2, and d_3 in hand, we have solved the inverse position problem to the wrist center (x_c, y_c, z_c). We now will solve the inverse orientation problem to find joint variables $\theta_4, \theta_5, \theta_6$ using the Euler solution for a spherical wrist.

As shown in Figures 4 and 5a the DH table for the MOM manipulator is as follows:

Joint	θ_i	d_i	a_i	α_i
1	θ_1^*	0	0	0
2	0	d_2^*	$-a_2$	$\frac{-\pi}{2}$
3	0	d_3^*	0	0
4	θ_4^*	0	0	$\frac{-\pi}{2}$
5	θ_5^*	0	0	$\frac{\pi}{2}$
6	θ_6^*	d_6	0	0

Table 2: DH table for manipulator shown in figures 4 and 5a.

Using the DH table, we begin by finding T_{03}:

Let the homogenous transformation matrix $A_i = T_{(i-1)i}$ be a function of the joint variable q_i. It follows that $T_{ij} = A_{i+1}A_{i+2}...A_{j-1}A_j$. Using the DH convention, each of these homogenous transformations can be represented as follows:

$$A_i = Rot_{z,\theta_i}\ Trans_{z,d_i}\ Trans_{x,a_i}\ Rot_{x,\alpha_i} = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where θ_i, d_i, a_i, and α_i are the i^{th} joint angle, link offset, link length, and link twist respectively. Using the values found in table 2, then,

$$\begin{aligned}
T_{03} &= A_1 A_2 A_3 \\
&= \begin{bmatrix} c\theta_1 & -s\theta_1 c\alpha_1 & s\theta_1 s\alpha_1 & a_1 c\theta_1 \\ s\theta_1 & c\theta_1 c\alpha_1 & -c\theta_1 s\alpha_1 & a_1 s\theta_1 \\ 0 & s\alpha_1 & c\alpha_1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_2 & -s\theta_2 c\alpha_2 & s\theta_2 s\alpha_2 & a_2 c\theta_2 \\ s\theta_2 & c\theta_2 c\alpha_2 & -c\theta_2 s\alpha_2 & a_2 s\theta_2 \\ 0 & s\alpha_2 & c\alpha_2 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_3 & -s\theta_3 c\alpha_3 & s\theta_3 s\alpha_3 & a_3 c\theta_3 \\ s\theta_3 & c\theta_3 c\alpha_3 & -c\theta_3 s\alpha_3 & a_3 s\theta_3 \\ 0 & s\alpha_3 & c\alpha_3 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c\theta_1 & -s\theta_1 c(0) & s\theta_1 s(0) & (0)c\theta_1 \\ s\theta_1 & c\theta_1 c(0) & -c\theta_1 s(0) & (0)s\theta_1 \\ 0 & s(0) & c(0) & (0) \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c(0) & -s(0)c\frac{-\pi}{2} & s(0)s\frac{-\pi}{2} & -a_2 c(0) \\ s(0) & c(0)c\frac{-\pi}{2} & -c(0)s\frac{-\pi}{2} & -a_2 s(0) \\ 0 & s\frac{-\pi}{2} & c\frac{-\pi}{2} & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c(0) & -s(0)c(0) & s(0)s(0) & (0)c(0) \\ s(0) & c(0)c(0) & -c(0)s(0) & (0)s(0) \\ 0 & s(0) & c(0) & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -a_2 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -a_2 \\ 0 & 0 & 1 & d_3 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c\theta_1 & 0 & -s\theta_1 & -a_2 c\theta_1 - d_3 s\theta_1 \\ s\theta_1 & 0 & c\theta_1 & d_3 c\theta_1 - a_2 s\theta_1 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

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Hence,

$$R_{03} = \begin{bmatrix} c\theta_1 & 0 & -s\theta_1 \\ s\theta_1 & 0 & c\theta_1 \\ 0 & -1 & 0 \end{bmatrix}, \text{ and } \quad R_{03}^{-1} = R_{03}^T = \begin{bmatrix} c\theta_1 & s\theta_1 & 0 \\ 0 & 0 & -1 \\ -s\theta_1 & c\theta_1 & 0 \end{bmatrix}$$

