Supplemental Material for "Lattice supersymmetry and order-disorder coexistence in the tricritical Ising model"

Ground states of the frustration-free model

In the main text it was stated that there are exactly three ground states at $\lambda_3 = \lambda_I$ (and $\lambda_c = 0$). The purpose of this appendix is to prove this assertion. At this point the Hamiltonian becomes $H = \sum_j H_{j,j+1,j+2}$, where

$$H_{j,j+1,j+2} = \left(\mathbb{1}_j - \sigma^x_j\right)\left(\mathbb{1}_{j+1,j+2} - \sigma^z_{j+1}\sigma^z_{j+2}\right) + \left(\mathbb{1}_{j,j+1} - \sigma^z_j\sigma^z_{j+1}\right)\left(\mathbb{1}_{j+2} - \sigma^x_{j+2}\right). \quad (1)$$

As $H_{j,j+1,j+2}$ is a sum of projectors, its eigenvalues are non-negative and hence the eigenvalues of H are also non-negative. Any state annihilated by H is then necessarily a ground state. Conversely, all zero-energy ground states must necessarily be annihilated by each $H_{j,j+1,j+2}$ individually.

Of the eight basis states on the three sites $j, j+1$ and $j+2$, four of them are annihilated by $H_{j,j+1,j+2}$. The eigenstates corresponding to the zero eigenvalues are

$$\left|\mathcal{G}^{(3)}_{j,\uparrow}\right\rangle = |\uparrow\uparrow\uparrow\rangle\,, \quad \left|\mathcal{G}^{(3)}_{j,\downarrow}\right\rangle = |\downarrow\downarrow\downarrow\rangle\,, \quad \left|\mathcal{G}^{(3)}_{j,\updownarrow}\right\rangle = |\updownarrow\updownarrow\updownarrow\rangle\,,$$
$$\left|\mathcal{G}^{(3)}_{j,s}\right\rangle = |\uparrow\uparrow\downarrow\rangle + |\downarrow\uparrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle - |\uparrow\downarrow\uparrow\rangle - |\uparrow\downarrow\downarrow\rangle - |\downarrow\downarrow\uparrow\rangle\,, \quad (2)$$

where $\left|\updownarrow\right\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$. The state $\left|\mathcal{G}^{(3)}_{j,\updownarrow}\right\rangle$ is not orthogonal to $\left|\mathcal{G}^{(3)}_{j,\uparrow}\right\rangle$ or $\left|\mathcal{G}^{(3)}_{j,\downarrow}\right\rangle$, but these three are linearly independent and are orthogonal to $\left|\mathcal{G}^{(3)}_{j,s}\right\rangle$. Thus the three states $\left|\mathcal{G}_{\downarrow}\right\rangle$, $\left|\mathcal{G}_{\uparrow}\right\rangle$ and $|\mathcal{G}_0\rangle$ described in the main text are annihilated by all $H_{j,j+1,j+2}$, and so indeed are zero-energy ground states.

To determine if there are further ground states now consider the four sites $j-1, j, j+1, j+2$. From (2), it follows that such ground states can be written as a linear combination of the eight configurations $|\uparrow \mathcal{G}^{(3)}_{j,r}\rangle$ and $|\downarrow \mathcal{G}^{(3)}_{j,r}\rangle$ on these four sites, where r is one of $\uparrow, \downarrow, \updownarrow, s$. Since all zero-energy ground states must be annihilated by $H_{j-1,j,j+1}$ as well, they must also be linear combinations of $|\mathcal{G}^{(3)}_{j-1,r'}\uparrow\rangle$ and $|\mathcal{G}^{(3)}_{j-1,r'}\downarrow\rangle$. Thus we need to look for non-vanishing coefficients $\alpha_{r,\uparrow}, \alpha_{r,\downarrow}, \beta_{r',\uparrow}, \beta_{r',\downarrow}$ satisfying

$$\sum_r \left(\alpha_{r,\uparrow}\left|\uparrow \mathcal{G}^{(3)}_{j,r}\right\rangle + \alpha_{r,\downarrow}\left|\downarrow \mathcal{G}^{(3)}_{j,r}\right\rangle\right) = \sum_{r'} \left(\beta_{r',\uparrow}\left|\mathcal{G}^{(3)}_{j-1,r'}\uparrow\right\rangle + \beta_{r',\downarrow}\left|\mathcal{G}^{(3)}_{j-1,r'}\downarrow\right\rangle\right).$$

