Nuclear Physics B275 [FS17] (1986) 200–218
North-Holland, Amsterdam

EFFECT OF BOUNDARY CONDITIONS ON THE OPERATOR CONTENT OF TWO-DIMENSIONAL CONFORMALLY INVARIANT THEORIES

John L. CARDY

Department of Physics, University of California, Santa Barbara CA 93106, USA

Received 6 May 1986

The operator content of unitary conformally invariant theories with $c < 1$ is further analysed by deriving the spectrum of the transfer matrix for finite width strips and a variety of boundary conditions: antiperiodic, cyclic, twisted, free, fixed and a mixture of the last two. Complete results are obtained for the Ising model and for the three-state Potts model, as illustrations of the method. They demonstrate how the internal symmetries of these theories are tied in with their conformal properties.

1. Introduction

Recently there has been considerable progress in using the principle of conformal invariance [1] to classify possible universality classes of critical behavior in two dimensions. Friedan, Qiu and Shenker [2] showed that for unitary models with conformal anomaly number c less than unity, c must be quantized according to

$$c = 1 - \frac{6}{m(m+1)}, \qquad \text{for } m = 3, 4, \ldots, \tag{1.1}$$

and that the allowed values of the scaling dimensions of the primary scaling operators are $(h_{p,q}, h_{\bar{p},\bar{q}})$, where

$$h_{p,q} = \frac{(p(m+1) - qm)^2 - 1}{4m(m+1)}, \tag{1.2}$$

with $1 \leqslant q \leqslant p \leqslant m-1$. For a given m, different universality classes correspond to different subsets of the allowed operators. In a recent paper [3], hereinafter referred to as I, we showed that these subsets can be found by requiring the modular invariance of the partition function on a torus. This result rests on the observation that conformal invariance implies a one-to-one correspondence between the scaling operators and the eigenstates of the transfer matrix of the theory defined on a cylinder. The fact that the partition function on a torus can be calculated in terms of the cylinder transfer matrix in two possible ways gives rise to a system of linear Diophantine equations for the number $\mathcal{N}_{\mathrm{P}}(p, q; \bar{p}, \bar{q})$ of primary scaling operators with given scaling dimensions.

For each m, there is always the solution $\mathcal{N}_{\mathrm{P}}(p, q; \bar{p}, \bar{q}) = \delta_{p\bar{p}}\delta_{q\bar{q}}$, where each allowed scalar operator is present just once (and no others appear.) These were identified with the universality classes of critical, tricritical, tetracritical, etc. points in Ising models. In addition, for $m = 5$ and $m = 6$, two further solutions were found [3, 4] identified with the critical and tricritical 3-state Potts universality classes respectively. For $m \leqslant 6$, these are the only solutions of the modular invariance constraint [3, 4]. However, Itzykson and Zuber [5] have found additional solutions generalizing the above for $m \equiv 1, 2 \pmod 4$.

The above solutions give the full scaling spectrum of the cylinder transfer matrix with periodic boundary conditions, corresponding to operators with integer spin [3]. To expose the full operator content of a given universality class, it is necessary to consider other types of boundary condition. This is carried out in the present paper. The strategy is an extension of that used in I.

To establish notation, let us denote the transfer matrix for an infinitely long strip of width l and boundary conditions of type X by $\exp(-a\hat{H}_X)$ (where a is the lattice spacing) and the partition function for an $l \times l'$ rectangle, with boundary conditions of types X, Y on the two pairs of opposite sides respectively, by Z_{XY}. The types of boundary condition we consider are:

Periodic, P. In I it was shown how, by considering Z_{PP} (generalized to the case of a nonrectilinear torus), one can determine the operator content of $\hat{H}_{\mathrm{P}}$. This will be the input for the other cases.

Antiperiodic, A (Z_2 models); *Cyclic*, C and *Twisted*, T (3-state Potts model). Z_{PA} can be constructed from the knowledge of $\hat{H}_{\mathrm{P}}$, together with the Z_2 transformation properties of the eigenstates. From this the operator content of $\hat{H}_{\mathrm{A}}$ follows. The Z_2 properties of these operators follow by considering Z_{AA}. For the 3-state Potts model, a similar result holds for "cyclic" and "twisted" boundary conditions, as defined in sect. 4.

Free, F. In this case the order parameter is unconstrained at the boundary. In the continuum limit, this means that it actually vanishes there [6]. Z_{PF} and Z_{AF} can be constructed from $\hat{H}_{\mathrm{P}}$ and $\hat{H}_{\mathrm{A}}$, but unknown matrix elements are involved. It turns out, however, that with some additional physical information, the spectrum of $\hat{H}_{\mathrm{F}}$ can nevertheless be determined. It was shown in ref. [7] that conformal invariance relates this geometry to that of a semi-infinite system, and that the eigenvalues of $\hat{H}_{\mathrm{F}}$ determine the surface critical exponents describing the decay of correlations along the boundary. We thus obtain the complete set of surface exponents (at the ordinary transition [9]) by this method. In ref. [8], only the leading magnetic exponents of certain models were identified.

Fixed, $++$ *or* $+-$. (Z_2 models.) In this case the order parameter is fixed to equal (or opposite) values on either side of the strip. For self-dual models, Z_{PF} is related to $Z_{\mathrm{P},++} + Z_{\mathrm{P},+-}$ by duality. Further symmetry considerations allow us to disentangle the two terms, and hence determine the spectrum of $\hat{H}_{++}$ and $\hat{H}_{+-}$. The former gives the complete set of surface exponents at the extraordinary transition [9]. A simple generalization then applies to the 3-state Potts model and to other models with other symmetry groups.

Mixed, M. This corresponds to free boundary conditions on one side of the strip, and fixed boundary conditions on the other. It will be shown that the information obtained about various matrix elements in the last two cases fixes the spectrum of $\hat{H}_{\mathrm{M}}$ also.

The first two cases involve the full two-dimensional conformal algebra, so that primary scaling operators correspond to a representation of a direct product of two Virasoro algebras [2, 10]. As a consequence, the operator content is specified by the non-negative integers $\mathcal{N}_X(p, q; \bar{p}, \bar{q})$. For the last three cases, there is only one Virasoro algebra, and the operator content is specified by the integers $\mathcal{N}_X(p, q)$. This is discussed in greater detail in the appendix. In both cases, the Kac formula (1.2) gives the *allowed* values of the scaling dimensions, but, as we shall show, in each sector a given model uses only a subset of these. In addition, values of p and q which correspond to energy-like operators in one sector may correspond to magnetization operators in another, and *vice versa*. Since the differential equations obeyed by correlation functions in these models depend only on the values of p and q [10], we see that spin correlations in one sector may resemble energy correlations in another, although the boundary conditions on the equations may be different.

Previous work in this area has mostly been a combination of numerical analysis and some guesswork. Gehlen and Rittenberg [11] have identified the operator content of the 3-state Potts model for the cases P, C and F by numerically diagonalizing the transfer matrix, or rather the equivalent quantum hamiltonian, in finite-width strips. Burkhardt [12] has obtained analytically the leading magnetization and energy eigenvalues for the case $++$ for self-dual models, and Burkhardt and Guim [13] have obtained results for the cases P, A, F and M for the Ising model. Our results agree with and extend those of the above authors.

