Problem 1: [15 points]

Show that the characteristic polynomial, $det(R - \lambda I)$, of a rotation matrix R is given by:

$$\lambda^3 - Tr(R)\lambda^2 + Tr(R)\lambda - 1 = 0$$

where $Tr(R)$ is the trace of the rotation matrix R.

Let the rotation matrix R be represented as follows:

$$R = \begin{bmatrix} r_{11} & r_{11} & r_{11} \\ r_{21} & r_{22} & r_{22} \\ r_{31} & r_{33} & r_{33} \end{bmatrix}$$

Observe,

$$\begin{aligned} det(R - \lambda I) &= det\left(\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \\ &= det\left(\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) \\ &= det\left(\begin{bmatrix} r_{11} - \lambda & r_{12} & r_{13} \\ r_{21} & r_{22} - \lambda & r_{23} \\ r_{31} & r_{32} & r_{33} - \lambda \end{bmatrix}\right) \end{aligned}$$

Note that by definition, the determinant of an $n \times n$ matrix is as follows:

$$det(A) = \sum_{j=1}^{n} (-1)^{1+j}\ a_{1j}\ det(A_{1j}) \tag{1}$$

where A_{1j} is the $(n-1) \times (n-1)$ matrix made from A with row 1 and column j removed. From this point onwards, the following notation for determinants will be adopted: $det(A) = |A|$. It follows that,

$$\begin{aligned} &\begin{vmatrix} r_{11} - \lambda & r_{12} & r_{13} \\ r_{21} & r_{22} - \lambda & r_{23} \\ r_{31} & r_{32} & r_{33} - \lambda \end{vmatrix} \\ &\quad = (r_{11} - \lambda)\begin{vmatrix} r_{22} - \lambda & r_{23} \\ r_{32} & r_{33} - \lambda \end{vmatrix} - r_{12}\begin{vmatrix} r_{21} & r_{23} \\ r_{31} & r_{33} - \lambda \end{vmatrix} + r_{13}\begin{vmatrix} r_{21} & r_{22} - \lambda \\ r_{31} & r_{32} \end{vmatrix} \\ &\quad = (r_{11} - \lambda)\big[(r_{22} - \lambda)(r_{33} - \lambda) - r_{23}r_{32}\big] - r_{12}\big[r_{21}(r_{33} - \lambda) - r_{23}r_{31}\big] + r_{13}\big[r_{21}r_{32} - (r_{22} - \lambda)r_{31}\big] \\ &\quad = (r_{11} - \lambda)\big[r_{22}r_{33} - (r_{22} + r_{33})\lambda + \lambda^2 - r_{23}r_{32}\big] - r_{12}\big[r_{21}r_{33} - r_{21}\lambda - r_{23}r_{31}\big] + r_{13}\big[r_{21}r_{32} - r_{22}r_{31} + r_{31}\lambda\big] \\ &\quad = (r_{11} - \lambda)\big[\lambda^2 - (r_{22} + r_{33})\lambda + r_{22}r_{33} - r_{23}r_{32}\big] - r_{12}r_{21}r_{33} + r_{12}r_{21}\lambda + r_{12}r_{23}r_{31} + r_{13}r_{21}r_{32} - r_{13}r_{21}r_{31} + r_{13}r_{31}\lambda \\ &\quad = -\lambda^3 + \big(r_{11} + r_{22} + r_{33}\big)\lambda^2 + \big(-r_{11}r_{22} - r_{11}r_{33} + r_{12}r_{21} + r_{13}r_{31} - r_{22}r_{33} + r_{23}r_{32}\big)\lambda \\ &\qquad + \big(r_{11}r_{22}r_{33} - r_{11}r_{23}r_{32} - r_{12}r_{21}r_{33} + r_{12}r_{23}r_{31} + r_{13}r_{21}r_{32} + r_{13}r_{22}r_{31}\big) \end{aligned}$$

Notice that the constant term in the expression above can be rewritten as follows:

$$\begin{aligned} &r_{11}r_{22}r_{33} - r_{11}r_{23}r_{32} - r_{12}r_{21}r_{33} + r_{12}r_{23}r_{31} + r_{13}r_{21}r_{32} + r_{13}r_{22}r_{31} \\ &\quad = r_{11}\big(r_{22}r_{33} - r_{23}r_{32}\big) - r_{12}\big(r_{21}r_{33} + r_{23}r_{31}\big) + r_{13}\big(r_{21}r_{32} + r_{22}r_{31}\big) \\ &\quad = r_{11}\begin{vmatrix} r_{22} & r_{23} \\ r_{32} & r_{33} \end{vmatrix} - r_{12}\begin{vmatrix} r_{21} & r_{23} \\ r_{31} & r_{33} \end{vmatrix} + r_{13}\begin{vmatrix} r_{21} & r_{22} \\ r_{31} & r_{32} \end{vmatrix} \end{aligned}$$

Continued on next page ⟶

Which, by equation 1, is equal to $det(R)$. R is a rotation matrix, however, which means that $det(R)$ is necessarily equal to 1. Therefore:

$$\begin{aligned}
&-\lambda^3 + \big(r_{11} + r_{22} + r_{33}\big)\lambda^2 + \big(-r_{11}r_{22} - r_{11}r_{33} + r_{12}r_{21} + r_{13}r_{31} - r_{22}r_{33} + r_{23}r_{32}\big)\lambda \\
&\quad + \big(r_{11}r_{22}r_{33} - r_{11}r_{23}r_{32} - r_{12}r_{21}r_{33} + r_{12}r_{23}r_{31} + r_{13}r_{21}r_{32} + r_{13}r_{22}r_{31}\big) \\
&\quad = -\lambda^3 + \big(r_{11} + r_{22} + r_{33}\big)\lambda^2 + \big(-r_{11}r_{22} - r_{11}r_{33} + r_{12}r_{21} + r_{13}r_{31} - r_{22}r_{33} + r_{23}r_{32}\big)\lambda + det(R) \\
&\quad = -\lambda^3 + \big(r_{11} + r_{22} + r_{33}\big)\lambda^2 + \big(-r_{11}r_{22} - r_{11}r_{33} + r_{12}r_{21} + r_{13}r_{31} - r_{22}r_{33} + r_{23}r_{32}\big)\lambda + 1 \\
&\quad = -\lambda^3 + Tr(R)\ \lambda^2 + \big(-r_{11}r_{22} - r_{11}r_{33} + r_{12}r_{21} + r_{13}r_{31} - r_{22}r_{33} + r_{23}r_{32}\big)\lambda + 1 \\
&\quad = -\lambda^3 + Tr(R)\ \lambda^2 + \big(-(r_{22}r_{33} - r_{23}r_{32}) - (r_{11}r_{33} - r_{13}r_{31}) - (r_{11}r_{22} - r_{12}r_{21})\big)\lambda + 1
\end{aligned}$$

