SUPERCONFORMAL INVARIANCE IN TWO DIMENSIONS AND THE TRICRITICAL ISING MODEL

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Received 25 October 1984

We discuss the realization of superconformal invariance in two dimensional quantum field theory. The Hilbert space of a superconformal theory splits into two sectors; one a representation of the Neveu–Schwarz algebra, the other of the Ramond algebra. We introduce the spin fields which intertwine the two sectors and correspond to the irreducible representations of the Ramond algebra. We give the determinant formula for the Ramond algebra and the discrete list of possible unitary representations. We have previously noted that the Z_2 even sector of the tricritical Ising model is a representation of the Neveu–Schwarz algebra. Here we complete the picture by showing that the Z_2 odd sector forms a representation of the Ramond algebra. This system is the first experimentally realizable supersymmetric field theory.

Superconformal field theories are the supersymmetric generalizations of conformally invariant field theories. Every supersymmetric local field theory is superconformally invariant at short distances, so the realizations of superconformal invariance completely determine the possible supersymmetric field theories. In statistical mechanics superconformal field theories describe special critical points at which the macroscopic physics is supersymmetric.

In two dimensions the superconformal algebra has infinitely many generators. The structure of the algebra is rigid enough to place significant restrictions on the field theories in which it is realized, extending the restrictions imposed by the ordinary infinite dimensional conformal algebra, the Virasoro algebra [1,2].

The first examples of supersymmetry appeared in string theory [3,4]. In fact, they were examples of two dimensional superconformal invariance – theories of free massless superfields on the world-surface of the string. The two superconformal algebras, the Ramond [3] and Neveu–Schwarz [4] algebras, act as gauge symmetries of the string.

In ordinary conformal field theory the scaling dimensions of fields are determined by the unitary representations of the Virasoro algebra [2]. In superconformal field theory the superfields correspond to irreducible representations of the Neveu–Schwarz algebra. In ref. [2] we gave the discrete list of possible unitary representations of the Neveu–Schwarz algebra and thus the possible dimensions of superfields in two dimensions. We pointed out that the simplest nontrivial examples in the discrete list corresponded to the Z_2 invariant sector of the tricritical Ising model, which describes generic tricritical phenomena in two dimensions [5]. The supersymmetry of the corresponding conformal field theory was also noted by Zamolodchikov [6].

In this letter we point out that a superconformal field theory has a richer structure than given by its superfields alone. The Hilbert space of the theory also contains irreducible representations of the Ramond algebra, corresponding to conformal fields distinct from the superfields and in fact nonlocal (i.e., double-valued) with respect to the fermionic parts of the superfields. We call these the *spin fields.* The Z_2 odd operators of the tricritical Ising model are spin fields.

The fermion parity $(-1)^F$ is multiplicatively conserved so we can project on the sector of even fermion number. This selects the bosonic parts of the superfields and a subset of the spin fields. The result is a new local bosonic field theory. We call this the *spin model* corresponding to the original superconformal theory. The construction of the spin model from a superconformal field theory is a direct generalization of the construction of the superstring from the free fields of the Ramond–Neveu–Schwarz model [3,4,7], which is based on the construction of the critical Ising model from the massless Majorana fermion.

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(North-Holland Physics Publishing Division)

The infinite superconformal algebra is generated by the super stress–energy tensor $T(\mathbf{z}) = T_F(z) + \theta T_B(z)$ where $\mathbf{z} = (z, \theta)$ is a complex super coordinate, $T_B(z)$ is the ordinary stress energy tensor (spin = dimension = 2), and $T_F(z)$ is its fermionic superpartner (spin = dimension = 3/2). We omit discussion of the $\bar{\mathbf{z}}$ dependence since it is equivalent to the $\mathbf{z}$ dependence [1,2]. The correlation functions of $T_F(z)$ can be double-valued because its spin is half-integral. Thus the Hilbert space of the theory divides into two subspaces: the Neveu–Schwarz subspace on which $T_F(\exp(2\pi i) z) = T_F(z)$, and the Ramond subspace on which $T_F(\exp(2\pi i)z) = -T_F(z)$.