Obviously, three solutions of this equation correspond to the states

$$|\uparrow\uparrow\uparrow\uparrow\rangle\,, \quad |\downarrow\downarrow\downarrow\downarrow\rangle\,, \quad |\updownarrow\updownarrow\updownarrow\updownarrow\rangle\,, \quad (3)$$

Any other solutions must involve non-vanishing coefficients with $r = s$ and $r' = s$. Comparing configurations $|\uparrow\downarrow\downarrow\uparrow\rangle$ on the two sides means $\alpha_{\uparrow} = \beta_{s,\uparrow}$, and likewise comparing $|\downarrow\uparrow\uparrow\downarrow\rangle$ gives $\alpha_{\downarrow} = \beta_{s,\downarrow}$. However, comparing $|\uparrow\downarrow\uparrow\uparrow\rangle$ on the two sides means $\alpha_{\uparrow} = -\beta_{s,\uparrow}$, and comparing $|\downarrow\uparrow\downarrow\downarrow\rangle$ gives $\alpha_{\downarrow} = -\beta_{s,\downarrow}$. These coefficients must vanish.

Thus the three states in (3) are the only ground states possible on four sites. This argument can then be rerun, increasing the number of sites by one each time. Thus for any number of sites greater than 3, there are exactly three zero-energy ground states.

Other 4-Majorana perturbations

In terms of Majorana fermion operators our model in the chirally invariant case can be written as

$$H = \sum_j \left(2i\lambda_I \gamma_j\gamma_{j+1} - \lambda_3\gamma_{j-2}\gamma_{j-1}\gamma_{j+1}\gamma_{j+2}\right). \quad (4)$$

The latter are not the only four-Majorana terms both chirally invariant and involving nearest- and next-nearest-neighbor interactions. Indeed, Refs. 1 and 2 studied at length the effect of adding

$$H_R = \sum_j \gamma_{j-1}\gamma_j\gamma_{j+1}\gamma_{j+2} = -\sum_j \left(\sigma^z_{j-1}\sigma^z_{j+1} + \sigma^x_j\sigma^x_{j+1}\right) \quad (5)$$

to the Ising chain. Another such four-Majorana term is

$$H_y = \sum_j \left(\gamma_{j-2}\gamma_j\gamma_{j+1}\gamma_{j+2} - \gamma_{j-2}\gamma_{j-1}\gamma_j\gamma_{j+2}\right) = \sum_j \Big(\sigma^z_{j-1}\sigma^y_j\sigma^x_{j+1} + \sigma^x_{j-1}\sigma^y_j\sigma^z_{j+1} - \sigma^y_{j-1}\sigma^z_{j+1} - \sigma^z_{j-1}\sigma^y_{j+1}\Big). \quad (6)$$

We thus consider the Hamiltonian

$$H = 2\lambda_I H_I + \lambda_3 H_3 + \lambda_R H_R + \lambda_y H_y\,. \quad (7)$$

All self-dual perturbations of the Ising conformal field theory are irrelevant, with the leading irrelevant operator

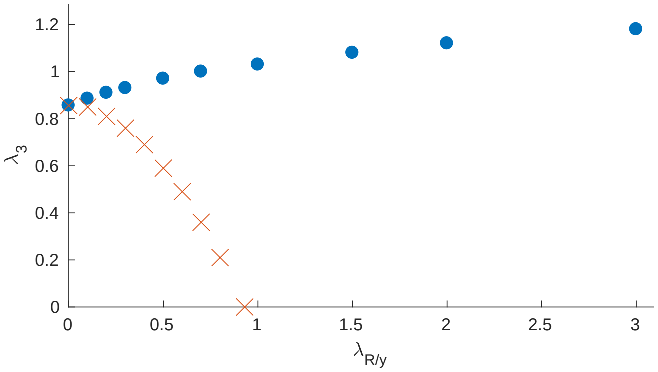

FIG. 1. Locations of the TCI transition in phase space for λ_3 as a function of λ_R with $\lambda_y = 0$ (blue dots), and a function of λ_y with λ_R (orange crosses).

given by $T\overline{T}$. Thus naively, one would expect that each of the operators H_3, H_R and H_y would correspond to $T\overline{T}$, and so tuning any or all of the corresponding coefficient sufficiently will result in reaching the tricriticial point. Ref. 1 found that for $\lambda_3 = \lambda_y = 0$, the tricritical Ising point is at $\lambda_R \approx 250\lambda_I$, a (still) mysteriously large value. In the text, we located the tricritical point for $\lambda_R = \lambda_y = 0$ at $\lambda_3 \approx 0.856\lambda_I$. In Fig. 1, we locate approximately the tricritical point in the two cases $\lambda_R = 0$ and $\lambda_y = 0$. When both couplings are included the transition is found to be between the two cases. We thus confirm that indeed the tricritical point occurs with any or all of the perturbations, with λ_y and λ_3 behaving qualitatively similarly, causing the transition at couplings of order 1. In keeping with the mystery of the large coupling found in Ref. 1, making λ_R small and positive requires increases the other couplings further to reach the tricritical point.