Additionally,

$$R_{03}^T R = \begin{bmatrix} c\theta_1 & s\theta_1 & 0 \\ 0 & 0 & -1 \\ -s\theta_1 & c\theta_1 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\theta_1 r_{11} + s\theta_1 r_{21} & c\theta_1 r_{12} + s\theta_1 r_{22} & c\theta_1 r_{13} + s\theta_1 r_{23} \\ -r_{31} & -r_{32} & -r_{33} \\ c\theta_1 r_{21} - s\theta_1 r_{11} & c\theta_1 r_{22} - s\theta_1 r_{12} & c\theta_1 r_{23} - s\theta_1 r_{13} \end{bmatrix}$$

Next, note that since joints 4, 5, and 6 make up a standard spherical wrist, T_{36} is nothing more than an Euler angle transformation on the set of Euler angles $\phi = \theta_4, \theta = \theta_5, \psi = \theta_6$:

$$T_{36} = \begin{bmatrix} c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6 & -c\theta_4 c\theta_5 s\theta_6 - s\theta_4 c\theta_6 & c\theta_4 s\theta_5 & c\theta_4 s\theta_5 d_6 \\ s\theta_4 c\theta_5 c\theta_6 + c\theta_4 s\theta_6 & -s\theta_4 c\theta_5 s\theta_6 + c\theta_4 c\theta_6 & s\theta_4 s\theta_5 & s\theta_4 s\theta_5 d_6 \\ -s\theta_5 c\theta_6 & s\theta_5 s\theta_6 & c\theta_5 & c\theta_5 d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence,

$$R_{36} = \begin{bmatrix} c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6 & -c\theta_4 c\theta_5 s\theta_6 - s\theta_4 c\theta_6 & c\theta_4 s\theta_5 \\ s\theta_4 c\theta_5 c\theta_6 + c\theta_4 s\theta_6 & -s\theta_4 c\theta_5 s\theta_6 + c\theta_4 c\theta_6 & s\theta_4 s\theta_5 \\ -s\theta_5 c\theta_6 & s\theta_5 s\theta_6 & c\theta_5 \end{bmatrix}$$

Using the fact that $R_{36} = R_{03}^T R$, the following expressions can be derived:

$$\begin{aligned} c\theta_4 s\theta_5 &= c\theta_1 r_{13} + s\theta_1 r_{23} \\ s\theta_4 s\theta_5 &= -r_{33} \\ c\theta_5 &= c\theta_1 r_{23} - s\theta_1 r_{13} \\ -s\theta_5 c\theta_6 &= c\theta_1 r_{21} - s\theta_1 r_{11} \\ s\theta_5 s\theta_6 &= c\theta_1 r_{22} - s\theta_1 r_{12} \end{aligned}$$

Recall that the Euler solution equations are as follows for a ZYZ-Euler transformation R' when r'_{13} and r'_{23} are not both zero is either

$$\begin{aligned} \theta &= Atan2(r'_{33}, \sqrt{1 - (r'_{33})^2}) \\ \phi &= Atan2(r'_{13}, r'_{23}) \\ \psi &= Atan2(-r'_{31}, r'_{32}) \end{aligned}$$

or

$$\begin{aligned} \theta &= Atan2(r'_{33}, -\sqrt{1 - (r'_{33})^2}) \\ \phi &= Atan2(-r'_{13}, -r'_{23}) \\ \psi &= Atan2(r'_{31}, -r'_{32}) \end{aligned}$$

depending on which sign is chosen for θ. Alternatively, if $r'_{13} = r'_{23} = 0$ we can further assert that the Euler transformation R' is of the form:

$$R' = \begin{bmatrix} r'_{11} & r'_{12} & 0 \\ r'_{21} & r'_{22} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

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In this case, there exist infintely many solutions for ϕ and ψ so long as they satisfy:

$$\begin{aligned}
\text{if } r'_{33} = -1 \text{ then}& \\
\theta &= -\pi \\
\phi - \psi &= Atan2(-r'_{11}, -r'_{12}) \\
\text{else if } r'_{33} = 1 \text{ then}& \\
\theta &= 0 \\
\phi + \psi &= Atan2(r'_{11}, r'_{21})
\end{aligned}$$