Given that R is a rotation matrix, we know that its columns are both mutally orthogonal, and of unit length. As such, the cross product of any two column vectors from R will result in a vector parallel in direction and equal in magnitude to the unused column vector. Since rotation matrices are restricted (by convention) to right-handed coordinate frames, we can further assert that column 1 $\times$ column 2 will yield column 3, column 2 $\times$ column 3 will yield column 1, and column 3 $\times$ column 1 will yield column 2. Therefore,

$$\begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix} \times \begin{bmatrix} r_{12} \\ r_{22} \\ r_{32} \end{bmatrix} = \begin{bmatrix} r_{21}r_{32} - r_{22}r_{31} \\ r_{11}r_{32} - r_{12}r_{31} \\ r_{11}r_{22} - r_{12}r_{21} \end{bmatrix} = \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix},$$

$$\begin{bmatrix} r_{12} \\ r_{22} \\ r_{32} \end{bmatrix} \times \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix} = \begin{bmatrix} r_{22}r_{33} - r_{23}r_{32} \\ r_{12}r_{33} - r_{13}r_{32} \\ r_{12}r_{23} - r_{13}r_{22} \end{bmatrix} = \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix},$$

and $$\begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix} \times \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix} = \begin{bmatrix} r_{23}r_{31} - r_{21}r_{33} \\ r_{13}r_{31} - r_{11}r_{33} \\ r_{13}r_{21} - r_{11}r_{23} \end{bmatrix} = \begin{bmatrix} r_{12} \\ r_{22} \\ r_{32} \end{bmatrix}$$

Importantly, using the above cross product expressions, we have found that:

$$\begin{aligned}
r_{11} &= r_{22}r_{33} - r_{23}r_{32} \\
r_{22} &= r_{13}r_{31} - r_{11}r_{33} \\
r_{33} &= r_{11}r_{22} - r_{12}r_{21}
\end{aligned}$$

As such,

$$\begin{aligned}
&-\lambda^3 + Tr(R)\ \lambda^2 + \big(-(r_{22}r_{33} - r_{23}r_{32}) - (r_{11}r_{33} - r_{13}r_{31}) - (r_{11}r_{22} - r_{12}r_{21})\big)\lambda + 1 \\
&\quad = -\lambda^3 + Tr(R)\ \lambda^2 + \big(-r_{11} - r_{22} - r_{33}\big)\lambda + 1 \\
&\quad = -\lambda^3 + Tr(R)\ \lambda^2 - Tr(R)\ \lambda + 1 \\
&\quad = -\big(\lambda^3 - Tr(R)\ \lambda^2 + Tr(R)\ \lambda - 1\big)
\end{aligned}$$

Hence,

$$det(R - \lambda I) = -\big(\lambda^3 - Tr(R)\ \lambda^2 + Tr(R)\ \lambda - 1\big)$$

For λ to be an eigenvalue of R, it must satisfy the characteristic equation $det(R - \lambda I) = 0$. As such, λ is an eigenvalue of $R \iff -\big(\lambda^3 - Tr(R)\ \lambda^2 + Tr(R)\ \lambda - 1\big) = 0 \iff \lambda^3 - Tr(R)\ \lambda^2 + Tr(R)\ \lambda - 1 = 0$.

Therefore, the characteristic equation for a rotation matrix R is as follows:

$$\lambda^3 - Tr(R)\ \lambda^2 + Tr(R)\ \lambda - 1 = 0$$

Problem 2: [15 points]

Let $\hat{r} = (r_x, r_y, r_z)^T$, where $\hat{r}$ is a unit vector specifying the axis of rotation of ϕ. Show that

$$R_{\hat{r},\phi} = \begin{bmatrix} r_x^2 v\phi + c\phi & r_x r_y v\phi - r_z s\phi & r_x r_z v\phi + r_y s\phi \\ r_x r_y v\phi + r_z s\phi & r_y^2 v\phi + c\phi & r_y r_z v\phi - r_x s\phi \\ r_x r_z v\phi - r_y s\phi & r_y r_z v\phi + r_x s\phi & r_z^2 v\phi + c\phi \end{bmatrix}$$

where $c\phi = \cos\phi$, $s\phi = \sin\phi$, and $v\phi = 1 - c\phi$.

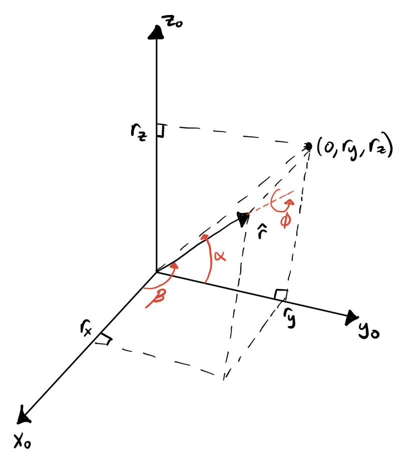
Using a similarity transform, let $R_{\hat{r},\phi} = R\ R_{x,\phi}\ R^{-1}$, where R is defined by a sequence of two current-frame rotations: $R = R_{x,\alpha} R_{z,\beta}$. The basic rotation matrices about the x-axis and z-axis are as follows:

$$R_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix},$$

$$\text{and} \qquad R_{x,\phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix},$$

$$\text{and} \qquad R_{z,\beta} = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note the following geometric relationships which can be observed in the figure to the right,



$$\sin\alpha = \frac{r_z}{\sqrt{r_y^2 + r_z^2}} \quad \text{, and} \quad \cos\alpha = \frac{r_y}{\sqrt{r_y^2 + r_z^2}}$$

$$\sin\beta = \frac{\sqrt{r_y^2 + r_z^2}}{||\hat{r}||} = \sqrt{r_y^2 + r_z^2} \quad \text{, and} \quad \cos\beta = \frac{r_x}{||\hat{r}||} = r_x$$