The incommensurate phase

For $|\lambda_I| > \lambda_c$ the model is in an incommensurate phase. In this phase the ground state changes very rapidly as λ_c/λ_I is varied. We have verified this numerically for non-zero λ_3. For $\lambda_3 = 0.856\lambda_I$ the lowest energy states in the $\mathcal{F} = 1$ sector are plotted against λ_c/λ_I for each momentum sector with $L = 19$ in Fig. 2. The ground state is in the zero momentum sector up to $\lambda_c/\lambda_I = 1$. After this there are many level crossings. This behaviour is reproduced all along the line and becomes more striking with larger lattice length, L. In the $L \to \infty$ limit an infinitesimal change in λ_c/λ_I will mean the ground state is now a completely different state in a completely different momentum sector.

The special case $\lambda_3 = 0$ can be solved exactly using Majorana fermions even with $\lambda_c \neq 0$. This calculation is standard, and we are sure it is presented somewhere in the literature. However, we do not know of a reference. Thus we present this calculation here, in perhaps

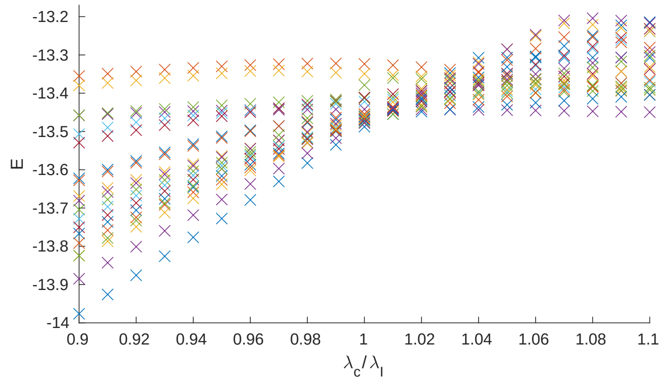

FIG. 2. Lowest energy levels in each momentum sector for $\lambda_3/\lambda_I = 0.856$, $L = 19$ and $\mathcal{F} = 1$ as λ_c/λ_I is taken across the chiral transition to the incommensurate phase. The Hamiltonian is scaled so that $\lambda_I + \lambda_3 + \lambda_c = 1$ for all values of λ_c/λ_I. The blue crosses (lowest for $\lambda_c/\lambda_I = 1$) are in the zero momentum sector. The purple crosses (lowest for $\lambda_c/\lambda_I = 1.1$) are in the $k = 6\pi/19$ sector.

a slightly different way than the standard Bogoliubov transformation. Throughout this we will be following the method introduced in [3] and written out explicitly for Ising in [4]. We show that indeed there is a transition from a critical Ising phase to an incommensurate phase at the supersymmetric point. This also has the advantage of allowing for disordered couplings.

We first write down the Hamiltonian generally in terms of Majoranas, where we allow arbitrary couplings at each site:

$$H = i\sum_{j=1}^{2L}\left(2\lambda_j\gamma_j\gamma_{j+1} - \kappa_j\gamma_{j-1}\gamma_{j+1}\right) \ . \tag{8}$$

Here L is the length of the lattice of spins, γ_j is a Majorana obeying $\{\gamma_j, \gamma_k\} = 2\delta_{jk}$, and the coefficients λ_j, κ_j are chosen such that we have the Ising model (with different couplings per site) for $\kappa_j = 0\ \forall j$, and the supersymmetric model for $\lambda_j = \lambda = \kappa_j\ \forall j$. Note that we define $\gamma_{j+2L} = \gamma_j$ and so we have periodic boundary conditions on the fermions. To get antiperiodic boundary conditions we simply change the sign of $\lambda_{2L}, \kappa_{2L}$ and κ_1.