The natural operator representation of superconformal field theory comes from the radial quantization. We write $z = \exp(\tau + i\sigma)$ and take τ to be euclidean time, σ to be periodic space. The complex super coordinate $(\log z, z^{-1/2}\theta)$ describes a cylindrical superspace. The super stress–energy tensor can be expanded in operator Fourier coefficients

$$T_F(z) + \theta T_B(z) = \sum_n z^{-n-3/2} T_{F,n} + \theta z^{-n-2} T_{B,n}. \tag{1}$$

We will use the more conventional notation $G_n = 2T_{F,n}$, $L_n = T_{B,n}$. In the Ramond sector the indices of the G_n are integers; in the Neveu–Schwarz sector they are half-integers. The Fourier coefficients satisfy

$$L_n^\dagger = L_{-n}, \qquad G_n^\dagger = G_{-n},$$

$$[L_m, L_n] = (m-n) L_{m+n} + \tfrac{1}{8}\hat{c}\,(m^3 - m)\delta_{m+n},$$

$$[G_m, G_n]_+ = 2L_{m+n} + \tfrac{1}{2}\hat{c}(m^2 - \tfrac{1}{4})\delta_{m+n},$$

$$[L_m, G_n] = (\tfrac{1}{2}m - n)\, G_{m+n}. \tag{2}$$

The hermiticity conditions follow from reality of the super stress–energy tensor. The (anti-)commutation relations are equivalent to the operator product expansion

$$T(\mathbf{z}_1)T(\mathbf{z}_2) \sim \tfrac{1}{4}\hat{c}\, z_{12}^{-3} + (2\theta_{12} z_{12}^{-2} + \tfrac{1}{2} z_{12}^{-1} \mathrm{D}_2 + \theta_{12} z_{12}^{-1} \partial_2)\, T(\mathbf{z}_2), \tag{3}$$

where $z_{12} = z_1 - z_2 - \theta_1\theta_2$, $\theta_{12} = \theta_1 - \theta_2$, and $\mathrm{D} = \partial_\theta + \theta\partial_z$.

The coefficient $\hat{c}$ is a number which characterizes the operator implementation of superconformal invariance in a particular theory. The usual central charge of the Virasoro algebra is $c = 3\hat{c}/2$. A single free superfield has $\hat{c} = 1$. It consists of a free scalar field with $c = 1$ and a Majorana fermion with $c = 1/2$.

A superfield $\phi(\mathbf{z}) = \phi_0(z) + \theta\phi_1(z)$ is a *conformal* superfield if it obeys the operator product expansion

$$T(\mathbf{z}_1)\,\phi(\mathbf{z}_2) \sim h\,\theta_{12} z_{12}^{-2}\phi(\mathbf{z}_2) + \tfrac{1}{2} z_{12}^{-1}\, \mathrm{D}_2\phi + \theta_{12} z_{12}^{-1} + \partial_2\phi. \tag{4}$$

This is equivalent to the commutation relations

$$[T_v, \phi(\mathbf{z})] = v\partial\phi + \tfrac{1}{2}(\mathrm{D}v)\,\mathrm{D}\phi + h(\partial v)\,\phi, \tag{5}$$

where

$$T_v = \frac{1}{2\pi i}\oint \mathrm{d}z\, \mathrm{d}\theta\; v(\mathbf{z})\, T(\mathbf{z}) \tag{6}$$

is the generator of the infinitesimal superconformal transformation $\delta\theta = \mathrm{D}v/2$, $\delta z = v - \theta\delta\theta$. The complete algebra of superfields is generated by products of the super stress–energy tensor with the conformal superfields.

In radial quantization the dilation operator L_0 acts as hamiltonian. The L_n and G_n are lowering operators when $n > 0$ and raising operators when $n < 0$. A state $|h\rangle$ of energy h is called a *ground state* if it is killed by all the lowering operators. The raising operators acting on a ground state generate an irreducible representation of the algebra. The eigenspace $L_0 = h + n$ is called level n.

