Problem 1: [$5 \times 3 = 15$ points]

(a) **Given a fixed frame F_0 and a moving frame F_1 initially aligned with F_0, perform the following sequence of rotations on F_1:**

i. Rotate F_1 about the F_0 frame x-axis by α; call this new frame F_2.

ii. Rotate F_2 about the F_0 frame y-axis by β; call this new frame F_3.

iii. Rotate F_3 about the F_0 frame z-axis by γ; call this new frame F_4.

Show the expression (in terms of rotation matrices) for the final orientation R_{04}.

Notice that in the sequence of rotations above, each is done with respect to the fixed frame F_0. Initially, F_0 and F_1 are aligned. Therefore, the first rotation of F_1 about the F_0 x-axis by α (which gives F_2) is no different from a current frame rotation about the current x-axis by α which is represented by the following basic rotation matrix:

$$R_{02} = R_{12} = R_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{bmatrix}$$

Next in the sequence of rotations is the rotation of F_2 about the fixed frame's y-axis by β, obtaining F_3. We will now use a similarity transformation to find R_{23} (recall that the inverse of a rotation matrix is its transpose):

$$\begin{aligned} R_{23} = R_{02}^{-1} R_{y,\beta} R_{02} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{bmatrix}^T \begin{bmatrix} c\alpha & 0 & -s\alpha \\ 0 & 1 & 0 \\ s\alpha & 0 & c\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{bmatrix}^T \begin{bmatrix} c\beta & -s\alpha\ s\beta & -c\alpha\ s\beta \\ 0 & c\alpha & -s\alpha \\ s\beta & s\alpha\ c\beta & c\alpha\ c\beta \end{bmatrix} \\ &= \begin{bmatrix} c\beta & -s\alpha\ s\beta & -c\alpha\ s\beta \\ s\alpha\ s\beta & c^2\alpha + s^2\alpha\ c\beta & s\alpha\ c\alpha(c\beta - 1) \\ c\alpha\ s\beta & s\alpha\ c\alpha\ (c\beta - 1) & c^2\alpha\ c\beta + s^2\alpha \end{bmatrix} \end{aligned}$$

With R_{23} and R_{02} in hand, we can now find R_{03}:

$$\begin{aligned} R_{03} = R_{02} R_{23} = R_{02} R_{02}^{-1} R_{y,\beta} R_{02} = I\ R_{y,\beta} R_{02} &= R_{y,\beta} R_{02} \\ &= \begin{bmatrix} c\alpha & 0 & -s\alpha \\ 0 & 1 & 0 \\ s\alpha & 0 & c\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{bmatrix} \\ &= \begin{bmatrix} c\beta & -s\alpha\ s\beta & -c\alpha\ s\beta \\ 0 & c\alpha & -s\alpha \\ s\beta & s\alpha\ c\beta & c\alpha\ c\beta \end{bmatrix} \end{aligned}$$

The final rotation to be performed is a rotation of F_3 about the fixed frame z-axis by γ, to obtain F_4. Using a similarity transformation, again, we will now find R_{34}:

$$\begin{aligned} R_{34} = R_{03}^{-1} R_{z,\gamma} R_{03} &= \begin{bmatrix} c\beta & -s\alpha\ s\beta & -c\alpha\ s\beta \\ 0 & c\alpha & -s\alpha \\ s\beta & s\alpha\ c\beta & c\alpha\ c\beta \end{bmatrix}^T \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & -s\alpha\ s\beta & -c\alpha\ s\beta \\ 0 & c\alpha & -s\alpha \\ s\beta & s\alpha\ c\beta & c\alpha\ c\beta \end{bmatrix} \\ &= \begin{bmatrix} c\beta & -s\alpha\ s\beta & -c\alpha\ s\beta \\ 0 & c\alpha & -s\alpha \\ s\beta & s\alpha\ c\beta & c\alpha\ c\beta \end{bmatrix}^T \begin{bmatrix} c\beta\ c\gamma & -c\alpha\ s\gamma - s\alpha\ s\beta\ c\gamma & s\alpha\ s\gamma - c\alpha\ s\beta\ c\gamma \\ c\beta\ s\gamma & c\alpha\ c\gamma - s\alpha\ s\beta\ s\gamma & -c\alpha\ s\beta\ s\gamma - s\alpha\ c\gamma \\ s\beta & s\alpha\ c\beta & c\alpha\ c\beta \end{bmatrix} \\ &= \begin{bmatrix} c^2\beta\ c\gamma + s^2\beta & -(c\alpha\ s\gamma + s\alpha\ s\beta\ (c\gamma - 1))\ c\beta & -(c\alpha\ s\beta(c\gamma - 1) - s\alpha\ s\gamma)\ c\beta \\ c\alpha\ c\beta\ s\gamma + s\alpha\ s\beta\ c\beta\ (1 - c\gamma) & c^2\alpha\ c\gamma + s^2\alpha\ (c^2\beta + s^2\beta\ c\gamma) & s\alpha\ c\alpha\ c^2\beta\ (1 - c\gamma) - s\beta\ s\gamma \\ c\alpha\ s\beta\ c\beta\ (1 - c\gamma) - s\alpha\ c\beta\ s\gamma & s\alpha\ c\alpha\ c^2\beta\ (1 - c\gamma) + s\beta\ s\gamma & c^2\alpha\ (c^2\beta + s^2\beta\ c\gamma) + s^2\alpha\ c\gamma \end{bmatrix} \end{aligned}$$

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With R_{34} and R_{03} in hand, we can now find R_{04}:

$$
\begin{aligned}
R_{04} = R_{03}R_{34} = R_{03}R_{03}^{-1}R_{z,\gamma}R_{03} = I\ R_{z,\gamma}R_{03} = R_{z,\gamma}R_{03}& \\
&= \begin{bmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & -s\alpha\ s\beta & -c\alpha\ s\beta \\ 0 & c\alpha & -s\alpha \\ s\beta & s\alpha\ c\beta & c\alpha\ c\beta \end{bmatrix} \\
&= \begin{bmatrix} c\beta\ c\gamma & -c\alpha\ s\gamma - s\alpha\ s\beta\ c\gamma & s\alpha\ s\gamma - c\alpha\ s\beta\ c\gamma \\ c\beta\ s\gamma & c\alpha\ c\gamma - s\alpha\ s\beta\ s\gamma & -c\alpha\ s\beta\ s\gamma - s\alpha\ c\gamma \\ s\beta & s\alpha\ c\beta & c\alpha\ c\beta \end{bmatrix}
\end{aligned}
$$