We next construct raising and lowering operators. If we can find μ_j such that $[H, \Psi] = 2\epsilon\Psi$, then taking any eigenstate $|E\rangle$ of H with energy E and applying Ψ we either have $H\Psi|E\rangle = (E + 2\epsilon)\Psi|E\rangle$ or $H\Psi|E\rangle = 0$, hence $\Psi|E\rangle$ is either an eigenstate of the Hamiltonian with energy $E + 2\epsilon$ or is a null vector. We consider the operator

$$\Psi = \sum_{j=1}^{2L}\mu_j\gamma_j \ , \tag{9}$$

where μ_j are arbitrary so far. The commutation relation

$$\left[\gamma_j\gamma_k, \gamma_l\right] = \gamma_j\delta_{kl} - \gamma_k\delta_{jl}, \tag{10}$$

gives

$$[H,\Psi] = \sum_j \mu'_j \gamma_j \ , \tag{11}$$

$$\mu'_j = i\big(2\lambda_j\mu_{j+1} - 2\lambda_{j-1}\mu_{j-1} - \kappa_{j+1}\mu_{j+2} + \kappa_{j-1}\mu_{j-2}\big) \ .$$

Using Equation 11 we can then find every creation operator. The ground state is the state annihilated by all operators which lower the energy and not to be annihilated by any which raise the energy.

Going to the self-dual and translation-invariant model with periodic boundary conditions, we set $\lambda_j = \lambda_I$, $\kappa_j = \lambda_c$, $\forall j$ and hence the equations are trivial to solve with the Ansatz

$$\mu_j = \mu e^{ikj}, \tag{12}$$

where $k = n\pi/L$, $n = -L+1, -L+2, ..., -1, 0, 1, ..., L-1, L$ to ensure $\mu_j = \mu_{j+2L}$. These indeed satisfy the recursion relations and give

$$2\epsilon_k = -4\sin(k)\left(\lambda_I - \lambda_c\cos(k)\right) \ . \tag{13}$$

We then see that $\epsilon_{-k} = -\epsilon_k$. We also note that

$$\Psi_{-k} = \Psi_k^\dagger, \quad \{\Psi_k, \Psi'_k\} = 2L\delta_{k',-k} \ , \tag{14}$$

and hence we can identify Ψ_k as an annihilation operator for $0 < k < \pi$ with Ψ_{-k} the corresponding creation operator assuming that $\lambda_c \leq \lambda_I$.

There is a slight subtlety if $k = 0$ or $k = \pi$, as here the energies are zero and the creation and annihilation operators are the same. To resolve this we first realise that if Ψ_0 is an allowed operator with the total number of states and boundary conditions then Ψ_π must be too. We therefore consider the two combinations

$$\Psi_+ = \Psi_0 + i\Psi_\pi, \quad \Psi_- = \Psi_0 - i\Psi_\pi \ . \tag{15}$$

These operators now satisfy $\{\Psi_+, \Psi_+\} = \{\Psi_-, \Psi_-\} = 0$, $\{\Psi_+, \Psi_-\} = 0$ and $[H, \Psi_+] = [H, \Psi_-] = 0$ (note that what we have done is only sensible as Ψ_0 and Ψ_π have the same eigenvalue under commutation with H). We now make the arbitrary choice that Ψ_+ is one of the creation operators (along with $-0 < k < \pi$) and Ψ_- is an annihilation operator. As we have now ensured that we always have L independent creation operators with corresponding annihilation operators we have covered the whole space.

For antiperiodic boundary conditions we consider the same but this time $\lambda_{2L} = -\lambda_I$ and $\kappa_{2L}, \kappa_1 = -\lambda_c$. Our Ansatz is then adapted to

$$\tilde{\mu}_j = \tilde{\mu} e^{ipj}, \tag{16}$$

where $p = q\pi/L$, $q = -L + 1/2, -L + 3/2, ..., -1/2, 1/2, ..., L-3/2, L-1/2$ (so $\tilde{\mu}_j = -\tilde{\mu}_{j+2L}$). We therefore get the same formula for ϵ_p as ϵ_k above but the allowed values of k differ from those of p. It is then clear that the ground state remains constant for $-\lambda_I \leq \lambda_c \leq \lambda_I$ as the creation and annihilation operators stay the same for both periodic and antiperiodic boundary conditions. The low-lying excited states within each sector do change slightly, but this is due to the chiral term imposing a different Fermi velocity in the left- and right-moving directions so the underlying CFT is unchanged. For $\lambda_c > \lambda_I$ the ground state changes rapidly as λ_c is changed as the set of creation operators also varies, signalling the incommensurate phase. As an interesting side point, note that for $\lambda_I = 0$, $\lambda_c > 0$ we in fact have two Ising models, one corresponding to creation operators with $0 < k < \pi/2$ and one with $-\pi < k < -\pi/2$.

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