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SUPERSYMMETRY, TWO-DIMENSIONAL CRITICAL PHENOMENA AND THE TRICRITICAL ISING MODEL*†

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We discuss the general properties of supersymmetrical two-dimensional critical phenomena, i.e. superconformal field theory. We first review the consequences of ordinary conformal invariance. We then discuss general features of superconformal invariance and the superdifferential equation that correlation functions in certain theories obey. We give the explicit form of this differential equation in the even sector of the tricritical Ising model at its tricritical point. We find the solution of this equation. The physical realizations of this model, e.g. helium adsorbed on krypton-plated graphite, are the first observable supersymmetric field theories in nature.

1. Introduction

When a physical system is near its critical point the physics of the system is characterized by its correlation length – which is much larger than all microscopic lengths. This independence of microscopic scales makes many systems look identical – the principle of universality [1]. All universal quantities are related to the correlation length by a set of critical exponents. As the system approaches its critical point the correlation length becomes infinite. The system loses its only scale, becoming globally scale invariant. Moreover, if the system we are dealing with has local interactions, the locality of these interactions implies that the system responds simply to local scale transformations. Local scale transformations are conformal transformations; these are mappings that preserve angles but change length.

In this paper we will restrict ourselves to two dimensions because the group of conformal transformations is infinite-dimensional in two dimensions. Requiring the system to respond simply to conformal transformations puts powerful constraints on

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critical systems. Two-dimensional conformal invariance also plays an important role in string theory where it acts as a gauge symmetry. The vertex operators on the world surface of the string form a conformal field theory.

Recent work by Belavin, Polyakov and Zamolodchikov (BPZ) [2] initiated a new approach to two-dimensional conformal field theory. They showed that there is a special class of such theories, so-called degenerate conformal theories, in which all the correlation functions satisfy linear differential equations. In particular, there are special conformal theories where there are only a finite number of conformal fields. BPZ also suggested that there is a close connection between these degenerate conformal field theories and statistical mechanics.

Two-dimensional statistical mechanical systems can also be interpreted as $1+1$ dimensional operator field theories, via the well-known transfer matrix approach. Systems with local couplings, spatial isotropy and genuine thermal Boltzmann weights will have an especially nice operator interpretation near their critical point. They will be described by quantum field theories (QFT). The distinctive feature of such a QFT is that all the states must have nonnegative norm, i.e. the state space is a Hilbert space. In other words the theory must be unitary. It is remarkable that unitarity along with the infinite conformal group in two dimensions puts powerful constraints on the allowed physical systems. Friedan, Shenker and the author (FQS) [3] have studied the important role of unitarity in conformal field theories. In two-dimensional conformal QFTs the scaling dimensions of fields are determined by the unitary representations of the Virasoro algebra. By classifying all possible unitary representations of the Virasoro algebra, we are able to find all possible scaling dimensions in two dimensions. In particular for the region of $c<1$ (c is a real number which parameterizes conformal theories) there is a discrete list of possible conformal QFTs. In this class of theories there are only a finite number of conformal fields for each c. The first few theories in this class of theories are well-known statistical mechanical models, the Ising model, the tricritical Ising model, the three-state Potts model, and the tricritical three-state Potts model. In fact, all of these models have their corresponding statistical mechanics systems. This class of conformal QFTs is the subset of degenerate conformal field theories BPZ discussed. Therefore all the correlation functions in this class of conformal QFTs satisfy differential equations.

More recently FQS studied superconformal field theories in two dimensions [3, 4]. Superconformal symmetry is the symmetry combining conformal symmetry and supersymmetry – the symmetry between fermionic fields and bosonic fields. Superconformal field theories are the supersymmetric generalizations of conformally invariant field theories. Every supersymmetric local field theory is superconformally invariant at short distances, so the realizations of superconformal invariance completely determine the possible supersymmetric field theories. In superconformal theories superfields correspond to irreducible representations of the Neveu-Schwarz algebra [3]. The superconformal field theories also contain spin fields which correspond to the irreducible representations of the Ramond algebra [4]. The spin fields are non-local with respect to the fermionic part of the superfields. The fermion parity $\Gamma = (-1)^F$ is multiplicatively conserved so we can project on the sector of even fermion number. This projection selects the bosonic parts of the superconformal field and a subset of the spin fields. The resulting theory is a new local bosonic field theory, called the spin model. The construction of the spin model is a direct generalization of the construction of the superstring from the Neveu-Schwarz-Ramond model. The spin model construction is of particular interest in the superstring theory. In statistical mechanics superconformal field theories describe special critical points at which the macroscopic physics is supersymmetric. FQS pointed out that the tricritical Ising (TIM) model is a superconformal theory. Physical systems described by the TIM are the first known realizations of a supersymmetric QFT in nature. Superconformal fields in the superconformal theory have also been studied independently by Bershadsky, Knizhnik and Teitelman [5]. In this paper we give some more detailed discussions of the Neveu-Schwarz sector of superconformal theories. We derive a superdifferential equation as a result of the null state present in the representation of the Neveu-Schwarz algebra. We use the superconformal formulation to study the TIM. The complete operator algebra of the even sector of the TIM is presented. We then discuss some consequences of supersymmetry in the TIM. Since the TIM can be realized in physical systems, some of the results may be verifiable experimentally.

2. Review of the conformal field theory

The group of conformal transformations is infinite-dimensional in two dimensions. Any analytic function mapping the complex plane to itself is a conformal transformation. It is convenient to use complex coordinates $z = x + iy$ and $\bar{z} = x - iy$. The central object to study is the stress-energy tensor $T_{\mu\nu}$. In QFT, the stress-energy tensor is the operator which generates local coordinate transformations. It is the operator that locally stresses the system, weakening couplings in one direction and strengthening them in the other. In two dimensions the stress-energy tensor has four components. It is symmetric, and traceless if the system we are dealing with is globally scaling invariant. In scale invariant systems the stress-energy tensor thus has two independent components which generate local scale transformations, i.e. conformal transformations. From the conservation law $\partial_\mu T_{\mu\nu} = 0$, the two independent combinations $T = T_{11} + iT_{12}$ and $\bar{T} = T_{11} - iT_{12}$ only depend on z and $\bar{z}$ respectively. We will use $T(z)$ and $\bar{T}(\bar{z})$ as stress-energy tensors in this paper.

Because we are interested in scale invariant systems it is natural to single out the scaling (dilation) operator. To do this we use a nonstandard quantization, radial quantization, in our discussion. This quantization is most easily realized by making the conformal transformation $z = \exp(\tau + i\sigma)$. It transforms the whole complex plane to a strip, $-\infty < \tau < +\infty$, $0 \le \sigma < 2\pi$, with periodic boundary conditions in

σ. We interpret τ as the (euclidean) time and σ as the space coordinate. Translation in the τ direction corresponds to dilation in the z plane. Therefore the operator which generates dilation is the hamiltonian in radial quantization. Because T and $\overline{T}$ are periodic in σ, we can expand them in operator Fourier expansions

$$T(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2}, \qquad \overline{T}(\bar{z}) = \sum_{n=-\infty}^{\infty} \overline{L}_n \bar{z}^{-n-2},$$

$$L_n = \frac{1}{2\pi i}\oint_C \mathrm{d}z\, T(z) z^{n+1}, \qquad \overline{L}_n = \frac{1}{2\pi i}\oint_C \mathrm{d}\bar{z}\, \overline{T}(\bar{z}) \bar{z}^{n+1}, \tag{1}$$

where C is a contour winding around the origin. It is important to note that operators L_n, $\overline{L}_n$ are independent of the contour used.

We now discuss how the stress-energy tensor generates local conformal transformations. Suppose $\phi(z, \bar{z})$ is some local field. We make an infinitesimal conformal transformation

$$z \to z + \varepsilon(z), \qquad \bar{z} \to \bar{z} + \bar{\varepsilon}(\bar{z}), \tag{2}$$

where

$$\varepsilon(z) = \sum_{n=-\infty}^{\infty} \varepsilon_n z^n, \qquad \bar{\varepsilon}(\bar{z}) = \sum_{n=-\infty}^{\infty} \bar{\varepsilon}_n \bar{z}^n.$$

The change of the field ϕ under this transformation is

$$\delta\phi(w, \overline{w}) = \frac{1}{2\pi i}\oint_C \mathrm{d}z\, \varepsilon(z) T(z) \phi(w, \overline{w}) + \frac{1}{2\pi i}\oint_C \mathrm{d}\bar{z}\, \bar{\varepsilon}(\bar{z}) \overline{T}(\bar{z}) \phi(w, \overline{w}) = I_1 + I_2, \tag{3}$$

where the contour C winds once around w. This change is only dependent on the local properties of the integrand near $z = w$. Eq. (3) can be viewed as the definition of the stress-energy tensor. We can rewrite (3) as equal-time commutation relations in radial quantization. Let's consider the term

$$I_1 = \frac{1}{2\pi i}\oint_C \mathrm{d}z\, \varepsilon(z) T(z) \phi(w, \overline{w}).$$

Since the integrand is nonsingular except at $z = w$, we can change the integration to the following

$$I_1 = \frac{1}{2\pi i}\oint_{C_+} \mathrm{d}z\, \varepsilon(z) T(z) \phi(w, \overline{w}) - \frac{1}{2\pi i}\oint_{C_-} \mathrm{d}z\, \varepsilon(z) T(z) \phi(w, \overline{w}),$$

where contours C_+ and C_- are contours $|z| > |w|$ and $|z| < |w|$ respectively and the minus sign comes from changing the direction of the integration along C_-. Remember that we have to time order operators in the quantization. Along the contour C_+ the operator $T(z)$ is later than the operator ϕ so $T(z)$ should be on the left of the ϕ. Along the contour C_- $T(z)$ should be on the right of the ϕ. Therefore as operators we should write I_1 as

$$I_1 = \frac{1}{2\pi i}\oint_{C_+} \mathrm{d}z\, \varepsilon(z) T(z) \phi(w, \bar{w}) - \frac{1}{2\pi i}\oint_{C_-} \mathrm{d}z\, \varepsilon(z) \phi(w, \bar{w}) T(z).$$

Using the definition of the operator L we obtain the following commutation relations

$$I_1 = \sum_{n=-\infty}^{\infty} \varepsilon_{n+1} [L_n, \phi(w, \bar{w})].$$

Similarly we obtain

$$\delta\phi(w, \bar{w}) = \sum_{n=-\infty}^{\infty} \varepsilon_{n+1} [L_n, \phi(w, \bar{w})] + \sum_{n=-\infty}^{\infty} \bar{\varepsilon}_{n+1} [\bar{L}_n, \phi(w, \bar{w})]. \tag{4}$$

It is therefore easy to see that L_n and $\bar{L}_n$ are the operators which generate the infinitesimal conformal transformations $z \to z + \varepsilon z^{n+1}$, $\bar{z} \to \bar{z} + \bar{\varepsilon}\bar{z}^{n+1}$ with ε a complex number.

