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Entanglement entropy and quantum field theory

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Abstract. We carry out a systematic study of entanglement entropy in relativistic quantum field theory. This is defined as the von Neumann entropy $S_A = -\mathrm{Tr}\,\rho_A \log \rho_A$ corresponding to the reduced density matrix ρ_A of a subsystem A. For the case of a 1+1-dimensional critical system, whose continuum limit is a conformal field theory with central charge c, we re-derive the result $S_A \sim (c/3)\log \ell$ of Holzhey *et al* when A is a finite interval of length ℓ in an infinite system, and extend it to many other cases: finite systems, finite temperatures, and when A consists of an arbitrary number of disjoint intervals. For such a system away from its critical point, when the correlation length ξ is large but finite, we show that $S_A \sim \mathcal{A}(c/6)\log \xi$, where $\mathcal{A}$ is the number of boundary points of A. These results are verified for a free massive field theory, which is also used to confirm a scaling ansatz for the case of finite size off-critical systems, and for integrable lattice models, such as the Ising and XXZ models, which are solvable by corner transfer matrix methods. Finally the free field results are extended to higher dimensions, and used to motivate a scaling form for the singular part of the entanglement entropy near a quantum phase transition.

Keywords: conformal field theory, integrable spin chains (vertex models), entanglement in extended quantum systems (theory), quantum phase transitions (theory), new applications of statistical mechanics

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Contents

1. Introduction

Recently there has been considerable interest in formulating measures of quantum entanglement and applying them to extended quantum systems with many degrees of freedom, such as quantum spin chains.

One of these measures [1] is entanglement *entropy*. Suppose the whole system is in a pure quantum state $|\Psi\rangle$, with density matrix $\rho = |\Psi\rangle\langle\Psi|$, and an observer A measures only a subset A of a complete set of commuting observables, while another observer B may measure the remainder. A's reduced density matrix is $\rho_A = \mathrm{Tr}_B \rho$. The entanglement entropy is just the von Neumann entropy $S_A = -\mathrm{Tr}_A \rho_A \log \rho_A$ associated with this reduced density matrix. It is easy to see that $S_A = S_B$. For an unentangled product state, $S_A = 0$. Conversely, S_A should be a maximum for a maximally entangled state.

For example, for a system with two binary (spin-1/2) degrees of freedom, with $|\Psi\rangle = \cos\theta|\uparrow\downarrow\rangle + \sin\theta|\downarrow\uparrow\rangle$, where A observes only the first spin and B the second, S_A takes its maximum value of $\log 2$ when $\cos^2\theta = 1/2$, which agrees with our intuitive idea of maximal entanglement. For this system it has been shown [1], even in the partially entangled case, that if there are M copies of the state available, by making only local

operations A can produce $M' < M$ states which are maximally entangled. The optimal conversion ratio M'/M is given, for large M, by S_A.[4]

Although there are other measures of entanglement [2], the entropy is most readily suited to analytic investigation. In several papers [3]–[6], the concept has been applied to quantum spin chains. Typically, the subset A consists of a commuting set of components of the spin degrees of freedom in some interval of length ℓ, in an infinitely long chain. It is found that the entanglement entropy generally tends to a finite value as ℓ increases, but that this value diverges as the system approaches a quantum critical point. At such a critical point, the entropy grows proportionally to $\log \ell$ for large ℓ.

Close to a quantum critical point, where the correlation length ξ is much larger than the lattice spacing a, the low-lying excitations and the long-distance behaviour of the correlations in the ground state of a quantum spin chain are believed to be described by a quantum field theory in $1+1$ dimensions. If the dispersion relation of the low-lying excitations is linear for wavenumbers k such that $\xi^{-1} \ll |k| \ll a^{-1}$, the field theory is relativistic. We shall consider only those cases in this paper. At the critical point, where $\xi^{-1} = 0$, the field theory is massless, and is a *conformal* field theory (CFT).

In this case, the von Neumann entropy of the subsystem corresponding to an interval of length ℓ was calculated some time ago by Holzhey *et al* [7], in the context of black hole physics (although that connection has been questioned), where it was termed 'geometric' entropy. Using methods of conformal field theory, based in part on earlier work of Cardy and Peschel [8], they found $S_A \sim (c/3)\log(\ell/a)$, where c is the conformal anomaly number (sometimes called the central charge) of the corresponding CFT.