Applying these Euler solution equations to the problem at hand results in the following closed form solution to the inverse orientation problem:

If $c\theta_4 s\theta_5 \neq 0$ and $s\theta_4 s\theta_5 \neq 0$ then

$$\begin{aligned}
\theta_5 &= Atan2\big(c\theta_1 r_{23} - s\theta_1 r_{13}, \sqrt{1 - (c\theta_1 r_{23} - s\theta_1 r_{13})^2}\big) \\
\theta_4 &= Atan2(c\theta_1 r_{13} + s\theta_1 r_{23}, -r_{33}) \\
\theta_6 &= Atan2(-c\theta_1 r_{21} + s\theta_1 r_{11}, c\theta_1 r_{22} - s\theta_1 r_{12})
\end{aligned}$$

or

$$\begin{aligned}
\theta_5 &= Atan2\big(c\theta_1 r_{23} - s\theta_1 r_{13}, -\sqrt{1 - (c\theta_1 r_{23} - s\theta_1 r_{13})^2}\big) \\
\theta_4 &= Atan2(-c\theta_1 r_{13} - s\theta_1 r_{23}, r_{33}) \\
\theta_6 &= Atan2(c\theta_1 r_{21} - s\theta_1 r_{11}, -c\theta_1 r_{22} + s\theta_1 r_{12})
\end{aligned}$$

Else If $c\theta_4 s\theta_5 = s\theta_4 s\theta_5 = 0$ and $c\theta_5 = 1$ then there are infinetely many solutions so long as they satisfy:

$$\begin{aligned}
\theta_5 &= 0 \\
\theta_4 + \theta_6 &= Atan2(c\theta_1 r_{11} + s\theta_1 r_{21}, -r_{31})
\end{aligned}$$

Else If $c\theta_4 s\theta_5 = s\theta_4 s\theta_5 = 0$ and $c\theta_5 = -1$ then there are infinetely many solutions so long as they satisfy:

$$\begin{aligned}
\theta_5 &= -\pi \\
\theta_4 - \theta_6 &= Atan2(-c\theta_1 r_{11} - s\theta_1 r_{21}, -c\theta_1 r_{12} - s\theta_1 r_{22})
\end{aligned}$$

In conclusion, Given a homogenous transformation and wrist center

$$\begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & o_x \\ r_{21} & r_{22} & r_{23} & o_y \\ r_{31} & r_{32} & r_{33} & o_z \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$

for the MOM manipulator shown in figure 4, we have shown that:

$$\begin{aligned}
\theta_1 &= \cos^{-1}\left(\frac{a_2}{\sqrt{(x_c)^2 + (y_c)^2}}\right) - Atan2(x_c, y_c) \\
d_2 &= z_c \\
d_3 &= \sqrt{(y_c)^2 + y_c \tan(\theta_1)} + \sqrt{\left(\frac{a_2}{\cos(\theta_1)}\right)^2 - (a_2)^2}
\end{aligned}$$

In addition to all the possible solutions outlined at the top of this page for joint variables $\theta_4, \theta_5, \theta_6$.

Problem 4: [25 points]

Again, using either the algebraic or geometric method, along with the Euler solution for a shperical wrist, find the closed-form inverse kinematics solution for the Stanford manipulator shown in Figure 6a.

