

POLYGON ORIENTATION

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POLYGON ORIENTATION

We define a polygon as a polygon without any intersecting edges or collinear vertices. If given the order of the vertices of the polygon, can we determine the orientation (whether clockwise or counter-clockwise)?

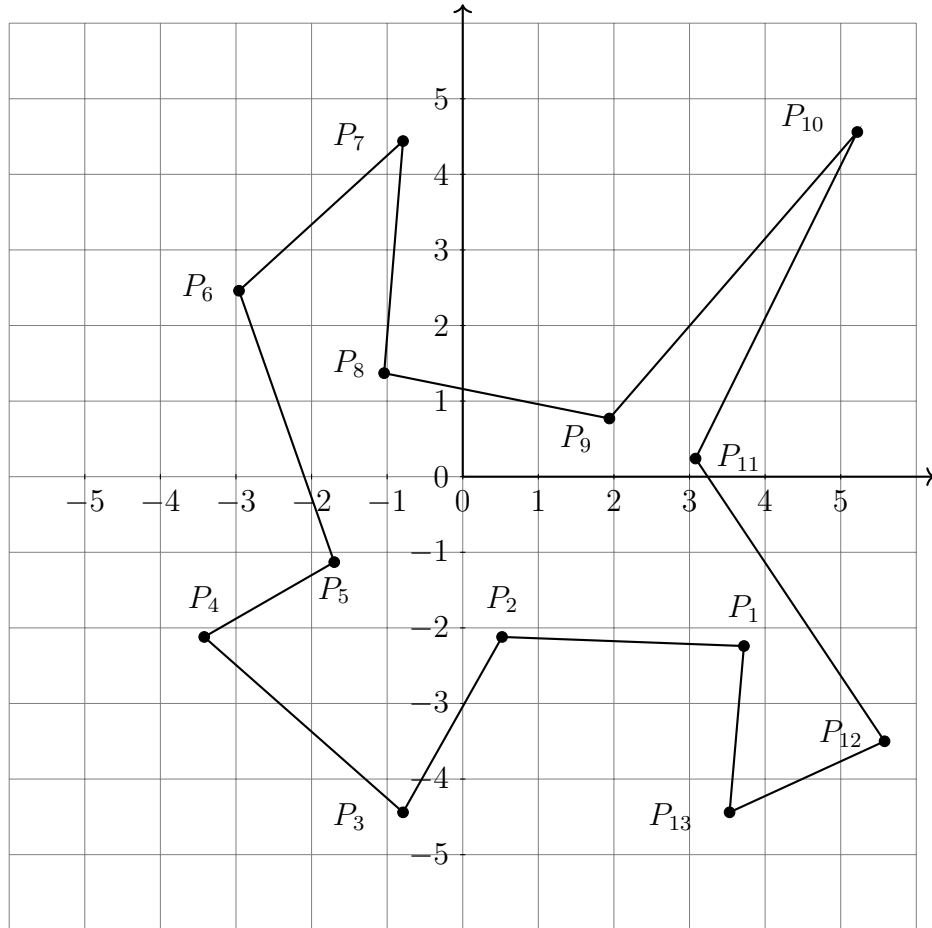


FIGURE 1. Polygon, with vertices ordered from $P_1 \dots P_{13}$

From observation, it can be seen that this polygon is in a clockwise orientation, but how do we mathematically come to this conclusive (beyond an "eye test").

I believe that using the orientation of the bottom vertex (and its adjacent edges), we can determine the orientation of the whole polygon. For this particular polygon, these vertices are P_3 and P_{13} .