The vacuum $|0\rangle$ is the ground state of lowest energy $h = 0$. It belongs to the Neveu–Schwarz sector. It is invariant under the global superconformal group OSP (2|1), i.e. it is annihilated by the five generators L_{-1}, L_0, $L_1, G_{1/2}$ and $G_{-1/2}$. The conformal superfields of the theory are in one to one correspondence with the ground states of the Neveu–Schwarz algebra, the superfield $\phi(z, \theta)$ of dimension h being associated with the ground state $|h\rangle = \phi(0,0)|0\rangle$ of energy h.

The relation between ground states of the Ramond algebra and conformal fields is more complicated. G_0 commutes with L_0, so it acts on the ground states, which therefore come in orthogonal pairs $|h^+\rangle$ and $|h^-\rangle = G_0|h^+\rangle$. By the general principle that ground states for the Virasoro algebra are associated with ordinary conformal fields [1,2], the Ramond ground states $|h^\pm\rangle$ are created from the vacuum by *spin fields* $\Theta^\pm(z)$ which are ordinary conformal fields of dimension h: $|h^\pm\rangle = \Theta^\pm(0)|0\rangle$. From the action of $T_F(z)$ on the Ramond ground states we find the operator product

$$T_F(z)\,\Theta^\pm(w) \sim \tfrac{1}{2}(z-w)^{-3/2} a_\pm \Theta^\mp(w), \tag{7}$$

where $a_+ = 1, a_- = h - \hat{c}/16$. Thus $T_F(z)$ is double-valued with respect to the spin fields. In fact, all of the fermionic parts of the superfields are double-valued with respect to the spin fields, and all of the bosonic parts are single-valued. Ramond boundary conditions can be regarded as due to a cut connecting spin fields at $z = 0$ and ∞. The operator product (7) is equivalent to the relations

$$K^\dagger_{-n-1/2}\Theta^\pm(1) \pm \mathrm{i}\Theta^\pm(1) K_n = a_\pm \Theta^\mp(1), \tag{8}$$

where

$$K_n = \frac{1}{2\pi\mathrm{i}} \oint_{|z|<1} \mathrm{d}z\, z^{n+1/2}(1-z)^{1/2}\, 2T_F(z)$$
$$= G_n - \tfrac{1}{2} G_{n+1} \cdots . \tag{9}$$

These nonlocal relations between spin fields and the fermionic part of the super stress–energy tensor can be regarded as the defining property of spin fields.

If we make the conformal transformation from the plane to the cylinder we find that in the Ramond sector all of the fermionic parts of the superfields become periodic in space. The superfields acting in the Ramond sector form a self-contained supersymmetric field theory on the cylinder, with periodic spatial boundary conditions. The global supersymmetry is expressed by $G_0^2 = L_0 - \hat{c}/16$. Supersymmetry is unbroken if and only if there are ground states with $G_0 = 0$, i.e. with $h = \hat{c}/16$.

The fermion parity operator $\Gamma = (-1)^F$ anti-commutes with all the fermionic parts of the superfields and commutes with the bosonic parts. Because of its role in the fermionic string theory we refer to Γ as the *chirality operator*. Since G_0 reverses chirality, the paired Ramond ground states have opposite chirality. If $h = \hat{c}/16$ then $|h^-\rangle$ is a null state and decouples, leaving a chirally asymmetric ground state. Witten's index [8] $\mathrm{tr}\,(-1)^F$ for the supersymmetric theory on the cylinder with periodic spatial boundary conditions counts the chiralities of the $h = \hat{c}/16$ Ramond ground states. If the index is nonzero then supersymmetry cannot be broken and there must be some chirally asymmetric $h = \hat{c}/16$ ground states. If the index is zero then there is no chiral asymmetry and in the generic case the supersymmetry will be broken.

