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BOUNDARY CONDITIONS, FUSION RULES AND THE VERLINDE FORMULA

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Boundary operators in conformal field theory are considered as arising from the juxtaposition of different types of boundary conditions. From this point of view, the operator content of the theory in an annulus may be related to the fusion rules. By considering the partition function in such a geometry, we give a simple derivation of the Verlinde formula.

1. Introduction

Recently there has been considerable progress made in understanding the problem of classifying conformal theories, following the observation of E. Verlinde [1] that the fusion rules of the underlying algebra are related by formula

$$\sum_i S_i^j N_{kl}^i = S_k^j S_l^j / S_0^j \tag{1}$$

to the elements S_i^j of the matrix which represents the modular transformation $\tau \to -1/\tau$ acting on the Virasoro characters.

A conformal field theory defined on a manifold without boundaries has as its underlying symmetry two algebras $\mathscr{A}$ and $\bar{\mathscr{A}}$ which act respectively on the holomorphic (z) and antiholomorphic ($\bar{z}$) dependences of the physical fields of the theory. In a rational conformal field theory, the irreducible representations of these algebras are constructed by acting on a highest weight vector with all possible lowering operators, and then projecting out null states. The fusion rule coefficients N_{jk}^i of the algebra $\mathscr{A}$ give the number of distinct ways that the representation i occurs in the "fusion" of two fields transforming according to the representations j, k respectively. This process of fusion corresponds to considering only the holomorphic, or only the antiholomorphic part of the operator product expansion of two physical operators. The decoupling of the null states after this fusion process then gives strong constraints on the N_{jk}^i, first analysed for the case of the Virasoro algebra by Belavin et al. [2].

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This description is somewhat heuristic, however, because the holomorphic part of a physical field is not itself physical. Its correlation functions are multivalued, and are not defined, even heuristically, by a path integral. In order to make the above procedure well-defined, it has proved necessary [3] to adopt an axiomatic approach to the study of these objects, abstracting from known examples such properties which will lead to sensible physical correlation functions when the holomorphic and antiholomorphic parts are sewn together appropriately. While this procedure is in the end justified by its results, it is desirable to have a more physical way of understanding such results as eq. (1).

When conformal field theory is defined on a manifold with a boundary, there is only one algebra, rather than two. This is because, as will be explained later, the holomorphic and antiholomorphic fields $W^{(r)}$ and $\overline{W}^{(r)}$ which generate the left and right algebras are not independent of each other on the boundary. As a result, a single algebra is realized directly on the physical fields of the theory. The partition function of the theory defined on an annulus turns out to be expressible as a linear combination of characters of the algebra (rather than a sesquilinear combination as in the case of the torus), and the modular transformation $\tau \to -1/\tau$ has a simple physical interpretation. When boundaries are present, we have some freedom in choosing the boundary condition consistent with the extended algebra. As we shall discuss, there is an isomorphism between the set of possible boundary conditions and the space of conformal blocks. By considering annuli with different types of boundary conditions it is then possible to relate their partition functions to the fusion rules. In fact we shall show that* for boundary conditions of type j and k on either boundary of the annulus, the partition function is simply

$$Z = \sum_i N_{jk}^i X_i(q), \tag{2}$$

where q is the modular parameter of the annulus. Combining this result with the modular properties of Z then leads to the Verlinde formula eq. (1). Actually, as will become apparent, our arguments are not complete in the general case when different representations of $\mathscr{A}$ have the same Virasoro character. This is because we work only at the level of the partition function, which is color-blind to such distinctions. Nevertheless, as will be discussed in sect. 3, we believe that a deeper discussion of the operator content will lead to a derivation covering all cases.

There are other reasons for studying conformal field theory on manifolds with boundaries. One is the obvious connection with open string theory. Another relates to the connection with statistical mechanics. Given a conformal field theory, it is useful to know an underlying lattice model whose continuum limit at criticality the theory represents. There are indications that for every conformal field theory

* In fact, eq. (2) is slightly modified for non-selfconjugate representations, as discussed in sect. 3.

obtained by the GKO coset construction [4], there exists some lattice model which is integrable not only at the critical point, but also on some one-parameter curve in the coupling constant space away from criticality [5]. Thus in general one would like to solve the inverse problem: starting from some conformal field theory, construct the solution of some integrable lattice model away from criticality, and then perhaps even calculate the Boltzmann weights. The first step in this program would be to identify the appropriate microscopic degrees of freedom. One may then hope that fixing the boundary degrees of freedom into one of these microscopic states will lead, in the continuum limit, to a conformally invariant boundary condition of the type we consider in this paper. This has been already understood for simple models like the Ising model and the 3-state Potts model [6] and, more recently, Saleur and Bauer [7], and Saleur [8], have obtained results for the A-D-E series of models with $c<1$, and for the $\mathrm{SU}(N)$ vertex models. In such cases, we can show that the result (2) is consistent with their findings. In general, then, one may hope to build appropriate models by choosing microscopic states labelled by highest weight representations of the algebra of the conformal field theory.

The layout of this paper is as follows. In sect. 2 we recall some basic facts about conformal field theories on manifolds with boundaries, and establish our notation. In sect. 3, we discuss conformal field theory on an annulus, analysing more carefully the idea of a boundary state, discussing modular properties, and finally deriving eq. (1). In sect. 3, we compare eq. (2) with known results for lattice models.