It follows that:

$$R_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{r_y}{\sqrt{r_y^2+r_z^2}} & -\frac{r_z}{\sqrt{r_y^2+r_z^2}} \\ 0 & \frac{r_z}{\sqrt{r_y^2+r_z^2}} & \frac{r_y}{\sqrt{r_y^2+r_z^2}} \end{bmatrix},$$

$$\text{and} \qquad R_{z,\beta} = \begin{bmatrix} r_x & -\sqrt{r_y^2 + r_z^2} & 0 \\ \sqrt{r_y^2 + r_z^2} & r_x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let the rotation matrices $R, R_{x,\alpha}$, and $R_{z,\beta}$ be represented as follows:

$$R = \begin{bmatrix} r_{11} & r_{11} & r_{11} \\ r_{21} & r_{22} & r_{22} \\ r_{31} & r_{33} & r_{33} \end{bmatrix},\ R_{x,\alpha} = \begin{bmatrix} a_{11} & a_{11} & a_{11} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{33} & a_{33} \end{bmatrix},\ R_{z,\beta} = \begin{bmatrix} b_{11} & b_{11} & b_{11} \\ b_{21} & b_{22} & b_{22} \\ b_{31} & b_{33} & b_{33} \end{bmatrix}$$

By the definition of matrix multplication, then, $r_{ij} = \sum_{k=1}^{3} a_{ik} b_{kj}$, for $i = 1, ..., 3$ and $j = 1, ..., 3$.

Continued on next page ⟶

Thus,

$$
\begin{aligned}
R = R_{x,\alpha}R_{z,\beta} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{r_y}{\sqrt{r_y^2+r_z^2}} & -\frac{r_z}{\sqrt{r_y^2+r_z^2}} \\ 0 & \frac{r_z}{\sqrt{r_y^2+r_z^2}} & \frac{r_y}{\sqrt{r_y^2+r_z^2}} \end{bmatrix} \begin{bmatrix} r_x & -\sqrt{r_y^2+r_z^2} & 0 \\ \sqrt{r_y^2+r_z^2} & r_x & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} r_x & -\sqrt{r_y^2+r_z^2} & 0 \\ \frac{r_y}{\sqrt{r_y^2+r_z^2}}\sqrt{r_y^2+r_z^2} & r_x\frac{r_y}{\sqrt{r_y^2+r_z^2}} & -\frac{r_z}{\sqrt{r_y^2+r_z^2}} \\ \frac{r_z}{\sqrt{r_y^2+r_z^2}}\sqrt{r_y^2+r_z^2} & r_x\frac{r_z}{\sqrt{r_y^2+r_z^2}} & \frac{r_y}{\sqrt{r_y^2+r_z^2}} \end{bmatrix} \\
&= \begin{bmatrix} r_x & -\sqrt{r_y^2+r_z^2} & 0 \\ r_y & \frac{r_x r_y}{\sqrt{r_y^2+r_z^2}} & -\frac{r_z}{\sqrt{r_y^2+r_z^2}} \\ r_z & \frac{r_x r_z}{\sqrt{r_y^2+r_z^2}} & \frac{r_y}{\sqrt{r_y^2+r_z^2}} \end{bmatrix}
\end{aligned}
$$

Since R is a rotation matrix, it follows that $R^{-1} = R^T$. Thus,

$$
R^{-1} = \begin{bmatrix} r_x & r_y & r_z \\ -\sqrt{r_y^2+r_z^2} & \frac{r_x r_y}{\sqrt{r_y^2+r_z^2}} & \frac{r_x r_z}{\sqrt{r_y^2+r_z^2}} \\ 0 & -\frac{r_z}{\sqrt{r_y^2+r_z^2}} & \frac{r_y}{\sqrt{r_y^2+r_z^2}} \end{bmatrix}
$$

Continuing,

$$
\begin{aligned}
R\ R_{x\phi} &= \begin{bmatrix} r_x & -\sqrt{r_y^2+r_z^2} & 0 \\ r_y & \frac{r_x r_y}{\sqrt{r_y^2+r_z^2}} & -\frac{r_z}{\sqrt{r_y^2+r_z^2}} \\ r_z & \frac{r_x r_z}{\sqrt{r_y^2+r_z^2}} & \frac{r_y}{\sqrt{r_y^2+r_z^2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \\
&= \begin{bmatrix} r_x & -c\phi\sqrt{r_y^2+r_z^2} & s\phi\sqrt{r_y^2+r_z^2} \\ r_y & \left(c\phi\frac{r_x r_y}{\sqrt{r_y^2+r_z^2}} - s\phi\frac{r_z}{\sqrt{r_y^2+r_z^2}}\right) & \left(-s\phi\frac{r_x r_y}{\sqrt{r_y^2+r_z^2}} - c\phi\frac{r_z}{\sqrt{r_y^2+r_z^2}}\right) \\ r_z & \left(c\phi\frac{r_x r_z}{\sqrt{r_y^2+r_z^2}} + s\phi\frac{r_y}{\sqrt{r_y^2+r_z^2}}\right) & \left(-s\phi\frac{r_x r_z}{\sqrt{r_y^2+r_z^2}} + c\phi\frac{r_y}{\sqrt{r_y^2+r_z^2}}\right) \end{bmatrix}
\end{aligned}
$$