The layout of this paper is as follows. In sect. 2 we sketch the modular invariance arguments for Z_{PP} as given in I, and show how they generalize to Z_{PF}. In sect. 3, we use these results to determine the operator content for the five cases P through M described above, for the critical Ising model, as a first simple example. Next, in sect. 4, we discuss the 3-state Potts model, where several other features arise. At this stage it should be clear how to obtain similar results for any other model in the Friedan, Qiu and Shenker [2] classification, and in sect. 5, we summarize partial results for some of these other models. In particular, we obtain results for the surface exponents of the tricritical Ising model which are new.

2. Inversion sum rules

In I it was shown how the eigenvalues E_n of $\hat{H}_{\mathrm{P}}$ are related to the scaling dimensions of the scaling operators of a given theory. This is expressed simply in terms of the observation [5,14] that under the conformal mapping $w=(l/2\pi)\ln z$, which maps the whole z-plane onto the cylinder,

$$L_0+\bar{L}_0 \to \frac{l}{2\pi}\left(\hat{H}_{\mathrm{P}}+\frac{\pi c}{6l}\right). \tag{2.1}$$

The operator on the left generates dilatations in the plane, and its eigenvalues are the scaling dimensions x_n of the scaling operators. The last term on the r.h.s. comes from the anomalous transformation properties of the stress tensor [10]. This term was first obtained by other, equivalent, means by Blöte, Cardy and Nightingale [15], and by Affleck [16].

The partition function Z_{PP} for a rectangular torus of dimensions $l\times l'$ is then

$$Z_{\mathrm{PP}}=\mathrm{Tr}\,\mathrm{e}^{-l'\hat{H}_{\mathrm{P}}}=\mathrm{e}^{\pi c\delta/6}\sum_n \mathrm{e}^{-2\pi x_n\delta}, \tag{2.2}$$

where $\delta=l'/l$. Here, as throughout this paper, we have set the nonuniversal bulk free energy equal to zero. The set of scaling operators decomposes into conformal blocks, each associated with a primary operator whose scaling dimensions are given by eq. (1.2). The scaling dimension of an operator in a given block is of the form $h_{pq}+h_{\bar{p}\bar{q}}+N+\bar{N}$, where N and $\bar{N}$ denote the level in the block. The sum in eq. (2.2) may then be written as

$$\sum_{pq;\,\bar{p}\bar{q}} \mathcal{N}_{\mathrm{P}}(p,q;\bar{p},\bar{q})\,d_{pq}(N)\,d_{\bar{p}\bar{q}}(\bar{N})\exp\left(-2\pi\delta\left(h_{pq}+h_{\bar{p}\bar{q}}+N+\bar{N}\right)\right), \tag{2.3}$$

where the factors d denote the degeneracy at the appropriate level in the conformal block. The generating function for these was given by Rocha-Caridi [17], from which we find

$$Z_{\mathrm{PP}}(\delta)=\mathrm{e}^{\pi c\delta/6}\sum_{pq;\,\bar{p}\bar{q}}\mathcal{N}_{\mathrm{P}}(p,q;\bar{p},\bar{q})\,\chi_{pq}(\delta)\,\chi_{\bar{p}\bar{q}}(\delta), \tag{2.4}$$

where

$$\chi_{pq}(\delta) = \prod_{n=1}^{\infty}(1-e^{-2\pi n\delta})^{-1} \times \sum_{k=-\infty}^{\infty}\left(\exp\left(-\frac{\pi\delta}{2m(m+1)}\left[(2m(m+1)k+(m+1)p-mq)^2-1\right]\right) - \{q\to -q\}\right). \quad (2.5)$$

In I it was shown that an application of the Poisson sum formula leads to

$$\chi_{pq}(\delta) = \left(\frac{2}{m(m+1)}\right)^{1/2}\exp\left(\left(\tfrac{1}{12}\pi c\right)(\delta^{-1}-\delta)\right)\prod_{n=1}^{\infty}(1-e^{-2\pi n\delta^{-1}})^{-1} \times \sum_{r=-\infty}^{\infty}\exp\left(-\frac{\pi(r^2-1)}{2\delta m(m+1)}\right)\sin\frac{\pi rp}{m}\sin\frac{\pi rq}{m+1}. \quad (2.6)$$

Further manipulation [3] then shows that (2.4) may be rewritten as

$$Z_{\mathrm{PP}}(\delta) = e^{\pi c/6\delta}\sum_{pq;\,\bar{p}\bar{q}}\ \sum_{p'q';\,\bar{p}'\bar{q}'} M_{pq}^{p'q'}M_{\bar{p}\bar{q}}^{\bar{p}'\bar{q}'}\mathcal{N}_{\mathrm{P}}(p',q';\bar{p}',\bar{q}')\chi_{pq}(\delta^{-1})\chi_{\bar{p}\bar{q}}(\delta^{-1}), \quad (2.7)$$

where the matrix $\boldsymbol{M}$ has elements

$$M_{pq}^{p'q'} = \left(\tfrac{1}{8}m(m+1)\right)^{1/2}(-1)^{(p+q)(p'+q')}\sin\frac{\pi pp'}{m}\sin\frac{\pi qq'}{m+1}. \quad (2.8)$$

If we now impose the condition that $Z_{\mathrm{PP}}(\delta) = Z_{\mathrm{PP}}(1/\delta)$ and equate powers* of $e^{-1/\delta}$, we obtain the inversion sum rules [3], which are linear equations of the form

$$[\boldsymbol{M}\otimes\boldsymbol{M}]\mathcal{N}_{\mathrm{P}} = \mathcal{N}_{\mathrm{P}}. \quad (2.9)$$

The solutions of these equations have been discussed in refs. [3, 5]. Since periodic boundary conditions are used, only the $\mathcal{N}_{\mathrm{P}}(p,q;\bar{p},\bar{q})$ such that the spin $h_{pq}-h_{\bar{p}\bar{q}}$ is an integer are nonzero. This fact implies that $Z_{\mathrm{PP}}(\delta)$ is invariant under the full modular group generated by $\delta\to 1/\delta$ and $\delta\to\delta+i$.

* In fact, to disentangle the dependence, it is necessary to generalize the calculation to a general nonrectilinear torus, so that δ becomes complex [3]. Since this is unnecessary for the extensions to be discussed in this paper, we have not included this complication.