2. Boundary operators in conformal field theory

In this section we recall some basic ideas of conformal field theory on a manifold with a boundary [9, 6]. For this purpose, the prototype geometry is that of the upper-half plane $\mathrm{Im}\, z>0$. We suppose that some boundary condition, labelled by (α), is prescribed on the real axis. The action of the conformal field theory is locally invariant under infinitesimal conformal transformations $z \to z+a(z)$, where, in order to preserve the geometry, $a(z)$ must be real when z lies on the real axis. Thus, if $a(z)=\Sigma_n a_n z^{n+1}$, the parameters a_n should be real. As we shall see, this has the consequence that there is only one set of corresponding Virasoro generators L_n, rather than $(L_n, \bar{L}_n)$ as in the usual case.

In deriving the analog of the conformal Ward identity, the global variation of the action is written as usual as a contour integral

$$\frac{1}{2\pi i}\oint_{\mathrm{C}} a(z)T(z)\,\mathrm{d}z-\frac{1}{2\pi i}\oint_{\mathrm{C}} a(\bar{z})\bar{T}(\bar{z})\,\mathrm{d}\bar{z}, \tag{3}$$

where C is a sufficiently large contour, which surrounds all the points at which correlation functions are to be evaluated. It is convenient to take C to be a large semicircle C_+ in the upper-half plane, together with a portion of the real axis. In order that the conformal Ward identity continues to be valid as the arguments of the correlation functions approach the real axis, the contribution to eq. (3) from this portion of the contour should vanish. Thus we insist that $T = \bar{T}$ on the real axis. This is a necessary condition on the type of boundary condition (α) we consider. In cartesian components, it is equivalent to $T_{xy} = 0$, expressing in field theory terms the fact that no momentum flows across the boundary.

The theory is now developed in analogy with that in the plane. The Virasoro generators are

$$L_n = \frac{1}{2\pi i}\int_{C_+} z^{n+1}T(z)\,dz - \frac{1}{2\pi i}\int_{C_+} \bar{z}^{n+1}\bar{T}(\bar{z})\,d\bar{z}. \tag{4}$$

From the boundary condition we see that $\bar{T}(\bar{z})$ may be taken as the analytic continuation of $T(z)$ into the lower-half plane, so that the terms in eq. (4) may be combined as

$$L_n = \frac{1}{2\pi i}\oint_C z^{n+1}T(z)\,dz, \tag{5}$$

where C is a full circle. It is then clear, by analogy with the usual case, that the L_n satisfy the Virasoro algebra.

In radial quantization, the fact that $T(z)$ is non-singular as $z \to 0$ implies that the vacuum state $|0\rangle$ satisfies $L_n|0\rangle = 0$ for $n \geqslant -1$. Highest weight states of irreducible representations of the Virasoro algebra are created by primary operators $\phi(0)$ acting on $|0\rangle$. Note that ϕ is a *boundary* operator. Its L_0 eigenvalue is a boundary, or surface, scaling dimension, which determines, for example, how its two-point function decays as a function of distance along the boundary. A boundary operator and its bulk counterpart have, in general, different scaling dimensions. For example, in the Ising model with free boundary conditions, the spin operator has a boundary scaling dimension of $\frac{1}{2}$, while in the bulk its dimensions are $(\frac{1}{16}, \frac{1}{16})$. An operator which is primary with respect to the usual bulk Virasoro algebras may have no primary counterpart in the set of boundary operators. For example, in the same model the energy operator is primary in the bulk, but appears as a descendant of the identity operator when classified according to the irreducible representations of the boundary Virasoro algebra.

Now consider a related geometry, that of an infinitely long strip, whose width we take to be π in suitable units. This is related to the upper half plane by the conformal mapping $w = \ln z$. Writing $w = t + i\sigma$, the generator of t-translations, or hamiltonian, H, is given in terms of the generator of scale transformations in the upper half plane, L_0, by [6, 10]

$$H = \int_0^{\pi} T_{tt}\,d\sigma = L_0 - \frac{c}{24}, \tag{6}$$

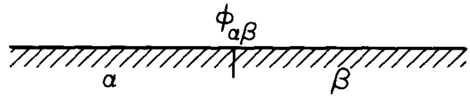

Fig. 1. Discontinuity in the boundary conditions corresponds to the insertion of a boundary operator.

where the second term comes from the schwartzian derivative in the transformation law for T. Likewise, the other Virasoro generators may be identified with appropriate Fourier components of T_{tt} and $T_{t\sigma}$. As in the upper-half plane, they satisfy a Virasoro algebra. In the case we have described, the boundary conditions on either side of the strip will be of the same type (α). However, the existence of the Virasoro algebra depends only on the fact that the boundary conditions are conformally invariant, that is $T_{t\sigma} = 0$ on $\sigma = 0, \pi$. Thus we are free to consider the more general case when the boundary conditions on either side of the strip are not necessarily the same. We shall label this pair of boundary conditions by $(\alpha\beta)$, and call the corresponding hamiltonian $H_{\alpha\beta}$. The eigenstates of $H_{\alpha\beta}$ will fall into irreducible representations of the Virasoro algebra. Denote by $n^h_{\alpha\beta}$ the number of times that the irreducible representation with highest weight h occurs in the spectrum of $H_{\alpha\beta}$. Note that $n^0_{\alpha\alpha} = 1$.