Finally,

$$
R\ R_{x\phi}\ R^{-1} = \begin{bmatrix} r_x & -c\phi\sqrt{r_y^2+r_z^2} & s\phi\sqrt{r_y^2+r_z^2} \\ r_y & \left(c\phi\frac{r_x r_y}{\sqrt{r_y^2+r_z^2}} - s\phi\frac{r_z}{\sqrt{r_y^2+r_z^2}}\right) & \left(-s\phi\frac{r_x r_y}{\sqrt{r_y^2+r_z^2}} - c\phi\frac{r_z}{\sqrt{r_y^2+r_z^2}}\right) \\ r_z & \left(c\phi\frac{r_x r_z}{\sqrt{r_y^2+r_z^2}} + s\phi\frac{r_y}{\sqrt{r_y^2+r_z^2}}\right) & \left(-s\phi\frac{r_x r_z}{\sqrt{r_y^2+r_z^2}} + c\phi\frac{r_y}{\sqrt{r_y^2+r_z^2}}\right) \end{bmatrix} \begin{bmatrix} r_x & r_y & r_z \\ -\sqrt{r_y^2+r_z^2} & \frac{r_x r_y}{\sqrt{r_y^2+r_z^2}} & \frac{r_x r_z}{\sqrt{r_y^2+r_z^2}} \\ 0 & -\frac{r_z}{\sqrt{r_y^2+r_z^2}} & \frac{r_y}{\sqrt{r_y^2+r_z^2}} \end{bmatrix}
$$

Continued on next page ⟶

$$
= \left[\begin{matrix} \left(r_x^2 + c\phi(r_y^2 + r_z^2)\right) \\ \left(r_x r_y + \left(c\phi\frac{r_x r_y}{\sqrt{r_y^2+r_z^2}} - s\phi\frac{r_z}{\sqrt{r_y^2+r_z^2}}\right)\left(-\sqrt{r_y^2+r_z^2}\right)\right) \\ \left(r_x r_z - \left(c\phi\frac{r_x r_z}{\sqrt{r_y^2+r_z^2}} + s\phi\frac{r_y}{\sqrt{r_y^2+r_z^2}}\right)\sqrt{r_y^2+r_z^2}\right) \end{matrix} \right.
$$

$$
\begin{matrix} \left(r_x r_y + -c\phi\sqrt{r_y^2+r_z^2}\frac{r_x r_y}{\sqrt{r_y^2+r_z^2}} - s\phi\sqrt{r_y^2+r_z^2}\frac{r_z}{\sqrt{r_y^2+r_z^2}}\right) \\ \left(r_y^2 + \left(c\phi\frac{r_x r_y}{\sqrt{r_y^2+r_z^2}} - s\phi\frac{r_z}{\sqrt{r_y^2+r_z^2}}\right)\frac{r_x r_y}{\sqrt{r_y^2+r_z^2}} - \left(-s\phi\frac{r_x r_y}{\sqrt{r_y^2+r_z^2}} - c\phi\frac{r_z}{\sqrt{r_y^2+r_z^2}}\right)\frac{r_z}{\sqrt{r_y^2+r_z^2}}\right) \\ \left(r_y r_z + \left(c\phi\frac{r_x r_y}{\sqrt{r_y^2+r_z^2}} + s\phi\frac{r_y}{\sqrt{r_y^2+r_z^2}}\right)\frac{r_x r_y}{\sqrt{r_y^2+r_z^2}} + \left(-s\phi\frac{r_x r_z}{\sqrt{r_y^2+r_z^2}} + c\phi\frac{r_y}{\sqrt{r_y^2+r_z^2}}\right)\frac{-r_z}{\sqrt{r_y^2+r_z^2}}\right) \end{matrix}
$$

$$
\left. \begin{matrix} \left(r_x r_z + -c\phi\sqrt{r_y^2+r_z^2}\frac{r_x r_z}{\sqrt{r_y^2+r_z^2}} - s\phi\sqrt{r_y^2+r_z^2}\frac{r_y}{\sqrt{r_y^2+r_z^2}}\right) \\ \left(r_y r_z + \left(c\phi\frac{r_x r_y}{\sqrt{r_y^2+r_z^2}} - s\phi\frac{r_z}{\sqrt{r_y^2+r_z^2}}\right)\frac{r_x r_z}{\sqrt{r_y^2+r_z^2}} + \left(-s\phi\frac{r_x r_y}{\sqrt{r_y^2+r_z^2}} - c\phi\frac{r_z}{\sqrt{r_y^2+r_z^2}}\right)\frac{r_y}{\sqrt{r_y^2+r_z^2}}\right) \\ \left(r_z^2 + \left(c\phi\frac{r_x r_z}{\sqrt{r_y^2+r_z^2}} + s\phi\frac{r_y}{\sqrt{r_y^2+r_z^2}}\right)\frac{r_x r_z}{\sqrt{r_y^2+r_z^2}} + \left(-s\phi\frac{r_x r_z}{\sqrt{r_y^2+r_z^2}} + c\phi\frac{r_y}{\sqrt{r_y^2+r_z^2}}\right)\frac{r_y}{\sqrt{r_y^2+r_z^2}}\right) \end{matrix} \right]
$$

Going term by term, I will now simplify each component of the above matrix. Observe,

Element 1,1:

$$
\begin{aligned} r_x^2 + c\phi(r_y^2 + r_z^2) &= r_x^2 + c\phi(1 - r_x^2) \qquad \text{b/c } \sqrt{r_x^2+r_y^2+r_z^2} = 1 \text{ since } \hat{r} \text{ is a unit vector} \\ &= r_x^2 + c\phi - r_x^2 c\phi \\ &= r_x^2 - r_x^2 c\phi + c\phi \\ &= r_x^2(1 - c\phi) + c\phi \\ &= r_x^2 v\phi + c\phi \end{aligned}
$$

Element 2,1:

$$
\begin{aligned} & r_x r_y + \left(c\phi\frac{r_x r_y}{\sqrt{r_y^2+r_z^2}} - s\phi\frac{r_z}{\sqrt{r_y^2+r_z^2}}\right)\left(-\sqrt{r_y^2+r_z^2}\right) \\ &= r_x r_y - r_x r_y c\phi + r_z s\phi \\ &= r_x r_y(1 - c\phi) + r_z s\phi \\ &= r_x r_y v\phi + r_z s\phi \end{aligned}
$$