In conclusion,

$$
\begin{aligned}
R_{04} = R_{03}R_{34} &= R_{z,\gamma}R_{03} \\
&= R_{z,\gamma}R_{02}R_{23} \\
&= R_{z,\gamma}R_{y,\beta}R_{02} \\
&= R_{z,\gamma}R_{y,\beta}R_{12} \\
&= R_{z,\gamma}R_{y,\beta}R_{x,\alpha} \\
&= \begin{bmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & -s\beta \\ 0 & 1 & 0 \\ s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{bmatrix} \\
&= \begin{bmatrix} c\beta\ c\gamma & -c\alpha\ s\gamma - s\alpha\ s\beta\ c\gamma & s\alpha\ s\gamma - c\alpha\ s\beta\ c\gamma \\ c\beta\ s\gamma & c\alpha\ c\gamma - s\alpha\ s\beta\ s\gamma & -c\alpha\ s\beta\ s\gamma - s\alpha\ c\gamma \\ s\beta & s\alpha\ c\beta & c\alpha\ c\beta \end{bmatrix}
\end{aligned}
$$

(b) **Suppose that the third step above is replaced by the following: "Rotate F_3 about the z-axis of frame F_3 by γ; call this new frame F_4." What is the final orientation (again expressed in terms of other roation matrices) R_{04}?**

First, observe that a rotation of F_3 about the z-axis of frame F_3 by γ is simply a current frame rotation about the z-axis by γ. It follows that:

$$
R_{34} = R_{z,\gamma} = \begin{bmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

Additionally, note that R_{03} and R_{02} remain unchanged from part (a) above, since only the third rotation of the sequence was modified. As such,

$$
\begin{aligned}
R_{04} &= R_{03}R_{34} \\
&= R_{03}R_{z,\gamma} \\
&= R_{02}R_{23}R_{z,\gamma} \\
&= R_{02}R_{02}^{-1}R_{y,\beta}R_{02}R_{z,\gamma} \\
&= I\ R_{y,\beta}R_{02}R_{z,\gamma} \\
&= R_{y,\beta}R_{x,\alpha}R_{z,\gamma} \\
&= \begin{bmatrix} c\beta & 0 & -s\beta \\ 0 & 1 & 0 \\ s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{bmatrix} \begin{bmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c\beta & -s\alpha\ s\beta & -c\alpha\ s\beta \\ 0 & c\alpha & -s\alpha \\ s\beta & s\alpha\ c\beta & c\alpha\ c\beta \end{bmatrix} \begin{bmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c\beta\ c\gamma - s\alpha\ s\beta\ s\gamma & -s\alpha\ s\beta\ c\gamma - c\beta\ s\gamma & -c\alpha\ s\beta \\ c\alpha\ s\gamma & c\alpha\ c\gamma & -s\alpha \\ s\alpha\ c\beta\ s\gamma + s\beta\ c\gamma & s\alpha\ c\beta\ c\gamma - s\beta\ s\gamma & c\alpha\ c\beta \end{bmatrix}
\end{aligned}
$$

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(c) **Find T_{ca} for the following transformations:**

$$T_{ab} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; T_{cb} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Observe,

$$T_{ca} = T_{cb}T_{ba} = T_{cb}T_{ab}^{-1}$$

To find T_{ab}^{-1} recall that the definition of the inverse of some homogenous transformation H where

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \qquad \text{is} \qquad H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}$$

In the case of T_{ab} where

$$R = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad d = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

we first determine that

$$R^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad -R^T d = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ -1 \end{bmatrix}$$

which then means that

$$T_{ab}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

With T_{ab}^{-1} in hand, we can now find T_{ca} as follows:

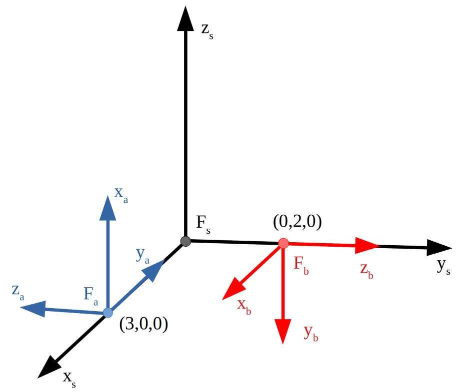
$$\begin{aligned} T_{ca} &= T_{cb}T_{ab}^{-1} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} - \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 1 - \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} - \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Problem 2: $[4 \times 5 + 1 \times 10 = 30 \text{ points}]$

In terms of the (x_s, y_s, z_s) coordinates of a fixed space frame F_s, frame F_a has its x_a-axis pointing in the direction (0,0,1) and its y_a-axis pointing in the direction (-1,0,0), and frame F_b has its x_b-axis pointing in the direction (1,0,0) and its y_b-axis pointing in the direction (0,0,-1). The origin of F_a is at (3,0,0) in F_s and the origin of F_b is at (0,2,0) is F_s.

(a) **Draw by hand a diagram showing F_a and F_b relative to F_s.**

The direction of the z-axis for the F_b and F_a frames can be found using the right-hand rule. Note that the figure below is not necessarily to scale (though all points and axis vectors are labeled). Also, all of the axis vectors are either parallel or orthogonal.



(b) **Write down the rotation matrices R_{sa} and R_{sb} and the transformation matrices T_{sa} and T_{sb}.**

Let the fixed frame F_s axis-vectors be defined as follows:

$$x_s = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; y_s = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; z_s = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Per the problem description, the x-axis and y-axis vectors of frame F_a relative to the fixed frame F_s are as follows:

$$x_a^s = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; y_a^s = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

Using the right hand rule, we can further assert that $z_a^s = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$. To find R_{sa}, each axis of frame F_a is projected onto the fixed frame F_s:

$$R_{sa} = \begin{bmatrix} x_a^s \cdot x_s & y_a^s \cdot x_s & z_a^s \cdot x_s \\ x_a^s \cdot y_s & y_a^s \cdot y_s & z_a^s \cdot y_s \\ x_a^s \cdot z_s & y_a^s \cdot z_s & z_a^s \cdot z_s \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

Looking at the figure provided as a solution to part (a), this result for R_{sa} makes sense geometrically. Specifically, from the figure, we can see that the orientation of frame F_a can be obtained from F_s by a sequence of two current-frame rotations:

$$R_{sa} = R_{x,\frac{\pi}{2}} R_{z,\frac{\pi}{2}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\frac{\pi}{2} & -s\frac{\pi}{2} \\ 0 & s\frac{\pi}{2} & c\frac{\pi}{2} \end{bmatrix} \begin{bmatrix} c\frac{\pi}{2} & -s\frac{\pi}{2} & 0 \\ s\frac{\pi}{2} & c\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