If we transform this more general boundary condition $(\alpha\beta)$ back to the upper-half plane, we have the situation shown in fig. 1. There is a discontinuity in the boundary condition at $z = 0$. In radial quantization, this corresponds to a "vacuum" state which is no longer annihilated by L_{-1}. We may consider this state as equivalent to the action of a boundary operator $\phi_{\alpha\beta}(0)$ acting on the true vacuum $|0\rangle$. It is a highest weight state with weight $h_{\alpha\beta}$ equal to the lowest value of h for which $n^h_{\alpha\beta} \neq 0$. If we act on this state with other local operators, we obtain the other representations with non-zero $n^h_{\alpha\beta}$.

Thus we arrive at one of the fundamental ideas underlying the arguments of this paper: juxtaposition of different (conformally invariant) boundary conditions is equivalent to the insertion of boundary operators. This is similar to the definition of disorder and twist operators in the bulk. It is not necessarily true that all boundary operators are of this form, but it will be seen later that all possible highest weights h may be realized by an appropriate choice of $(\alpha\beta)$.

To use the Ising model once more as an example, there are three distinct conformally invariant boundary conditions: when the boundary spins are all in the $s = 1$ state, all in the $s = -1$ state, or are all free. These we denote by $(+), (-)$ and (f). The non-zero $n^h_{\alpha\beta} = n^h_{\beta\alpha}$ are then [6]

$$\begin{aligned} (++) \quad \text{or} \quad (--): &\quad h = 0 \\ (ff): &\quad h = 0, \tfrac{1}{2} \\ (+-): &\quad h = \tfrac{1}{2} \\ (+f) \quad \text{or} \quad (-f): &\quad h = \tfrac{1}{16}\,. \end{aligned} \tag{7}$$

In the case (ff), the lowest representation has $h=0$, but by inserting a spin operator at $z=0$, we get the representation with $h=1/2$. With the other boundary conditions, insertion of a spin operator does not change the representation. This is because the boundary conditions already break the Z_2 spin symmetry of the bulk theory.

Note that since the two-point function $\langle\phi_{\alpha\beta}(x_1)\phi_{\beta\alpha}(x_2)\rangle$ is evidently non-zero, $\phi_{\alpha\beta}$ and $\phi_{\beta\alpha}$ should be considered as conjugate operators. If the discontinuity in the boundary condition does not occur at the origin, but rather at $z=x_1$, then the corresponding state $\phi_{\alpha\beta}(x_1)|0\rangle = e^{x_1 L_{-1}}\phi_{\alpha\beta}|0\rangle$ is no longer an eigenstate of L_0, but nevertheless lies in the representation corresponding to the highest weight $h_{\alpha\beta}$. Similarly, other local operators acting at z_1 will give states lying in the other representations with $n^h_{\alpha\beta}\neq 0$, although these will not be highest weight states.

So far, we have discussed only the Virasoro algebra. In many interesting conformal field theories there also exists an extended algebra, generated by currents in addition to the stress tensor. The above considerations may be extended to this case, if we consider boundary conditions which are also invariant under this larger symmetry. In the upper-half plane, this is equivalent to requiring that $W^{(r)}=\overline{W}^{(r)}$ on the real axis, where $\{W^{(r)},\overline{W}^{(r)}\}$ represents the set of conserved currents. In this case, the eigenstates of $H_{\alpha\beta}$ will be organized into highest weight representations of the extended algebra. These representations will be labelled by an index i whose specification includes the L_0-eigenvalue of the highest weight state. We then define the non-negative integer $n^i_{\alpha\beta}$ to be the number of times that the representation i occurs in the spectrum of $H_{\alpha\beta}$. In the case of extended algebras, it is possible to have representations which are not self-conjugate. In this case, we denote the representation conjugate to i by $i^\vee$. From the previous argument we see that $n^i_{\alpha\beta}=n^{i^\vee}_{\beta\alpha}$. It is, of course, also possible, and indeed interesting, to consider boundary conditions which break the extended symmetry, but which are nevertheless conformally invariant. We shall not consider such a situation in this paper.

3. Conformal field theory on an annulus

Suppose that the strip is now made periodic in the t-direction, so that different values of t are identified modulo $2\pi\,\mathrm{Im}\,\tau$. Here τ is a complex number which, for the purposes of this paper, will always be purely imaginary. The manifold is now topologically an annulus. The modular parameter is $q\equiv e^{2\pi i\tau}$. The partition function in this geometry is then

$$Z_{\alpha\beta}(q)=\mathrm{Tr}\,e^{-(2\pi\,\mathrm{Im}\,\tau)H_{\alpha\beta}}=\sum_i n^i_{\alpha\beta}\chi_i(q), \tag{8}$$

where $\chi_i(q)\equiv q^{-c/24}\,\mathrm{Tr}_i q^{L_0}$ is the Virasoro character of the representation i.

The partition function on a torus may be written as a sesquilinear form in the characters [11]. For a rational conformal field theory, when the number of characters is finite, the invariance of this partition function under modular transformations implies that the characters themselves transform linearly. Under the modular transformation $\tau \to -1/\tau$,

$$\chi_i(q) = \sum_j S_i^j \chi_j(\tilde{q}), \tag{9}$$

where $\tilde{q} = \mathrm{e}^{-2\pi i/\tau}$, so that

$$Z_{\alpha\beta}(q) = \sum_{i,j} n^i_{\alpha\beta} S_i^j \chi_j(\tilde{q}). \tag{10}$$