Element 3,1:

$$
\begin{aligned} & r_x r_z - \left(c\phi\frac{r_x r_z}{\sqrt{r_y^2+r_z^2}} + s\phi\frac{r_y}{\sqrt{r_y^2+r_z^2}}\right)\sqrt{r_y^2+r_z^2} \\ &= r_x r_z - r_x r_z c\phi - r_y s\phi \\ &= r_x r_z(1 - c\phi) - r_y s\phi \\ &= r_x r_z v\phi - r_y s\phi \end{aligned}
$$

Continued on next page ⟶

Element 1,2:

$$\begin{aligned}
&r_x r_y + -c\phi\sqrt{r_y^2 + r_z^2}\frac{r_x r_y}{\sqrt{r_y^2 + r_z^2}} - s\phi\sqrt{r_y^2 + r_z^2}\frac{r_z}{\sqrt{r_y^2 + r_z^2}} \\
&= r_x r_y - r_x r_y c\phi - r_z s\phi \\
&= r_x r_y(1 - c\phi) - r_z s\phi \\
&= r_x r_y v\phi - r_z s\phi
\end{aligned}$$

Element 2,2:

$$\begin{aligned}
&r_y^2 + \left(c\phi\frac{r_x r_y}{\sqrt{r_y^2 + r_z^2}} - s\phi\frac{r_z}{\sqrt{r_y^2 + r_z^2}}\right)\frac{r_x r_y}{\sqrt{r_y^2 + r_z^2}} - \left(-s\phi\frac{r_x r_y}{\sqrt{r_y^2 + r_z^2}} - c\phi\frac{r_z}{\sqrt{r_y^2 + r_z^2}}\right)\frac{r_z}{\sqrt{r_y^2 + r_z^2}} \\
&= r_y^2 + \frac{r_x^2 r_y^2 c\phi - r_x r_y r_z s\phi}{r_y^2 + r_z^2} + \frac{r_x r_y r_z s\phi + r_z^2 c\phi}{r_y^2 + r_z^2} \\
&= r_y^2 + \frac{r_x^2 r_y^2 c\phi - r_x r_y r_z s\phi + r_x r_y r_z s\phi + r_z^2 c\phi}{r_y^2 + r_z^2} \\
&= r_y^2 + \frac{r_x^2 r_y^2 + r_z^2}{r_y^2 + r_z^2}c\phi \\
&= r_y^2 + \frac{r_x^2 r_y^2 + r_z^2}{1 - r_x^2}c\phi
\end{aligned}$$

To further simplify this expression, observe the following derivation:

$$\begin{aligned}
1 &= \sqrt{r_x^2 + r_y^2 + r_z^2} \qquad \text{b/c } \hat{r} \text{ is a unit vector} \\
\iff 1 &= r_x^2 + r_y^2 + r_z^2 \\
\iff r_z^2 &= 1 - r_x^2 - r_y^2 \\
\iff r_z^2 + r_x^2 r_y^2 &= 1 - r_x^2 - r_y^2 + r_x^2 r_y^2 \\
\iff r_z^2 + r_x^2 r_y^2 &= (1 - r_y^2)(1 - r_x^2)
\end{aligned}$$

Thus,

$$\begin{aligned}
r_y^2 + \frac{r_x^2 r_y^2 + r_z^2}{1 - r_x^2}c\phi &= r_y^2 + \frac{(1 - r_y^2)(1 - r_x^2)}{1 - r_x^2}c\phi \\
&= r_y^2 + (1 - r_y^2)c\phi \\
&= r_y^2 - r_y^2 c\phi + c\phi \\
&= r_y^2(1 - c\phi) + c\phi \\
&= r_y^2 v\phi + c\phi
\end{aligned}$$

Therefore, element 2,2:

$$r_y^2 + \left(c\phi\frac{r_x r_y}{\sqrt{r_y^2 + r_z^2}} - s\phi\frac{r_z}{\sqrt{r_y^2 + r_z^2}}\right)\frac{r_x r_y}{\sqrt{r_y^2 + r_z^2}} - \left(-s\phi\frac{r_x r_y}{\sqrt{r_y^2 + r_z^2}} - c\phi\frac{r_z}{\sqrt{r_y^2 + r_z^2}}\right)\frac{r_z}{\sqrt{r_y^2 + r_z^2}} = r_y^2 v\phi + c\phi$$

 Continued on next page ⟶

Element 3,2:

$$
\begin{aligned}
&r_y r_z + \left(c\phi \frac{r_x r_y}{\sqrt{r_y^2+r_z^2}} + s\phi \frac{r_y}{\sqrt{r_y^2+r_z^2}}\right) \frac{r_x r_y}{\sqrt{r_y^2+r_z^2}} + \left(-s\phi \frac{r_x r_z}{\sqrt{r_y^2+r_z^2}} + c\phi \frac{r_y}{\sqrt{r_y^2+r_z^2}}\right) \frac{-r_z}{\sqrt{r_y^2+r_z^2}} \\
&= r_y r_z + \frac{r_x^2 r_y r_z c\phi + r_x r_y^2 s\phi}{r_y^2+r_z^2} + \frac{r_x r_z^2 S\phi - r_y r_z c\phi}{r_y^2+r_z^2} \\
&= r_y r_z + \frac{r_x^2 r_y r_z c\phi - r_y r_z c\phi + r_x r_y^2 s\phi + r_x r_z^2 S\phi}{r_y^2+r_z^2} \\
&= r_y r_z + \frac{(r_x^2-1) r_y r_z c\phi + (r_y^2+r_z^2) r_x s\phi}{r_y^2+r_z^2} \\
&= r_y r_z + \frac{-(r_y^2+r_z^2) r_y r_z c\phi + (r_y^2+r_z^2) r_x s\phi}{r_y^2+r_z^2} \\
&= r_y r_z + r_y r_z c\phi + r_x s\phi \\
&= r_y r_z (1 - c\phi) + r_x s\phi \\
&= r_y r_z v\phi + r_x s\phi
\end{aligned}
$$

Element 1,3:

$$
\begin{aligned}
&r_x r_z + -c\phi \sqrt{r_y^2+r_z^2} \frac{r_x r_z}{\sqrt{r_y^2+r_z^2}} - s\phi \sqrt{r_y^2+r_z^2} \frac{r_y}{\sqrt{r_y^2+r_z^2}} \\
&= r_x r_z - r_x r_z c\phi + r_y s\phi \\
&= r_x r_z (1 - c\phi) + r_y s\phi \\
&= r_x r_z v\phi + r_y s\phi
\end{aligned}
$$