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To find T_{sa} all that is needed is to combine R_{sa} with the displacement vector

$$d_{sa} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

Where

$$T_{sa} = \begin{bmatrix} R_{sa} & d_{sa} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Following a similar process, we will now find R_{sb} and T_{sb}. Per the problem description, the x-axis and y-axis vectors of frame F_b relative to the fixed frame F_s are as follows:

$$x_b^s = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ y_b^s = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Using the right hand rule, we can further assert that $z_b^s = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. To find R_{sb}, each axis of frame F_b is projected onto the fixed frame F_s:

$$R_{sb} = \begin{bmatrix} x_b^s \cdot x_s & y_b^s \cdot x_s & z_b^s \cdot x_s \\ x_b^s \cdot y_s & y_b^s \cdot y_s & z_b^s \cdot y_s \\ x_b^s \cdot z_s & y_b^s \cdot z_s & z_b^s \cdot z_s \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Looking at the figure provided as a solution to part (a), this result for R_{sb} makes sense geometrically. Specifically, from the figure, we can see that the orientation of frame F_b can be obtained from F_s by a single current-frame rotation:

$$R_{sb} = R_{x,\frac{-\pi}{2}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\frac{-\pi}{2} & -s\frac{-\pi}{2} \\ 0 & s\frac{-\pi}{2} & c\frac{-\pi}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

To find T_{sb} all that is needed is to combine R_{sb} with the displacement vector

$$d_{sb} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

Where

$$T_{sb} = \begin{bmatrix} R_{sb} & d_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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(c) **Given T_{sb}, how do you calculate T_{sb}^{-1} without using a matrix inverse? Write T_{sb}^{-1} and verify its correctness using your drawing.**

To find T_{sb}^{-1} recall that the definition of the inverse of some homogenous transformation H where

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \quad \text{is} \quad H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}$$

In the case of T_{sb} where

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad d = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

we first determine that

$$R^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad -R^T d = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

which then means that

$$T_{sb}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To verify the correctness of T_{sb}^{-1} let the point p as seen from the fixed frame F_s be $p^s = (3, 3, 3)$. From the figure provided as a solution to part (a), we can see that this point, as seen from F_b should be $p^b = (3, -3, 1)$. Let P^s and P^b be augmented vectors pointing from each frame's respective origin's to point p. It follows that

$$P^s = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 1 \end{bmatrix}, \text{ and } P^b = \begin{bmatrix} 3 \\ -3 \\ 1 \\ 1 \end{bmatrix}$$

We expect that $T_{sb}P^b = P^s$, and $T_{sb}^{-1}P^s = P^b$. Observe,

$$T_{sb}P^b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 1 \end{bmatrix}$$

$$T_{sb}^{-1}P^s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 1 \\ 1 \end{bmatrix}$$

Alternatively, we can also contruct T_{sb}^{-1} by recognizing that the displacement vector from the origin of frame F_b to the origin of F_s is

$$d_{bs} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

and only a single current frame rotation is needed to orient frame F_b with the fixed frame F_s:

$$R_{bs} = R_{x,\frac{\pi}{2}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\frac{\pi}{2} & -s\frac{\pi}{2} \\ 0 & s\frac{\pi}{2} & c\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Thus

$$T_{bs} = T_{sb}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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(d) **Use T_{sb} to change the representation of the point $p_b = (1, 2, 3)$ in F_b coordinates to F_s coordinates.**

Let P_b be augmented vector pointing from the origin of frame F_b to point p. Additionally, let P_s be the augmented vector pointing from the origin of frame F_s to the point p:

$$P_b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

Observe,

$$P_s = T_{sb}P_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -2 \\ 1 \end{bmatrix}$$

Thus the coordinates of p as seen from frame F_s is $p_s = (1, 5, -2)$. This result matches what we would expect given the figure in part (a).

(e) **Choose a point p represented by $p_s = (1, 2, 3)$ in F_s coordinates. Calculate $p' = T_{sb}p_s$ and $p'' = T_{sb}^{-1}p_s$. For each operation, should the result be interpreted as changing coordinates (from the F_s frame to F_b) without moving the point p, or as moving the location of the point without changing the reference frame of the representation?**

Recall that

$$T_{sb} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } T_{sb}^{-1} = T_{bs} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let P', P'', P_s, and P_b be the agumented vector coordinates of p', p'', p_s, and p_b respectively. Observe,

$$P'' = T_{sb}^{-1}P_s = T_{bs}P_s = P_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Thus, $p'' = (1, -3, 0)$. This result is just the coordinates of the point p specified according to frame F_b instead of frame F_s.

Next, consider the following:

$$P' = T_{sb}P_s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -2 \\ 1 \end{bmatrix}$$

Thus, $p' = (1, 5, -2)$. It would be a mistake to interpret this result as coordinates of point p with respect to frame F_b. Instead, it is more accurate to describe this result as a tranformation of the point p to a new point p' in the same reference frame F_s. Alternatively, p' can be thought of as the coordinates of point p with respect to a new frame F_c whose origin is at $(0, 0, -2)$ with respect to F_s, and whose orientation can be obtained by rotating F_s by $\theta = \frac{-\pi}{2}$ about its x-axis.

Problem 3: [10 + 10 = 20 points]

Two satellites are circling the Earth as shown in Figure 1. Frames 1 and 2 are rigidly attached to the satellites in such a way that their x-axes always point toward the Earth. Satellite 1 moves at a constant speed v_1, while satellite 2 moves at a constant speed v_2. To simplify matters, ignore the roation of the Earth about its own axis. The fixed frame {0} is located at the center of the Earth. Figure 1 shows the position of the two satellites at $t = 0$.

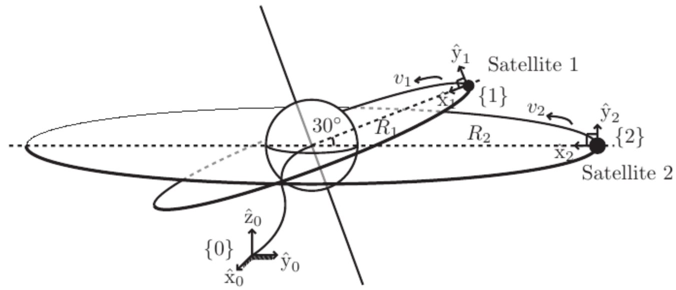

Figure 1: Figure for problem 3.