On the other hand, we may calculate this partition function using the hamiltonian acting in the σ-direction. This will be the hamiltonian $H^{(P)}$ for the cylinder, which is related by the exponential mapping $\zeta = \mathrm{e}^{-i(t+i\sigma)}$ to the Virasoro generators in the whole ζ-plane by

$$H^{(P)} = (\mathrm{Im}\tau)^{-1}\left(L_0^{(P)} + \overline{L}_0^{(P)} - \frac{c}{12}\right), \tag{11}$$

where we have used the superscript (P) to stress that they are not the same as the generators of the boundary Virasoro algebra. The partition function is then

$$Z_{\alpha\beta}(q) = \langle\alpha|\mathrm{e}^{-\pi H^{(P)}/\mathrm{Im}\tau}|\beta\rangle = \langle\alpha|\left(\tilde{q}^{1/2}\right)^{L_0^{(P)} + \overline{L}_0^{(P)} - c/12}|\beta\rangle, \tag{12}$$

where $|\alpha\rangle$ and $|\beta\rangle$ are boundary states. Since we now have a well-defined Hilbert space, on which the $(L_n^{(P)}, \overline{L}_n^{(P)})$ act, we may describe these states more precisely. The boundary conditions are $\mathrm{e}^{i\pi s/2}W^{(r)} = \mathrm{e}^{-i\pi s/2}\overline{W}^{(r)}$ on $\sigma = 0, \pi$. Here s is the spin of W. The phase factors arise from the factor of i in the exponential mapping. The generators $(W_n^{(r)}, \overline{W}_n^{(r)})$ are Fourier components of these operators with respect to t. Then it is straightforward to show that the boundary condition is equivalent to

$$\left(W_n^{(r)} - (-1)^s \overline{W}_{-n}^{(r)}\right)|\alpha\rangle = 0 \tag{13}$$

together with a similar condition on $|\beta\rangle$. Note in particular that $(L_n - \overline{L}_{-n})|\alpha\rangle = 0$, reflecting the reparametrization invariance of the boundary state.

The solution of the conditions in eq. (13) has been analysed by Ishibashi [12] and Onogi and Ishibashi [13]. Consider the representation j of the antiholomorphic algebra $\{\overline{W}^{(r)}\}$. Its states are linear combinations of states of the form $\Pi_l \overline{W}_{-n_l}^{(r_l)} \overline{|j;0\rangle}$, where the $\overline{W}_{-n_l}^{(r_l)}$ are lowering operators and $\overline{|j;0\rangle}$ is the highest weight state. Define the (anti-unitary) operator U acting on this space by

$$U\overline{|j;0\rangle}=\overline{|j;0\rangle}^*, \qquad U\overline{W}^{(r_l)}_{-n_l}=(-1)^{s_{r_l}}\overline{W}^{r_l}_{-n_l}U. \tag{14}$$

Note that this definition is consistent with projecting out the null states to obtain the irreducible representation. Let us denote an orthonormal basis of the representation j of the holomorphic algebra by $|j;N\rangle$, and the corresponding basis of the isomorphic representation of the antiholomorphic algebra by $\overline{|j;N\rangle}$. Then the state

$$|j\rangle\equiv\sum_N|j;N\rangle\otimes U\overline{|j;N\rangle} \tag{15}$$

is a solution of eq. (13). To see that eq. (15) satisfies eq. (13), consider the vectors $\langle k;N_1|\otimes U\overline{\langle l;N_2|}$, which form a basis for the dual space. Then

$$\begin{aligned}
&\langle k;N_1|\otimes U\overline{\langle l;N_2|}\left(W_n^{(r)}-(-1)^s\overline{W}^{(r)}_{-n}\right)|j\rangle\\
&=\sum_N\langle k;N_1|W_n^{(r)}|j;N\rangle\overline{\langle l;N_2|j;N\rangle}^*\\
&\quad-(-1)^s\sum_N\langle k;N_1|j;N\rangle\overline{\langle l;N_2|}U^\dagger\overline{W}^{(r)}_{-n}U\overline{|j;N\rangle}\\
&=\delta_{kj}\delta_{lj}\left(\langle j;N_1|W_n^{(r)}|j;N_2\rangle-\langle j;N_2|W^{(r)}_{-n}|j;N_1\rangle^*\right)\\
&=0,
\end{aligned} \tag{16}$$

where we have used the orthonormality of the basis, and the hermiticity property $W_n^{(r)\dagger}=W^{(r)}_{-n}$.

So far, we have not specified which particular theory we are considering. For a given extended algebra, with a given value of the central charge c, there are in general several different theories corresponding to different modular invariant combinations of characters on the torus. Now we specialize to the "diagonal" theories, that is, to those whose partition function on the torus is the diagonal combination $Z_{\rm torus}=\Sigma_i\chi_i(q)\chi_i(\bar q)$. For these theories, each representation j appears just once in the spectrum of $H^{(P)}$ in the sector in which $L_0=\overline{L}_0$. Therefore the states $|j\rangle$ defined in eq. (15) form a complete set in the space of conformal blocks. Writing, therefore, $\langle\alpha|=\langle\alpha|j\rangle\langle j|$ and similarly for $|\beta\rangle$, and substituting into eq. (12), we see that

$$Z_{\alpha\beta}(q)=\sum_j\langle\alpha|j\rangle\langle j|\beta\rangle\chi_j(\tilde q). \tag{17}$$

In the case when the characters are linearly independent, which happens if no two representations have the same Virasoro character, we have immediately, comparing eq. (10) and eq. (17)

$$\sum_i S_i^j n_{\alpha\beta}^i = \langle\alpha\,|\,j\rangle\langle j\,|\,\beta\rangle. \tag{18}$$

This is the most important equation of this paper. Since, by reversing the arguments of the rest of this section, it may be deduced from the Verlinde formula, its validity must be more general than the case of independent Virasoro characters for which we have derived it. To understand this, however, it is necessary to define the matrix S not by eq. (9), but at a more fundamental level as representing a map from the space of conformal blocks (isomorphic to the space of allowed boundary conditions) to the space of conformal blocks (isomorphic to the space of highest weight operators corresponding to discontinuities in the boundary conditions). While several reasonable arguments thus suggest themselves, we have not so far succeeded in constructing a convincing one without invoking further assumptions. Therefore, we shall continue with the argument with this proviso in mind.