This result has been verified by analytic and numerical calculations on integrable quantum spin chains corresponding to CFTs with $c = 1/2$ and 1 [3]–[5].

In this paper, we first put the CFT arguments of Holzhey *et al* [7] on a more systematic basis, and generalize their result in a number of ways. Our methods are based on a formula for the entropy in terms of the partition function in the path integral formulation of the quantum theory as a Euclidean field theory on an n-sheeted Riemann surface, in the limit $n \to 1$.

For a $1+1$-dimensional theory at a critical point, we derive analogous formulae for the entropy in the cases when the subsystem A consists of an *arbitrary* number of disjoint intervals of the real line (see equation (3.32)), and when the whole system has itself a finite length L. For example, for the case when A is a single interval of length ℓ, and periodic boundary conditions are imposed on the whole system, we find

$$S_A = (c/3)\log((L/\pi a)\sin(\pi\ell/L)) + c_1'. \tag{1.1}$$

On the other hand, for a finite system of total length L with open boundaries, divided at some interior point into an interval A of length ℓ and its complement, we find

$$S_A = (c/6)\log((2L/\pi a)\sin(\pi\ell/L)) + g + c_1', \tag{1.2}$$

where g is the boundary entropy of Affleck and Ludwig [9]. We also treat the case when the system is infinitely long but is in a thermal mixed state at finite temperature β^{-1}:

$$S_A = (c/3)\log((\beta/\pi a)\sinh(\pi\ell/\beta)) + c_1'. \tag{1.3}$$

[4] As far as we are aware, this analysis has not been extended to systems with many degrees of freedom such as we consider in this paper. Indeed, we shall see that for such systems S_A can be much larger than unity, so cannot have such a simple interpretation.

In all these cases, the constant c_1' is the same, but non-universal.

For a massive $1+1$-dimensional relativistic QFT (which corresponds to an off-critical quantum spin chain where the correlation length $\xi \gg a$) the simplest results are for an infinite system divided at some point into two semi-infinite pieces. In this case we verify that the entanglement entropy is finite, and derive the universal formula

$$S_{\rm A} \sim (c/6)\log(\xi/a). \tag{1.4}$$

In the more general case when A consists of a collection of disjoint intervals, each of length $\gg\xi$, we expect (1.4) to be multiplied by a factor $\mathcal{A}$ which counts the number of boundary points between A and B (the 1D analogue of surface area).

For $\mathcal{A}=1$ the entropy is exactly calculable in the case of a free field theory. We verify the above formula, and exhibit the finite size crossover which occurs when ξ is of the size L of the system. For a lattice model in this geometry, with L infinite, we point out that $\rho_{\rm A}$ is simply related to Baxter's corner transfer matrix, and thus, for integrable models whose weights satisfy a Yang–Baxter relation, all its eigenvalues can be determined exactly. We treat explicitly the case of the Ising model and its anisotropic limit, the transverse Ising spin chain, and also the XXZ model, computing exactly at all values of the coupling the finite part of the entropy S. This agrees with our continuum result (1.4) for $c=1/2, 1$ when the correlation length ξ is large.

The analysis for the free theory is straightforward to extend to higher dimensions, at least in suitable geometries. This leads to the well-known law, first found by Srednicki [10], that the entropy should be proportional to the surface area $\mathcal{A}$ of the subsystem A. As pointed out by Srednicki, the coefficient of this term, for $d>1$, depends on the UV cut-off and is therefore non-universal. However, on the basis of our calculations, we propose that there should be a non-leading piece in $S_{\rm A}/\mathcal{A}$, proportional to $\xi^{-(d-1)}$, which depends in a singular way on the couplings near a quantum phase transition, and whose form is, moreover, *universal*.

The layout of this paper is as follows. In the next section, we discuss the entropy in terms of the Euclidean path integral on an n-sheeted Riemann surface. In section 3 we consider the $1+1$-dimensional conformal case. We use the powerful methods of CFT to show that the partition function on the Riemann surface of interest is given, up to a constant, by a calculable correlation function of vertex operators in a $c=1$ CFT in the complex plane. Similar results apply to a system with boundaries. This general result allows us to derive all the special cases described above. In section 4 we consider the case of a massive $1+1$-dimensional field theory. We derive the result $S\sim(c/6)\log(\xi/a)$ from completely general properties of the stress tensor in the relevant geometry. This is then verified for a free bosonic massive field. In the last part of this section we relate the lattice version of this problem to the corner transfer matrix, and compute the off-critical entropy for the case of the Ising and XXZ Heisenberg spin chains. In section 5 we study the off-critical case in a finite size system, propose a scaling law, and verify it for the case of a free massive field theory. We compute the relevant scaling function in a systematic sequence of approximations. Section 6 is devoted to the discussion of higher dimensions.