(a) **Derive the frames T_{01}, T_{02} as a function of t.**

Observe figure 2 which highlights some important lengths measured at some arbitrarty time t:

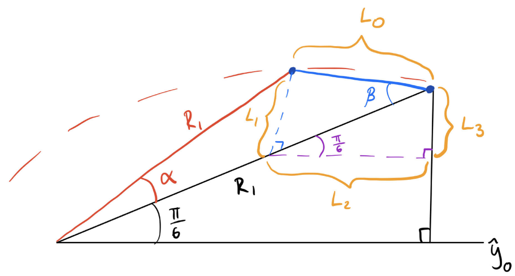

Figure 2: Used to find $T_{01}(t)$

We will begin by finding all of the unknown values in figure 2 starting with α. Note that because the orbits are circular, the distance traveled by Satellite 1 during the time period $[0, t]$ (distance = rate · time) is related to α by the arclength which spans the chord L_0. Specifically,

$$\frac{\alpha}{2\pi} = \frac{v_1 t}{2\pi R_1}$$
$$\iff \alpha = \frac{v_1 t}{R_1}$$

β can be found by noticing that it is a base interior angle of an isosceles triangle. Using some simple geometric relationships, then, we see that $\beta = \frac{\pi - \alpha}{2} = \frac{\pi}{2} - \frac{\alpha}{2}$.

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The following expressions for L_0, L_1, L_2, and L_3 can all be found using simple geometric relatioships which can be observed in figure 2:

$$L_0 = 2R_1 \sin(\frac{\alpha}{2})$$

$$\begin{aligned} L_1 &= L_0 \sin(\beta) \\ &= 2R_1 \sin(\frac{\alpha}{2}) \sin(\frac{\pi}{2} - \frac{\alpha}{2}) \\ &= 2R_1 \sin(\frac{\alpha}{2}) \cos(\frac{\alpha}{2}) \\ &= R_1 \sin(\alpha) \qquad \text{// double-angle identity} \end{aligned}$$

$$\begin{aligned} L_2 &= L_0 \cos(\beta) \cos(\frac{\pi}{6}) \\ &= L_0 \cos(\frac{\pi}{2} - \frac{\alpha}{2}) \frac{\sqrt{3}}{2} \\ &= 2R_1 \sin(\frac{\alpha}{2}) \sin(\frac{\alpha}{2}) \frac{\sqrt{3}}{2} \\ &= \sqrt{3}\ R_1 \sin^2(\frac{\alpha}{2}) \end{aligned}$$

$$\begin{aligned} L_3 &= L_0 \cos(\beta) \sin(\frac{\pi}{6}) \\ &= L_0 \cos(\frac{\pi}{2} - \frac{\alpha}{2}) \frac{1}{2} \\ &= L_0 \sin(\frac{\alpha}{2}) \frac{1}{2} \\ &= 2R_1 \sin(\frac{\alpha}{2}) \sin(\frac{\alpha}{2}) \frac{1}{2} \\ &= R_1 \sin^2(\frac{\alpha}{2}) \end{aligned}$$

To find $T_{01}(t)$, recall that the general form of some homogenous transformation H_{01} is

$$\begin{bmatrix} n & s & a & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where n, s, and a are unit vectors representing the direction of the x-axis, y-axis, and z-axis, respectively, of frame F_1 with respect to frame F_0, and d is the displacement vector from the origin of frame F_0 to the origin of frame F_1.

First lets find the displacement vector $d_1^0(0)$ which goes from the origin of frame F_0 to the origin of frame F_1 at time $t = 0$. From figure 1, we can see that $d_{1,x}^0(0) = 0$, $d_{1,y}^0(0) = R_1 \cos \frac{\pi}{6} = R_1 \frac{\sqrt{3}}{2}$, and $d_{1,z}^0(0) = R_1 \sin \frac{\pi}{6} = R_1 \frac{1}{2}$. That is,

$$d_1^0(t) = \begin{bmatrix} 0 \\ \frac{\sqrt{3}}{2} R_1 \\ \frac{1}{2} R_1 \end{bmatrix}$$

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With $d_1^0(0)$ in hand, we will now find $d_1^0(t)$ one component at a time:

$$\begin{aligned}
d_{1,x}^0(t) &= d_{1,x}^0(0) - L_1 \\
&= 0 - R_1 \sin(\alpha) \\
&= -R_1 \sin(\frac{v_1 t}{R_1})
\end{aligned}$$

$$\begin{aligned}
d_{1,y}^0(t) &= d_{1,y}^0(0) - L_2 \\
&= \frac{\sqrt{3}}{2} R_1 - \sqrt{3}\ R_1 \sin^2(\frac{\alpha}{2}) \\
&= \frac{\sqrt{3}}{2} R_1 \left(1 - 2\sin^2(\frac{\alpha}{2})\right) \\
&= \frac{\sqrt{3}}{2} R_1 \cos(\alpha) \qquad \text{// double-angle identity} \\
&= \frac{\sqrt{3}}{2} R_1 \cos(\frac{v_1 t}{R_1})
\end{aligned}$$

$$\begin{aligned}
d_{1,z}^0(t) &= d_{1,z}^0(0) - L_3 \\
&= \frac{1}{2} R_1 - R_1 \sin^2(\frac{\alpha}{2}) \\
&= \frac{1}{2} R_1 \left(1 - 2\sin^2(\frac{\alpha}{2})\right) \\
&= \frac{1}{2} R_1 \cos(\alpha) \qquad \text{// double-angle identity} \\
&= \frac{1}{2} R_1 \cos(\frac{v_1 t}{R_1})
\end{aligned}$$

Therefore,

$$d_1^0(t) = \begin{bmatrix} -R_1 \sin(\frac{v_1 t}{R_1}) \\ \frac{\sqrt{3}}{2} R_1 \cos(\frac{v_1 t}{R_1}) \\ \frac{1}{2} R_1 \cos(\frac{v_1 t}{R_1}) \end{bmatrix}$$

Next, we find $n_1^0(t)$, $s_1^0(t)$, and $a_1^0(t)$. Note that $n_1^0(t)$ will just be the unit vector opposite the displacement vector $d_1^0(t)$ since the x-axis of F_1 always points to origin of F_0.

$$n_1^0(t) = -\frac{d_1^0(t)}{R_1} = \begin{bmatrix} \sin(\frac{v_1 t}{R_1}) \\ -\frac{\sqrt{3}}{2} \cos(\frac{v_1 t}{R_1}) \\ -\frac{1}{2} \cos(\frac{v_1 t}{R_1}) \end{bmatrix}$$