Element 2,3:

$$
\begin{aligned}
&r_y r_z + \left(c\phi \frac{r_x r_y}{\sqrt{r_y^2+r_z^2}} - s\phi \frac{r_z}{\sqrt{r_y^2+r_z^2}}\right) \frac{r_x r_z}{\sqrt{r_y^2+r_z^2}} + \left(-s\phi \frac{r_x r_y}{\sqrt{r_y^2+r_z^2}} - c\phi \frac{r_z}{\sqrt{r_y^2+r_z^2}}\right) \frac{r_y}{\sqrt{r_y^2+r_z^2}} \\
&= r_y r_z + \frac{r_x^2 r_y r_z c\phi - r_x r_z^2 s\phi - r_x r_y^2 s\phi - r_y r_z c\phi}{r_y^2+r_z^2} \\
&= r_y r_z + \frac{r_x^2 r_y r_z c\phi - r_y r_z c\phi - r_x r_z^2 s\phi - r_x r_y^2 s\phi}{r_y^2+r_z^2} \\
&= r_y r_z + \frac{(r_x^2-1) r_y r_z c\phi - (r_z^2+r_y^2) r_x s\phi}{r_y^2+r_z^2} \\
&= r_y r_z + \frac{-(r_y^2+r_z^2) r_y r_z c\phi - (r_z^2+r_y^2) r_x s\phi}{r_y^2+r_z^2} \\
&= r_y r_z - r_y r_z c\phi - r_x s\phi \\
&= r_y r_z (1 - c\phi) - r_x s\phi \\
&= r_y r_z v\phi - r_x s\phi
\end{aligned}
$$

 Continued on next page ⟶

Element 3,3:

$$
\begin{aligned}
& r_z^2 + \left(c\phi \frac{r_x r_z}{\sqrt{r_y^2 + r_z^2}} + s\phi \frac{r_y}{\sqrt{r_y^2 + r_z^2}}\right) \frac{r_x r_z}{\sqrt{r_y^2 + r_z^2}} + \left(-s\phi \frac{r_x r_z}{\sqrt{r_y^2 + r_z^2}} + c\phi \frac{r_y}{\sqrt{r_y^2 + r_z^2}}\right) \frac{r_y}{\sqrt{r_y^2 + r_z^2}} \\
&= r_z^2 + \frac{r_x^2 r_z^2 c\phi + r_x r_y r_z s\phi - r_x r_y r_z s\phi + r_y^2 c\phi}{r_y^2 + r_z^2} \\
&= r_z^2 + \frac{r_x^2 r_z^2 c\phi + r_y^2 c\phi}{r_y^2 + r_z^2} \\
&= r_z^2 + \frac{(r_x^2 r_z^2 + r_y^2) c\phi}{r_y^2 + r_z^2}
\end{aligned}
$$

To further simplify this expression, observe the following derivation:

$$
\begin{aligned}
1 &= \sqrt{r_x^2 + r_y^2 + r_z^2} \qquad \text{b/c } \hat{r} \text{ is a unit vector} \\
\iff 1 &= r_x^2 + r_y^2 + r_z^2 \\
\iff r_y^2 &= 1 - r_x^2 - r_z^2 \\
\iff r_y^2 + r_x^2 r_z^2 &= 1 - r_x^2 - r_z^2 + r_x^2 r_z^2 \\
\iff r_y^2 + r_x^2 r_z^2 &= (1 - r_x^2)(1 - r_z^2)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& r_z^2 + \frac{(r_x^2 r_z^2 + r_y^2) c\phi}{r_y^2 + r_z^2} \\
&= r_z^2 + \frac{(1 - r_x^2)(1 - r_z^2) c\phi}{r_y^2 + r_z^2} \\
&= r_z^2 + \frac{(r_y^2 + r_z^2)(1 - r_z^2) c\phi}{r_y^2 + r_z^2} \\
&= r_z^2 + (1 - r_z^2) c\phi \\
&= r_z^2 + c\phi - r_z^2 c\phi \\
&= r_z^2 - r_z^2 c\phi + c\phi \\
&= r_z^2 (1 - c\phi) + c\phi \\
&= r_z^2 v\phi + c\phi
\end{aligned}
$$

Therefore, element 3,3:

$$
r_z^2 + \left(c\phi \frac{r_x r_z}{\sqrt{r_y^2 + r_z^2}} + s\phi \frac{r_y}{\sqrt{r_y^2 + r_z^2}}\right) \frac{r_x r_z}{\sqrt{r_y^2 + r_z^2}} + \left(-s\phi \frac{r_x r_z}{\sqrt{r_y^2 + r_z^2}} + c\phi \frac{r_y}{\sqrt{r_y^2 + r_z^2}}\right) \frac{r_y}{\sqrt{r_y^2 + r_z^2}} = r_z^2 v\phi + c\phi
$$

Combining all of the simplified elements of $R\ R_{x\phi}\ R^{-1}$, we find that:

$$
R_{\hat{r},\phi} = R\ R_{x\phi}\ R^{-1} = \begin{bmatrix} r_x^2 v\phi + c\phi & r_x r_y v\phi - r_z s\phi & r_x r_z v\phi + r_y s\phi \\ r_x r_y v\phi + r_z s\phi & r_y^2 v\phi + c\phi & r_y r_z v\phi - r_x s\phi \\ r_x r_z v\phi - r_y s\phi & r_y r_z v\phi + r_x s\phi & r_z^2 v\phi + c\phi \end{bmatrix}
$$

Problem 3: [15 points]

Following up from question 2 above, show that

$$\hat{r} = \frac{1}{2\sin\phi}\begin{bmatrix} r_{32} - r_{23} \\ r_{12} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \tag{2}$$

with $\hat{r}$ being the axis of roation for ϕ, as in question 2, and r_{ij}; with $i, j = 1, 2, 3$ are elements of rotation matrix $R_{\hat{r},\phi}$.