There are several conformal transformations which are of particular interest because they can be defined globally. The operators L_0 and $\bar{L}_0$ generate the dilations in z and $\bar{z}$ respectively

$$z \to z + \varepsilon z, \qquad \bar{z} \to \bar{z} + \bar{\varepsilon}\bar{z}.$$

The operator $L_0 + \bar{L}_0$ generates the dilation of the two-dimensional plane

$$z \to z + \lambda z, \qquad \bar{z} \to \bar{z} + \lambda \bar{z},$$

where λ is real number. In terms of the (x, y) coordinates,

$$x \to x + \lambda x, \qquad y \to y + \lambda y.$$

Similarly, we can see that the operator $L_0 - \bar{L}_0$ generates rotations in the two-dimensional plane. The operators L_{-1} and $\bar{L}_{-1}$ generate translations. Later we will see that operators L_0, $\bar{L}_0$, L_{-1}, $\bar{L}_{-1}$ together with L_1 and $\bar{L}_1$ form the group of global conformal transformations.

The stress-energy tensor is a special local field which has the following operator product expansions

$$T(z)T(z) \sim \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \text{nonsingular terms},$$

$$\bar{T}(\bar{z})\bar{T}(\bar{w}) \sim \frac{c}{2(\bar{z}-\bar{w})^4} + \frac{2\bar{T}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\partial_{\bar{w}} \bar{T}(\bar{w})}{(\bar{z}-\bar{w})} + \text{nonsingular terms},$$

$$T(z)\bar{T}(\bar{w}) \sim \text{nonsingular terms}. \tag{5}$$

Operator product expansions like (5) mean that inside any correlation function the left-hand side is equivalent to the right. The form of the right-hand side is found by euclidean invariance, i.e. it is translation invariant and has the right scaling dimension. In particular eq. (5) is the statement that the commutator of two successive conformal transformations is a conformal transformation.

The change of the stress-energy tensor under conformal transformations (2) is given by (3). We can do the integration (3) directly by using the operator product expansions (5). We obtain

$$[L_n, T(z)] = z^{n+1}\partial_z T(z) + 2(n+1)z^n T(z) + \tfrac{1}{12}cn(n^2-1)z^{n-2},$$

$$[\bar{L}_n, \bar{T}(\bar{z})] = \bar{z}^{n+1}\partial_{\bar{z}} \bar{T}(\bar{z}) + 2(n+1)\bar{z}^n \bar{T}(\bar{z}) + \tfrac{1}{12}cn(n^2-1)\bar{z}^{n-2},$$

$$[L_n, \bar{T}(\bar{z})] = 0, \qquad [\bar{L}_n, T(z)] = 0. \tag{6}$$

We should note that these commutation relations are derived only from the singular part of the operator product expansions (5). From eqs. (1), (6) and the reality of $T_{\mu\nu}$, L_n and $\bar{L}_n$ satisfy the following hermiticity and commutation relations

$$L_n^\dagger = L_{-n}, \qquad \bar{L}_n^\dagger = \bar{L}_{-n},$$

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{12}cm(m^2-1)\delta_{m+n,0},$$

$$[\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n} + \tfrac{1}{12}cm(m^2-1)\delta_{m+n,0},$$

$$[L_m, \bar{L}_n] = 0. \tag{7}$$

The algebra formed by the $L_n(\bar{L}_n)$ is called the Virasoro algebra [6]. Thus the algebra of local conformal transformations is a product of two commuting Virasoro algebras.

The central term c is called the central charge which comes from quantization of the theory. For each conformal theory there is one corresponding c value. So it parameterizes the conformal theory. Physically, c measures the response of the system to changes of the background geometry – the trace anomaly [7].

The Hilbert space of the conformal theory can be decomposed into irreducible representations of the two Virasoro algebras. The dilation operator $L_0 + \overline{L}_0$ is the hamiltonian and so its eigenvalues should be bounded below. Since the L_n and $\overline{L}_n$ algebras commute with each other, each representation of the two Virasoro algebras can be decomposed into a representation of the L_n algebra tensoring a representation of the $\overline{L}_n$ algebra. We can treat the representation of the L_n Virasoro algebra separately. The discussion can also be carried out for the $\overline{L}_n$ algebra in parallel. A *state* is an eigenstate of L_0 with eigenvalue h. The eigenvalue of L_0 is the *energy* of that state. From the commutation relations (7)

$$[L_0, L_n] = -nL_n .$$

Therefore L_n is a raising operator if $n < 0$ and a lowering operator if $n > 0$. It raises (lowers) the energy of the state by n. A state $|h\rangle$ is a *ground* state if it is annihilated by all the lowering operators

$$L_0|h\rangle = h|h\rangle , \qquad L_n|h\rangle = 0 \qquad \text{for } n > 0 .$$

The whole Hilbert space is generated from the ground states by applying the raising operator.

The vacuum $|0\rangle$ is the ground state of lowest energy. It is annihilated by L_n and $\overline{L}_n$ with $n \geq -1$, because of the requirement that the stress-energy tensor be nonsingular at $z = 0$. In particular, the vacuum is invariant under the global conformal transformations generated by L_n and $\overline{L}_n$, $n = -1, 0, 1$. These operators generate the well-known fractional linear transformations in complex analysis

$$z \to \frac{az + b}{cz + d} , \qquad ad - bc = 1 ,$$

which are the global conformal transformations of the plane. Consequently all the correlation functions are invariant under global conformal transformations.

The fundamental fluctuating fields of the conformal theory are *conformal fields*. The conformal field $\phi(z, \bar{z})$ is an operator such that $\phi(z, \bar{z})\mathrm{d}z^h \mathrm{d}\bar{z}^{\bar{h}}$ is invariant under local conformal transformations. This is equivalent to the following operator product expansions

$$T(z)\phi(w, \overline{w}) \sim \frac{h\phi(w, \overline{w})}{(z - w)^2} + \frac{\partial_w \phi(w, \overline{w})}{z - w} ,$$

$$\overline{T}(\bar{z})\phi(w, \overline{w}) \sim \frac{\bar{h}\phi(w, \overline{w})}{(\bar{z} - \overline{w})^2} + \frac{\partial_{\overline{w}} \phi(w, \overline{w})}{\bar{z} - \overline{w}} . \qquad (8)$$

By using (3) and (4) we can express these operator product expansions as the following commutation relations

$$[L_n, \phi(z,\bar{z})] = z^{n+1}\partial_z\phi(z,\bar{z}) + h(n+1)z^n\phi(z,\bar{z}),$$
$$[\bar{L}_n, \phi(z,\bar{z})] = \bar{z}^{n+1}\partial_{\bar{z}}\phi(z,\bar{z}) + \bar{h}(n+1)\bar{z}^n\phi(z,\bar{z}). \tag{9}$$

The physical meaning of h and $\bar{h}$ can be understood by studying correlation functions of conformal fields. Correlation functions of conformal fields are invariant under global conformal transformations. In particular, the form of the two-point function is determined:

$$\langle 0|\phi(z,\bar{z})\phi(0,0)|0\rangle = \langle 0|\phi(re^{i\theta}, re^{-i\theta})\phi(0,0)|0\rangle = r^{-2(h+\bar{h})}e^{2i(h-\bar{h})\theta}$$

(One uses the identity $\phi(z,\bar{z}) = z^{L_0}\bar{z}^{\bar{L}_0}\phi(1,1)z^{L_0-h}\bar{z}^{\bar{L}_0-\bar{h}}$.) So we see that ϕ has scaling dimension $h+\bar{h}$ and spin $h-\bar{h}$. Moreover the three-point correlation functions of conformal fields are also determined by the global conformal transformations:

$$\begin{aligned}
&\langle 0|\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)|0\rangle \\
&\quad = C_{123}(z_1-z_2)^{h_3-h_1-h_2}(z_2-z_3)^{h_1-h_2-h_3} \\
&\qquad \times (z_3-z_1)^{h_2-h_3-h_1}(\bar{z}_1-\bar{z}_2)^{\bar{h}_3-\bar{h}_1-\bar{h}_2}(\bar{z}_2-\bar{z}_3)^{\bar{h}_1-\bar{h}_2-\bar{h}_3}(\bar{z}_3-\bar{z}_1)^{\bar{h}_2-\bar{h}_3-\bar{h}_1},
\end{aligned} \tag{10}$$

where the constant C_{123} is called an operator product coefficient.

The conformal field $\phi(z,\bar{z})$ generates another state from the vacuum

$$L_0\phi(0,0)|0\rangle = [L_0, \phi(0,0)]|0\rangle = h\phi(0,0)|0\rangle,$$
$$\bar{L}_0\phi(0,0)|0\rangle = [\bar{L}_0, \phi(0,0)]|0\rangle = \bar{h}\phi(0,0)|0\rangle, \tag{11}$$

where we have used the commutation relation (9). Next we show that the state $\phi(0,0)|0\rangle$ is also a ground state, i.e. it is annihilated by all the lowering operators, L_n and $\bar{L}_n$ with $n>0$. Since lowering operators annihilate the vacuum

$$L_n\phi(0,0)|0\rangle = [L_n, \phi(0,0)]|0\rangle = 0,$$
$$\bar{L}_n\phi(0,0)|0\rangle = [\bar{L}_n, \phi(0,0)]|0\rangle = 0, \qquad \text{for } n>0, \tag{12}$$

where the second step is the result of (9) by putting $z=0$ and $\bar{z}=0$. Therefore there is a correspondence between the conformal field and the ground state of the

Virasoro algebra. The state $|0\rangle$ can be regarded as the state generated from the vacuum by the identity operator. The states generated from the ground states by the L_n and $\bar{L}_n$ are called *descendant* states. The corresponding fields are called *descendant* fields. All the descendant fields can be represented by applying the raising operators on the conformal field. Therefore the conformal fields and the stress-energy tensor completely determine the operator structure of the conformal theory. We need to find all the ground states, or equivalently, to find all the conformal fields which generate all the ground states from the vacuum.

BPZ study what they call degenerate theories. The dimensions of the conformal fields in this class of theories are given by simple algebraic equations. The correlation functions of these fields satisfy differential equations arising from the constraints of conformal symmetry.