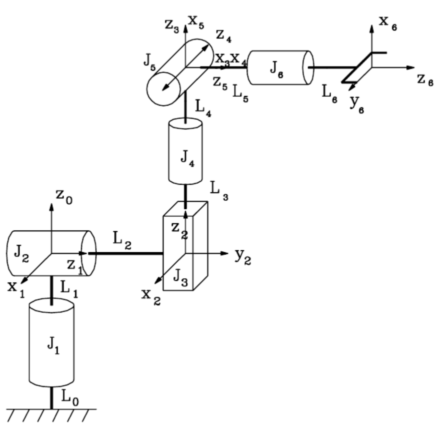
Given a homogenous transformation

$$\begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & o_x \\ r_{21} & r_{22} & r_{23} & o_y \\ r_{31} & r_{32} & r_{33} & o_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

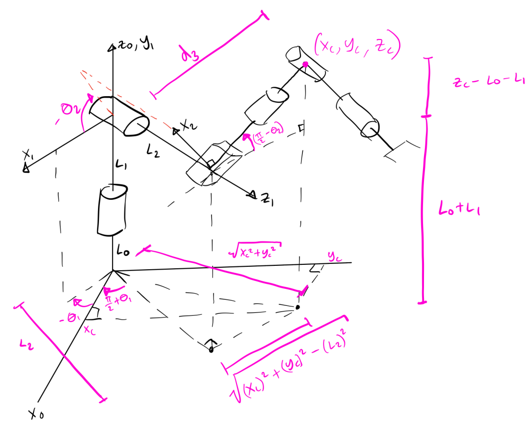
we aim to find closed form expressions for each of the six joint variables of the Stanford manipulator shown in Figure 6a. To assist in kinemeatic decoupling, let the point (x_c, y_c, z_c) represent the spherical wrist center as seen from the base frame. Specifically, let

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$

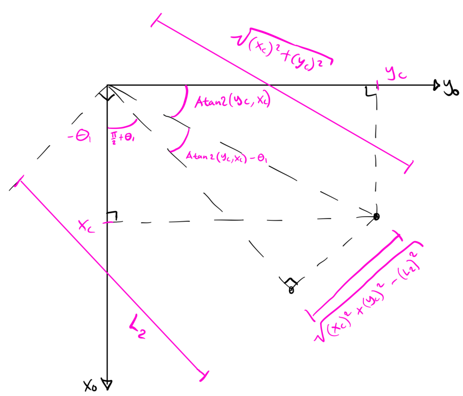
Observe the following figures which will be used in solving the inverse position problem:



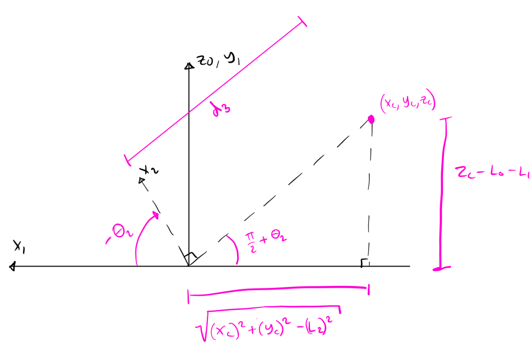
(a) The Stanford manipulator.



(b) Alternate perspecive of Standford manipulator shown in Figure 6a.



(c) View of Stanford manipular projected onto the x_0, y_0 plane.



(d) View of Stanford manipular projected onto the x_1, y_1 plane.

Figure 6: Four perspectives of the Stanford manipulator.

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To solve this inverse kinematics problem, we will first solve the inverse position problem before solving the inverse orientation problem. Looking at Figure 6c above, we begin by finding θ_1:

$$\begin{aligned} & Atan2(y_c, x_c) - \theta_1 = Atan2(\sqrt{(x_c)^2 + (y_c)^2 - (L_2)^2}, L_2) \\ \iff \quad & \theta_1 = Atan2(y_c, x_c) - Atan2(\sqrt{(x_c)^2 + (y_c)^2 - (L_2)^2}, L_2) \end{aligned}$$

Next, we will find joint variables θ_2 and d_3 using Figure 6d. Note that $\cot(\pi/2 + \theta) = -\tan(\theta)$:

$$\begin{aligned} & \cot\left(\frac{\pi}{2} + \theta_2\right) = \frac{\sqrt{(x_c)^2 + (y_c)^2 - (L_2)^2}}{z_c - L_0 - L_1} \\ \iff \quad & -\tan(\theta_2) = \frac{\sqrt{(x_c)^2 + (y_c)^2 - (L_2)^2}}{z_c - L_0 - L_1} \\ \iff \quad & \theta_2 = Atan2\left(L_0 + L_1 - z_c, \sqrt{(x_c)^2 + (y_c)^2 - (L_2)^2}\right) \end{aligned}$$