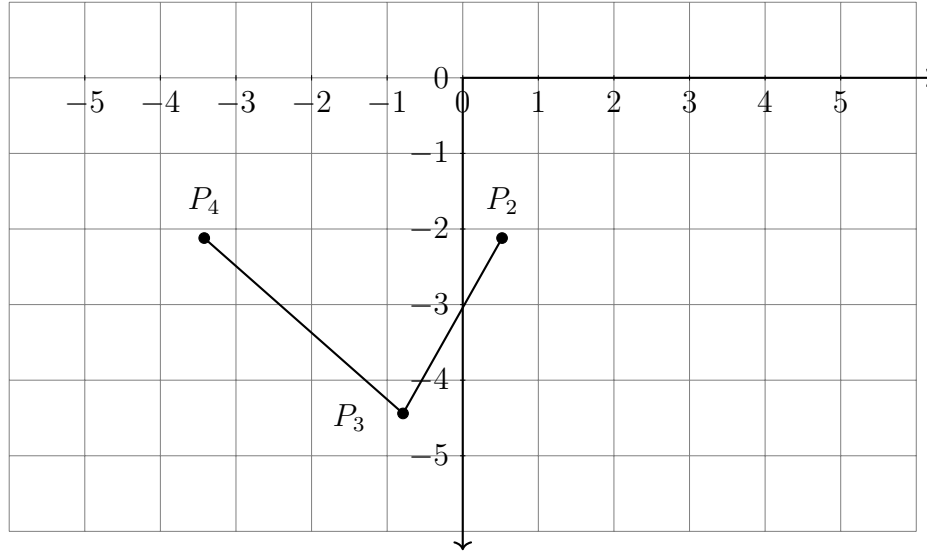


FIGURE 2. P_3 , with its adjacent edges to vertices P_2 and P_4

In regards to vertex P_3 , all other vertices either must have an equal or higher y-value because P_3 is the bottom-most vertex by definition. For this polygon (or any n-sided polygon, in general), there must be an edge between the last vertex to the first vertex (to complete and close the polygon). As a result, the rest of the vertices will complete the cycle above P_3 . This "finishing of cycle" from P_3 to P_3 will be in the same orientation as the current cycle of $P_2 - P_4$ because the orientation of how the cycle approaches P_3 cannot change.

Using our current example, the orientation of $P_2 - -P_4$ is clockwise. As a result, the polygon is oriented clockwise and will finish in a clockwise direction. If it were to finish in a counter-clockwise direction, the cycle would have to "loop around" to reach P_3 . Because our polygon cannot have intersecting edges and the cycle cannot go below the y-axis of P_3 (due to P_3 's bottom-most property), a counter-clockwise orientation is not possible. Thus, the polygon's orientation must be clockwise.

One particular example of note is the "Crescent Moon" polygon, shown below.

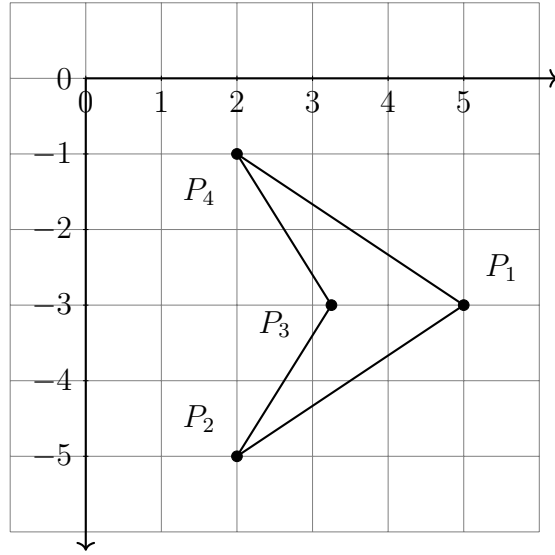


FIGURE 3. Clockwise "Crescent Moon" Polygon

Although $P_2 - -P_3$ appears to be a counter-clockwise vector, the clockwise orientation of the cycle of $P_1 - -P_3$ matches the clockwise orientation of this entire polygon.

This hypothesis also works in determining counter-clockwise polygons.

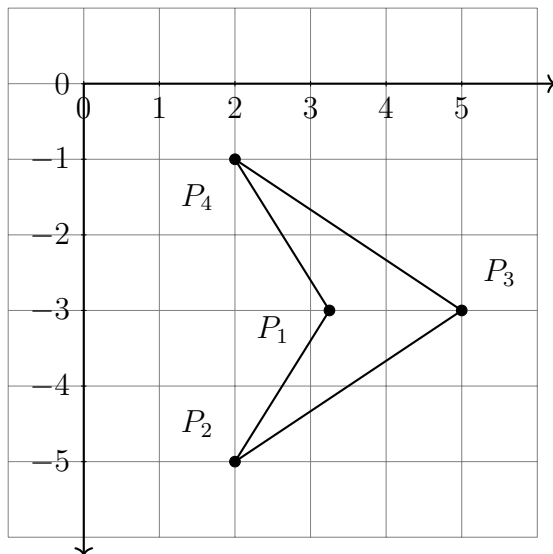


FIGURE 4. Counter-Clockwise "Crescent Moon" Polygon

Now, we know that the orientation of the bottom-most vertex matches the orientation of the entire polygon. From here, however, how can we mathematically determine the orientation of the bottom-most vertex?

To do so, we use the sign of the magnitude of the z-axis of the cross-product of the two vectors emerging from this particular vertex, from the polygon in Figure 1.