The fermionic string theory in D space–time dimensions is described by D free superfields $X^\mu(z) = x^\mu(z) + \theta\psi^\mu(z)$ on the world-surface of the string. The super stress–energy tensor is $T = \mathrm{D}X_\mu \mathrm{D}^2 X^\mu/2$, and the operator product (3), with $\hat{c} = D$, can be verified using the two point function $\langle X^\mu(z_1) X_\nu(z_2)\rangle = \delta^\mu_\nu \ln z_{12}$. For a single free superfield, the Majorana fermion ψ is equivalent to the Ising model at its critical point. The ground state energy with Ramond boundary conditions is 1/16. The spin fields are the dimension 1/16 order and disorder operators of the Ising model. For arbitrary D the spin fields are products of D Ising order or disorder operators and have dimension $h = D/16 = \hat{c}/16$. They form the fermionic vertex constructed by Corrigan, Goddard and Olive [9], who wrote the relations (8) for the degenerate case $h = \hat{c}/16$. The connection between the fermionic string and the Ising model was first pointed out by Aharonov, Casher and Susskind [10].

With Ramond boundary conditions, the fields $\psi^\mu(z)$ contain zero modes ψ_0^μ which obey the anti-commutation relations of the Clifford algebra γ^μ. They commute with L_0 and therefore act on the Ramond ground states. Thus for D even the spin field Θ^α transforms as a space–time Dirac spinor. For D odd, e.g. $D = 1$, the spin fields must transform as a pair of spinors of opposite chirality in order for Γ to be defined. For all D, the chirality operator acts by $\Gamma\Theta\Gamma = \gamma_{D+1}\Theta$. Projection on the $\Gamma = 1$ sector of the $D = 10$ fermionic string yields the superstring [7]. The spin field Θ^α becomes a Weyl spinor.

In the general superconformal theory, spin fields of opposite chirality are nonlocal with respect to each other, since their operator products contain fermionic fields, while spin fields of the same chirality are mutually local. Therefore, projecting on the sector $\Gamma = 1$ in any superconformal field theory gives a local field theory, the *spin model*. Of course, in a complete treatment of superconformal field theory we must take into account the $(\bar{z}, \bar{\theta})$ dependence of the fields. The full superconformal algebra is the tensor product of two algebras, one describing transformations in (z, θ), the other in $(\bar{z}, \bar{\theta})$. The natural chirality operator is $\Gamma = (-1)^F$ where F is the fermion number of the full theory. It is interesting to note, however, that in the fermionic string two chirality operators, Γ and $\bar{\Gamma}$, are defined, one in the (z, θ) sector, the other in the $(\bar{z}, \bar{\theta})$ sector. The superstring requires projection on $\Gamma = \bar{\Gamma} = 1$.

In summary, the Hilbert space of a superconformal theory divides into two subspaces, the Neveu–Schwarz and Ramond sector. The vacuum is in the Neveu–Schwarz sector. The operators of the theory can be pictured in two by two block form. The superfields of the theory are block diagonal, taking the Neveu–Schwarz sector to the Neveu–Schwarz sector and the Ramond sector to the Ramond sector. The spin fields are block off-diagonal. The dimensions of the superfields are given by the ground state energies in the Neveu–Schwarz sector, since they make Neveu–Schwarz states from the vacuum. The dimensions of the spin fields are given by the ground state energies in the Ramond sector, since the spin fields make Ramond states from the Neveu–Schwarz vacuum. The complete superconformal field theory is not local, since the fermionic operators are double-valued with respect to the spin fields. There are two ways to project onto a local field theory. The first is to restrict to the Neveu–Schwarz sector, giving the usual algebra of superfields. The second is to restrict to the $\Gamma = 1$ sector, giving the spin model. The complete superconformal theory can be reconstructed from the superfields by representing a pair of spin fields as endpoints of a cut in the plane across which the fermionic fields change sign. The fermionic fields can presumably be reconstructed from the $\Gamma = 1$ spin fields by a generalization of the Jordan–Wigner construction of the fermion field from the Ising order parameter. It is not yet clear how to characterize those local field theories which are spin models of superconformal field theories.