We will discuss the simplest example of these. Let $|h\rangle$ be a ground state normalized by $\langle h|h\rangle = 1$. We can create a ladder of states from it by using raising operators. The states at level n are those with L_0 eigenvalue $n+h$. At level 1 there is only one such state, $L_{-1}|h\rangle$. The matrix of the inner product of states at this level is

$$M_1(h,c) = \langle h|L^{\dagger}_{-1}L_{-1}|h\rangle = \langle h|2L_0|h\rangle = 2h\,.$$

At level 2 there are two states $L^2_{-1}|h\rangle$ and $L_{-2}|h\rangle$. The inner product matrix is given by

$$M_2(h,c) = \begin{bmatrix} 4h(2h+1) & 6h \\ 6h & 4h+\frac{1}{2}c \end{bmatrix}.$$

In general this inner product matrix can be calculated level by level using the commutation and hermiticity relations (7). We do not need the exact form of this matrix in conformal theory. What we really need is the determinant of this matrix. The determinants at level one and two are

$$\det M_1(h,c) = 2h\,,$$

$$\det M_2(h,c) = 4h\left(8h^2 + h(c-5) + \tfrac{1}{2}c\right) = 32h(h-h_{1,2})(h-h_{2,1})\,,$$

where $h_{1,2}$ and $h_{2,1}$ are the two roots of the quadratic equation. If $h = h_{1,2}$ or $h = h_{2,1}$, there will be some combination of the two states at level 2 which has zero norm. In fact the state $|\text{null}\rangle = (L_{-2} - (3h/2(2h+1))L^2_{-1})|h\rangle$ will have zero norm when $h = h_{1,2}$ or $h = h_{2,1}$. The state $|\text{null}\rangle$ is orthogonal to both level-two states generated from the ground state $|h\rangle$. It is easy to show that states at different levels and from different ground states are orthogonal to each other. Therefore $|\text{null}\rangle$ is orthogonal to all the states.

Let ϕ_1 be a conformal field with dimension h. We will use this null state to derive differential equations for the correlation functions $\langle 0|\phi_n(z_n)\ldots\phi_2(z_2)\phi_1(z_1)|0\rangle$

(dependence on $\bar{z}_i$ is implicitly understood). We make a translation $z \to z' = z - z_1$ which leaves the correlation function invariant

$$\langle 0|\phi_n(z_n')\dots\phi_2(z_2')\phi_1(z_1')|0\rangle = \langle 0|\phi_n(z_n)\dots\phi_2(z_2)\phi_1(z_1)|0\rangle ,$$

where $z_1' = z_1 - z_1 = 0$. The state $|\text{null}\rangle$ is orthogonal to all the states so

$$\langle 0|\phi_n(z_n')\dots\phi_2(z_2')\left(L_{-2} - \frac{3h}{2(2h+1)}L_{-1}^2\right)\phi_1(0)|0\rangle = 0 .$$

Using commutation relations (5), the fact that L_{-2} and L_{-1} annihilate the state $\langle 0|$ and global conformal invariance of the correlation function is achieved, we arrive at the following differential equation

$$O\langle 0|\phi_n(z_n)\dots\phi_2(z_2)\phi_1(z_1)|0\rangle = 0 ,$$

with differential operator

$$O = \left\{ \sum_{i=2}^{n}\left[z_i'^{-1}\frac{\partial}{\partial z_i'} - h_i z_i'^{-2}\right] + \frac{3h}{2(2h+1)}\sum_{i=2}^{n}\sum_{j=2}^{n}\frac{\partial}{\partial z_i'}\frac{\partial}{\partial z_j'}\right\} .$$

This equation can be rewritten, using global conformal invariance of the correlation function, as

$$\left\{\frac{3h}{2(2h+1)}\frac{\partial^2}{\partial z_1^2} - \sum_{i=2}^{n}\left[h_i(z_i - z_1)^{-2} - (z_i - z_1)^{-1}\frac{\partial}{\partial z_i}\right]\right\}$$
$$\times\langle 0|\phi_n(z_n)\dots\phi_2(z_2)\phi_1(z_1)|0\rangle = 0 . \tag{13}$$

Kac wrote the determinant at level n [8]:

$$\det M_n(h, c) = \text{constant} \prod_{pq \le n}(h - h_{pq})^{P(n-pq)} , \tag{14}$$

where $P(i)$ is the number of states generated from a ground state at level i, i.e. the classical partition function, defined by

$$\prod_{n=1}^{\infty}(1 - t^n)^{-1} = \sum_{i=0}^{\infty}P(i)t^i . \tag{15}$$

The quantities $h_{p,q}$ are defined by

$$h_{p,q} = \frac{[p(m+1) - qm]^2 - 1}{4m(m+1)} ,$$

$$c = 1 - \frac{6}{m(m+1)} , \tag{16}$$

where p, q are positive integers. The conformal field $\phi_{p,q}$ is an operator of dimension $h_{p,q}$. In general there is a null state at level pq generated from ground state $|h_{p,q}\rangle$ because there is a zero eigenvalue in Kac's determinant at that level. The null state at that level is

$$\sum_{\Sigma i\lambda_i = pq} a_{\lambda_1\lambda_2}\ldots L_{-1}^{\lambda_1}L_{-2}^{\lambda_2}\ldots|h_{p,q}\rangle,$$

where $a_{\lambda_1\lambda_2}\ldots$ are constants which can be determined by the condition that the null state is orthogonal to all the other states at the same level. Using the same procedure as before we can show that the correlation functions containing the conformal field $\phi_{p,q}$ satisfy differential equations of order pq. In the degenerate conformal theories of BPZ [2], all the conformal fields are of the form $\phi_{p,q}$. From (16) we can easily see that the degenerate theories correspond to the region $c < 1$ (or $h < 0$).

The other important property of degenerate conformal theory is that the conformal fields of the theory have very simple operator product expansions

$$\phi_{p_1,p_2}\phi_{q_1,q_2} \sim \sum_{k_1=|p_1-q_1|+1}^{p_1+q_1+1} \quad \sum_{k_2=|p_2-q_2|+1}^{p_2+q_2+1} C_{(pqk)}\phi_{k_1,k_2}, \tag{17}$$

where constants $C_{(pqk)}$ are the only nonzero operator product coefficients and the coordinate dependence of (17) can be easily put back by dimensional argument. The operator product coefficients along with the dimensions of conformal fields are the parameters that specify a conformal field theory. We can calculate them from the correlation functions of conformal fields and vice versa.

We now discuss how unitarity constrains the possible conformal QFT. In ref. [3], FQS have discussed this problem with the help of Kac's determinant. We only give the result here. For $c \geq 1$ all the theories are unitary provided $h \geq 0$. For $c < 1$ the only possible unitary theories are those in (9) with

$$m = 2, 3, \ldots, \qquad 0 \leq p < m, \qquad 1 \leq q \leq m. \tag{18}$$

FQS showed that (18) is necessary for unitarity but did not show that the representations in (18) are unitary. That all the representations in (18) are indeed unitary has since been shown by explicit construction [9]. We will return to this point later.

The first few of the theories in (18) correspond to well-known statistical mechanical models. They are the Ising model ($m = 3$), the tricritical Ising model ($m = 4$), the three-state Potts model ($m = 5$) and the tricritical three-state Potts model ($m = 6$). Recent results of Andrews, Baxter and Forrester provide explicit realizations of the remaining models [10]. Other realizations of (18) are q-state Potts models with irrational values of q. There are some common features for this class of theories. All the ground states lie on the vanishing curves $h = h_{p,q}$, so the results of BPZ apply to them. Unitarity provides a physical reason to study such theories.

There is now a complete understanding of two-dimensional QFT with $c < 1$. For each m in (18), i.e. each c, there is a set of allowed values from representations of the Virasoro algebra, $\{v_i\}$. All the conformal fields in the theory must have $h = v_i$ and $\bar{h} = v_j$. All the correlation functions of conformal fields satisfy differential equations. From solutions of differential equations we can calculate all the operator product coefficients. We should mention that a conformal QFT does not have to use all the $\{v_i\}$ as long as we can find a closed set of conformal fields. Therefore there could be several theories at each c value.

One other systematic method to calculate the correlation functions of conformal fields of degenerate conformal theories is called the Feigin-Fuchs integral representation [11], which is basically a kind of Coulomb gas representation. In this method, one starts with one free scalar field ϕ, which has $c = 1$. One modifies the boundary conditions by putting the operator $\exp(2\alpha_0\phi)$ at infinity. The effect of this background charge is to change the trace anomaly to $c = 1 - 24\alpha_0^2$. By putting in appropriate screening charges, with dimensions $h = \bar{h} = 1$, we can represent correlation functions of conformal fields in degenerate conformal theories as multiple integrals. These multiple integrals are the solutions of the BPZ differential equations discussed earlier. By requiring crossing symmetry (which is the associativity in applying operator product expansions to calculate the correlation functions) we can obtain the complete (including z and $\bar{z}$ dependence) correlation function.

For $c > 1$, conformal invariance and unitarity are not strong enough to completely specify the theory. There is a large set of possible conformal QFTs which we know little about. There are several subsets which are very interesting. One is the sigma model with Wess-Zumino (WZ) term [12], which has a nonabelian continuous symmetry in addition to conformal symmetry. In particular, the sigma model with WZ term based on a group manifold has Kac-Moody symmetry in addition to conformal symmetry [13]. There is a natural construction of the Virasoro algebra from the Kac-Moody algebra. The correlation functions in these theories satisfy a set of differential equations arising from the relations between the Virasoro algebra and the Kac-Moody algebra [14]. Therefore these theories can be solved in the same sense as we discussed earlier. Another application of these theories is that we can construct all the unitary representations (18) of the Virasoro algebra. Goddard, Olive and Kent [9] use this method to prove the unitarity of (18).

Another class of conformal QFTs with $c > 1$ are sigma models defined on the Calabi-Yau manifolds, which are compact manifolds with vanishing Ricci tensor. These conformal QFTs are of particular importance in the string compactifications [4, 15].

3. Supersymmetry

In many statistical mechanical systems, there are not only scalar and tensor operators ($h - \bar{h} =$ integer) but also fermionic operators ($h - \bar{h} =$ integer $+ \frac{1}{2}$). For example, we can describe the Ising model by one free fermion field. Theories with an equal number of fermionic degree of freedom and bosonic degree of freedom are of particular interest. These theories can have a symmetry which changes fermionic fields into bosonic fields and vice versa. This symmetry is called a supersymmetry, and a QFT which has such a symmetry is called a supersymmetric QFT. Supersymmetric QFTs have been extensively studied in particle physics [16]. In an earlier paper [4], FQS have pointed out that supersymmetry is also a property of certain statistical mechanical systems. FQS studied the supersymmetric generalization of conformal field theory, superconformal field theory. We found that by combining superconformal invariance and unitary, superconformal QFTs are severely limited in the region $c < \frac{3}{2}$. In the rest of this paper we will discuss such theories in detail.