Now we show that there exists a boundary state $|\tilde{0}\rangle$ such that $n_{\tilde{0}\tilde{0}}^i = \delta_0^i$, that is, the only representation which occurs in $H_{\tilde{0}\tilde{0}}$ is the identity representation, corresponding to the conformal block of the unit operator. From eq. (18), such a state must satisfy

$$|\langle\tilde{0}|j\rangle|^2 = S_0^j. \tag{19}$$

Taking the limit $\tilde{q} \to 0$ in eq. (9), we see that $\chi_i(q) \sim \tilde{q}^{-c/24} S_j^0$, and hence $S_j^0 > 0$. Then, using the relation $\mathbf{S} = \mathbf{S}^2\mathbf{S}^\dagger$, it follows that $S_0^j = (S^2)_0^k (S_j^k)^* = S_j^0$, since only the identity operator corresponds to the character χ_0. Hence the right-hand side of eq. (19) is positive, which shows that if we define the state

$$|\tilde{0}\rangle = \sum_j \left(S_0^j\right)^{1/2} |j\rangle \tag{20}$$

it will have the required properties. Of course eq. (20) is not unique, for we could choose different relative phases of the terms in the sum.

In a similar way, we may define a boundary state $|\tilde{l}\rangle$, with the property that $n_{\tilde{0}\tilde{l}}^i = \delta_l^i$, that is, only the representation l appears in the spectrum of $H_{\tilde{0}\tilde{l}}$. From eq. (18), such a state is

$$|\tilde{l}\rangle = \sum_j \frac{S_l^j}{\left(S_0^j\right)^{1/2}} |j\rangle. \tag{21}$$

Notice that $n_{\tilde{l}\tilde{0}}^i = \delta_{l^\vee}^i$. This corresponds to choosing a bra boundary state

$$\langle\tilde{l}| = \sum_j \frac{\left(S_l^j\right)^*}{\left(S_0^j\right)^{1/2}} \langle j|\,. \tag{22}$$

In order to have the representation l rather than $l^{\vee}$ running in the t-direction, the correct bra state is

$$\langle \tilde{l}^{\vee} | = \sum_j \frac{S_l^j}{\left(S_0^j\right)^{1/2}} \langle j | . \tag{23}$$

Now consider the boundary conditions $(\tilde{k}^{\vee}\tilde{l})$. Applying eq. (18) once again,

$$\sum_i S_i^j n_{\tilde{k}^{\vee}\tilde{l}}^i = \langle \tilde{k}^{\vee} | j \rangle \langle j | \tilde{l} \rangle = \frac{S_k^j S_l^j}{S_0^j} . \tag{24}$$

However, $n_{\tilde{k}^{\vee}\tilde{l}}^i$ may be related to the fusion rules. To see this, consider the geometry illustrated in fig. 2. This consists of a very long strip. For "time" $t < t_1$, the boundary conditions on both sides of the strip are $(\tilde{0}\tilde{0})$, so that only states in the representation of the identity propagate. At time t_1, the boundary condition on the right-hand side changes to $(\tilde{l})$, so that for $t_1 < t < t_2$, the boundary conditions are $(\tilde{0}\tilde{l})$. The only states which may propagate then belong to representation l. We may see this in another way: the change of boundary condition at $t = t_1$ corresponds to the insertion of a boundary operator $\phi_{\tilde{0}\tilde{l}}$. This transforms according to the representation l, by the arguments of sect. 2. Thus, when it acts on a state in the representation 0, it will give only states in the representation l. In terms of the fusion rules, this is just the statement that $N_{0l}^i = \delta_l^i$. Referring to fig. 2 once again, the boundary condition on the left-hand side changes to $(\tilde{k}^{\vee})$ at time t_2, corresponding to the insertion of an operator $\phi_{\tilde{k}^{\vee}\tilde{0}} = \phi_{\tilde{0}\tilde{k}}$, transforming according to the representation k. The number of times the representation i can occur when this operator acts on the space of states in the representation k is precisely the same

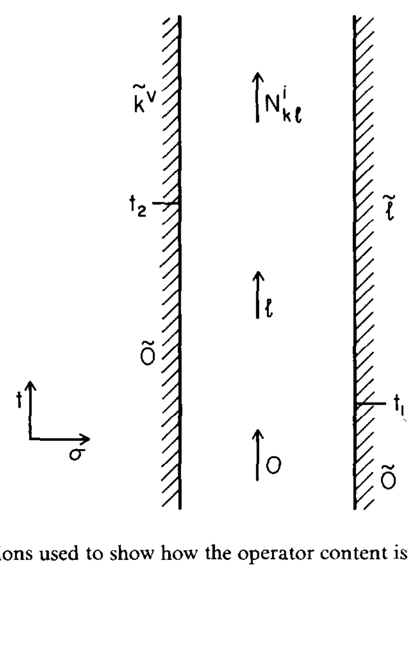

Fig. 2. Boundary conditions used to show how the operator content is related to the fusion rules.