We focus now on unitary superconformal theories. Unitarity requires that the Hilbert space of a theory contain no negative metric states. Irreducible representations of the conformal algebras can be free from negative metric states only for certain values of the superconformal anomaly $\hat{c}$ and the ground state energy h. If $\hat{c} \geqslant 1$ then all representations with $h \geqslant 0$ are unitary. For $\hat{c} < 1$ there is a discrete list of possible unitary representations. For p and q positive integers, define $\hat{m}$ and $h_{p,q}$ by

$$\hat{c}(\hat{m}) = 1 - 8/\hat{m}(\hat{m}+2),$$

$$h_{p,q} = \{[(\hat{m}+2)p - \hat{m}q]^2 - 4\}/8\hat{m}(\hat{m}+2) + \tfrac{1}{32}[1-(-1)^{p-q}]. \tag{10}$$

The possible unitary representations of the superconformal algebras with $\hat{c} < 1$ are

$$\hat{c} = \hat{c}(\hat{m}), \quad \hat{m} = 2, 3, 4, \ldots$$

$$h = h_{p,q}(\hat{m}), \quad 1 \leqslant p < \hat{m}, \quad 1 \leqslant q < \hat{m}+2. \tag{11}$$

The Neveu–Schwarz representations are given by the $h_{p,q}$ with $p-q$ even [2]; the Ramond representations by the $h_{p,q}$ with $p-q$ odd.

Each unitary field theory with $\hat{c} < 1$ must be built from the possible representations on the list for some fixed $\hat{m}$. The striking feature of the list (11) is that the possible representations of the two superconformal algebras occur at the same values of $\hat{c}$. This strongly indicates the existence of a discrete series of superconformal models in the sense described above.

For $\hat{m}$ odd, $\hat{c}/16$ is not an allowed ground state energy, so the supersymmetry of the periodic model on the cylinder is broken. When $\hat{m}$ is even, $h_{\hat{m}/2,\,\hat{m}/2+1} = \hat{c}/16$, so the supersymmetry on the cylinder is unbroken.

The constraint of unitarity applies level by level, since the eigenspaces of L_0 are orthogonal The nth level of an irreducible representation is spanned by the vectors

$$G_{-m_1}G_{-m_2}\ldots L_{-n_1}L_{-n_2}\ldots|h\rangle,$$

$$0 < m_1 < m_2 \ldots, \quad 0 < n_1 \leqslant n_2 \ldots,$$

$$n = \textstyle\sum m_i + \sum n_i, \tag{12}$$

where $|h\rangle$ is a ground state vector. In the Neveu–Schwarz sector the m_i are half-integers; in the Ramond sector the m_i are integers. A chirality operator can be defined by

$$\Gamma|h\rangle = |h\rangle \text{ or } \Gamma|h^{\pm}\rangle = \pm|h^{\pm}\rangle,$$

$$[\Gamma, G_n]_+ = [\Gamma, L_n] = 0. \tag{13}$$

Each Ramond level splits into $\Gamma = \pm 1$ eigenspaces, while each Neveu–Schwarz level is itself a $\Gamma = \pm 1$ eigenspace. In each sector, at each level n and in each $\Gamma = \pm 1$ eigenspace we can calculate the matrix $M_n^{\pm}$ of inner products of the states (12). Unitarity requires that all of these matrices be nonnegative.