While all this work was being carried out, some other related papers have appeared in the literature. In [11], the result $S_{\rm A}\sim(c/3)\log(\ell/a)$, first found by Holzhey *et al* [7], was obtained by the following argument: it was assumed that the entropy should be conformally invariant and therefore some function $F(x)$ of the variable

$x = (\beta/\pi)\sinh(\pi\ell/\beta)$; by comparing with known case $\ell \gg \beta$ [12, 13] it was observed that $F(x)$ should behave as $(c/3)\log x$ as $x \to \infty$; finally it was assumed (with no justification being given) that the $\log x$ form for F is valid for all values of x; finally the limit $\beta \gg \ell$ was taken. In the present paper, we should stress, we have derived all these statements from first principles of CFT.

Very recently Casini and Huerta [6] have considered the case of two intervals A and B, and argued that the quantity $F(A,B) \equiv S(A) + S(B) - S(A \cap B) - S(A \cup B)$ is UV finite as $a \to 0$, and is given by a universal logarithmic function of the cross-ratio of the four end points. This corresponds to, and agrees with, our case $N = 2$. These authors, however, assume the conformal invariance of the entropy, while, once again, we stress that in the present paper we derive this from fundamental properties of the stress tensor. These authors also give a very nice alternative derivation of Zamolodchikov's c-theorem [14] based on this quantity $F(A,B)$.

2. von Neumann entropy and Riemann surfaces

Consider a lattice quantum theory in one space and one time dimension, initially on the infinite line. The lattice spacing is a, and the lattice sites are labelled by a discrete variable x. The domain of x can be finite, i.e. some interval of length L, semi-infinite, or infinite. Time is considered to be continuous. A complete set of local commuting observables will be denoted by $\{\hat{\phi}(x)\}$, and their eigenvalues and corresponding eigenstates by $\{\phi(x)\}$ and $\otimes_x|\{\phi(x)\}\rangle$ respectively. For a bosonic lattice field theory, these will be the fundamental bosonic fields of the theory; for a spin model some particular component of the local spin. The dynamics of the theory is described by a time-evolution operator $\hat{H}$. The density matrix ρ in a thermal state at inverse temperature β is

$$\rho(\{\phi(x'')''\}|\{\phi(x')'\}) = Z(\beta)^{-1}\langle\{\phi(x'')''\}|\mathrm{e}^{-\beta\hat{H}}|\{\phi(x')'\}\rangle, \tag{2.1}$$

where $Z(\beta) = \mathrm{Tr}\,\mathrm{e}^{-\beta\hat{H}}$ is the partition function.

This may be expressed in the standard way as a (Euclidean) path integral:

$$\rho = Z^{-1}\int[\mathrm{d}\phi\,(x,\tau)]\prod_x \delta(\phi(x,0) - \phi(x')')\prod_x \delta(\phi(x,\beta) - \phi(x'')'')\mathrm{e}^{-S_\mathrm{E}}, \tag{2.2}$$

where $S_\mathrm{E} = \int_0^\beta L_\mathrm{E}\,\mathrm{d}\tau$, with L_E the Euclidean Lagrangian. (For a spin model this would be replaced by a coherent state path integral.)

The normalization factor of the partition function ensures that $\mathrm{Tr}\,\rho = 1$, and is found by setting $\{\phi(x)''\} = \{\phi(x)'\}$ and integrating over these variables. This has the effect of sewing together the edges along $\tau = 0$ and β to form a cylinder of circumference β.

Now let A be a subsystem consisting of the points x in the disjoint[5] intervals $(u_1, v_1), \ldots, (u_N, v_N)$. An expression for the reduced density matrix ρ_A may be found from (2.2) by sewing together only those points x which are not in A. This will have the effect of leaving open cuts, one for each interval (u_j, v_j), along the line $\tau = 0$.

