Journal of Physics A: Mathematical and General

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To cite this article: Ian Affleck 2000 *J. Phys. A: Math. Gen.* **33** 6473

J. Phys. A: Math. Gen. **33** (2000) 6473–6479. Printed in the UK PII: S0305-4470(00)13944-7

Edge critical behaviour of the two-dimensional tri-critical Ising model

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Received 12 May 2000

Abstract. Using previous results from boundary conformal field theory and integrability, a phase diagram is derived for the two-dimensional Ising model at its bulk tri-critical point as a function of boundary magnetic field and boundary spin-coupling constant. A boundary tri-critical point separates phases where the spins on the boundary are ordered or disordered. In the latter range of coupling constant, there is a non-zero critical field where the magnetization is singular. In the former range, as the temperature is lowered, the boundary undergoes a first-order transition while the bulk simultaneously undergoes a second-order transition.

Conformal field theory has led to many exact results on two-dimensional critical pheomena with regard to both bulk behaviour and edge or boundary behaviour. (For a review see [1].) Assuming the bulk system is at a critical point, one can consider critical behaviour at the boundary as a function of various fields and interactions applied near the boundary. In general, various boundary phases and critical points exist for a given bulk critical point. These models can be used to describe either two-dimensional classical systems at bulk critical points or else semi-infinite quantum chains at zero temperature. Some of these latter systems find experimental application to strongly correlated electron impurity problems. While the boundary phase diagram of the critical Ising model is well understood [2], surprisingly, this is not so for the next simplest case, the tri-critical Ising model. Six conformally invariant boundary conditions (b.c.'s) have been constructed using the fusion method by Cardy [2], which should correspond to boundary critical points. Certain integrable renormalization group (RG) flows between these critical points have been constructed by Chim [3]. The purpose of this paper is simply to connect the points with a phase diagram written in terms of microscopic parameters. This is shown schematically in figure 1. The most surprising conclusion is perhaps the existence of a phase, in zero magnetic field, where the spins on the boundary exhibit long-range order while those in the bulk do not. This is the physical interpretation of a b.c. for which the corresponding boundary state is a sum of boundary states corresponding to a spin-up or spin-down b.c.

We first briefly review the simpler case of the ordinary critical Ising model. In that case, Cardy identified only three conformally invariant b.c.'s corresponding to spin up, spin down and free. There are no relevant boundary operators at the spin-up/down critical points, indicating

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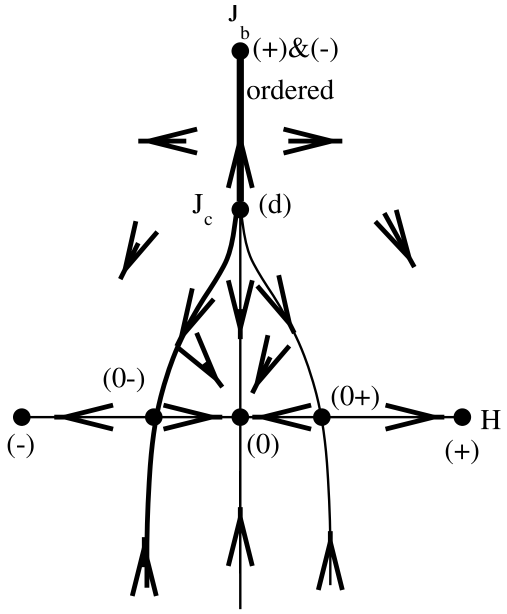

Figure 1. Schematic phase diagram of the boundary tri-critical Ising model. Arrows indicate directions of RG flows as the length scale is increased. Along the thick line the spins on the boundary are ordered.

that they are stable against the addition to the Hamiltonian of arbitrary perturbations located near the boundary. On the other hand, the free b.c. has one relevant operator, of dimension $x = \frac{1}{2}$. (Boundary operators are relevant if they have scaling dimension $x < 1$.) The corresponding relevant coupling constant is naturally interpreted as a boundary magnetic field. By standard scaling arguments we expect the boundary magnetization to scale with boundary field as

$$|m| \propto |H|^{x/(1-x)} \propto |H|. \tag{1}$$

This behaviour is expected to be independent of the details of the spin-coupling constants near the boundary. (Only the constant of proportionality in equation (1) will depend on these coupling constants, not the exponent.) No additional critical points are expected at any finite field. The RG flow between free and fixed b.c.'s was shown to be integrable by Ghoshal and Zamalodchikov [4] in a pioneering paper.