The essential tool for finding negative metric states is a formula for $\det(M_n^{\pm})$. Kac [11] gave the formula for the Neveu–Schwarz algebra:

$$\det(M_n) = \prod [h - h_{p,q}(\hat{c})]^{P_{\mathrm{NS}}(n-pq/2)}, \tag{14}$$

where the product runs over positive integers p, q with $pq/2 \leqslant n$ and $p-q$ even. $P_{\mathrm{NS}}(k)$ is the dimension of level k:

$$\sum_{k=0} t^k P_{\mathrm{NS}}(k) = \prod_{k=1} (1+t^{k-1/2})/(1-t^k). \qquad (15)$$

We arrived at the determinant formula for the Ramond algebra by numerical calculations:

$$\det(M_0^+) = 1, \quad \det(M_0^-) = h - \hat{c}/16,$$

$$\det(M_n^+) = \det(M_n^-) = (h - \hat{c}/16)^{P_{\mathrm{R}}(n)/2}$$

$$\times \prod [h - h_{p,q}(\hat{c})]^{P_{\mathrm{R}}(n-pq/2)} \quad \text{for } n > 0, \qquad (16)$$

where the product runs over positive integers p, q with $pq/2 \leqslant n$ and $p-q$ odd. $P_{\mathrm{R}}(k)$ is half the dimension of the kth level:

$$\sum_{k=0} t^k P_{\mathrm{R}}(k) = \prod_{k=1} (1+t^k)/(1-t^k). \qquad (17)$$

Both determinant formulas (14), (16) have now been proved by Meurman and Rocha-Caridi [12] using the technique that Feigin and Fuchs [13] applied to prove the Kac determinant formula for the Virasoro algebra. Curtwright and Thorn [14] have independently proved these determinant formulas.

As with the Virasoro algebra, $M_n^{\pm}$ becomes diagonal and positive as $h \to \infty$, and $\det(M_n^{\pm})$ has no zeros in the quadrant $\hat{c} > 1, h > 0$. Thus there are no negative metric states for $\hat{c} \geqslant 0, h \geqslant 0$. On the other hand, for $\hat{c} < 1$ and any h, there is always some n for which $\det(M_n^{\pm}(\hat{c}, h)) \leqslant 0$. A negative determinant indicates at least one negative metric state, so every unitary representation with $\hat{c} < 1$ must lie on a vanishing curve $h = h_{p,q}(\hat{c})$ or, in the Ramond case, $h = \hat{c}/16$. The rest of the argument which gives the list (11) is essentially the same as used in ref. [2] for the Virasoro algebra. The list (11) consists of the so-called *first intersections* [2] of the vanishing curves $h = h_{p,q}(\hat{c})$. It can be shown that there is a negative metric state between first intersections on each vanishing curve, which leaves only the first intersections as possible unitary representations. For the Ramond algebra there is a slight variation in the argument, because the collection of curves $\{h_{p,p+1}\}$ makes first intersections with the single curve $h = \hat{c}/16$ while for $k \neq 1$ the collection $\{h_{p,p+k}\}$ makes first intersections with the collection $\{h_{p,p+2-k}\}$.

The first nontrivial model has $\hat{m} = 3, \hat{c} = 7/15, c = 7/10$. This is the only value of c which occurs in the discrete lists for both the superconformal algebras and the Virasoro algebra [2]. In ref. [2] we identified $c = 7/10$ with the tricritical Ising model and Kadanoff [15] confirmed the identification using correlation functions calculated by himself and Nienhuis [16]. The tricritical Ising model [5] is an Ising model with annealed vacancies. It has a tricritical point at a certain temperature and vacancy chemical potential. The model is invariant under the Z_2 symmetry which flips both the order operators, the Ising spins, and the disorder operators [17]. We pointed out in ref. [2] that the Z_2 even sector is described by representations of the Neveu–Schwarz algebra. Here we complete the picture by pointing out that the Z_2 odd sector is described by representations of the Ramond algebra. Thus the tricritical Ising model is a superconformal field theory.