Note that $s_1^0(t)$ does not change with t since the y-axis of frame F_1 is perpendicular to the plane on which Satellite 1 orbits. Using figure 1, then, we can see that

$$s_1^0(t) = s_1^0(0) = \begin{bmatrix} 0 \\ -\cos(\frac{\pi}{3}) \\ \sin(\frac{\pi}{3}) \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

Finally, $a_1^0(t)$ can be found by taking the cross product of $n_1^0(t)$ and $s_1^0(t)$:

$$a_1^0(t) = n_1^0(t) \times s_1^0(t) = \begin{bmatrix} \sin(\frac{v_1 t}{R_1}) \\ -\frac{\sqrt{3}}{2} \cos(\frac{v_1 t}{R_1}) \\ -\frac{1}{2} \cos(\frac{v_1 t}{R_1}) \end{bmatrix} \times \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2} \cos(\frac{v_1 t}{R_1}) \frac{\sqrt{3}}{2} - \left(-\frac{1}{2} \cos(\frac{v_1 t}{R_1}) \left(-\frac{1}{2}\right)\right) \\ -\left(\sin(\frac{v_1 t}{R_1}) \frac{\sqrt{3}}{2} - 0\right) \\ \sin(\frac{v_1 t}{R_1})(\frac{-1}{2}) - 0 \end{bmatrix} = \begin{bmatrix} -\cos(\frac{v_1 t}{R_1}) \\ -\frac{\sqrt{3}}{2} \sin(\frac{v_1 t}{R_1}) \\ -\frac{1}{2} \sin(\frac{v_1 t}{R_1}) \end{bmatrix}$$

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With $d_1^0(t)$, $n_1^0(t)$, $s_1^0(t)$, and $a_1^0(t)$, we can finally write the expression for $T_{01}(t)$:

$$T_{01}(t) = \begin{bmatrix} \sin(\frac{v_1 t}{R_1}) & 0 & -\cos(\frac{v_1 t}{R_1}) & -R_1\sin(\frac{v_1 t}{R_1}) \\ -\frac{\sqrt{3}}{2}\cos(\frac{v_1 t}{R_1}) & -\frac{1}{2} & -\frac{\sqrt{3}}{2}\sin(\frac{v_1 t}{R_1}) & \frac{\sqrt{3}}{2}R_1\cos(\frac{v_1 t}{R_1}) \\ -\frac{1}{2}\cos(\frac{v_1 t}{R_1}) & \frac{\sqrt{3}}{2} & -\frac{1}{2}\sin(\frac{v_1 t}{R_1}) & \frac{1}{2}R_1\cos(\frac{v_1 t}{R_1}) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Next we follow a very similar process to find $T_{02}(t)$. Observe figure 3 which highlights some important lengths measured at some arbitrarty time t:

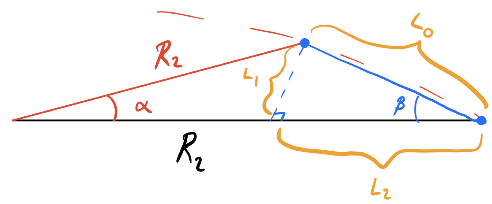

Figure 3: Used to find $T_{02}(t)$

We will begin by finding all of the unknown values in figure 3 starting with α. Note that because the orbits are circular, the distance traveled by Satellite 2 during the time period $[0, t]$ (distance = rate · time) is related to α by the arclength which spans the chord L_0. Specifically,

$$\frac{\alpha}{2\pi} = \frac{v_2 t}{2\pi R_2}$$
$$\iff \alpha = \frac{v_2 t}{R_2}$$

β can be found by noticing that it is a base interior angle of an isosceles triangle. Using some simple geometric relationships, then, we see that $\beta = \frac{\pi - \alpha}{2} = \frac{\pi}{2} - \frac{\alpha}{2}$.

The following expressions for L_0, L_1, and L_2 can all be found using simple geometric relatioships which can be observed in figure 3:

$$L_0 = 2R_2\sin(\frac{\alpha}{2})$$

$$\begin{aligned} L_1 &= L_0\sin(\beta) \\ &= 2R_2\sin(\frac{\alpha}{2})\sin(\frac{\pi}{2} - \frac{\alpha}{2}) \\ &= 2R_2\sin(\frac{\alpha}{2})\cos(\frac{\alpha}{2}) \\ &= R_2\sin(\alpha) \qquad \text{// double-angle identity} \end{aligned}$$

$$\begin{aligned} L_2 &= L_0\cos(\beta) \\ &= 2R_2\sin(\frac{\alpha}{2})\cos(\frac{\pi}{2} - \frac{\alpha}{2}) \\ &= 2R_2\sin(\frac{\alpha}{2})\sin(\frac{\alpha}{2}) \\ &= 2R_2\sin^2(\frac{\alpha}{2}) \end{aligned}$$

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To find $T_{02}(t)$, recall that the general form of some homogenous transformation H_{01} is

$$\begin{bmatrix} n & s & a & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where n, s, and a are unit vectors representing the direction of the x-axis, y-axis, and z-axis, respectively, of frame F_1 with respect to frame F_0, and d is the displacement vector from the origin of frame F_0 to the origin of frame F_1.

First lets find the displacement vector $d_2^0(0)$ which goes from the origin of frame F_0 to the origin of frame F_2 at time $t = 0$. From figure 1, we can see that $d_{2,x}^0(0) = 0$, $d_{2,y}^0(0) = R_2$, and $d_{2,z}^0(0) = 0$. That is,

$$d_2^0(0) = \begin{bmatrix} 0 \\ R_2 \\ 0 \end{bmatrix}$$

With $d_2^0(0)$ in hand, we will now find $d_2^0(t)$ one component at a time:

$$\begin{aligned} d_{2,x}^0(t) &= d_{2,x}^0(0) - L_1 \\ &= 0 - R_2\sin(\alpha) \\ &= -R_2\sin(\frac{v_2 t}{R_2}) \end{aligned}$$

$$\begin{aligned} d_{2,y}^0(t) &= d_{2,y}^0(0) - L_2 \\ &= R_2 - 2R_2\sin^2(\frac{\alpha}{2}) \\ &= R_2\left(1 - 2\sin^2(\frac{\alpha}{2})\right) \\ &= R_2\cos(\alpha) \qquad \text{// double-angle identity} \\ &= R_2\cos(\frac{v_2 t}{R_2}) \end{aligned}$$

$$\begin{aligned} d_{2,z}^0(t) &= d_{2,z}^0(0) - 0 \\ &= 0 \end{aligned}$$

Therefore,

$$d_2^0(t) = \begin{bmatrix} -R_2\sin(\frac{v_2 t}{R_2}) \\ R_2\cos(\frac{v_2 t}{R_2}) \\ 0 \end{bmatrix}$$