The phase diagram for the tri-critical Ising model is more interesting. This model can be defined as a spin-1 Ising model with a crystal field term which favours the $S = 0$ state over $S = \pm 1$ or equivalently a diluted spin-$\frac{1}{2}$ Ising model. The (classical) Hamiltonian is

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} S_i S_j - \mu \sum_i S_i^2 \qquad (S_i = -1, 0, 1). \tag{2}$$

The schematic (bulk) phase diagram is drawn in figure 2. There is a second-order phase transition line in the Ising universality class and also a first-order phase transition line separating phases with unbroken and broken symmetry. These lines join at a tri-critical point. By an approximate transfer matrix mapping, one can show that this classical model exhibits the same critical behaviour as a quantum chain at $T = 0$. This model has the Hamiltonian

$$\mathcal{H} = -\sum_i [S_i^z S_{i+1}^z - D(S_i^z)^2 + H_{\mathrm{T}} S_i^x]. \tag{3}$$

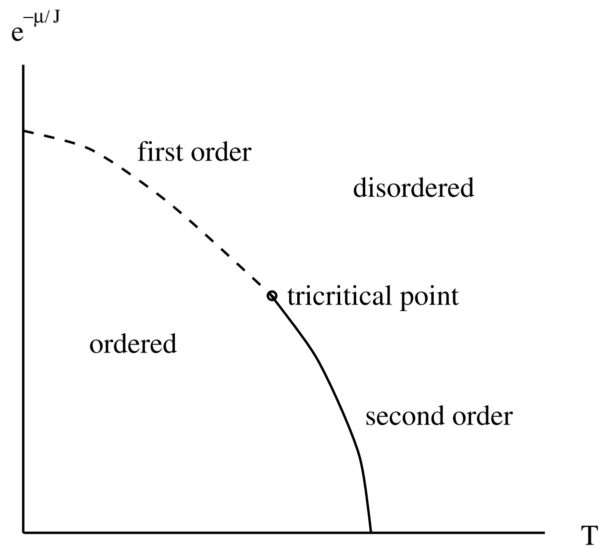

Figure 2. Schematic bulk phase diagram of the spin-1 Ising model of equation (2).

S_j^a now label quantum $S = 1$ operators. The transverse field H_T now controls the temperature in the classical model. While the ordinary Ising critical point corresponds to the simplest conformal field theory with central charge, $c = \frac{1}{2}$, the tri-critical point corresponds to the next unitary minimal model with $c = \frac{7}{10}$.

We now wish to consider the classical model on a semi-infinite half-plane with a boundary as shown in figure 3. Although various microscopic boundary interactions could be considered, for our purposes it is enough to consider a boundary field, H, and a modified interaction, J_b, along the boundary. These couplings are indicated in figure 3. In the corresponding quantum chain, the Hamiltonian is

$$\mathcal{H} = -\sum_{i=0}^{\infty} S_i^z S_{i+1}^z + \sum_{i=1}^{\infty}[-H_T S_i^x + D(S_i^z)^2] - H_{Tb} S_0^x + D_b (S_0^z)^2 - H S_0^z. \quad (4)$$

Roughly speaking, increasing the boundary interaction, J_b, in the classical model corresponds to decreasing $|H_{Tb}|$ and D_b in the quantum model, thus enhancing the tendency for the spins to order at the boundary.