The ground state energies allowed by unitarity and ordinary conformal invariance are [2] $h = 0, 3/80, 1/10, 7/16, 3/5$, and $3/2$. The ground state energies allowed by unitarity and superconformal invariance in the Neveu–Schwarz sector are $h_{1,1} = 0$ and $h_{2,2} = 1/10$. They decompose into irreducible representations of the Virasoro algebra:

$$(0)_{\mathrm{NS}} = (0)_{\mathrm{VIR}} \oplus (3/2)_{\mathrm{VIR}},$$

$$(1/10)_{\mathrm{NS}} = (1/10)_{\mathrm{VIR}} \oplus (6/10)_{\mathrm{VIR}}. \qquad (18)$$

These are the dimensions of the Z_2 even operators of the tricritical Ising model [18]. The allowed representations of the Ramond algebra are $h_{1,2} = 3/80$ and $h_{2,1} = 7/16$, each consisting of exactly one irreducible representation of the Virasoro algebra [‡1]. These are the dimensions of the Z_2 odd operators of the tricritical Ising model, i.e., the leading and subleading magnetic spin operators [19]. These operators take Z_2 even states to Z_2 odd and vice versa and so intertwine the Ramond and Neveu–Schwarz sectors of the theory. The disorder operators are the $\Gamma = -1$ spin fields. The $\Gamma = 1$ projection of the superconformal model is a maximal mutually local algebra of operators. This picture of the tricritical Ising model is supported by the operator product relations of ordinary conformal field theory [1].

All of the Z_2 even operators can be constructed from the super stress–energy tensor and a single conformal superfield. To do this we use the complex conjugate coordinates $(\bar{z}, \bar{\theta})$ and the $(h, \bar{h})$ notation which gives the dimension $(h + \bar{h})$ and the spin $(h - \bar{h})$ of a

[‡1] A superconformal representation does not typically decompose into only a finite number of Virasoro representations.

conformal field [1,2]. The Z_2 invariant operators consists [18] of the energy operator ϵ with $(h, \bar{h}) = (1/10, 1/10)$, the vacancy operator t with $(6/10, 6/10)$, and an irrelevant operator with $(3/2, 3/2)$. There are also Z_2 even fermionic operators ψ with $(6/10, 1/10)$ and T_F with $(3/2, 0)$. The ordinary stress–energy tensor is combined with T_F to form the super stress–energy tensor, and the $(3/2, 3/2)$ operator is interpreted as $T_F \bar{T}_F$. The remaining even operators are combined to form a superfield $\Phi = \epsilon + \theta\psi + \bar{\theta}\bar{\psi} + \theta\bar{\theta}t$ with $(h, \bar{h}) = (1/10, 1/10)$. All of the correlation functions of the Z_2 even operators can be derived from the correlations of the superfield Φ, since correlation functions of the super stress–energy tensor are completely determined by the commutation relations (2), as in ordinary conformal field theory [1].

We have calculated correlation functions of the superfield of the tricritical Ising model by a supersymmetric version of the method of null vectors and linear differential equations [1,20]. Here we present the result for the four point function. Derivations and additional results will be presented elsewhere [21]. The null vector $[G_{-3/2} - (5/3) L_{-1} G_{-1/2}]\, |1/10\rangle$ gives rise to a linear super differential equation on correlation functions of Φ which, along with the appropriate singularity and monodromy conditions, determines the four point function:

$$\langle \Phi(1)\, \Phi(2)\, \Phi(3)\, \Phi(4) \rangle = |z_{12} z_{23} z_{34} z_{41}|^{-1/5} \times (|f(\eta, \zeta)|^2 + A |g(\eta, \zeta)|^2), \tag{19}$$

where $\eta = z_{12} z_{34} / z_{13} z_{24}$, $\zeta = 1 - \eta - z_{14} z_{23} / z_{13} z_{24}$ and

$$f = (1 + \zeta\eta\, \mathrm{d}/\mathrm{d}\eta)\, ([\eta(1-\eta)]^{-1/10} F_1(\tfrac{1}{5}, -\tfrac{2}{5}, \tfrac{2}{5}, \eta)) + \tfrac{3}{35}\, \zeta [\eta(1-\eta)]^{9/10} F_1(\tfrac{8}{5}, \tfrac{11}{5}, \tfrac{12}{5}, \eta),$$