Supersymmetry is basically the square root of a translation, i.e. performing two supersymmetry transformations in succession is equivalent to a translation. In this paper we will consider two space-time dimensions and $N = 1$ or $(1,1)$ supersymmetry. It is convenient to work in superspace. In the superspace formalism one introduces fermionic coordinates, θ and $\bar{\theta}$, in addition to the normal coordinates, z and $\bar{z}$. The coordinate $\theta(\bar{\theta})$ has one-half the scaling dimension of $z(\bar{z})$. They are anti-commuting numbers, $\theta_1\theta_2 = -\theta_2\theta_1$, etc., in particular $\theta^2 = 0$. We will use the notations $\mathfrak{z} = (z, \theta)$ and $\bar{\mathfrak{z}} = (\bar{z}, \bar{\theta})$. A superfield $\Phi(\mathfrak{z}, \bar{\mathfrak{z}})$ is a function defined on superspace. It can be expanded as a power series in θ and $\bar{\theta}$

$$\Phi(\mathfrak{z}, \bar{\mathfrak{z}}) = \phi(z, \bar{z}) + \theta\psi(z, \bar{z}) + \bar{\theta}\bar{\psi}(z, \bar{z}) + \theta\bar{\theta}F(z, \bar{z}). \tag{19}$$

The supersymmetry transformation can be written as

$$(z, \theta) \to (z - \hat{\varepsilon}\theta, \theta + \hat{\varepsilon}), \qquad (\bar{z}, \bar{\theta}) \to (\bar{z} - \hat{\bar{\varepsilon}}\bar{\theta}, \bar{\theta} + \hat{\bar{\varepsilon}}),$$

where $\hat{\varepsilon}$ and $\hat{\bar{\varepsilon}}$ are anti-commuting c-numbers which are parameters of the transformation. Note that as in conformal transformations, the bar part and the unbarred part are independent. The transformations of the $\mathfrak{z}$ and $\bar{\mathfrak{z}}$ parts have the same structure, and so we will only deal with the $\mathfrak{z}$ part in most of the following. Restricted to $\mathfrak{z}(\bar{\mathfrak{z}})$ the symmetry is called an $N = \frac{1}{2}$ or $(1,0)$ $((0,1))$ supersymmetry. So a $N = 1$ supersymmetry consists of two $N = \frac{1}{2}$ supersymmetries. An $N = \frac{1}{2}$ superfield can be written as $\Phi(\mathfrak{z}) = \phi(z) + \theta\psi(z)$. Under a supersymmetry transformation parameterized by $\hat{\varepsilon}$

$$\begin{aligned}\Phi(z, \theta) \to \Phi(z - \hat{\varepsilon}\theta, \theta + \hat{\varepsilon}) &= \phi(z - \hat{\varepsilon}\theta) + (\theta + \hat{\varepsilon})\psi(z - \hat{\varepsilon}\theta)\\ &= \phi(z) - \hat{\varepsilon}\theta\, \partial_z\phi(z) + \theta\psi(z) + \hat{\varepsilon}\psi(z).\end{aligned}$$

The changes in the component fields are

$$\delta_\varepsilon\phi(z) = \hat{\varepsilon}\psi(z), \qquad \delta_\varepsilon\psi(z) = \hat{\varepsilon}\, \partial_z\phi(z).$$

We now make another supersymmetry transformation parameterized by $\hat{\eta}$

$$\delta_\eta \delta_\varepsilon \phi(z) = \delta_\eta \hat{\varepsilon} \psi(z) = -\hat{\varepsilon}\hat{\eta}\, \partial_z \phi(z),$$

$$\delta_\eta \delta_\varepsilon \psi(z) = \delta_\eta \hat{\varepsilon}\, \partial_z \phi(z) = -\hat{\varepsilon}\hat{\eta}\, \partial_z \psi(z).$$

The change of all the component fields is the same as under the translation $z \to z + \hat{\eta}\hat{\varepsilon}$.

To further illustrate how the superspace formulation is related to familiar notations in field theory, we give the action of one free superfield

$$S = \int \mathrm{d}z\, \mathrm{d}\bar{z}\, \mathrm{d}\theta\, \mathrm{d}\bar{\theta}\, \bar{D}\Phi(z,\bar{z})\, D\Phi(z,\bar{z}) \tag{20}$$

where $D = \partial_\theta + \theta\, \partial_z$ and $\bar{D} = \partial_{\bar{\theta}} + \bar{\theta}\, \partial_{\bar{z}}$ are super derivatives. Integrations over θ and $\bar{\theta}$ are carried out by the following rules

$$\int \mathrm{d}\theta = 0, \qquad \int \mathrm{d}\theta\, \theta = 1, \qquad \int \mathrm{d}\bar{\theta} = 0, \qquad \int \mathrm{d}\bar{\theta}\, \bar{\theta} = 1.$$

The resulting action is

$$S = \int \mathrm{d}z\, \mathrm{d}\bar{z} \left[\partial_{\bar{z}}\phi\, \partial_z \phi - \bar{\psi}\, \partial_z \bar{\psi} - \psi\, \partial_{\bar{z}} \psi + F^2 \right]. \tag{21}$$

The first term is the action of one free scalar field. The second and third terms are the action of one free Majorana fermion field, which is equivalent to the Ising model. The fields ψ and $\bar{\psi}$ are related to the usual two-component form of the fermion by

$$\psi = \psi_1 + i\psi_2, \qquad \bar{\psi} = \psi_1 - i\psi_2.$$

The fourth term can be eliminated by its equation of motion because it has no dynamics and can be integrated out. So we see that one free superfield can be used to describe a gaussian model combined with an Ising model.

We should note that in the free field case the scaling dimension of the fermion field is one-half and its spin is one-half, which is equivalent to saying that ψ has $h = \frac{1}{2}$ and $\bar{h} = 0$. So the supersymmetry transformation increases h by one half. We use $G_{-1/2}$ to represent the supersymmetry transformation. The fact that the square of a supersymmetry transformation is a translation can be represented as

$$G_{-1/2}^2 = \tfrac{1}{2}\left[G_{-1/2}, G_{-1/2}\right]_+ = L_{-1}.$$

$G_{-1/2}$ increases the eigenvalue of L_0 by one half. This is equivalent to the commutation relation

$$\left[L_0, G_{-1/2}\right] = \tfrac{1}{2}G_{-1/2}.$$

The full superconformal transformations are a closed set of generators including the Virasoro algebra and the supersymmetry transformation which have the following (anti-) commutation relations

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \tfrac{1}{8}\hat{c}n(n^2-1)\delta_{n+m,0},\\
[L_n, G_m] &= \left(\tfrac{1}{2}n - m\right)G_{n+m},\\
[G_n, G_m]_+ &= 2L_{n+m} + \tfrac{1}{2}\hat{c}\left(n^2 - \tfrac{1}{4}\right)\delta_{n+m,0}.
\end{aligned} \tag{22}$$

$\hat{c} = \frac{2}{3}c$ and all the G_n with $n \neq -\frac{1}{2}$ are the commutators of conformal transformations with the supersymmetry transformation

$$\left(\tfrac{1}{2}n + \tfrac{1}{2}\right)G_{n'} = \left[L_n, G_{-1/2}\right], \qquad n' = n - \tfrac{1}{2}.$$

The algebra (22) is called the Neveu-Schwarz algebra [17]. Later we will see that there is a second superconformal algebra, the Ramond algebra [18].

4. Superconformal theory

The general structure of superconformal theory has been discussed in ref. [4]. We will give a brief review in this section and some further discussion. The superconformal transformations are the generalization of the conformal transformations to superspace. The infinitesimal superconformal transformations can be written as

$$\begin{aligned}
&\theta \to \theta + \tfrac{1}{2}Dv(z), \qquad z \to z + v(z) - \tfrac{1}{2}\theta\, Dv(z),\\
&\bar{\theta} \to \bar{\theta} + \tfrac{1}{2}\bar{D}\bar{v}(\bar{z}), \qquad \bar{z} \to \bar{z} + \bar{v}(\bar{z}) - \tfrac{1}{2}\bar{\theta}\,\bar{D}\bar{v}(\bar{z}),
\end{aligned} \tag{23}$$

where $v(z)$ is given by

$$v(z) = v_0(z) + \theta v_1(z) = \sum_{n=-\infty}^{\infty} v_{0,n}z^n + \theta \sum_{m=-\infty}^{\infty} v_{1,m}z^m,$$

where v_1 is a Grassmann function. The superconformal transformations are generated by the super stress-energy tensor, $T(z) = T_F(z) + \theta T_B(z)$ and $\bar{T}(\bar{z}) = \bar{T}_F(\bar{z}) + \bar{\theta}T_B(\bar{z})$, as in the conformal theories. T_B and $\bar{T}_B$ are the ordinary stress-energy tensor. T_F and $\bar{T}_F$ are their fermionic partners.

Suppose $\Phi(z, \bar{z})$ is a local superfield. Under superconformal transformations (23), the change of Φ is given by

$$\delta\Phi(w) = \frac{1}{2\pi i}\oint_C \mathrm{d}z\,\mathrm{d}\theta\, v(z)T(z)\Phi(w,\bar{w}) + \frac{1}{2\pi i}\oint_C \mathrm{d}\bar{z}\,\mathrm{d}\bar{\theta}\,\bar{v}(\bar{z})\bar{T}(\bar{z})\Phi(w,\bar{w}), \quad (24)$$

where the contour C is around $z = 0$ and $\theta, \bar{\theta}$ integrations are performed by the rules we discussed earlier. We can regard (24) as the definition of super stress-energy tensor.

We again use the radial quantization by transforming from planar superspace (z, θ) to the cylindrical superspace $(\log z, z^{-1/2}\theta) = (\tau + i\sigma, z^{-1/2}\theta)$. Fermionic fields on the cycle can be periodic or anti-periodic when we rotate them around the cylinder once. So there are two natural boundary conditions for the T_{F} under transformation $z \to \mathrm{e}^{2\pi i}z$, $T_{\mathrm{F}}(z\mathrm{e}^{2\pi i}) = \pm T_{\mathrm{F}}(z)$. We only presented one case in the previous discussion about the free superfield. The most general superconformal theory includes both boundary conditions. The Hilbert space of superconformal QFTs contain two subspaces: the Neveu-Schwarz subspace with $T_{\mathrm{F}}(z\mathrm{e}^{2\pi i}) = T_{\mathrm{F}}(z)$ and the Ramond subspace with $T_{\mathrm{F}}(z\mathrm{e}^{2\pi i}) = -T_{\mathrm{F}}(z)$. As in the conformal field theory we can restrict discussion to the z dependence in most of the following. When adding $\bar{z} = (\bar{z}, \bar{\theta})$ part we will say so explicitly. The operator Fourier expansion of $T(z)$ is

$$T_{\mathrm{F}}(z) + \theta T_{\mathrm{B}}(z) = \sum_n z^{-n-3/2}T_{\mathrm{F},n} + \theta\sum_n z^{-n-2}T_{\mathrm{B},n}. \quad (25)$$

The conventional notations are $G_n = 2T_{\mathrm{F},n}$ and $L_n = T_{\mathrm{B},n}$.