The phase diagram in figure 1 can be deduced rather straightforwardly from the properties of the six conformally invariant boundary states found by Cardy and discussed by Chim, and from the integrable RG flows discussed by Chim. We first review these boundary states. Using the fusion approach, one finds six boundary states corresponding to the six primary fields in the (bulk) tri-critical Ising model. Their physical properties have been elucidated, to some extent, by Chim, and we use his notation for them. There are two states corresponding to spin-up and spin-down b.c.'s (±). There are two more b.c.'s (0±) which also break the Z_2 symmetry but appear to have the spins at the boundary only partially polarized. (Chim actually labelled the negative polarization b.c. (−0) rather than (0−) but we prefer the latter notation.) (0) is the free b.c. [5]. Finally there is one more b.c. which does not break the Z_2 symmetry and is labelled (d) (for degenerate). The correspondence between the fusion label and the physical

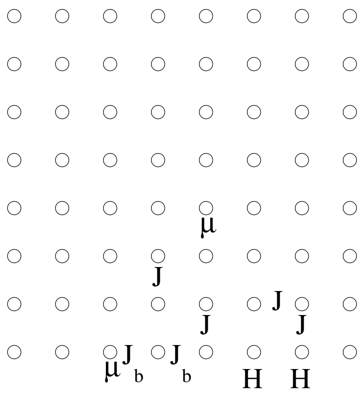

Figure 3. Couplings and field for the boundary tri-critical Ising model. The bulk parameters, J and μ, are adjusted to the tri-critical point. The magnetic field is applied only at the boundary.

label for the corresponding boundary states is

$$\begin{aligned}
|\tilde{0}\rangle &= |(-)\rangle \\
|\tilde{\tfrac{3}{2}}\rangle &= |(+)\rangle \\
|\tilde{\tfrac{1}{10}}\rangle &= |(0-)\rangle \\
|\tilde{\tfrac{3}{5}}\rangle &= |(0+)\rangle \\
|\tilde{\tfrac{7}{16}}\rangle &= |(0)\rangle \\
|\tilde{\tfrac{3}{80}}\rangle &= |(\mathrm{d})\rangle.
\end{aligned} \tag{5}$$

By the standard fusion rules, the (primary) boundary operator content with the b.c. corresponding to the state $|\tilde{a}\rangle$ is the set of operators appearing in the (bulk) operator product expansion (OPE) of $\mathcal{O}_a \times \mathcal{O}_a$. It thus follows that the ($\pm$) b.c.'s admit no relevant operators; they are completely stable. (The only operator appearing in the OPE is the identity operator which just corresponds to the possibility of adding a c-number to the quantum Hamiltonian, having no effect on the critical behaviour.) The only primary operator at the free b.c. (0) has dimension $\frac{3}{2}$. This follows from the OPE

$$\sigma' \times \sigma' = I + \epsilon'' \tag{6}$$

where σ' and ϵ'' are the primary fields of dimension $\frac{7}{16}$ and $\frac{3}{2}$ respectively. Thus the free b.c. is also a stable fixed point! This is a somewhat surprising result since it implies that adding a boundary magnetic field does not destabilize the free fixed point in the tri-critical Ising model, unlike what happens in the ordinary Ising model. The partially polarized b.c.'s ($0\pm$) have one relevant boundary operator with $x = \frac{3}{5}$. Thus, there should be one unstable direction and one stable direction in the RG flow in the (J_b, H) plane at the corresponding fixed points. The (d) b.c. has two relevant boundary operators of dimension $x = \frac{1}{10}$ and $\frac{3}{5}$, so both directions should be unstable at this fixed point. Finally, it is important to consider the b.c. labelled (+)&(−) by Chim. The corresponding boundary state is

$$|(+)\&(-)\rangle = |(+)\rangle + |(-)\rangle = |\tilde{0}\rangle + |\tilde{\tfrac{3}{2}}\rangle. \tag{7}$$

The corresponding OPE is

$$[I + \epsilon''] \times [I + \epsilon''] = 2[I + \epsilon'']. \tag{8}$$

Thus there should be only one relevant boundary operator, of dimension $x = 0$, at this critical point (disgarding the identity operator, which is always present and has no effect). The presence of a non-trivial boundary operator with dimension zero is the hallmark of an ordered phase, or equivalently a first-order phase transition with magnetic field. It is natural to associate this operator with a boundary magnetic field. The usual scaling law,