Finally, Joint variable d_3 can be found by using the pythagorean theorem on the same triangle used in determining θ_2:

$$\begin{aligned} & (d_3)^2 = \left((x_c)^2 + (y_c)^2 - (L_2)^2\right) + (z_c - L_0 - L_1)^2 \\ \iff \quad & d_3 = \sqrt{(x_c)^2 + (y_c)^2 - (L_2)^2 + (z_c - L_0 - L_1)^2} \end{aligned}$$

With expressions for joint variables θ_1, θ_2, and d_3 in hand, we have solved the inverse position problem to the wrist center (x_c, y_c, z_c). We now will solve the inverse orientation problem to find joint variables $\theta_4, \theta_5, \theta_6$ using the Euler solution for a spherical wrist.

As shown in Figures 6a and 6b the DH table for the stanford manipulator is as follows:

Joint	θ_i	d_i	a_i	α_i
1	θ_1^*	$L_0 + L_1$	0	$\frac{-\pi}{2}$
2	θ_2^*	L_2	0	$\frac{\pi}{2}$
3	0	d_3^*	0	0
4	θ_4^*	0	0	$\frac{-\pi}{2}$
5	θ_5^*	0	0	$\frac{-\pi}{2}$
6	θ_6^*	$L_5 + L_6$	0	0

Table 3: DH table for stanford manipulator shown in Figures 6a and 6b.

Using the DH table, we begin by finding T_{03}:

Let the homogenous transformation matrix $A_i = T_{(i-1)i}$ be a function of the joint variable q_i. It follows that $T_{ij} = A_{i+1}A_{i+2}...A_{j-1}A_j$. Using the DH convention, each of these homogenous transformations can be represented as follows:

$$A_i = Rot_{z,\theta_i}\ Trans_{z,d_i}\ Trans_{x,a_i}\ Rot_{x,\alpha_i} = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where θ_i, d_i, a_i, and α_i are the i^{th} joint angle, link offset, link length, and link twist respectively.

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Using the values found in table 3, then,

$$
\begin{aligned}
T_{03} &= A_1A_2A_3 \\
&= \begin{bmatrix} c\theta_1 & -s\theta_1 c\alpha_1 & s\theta_1 s\alpha_1 & a_1 c\theta_1 \\ s\theta_1 & c\theta_1 c\alpha_1 & -c\theta_1 s\alpha_1 & a_1 s\theta_1 \\ 0 & s\alpha_1 & c\alpha_1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_2 & -s\theta_2 c\alpha_2 & s\theta_2 s\alpha_2 & a_2 c\theta_2 \\ s\theta_2 & c\theta_2 c\alpha_2 & -c\theta_2 s\alpha_2 & a_2 s\theta_2 \\ 0 & s\alpha_2 & c\alpha_2 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_3 & -s\theta_3 c\alpha_3 & s\theta_3 s\alpha_3 & a_3 c\theta_3 \\ s\theta_3 & c\theta_3 c\alpha_3 & -c\theta_3 s\alpha_3 & a_3 s\theta_3 \\ 0 & s\alpha_3 & c\alpha_3 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c\theta_1 & -s\theta_1 c\frac{-\pi}{2} & s\theta_1 s\frac{-\pi}{2} & (0)c\theta_1 \\ s\theta_1 & c\theta_1 c\frac{-\pi}{2} & -c\theta_1 s\frac{-\pi}{2} & (0)s\theta_1 \\ 0 & s\frac{-\pi}{2} & c\frac{-\pi}{2} & L_0+L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_2 & -s\theta_2 c\frac{\pi}{2} & s\theta_2 s\frac{\pi}{2} & (0)c\theta_2 \\ s\theta_2 & c\theta_2 c\frac{\pi}{2} & -c\theta_2 s\frac{\pi}{2} & (0)s\theta_2 \\ 0 & s\frac{\pi}{2} & c\frac{\pi}{2} & L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c(0) & -s(0)c(0) & s(0)s(0) & (0)c(0) \\ s(0) & c(0)c(0) & -c(0)s(0) & (0)s(0) \\ 0 & s(0) & c(0) & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c\theta_1 & 0 & -s\theta_1 & 0 \\ s\theta_1 & 0 & c\theta_1 & 0 \\ 0 & -1 & 0 & L_0+L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_2 & 0 & s\theta_2 & 0 \\ s\theta_2 & 0 & -c\theta_2 & 0 \\ 0 & 1 & 0 & L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c\theta_1 & 0 & -s\theta_1 & 0 \\ s\theta_1 & 0 & c\theta_1 & 0 \\ 0 & -1 & 0 & L_0+L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_2 & 0 & s\theta_2 & d_3 s\theta_2 \\ s\theta_2 & 0 & -c\theta_2 & -d_3 c\theta_2 \\ 0 & 1 & 0 & L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c\theta_1 c\theta_2 & -s\theta_1 & s\theta_2 c\theta_1 & d_3 s\theta_2 c\theta_1 - L_2 s\theta_1 \\ s\theta_1 c\theta_2 & c\theta_1 & s\theta_1 s\theta_2 & L_2 c\theta_1 + d_3 s\theta_1 s\theta_2 \\ -s\theta_2 & 0 & c\theta_2 & d_3 c\theta_2 + L_0 + L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Hence,