We now discuss the extension of the above to the calculation of Z_{FP} in terms of the eigenstates of $\hat{H}_{\mathrm{F}}$. As shown in ref. [15], the lowest eigenvalue has the form

$$E_0 = 2b - \frac{\pi c}{24l} + \cdots . \tag{2.10}$$

The first term on the r.h.s. is nonuniversal, and corresponds to twice the surface free energy per unit length. As discussed in the appendix, the eigenstates of $\hat{H}_{\mathrm{F}}$ are in 1-1 correspondence with those of L_0, and, as shown in ref. [7], the eigenvalues E_n are related to the surface scaling dimensions $\tilde{x}_n$ by

$$E_n - E_0 \sim \frac{\pi \tilde{x}_n}{l} . \tag{2.11}$$

Hence, if we denote the number of primary operators with scaling dimension $\tilde{x}_n = h_{pq}$ by $\mathcal{N}_{\mathrm{F}}(p,q)$, we can write, analogously to eq. (2.4),

$$Z_{\mathrm{FP}}(l,l') = \exp\left(2bl' + \tfrac{1}{24}\pi c\delta\right) \sum_{pq} \mathcal{N}_{\mathrm{F}}(p,q) \chi_{pq}\left(\tfrac{1}{2}\delta\right) . \tag{2.12}$$

Using eq. (2.6) this can be rewritten as

$$\begin{aligned} &\left(\frac{2}{m(m+1)}\right)^{1/2} \exp(2bl' + \pi c/6\delta) \prod_{n=1}^{\infty} \left(1 - \mathrm{e}^{-4\pi n/\delta}\right)^{-1} \\ &\times \sum_{pq} \mathcal{N}_{\mathrm{F}}(p,q) \sum_{r=-\infty}^{\infty} \exp\left(-\frac{\pi(r^2-1)}{\delta m(m+1)}\right) \sin\frac{\pi rp}{m} \sin\frac{\pi rq}{m+1} . \end{aligned} \tag{2.13}$$

The fact that this can be equated to $Z_{\mathrm{PF}}(l', l)$ will be the starting point for the analysis of the spectrum of $\hat{H}_{\mathrm{F}}$ in the subsequent sections. Note that the fact that the leading dependence as $\delta \to \infty$ cancels on using eq. (2.6) is a check on the result (2.9).

3. Ising model

As discussed in I, there is only one solution of the inversion sum rules for $m = 3$, corresponding to, in the periodic sector,

$$\mathcal{N}_{\mathrm{P}}(1,1;1,1) = \mathcal{N}_{\mathrm{P}}(2,1;2,1) = \mathcal{N}_{\mathrm{P}}(2,2;2,2) = 1 , \tag{3.1}$$

with all other $\mathcal{N}_{\mathrm{P}}(p,q;\bar{p},\bar{q})$ zero. Denoting the corresponding operators by $\mathbb{1}$, ε, σ respectively, the selection rules [10, 1] show that the operator product expansion has the form

$$\begin{aligned} \varepsilon\varepsilon &\sim \mathbb{1} , \\ \sigma\sigma &\sim \mathbb{1} + \varepsilon , \\ \varepsilon\sigma &\sim \sigma . \end{aligned} \tag{3.2}$$

This is consistent with the model possessing a Z_2 symmetry, with ε and σ being respectively even and odd under this symmetry. The theory is identified with the universality class of the critical Ising model, and many pieces of information support this identification [10].

We first consider Z_{PA}. It is useful to have a specific lattice form for the transfer matrix in mind, for example

$$\hat{T}_P = \exp\left(J \sum_{j=1}^{n} \sigma_j^z \sigma_{j+1}^z \right) \exp\left(J \sum_{j=1}^{n} \sigma_j^x \right), \tag{3.3}$$

where $\sigma_{n+1}^z = \sigma_1^z$. This commutes with the operator $\Sigma \equiv \prod_j \sigma_j^x$, which implements the Z_2 symmetry. The eigenstates of $\hat{H}_P$ are also eigenstates of Σ. The partition function can be written

$$Z_{PA}(l, l') = \mathrm{Tr}\, \Sigma\, e^{-l' \hat{H}_P}. \tag{3.4}$$

Since the magnetization operator corresponds to q even, we simply insert a factor $(-1)^{q-1}$ into eq. (2.4) to obtain Z_{PA}. Following through the analysis that leads to (2.8), (2.9) we then find that the operator content of $\hat{H}_A$ is

$$\begin{aligned} \mathcal{N}_A(p, q; \bar{p}, \bar{q}) &= \sum_{p'q'} (-1)^{q'-1} M_{pq}^{p'q'} M_{\bar{p}\bar{q}}^{p'q'} \\ &= \mathcal{N}_P(p, q; \bar{p}, m+1-\bar{q}) = \delta_{p\bar{p}} \delta_{q, m+1-\bar{q}}, \end{aligned} \tag{3.5}$$

where the second equality follows from the definition of $\boldsymbol{M}$, and the fact that m is odd. We have adopted the convention that

$$\mathcal{N}_X(p, q; m-\bar{p}, m+1-\bar{q}) = \mathcal{N}_X(m-p, m+1-q; \bar{p}, \bar{q}) = \mathcal{N}_X(p, q; \bar{p}, \bar{q}). \tag{3.6}$$

Using (3.5) we can read off the scaling dimensions of the primary operators in the A sector. They are $(\frac{1}{16}, \frac{1}{16})$, $(\frac{1}{2}, 0)$, and $(0, \frac{1}{2})$. To determine their Z_2 transformation properties, we consider Z_{AA}. This will be given by the same form as Z_{AP} with appropriate minus signs multiplying the Z_2 odd terms, but it must be symmetric under $l \leftrightarrow l'$. It is straightforward to show that

$$(-1)^q \delta_{p\bar{p}} \delta_{q, m+1-\bar{q}} \tag{3.7}$$

satisfies the inversion sum rules (2.9) if m is odd. Thus the Z_2 odd(even) operators correspond to q odd(even) in this sector, the opposite of what happens in the P sector.

The above operators correspond to the disorder and fermion operators of the Ising model [18, 10]. That they should appear in the spectrum of $\hat{H}_{\mathrm{A}}$ was pointed out in ref. [19]. However, the above result exhibits all the local primary operators in this sector. One feature of this is that there is only one Z_2 even primary operator, which is the ground state of $\hat{H}_{\mathrm{A}}$. Thus the correlation length of energy-energy correlations in the strip is determined by the next lowest operator in the same conformal block as the ground state. The lowest ones are $L_{-1}(\frac{1}{16}, \frac{1}{16})$ and $\bar{L}_{-1}(\frac{1}{16}, \frac{1}{16})$, which have gaps of $2\pi/l$, in agreement with ref. [13].

Next we discuss the spectrum of $\hat{H}_{\mathrm{F}}$. If we calculate Z_{PF} in terms of $\hat{H}_{\mathrm{P}}$ it has the form

$$Z_{\mathrm{PF}}(l', l) = \sum_{n} |\langle F|n\rangle|^2 \mathrm{e}^{-lE_n}, \tag{3.8}$$

where $|F\rangle$ denotes the "boundary" state. For the specific model of eq. (3.3), for example, this would be

$$|F\rangle = \prod_{j} \sqrt{\tfrac{1}{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}_j . \tag{3.9}$$

The leading terms on the r.h.s. of eq. (3.8) may be rewritten as

$$\sum_{pq;\,\bar{p}\bar{q}} \mathcal{N}_{\mathrm{P}}(p, q; \bar{p}, \bar{q}) |\langle F|pq; \bar{p}\bar{q}\rangle|^2 \exp\left(-2\pi\left(h_{pq} + h_{\bar{p}\bar{q}}\right)/\delta\right), \tag{3.10}$$

where we have kept only the lowest state in each conformal block. Since $|F\rangle$ is translationally invariant, only $k = 0$ states, corresponding to $(pq) = (\bar{p}\bar{q})$, enter this sum, so it simplifies to

$$\sum_{pq} \mathcal{N}_{\mathrm{P}}(p, q; p, q) |\langle F|pq\rangle|^2 \exp\left(-\frac{\pi\left[(p(m+1) - qm)^2 - 1\right]}{\delta m(m+1)}\right). \tag{3.11}$$