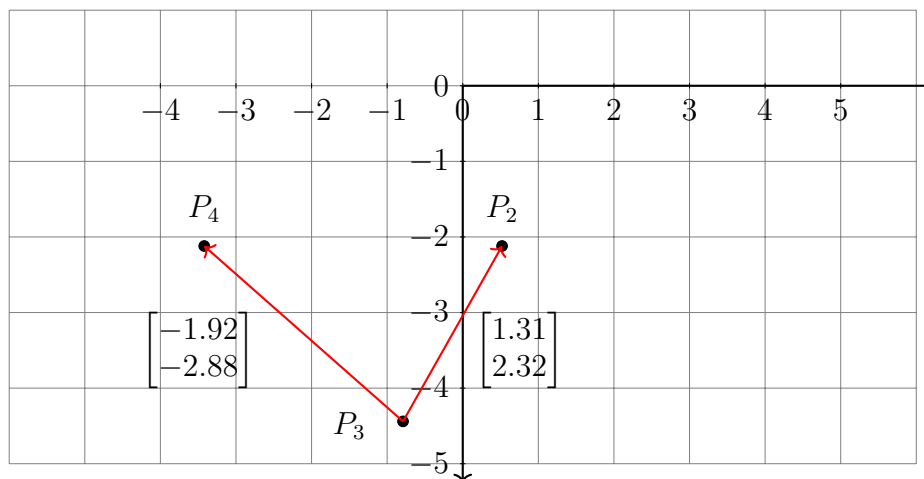


FIGURE 5. P_3 and its emerging vectors

Then, we determine the magnitude of the z-axis of their cross-product, in the order of the vector toward the previous vertex to the vector toward the next vertex.

$$\begin{aligned}
 \overrightarrow{P_3P_2} \times \overrightarrow{P_3P_4} &= \begin{vmatrix} X & Y & Z \\ 1.31 & 2.32 & 0 \\ -1.92 & -2.88 & 0 \end{vmatrix} \\
 &= (0, 0, 0.6816) \\
 &\rightarrow 0.6816 \hat{z}
 \end{aligned}$$

For the case of P_3 , the z-axis is observed to be positive.

$$\begin{aligned}
 \overrightarrow{P_3P_2} \times \overrightarrow{P_3P_4} &= \begin{vmatrix} X & Y & Z \\ -1.92 & -2.88 & 0 \\ 1.31 & 2.32 & 0 \end{vmatrix} \\
 &= (0, 0, -0.6816) \\
 &\rightarrow -0.6816 \hat{z}
 \end{aligned}$$

If we were to reverse the order of the vertices adjacent to P_3 to create a counter-clockwise orientation, the z-axis is observed to be negative.

Since P_3 and P_{13} have the same y-axis value, we can examine another case for clockwise orientation.

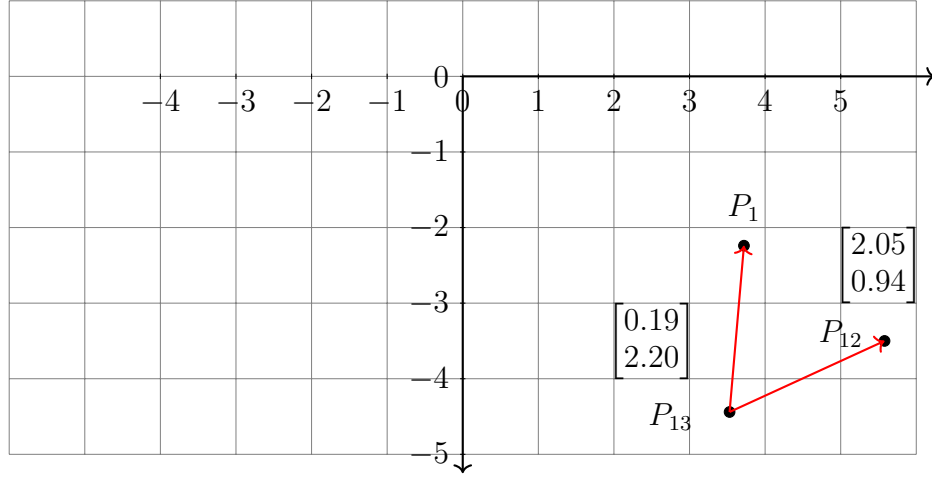
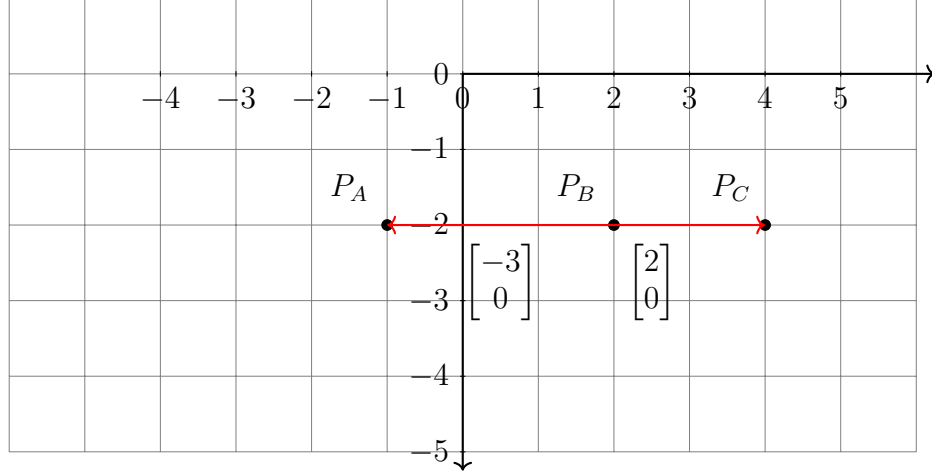