$$g = (1 + \zeta\eta\, \mathrm{d}/\mathrm{d}\eta)\, ([\eta(1-\eta)]^{1/2} F_1(\tfrac{7}{5}, \tfrac{4}{5}, \tfrac{8}{5}, \eta)) + \tfrac{1}{5}\, \zeta [\eta(1-\eta)]^{-1/2} F_1(-\tfrac{6}{5}, -\tfrac{3}{5}, -\tfrac{2}{5}, \eta),$$

$$A = \tfrac{4}{9}\, \Gamma(4/5)\, \Gamma(2/5)^3 / \Gamma(1/5)\, \Gamma(3/5)^3. \tag{20}$$

It is also possible to construct correlation functions for conformal superfields by a straightforward supersymmetric extension of the Feigin–Fuchs Coulomb gas integral representation [22] [‡2]. Conformal superfields are represented by exponentials of a free superfield. An extra supercharge is placed at infinity and the net charge is neutralized by superconformally invariant contour integrals of $h = 1/2$ exponentials of the free superfield. To obtain nonvanishing correlation functions of bosonic fields there must be an even number of neutralizing supercharges. By this method it is possible to describe correlation functions of all conformal superfields corresponding to degenerate representations of the Neveu–Schwarz algebra (as well as certain others). Bershadsky, Knizhnik, and Teitelman [23] have independently constructed this integral representation and used it to study the operator product structure of conformal superfields.

The correlation functions of the tricritical Ising model can also be calculated using ordinary conformal properties. The results of Dotsenko and Fateev [22] show that there is a unique conformally invariant theory with the appropriate field content (at least in the Z_2 even sector), so it must be the supersymmetric theory. Relations like eq. (7) can be checked by the ordinary conformal techniques. We are currently constructing explicitly supersymmetric representations of combined spin operator and superfield correlation functions.

We should note that Rocha–Caridi [24] calculated the partition functions of the representations with $c = 7/10$. We should also mention interesting recent work by Goddard and Olive [25]. Using Kac–Moody algebra techniques they have found a fermionic oscillator realization for the $c = 7/10$ representations, among others, thus proving their unitarity.

There are several applications of superconformal field theory in superstring theory. The Faddeev–Popov ghosts of the fermionic string [26] form a superconformal system. The ghost spin fields complete the covariant construction of the fermionic vertex [27]. Ising model techniques can be used to calculate multi-fermion amplitudes [28]. Nontrivial superconformal field theories with $\hat{c} = 6$ give possible compactifications of superstring theories from ten to four dimensions [29]. In particular, the $N = 1$ supersymmetric nonlinear sigma models with Ricci-flat internal spaces are conjectured to have zero beta function [30], which implies superconformal invariance. Such models can be defined on the world-surface of the open type I superstring if they are extended by boundary terms which depend on a Yang–Mills gauge field on the internal space. The Ramond representation of the superconformal sigma model determines the fermionic spectrum of the compactified superstring,

[‡2] For related works, see refs. [18,19].

and the index tr(Γ) gives the chiral fermion content. Another set of superconformal models which can provide compactifications of the string are the $N = 1$ supersymmetric generalizations of the nonlinear sigma models with Wess–Zumino term, at zeroes of their beta functions [31]. The super Kac–Moody current algebras of these models can be used to show the absence of massless fermions in the closed string sector, and thus the breaking of supersymmetry in the compactified string.

Finally, we remark that the tricritical Ising model can be realized experimentally (for example by adsorbing helium-4 on krypton plated graphite [32]) and so provides the first instance of a supersymmetric field theory in nature.

One of us (D.F.) is grateful to the Landau Institute and Nordita for the invitation to present this work at the Landau–Nordita Conference on Field Theory and Statistical Mechanics, Moscow, USSR, May 23–June 2, 1984. This work was supported in part by U.S. Department of Energy contract DE-AC02-81ER-10957, National Science Foundation grant NSF-DMR-82-16892 and the Alfred P. Sloan Foundation.

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