$$
R_{03} = \begin{bmatrix} c\theta_1 c\theta_2 & -s\theta_1 & s\theta_2 c\theta_1 \\ s\theta_1 c\theta_2 & c\theta_1 & s\theta_1 s\theta_2 \\ -s\theta_2 & 0 & c\theta_2 \end{bmatrix}, \text{ and } \quad R_{03}^{-1} = R_{03}^T = \begin{bmatrix} c\theta_1 c\theta_2 & s\theta_1 c\theta_2 & -s\theta_2 \\ -s\theta_1 & c\theta_1 & 0 \\ s\theta_2 c\theta_1 & s\theta_1 s\theta_2 & c\theta_2 \end{bmatrix}
$$

Additionally,

$$
\begin{aligned}
R_{03}^T R &= \begin{bmatrix} c\theta_1 c\theta_2 & s\theta_1 c\theta_2 & -s\theta_2 \\ -s\theta_1 & c\theta_1 & 0 \\ s\theta_2 c\theta_1 & s\theta_1 s\theta_2 & c\theta_2 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \\
&= \begin{bmatrix} c\theta_1 c\theta_2 r_{11} + c\theta_2 s\theta_1 r_{21} - s\theta_2 r_{31} & c\theta_1 c\theta_2 r_{12} + c\theta_2 s\theta_1 r_{22} - s\theta_2 r_{32} & c\theta_1 c\theta_2 r_{13} + c\theta_2 s\theta_1 r_{23} - s\theta_2 r_{33} \\ c\theta_1 r_{21} - s\theta_1 r_{11} & c\theta_1 r_{22} - s\theta_1 r_{12} & c\theta_1 r_{23} - s\theta_1 r_{13} \\ s\theta_2 c\theta_1 r_{11} + s\theta_1 s\theta_2 r_{21} + c\theta_2 r_{31} & s\theta_2 c\theta_1 r_{12} + s\theta_1 s\theta_2 r_{22} + c\theta_2 r_{32} & s\theta_2 c\theta_1 r_{13} + s\theta_1 s\theta_2 r_{23} + c\theta_2 r_{33} \end{bmatrix}
\end{aligned}
$$