This is to be compared with eq. (2.13), the leading terms of which correspond to $\pm r = p(m+1) - qm$. The result is

$$|\langle F|pq\rangle|^2 = \frac{\mathrm{e}^{2bl'}}{\left(\frac{1}{8}m(m+1)\right)^{1/2}} \sum_{p'q'} \mathcal{N}_{\mathrm{F}}(p', q')(-1)^{(p+q)(p'+q')} \sin\frac{\pi p p'}{m} \sin\frac{\pi q q'}{m+1}. \tag{3.12}$$

Thus far the discussion has been quite general. We now use the fact that $|F\rangle$ is a Z_2 even state, so that $\langle F|22\rangle = 0$. This leads to the result

$$\mathcal{N}_{\mathrm{F}}(2,1) = \mathcal{N}_{\mathrm{F}}(1,1) = 1, \tag{3.13}$$

where we have used the fact that the operator $\mathbb{1}$ corresponds to $(pq)=(11)$, and should appear only once in the spectrum of $\hat{H}_{\mathrm{F}}$. Further information can be obtained by considering Z_{AF}. An equation similar to (3.12) will be valid, with $|pq\rangle$ replaced by a $k=0$ eigenstate of $\hat{H}_{\mathrm{A}}$, and a factor Σ included on the r.h.s. In fact, we need only the information that (11; 11) is absent from the spectrum of $\hat{H}_{\mathrm{A}}$. This leads to

$$\mathscr{N}_{\mathrm{F}}(1,1)+\Sigma\mathscr{N}_{\mathrm{F}}(2,1)+\sqrt{2}\,\Sigma\mathscr{N}_{\mathrm{F}}(2,2)=0\,, \tag{3.14}$$

where $\Sigma=\pm 1$ depending on whether the corresponding state is Z_2 even or odd. This can only be satisfied if $\mathscr{N}_{\mathrm{F}}(2,2)=0$ and (21) is Z_2 odd. We conclude that the scaling operators of the primary operators in the F sector are 0 and $\frac{1}{2}$, with the latter corresponding to the magnetization. This is consistent with the result of ref. [7], and the fact that $\eta_{\parallel}=1$ for the Ising model [20]. Since there is no independent primary operator coupling to the energy density, the decay of energy-energy correlations is governed by the first excited state in the conformal block of $\mathbb{1}$. This is $L_{-2}\mathbb{1}$, with a gap of $2\pi/l$, in agreement with the result of Burkhardt [12].

Note that many other conditions follow from the non-negativity of the l.h.s. of eq. (3.12) for arbitrary $|pq\rangle$. It is tedious but straightforward to verify that they are all satisfied. In particular we see from (3.12) that the l'-dependence of the inner products is independent of the particular state. Ratios of inner products are universal. In particular, we find

$$|\langle F|11\rangle|^2=|\langle F|21\rangle|^2=\mathrm{e}^{2bl'}\,, \tag{3.15}$$

from which we see that $b<0$ if $\langle F|F\rangle=1$. Note that eq. (3.12) followed by equating only the coefficients of $\mathrm{e}^{-1/\delta}$ corresponding to the lowest states in each block. Equating higher powers will lead to formulae for the inner products between $|F\rangle$ and higher states.

For the Ising model, the cases of free and fixed boundary conditions are related by duality. For the microscopic model whose transfer matrix in the F sector is

$$\hat{T}_{\mathrm{F}}=\exp\left(J\sum_{j=1}^{n-1}\sigma_j^z\sigma_{j+1}^z\right)\exp\left(J\sum_{j=1}^{n}\sigma_j^x\right), \tag{3.16}$$

the duality transformation is implemented by defining the disorder variables [18]

$$\begin{aligned}\mu_j^x&=\sigma_j^z\sigma_{j+1}^z \qquad (1\leqslant j\leqslant n-1)\,,\\ \mu_j^z&=\sigma_1^x\sigma_2^x\cdots\sigma_j^x\,,\end{aligned} \tag{3.17}$$

in terms of which*

$$\hat{T}_{\mathrm{F}} = \exp\left(\mu_1^z + \mu_1^z\mu_2^z + \cdots + \mu_{n-1}^z\Sigma\right)\exp\left(\sum_{j=1}^{n-1}\mu_j^x\right). \tag{3.18}$$

From this we see that the $\Sigma = +1$ sector of $\hat{H}_{\mathrm{F}}$ is equivalent to $\hat{H}_{++}$, and the $\Sigma = -1$ sector is equivalent to $\hat{H}_{+-}$, in the sense that they have the same spectrum. Thus each of these sectors have only one primary operator. Both energy-energy and spin-spin correlations in the strip with boundary conditions $(++)$ are governed by the state $L_{-2}\mathbb{1}$ with a gap $2\pi/l$, corresponding to an r^{-4} decay of correlations along the surface at the extraordinary transition, as found by Burkhardt [12]. In the $(+-)$ sector, the lowest available state is $L_{-1}(\frac{1}{2})$, with a gap π/l. The difference in ground state energies of the two sectors, which is one way of defining the interfacial tension, is

$$E_0^{+-} - E_0^{++} = \frac{\pi}{2l}. \tag{3.19}$$

If we now consider $Z_{\mathrm{P},++}$ and $Z_{\mathrm{P},+-}$, we have a similar result to eq. (3.12). This leads to the inner products

$$\begin{aligned} \langle + |11\rangle &= \langle - |11\rangle = 2^{-1/2}, \\ \langle + |21\rangle &= \langle - |21\rangle = 2^{-1/2}, \\ -\langle + |22\rangle &= \langle - |22\rangle = 2^{-1/4}, \end{aligned} \tag{3.20}$$

where we have set the nonuniversal constant $b = 0$. From eq. (3.15) we also have $\langle F|11\rangle = 1$ and $\langle F|21\rangle = -1$. The relative minus sign between $\langle + |21\rangle$ and $\langle F|21\rangle$ is due to the fact that the energy operator (21) changes sign under duality. We can now construct Z_{PM}, and hence determine the spectrum of $\hat{H}_{\mathrm{M}}$, using the analog of eq. (3.12)

$$\langle + |pq\rangle\langle pq|F\rangle = \left(\tfrac{2}{3}\right)^{1/2}\sum_{p'q'}\mathcal{N}_{\mathrm{M}}(p',q')(-1)^{(p+q)(p'+q')}\sin\tfrac{1}{3}\pi pp'\sin\tfrac{1}{4}\pi qq'. \tag{3.21}$$

We find that the only primary operator in the spectrum of $\hat{H}_{\mathrm{M}}$ has scaling dimension $\frac{1}{16}$ corresponding to $(pq) = (22)$. The correlation length in the strip is given by the gap π/l to the first excited state $L_{-1}(\frac{1}{16})$. This is in agreement with the numerical result of Burkhardt and Guim [13].

* Actually the factors appear in the reverse order, but the result is equivalent to (3.18) by a similarity transformation.