number which arises in the computation by Belavin et al. [2] of the four-point function in the plane using the operator product expansion, which they identify as the fusion rule coefficient N^i_{kl}. Thus $n^i_{\tilde{k}^\vee \tilde{l}} = N^i_{kl}$. Substituting this into eq. (24) then gives the Verlinde formula

$$\sum_i S_i^j N^i_{kl} = S_k^j S_l^j / S_0^j. \tag{25}$$

In deriving eq. (25) we have been careful to consider only states and operators transforming according to a single irreducible representation. It is interesting to analyse the situation of fig. 2 when we consider "time" t as running in the opposite direction. This is illustrated by rotating fig. 2 through 180°. For $-t < -t_2$ the representation content is given by $n^i_{\tilde{l}\tilde{k}^\vee} = N^i_{l^\vee k^\vee}$. At time $-t_2$ the operator $\phi_{\tilde{k}^\vee \tilde{0}}$, transforming according to the representation k, acts. Since $N^i_{l^\vee k^\vee}$ may be non-zero for more than one value of k, if we try to apply the same argument as above cancellations may occur, and we may conclude only that the representation content for $-t_2 < -t < -t_1$ satisfies

$$n^j_{\tilde{l}\tilde{0}} \leqslant N^j_{ik} N^{i^\vee}_{l^\vee k} = N^j_{il^\vee} N^i_{kk^\vee}, \tag{26}$$

where we have used the associativity property of the fusion algebra to obtain the last expression. Now $N^0_{kk^\vee} = 1$ and $N^j_{0l^\vee} = \delta^j_{l^\vee}$, so that eq. (26) is consistent with the result $n^j_{\tilde{l}\tilde{0}} = \delta^j_{l^\vee}$, but we may draw no stronger conclusion by running the argument in this direction.

4. Examples

In comparing our results with calculations on specific lattice models, we do not know a priori which boundary conditions on the microscopic model will lead to boundary states satisfying eq. (13). In general, this can only be settled by a detailed analysis of the spectrum of $H_{\alpha\beta}$ to see whether its states fall into irreducible representations of the extended algebra. In certain simple cases, however, it is clear how to procede.

4.1. ISING MODEL

The matrix $\mathbf{S}$ for the $c = \frac{1}{2}$ representations of the Virasoro algebra is

$$\mathbf{S} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \tag{27}$$

where the rows and columns are labelled by the highest weights $(0, \frac{1}{2}, \frac{1}{16})$. Thus the boundary states are

$$|\tilde{0}\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|\varepsilon\rangle + \frac{1}{\sqrt[4]{2}}|\sigma\rangle,$$

$$|\tilde{\tfrac{1}{2}}\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|\varepsilon\rangle - \frac{1}{\sqrt[4]{2}}|\sigma\rangle,$$

$$|\tilde{\tfrac{1}{16}}\rangle = |0\rangle - |\varepsilon\rangle, \tag{28}$$

where we have labelled the states $|j\rangle$ by the corresponding physical scalar bulk operators. Note that $|\tilde{0}\rangle$ and $|\tilde{\frac{1}{2}}\rangle$ differ in the sign of the coefficient of the Z_2-odd state $|\sigma\rangle$. It is then natural to identify them with $|+\rangle$ and $|-\rangle$ respectively. In fact $|-\rangle$, since it only differs from $|+\rangle$ in the assignment of a phase factor, is an equally good candidate for $|\tilde{0}\rangle$. There are other possibilities for $|\tilde{0}\rangle$, obtained from the above two by changing the sign of the coefficient of $|\varepsilon\rangle$. These correspond to fixing the *dual* spins to ± 1 respectively on the boundary. Under duality, the energy-like state $|\varepsilon\rangle$ changes sign. Also, the free boundary state $|f\rangle$ goes into an equal superposition of the $|+\rangle$ and $|-\rangle$ states. Thus we may identify $|f\rangle$ with $|\tilde{\frac{1}{16}}\rangle$.

Note that, although the states in eq. (28) should be orthogonal, this is not obvious from their definition. However, it should be remembered that the states $|j\rangle$ are not normalizable. If the inner product is defined e.g., as

$$(+|-) \equiv \lim_{q\to 1} \frac{\langle +|q^{L_0+\bar{L}_0}|-\rangle}{\left(\langle +|q^{L_0+\bar{L}_0}|+\rangle\langle -|q^{L_0+\bar{L}_0}|-\rangle\right)^{1/2}}, \tag{29}$$

we obtain zero*.

We have therefore reproduced the cases $(++)$, $(+-)$ and $(+f)$ of eq. (7). The remaining case (ff) then follows from the fusion rules

$$\left[\tfrac{1}{16}\right] \times \left[\tfrac{1}{16}\right] = [0] + \left[\tfrac{1}{2}\right]. \tag{30}$$

This verifies the results stated in eq. (7), which were first derived in ref. [6] by a slightly different argument, and confirmed by direct calculation by Saleur and Bauer [7].

4.2. 3-STATE POTTS MODEL

In the simplest realization of this model, there are three possible microscopic states A, B and C, related by an S_3 symmetry. The model corresponds to the case

* In general, states $|\tilde{k}\rangle$ and $|\tilde{l}\rangle$ are orthogonal in this sense if $N_{k^\vee l}^{0} = 0$.