For reference, problem 2 found that:

$$R_{\hat{r},\phi} = \begin{bmatrix} r_x^2 v\phi + c\phi & r_x r_y v\phi - r_z s\phi & r_x r_z v\phi + r_y s\phi \\ r_x r_y v\phi + r_z s\phi & r_y^2 v\phi + c\phi & r_y r_z v\phi - r_x s\phi \\ r_x r_z v\phi - r_y s\phi & r_y r_z v\phi + r_x s\phi & r_z^2 v\phi + c\phi \end{bmatrix}$$

Beginning with an undoubtedly true expression $(1 = 1)$, I will use algebraic equivalences to build up, through a series of biconditional connectives, each component of $\hat{r}$ to show that the provided Eq. 2 holds.

Observe,

$$\begin{aligned}
& 1 = 1 \\
\iff\ & 2r_x s\phi = 2r_x s\phi \\
\iff\ & 2r_x s\phi = 2r_x s\phi + 0 \\
\iff\ & 2r_x s\phi = 2r_x s\phi + (r_y r_z v\phi - r_y r_z v\phi) \\
\iff\ & 2r_x s\phi = (r_y r_z v\phi + r_x s\phi) + (-r_y r_z v\phi + r_x s\phi) \\
\iff\ & 2r_x s\phi = (r_y r_z v\phi + r_x s\phi) - (r_y r_z v\phi - r_x s\phi) \\
\iff\ & r_x = \frac{1}{2s\phi}\big((r_y r_z v\phi + r_x s\phi) - (r_y r_z v\phi - r_x s\phi)\big) \qquad \text{// as long as } \phi \text{ is not } 0 \\
\iff\ & r_x = \frac{1}{2s\phi}(r_{32} - r_{23})
\end{aligned}$$

Similarily,

$$\begin{aligned}
& 1 = 1 \\
\iff\ & 2r_y s\phi = 2r_y s\phi \\
\iff\ & 2r_y s\phi = 2r_y s\phi + 0 \\
\iff\ & 2r_y s\phi = 2r_y s\phi + (r_x r_z v\phi - r_x r_z v\phi) \\
\iff\ & 2r_y s\phi = (r_x r_z v\phi + r_y s\phi) + (-r_x r_z v\phi + r_y s\phi) \\
\iff\ & 2r_y s\phi = (r_x r_z v\phi + r_y s\phi) - (r_x r_z v\phi - r_y s\phi) \\
\iff\ & r_y = \frac{1}{2s\phi}\big((r_x r_z v\phi + r_y s\phi) - (r_x r_z v\phi - r_y s\phi)\big) \qquad \text{// as long as } \phi \text{ is not } 0 \\
\iff\ & r_y = \frac{1}{2s\phi}(r_{13} - r_{31})
\end{aligned}$$

Finally,

$$\begin{aligned}
& 1 = 1 \\
\iff\ & 2r_z s\phi = 2r_z s\phi \\
\iff\ & 2r_z s\phi = 2r_z s\phi + 0 \\
\iff\ & 2r_z s\phi = 2r_z s\phi + (r_x r_y v\phi - r_x r_y v\phi) \\
\iff\ & 2r_z s\phi = (r_x r_y v\phi + r_z s\phi) + (-r_x r_y v\phi + r_z s\phi) \\
\iff\ & 2r_z s\phi = (r_x r_y v\phi + r_z s\phi) - (r_x r_y v\phi - r_z s\phi) \\
\iff\ & r_z = \frac{1}{2s\phi}\big((r_x r_y v\phi + r_z s\phi) - (r_x r_y v\phi - r_z s\phi)\big) \qquad \text{// as long as } \phi \text{ is not } 0 \\
\iff\ & r_z = \frac{1}{2s\phi}(r_{21} - r_{12})
\end{aligned}$$

 Continued on next page ⟶

Note that if $\phi = 0$, the axis of rotation is indeterminate. Therefore, for any ϕ not equal to zero,

$$\hat{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} \frac{1}{2s\phi}(r_{32} - r_{23}) \\ \frac{1}{2s\phi}(r_{12} - r_{31}) \\ \frac{1}{2s\phi}(r_{21} - r_{12}) \end{bmatrix} = \frac{1}{2\sin\phi} \begin{bmatrix} r_{32} - r_{23} \\ r_{12} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

Thus Eq. 2 holds.

Problem 4: [15 points]

Derive the rotation matrix corresponding to the $Z_\phi X_\theta Z_\psi$ Euler angles.

The rotation matrix R corresponding to a sequence of three current frame rotatations, $R_{z,\phi}\ R_{x,\theta}\ R_{z,\psi}$ known as the ZXZ-Euler angle transformation can be derived from the basic rotation matrices as follows:

$$\begin{aligned} R_{ZXZ} = R_{z,\phi}\ R_{x,\theta}\ R_{z,\psi} &= \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix} \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\psi & -s\phi & 0 \\ c\theta s\psi & c\theta c\psi & -s\theta \\ s\theta s\psi & s\theta c\psi & c\theta \end{bmatrix} \\ &= \begin{bmatrix} c\phi c\psi - c\theta s\phi s\psi & -c\phi s\psi - c\theta c\psi s\phi & s\phi s\theta \\ c\psi s\phi + c\phi c\theta s\psi & c\phi c\theta c\psi - s\phi s\psi & -c\phi s\theta \\ s\theta s\psi & c\psi s\theta & c\theta \end{bmatrix} \end{aligned}$$

Problem 5: $[10 + 10 = 20 \text{ points}]$

Consider a robot arm mounted on a spacecraft as shown in Figure 1, in which frames are attached to the Earth $\{e\}$, a satellite $\{s\}$, the spacecraft $\{a\}$, and the robot arm $\{r\}$, respectively.

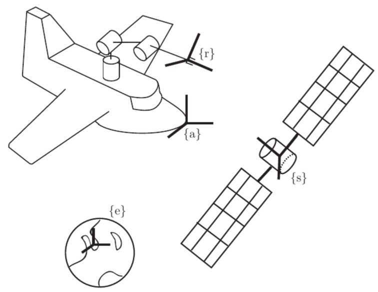


Figure 1: Figure for Problem 5.