$$|m| \propto |H|^{x/(1-x)} \tag{9}$$

implies $|m| \propto |H|^0$, i.e. a discontinuous jump in m as H passes through zero. This in turn implies long-range order in zero field. It is also noteworthy that there are no additional relevant operators with the (+)&(−) b.c. Thus we expect it is stable against small variations of J_b at $H = 0$. This is different from the other combination (0+) and (0−) which has, in addition to an $x = 0$ boundary operator, two other relevant boundary operators with $x = \frac{2}{5}$. It is also different from the situation in the ordinary Ising model, where the combination of spin up and down gives three relevant boundary operators. The identification of a boundary state which is a sum of two or more other boundary states with long-range order was also made in the context of a critical line separating two semi-infinite Ising planes [6] and in the boundary three-state Potts model [7]. However, in both those cases it is an unstable fixed point, even in zero field. The somewhat unusual feature of the tri-critical Ising model is that the broken-symmetry phase is stable. Related phenomena also occur in quantum Brownian motion on a triangular lattice [8].

It is now a relatively straightforward matter to connect the points to obtain the schematic phase diagram of figure 1. Several comments about this phase diagram are in order. The flows from (d) to (0) and (+)&(−) and from (0+) to (0) and (+) are integrable. Since there is only one relevant operator at the (0±) fixed points there must be lines in the (J_b, H) plane which flow towards them. These lines must end at the tri-critical point (d) since it is the only fixed point with two relevant operators. The values of J_b at the fixed points are in general unknown (and are presumably *not* equal at five of the fixed points as drawn in figure 1). However, we do expect that J_b is smaller at (0) than at (d). It is natural to place the (+)&(−) fixed point at $J_b = \infty$ since there the spins along the boundary are perfectly ordered. The values of J_b at the (±) and (0±) fixed points are relatively insignificant and simply correspond to points where the leading irrelevant coupling constant vanishes. Although a boundary field is irrelevant in the central phase of the phase diagram this *does not* imply that the boundary magnetization is zero in the presence of a non-zero field. Irrelevant operators will still lead to a non-zero magnetization which should be an *analytic* function of H, thus being linear at small H. At the (0±) fixed points, we expect $m(H)$ to be singular, behaving as

$$m - m_c - a(H - H_c) \to b_\pm |H - H_c|^{x/(1-x)} = b_\pm |H - H_c|^{3/2} \tag{10}$$

since $x = \frac{3}{5}$. Here the amplitudes, $b_\pm$, are presumably different for $H > H_c$ and $H < H_c$. The linear term, $\propto a$, is non-singular. This is a relatively mild singularity since both m and its first derivative remain finite and continuous, while the second derivative diverges. The shape of the phase boundary near (d) is determined from the scaling dimension of H and $J_b - J_c$ to be

$$J_c - J_b(H) \propto |H|^{4/9}. \tag{11}$$

At the tri-critical point (d),

$$|m| \propto |H|^{1/9} \tag{12}$$

and, for $J_b > J_c$, m has a first-order jump at $H = 0$.

These results are all consistent with the 'g-theorem' [9] that states that the ground-state degeneracy, g, always decreases during a boundary RG flow. The g-values are given by

$$\begin{aligned} g_{(\pm)} &= C \\ g_{(0\pm)} &= C\eta^2 \\ g_{(0)} &= \sqrt{2}C \\ g_{(\mathrm{d})} &= \sqrt{2}\eta^2 C \\ g_{(+)\&(-)} &= 2C \end{aligned} \tag{13}$$

where

$$C = \sqrt{\frac{\sin\frac{\pi}{5}}{\sqrt{5}}} \qquad \eta = \sqrt{\frac{\sin\frac{2\pi}{5}}{\sin\frac{\pi}{5}}}. \tag{14}$$

Noting that $\eta^2 \approx 1.618\,03 > \sqrt{2}$ we see that all RG flows in figure 1 are consistent with the g-theorem. This was observed by Chim in the special cases of integrable flows.