FIGURE 6. P_{13} and its emerging vectors

$$\begin{aligned}
 \overrightarrow{P_{13}P_{12}} \times \overrightarrow{P_{13}P_1} &= \begin{vmatrix} X & Y & Z \\ 2.05 & 0.94 & 0 \\ 0.19 & 2.20 & 0 \end{vmatrix} \\
 &= (0, 0, 4.3314) \\
 &\rightarrow 4.3314 \hat{z}
 \end{aligned}$$

Using this cross-product, a positive value for the z-axis implies a clockwise orientation. Conversely, a negative value for the z-axis implies a counter-clockwise orientation.

One particular issue, however, would be the case where the cross-product was equal to 0. This happens when the three chosen points (from the set of vertices with the lowest y-value) are collinear.

FIGURE 7. P_B and its emerging collinear vectors

$$\begin{aligned}
 \overrightarrow{P_B P_A} \times \overrightarrow{P_B P_C} &= \begin{vmatrix} X & Y & Z \\ -3 & 0 & 0 \\ 2 & 0 & 0 \end{vmatrix} \\
 &= (0, 0, 0) \\
 &\rightarrow 0 \hat{z}
 \end{aligned}$$

To solve this, we perform a sorting algorithm on this set of bottom-most points, then choosing the vertex with the highest x-value. As a result, one of its adjacent vertices would have a y-value greater than the lowest value in the polygon. Thus, the cross-product of its adjacent vectors would no longer be 0.

Then, we created a Python simulation, where random polygons would be generated under a randomly selected orientation. From here, we would run this particular algorithm on it, predicting the orientation without utilizing prior knowledge of the orientation from the polygon's creation.

One hypothesis that we would like to observe is the case where we can get our result by running this "cross-product" algorithm on all vertices. Here, each vertex would put a "vote" for either type of orientation (CW vs CCW). Then, whatever side had the greater amount of "votes" would be the determined orientation.

This would not work, as observed with the following polygon.

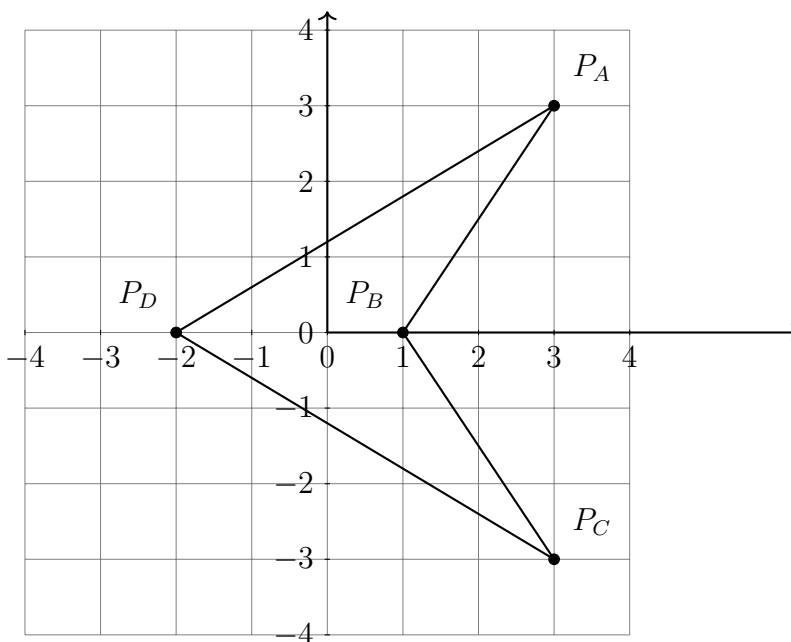


FIGURE 8. P_9

If we apply the cross-product algorithm to each vertex, we can observe 2 votes for Clockwise and 2 votes for Counterclockwise for this clockwise polygon. As a result, this method of voting would not be practical.

GitHub Repository: <https://github.com/anthony-nguyen-04/PolygonOrientation>