TABLE 1
Operator content of the Ising model in the sectors discussed in the text

Sector	E_0	Quantum numbers	Dimensions	Gap	Symmetry
P	$-\frac{1}{12}$	(11; 11) (21; 21)	$(0,0)\ (\frac{1}{2},\frac{1}{2})$	2	+
		(22; 22)	$(\frac{1}{16},\frac{1}{16})$	$\frac{1}{4}$	−
A	$\frac{1}{6}$	(22; 22)	$(\frac{1}{16},\frac{1}{16})$	2	+
		(21; 11) (11; 21)	$(\frac{1}{2},0)\ (0,\frac{1}{2})$	$\frac{3}{4}$	−
F	$-\frac{1}{48}$	(11)	0	2	+
		(21)	$\frac{1}{2}$	$\frac{1}{2}$	−
+ +	$-\frac{1}{48}$	(11)	0	2	
+ −	$\frac{23}{48}$	(21)	$\frac{1}{2}$	1	
M	$\frac{1}{24}$	(22)	$\frac{1}{16}$	1	

Column 2 gives the finite-size correction to the free energy per unit length of a strip of width l, in units of π/l. Column 3 shows the quantum numbers $(pq;\bar{p}\bar{q})$ of the primary operators, and column 4 gives their scaling dimensions. The last column gives their Z_2 symmetry properties, and column 5 shows the gap to the first excited state in that Z_2 sector, in units of π/l.

This completes our analysis of the operator content of the Ising model in sectors P through M. The results are summarized in table 1.

4. 3-state Potts model

As pointed out by Dotsenko [21], the exponents previously known for this model are accounted for by the Kac formula (1.2) with $m=5$. In I we gave the complete operator content in the P sector:

$$\begin{aligned}&\mathcal{N}_{\mathrm{P}}(1,1;1,1)=\mathcal{N}_{\mathrm{P}}(2,1;2,1)=\mathcal{N}_{\mathrm{P}}(3,1;3,1)=\mathcal{N}_{\mathrm{P}}(4,1;4,1)=1\,,\\&\mathcal{N}_{\mathrm{P}}(4,1;1,1)=\mathcal{N}_{\mathrm{P}}(1,1;4,1)=\mathcal{N}_{\mathrm{P}}(2,1;3,1)=\mathcal{N}_{\mathrm{P}}(3,1;2,1)=1\,,\\&\mathcal{N}_{\mathrm{P}}(3,3;3,3)=\mathcal{N}_{\mathrm{P}}(4,3;4,3)=2\,,\end{aligned}\tag{4.1}$$

where the first two lines correspond to operators invariant under the Z_3 symmetry, and the last line corresponds to the magnetization operators. It is useful to have in mind the microscopic model defined by the transfer matrix

$$\hat{T}_{\mathrm{P}}=\exp\Big(J\sum_j\big(\sigma_j^{\dagger}\sigma_{j+1}+\text{h.c.}\big)\Big)\exp\Big(J\sum_j\big(\Gamma_j+\text{h.c.}\big)\Big)\,,\tag{4.2}$$

where

$$\sigma_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}_j \tag{4.3}$$

$$\Gamma_j = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}_j , \tag{4.4}$$

where $\omega = e^{2\pi i/3}$. In analogy to sect. 3 we can define an operator $\Sigma = \prod_j \Gamma_j$ which commutes with $\hat{H}_P$ and implements the Z_3 symmetry of the model. The model is in fact invariant under the larger permutation group S_3. One of the Z_2 subgroups is implemented by the operator

$$\tau = \prod_j \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}_j , \tag{4.5}$$

which, in the present basis, corresponds to complex conjugation. There are, of course, two other such operators related to τ by Z_3 rotations.

The Z_3 invariance allows us to consider cyclic* boundary conditions, where spins on opposite boundaries differ by a factor of either ω or of ω^2. These two cases are related by a parity transformation, so that the spectra are identical. We will refer to either of these sectors as the C sector. By considering Z_{PC} we then have (cf. (3.5))

$$\mathcal{N}_C(p,q;\bar{p},\bar{q}) = \sum_{pq\bar{p}\bar{q}} \Sigma_q M_{pq}^{p'q'} M_{\bar{p}\bar{q}}^{\bar{p}'\bar{q}'} \mathcal{N}_P(p',q';\bar{p}',\bar{q}') , \tag{4.6}$$

where $\Sigma_q = 1$ for $q=1$, and $\frac{1}{2}(\omega+\omega^2)$ for $q=3$. The nonzero values are $\mathcal{N}_C(3,3;3,3) = \mathcal{N}_C(4,3;4,3) = 1$, and $\mathcal{N}_C(p,1;p,3) = \mathcal{N}_C(p,3;p,1) = 1$ for $p = 1,2,3,4$. Disentangling the Z_3 transformation properties of these states by consideration of Z_{CC} is somewhat tedious. It is possible to show that

$$\omega^{q-\bar{q}} \mathcal{N}_C(p,q;p,\bar{q}) \tag{4.7}$$

satisfies the inversion sum rules (2.9). This means that the states with $q = \bar{q} = 1$ correspond to energy operators, and the remaining ones correspond to the two components of the magnetization, which transform by either ω or ω^2 under a Z_3 rotation. These results are summarized in table 2. The states in this sector correspond to the disorder and parafermion operators of the Z_3 model [21, 22]. That they should appear in this sector was discussed in ref. [19].

* In previous work, these have been called twisted boundary conditions. However, it is more appropriate to reserve that name for the case to follow.

TABLE 2
Operator content of the 3-state Potts model in the sectors discussed in the text

Sector	E_0	Quantum numbers	Dimensions	Gap	Symmetry
P	$-\frac{2}{15}$	(11;11) (21;21)	$(0,0)\ (\frac{2}{5},\frac{2}{5})$	$\frac{8}{5}$	1
		(31;31) (41;41)	$(\frac{7}{5},\frac{7}{5})\ (3,3)$		1
		(41;11) (11;41)	$(3,0)\ (0,3)$		1
		(31;21) (21;31)	$(\frac{2}{5},\frac{7}{5})\ (\frac{7}{5},\frac{2}{5})$		1
		(33;33) × 2	$(\frac{1}{15},\frac{1}{15})$	$\frac{4}{15}$	$\omega,\ \omega^2$
		(43;43) × 2	$(\frac{2}{3},\frac{2}{3})$		$\omega,\ \omega^2$
C	$\frac{2}{15}$	(33;33) (43;43)	$(\frac{1}{15},\frac{1}{15})\ (\frac{2}{3},\frac{2}{3})$	2	1
		(13;11) (11;31)	$(\frac{2}{3},0)\ (0,\frac{2}{3})$		$\omega,\ \omega^2$
		(23;21) (21;23)	$(\frac{2}{5},\frac{1}{15})\ (\frac{1}{15},\frac{2}{5})$	$\frac{2}{3}$	$\omega,\ \omega^2$
		(33;31) (31;33)	$(\frac{7}{5},\frac{1}{15})\ (\frac{1}{15},\frac{7}{5})$		$\omega,\ \omega^2$
		(43;41) (41;43)	$(3,\frac{2}{3})\ (\frac{2}{3},3)$		$\omega,\ \omega^2$
T	$-\frac{1}{30}$	(12;12) (22;22)	$(\frac{1}{8},\frac{1}{8})\ (\frac{1}{40},\frac{1}{40})$	$\frac{2}{5}$	+
		(32;32) (42;42)	$(\frac{21}{40},\frac{21}{40})\ (\frac{13}{8},\frac{13}{8})$		+
		(42;12) (12;42)	$(\frac{13}{8},\frac{1}{8})\ (\frac{1}{8},\frac{13}{8})$		−
		(32;22) (22;32)	$(\frac{21}{40},\frac{1}{40})\ (\frac{1}{40},\frac{21}{40})$	1	−
F	$-\frac{1}{30}$	(11) (41)	0 3	2	1
		(43) × 2	$\frac{2}{3}$	$\frac{2}{3}$	$\omega,\ \omega^2$
11	$-\frac{1}{30}$	(11)	0	2	+
		(41)	3	3	−
1ω	$\frac{19}{30}$	(43)	$\frac{2}{3}$	1	
M	$\frac{11}{120}$	(12)	$\frac{1}{8}$	1	+
		(42)	$\frac{13}{8}$	$\frac{3}{2}$	−

Notation as in table 1. The last column gives the transformation properties of the operators under either the Z_3 subgroup $(1, \omega, \omega^2)$, or under one of the Z_2 subgroups $(\pm)$.