(a) Given T_{ea}, T_{ar}, and T_{es}, find T_{rs}.

$$T_{rs} = T_{ra}\ T_{ae}\ T_{es} = T_{ar}^{-1}\ T_{ea}^{-1}\ T_{es}$$

(b) Suppose that the frame $\{s\}$ origin as seen from $\{e\}$ is $(1, 1, 1)$ and that

$$T_{er} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Write down the coordinates of the frame $\{s\}$ origin as seen from frame $\{r\}$.

Let o_s represent the origin of reference frame $\{s\}$. Additionally, let o_s^e represent the origin of reference frame $\{s\}$ as seen from $\{e\}$, and let O_s^e be the homogenous representation of o_s^e. Thus,

$$o_s^e = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad O_s^e = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

We aim to find the coordinates of o_s as seen from reference frame $\{r\}$, namely, o_s^r. First, note that $O_s^e = T_{er} O_s^r$, meaning that $O_s^r = T_{er}^{-1}\ O_s^e$. To find T_{er}^{-1}, recall the definition of the inverse of some homogenous transformation H where

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \qquad \text{is} \qquad H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}$$

In the case of T_{er} where

$$R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \text{and} \qquad d = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

we first determine that

$$R^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \text{and} \qquad -R^T d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Continued on next page ⟶

which then means that

$$T_{er}^{-1} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

With T_{er}^{-1} in hand, we can now find o_s^r as follows:

$$O_s^r = T_{er}^{-1}\ O_s^e = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore,

$$o_s^r = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In other words, the origin of frame $\{s\}$ as seen from frame $\{r\}$ is $(0, 0, 0)$ meaning that the origin of frame $\{s\}$ and the origin of frame $\{r\}$ coincide. This result makes sense when we consider that the displacement vector from the origin of frame $\{e\}$ to frame $\{r\}$, as determined from the fourth column of T_{er} is

$$d_r^e = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

which is equal to the given displacement vector from the origin of frame $\{e\}$ to frame $\{s\}$

$$d_s^e = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Problem 6: $[10 + 10 = 20 \text{ points}]$

Four reference frames are shown in the robot workspace of Figure 2: the fixed frame $\{a\}$, the end-effector frame effector $\{b\}$, the camera frame $\{c\}$, and the workpiece frame $\{d\}$

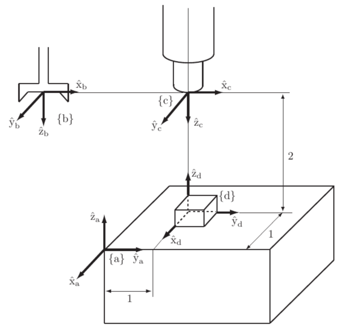


Figure 2: Figure for Problem 6.

(a) Find T_{ad} and T_{cd} in terms of the dimensions given in the figure.

Let R_d^a be the rotation matrix that specifies the orientation of frame $\{d\}$ from frame $\{a\}$, and let d_d^a be the displacement vector from the origin of frame $\{a\}$ to the origin of frame $\{d\}$. It follows that:

$$T_{ad} = \begin{bmatrix} R_d^a & d_d^a \\ 0 & 1 \end{bmatrix}$$

From figure 2, we can see that their is no difference in the orientations of frames $\{a\}$ and $\{d\}$, which is to say $R_d^a = I$. There is, however, a displacement of

$$d_d^a = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Therefore,

$$T_{ad} = \begin{bmatrix} R_d^a & d_d^a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Next, let R_d^c be the rotation matrix that specifies the orientation of frame $\{d\}$ from frame $\{c\}$, and let d_d^c be the displacement vector from the origin of frame $\{c\}$ to the origin of frame $\{d\}$. It follows that:

$$T_{cd} = \begin{bmatrix} R_d^c & d_d^c \\ 0 & 1 \end{bmatrix}$$

From figure 2, we can see that a rotation of $\phi = \frac{\pi}{2}$ about the z-axis of frame $\{c\}$ followed by a current frame rotation of $\theta = \pi$ about the x-axis of frame $\{c_1\}$ suffices to align the orientation of frame $\{c_2\}$ with frame $\{d\}$, which is to say:

$$\begin{aligned} R_d^c = R_{z,\frac{\pi}{2}}\ R_{x,\pi} &= \begin{bmatrix} c(\frac{\pi}{2}) & -s(\frac{\pi}{2}) & 0 \\ s(\frac{\pi}{2}) & c(\frac{\pi}{2}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c(\frac{\pi}{2}) & -s(\frac{\pi}{2}) \\ 0 & s(\frac{\pi}{2}) & c(\frac{\pi}{2}) \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

 Continued on next page ⟶

From figure 2, we can also determine that

$$d_d^c = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

Therefore,

$$T_{cd} = \begin{bmatrix} R_d^c & d_d^c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Find T_{ab} given that

$$T_{bc} = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that $T_{ab} = T_{ac}\ T_{cb} = T_{ad}\ T_{dc}\ T_{cb} = T_{ad}\ T_{cd}^{-1}\ T_{bc}^{-1}$. Recall the definition of the inverse of some homogenous transformation H where

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \qquad \text{is} \qquad H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}$$

Thus for T_{cd}, where

$$R = R^T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \text{and} \qquad -R^T d = -1 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \text{ we can see that } \quad T_{cd}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Similarily for T_{bc}, where

$$R = R^T = I \quad \text{and} \quad -R^T d = -I \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix} \text{ we can see that } \quad T_{bc}^{-1} = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

As such,

$$\begin{aligned} T_{ab} = T_{ad}\ T_{cd}^{-1}\ T_{bc}^{-1} &= \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -4 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & -3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

We can quickly verify that this is the correct transformation matrix by transforming the coordinates of the origin of frame $\{c\}$ as seen from frame $\{b\}$, to the origin of frame $\{c\}$ as seen from frame $\{a\}$ which we can see in figure 2 should be $o_c^a = (-1, 1, 2)$. Observe,

$$O_c^a = T_{ab}\ O_c^b = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & -3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \qquad \text{thus,} \quad o_c^a = (-1, 1, 2) \quad \text{as expected.}$$