In the Neveu-Schwarz sector the index of G_n is half integral and in the Ramond sector the index of G_n is integral. The super stress-energy tensor has the following operator product expansion

$$T(z_1)T(z_2) \sim \tfrac{1}{4}\hat{c}z_{12}^{-3} + \left(2\theta_{12}z_{12}^{-2} + \tfrac{1}{2}z_{12}^{-1}D_2 + \theta_{12}z_{12}^{-1}\partial_2\right)T(z_2), \quad (26)$$

where $\theta_{12} = \theta_1 - \theta_2$, $z_{12} = z_1 - z_2 - \theta_1\theta_2$ and $\hat{c}$ is related to the central charge c by $\hat{c} = \frac{2}{3}c$. The terms on the right-hand side of (26) are the most general terms we can write which are euclidean invariant, supersymmetric and agree with (5) when restricted to the T_{B} part. Using the same procedure as in the conformal field theories we can obtain the (anti-) commutation relations (22) from (24)–(26). From the reality of T we have the following hermiticity conditions

$$L_n^{\dagger} = L_{-n}, \qquad G_n^{\dagger} = G_{-n}. \quad (27)$$

The algebras corresponding to the Neveu-Schwarz and the Ramond boundary conditions are called the Neveu-Schwarz and the Ramond algebras, respectively.

A superfield $\Phi(z)=\phi(z)+\theta\psi(z)$ is a conformal superfield if it satisfies the operator product expansion

$$T(z_1)\Phi(z_2)\sim h\theta_{12}z_{12}^{-2}\Phi(z_2)+\tfrac{1}{2}z_{12}^{-1}D_2\Phi(z_2)+\theta_{12}z_{12}^{-1}\partial_2\Phi(z_2). \tag{28}$$

We can reexpress the above operator product expansion in terms of the following commutation relations

$$\begin{aligned}\left[L_n,\Phi(z)\right]&=z^{n+1}\partial_z\Phi(z)+\tfrac{1}{2}(n+1)z^n\theta\,\partial_\theta\Phi(z)+h(n+1)z^n\Phi(z),\\ \left[\hat{\varepsilon}G_n,\Phi(z)\right]&=\hat{\varepsilon}\left[z^{n+1/2}(\partial_\theta-\theta\,\partial_z)-2h\left(n+\tfrac{1}{2}\right)\theta z^{n-1/2}\right]\Phi(z),\end{aligned} \tag{29}$$

where $\hat{\varepsilon}$ is an anti-commuting number.

Our discussion in the conformal field theory can easily be generalized to the superconformal theory. We will focus on the representations of $\{L_n,G_n\}$ algebra here. The same discussion can be carried out for $\{\bar{L}_n,\bar{G}_n\}$ algebra in parallel. We discuss eigenstates $|h\rangle$ of the L_0 with eigenvalue h. From the commutation relations

$$\left[L_0,L_n\right]=-nL_n,$$

$$\left[L_0,G_n\right]=-nG_n.$$

L_n and G_n are lowering operators for $n>0$ and raising operator for $n<0$. A state $|h\rangle$ is a ground state if it is annihilated by all the lowering operators. The raising operators acting on the ground state generate an irreducible representation of the superconformal algebra. The complete representation is generated by applying raising operators to the ground states.

The vacuum $|0\rangle$ is the ground state of energy $h=0$. The condition that the super stress-energy tensor is nonsingular at $z=0$ implies

$$G_m|0\rangle=0,\qquad L_n|0\rangle=0,\quad \text{when } n\geq-1 \text{ and } m\geq-\tfrac{1}{2}.$$

Therefore the vacuum belongs to the Neveu-Schwarz sector. It is invariant under the global superconformal group OSP(2|1), i.e. it is annihilated by the five generators $L_{-1}, L_0, L_1, G_{-1/2}, G_{1/2}$. There is a correspondence between superconformal fields and ground states of the Neveu-Schwarz algebra, $|h\rangle=\Phi(0,0)|0\rangle$.

The correspondence between the ground states and the representations of the Ramond algebra is much more complicated because of the existence of the operator G_0 which commutes with L_0. The ground states will be degenerate, $|h^+\rangle$ and $|h^-\rangle=G_0|h^+\rangle$. The Ramond algebra contains the Virasoro algebra as a subalgebra. The ground states of the Virasoro algebra are associated with ordinary conformal fields. Therefore the Ramond ground states $|h^\pm\rangle$ are created from the vacuum by conformal fields $\Theta^\pm(z)$, which are called spin fields, of dimension h, $|h^\pm\rangle=$

$\Theta^{\pm}(0)|0\rangle$. The action of T_F on the Ramond ground states is given by the operator product expansion

$$T_F(z)\Theta^{\pm}(w) \sim \tfrac{1}{2}(z-w)^{-3/2}a_{\pm}\Theta^{\mp}(w), \tag{30}$$

where $a_+ = 1$ and $a_- = h - \frac{1}{16}\hat{c}$. Thus T_F is double valued with respect to the spin fields. In fact all the fermionic fields are double valued with respect to the spin fields. The spin fields change the boundary conditions of the superconformal theory. Suppose the system starts with Neveu-Schwarz boundary condition at $\tau = -\infty$. Then the fermionic fields are anti-periodic on the cylinder. At some time τ_0 the appearance of a spin field changes the system from the Neveu-Schwarz boundary condition to the Ramond boundary condition. After time τ_0 all the fermionic operators become periodic. At some later time τ_1 another spin field will change the system back to the Neveu-Schwarz boundary condition. So the spin field acts like an operator which opens or closes a cut on the cylinder. This is identical to the effect of order-disorder operators in the Ising model. The system with the Ramond boundary condition can be realized by putting spin fields at $\tau = -\infty$ and $\tau = \infty$. In the cylindrical superspace we find that in the Ramond sector all of the fermionic parts of the superfield become periodic in the space σ. The superfields acting on the Ramond sector form a self-contained supersymmetric conformal theory on the cylinder. The global supersymmetry is generated by the operator G_0. Unbroken supersymmetry is equivalent to that there exists a ground state $|h\rangle$ in the Ramond sector such that $G_0|h\rangle = 0$.

$$G_0|h^-\rangle = G_0^2|h^+\rangle = \left(L_0 - \tfrac{1}{16}\hat{c}\right)|h^+\rangle.$$

Therefore the existence of the Ramond field with $h = \frac{1}{16}\hat{c}$ is equivalent to the condition of unbroken supersymmetry.

The Hilbert space of a superconformal theory is divided into two sectors. One sector contains states with the Neveu-Schwarz boundary condition, called the Neveu-Schwarz sector. Another sector contains states with the Ramond boundary condition, called the Ramond sector. Therefore states of a superconformal theory can be written in a two-component vector form with the first component in the Neveu-Schwarz sector and the second in the Ramond sector. The operators are two-by-two matrices. The superconformal fields are diagonal, giving a Neveu-Schwarz state when acting on a Neveu-Schwarz state and a Ramond state on a Ramond state. The spin fields are off-diagonal, changing Neveu-Schwarz states to Ramond states and vice-versa.

The fermionic part of a superconformal field is nonlocal with respect to a spin field as a consequence of the square root singularity in the operator product expansion of the spin field with the fermionic field. The pair of spin fields $\Theta^{\pm}$ are nonlocal with respect to each other since their operator product containing fermionic fields, which is the analogue of the order-disorder operators in the Ising model. We have to make a projection to get a local theory. There are two possibilities. We can project on the part of the theory which only contains the superconformal fields. The Hilbert space will only contain the Neveu-Schwarz sector. The more interesting projection is to project out all the fermionic parts of the superconformal fields and half of the Ramond fields. To define this projection we define the operator $\Gamma = (-1)^F$. F is the fermion number operator which counts the number of fermions in a state. Γ is called the chirality operator because of its role in superstring theory. Γ commutes with all the bosonic parts of superfields and anticommutes with the fermionic parts. The two spin fields $\Theta^{\pm}$ are of opposite chirality from (30). The projection $\Gamma = 1$, i.e. the part of Hilbert space with $\Gamma = 1$, will yield a local theory since spin fields with the same chirality are local with respect to each other (because there is no fermionic field in their operator products). The resulting theory is called the spin model, which is of particular importance in superstring theory [4, 19].

In parallel to our discussion of unitarity in conformal field theory, we will now discuss unitarity in superconformal theory. For p, q positive integers we define

$$\hat{c}(\hat{m}) = 1 - \frac{8}{\hat{m}(\hat{m}+2)},$$

$$h_{p,q} = \frac{[(\hat{m}+2)p - \hat{m}q]^2 - 4}{8\hat{m}(\hat{m}+2)} + \tfrac{1}{32}\left(1 - (-1)^{p-q}\right). \tag{31}$$

The determinant formula for the Neveu-Schwarz algebra has been found by Kac [8] and was proved by Meurman, Rocha-Carida [20] and Curtwright-Thorn [21]

$$\det(M_n) = \text{const} \prod \left[h - h_{p,q}\right]^{P_{\mathrm{NS}}(n - pq/2)}, \tag{32}$$

where the product runs over p, q with $p - q$ even and $P_{\mathrm{NS}}(k)$ is the number of states, arising from a ground state, at level k:

$$\sum_{k=0}^{\infty} t^k P_{\mathrm{NS}}(k) = \prod_{k=0}^{\infty} \frac{1 + t^{k-1/2}}{1 - t^k}.$$

The determinant formula for the Ramond algebra has been found by FQS [4] and was proven by Meurman, Rocha-Carida [20] and Curtwright-Thorn [21]

$$\det(M_0^+) = 1, \qquad \det(M_0^-) = h - \tfrac{1}{16}\hat{c},$$

$$\det(M_n^+) = \det(M_n^-) = \left(h - \tfrac{1}{16}\hat{c}\right)^{P_{\mathrm{R}}(n)/2} \prod \left[h - h_{p,q}(\hat{c})\right]^{P_{\mathrm{R}}(n - pq/2)}, \tag{33}$$

where the product runs over p, q with $p - q$ odd and $P_{\mathrm{R}}(k)$ is half the number of states at the kth level

$$\sum_{k=0}^{\infty} t^k P_{\mathrm{R}}(k) = \prod_{k=0}^{\infty} (1+t^k)/(1-t^k).$$

The $\pm$ indices refer to inner product matrices of states with positive and negative chirality. States with different chirality are orthogonal to each other. In refs. [3,4] FQS also found all the possible unitary representations of the Neveu-Schwarz and the Ramond algebras. If $\hat{c} \geq 1$ all the representations with $h \geq 0$ are unitary. For $\hat{c} < 1$ there is a discrete list of possible unitary representations

$$\hat{c} = \hat{c}(\hat{m}), \qquad \hat{m} = 2,3,4,\ldots, \qquad h = h_{p,q}(\hat{m}), \qquad 1 \leq p < \hat{m}, 1 \leq q < \hat{m}+2. \tag{34}$$

The Neveu-Schwarz representations are given by the $h_{p,q}$ with $p-q$ even [3], the Ramond representations by the $h_{p,q}$ with $p-q$ odd [4]. We use $\Phi_{p,q}$ to represent the field with dimension $h_{p,q}$. The striking feature of the list (34) is that the possible representations of the two superconformal algebras occur at the same values of $\hat{c}$. It was this coincidence that originally suggested that there is a spin model for every superconformal QFT.