Next, we find $n_2^0(t)$, $s_2^0(t)$, and $a_2^0(t)$. Note that $n_2^0(t)$ will just be the unit vector opposite the displacement vector $d_2^0(t)$ since the x-axis of F_2 always points to origin of F_0.

$$n_2^0(t) = -\frac{d_2^0(t)}{R_2} = \begin{bmatrix} \sin(\frac{v_2 t}{R_2}) \\ -\cos(\frac{v_2 t}{R_2}) \\ 0 \end{bmatrix}$$

Note that $s_2^0(t)$ does not change with t since the y-axis of frame F_2 is perpendicular to the plane on which Satellite 2 orbits. Thus,

$$s_2^0(t) = s_2^0(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Finally, $a_2^0(t)$ can be found by taking the cross product of $n_2^0(t)$ and $s_2^0(t)$:

$$a_2^0(t) = n_2^0(t) \times s_2^0(t) = \begin{bmatrix} \sin(\frac{v_2 t}{R_2}) \\ -\cos(\frac{v_2 t}{R_2}) \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\cos(\frac{v_2 t}{R_2}) \\ -\sin(\frac{v_2 t}{R_2}) \\ 0 \end{bmatrix}$$

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With $d_2^0(t)$, $n_2^0(t)$, $s_2^0(t)$, and $a_2^0(t)$, we can finally write the expression for $T_{02}(t)$:

$$T_{02}(t) = \begin{bmatrix} \sin(\frac{v_2 t}{R_2}) & 0 & -\cos(\frac{v_2 t}{R_2}) & -R_2\sin(\frac{v_2 t}{R_2}) \\ -\cos(\frac{v_2 t}{R_2}) & 0 & -\sin(\frac{v_2 t}{R_2}) & R_2\cos(\frac{v_2 t}{R_2}) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) **Using your results from part (a), find T_{21} as a function of t.**

First, note that

$$T_{21} = T_{20}T_{01} = T_{02}^{-1}T_{01}$$

Thus to find T_{21} we must first find T_{02}^{-1}. To find T_{02}^{-1} recall that the definition of the inverse of some homogenous transformation H where

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \qquad \text{is} \qquad H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}$$

In the case of T_{02} where

$$R = \begin{bmatrix} \sin(\frac{v_2 t}{R_2}) & 0 & -\cos(\frac{v_2 t}{R_2}) \\ -\cos(\frac{v_2 t}{R_2}) & 0 & -\sin(\frac{v_2 t}{R_2}) \\ 0 & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad d = \begin{bmatrix} -R_2\sin(\frac{v_2 t}{R_2}) \\ R_2\cos(\frac{v_2 t}{R_2}) \\ 0 \end{bmatrix}$$

we first determine that

$$R^T = \begin{bmatrix} \sin(\frac{v_2 t}{R_2}) & -\cos(\frac{v_2 t}{R_2}) & 0 \\ 0 & 0 & 1 \\ -\cos(\frac{v_2 t}{R_2}) & -\sin(\frac{v_2 t}{R_2}) & 0 \end{bmatrix}$$

and

$$\begin{aligned} -R^T d &= \begin{bmatrix} -\sin(\frac{v_2 t}{R_2}) & \cos(\frac{v_2 t}{R_2}) & 0 \\ 0 & 0 & -1 \\ \cos(\frac{v_2 t}{R_2}) & \sin(\frac{v_2 t}{R_2}) & 0 \end{bmatrix} \begin{bmatrix} -R_2\sin(\frac{v_2 t}{R_2}) \\ R_2\cos(\frac{v_2 t}{R_2}) \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} R_2\sin^2(\frac{v_2 t}{R_2}) + R_2\cos^2(\frac{v_2 t}{R_2}) \\ 0 \\ -R_2\cos(\frac{v_2 t}{R_2})\sin(\frac{v_2 t}{R_2}) + R_2\cos(\frac{v_2 t}{R_2})\sin(\frac{v_2 t}{R_2}) \end{bmatrix} \\ &= \begin{bmatrix} R_2\left(\sin^2(\frac{v_2 t}{R_2}) + \cos^2(\frac{v_2 t}{R_2})\right) \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} R_2 \\ 0 \\ 0 \end{bmatrix} \qquad \text{// pythagorean identity} \end{aligned}$$

This result is as expected since the x-axis of frame F_2 always points to the origin of frame F_0 which is R_2 units away irrespective of the time t. Combining R^T with $-R^T d$ gives:

$$T_{02}^{-1} = \begin{bmatrix} \sin(\frac{v_2 t}{R_2}) & -\cos(\frac{v_2 t}{R_2}) & 0 & R_2 \\ 0 & 0 & 1 & 0 \\ -\cos(\frac{v_2 t}{R_2}) & -\sin(\frac{v_2 t}{R_2}) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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In the interest of clarity, let $\phi = \frac{v_1 t}{R_1}$ and $\gamma = \frac{v_2 t}{R_2}$. Additionally, let s and c represent sin and cos respectively. It follows, then, that

$$\begin{aligned}
T_{21} &= T_{02}^{-1} T_{01} \\
&= \begin{bmatrix} s\gamma & -c\gamma & 0 & R_2 \\ 0 & 0 & 1 & 0 \\ -c\gamma & -s\gamma & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s\phi & 0 & -c\phi & -R_1 s\phi \\ -\frac{\sqrt{3}}{2}c\phi & -\frac{1}{2} & -\frac{\sqrt{3}}{2}s\phi & \frac{\sqrt{3}}{2}R_1 c\phi \\ -\frac{1}{2}c\phi & \frac{\sqrt{3}}{2} & -\frac{1}{2}s\phi & \frac{1}{2}R_1 c\phi \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sqrt{3}}{2}c\gamma c\phi + s\gamma s\phi & \frac{1}{2}c\gamma & \frac{\sqrt{3}}{2}c\gamma s\phi - s\gamma c\phi & R_2 - R_1 s\gamma s\phi - \frac{\sqrt{3}}{2}R_1 c\gamma c\phi \\ -\frac{1}{2}c\phi & \frac{\sqrt{3}}{2} & -\frac{1}{2}s\phi & \frac{1}{2}R_1 c\phi \\ \frac{\sqrt{3}}{2}s\gamma c\phi - c\gamma s\phi & \frac{1}{2}s\gamma & \frac{\sqrt{3}}{2}s\gamma s\phi + c\gamma c\phi & R_1 c\gamma s\phi - \frac{\sqrt{3}}{2}R_1 s\gamma c\phi \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Problem 4: [10 points]

Remember that a Quaternion is defined as the 4-tuple $\underline{Q} = q_0 + q_1 i + q_2 j + q_3 k$, where $i^2 = j^2 = k^2 = -1$ and $ijk = -1$. A rotation by θ about the unit vector $\hat{n} = [n_1, n_2, n_3]^T$ can be represented by the unit quarternion $\hat{\underline{Q}} = \cos\frac{\theta}{2} + n\ \sin\frac{\theta}{2}$. Show that $Norm\{\hat{\underline{Q}}\} = 1$.