$c=\frac{4}{5}$ of the Virasoro series. With respect to the Virasoro algebra, its partition function on the torus is given by a non-diagonal combination of characters [11]. However, this theory has a conserved spin-3 current, which generates an extended W-algebra, with respect to whose characters the torus partition function is diagonal [14]. Thus the analysis of sect. 3 applies. This is the simplest case where non-self-conjugate representations occur. The characters of the extended algebra, labelled by their highest Virasoro weights, are given in terms of the Virasoro characters χ_{rs} by

$$\begin{aligned} &\chi_0=\chi_{11}+\chi_{41}, \qquad \chi_{\frac{2}{5}}=\chi_{21}+\chi_{31}, \\ &\chi_{\frac{2}{3}}=\chi_{\frac{2}{3}^\vee}=\chi_{13}, \qquad \chi_{\frac{1}{15}}=\chi_{\frac{1}{15}^\vee}=\chi_{23}. \end{aligned} \tag{31}$$

With respect to the above basis $(0,\frac{2}{5},\frac{2}{3},\frac{1}{15},\frac{2}{3}^\vee,\frac{1}{15}^\vee)$, the matrix $\mathbf{S}$ has the direct product form

$$\mathbf{S}=\frac{1}{\sqrt{3}}\begin{pmatrix} \mathbf{s} & \mathbf{s} & \mathbf{s} \\ \mathbf{s} & \omega\mathbf{s} & \omega^2\mathbf{s} \\ \mathbf{s} & \omega^2\mathbf{s} & \omega\mathbf{s} \end{pmatrix}, \tag{32}$$

where $\omega=e^{2\pi i/3}$ and

$$\mathbf{s}=\frac{2}{\sqrt{5}}\begin{pmatrix} \sin\dfrac{\pi}{5} & \sin\dfrac{2\pi}{5} \\ \sin\dfrac{2\pi}{5} & -\sin\dfrac{\pi}{5} \end{pmatrix}. \tag{33}$$

The boundary states are then

$$\begin{aligned}
|\tilde{0}\rangle &= N\Big[|0\rangle+\lambda|\tfrac{2}{5}\rangle+|\tfrac{2}{3}\rangle+\lambda|\tfrac{1}{15}\rangle+|\tfrac{2}{3}^\vee\rangle+\lambda|\tfrac{1}{15}^\vee\rangle\Big], \\
|\tfrac{\tilde{2}}{3}\rangle &= N\Big[|0\rangle+\lambda|\tfrac{2}{5}\rangle+\omega|\tfrac{2}{3}\rangle+\omega\lambda|\tfrac{1}{15}\rangle+\omega^2|\tfrac{2}{3}^\vee\rangle+\omega^2|\tfrac{1}{15}^\vee\rangle\Big], \\
|\tfrac{\tilde{2}}{3}^\vee\rangle &= N\Big[|0\rangle+\lambda|\tfrac{2}{5}\rangle+\omega^2|\tfrac{2}{3}\rangle+\omega^2\lambda|\tfrac{1}{15}\rangle+\omega|\tfrac{2}{3}^\vee\rangle+\omega\lambda|\tfrac{2}{3}^\vee\rangle\Big], \\
|\tfrac{\tilde{2}}{5}\rangle &= N\Big[\lambda^2|0\rangle-\lambda^{-1}|\tfrac{2}{5}\rangle+\lambda^2|\tfrac{2}{3}\rangle-\lambda^{-1}|\tfrac{1}{15}\rangle+\lambda^2|\tfrac{2}{3}^\vee\rangle-\lambda^{-1}|\tfrac{1}{15}^\vee\rangle\Big] \\
|\tfrac{\tilde{1}}{15}\rangle &= N\Big[\lambda^2|0\rangle-\lambda^{-1}|\tfrac{2}{5}\rangle+\omega\lambda^2|\tfrac{2}{3}\rangle-\omega\lambda^{-1}|\tfrac{1}{15}\rangle+\omega^2\lambda^2|\tfrac{2}{3}^\vee\rangle-\omega^2\lambda^{-1}|\tfrac{1}{15}^\vee\rangle\Big] \\
|\tfrac{\tilde{1}}{15}^\vee\rangle &= N\Big[\lambda^2|0\rangle-\lambda^{-1}|\tfrac{2}{5}\rangle+\omega^2\lambda^2|\tfrac{2}{3}\rangle-\omega^2\lambda^{-1}|\tfrac{1}{15}\rangle+\omega\lambda^2|\tfrac{2}{3}^\vee\rangle-\omega\lambda^{-1}|\tfrac{1}{15}^\vee\rangle\Big]
\end{aligned} \tag{34}$$

where $N=(2/\sqrt{15})(\sin\frac{1}{5}\pi)^{1/2}$ and $\lambda=(\sin\frac{2}{5}\pi/\sin\frac{1}{5}\pi)^{1/2}$.

We see that the first three states are related by Z_3-rotations, as are the last three. Identifying therefore $|\tilde{0}\rangle$ with $|A\rangle$, which means that all sites on the boundary are in state A, the other two states in the first group may be identified with $|B\rangle$ and $|C\rangle$ respectively. Thus we see that

$$Z_{AA} = \chi_0, \qquad Z_{AB} = \chi_{\frac{2}{3}}, \qquad Z_{AC} = \chi_{\frac{2}{3}^\vee} \tag{35}$$

as was obtained in [6, 7]. Using the fusion rules

$$\left[\tfrac{2}{3}^\vee\right] \times \left[\tfrac{2}{3}\right] = [0], \qquad \left[\tfrac{2}{3}^\vee\right] \times \left[\tfrac{2}{3}^\vee\right] = \left[\tfrac{2}{3}\right] \tag{36}$$

we see that $Z_{BB} = Z_{CC} = Z_{AA}$ and $Z_{BC} = Z_{AB} = Z_{CA}$, as expected on the basis of the Z_3 symmetry.