Twisted boundary conditions correspond to $\sigma_{n+1} = \sigma_1^*$. The partition function Z_{PT} is given by $\mathrm{Tr}\, \tau \exp(-l\hat{H}_{\mathrm{P}})$. The eigenstates of $\hat{H}_{\mathrm{P}}$ corresponding to magnetization operators transform into each other under τ, so give no contribution to Z_{PT}. The transformation properties of the energy-like operators are most easily understood within the Landau-Ginzburg-Wilson formulation of the model, in which it is described by an action

$$\mathscr{H} = \int \left[|\nabla\phi|^2 + r|\phi|^2 + \lambda\left(\phi^3 + \phi^{*3}\right)\right] \mathrm{d}^d x\,. \tag{4.8}$$

Under τ, or complex conjugation, the scalar energy operators like $\phi^*\phi$ are invariant.

The odd-integer spin energy operators like $\phi^* \partial\phi - \phi\,\partial\phi^*$ and $\phi^*\partial^3\phi - \phi\,\partial^3\phi^*$, corresponding to the third line in eq. (4.1), change sign. We therefore have

$$\mathcal{N}_{\mathrm{T}}(p,q;\bar{p},\bar{q}) = \sum_{p'\bar{p}'} (-1)^{p'+\bar{p}'} M_{pq}^{p'1} M_{\bar{p}\bar{q}}^{\bar{p}'1} . \tag{4.9}$$

This gives the nonzero values $\mathcal{N}_{\mathrm{T}}(p,2;p,2) = 1$ for $p = 1,2,3,4$, and $\mathcal{N}_{\mathrm{T}}(1,2;4,2) = \mathcal{N}_{\mathrm{T}}(4,2;1,2) = \mathcal{N}_{\mathrm{T}}(2,2;3,2) = \mathcal{N}_{\mathrm{T}}(3,2;2,2) = 1$. By considering Z_{TT}, it is then straightforward to show that the scalar operators are even under τ, and the nonscalars, which have half-integral spin, are odd.

The appearance of these operators in the T sector is most interesting because it marks the first appearance of $q = 2$ operators in the theory. It is natural to call them twist operators. Such operators also appear in the gaussian model. Recently Friedan and Shenker [23] have shown that they correspond to the magnetization operators in the mapping [24] of the Ashkin-Teller model to the gaussian model. They also appear in string theory [25]. In the Potts model, they are analogous to the disorder operators. The lowest state in the T sector, with scaling dimensions $(\frac{1}{40}, \frac{1}{40})$, corresponds to the operator at the end of a defect line across which $\mathrm{Im}\,\phi$ changes sign. The half-integer spin operators are formed in the operator product expansion of this operator and ϕ. The existence of these operators, and the values of their scaling dimensions, have previously been derived by Nienhuis and Knops [25] from a mapping to the gaussian model. It is interesting to note that the sum $\frac{1}{2}(Z_{\mathrm{PP}} + Z_{\mathrm{PT}} + Z_{\mathrm{TP}} + Z_{\mathrm{TT}})$ is modular invariant, and in fact is equal to Z_{PP} for the other universality class with $m = 5$, the tetracritical Ising model*. This points to an interesting possible connection between these models.

Next we consider free boundary conditions. Using eq. (3.12) and the fact that $\langle F|33\rangle = 0$, we find

$$\begin{aligned}&\left(\mathcal{N}_{\mathrm{F}}(1,1)\sin\tfrac{3}{5}\pi + \mathcal{N}_{\mathrm{F}}(2,1)\sin\tfrac{6}{5}\pi + \mathcal{N}_{\mathrm{F}}(3,1)\sin\tfrac{9}{5}\pi + \mathcal{N}_{\mathrm{F}}(4,1)\sin\tfrac{12}{5}\pi\right)\sin\tfrac{3}{6}\pi \\ &\quad + \left(\mathcal{N}_{\mathrm{F}}(3,3)\sin\tfrac{9}{5}\pi + \mathcal{N}_{\mathrm{F}}(4,3)\sin\tfrac{12}{5}\pi\right)\sin\tfrac{9}{6}\pi = 0 ,\end{aligned} \tag{4.10}$$

which leads to

$$\begin{aligned}\mathcal{N}_{\mathrm{F}}(4,3) &= \mathcal{N}_{\mathrm{F}}(1,1) + \mathcal{N}_{\mathrm{F}}(4,1) ,\\ \mathcal{N}_{\mathrm{F}}(3,3) &= \mathcal{N}_{\mathrm{F}}(2,1) + \mathcal{N}_{\mathrm{F}}(3,1) .\end{aligned} \tag{4.11}$$

Similarly, the vanishing of $\langle F|22\rangle$ gives

$$\begin{aligned}&\mathcal{N}_{\mathrm{F}}(1,1) - \mathcal{N}_{\mathrm{F}}(4,1) - \mathcal{N}_{\mathrm{F}}(1,2) + \mathcal{N}_{\mathrm{F}}(4,2) = 0 ,\\ &\mathcal{N}_{\mathrm{F}}(2,1) - \mathcal{N}_{\mathrm{F}}(3,1) - \mathcal{N}_{\mathrm{F}}(2,2) + \mathcal{N}_{\mathrm{F}}(3,2) = 0 ,\end{aligned} \tag{4.12}$$

* This possibility was first pointed out to me by Stephen Shenker.

Further information comes from Z_{CF}. The absence of (11; 11) in the C sector implies

$$\Sigma\mathcal{N}_{\mathrm{F}}(1,1)+\Sigma\mathcal{N}_{\mathrm{F}}(4,1)+2\Sigma\mathcal{N}_{\mathrm{F}}(4,3)=0\,,$$

$$\Sigma\mathcal{N}_{\mathrm{F}}(2,1)+\Sigma\mathcal{N}_{\mathrm{F}}(3,1)+2\Sigma\mathcal{N}_{\mathrm{F}}(3,3)=0\,,$$

$$\Sigma\mathcal{N}_{\mathrm{F}}(1,2)+\Sigma\mathcal{N}_{\mathrm{F}}(4,2)=0\,,$$

$$\Sigma\mathcal{N}_{\mathrm{F}}(2,2)+\Sigma\mathcal{N}_{\mathrm{F}}(3,2)=0\,, \tag{4.13}$$