5. Superconformal formalism of the Neveu-Schwarz sector

Now we use the superconformal formalism to study the Neveu-Schwarz sector. The two-point function is completely determined by the OSP (2|1) invariance of the vacuum. The z dependence is given by $\langle 0|\Phi(z_1)\Phi(z_2)|0\rangle \sim z_{12}^{-2h}$. Suppose Φ is a scalar operator, i.e. $h = \bar{h}$. Adding the $\bar{z}$ dependence we obtain the two-point function

$$\langle 0|\Phi(z_1)\Phi(z_2)|0\rangle = |z_{12}|^{-4h}. \tag{35}$$

Using the component expansion $\Phi(z) = \phi(z,\bar{z}) + \theta\psi(z,\bar{z}) + \bar{\theta}\bar{\psi}(z,\bar{z}) + \theta\bar{\theta}F(z,\bar{z})$, all the two-point functions of the component fields can be calculated by expanding (35) and matching the $\theta, \bar{\theta}$ in the expansions of both sides

$$\langle 0|\phi(z_1)\phi(z_2)|0\rangle = |z_1 - z_2|^{-4h},$$

$$\langle 0|F(z_1)F(z_2)|0\rangle = -4h^2|z_1 - z_2|^{-4h-2},$$

$$\langle 0|\psi(z_1)\psi(z_2)|0\rangle = -2h|z_1 - z_2|^{-4h}(z_1 - z_2)^{-1},$$

$$\langle 0|\bar{\psi}(z_1)\bar{\psi}(z_2)|0\rangle = -2h|z_1 - z_2|^{-4h}(\bar{z}_1 - \bar{z}_2)^{-1}. \tag{36}$$

In the Neveu-Schwarz sector of superconformal field theory, there are five global symmetry transformations from OSP(2|1) invariance. The three-point function of superconformal field has six independent variables (z_i, θ_i). There is one combination of these variables which is invariant under all the five transformations. It is the anit-commuting combination

$$\hat{\eta} = (z_{12}z_{23}z_{31})^{-1/2}(\theta_1 z_{23} + \theta_2 z_{31} + \theta_3 z_{12} + \theta_1\theta_2\theta_3).$$

Since the square of an anti-commuting number is zero, the z dependence of the three-point function of superconformal fields can be written

$$\langle 0|\Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)|0\rangle \sim z_{12}^{h_3-h_1-h_2} z_{23}^{h_1-h_2-h_3} z_{31}^{h_2-h_3-h_1}(1+\hat{B}\hat{\eta}), \quad (37)$$

where $\hat{B}$ is an anti-commuting number.

We say an expression is even(odd) if it contains an even(odd) number of θ's or even(odd) number of $\bar{\theta}$'s. So the three point function (37) has an even and an odd part. When we combine the $\bar{z}$ dependence, a correlation function will have two parts, one is both even in θ and $\bar{\theta}$ another is both odd in θ and $\bar{\theta}$. The three-point function after combining the $\bar{z}$ dependent part, is

$$\langle 0|\Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)|0\rangle$$
$$= z_{12}^{h_3-h_1-h_2} z_{23}^{h_1-h_2-h_3} z_{31}^{h_2-h_3-h_1} \bar{z}_{12}^{\bar{h}_3-\bar{h}_1-\bar{h}_2} \bar{z}_{23}^{\bar{h}_1-\bar{h}_2-\bar{h}_3} \bar{z}_{31}^{\bar{h}_2-\bar{h}_3-\bar{h}_1}\left[a_1 + a_2|\hat{\eta}|^2\right]. \quad (38)$$

The reader might note that there is an ambiguity in $|\hat{\eta}|^2$. We will use the convention that the z dependent factor is always to the left of the $\bar{z}$ dependent factor, e.g. $|\hat{\eta}|^2 = \hat{\eta}\bar{\hat{\eta}}$.

We should also mention another important fact about superconformal theory. Since there are two independent constants in the three-point function (38), there are in general two operator product coefficients to be determined.

Now we are going to discuss the differential equations for superconformal field theory. Correlation functions of superconformal field theories in the region $\hat{c} < 1$ satisfy superdifferential equations, analogous to the situations in ordinary conformal field theories. A superdifferential equation is a generalization of an ordinary differential equation to superspace. It is equivalent to a set of differential equations in component form.

Here we derive one such equation arising from a null state generated from a representation of the Neveu-Schwarz algebra. We look at the null state at the lowest possible level. The first nontrivial null state is at level $\frac{3}{2}$.

$$|\text{null}\rangle = (G_{-3/2} - aL_{-1}G_{-1/2})|h\rangle, \qquad a = \frac{2}{2h+1},$$

where $h = h_{1,3}$ or $h_{3,1}$. The state $|h\rangle$ is associated with a superconformal field Φ by $|h\rangle = \Phi(0,0)|0\rangle$. The field Φ is either $\Phi_{1,3}$ or $\Phi_{3,1}$. Suppose Φ_1 is such a field and $\Phi_2, \Phi_3, \dots, \Phi_n$ are arbitrary superconformal fields. We can obtain the equation for the n-point function $\langle 0|\Phi_n(z_n)\dots\Phi_2(z_2)\Phi_1(z_1)|0\rangle$ as follows. First we make a translation

$$z \to z - z_1, \qquad \theta \to \theta$$

and then a supersymmetry transformation

$$\theta \to \theta - \theta_1, \qquad z \to z + \theta_1\theta .$$

Under these two transformations one has $z_i \to z_{i1}$ and $\theta_i \to \theta_{i1}$. Since the vacuum is invariant under these transformations, the whole correlation function does not change:

$$\langle 0|\Phi_n(z_n)\dots\Phi_2(z_2)\Phi_1(z_1)|0\rangle = \langle 0|\Phi_n(z_n')\dots\Phi_2(z_2')\Phi_1(z_1')|0\rangle .$$

Note that $z_1' = (0,0)$. Now by inserting the operator $(G_{-3/2} - aL_{-1}G_{-1/2})$, we get

$$\langle 0|\Phi_n(z_n')\dots\Phi_2(z_2')\left(G_{-3/2} - aL_{-1}G_{-1/2}\right)\Phi_1(z_1')|0\rangle = 0$$

since the null state is orthogonal to all the other states. Commuting all L's and G's to the left by the commutation relations (29) and noting that they annihilate the vacuum $\langle 0|$, we obtain

$$O_s\langle 0|\Phi_n(z_n)\dots\Phi_2(z_2)\Phi_1(z_1)|0\rangle = 0 ,$$

where

$$O_s = \sum_{i=2}^{n} z_{i1}^{-1}\left[\frac{\partial}{\partial\theta_{i1}} - \theta_{i1}\,\partial_{z_{i1}} - \theta_{i1}2h_i z_{i1}^{-1}\right] + a\sum_{i=2}^{n}\sum_{j=2}^{n}\left[\frac{\partial}{\partial\theta_{i1}} - \theta_{i1}\frac{\partial}{\partial z_{i1}}\right]\frac{\partial}{\partial z_{j1}} .$$

The superdifferential equation is of order $\frac{3}{2}$. This equation can be simplified by the OSP(2|1) invariance of the correlation functions. The result is

$$O_s'\langle 0|\Phi_n(z_n)\dots\Phi_2(z_2)\Phi_1(z_1)|0\rangle = 0 , \tag{39}$$

where

$$O_s' = \frac{2}{2h+1}\frac{\partial}{\partial z_1}\left(\frac{\partial}{\partial\theta_1} - \theta_1\frac{\partial}{\partial z_1}\right) - \sum_{i=2}^{n}\left[z_{i1}^{-1}\left(\frac{\partial}{\partial\theta_i} - \theta_{i1}\frac{\partial}{\partial z_i}\right) - 2h_i\theta_{i1}z_{i1}^{-2}\right].$$

In general there is a null state at level $\frac{1}{2}pq$ corresponding to the ground state $|h_{p,q}\rangle$. We can use this null state to derive superdifferential equations for correlation functions containing $\Phi_{p,q}$. The superdifferential equation is said to be of the order $\frac{1}{2}pq$.

6. Tricritical Ising model (TIM) as a superconformal theory

We now discuss the first nontrivial model in (34): $\hat{m}=3$, $\hat{c}=\frac{7}{15}$, $c=\frac{7}{10}$. This is the only value of c which occurs in the discrete lists for both the superconformal algebras and the Virasoro algebra [3,4]. In refs. [3,4] FQS identified $c=\frac{7}{10}$ with the tricritical Ising model. All the known operators in the TIM can be constructed from the representation of the Virasoro algebra at $c=\frac{7}{10}$. Kadanoff [22] confirmed the identification using correlation functions calculated by himself and Nienhuis [23].

The TIM is the Ising model with vacancies. It is defined by the following lattice hamiltonian [24]

$$H=\beta\sum_{(ij)}\sigma_i\sigma_j t_i t_j-\mu\sum_i t_i\,, \tag{40}$$

where the sum (ij) runs through all the nearest neighbors. The Ising spin σ takes values ± 1 and the density t takes values 0, 1. This system has a tricritical point at β_0 and μ_0.

Since the TIM is very similar to the Ising model, we give a very brief review of the Ising model here. In the Ising model we can construct the disorder operator [25] from the Ising spin operator by taking a chain of spin operators from infinity. We construct the fermionic operator by taking the product of order and disorder operators. If we move the fermionic operator along a closed loop in the plane, the fermionic operator will pick up a minus sign if there is a order or disorder operator inside the loop. The Ising model has a Z_2 symmetry which flips the order operator and disorder operator. The energy operator and the fermionic operators are the Z_2 even operators. The spin operator and the disorder operator are Z_2 odd operators. We can describe the Ising model, at least at its critical point, by one set of local operators because the model is local. This set can be the order operator and the energy operator, or the disorder operator and the energy operator, or the fermionic operator and the energy operator. Among other things, the description by the fermionic operator and the energy operator uses only Z_2 even operators.

Now in the TIM we can similarly construct disorder operators from the Ising spin operator. The TIM is invariant under the Z_2 symmetry which flips the Ising spins and the disorder operators. As FQS pointed out in refs. [3,4] the Z_2 even sector is described by representations of the Neveu-Schwarz algebra and the Z_2 odd sector is described by the representations of the Ramond algebra. The TIM can be described by the Z_2 even operators alone, therefore it is a superconformal field theory.