Recall that the norm of some quaternion $\underline{Q} = q_0 + q_1 i + q_2 j + q_3 k$ is defined as:

$$Norm\{\underline{Q}\} = \sqrt{\underline{Q}\ \underline{Q}^*} = \sqrt{(q_0)^2 + (q_1)^2 + (q_2)^2 + (q_3)^2}$$

For a derivation of the expression $\underline{Q}\ \underline{Q}^* = (q_0)^2 + (q_1)^2 + (q_2)^2 + (q_3)^2$ see problem 5.

It follows that for the quaternion $\hat{\underline{Q}} = \cos\frac{\theta}{2} + \hat{n}\ \sin\frac{\theta}{2} = \cos\left(\frac{\theta}{2}\right) + (\sin\left(\frac{\theta}{2}\right) n_1)i + (\sin\left(\frac{\theta}{2}\right) n_2)j + (\sin\left(\frac{\theta}{2}\right) n_3)k$:

$$\begin{aligned}
Norm\{\hat{\underline{Q}}\} &= \sqrt{\left(\cos\left(\frac{\theta}{2}\right)\right)^2 + \left(\sin\left(\frac{\theta}{2}\right)(n_1)\right)^2 + \left(\sin\left(\frac{\theta}{2}\right)(n_2)\right)^2 + \left(\sin\left(\frac{\theta}{2}\right)(n_3)\right)^2} \\
&= \sqrt{\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)(n_1)^2 + \sin^2\left(\frac{\theta}{2}\right)(n_2)^2 + \sin^2\left(\frac{\theta}{2}\right)(n_3)^2} \\
&= \sqrt{\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)\left((n_1)^2 + (n_2)^2 + (n_3)^2\right)} \\
&= \sqrt{\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)(1)} \qquad \text{// since } \hat{n} \text{ is a unit vector} \\
&= \sqrt{\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)} \\
&= \sqrt{1} \qquad \text{// pythagorean identity} \\
&= 1
\end{aligned}$$

Therefore, $Norm\{\hat{\underline{Q}}\} = 1$.

Problem 5: [10 points]

The conjugate $\underline{Q}^*$ of the quanterion $\underline{Q}$ is defined as $\underline{Q}^* = q_0 - q_1 i - q_2 j - q_3 k$. Show that $\underline{Q} * \underline{Q}^* = \underline{Q}^* * \underline{Q}$.

Note that the product of two quaternions $\underline{P} = p_0 + p_1 i + p_2 j + q_3 k$ and $\underline{Q} = q_0 + q_1 i + q_2 j + q_3 k$ is equal to:

$$\underline{P}\underline{Q} = (p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3) + (p_0q_1 + p_1q_0 + p_2q_3 - p_3q_2)i + (p_0q_2 - p_1q_3 + p_2q_0 + p_3q_1)j + (p_0q_3 + p_1q_2 - p_2q_1 + p_3q_0)k$$

The derivation of this expression can be found in problem 6. As such,

$$\begin{aligned}
\underline{Q}\ \underline{Q}^* &= \big(q_0q_0 - q_1(-q_1) - q_2(-q_2) - q_3(-q_3)\big) \\
&\quad + \big(q_0(-q_1) + q_1q_0 + q_2(-q_3) - q_3(-q_2)\big)i \\
&\quad + \big(q_0(-q_2) - q_1(-q_3) + q_2q_0 + q_3(-q_1)\big)j \\
&\quad + \big(q_0(-q_3) + q_1(-q_2) - q_2(-q_1) + q_3q_0\big)k \\
\\
&= \big((q_0)^2 + (q_1)^2 + (q_2)^2 + (q_3)^2\big) + \big(0\big)i + \big(0\big)j + \big(0\big)k \\
\\
&= (q_0)^2 + (q_1)^2 + (q_2)^2 + (q_3)^2
\end{aligned}$$

and...

$$\begin{aligned}
\underline{Q}^*\ \underline{Q} &= \big(q_0q_0 - (-q_1)q_1 - (-q_2)q_2 - (-q_3)q_3\big) \\
&\quad + \big(q_0q_1 + (-q_1)q_0 + (-q_2)q_3 - (-q_3)q_2\big)i \\
&\quad + \big(q_0q_2 - (-q_1)q_3 + (-q_2)q_0 + (-q_3)q_1\big)j \\
&\quad + \big(q_0q_3 + (-q_1)q_2 - (-q_2)q_1 + (-q_3)q_0\big)k \\
\\
&= \big((q_0)^2 + (q_1)^2 + (q_2)^2 + (q_3)^2\big) + \big(0\big)i + \big(0\big)j + \big(0\big)k \\
\\
&= (q_0)^2 + (q_1)^2 + (q_2)^2 + (q_3)^2
\end{aligned}$$

Therefore, both $\underline{Q}\ \underline{Q}^*$ and $\underline{Q}^*\ \underline{Q}$ are equal to $(q_0)^2 + (q_1)^2 + (q_2)^2 + (q_3)^2$. Thus $\underline{Q}\ \underline{Q}^* = \underline{Q}^*\ \underline{Q}$

Problem 6: [15 points]

Show with relevant details that the multiplication of two quaternions $\underline{P} = p_0 + p = p_0 + ip_1 + jp_2 + kp_3$ **and** $\underline{Q} = q_0 + q = q_0 + iq_1 + jq_2 + kq_3$ **yields:**

$$\underline{P}\underline{Q} = p_0q_0 - p \cdot q + p_0q + q_0p + p \times q$$

Per the problem description, $\underline{P} = p_0 + p = p_0 + ip_1 + jp_2 + kp_3$ and $\underline{Q} = q_0 + q = q_0 + iq_1 + jq_2 + kq_3$. It follows that:

$$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}, \text{ and } q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Using p and q, note the following expressions:

$$p \cdot q = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \cdot \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = p_1q_1 + p_2q_2 + p_3q_3$$

$$p_0q = p_0 \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} p_0q_1 \\ p_0q_2 \\ p_0q_3 \end{bmatrix}$$

$$q_0p = q_0 \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} q_0p_1 \\ q_0p_2 \\ q_0p_3 \end{bmatrix}$$

$$p \times q = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \times \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} p_2q_3 - p_3q_2 \\ p_3q_1 - p_1q_3 \\ p_1q_2 - p_2q_1 \end{bmatrix}$$