where $\Sigma=1$ if (pq) corresponds to an energy-like operator, and otherwise equals $-\frac{1}{2}$. The only way conditions (4.5)–(4.7) can be satisfied with non-negative integer values for $\mathcal{N}_{\mathrm{F}}(p,q)$ is to have $\mathcal{N}_{\mathrm{F}}(1,1)=\mathcal{N}_{\mathrm{F}}(4,1)=1$, $\mathcal{N}_{\mathrm{F}}(4,3)=2$, $\mathcal{N}_{\mathrm{F}}(p,2)=0$, and $\mathcal{N}_{\mathrm{F}}(2,1)=\mathcal{N}_{\mathrm{F}}(3,1)=K$, $\mathcal{N}_{\mathrm{F}}(3,3)=2K$, where (43) and (33) are magnetization operators. There are of course many more constraints which must be satisfied. Nevertheless we have not been able to pin down the value of K precisely, by this method. The best we can do is to show, from the positivity of $|\langle F|21\rangle|^2$, that $K=0$ or 1. The second possibility is in fact ruled out by other arguments. Having $\mathcal{N}_{\mathrm{F}}(2,1)=1$ would imply the existence of surface scaling operator with $\tilde{x}=\frac{2}{5}$, that is $\tilde{y}=\frac{3}{5}$. The existence of such a relevant energy-like operator would contradict the general theory of the ordinary surface transition [9]. In addition, the leading magnetization operator would correspond to $(pq)=(33)$, with $\tilde{x}=\frac{1}{15}$. This would contradict the result of Cardy [8] where a conjecture for the scaling dimension of the leading magnetic operator for the general Q-state Potts model was made. On the other hand, taking this operator to correspond to $(pq)=(43)=(13)$ agrees with ref. [8].

We conclude that the primary operators in the F sector have scaling dimensions 0, $\frac{2}{3}$, and 3, with the second being doubly degenerate and corresponding to the magnetization.

Under duality, Z_{PF} is equal to $Z_{\mathrm{P},11}+Z_{\mathrm{P},1\omega}+Z_{\mathrm{P},1\omega^2}$, where $Z_{\mathrm{P},xy}$ is the partition function for a cylinder with the order parameter fixed to x and y on either boundary. By studying the explicit properties of the duality transformation, as in sect. 3, it is easy to see that the scaling dimensions of the primary operators in the (11) sector are 0 and 3, while in the other two sectors there is only $\frac{2}{3}$. In the first case, $\hat{H}_{11}$ is invariant under τ twists. By considering $Z_{\mathrm{T},11}$ one can check that the scaling operator with scaling dimension 0 is even under τ, while the other is odd.

As in the Ising case, in order to determine the spectrum in the M sector, we must first determine various inner products of the eigenstates of $\hat{H}_{\mathrm{P}}$ with $|F\rangle$ and $|1\rangle$. These are readily evaluated using eq. (3.12):

$$\langle F|11\rangle=\langle F|41\rangle=\sqrt{3}\,\alpha\sin^{1/2}\left(\tfrac{1}{5}\pi\right),$$

$$\langle F|21\rangle=\langle F|31\rangle=\sqrt{3}\,\alpha\sin^{1/2}\left(\tfrac{2}{5}\pi\right), \tag{4.14}$$

and correspondingly

$$\langle 1|11\rangle = -\langle 1|41\rangle = \alpha \sin^{1/2}\left(\tfrac{1}{5}\pi\right),$$

$$\langle 1|21\rangle = -\langle 1|31\rangle = -\alpha \sin^{1/2}\left(\tfrac{2}{5}\pi\right), \tag{4.15}$$

where $\alpha = (\frac{4}{15})^{1/4}$, and the minus signs come from the fact that the operators (21) and (41) are odd under duality. When these matrix elements are used to construct Z_{PM}, and hence the spectrum of $\hat{H}_{\mathrm{M}}$, the important feature is the appearance of the $\sqrt{3}$ factors, which can only arise from terms proportional to $\sin(\frac{1}{6}\pi qq')$ with $qq' = 2, 4, \dots$. We therefore find that $q = 2$ operators play a role in this sector. In fact, the only nonzero values are $\mathcal{N}_{\mathrm{M}}(1,2) = \mathcal{N}_{\mathrm{M}}(4,2) = 1$. $\hat{H}_{\mathrm{M}}$ remains invariant under τ, and by considering Z_{MT} one may show that these two operators are respectively even(odd) under this symmetry.

The complete set of primary scaling operators and their scaling dimensions in all the above sectors are summarized in table 2.

5. Other models

Clearly it is possible to go on and consider other models in the Friedan, Qiu and Shenker [2] classification for which the operator content in the P sector is known [3, 5]. In this section we summarize some partial results we have obtained for the multicritical Ising models with $\mathcal{N}_{\mathrm{P}}(p, q; \bar{p}, \bar{q}) = \delta_{p\bar{p}}\delta_{q\bar{q}}$. These models possess a Z_2 symmetry, with q odd(even) giving the Z_2 even(odd) operators when m is odd, and with p odd(even) giving the Z_2 even(odd) operators when m is even. In carrying through the analysis of $\hat{H}_{\mathrm{A}}$ for the Ising model in sect. 3, we kept m an arbitrary odd integer. Thus eq. (3.5) holds in this more general case. For m even, we similarly find $\mathcal{N}_{\mathrm{A}}(p, q; \bar{p}, \bar{q}) = \delta_{p,m-\bar{p}}\delta_{q,\bar{q}}$. This sector contains operators with half-integral spin which are generalizations of the Ising fermions. The operators with q even(odd) (m odd), and with p even(odd) (m even) are respectively Z_2 even(odd) in the A sector.

In the F sector, we have performed calculations only for $m = 4$, the tricritical Ising model. As we found for the 3-state Potts model, the requirements of the inversion sum rules are not sufficient to determine completely the spectrum of $\hat{H}_{\mathrm{F}}$. By considering Z_{PF} and Z_{AF} we may conclude that the possible nonzero values are $\mathcal{N}_{\mathrm{F}}(1,1) = \mathcal{N}_{\mathrm{F}}(3,1) = 1$ and $\mathcal{N}_{\mathrm{F}}(1,2) = \mathcal{N}_{\mathrm{F}}(3,2) = K$, with $K = 0$ or 1, and (31) and (12) corresponding to Z_2 odd operators. The possibility $K = 1$ is eliminated because it would lead to an additional relevant Z_2 even surface operator, corresponding to (32) with $\tilde{x} = \frac{3}{5}$. Such an operator does not appear in the Landau-Ginzburg-Wilson approach. This means that the leading magnetic surface operator has $\tilde{x} = \frac{3}{2}$, corresponding to $\eta_{\parallel} = 3$. This result agrees with the relation $\eta_{\parallel} = 2/(3\nu - 1)$, derived in ref. [8] for the critical Q-state Potts model. This relation should also be valid for the tricritical model, since they are related [26] by an analytic continuation around $Q = 4$. It would be of interest to obtain further results on the tricritical Ising model, especially in relation to its superconformal invariance properties [27].

In this paper we have shown that the details of the full operator content of a given theory in its different sectors are closely connected with the symmetry properties of that theory. Such an analysis should be useful in clarifying the physical interpretation of other theories in the Friedan, Qiu and Shenker [2] classification, for example those corresponding to $m \equiv 1, 2 \pmod 4$ found in ref. [5].

I thank Stephen Shenker and others of the conformal group at the University of Chicago, as well as P. Kleban, for helpful comments, and G. v. Gehlen and V. Rittenberg for sending me their results before publication.