The ground state energies allowed by unitarity and ordinary conformal invariance are [3] $h=0, \frac{3}{80}, \frac{1}{10}, \frac{7}{16}, \frac{3}{5}$, and $\frac{3}{2}$. To describe the operators of the TIM we use the $(h, \bar{h})$ notation which gives the dimension $(h+\bar{h})$ and the spin $(h-\bar{h})$ of a conformal field. The Z_2 even operators of the TIM are [26]: the energy operator ε with $(h, \bar{h})=(\frac{1}{10}, \frac{1}{10})$; the vacancy operator t with $(\frac{6}{10}, \frac{6}{10})$; and an irrelevant operator $T_F \bar{T}_F$ with $(\frac{3}{2}, \frac{3}{2})$. The Z_2 odd operators of the TIM consist of the leading magnetic spin operator σ with $(\frac{3}{80}, \frac{3}{80})$ and the subleading magnetic operator with $(\frac{7}{16}, \frac{7}{16})$ [27]. The disorder operators are also in the Z_2 odd sector.

The ground state energies allowed by unitarity and superconformal invariance in the Neveu-Schwarz sector [3] are $h_{1,1}=0$ and $h_{2,2}=h_{3,1}=\frac{1}{10}$. They decompose into irreducible representations of the Virasoro algebra:

$$(0)_{\mathrm{NS}}=(0)_{\mathrm{Vir}} \oplus \left(\tfrac{3}{2}\right)_{\mathrm{Vir}},$$

$$\left(\tfrac{1}{10}\right)_{\mathrm{NS}}=\left(\tfrac{1}{10}\right)_{\mathrm{Vir}} \oplus \left(\tfrac{6}{10}\right)_{\mathrm{Vir}}.$$

This tells us that all the Z_2 even operators of the TIM can be combined into a single superfield with $(h, \bar{h})=(\frac{1}{10}, \frac{1}{10})$:

$$\Phi(z)=\varepsilon(z)+\theta\psi(z)+\bar{\theta}\bar{\psi}(z)+\theta\bar{\theta}t(z) \tag{41}$$

and the super stress-energy tensor

$$T(z)=T_F(z)+\theta T_B(z), \qquad \bar{T}(\bar{z})=\bar{T}_F(\bar{z})+\bar{\theta}\bar{T}_B(\bar{z}). \tag{42}$$

All fermionic fields $\psi, \bar{\psi}, T_F, \bar{T}_F$ are the fermionic operators we discussed earlier. They are constructed by taking products of order and disorder operators. To obtain all the operators of the TIM we define one projection operator $\Gamma\bar{\Gamma}$. The only operators left after the projection $\Gamma\bar{\Gamma}=1$ in the Neveu-Schwarz sector are ε, t, $T_F\bar{T}_F$. These are precisely the Z_2 even operators in the TIM.

The allowed representations of the Ramond algebra are $h_{1,2}=\frac{3}{80}$ and $h_{2,1}=\frac{7}{16}$, each consisting of exactly one irreducible representation of the Virasoro algebra. These are the energies of the leading and subleading magnetic operators and disorder operators. These operators take Z_2 even states to Z_2 odd and vice-versa, and so intertwine the Ramond and Neveu-Schwarz sectors of the theory. So the magnetic operators and the disorder operators are the spin fields. The magnetic operators are the $\Gamma\bar{\Gamma}=1$ spin fields and the disorder operators are the $\Gamma\bar{\Gamma}=-1$ spin fields.

The $\Gamma\bar{\Gamma}=1$ projection of the superconformal model is the maximal mutually local set of operators. It gives precisely all the usual local operators in the TIM. Therefore the TIM is a spin model.

Now let us concentrate on the Z_2 even sector of the TIM. We will study this sector by the superconformal theory formalism discussed earlier. The operator product of

$T_F \bar{T}_F$ with the superconformal field Φ is determined by eq. (28) or equivalently the commutation relations (29). The only other information we need is the operator product expansion of Φ with itself. The first thing we should notice is that the operator Φ is a $\Phi_{3,1}$ operator. Therefore all the correlation functions of the superfield Φ satisfy the superdifferential equation (39).

We start by examining the two-point function $\langle 0|\Phi(z_1)\Phi(z_2)|0\rangle$. From (36) we have

$$\begin{aligned}
\langle 0|\varepsilon(z_1)\varepsilon(z_2)|0\rangle &= |z_1 - z_2|^{-2/5}, \\
\langle 0|t(z_1)t(z_2)|0\rangle &= -\tfrac{1}{25}|z_1 - z_2|^{-12/5}, \\
\langle 0|\psi(z_1)\psi(z_2)|0\rangle &= -\tfrac{1}{5}|z_1 - z_2|^{-2/5}(z_1 - z_2)^{-1}, \\
\langle 0|\bar{\psi}(z_1)\bar{\psi}(z_2)|0\rangle &= -\tfrac{1}{5}|z_1 - z_2|^{-2/5}(\bar{z}_1 - \bar{z}_2)^{-1},
\end{aligned} \tag{43}$$

where the minus sign in front of some two-point functions is because we worked in the chiral basis. The physical operators are ψ_1, ψ_2 and it. The three-point function is given by (38). Since Φ is the $\Phi_{3,1}$ operator, the three-point function should satisfy the super differential equation (32), which provides additional information about the operator product expansion. Substituting the three-point function (38) into (39), we find that the even part of the three-point function vanishes unless $h = 0$. So the three-point function is given by

$$\langle 0|\Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)|0\rangle \sim (z_{12}z_{23}z_{31})^{-3/5}(\theta_1 z_{23} + \theta_2 z_{31} + \theta_3 z_{12} + \theta_1\theta_2\theta_3).$$

Next we combine z and $\bar{z}$ dependence together to get the whole three-point function

$$\langle 0|\Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)|0\rangle \sim \left|(z_{12}z_{23}z_{31})^{-3/5}(\theta_1 z_{23} + \theta_2 z_{31} + \theta_3 z_{12} + \theta_1\theta_2\theta_3)\right|^2. \tag{44}$$

Eq. (44) implies the following operator algebra

$$\begin{gathered}
\varepsilon\varepsilon \sim I - 25Ct, \qquad \varepsilon\psi \sim 5C\bar{\psi}, \qquad \varepsilon\bar{\psi} \sim 5C\psi, \qquad \varepsilon t \sim C\varepsilon, \\
tt \sim \tfrac{-1}{5}I + Ct, \qquad t\psi \sim -C\psi, \qquad t\bar{\psi} \sim C\bar{\psi}, \\
\psi\psi \sim -\tfrac{1}{5}I - 5t, \qquad \psi\bar{\psi} \sim -C\varepsilon, \qquad \bar{\psi}\bar{\psi} \sim -\tfrac{1}{5}I + 5Ct,
\end{gathered} \tag{45}$$

where C is a unknown constant. C is the only remaining unknown information about the theory. One should note that this operator algebra is consistent with the generic operator product formula of conformal theory (17) and that of the Neveu-Schwarz sector of the superconformal theory [5].

We should mention that all the four-point functions of scalar conformal fields ($h=\bar{h}$) in the TIM can be calculated by using the Feigin-Fuchs method [11] or differential equation techniques in conformal field theory. In particular all the operator product coefficients of the Z_2 even sector of the TIM can be determined from conformal field theory. But in conformal field theory, the conformal field t is the $\phi_{1,3}$ conformal field. Its correlation will satisfy a third-order differential equation. So the explicit form of such a correlation function only exists as a multiple contour integral even though the operator product coefficient can be calculated.

We will now present a different way of calculating all the correlation functions (including fermionic operators) of the even sector of the TIM by using superdifferential equations. The four-point correlation function of Φ satisfies the superdifferential equation

$$\left\{\frac{5}{3}\frac{\partial}{\partial z_1}\left(\frac{\partial}{\partial\theta_1}-\theta_1\frac{\partial}{\partial z_1}\right)-\sum_{i=2}^{4}\left[z_{i1}^{-1}\left(\frac{\partial}{\partial\theta_i}-\theta_{i1}\frac{\partial}{\partial z_i}\right)-2h_i\theta_{i1}z_{i1}^{-2}\right]\right\}$$

$$\times\langle 0|\Phi_4(z_4)\Phi(z_3)\Phi_2(z_2)\Phi_1(z_1)|0\rangle=0. \qquad (46)$$

The four-point function splits into even and odd parts. From the operator algebra (45) it is easy to show that the odd part of the four-point function vanishes. For example let us consider the part of the four-point function with only $\theta_1\bar{\theta}_2$. It corresponds to $\langle 0|\psi(z_4)\bar{\psi}(z_3)\varepsilon(z_2)\varepsilon(z_1)|0\rangle$. From (38) we have

$$\psi\bar{\psi}\sim -C\varepsilon, \qquad \varepsilon\varepsilon\sim I-25Ct.$$

Then it is obvious. Similarly we can show that all the other odd terms vanish.

There are eight independent variables (z_i,θ_i) and five constraints from global superconformal invariance. There are two even and one odd independent combinations which are invariant under global superconformal transformations. The even ones are

$$\eta=\frac{z_{12}z_{34}}{z_{13}z_{24}}, \qquad \xi=\frac{z_{14}z_{23}}{z_{13}z_{24}}-(1-\eta).$$

Since $\xi^2=0$, the general solution of (39) is of the form $S(\eta,\xi)=(z_{12}z_{23}z_{34}z_{41})^{-1/10}(f(\eta)+\xi g(\eta))$. Substituting this into (39) we get the following equations for $f(\eta)$ and $g(\eta)$

$$50\eta(1-\eta)f''-50(1-\eta)g'+(45-70\eta)f'+\left(6-\frac{3}{\eta}\right)f-\left(50-\frac{25}{\eta}\right)g=0,$$

$$50\eta(1-\eta)f''-50\eta(1-\eta)g'+(20-45\eta)f'+(5+20\eta)g-\left(\frac{7}{2\eta}+3\right)f=0. \qquad (47)$$

This equation has three independent solutions. Two of them are given by

$$S_1(\eta,\xi)=\left(1+\xi\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\right)\left(\left[\eta(1-\eta)\right]^{-1/10}F\left(\tfrac{1}{5},-\tfrac{2}{5},\tfrac{2}{5},\eta\right)\right)$$

$$+\tfrac{3}{35}\xi\left[\eta(1-\eta)\right]^{9/10}F\left(\tfrac{8}{5},\tfrac{11}{5},\tfrac{12}{5},\eta\right),$$

$$S_2(\eta,\xi)=\left(1+\xi\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\right)\left(\left[\eta(1-\eta)\right]^{1/2}F\left(\tfrac{7}{5},\tfrac{4}{5},\tfrac{8}{5},\eta\right)\right)$$

$$+\tfrac{1}{5}\xi\left[\eta(1-\eta)\right]^{-1/2}F\left(-\tfrac{6}{5},-\tfrac{3}{5},-\tfrac{2}{5},\eta\right), \tag{48}$$

where F is the hypergeometric function. We will show that they are the only general solutions consistent with (45). The argument goes as follows. Let all θ be zero, so that we are left only with the bosonic part of the correlation function $\langle 0|\varepsilon(z_1)\varepsilon(z_2)\varepsilon(z_3)\varepsilon(z_4)|0\rangle$. The singularities of this function are determined by the operator product expansion of the ε with itself. The two solutions in (41) are the only solutions that have the correct singularity structure, which is given by the operator product expansions (45). In terms of the differential equations (47), we can obtain a third-order differential equation of $f(\eta)$ alone. This differential equation factorizes. The similar thing happens in conformal field theories. For example in (18) $h_{p,q}=h_{m-p,m+1-q}$, so correlation functions containing a conformal field of dimension $h=h_{p,q}=h_{m-p,m+1-q}$ satisfy a differential equation of order pq and of order $(m-p)(m+1-q)$. The higher order differential equation will factorize into the lower one. All the extra solutions of the higher order equation have the wrong singularity structure and therefore have no contribution to the correlation functions.