The microscopic interpretation of the last three boundary states in eq. (34) is more difficult. Saleur and Bauer [7] have shown that they correspond to "mixed" boundary conditions. For example, the boundary sites may be in states B or C with equal probability. Neighboring sites along the boundary are not coupled and therefore are not constrained to be in the same state. Let us denote such a state by $|B + C\rangle$*. If we identify this with $|\tfrac{2}{5}\rangle$, then the Z_3 symmetry implies that the last two states in eq. (34) should be identified with $|C + A\rangle$ and $|A + B\rangle$ respectively. Thus

$$Z_{A,B+C} = \chi_{\frac{2}{5}} \qquad Z_{A,C+A} = \chi_{\frac{1}{15}} \qquad Z_{A,A+B} = \chi_{\frac{1}{15}^\vee}. \tag{37}$$

These are consistent with the findings of Saleur and Bauer [7]. From the fusion rules one may then predict, for example, that

$$Z_{C+A,A+B} = \chi_{\frac{2}{3}} + \chi_{\frac{1}{15}}. \tag{38}$$

In refs. [6, 7], free boundary conditions were also considered. The partition function in this case may be written combinations of Virasoro characters, which are not however characters of the W-algebra:

$$Z_{Af} = \chi_{12} + \chi_{42}, \qquad Z_{B+C,f} = \chi_{22} + \chi_{32}. \tag{39}$$

However Z_{ff} may be written in terms of W-characters:

$$Z_{ff} = \chi_0 + \chi_{\frac{2}{3}} + \chi_{\frac{2}{3}^\vee}. \tag{40}$$

* Note that this is not the same as $|B\rangle + |C\rangle$.

4.3. A_m RSOS MODELS

These lattice models, originally formulated and solved by Andrews et al. [15], correspond, in the continuum limit at criticality, to the diagonal modular invariants with $c = 1 - 6/m(m+1)$. The Virasoro characters $\chi_{r,s}$ of these theories are most easily labelled by the coordinates (r, s) of the corresponding highest weight $h_{r,s}$ in the Kac table, with $\chi_{r,s} = \chi_{m-r,m+1-s}$, where $1 \leqslant r \leqslant m-1$ and $1 \leqslant s \leqslant m$. The site variables of the lattice model take values on the vertices of the Dynkin diagram of A_m. Thus it is natural also to label them by s also, where $1 \leqslant s \leqslant m$. In fact, this correspondence is very natural, because, as was shown by explicit calculation by Saleur and Bauer [7], on the annulus we have

$$Z_{1s} = \chi_{1,s}. \tag{41}$$

Since $r = s = 1$ corresponds to the representation with $h = 0$, this result means that we should identify $|1\rangle$, the state where all the boundary spins are in state $s = 1$, with $|\tilde{0}\rangle$. In the same way, $|s\rangle$, where all the boundary spins are in state s, should be identified with $|\tilde{h}_{1,s}\rangle$. The partition functions when the spins are in states s and s' on either side of the strip are then, by the analysis of sect. 3, given by the fusion rules [2]:

$$Z_{ss'} = \sum_{p=0}^{p_{\max}} \chi_{1,\,|s-s'|\,+2p+1}, \tag{42}$$

where $p_{\max} = \min(s-1, s'-1, m-s, m-s')$. This may be shown to be in agreement with results of Saleur and Bauer [7]. They also identified another type of boundary condition, corresponding to varying r in the Kac table. In the RSOS models, neighboring sites must be in states corresponding to adjacent vertices on the Dynkin diagram. Thus it is appropriate to consider a boundary condition where the boundary spins are in state r, and their neighbors, one layer in from the boundary, in state $r+1$. In this case $1 \leqslant r \leqslant m-1$. Saleur and Bauer [7] conjectured that

$$Z_{r1} = \chi_{r,1}. \tag{43}$$

Note that $r = 1$ corresponds to $s = 1$, since the spins on the layer next to the boundary must then be in state 2. From the fusion rule

$$[h_{r,1}] \times [h_{1,s}] = [h_{r,s}] \tag{44}$$

we then see that

$$Z_{rs} = \chi_{r,s} \tag{45}$$

consistent with a further conjecture of Saleur and Bauer [7]. This result is important in their work which shows that these partition functions at criticality in finite-size geometry, and away from criticality in the infinite system, are both given by Virasoro characters, with, however, a different interpretation of q in the two cases.

In the above, we have identified boundary states $|\tilde{k}\rangle$ corresponding only to those highest weights in the first row and first column of the Kac table. It would be interesting to understand what microscopic boundary states correspond to the other entries. Saleur and Bauer [7] have also discussed the D_n models of Pasquier [16], of which we have treated only the simplest case D_4, equivalent to the 3-state Potts model, above. It would be interesting to consider in detail those of this series which have extended algebras.

In general, it should be clear from the above that the study of boundary states and boundary operators will give new insight into the properties of conformal field theories.

After this paper was completed, I was informed that the authors of ref. [7] have now derived the results (43) and (45).

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