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Having found $p \cdot q$, $p_0 q$, $q_0 p$, and $p \times q$, we will now expand the product $\underline{P}\ \underline{Q}$. Observe,

$$
\begin{aligned}
\underline{P}\ \underline{Q} &= \big(p_0 + p_1 i + p_2 j + p_3 k\big)\big(q_0 + q_1 i + q_2 j + q_3 k\big) \\[1em]
&= p_0 q_0 + p_0 q_1 i + p_0 q_2 j + p_0 q_3 k \\
&\quad + p_1 i q_0 + p_1 i q_1 i + p_1 i q_2 j + p_1 i q_3 k \\
&\quad + p_2 j q_0 + p_2 j q_1 i + p_2 j q_2 j + p_2 j q_3 k \\
&\quad + p_3 k q_0 + p_3 k q_1 i + p_3 k q_2 j + p_3 k q_3 k \\[1em]
&= \big(p_0 q_0 + p_1 i q_1 i + p_2 j q_2 j + p_3 k q_3 k\big) \\
&\quad + \big(p_1 i q_0 + p_0 q_1 i + p_3 k q_2 j + p_2 j q_3 k\big) \\
&\quad + \big(p_2 j q_0 + p_3 k q_1 i + p_0 q_2 j + p_1 i q_3 k\big) \\
&\quad + \big(p_3 k q_0 + p_2 j q_1 i + p_1 i q_2 j + p_0 q_3 k\big) \qquad \text{// rearrange terms} \\[1em]
&= \big(p_0 q_0 + p_1 q_1 ii + p_2 q_2 jj + p_3 q_3 kk\big) \\
&\quad + \big(p_1 q_0 i + p_0 q_1 i + p_3 q_2 kj + p_2 q_3 jk\big) \\
&\quad + \big(p_2 q_0 j + p_3 q_1 ki + p_0 q_2 j + p_1 q_3 ik\big) \\
&\quad + \big(p_3 q_0 k + p_2 q_1 ji + p_1 q_2 ij + p_0 q_3 k\big) \qquad \text{// note that symbols i, j, k are not commutative} \\[1em]
&= \big(p_0 q_0 + p_1 q_1(-1) + p_2 q_2(-1) + p_3 q_3(-1)\big) \\
&\quad + \big(p_1 q_0 i + p_0 q_1 i + p_3 q_2(-i) + p_2 q_3(i)\big) \\
&\quad + \big(p_2 q_0 j + p_3 q_1(j) + p_0 q_2 j + p_1 q_3(-j)\big) \\
&\quad + \big(p_3 q_0 k + p_2 q_1(-k) + p_1 q_2(k) + p_0 q_3 k\big) \qquad \text{// by quaternion symbol identities} \\[1em]
&= \big(p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3\big) \\
&\quad + \big(p_1 q_0 i + p_0 q_1 i - p_3 q_2 i + p_2 q_3 i\big) \\
&\quad + \big(p_2 q_0 j + p_3 q_1 j + p_0 q_2 j - p_1 q_3 j\big) \\
&\quad + \big(p_3 q_0 k - p_2 q_1 k + p_1 q_2 k + p_0 q_3 k\big) \\[1em]
&= \big(p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3\big) \\
&\quad + \big(p_1 q_0 + p_0 q_1 - p_3 q_2 + p_2 q_3\big) i \\
&\quad + \big(p_2 q_0 + p_3 q_1 + p_0 q_2 - p_1 q_3\big) j \\
&\quad + \big(p_3 q_0 - p_2 q_1 + p_1 q_2 + p_0 q_3\big) k \\[1em]
&= p_0 q_0 - \big(p_1 q_1 + p_2 q_2 + p_3 q_3\big) \\
&\quad + \big(p_0 q_1 + q_0 p_1 + p_2 q_3 - p_3 q_2\big) i \\
&\quad + \big(p_0 q_2 + q_0 p_2 + p_3 q_1 - p_1 q_3\big) j \\
&\quad + \big(p_0 q_3 + q_0 p_3 + p_1 q_2 - p_2 q_1\big) k \qquad \text{// reordering terms}
\end{aligned}
$$

Hence,

$$\underline{PQ} = p_0 q_0 - \big(p_1 q_1 + p_2 q_2 + p_3 q_3\big) + \big(p_0 q_1 + q_0 p_1 + p_2 q_3 - p_3 q_2\big) i + \big(p_0 q_2 + q_0 p_2 + p_3 q_1 - p_1 q_3\big) j + \big(p_0 q_3 + q_0 p_3 + p_1 q_2 - p_2 q_1\big) k$$

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Continuing,

$$\begin{aligned}
\underline{P}\underline{Q} = p_0q_0 &- \big(p_1q_1 + p_2q_2 + p_3q_3\big) \\
&+ \big(p_0q_1 + q_0p_1 + p_2q_3 - p_3q_2\big)i \\
&+ \big(p_0q_2 + q_0p_2 + p_3q_1 - p_1q_3\big)j \\
&+ \big(p_0q_3 + q_0p_3 + p_1q_2 - p_2q_1\big)k
\end{aligned}$$

$$= p_0q_0 - (p_1q_1 + p_2q_2 + p_3q_3) + \begin{bmatrix} p_0q_1 \\ p_0q_2 \\ p_0q_3 \end{bmatrix} + \begin{bmatrix} q_0p_1 \\ q_0p_2 \\ q_0p_3 \end{bmatrix} + \begin{bmatrix} p_2q_3 \\ p_3q_1 \\ p_1q_2 \end{bmatrix} + \begin{bmatrix} -p_3q_2 \\ -p_1q_3 \\ -p_2q_1 \end{bmatrix}$$

$$= p_0q_0 - (p_1q_1 + p_2q_2 + p_3q_3) + p_0 \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + q_0 \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + \begin{bmatrix} p_2q_3 - p_3q_2 \\ p_3q_1 - p_1q_3 \\ p_1q_2 - p_2q_1 \end{bmatrix}$$

$$= p_0q_0 - p \cdot q + p_0q + q_0p + p \times q \qquad \text{// by the expressions found on page 17}$$

Therefore, the product of two quaternions $\underline{P}$ and $\underline{Q}$ is equal to

$$\underline{P}\underline{Q} = p_0q_0 - p \cdot q + p_0q + q_0p + p \times q$$