The $\bar{z}$ dependence of the four-point function satisfied the same equation. Combining both z and $\bar{z}$ dependencies the four-point correlation function is

$$\langle 0|\Phi(z_1)\Phi(z_2)\Phi(z_3)\Phi(z_4)|0\rangle = A_{11}S_1(\eta,\xi)\overline{S_1(\eta,\xi)}+A_{12}S_1(\eta,\xi)\overline{S_2(\eta,\xi)}$$

$$+A_{21}S_2(\eta,\xi)\overline{S_1(\eta,\xi)}+A_{22}S_2(\eta,\xi)\overline{S_2(\eta,\xi)}.$$

We can determine all the constants A_{ij} by requiring single valuedness under monodromy transformations [28]. For simplicity we put all θ and $\bar{\theta}$ to zero. The resulting four-point function is

$$A=A_{11}I_1(\eta_0)\overline{I_1(\eta_0)}+A_{12}I_1(\eta_0)\overline{I_2(\eta_0)}$$

$$+A_{21}I_2(\eta_0)\overline{I_1(\eta_0)}+A_{22}I_2(\eta_0)\overline{I_2(\eta_0)},$$

where

$$I_1(\eta_0) = S_1(\eta, \xi)|_{\theta=0, \bar{\theta}=0} = \left([\eta_0(1-\eta_0)]^{-1/10} F(\tfrac{1}{5}, -\tfrac{2}{5}, \tfrac{2}{5}, \eta_0\right),$$

$$I_2(\eta_0) = S_2(\eta, \xi)|_{\theta=0, \bar{\theta}=0} = [\eta_0(1-\eta_0)]^{1/2} F(\tfrac{7}{5}, \tfrac{4}{5}, \tfrac{8}{5}, \eta_0)$$

and

$$\eta_0 = \eta|_{\theta=0, \bar{\theta}=0} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}.$$

These solutions have singularities at $\eta_0 = 0, 1, \infty$. We require that the solution should be single valued when we analytically continue it along any path in the complex plane (the so-called monodromy transformations). Here we only need to require that the solution is single valued when we take η_0 around $0, 1, \infty$. Using the properties of the hypergeometric functions and normalizing by two-point functions, the final four-point function is given by

$$\langle 0|\Phi(z_1)\Phi(z_2)\Phi(z_3)\Phi(z_4)|0\rangle = |z_{12}z_{23}z_{34}z_{41}|^{-1/5} \times\left(|S_1(\eta, \xi)|^2 + A|S_2(\eta, \xi)|^2\right), \tag{49}$$

where

$$A = \frac{4}{9}\frac{\Gamma(\frac{4}{5})\Gamma^3(\frac{2}{5})}{\Gamma(\frac{1}{5})\Gamma^3(\frac{3}{5})}.$$

We can use this result to finish the determination of the operator algebra (38). The unknown constant C can be determined from any four-point function e.g. $\langle 0|\varepsilon\varepsilon\varepsilon\varepsilon|0\rangle$

$$C = i\frac{1}{15}\sqrt{\frac{\Gamma(\frac{4}{5})\Gamma^3(\frac{2}{5})}{\Gamma(\frac{1}{5})\Gamma^3(\frac{3}{5})}}. \tag{50}$$

These results agree with those obtained for the scalar operators from ordinary conformal theory [11]. The uniqueness of the result ensures us that the $c = \frac{7}{10}$ conformal theory is automatically a superconformal theory. The superconformal techniques also yield new correlation functions involving fermionic operators.

7. Conclusion and discussions

We have discussed in detail the structure of the Neveu-Schwarz sector of superconformal field theories. The superdifferential equation enables us to have a complete understanding of the Z_2 even sector of the TIM. One important question remaining is the calculation of Ramond sector correlation functions. Since we can view operators in the Ramond sector as ordinary conformal operators, the operator algebra and correlation functions can be determined by the techniques of ordinary conformal theory. We use σ and X to represent the operator $(h, \bar{h}) = (\frac{3}{80}, \frac{3}{80})$ and $(\frac{7}{16}, \frac{7}{16})$, respectively. The operator product from conformal theory is (neglecting all the constants)

$$\sigma\sigma \sim I + \varepsilon + t + T_F\bar{T}_F, \qquad \sigma X \sim I + \varepsilon + t + T_F\bar{T}_F,$$

$$XX \sim I + T_F\bar{T}_F, \qquad \text{etc.}$$

All the operator product coefficients can be calculated from the solutions obtained by Dotsenko and Fateev [11]. The differential equation technique can be applied to the Ramond sector as well [29].

The physical systems described by the TIM, e.g. adsorbed helium on krypton-plated graphite [30], are the first known realizations of supersymmetric field theories in nature. The reader may ask what is special about supersymmetry in the TIM. First of all, the vanishing of certain operator product coefficients seems accidental from the point of view of conformal field theory. These relationships as well as those between correlation functions of operators that appear rather different (e.g. the energy and vacancy operators) are naturally explained by superconformal invariance. Since the TIM is supersymmetric at its tricritical point, supersymmetric perturbations can take the TIM away from the tricritical point but still preserve the supersymmetry. The physical systems described by the TIM can be used for experiments on supersymmetry in statistical mechanics. In statistical mechanics the supersymmetry would improve the behavior of the free energy near the critical point, making it less singular. It will be very interesting to find the manifest supersymmetric formulation of the TIM near its tricritical point.

In the list of refs. [3,4] or (34) there are an infinite number of superconformal field theories which have the same structure as the TIM. It would be interesting to find their physical realization. The family of supersymmetric critical systems will be very important for both theory and experiment. Using the Feigin-Fuchs integral representation techniques [5] the four-point functions in the Neveu-Schwarz sector can be represented as the sum of multiple contour integrations. So we will be able to calculate all the operator product coefficients in these theories. In the Ramond sector solutions of the differential equations will enable us to calculate the operator product coefficients for spin fields as well.

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References

[1] Phase transition and critical phenomena, ed. C. Domb and M.S. Green (Academic Press, London, New York)
[2] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Stat. Phys. 15 (Edinburgh, 1983) [J. Stat. Phys. 34 (1984) 763]; Nucl. Phys. B241 (1984) 333
[3] D. Friedan, Z. Qiu and S.H. Shenker, *in* Vertex operators in mathematics and physics, Proc. Conf. November 10–17, 1983, ed. J. Lepowsky (Springer, New York, 1984); Phys. Rev. Lett. 52 (1984) 1575
[4] D. Friedan, Z. Qiu and S.H. Shenker, Phys. Lett. 151B (1985) 37
[5] M.A. Bershadsky, V.G. Knizhnik and M.G. Teitelman, Phys. Lett. 151B (1985) 31
[6] M.A. Virasoro, Phys. Rev. (1970) 2933;
I.M. Gel'fand and D.B. Fuks, Funkts. Anal. Prilozhen. 2 (1968) 92
[7] D. Friedan, *in* Common trends in field theory and statistical mechanics, Les Houches, summer, 1982, ed. J.B. Zuber
[8] V.G. Kac, *in* Proc. Int. Congress of mathematicians (Helsinki, 1978); *in* Group theoretical methods in physics, ed. W. Beiglbock and A. Bohm, Lecture Notes in Physics, vol. 94 (Springer, Berlin, 1979) p. 441;
B.L. Feigin and K.B. Fuchs, Functs Anal. Prilozhen. 16 (1982) 47 [Funct. Anal. and Appl. 16 (1982) 114];
[9] P. Goddard, A. Kent and D. Olive, Phys. Lett. 152B (1985) 88
[10] G.E. Andrews, R.J. Baxter and P.J. Forrester, J. Stat. Phys. 35 Nos. 3/4 (1984) 193;
D.A. Huse, Phys. Rev. B30 (1984) 3908
[11] B.L. Feigin and D.B. Fuchs, Moscow preprint (1983);
Vl.S. Dotsenko and V.A. Fateev, Nucl. Phys. B240 (1984) 312
[12] T. Curtwright and C. Zachos, ANL preprint (1984);
E. Witten, Comm. Math. Phys. 92 (1984) 455
[13] P. Goddard and D. Olive, *in* Vertex operators in mathematics and physics, Proc. Conf. November 10–17, 1983, ed. J. Lepowsky (Springer, New York, 1984)
[14] V. Knizhnik and A.B. Zamolodchikov, Nucl. Phys. B247 (1984) 83
[15] D. Friedan and S. Shenker, Proc. Santa Fe meeting of the APS Division of Particle and Fields, October 29–November 3, 1984, to be published
P. Candelas, G.T. Horowitz, A. Strominger and E. Witten, Nucl. Phys. B258 (1985) 46
[16] J. Wess and J. Bagger, Supersymmetry and supergravity (Princeton Univ. Press)
[17] A. Neveu and J.H. Schwarz, Nucl. Phys. B31 (1971) 86; Phys. Rev. D4 (1971) 1109
[18] P. Ramond, Phys. Rev. D3 (1971) 2415
[19] D. Friedan, E. Martinec and S.H. Shenker, EFI preprint EFI 85-32
[20] A. Meurman and A. Rocha-Carida, in preparation
[21] T. Curtwright and C.B. Thorn, Univ. of Florida preprint (1984);
[22] L.P. Kadanoff, unpublished
[23] B. Nienhuis and L. Kadanoff, unpublished
[24] M.J. Blume, V.J. Emery and R.B. Griffiths, Phys. Rev. A4 (1971) 1071;
B. Nienhuis, A.N. Berker, E.K. Riedel and M. Schick, Phys. Rev. Lett. 43 (1979) 737
[25] L.P. Kadanoff and H. Ceva, Phys. Rev. B3 (1971) 3918
[26] B. Nienhuis, J. Phys. A15 (1982) 199;
D. Huse, J. Phys. A16 (1983) 4357
[27] M.P.M. den Nijs, Phys. Rev. B27 (1983) 1674
[28] Vl.S. Dotsenko, L.D. Stat. Phys. 15 [J. Stat. Phys. 34 (1984) 781]; Nucl. Phys. B241 (1984) 333
[29] D. Friedan, private communication
[30] M.J. Tejwani, O. Ferreira and O.E. Vilches, Phys. Rev. Lett. 44 (1980) 152;
W. Kinzel, M. Schick, and A.N. Berker, Ordering in two dimensions, ed. Sinha (North-Holland, 1980)