Appendix

In this appendix we discuss the role of the Virasoro algebra in the semi-infinite geometry, and its relation to the spectrum of the transfer matrix in strips with boundary conditions F through M. Conformal invariance in the half-plane was discussed in ref. [8]. Much of the analysis appropriate to the full plane [1, 10] goes through. Scale invariance, and local rotational and translational invariance, restrict the stress tensor to have two independent components $T(z)$ and $\overline{T}(\bar{z})$. The principal difference is that, if we choose the boundary to be the real axis, the system is invariant under only those conformal transformations which preserve the real axis. These correspond to mappings by real analytic functions $z \to f(z)$, where $\overline{f(z)} = f(\bar{z})$. As a result, the conformal Ward identity mixes up the z and $\bar{z}$ dependence. However, it was shown in ref. [8] that one can regard $\overline{T}(\bar{z})$ as the analytic continuation to the lower half-plane of $T(z)$. The Ward identity reads

$$\langle T(z)\phi_1(z_1, \bar{z}_1)\cdots\rangle = \sum_j \left[\frac{h_j}{(z-z_j)^2} + \frac{1}{z-z_j}\frac{\partial}{\partial z_j} + \frac{\bar{h}_j}{(z-\bar{z}_j)^2} + \frac{1}{z-\bar{z}_j}\frac{\partial}{\partial \bar{z}_j} \right] \times \langle \phi_1(z_1, \bar{z}_1)\cdots\rangle . \tag{A.1}$$

The Virasoro operators are defined in the usual way:

$$L_n = \oint \frac{\mathrm{d}z}{2\pi i} z^{n+1} T(z), \qquad \overline{L}_n = -\oint \frac{\mathrm{d}\bar{z}}{2\pi i} \bar{z}^{n+1} \overline{T}(\bar{z}), \tag{A.2}$$

where the contour is, for example, the unit circle. By the above result, we see that $L_n = \overline{L}_n$, so there is only one Virasoro algebra. We also have

$$L_0 = \int_0^{\pi} \frac{\mathrm{d}\theta}{2\pi} \left[\mathrm{e}^{2i\theta} T(\mathrm{e}^{i\theta}) + \mathrm{e}^{-2i\theta} \overline{T}(\mathrm{e}^{i\theta}) \right]. \tag{A.3}$$

Now, under the conformal transformation $z \to (l/\pi)\ln z$, which takes the half-plane into a strip of width l [10],

$$T \to \left(\frac{l}{\pi z}\right)^2 T + \frac{c}{24 z^2}, \tag{A.4}$$

with a similar result for $\bar{T}$, so that

$$L_0 \to \frac{l}{\pi}\left[\frac{1}{2\pi}\int_0^l (T+\bar{T})\,\mathrm{d}v\right] + \tfrac{1}{24}c, \tag{A.5}$$

where v measures distances across the strip. The expression in square brackets is just the integral of T_{00}, which equals the hamiltonian $\hat{H}_X$ of a quantum field theory defined on the line $0 \leqslant v \leqslant l$. This shows the 1-1 correspondence between eigenstates of $\hat{H}_X$ and scaling operators, which are eigenstates of L_0. The last term gives $-\pi c/24l$ for the ground state energy, in agreement with ref. [15].

Note that although $L_n = \bar{L}_n$, the scaling operators $(L_{-n}\phi)$ and $(\bar{L}_{-n}\phi)$ generated in the operator product expansion of T and $\bar{T}$ with ϕ are not identical. In fact, from the Ward identity (A.1) it follows that [10]

$$\langle [L_{-n}\phi(z,\bar{z})]\,\phi_1(z_1,\bar{z}_1)\cdots\rangle = \mathscr{L}_{-n}\langle\phi(z,\bar{z})\phi_1(z_1,\bar{z}_1)\cdots\rangle, \tag{A.6}$$

where

$$\mathscr{L}_{-n} = \frac{(1-n)\bar{h}}{(z-\bar{z})^n} - \frac{1}{(z-\bar{z})^{n-1}}\frac{\partial}{\partial\bar{z}} + \sum_j\left[\frac{(1-n)h_j}{(z-z_j)^n} + \frac{(1-n)\bar{h}_j}{(z-\bar{z}_j)^n} - \frac{1}{(z-z_j)^{n-1}}\frac{\partial}{\partial z_j} - \frac{1}{(z-\bar{z}_j)^{n-1}}\frac{\partial}{\partial\bar{z}_j}\right], \tag{A.7}$$

with a similar equation for $\bar{L}_{-n}$, with z and $\bar{z}$ interchanged.

References

[1] J.L. Cardy, *in* Phase transitions and critical phenomena, vol. 11, eds. C. Domb and J.L. Lebowitz (Academic Press, London) to be published.
[2] D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. 52 (1984) 1575; *in* Vertex operators in mathematics and physics, Proc. of a conference Nov. 10–17, 1983, ed. J. Lepowsky (Springer, New York, 1984), p. 419.
[3] J.L. Cardy, Nucl. Phys. B270 [FS16] (1986) 186
[4] J.L. Cardy, unpublished
[5] C. Itzykson and J.-B. Zuber, Saclay preprint (1986)
[6] H.W. Diehl and S. Dietrich, Z. Phys. B42 (1981) 65
[7] J.L. Cardy, J. Phys. A17 (1984) L385

[8] J.L. Cardy, Nucl. Phys. B240 [FS12] (1984) 514
[9] K. Binder, *in* Phase transitions and critical phenomena, vol. 8, ed. C. Domb and J.L. Lebowitz (Academic Press, London, 1983).
[10] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, J. Stat. Phys. 34 (1984) 763; Nucl. Phys. B241 (1984) 333
[11] G. v. Gehlen and V. Rittenberg, University of Bonn preprints (1986)
[12] T.W. Burkhardt, J. Phys. A18 (1985) L307
[13] T.W. Burkhardt and I. Guim, J. Phys. A18 (1985) L25, L33
[14] D. Friedan, private communication
[15] H.W. Blöte, J.L. Cardy and M.P. Nightingale, Phys. Rev. Lett. 56 (1986) 742
[16] I. Affleck, Phys. Rev. Lett. 56 (1986) 746
[17] A. Rocha-Caridi, *in* Vertex operators in mathematics and physics, ed. J. Lepowsky (Springer, Berlin, 1984), p. 451
[18] L.P. Kadanoff and H. Ceva, Phys. Rev. B3 (1971) 3918
[19] J.L. Cardy, J. Phys. A17 (1984) L961
[20] B. McCoy and T.T. Wu, Phys. Rev. 162 (1967) 436
[21] Vl.S. Dotsenko, Nucl. Phys. B235 [FS11] (1984) 54
[22] B. Nienhuis and H.J.F. Knops, Phys. Rev. B32 (1985) 1872
[23] D. Friedan and S. Shenker, in preparation
[24] L.P. Kadanoff and A.C. Brown, Ann. of Phys. 121 (1979) 318
[25] L. Dixon, J.A. Harvey, C. Vafa and E. Witten, Nucl. Phys. B261 (1985) 678
[26] B. Nienhuis, N.A. Berker, E.K. Riedel and M. Schick, Phys. Rev. Lett. 43 (1979) 737
[27] D. Friedan, Z. Qiu and S. Shenker, Phys. Lett. 151B (1985) 37