The existence of the ordered line may seem somewhat surprising since there is long-range order at finite temperature along the (one-dimensional) boundary, for $J_{\mathrm{b}} > J_{\mathrm{c}}$, even though the (two-dimensional) bulk is disordered (or, more accurately, is sitting at a critical point separating ordered and disordered phases). This is surely reasonable at $J_{\mathrm{b}} \to \infty$ but may be harder to swallow for finite J_{b}. In the quantum chain context this behaviour is not so unfamiliar. We may think of the $(+)\&(-)$ fixed points as corresponding to $H_{\mathrm{Tb}} = 0$. In this limit S_0^z commutes with the Hamiltonian and there are two degenerate ground states, with $S_0^z = \pm 1$. (We assume that D_{b} is sufficiently large and negative that these states have lower energy than the one with $S_0^z = 0$.) These ground states have non-zero, equal and opposite values of the magnetization, localized near the boundary. Applying an infinitesimal boundary field picks out one of these two ground states, leading to the discontinuity. The above RG analysis implies that a small transverse boundary field is *irrelevant* so the jump in the magnetization at $H = 0$ should persist for a range of non-zero H_{Tb}. A somewhat related phenomena occurs in the *ferromagnetic* Kondo problem. The Kondo coupling constant is irrelevant in this case so that the impurity spin decouples from the conduction electrons at the stable fixed point. Since the impurity spin operator then commutes with the Hamiltonian, there are degenerate ground states and a discontinuous magnetization in an applied field.

It is interesting to consider varying T through the bulk phase transition (with the Hamiltonian held fixed). Referring to figure 2, we see that for large μ the bulk transition is continuous and in the usual Ising universality class. In this case the boundary also orders continuously. (This follows from the fact that the $(+)\&(-)$ fixed point is unstable, even in zero field, in the ordinary Ising model.) On the other hand, for smaller μ the bulk transition is first order. We then expect the boundary transition to also be first order since critical behaviour of the boundary at a non-zero T is presumably impossible unless the bulk is also critical. Now consider what happens if μ is adjusted to its critical value so that the bulk transition is in the tri-critical universality class. In this case there are two possibilities for the boundary transition, depending on J_{b}. When $J_{\mathrm{b}} < J_{\mathrm{c}}$, the boundary transition is also second order. However, when $J_{\mathrm{b}} > J_{\mathrm{c}}$, the boundary undergoes a type of first-order transition while the bulk undergoes a second-order transition. This follows from observing that, infinitesimally below the critical temperature, the boundary magnetization is finite whereas the bulk magnetization is infinitesimal. On the other hand, infinitesimally above T_{c}, the correlation lengths in both bulk and boundary are presumably diverging together. We may understand the possibility of the boundary having a first-order transition at the bulk tri-critical point as being connected with the fact that the bulk system is at the end of a first-order transition line. When the bulk transition is 'almost first order' it becomes possible for the boundary transition to be truly first order.

It was recently observed [10] that the integrable RG flow from (d) to (+)&(−) has a generalization to all the minimal models with diagonal partition functions, which may be thought of as increasingly multi-critical Ising models and that, in all cases, the (+)&(−) fixed point is stable except for a dimension zero operator. This implies that all these models have an ordered phase at zero field, as discussed here for the tri-critical case.

Acknowledgments

I would like to thank M Oshikawa and H Saleur for numerous useful discussions on related subjects and H Saleur for drawing my attention to [10]. This research was supported by NSF grant PHY-94-07194 and by NSERC of Canada.

References

[1] Di Francesco P, Mathieu P and Sénéchal C 1997 *Conformal Field Theory* (Berlin: Springer)
[2] Cardy J L 1989 *Nucl. Phys.* B **324** 581
[3] Chim L 1996 *J. Math. Phys.* A **11** 4491
[4] Ghoshal S and Zamolodchikov A B 1994 *Int. J. Mod. Phys.* A **9** 3841
[5] Saleur H and Bauer M 1989 *Nucl. Phys.* B **320** 591
[6] Oshikawa M and Affleck I 1997 *Nucl. Phys.* B **495** 533
[7] Affleck I, Oshikawa M and Saleur H 1998 *J. Phys. A: Math. Gen.* **31** 5827
[8] Yi H and Kane C L 1998 *Phys. Rev.* B **57** R5579
Affleck I, Oshikawa M and Saleur H 2000 in preparation
[9] Affleck I and Ludwig A W W 1991 *Phys. Rev. Lett.* **67** 161
[10] Recknagel A, Roggenkamp D and Schomerus V 2000 *Preprint* hep-th/0003110 (*Nucl. Phys.* B, at press)