Next, note that since joints 4, 5, and 6 make up a spherical wrist, T_{36} is nothing more than an Euler angle transformation on the set of Euler angles $\phi = \theta_4, \theta = \theta_5, \psi = \theta_6$:

$$
T_{36} = \begin{bmatrix} c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6 & -c\theta_4 c\theta_5 s\theta_6 - s\theta_4 c\theta_6 & c\theta_4 s\theta_5 & c\theta_4 s\theta_5 d_6 \\ s\theta_4 c\theta_5 c\theta_6 + c\theta_4 s\theta_6 & -s\theta_4 c\theta_5 s\theta_6 + c\theta_4 c\theta_6 & s\theta_4 s\theta_5 & s\theta_4 s\theta_5 d_6 \\ -s\theta_5 c\theta_6 & s\theta_5 s\theta_6 & c\theta_5 & c\theta_5 d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Hence,

$$
R_{36} = \begin{bmatrix} c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6 & -c\theta_4 c\theta_5 s\theta_6 - s\theta_4 c\theta_6 & c\theta_4 s\theta_5 \\ s\theta_4 c\theta_5 c\theta_6 + c\theta_4 s\theta_6 & -s\theta_4 c\theta_5 s\theta_6 + c\theta_4 c\theta_6 & s\theta_4 s\theta_5 \\ -s\theta_5 c\theta_6 & s\theta_5 s\theta_6 & c\theta_5 \end{bmatrix}
$$

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Using the fact that $R_{36} = R_{03}^T R$, the following expressions can be derived:

$$
\begin{aligned}
c\theta_4 s\theta_5 &= c\theta_1 c\theta_2 r_{13} + c\theta_2 s\theta_1 r_{23} - s\theta_2 r_{33} \\
s\theta_4 s\theta_5 &= c\theta_1 r_{23} - s\theta_1 r_{13} \\
c\theta_5 &= s\theta_2 c\theta_1 r_{13} + s\theta_1 s\theta_2 r_{23} + c\theta_2 r_{33} \\
-s\theta_5 c\theta_6 &= s\theta_2 c\theta_1 r_{11} + s\theta_1 s\theta_2 r_{21} + c\theta_2 r_{31} \\
s\theta_5 s\theta_6 &= s\theta_2 c\theta_1 r_{12} + s\theta_1 s\theta_2 r_{22} + c\theta_2 r_{32}
\end{aligned}
$$

Recall that the Euler solution equations are as follows for a ZYZ-Euler transformation R' when r'_{13} and r'_{23} are not both zero is either

$$
\begin{aligned}
\theta &= Atan2(r'_{33}, \sqrt{1 - (r'_{33})^2}) \\
\phi &= Atan2(r'_{13}, r'_{23}) \\
\psi &= Atan2(-r'_{31}, r'_{32})
\end{aligned}
$$

or

$$
\begin{aligned}
\theta &= Atan2(r'_{33}, -\sqrt{1 - (r'_{33})^2}) \\
\phi &= Atan2(-r'_{13}, -r'_{23}) \\
\psi &= Atan2(r'_{31}, -r'_{32})
\end{aligned}
$$

depending on which sign is chosen for θ. Alternatively, if $r'_{13} = r'_{23} = 0$ we can further assert that the Euler transformation R' is of the form:

$$
R' = \begin{bmatrix} r'_{11} & r'_{12} & 0 \\ r'_{21} & r'_{22} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}
$$

In this case, there exist infintely many solutions for ϕ and ψ so long as they satisfy:

$$
\begin{aligned}
&\text{if } r'_{33} = -1 \text{ then} \\
&\qquad \theta = -\pi \\
&\qquad \phi - \psi = Atan2(-r'_{11}, -r'_{12}) \\
&\text{else if } r'_{33} = 1 \text{ then} \\
&\qquad \theta = 0 \\
&\qquad \phi + \psi = Atan2(r'_{11}, r'_{21})
\end{aligned}
$$

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Applying these Euler solution equations to the problem at hand results in the following closed form solution to the inverse orientation problem:

If $c\theta_1 c\theta_2 r_{13} + c\theta_2 s\theta_1 r_{23} - s\theta_2 r_{33} \neq 0$ and $c\theta_1 r_{23} - s\theta_1 r_{13} \neq 0$ then

$$\begin{aligned}
\theta_5 &= Atan2\big(s\theta_2 c\theta_1 r_{13} + s\theta_1 s\theta_2 r_{23} + c\theta_2 r_{33}, \sqrt{1 - (s\theta_2 c\theta_1 r_{13} + s\theta_1 s\theta_2 r_{23} + c\theta_2 r_{33})^2}\big) \\
\theta_4 &= Atan2(c\theta_1 c\theta_2 r_{13} + c\theta_2 s\theta_1 r_{23} - s\theta_2 r_{33}, c\theta_1 r_{23} - s\theta_1 r_{13}) \\
\theta_6 &= Atan2(-s\theta_2 c\theta_1 r_{11} - s\theta_1 s\theta_2 r_{21} - c\theta_2 r_{31}, s\theta_2 c\theta_1 r_{12} + s\theta_1 s\theta_2 r_{22} + c\theta_2 r_{32})
\end{aligned}$$

or

$$\begin{aligned}
\theta_5 &= Atan2\big(s\theta_2 c\theta_1 r_{13} + s\theta_1 s\theta_2 r_{23} + c\theta_2 r_{33}, -\sqrt{1 - (s\theta_2 c\theta_1 r_{13} + s\theta_1 s\theta_2 r_{23} + c\theta_2 r_{33})^2}\big) \\
\theta_4 &= Atan2(-c\theta_1 c\theta_2 r_{13} - c\theta_2 s\theta_1 r_{23} + s\theta_2 r_{33}, -c\theta_1 r_{23} + s\theta_1 r_{13}) \\
\theta_6 &= Atan2(s\theta_2 c\theta_1 r_{11} + s\theta_1 s\theta_2 r_{21} + c\theta_2 r_{31}, -s\theta_2 c\theta_1 r_{12} - s\theta_1 s\theta_2 r_{22} - c\theta_2 r_{32})
\end{aligned}$$

Else If $c\theta_1 c\theta_2 r_{13} + c\theta_2 s\theta_1 r_{23} - s\theta_2 r_{33} = c\theta_1 r_{23} - s\theta_1 r_{13} = 0$ and $s\theta_2 c\theta_1 r_{13} + s\theta_1 s\theta_2 r_{23} + c\theta_2 r_{33} = 1$ then there are infinetely many solutions so long as they satisfy:

$$\begin{aligned}
\theta_5 &= 0 \\
\theta_4 + \theta_6 &= Atan2(c\theta_1 c\theta_2 r_{11} + c\theta_2 s\theta_1 r_{21} - s\theta_2 r_{31}, c\theta_1 r_{21} - s\theta_1 r_{11})
\end{aligned}$$

Else If $c\theta_1 c\theta_2 r_{13} + c\theta_2 s\theta_1 r_{23} - s\theta_2 r_{33} = c\theta_1 r_{23} - s\theta_1 r_{13} = 0$ and $s\theta_2 c\theta_1 r_{13} + s\theta_1 s\theta_2 r_{23} + c\theta_2 r_{33} = -1$ then there are infinetely many solutions so long as they satisfy:

$$\begin{aligned}
\theta_5 &= -\pi \\
\theta_4 - \theta_6 &= Atan2(-c\theta_1 c\theta_2 r_{11} - c\theta_2 s\theta_1 r_{21} + s\theta_2 r_{31}, -c\theta_1 c\theta_2 r_{12} - c\theta_2 s\theta_1 r_{22} + s\theta_2 r_{32})
\end{aligned}$$

In conclusion, Given a homogenous transformation and wrist center

$$\begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & o_x \\ r_{21} & r_{22} & r_{23} & o_y \\ r_{31} & r_{32} & r_{33} & o_z \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$

for the Stanford manipulator shown in figure 6a, we have shown that:

$$\begin{aligned}
\theta_1 &= Atan2(y_c, x_c) - Atan2(\sqrt{(x_c)^2 + (y_c)^2 - (L_2)^2}, L_2) \\
\theta_2 &= Atan2\left(L_0 + L_1 - z_c, \sqrt{(x_c)^2 + (y_c)^2 - (L_2)^2}\right) \\
d_3 &= \sqrt{(x_c)^2 + (y_c)^2 - (L_2)^2 + (z_c - L_0 - L_1)^2}
\end{aligned}$$

In addition to all the possible solutions outlined at the top of this page for joint variables $\theta_4, \theta_5, \theta_6$.