We may then compute $\mathrm{Tr}\,\rho_\mathrm{A}^n$, for any positive integer n, by making n copies of the above, labelled by an integer k with $1 \le k \le n$, and sewing them together cyclically along

[5] This restriction is not necessary, but the set-up is easier to picture in this case.

the cuts so that $\phi(x)'_k = \phi(x)''_{k+1}$ (and $\phi(x)'_n = \phi(x)''_1$) for all $x \in A$. Let us denote the path integral on this n-sheeted structure by $Z_n(A)$. Then

$$\operatorname{Tr} \rho_{\mathrm{A}}^n = \frac{Z_n(A)}{Z^n}. \tag{2.3}$$

Now, since $\operatorname{Tr} \rho_{\mathrm{A}}^n = \sum_\lambda \lambda^n$, where λ are the eigenvalues of ρ_{A} (which lie in $[0,1)$), and since $\operatorname{Tr} \rho_{\mathrm{A}} = 1$, it follows that the left-hand side is absolutely convergent and therefore analytic for all $\operatorname{Re} n > 1$. The derivative wrt n therefore also exists and is analytic in the region. Moreover, if the entropy $\rho_{\mathrm{A}} = -\sum_\lambda \lambda \log \lambda$ is finite, the limit as $n \to 1+$ of the first derivative converges to this value.

We conclude that the right-hand side of (2.3) has a unique analytic continuation to $\operatorname{Re} n > 1$ and that its first derivative at $n = 1$ gives the required entropy:

$$S_{\mathrm{A}} = -\lim_{n\to 1} \frac{\partial}{\partial n} \frac{Z_n(A)}{Z^n}. \tag{2.4}$$

(Note that even before taking this limit, (2.3) gives an expression for the Tsallis entropy [15] $(\operatorname{Tr} \rho_{\mathrm{A}}^n - 1)/(1-n)$.)

So far, everything has been in the discrete space domain. We now discuss the continuum limit, in which $a \to 0$ keeping all other lengths fixed. The points x then assume real values, and the path integral is over fields $\phi(x,\tau)$ on an n-sheeted Riemann surface, with branch points at u_j and v_j. In this limit, S_{E} is supposed to go over into the Euclidean action for a quantum field theory. We shall restrict attention to the case when this is Lorentz invariant, since the full power of relativistic field theory can then be brought to bear. The behaviour of partition functions in this limit has been well studied. In two dimensions, the logarithm of a general partition function Z in a domain with total area $\mathcal{A}$ and with boundaries of total length $\mathcal{L}$ behaves as

$$\log Z = f_1 \mathcal{A} a^{-2} + f_2 \mathcal{L} a^{-1} + \cdots, \tag{2.5}$$

where f_1 and f_2 are the non-universal bulk and boundary free energies. Note, however, that these leading terms *cancel* in the ratio of partition functions in (2.3). However, as was argued by Cardy and Peschel [8] there are also *universal* terms proportional to $\log a$. These arise from points of non-zero curvature of the manifold and its boundary. In our case, these are conical singularities at the branch points. In fact, as we shall show, it is precisely these logarithmic terms which give rise to the non-trivial dependence of the final result for the entropy on the short-distance cut-off a. For the moment let us simply remark that, in order to achieve a finite limit as $a \to 0$, the right-hand side of (2.3) should be multiplied by some renormalization constant $\mathcal{Z}(A,n)$. Its dependence on a will emerge from the later analysis.

3. Entanglement entropy in 2D conformal field theory

Now specialize the discussion of the previous section to the case when the field theory is relativistic and massless, i.e. a conformal field theory (CFT), with central charge c, and initially consider the case of zero temperature.

We show that in this case the ratio of partition functions in (2.3) is the same as the correlation function arising from the insertion of primary scaling operators $\Phi_n(u_j)$

and $\Phi_{-n}(v_j)$, with scaling dimensions $X_n = 2\Delta_n = (c/12)(1-1/n^2)$, into each of the n (disconnected) sheets. Moreover, this $2N$-point correlation function is computable from the Ward identities of CFT.

In the language of string theory, the objects we consider are correlators of orbifold points in theories whose target space consists of n copies of the given CFT. We expect that some of our results may therefore have appeared in the literature of the subject.

3.1. Single interval

We first consider the case $N = 1$ and no boundaries, that is the case considered by Holzhey *et al* [7] of a single interval of length ℓ in an infinitely long 1D quantum system, at zero temperature. The conformal mapping $w \to \zeta = (w-u)/(w-v)$ maps the branch points to $(0,\infty)$. This is then uniformized by the mapping $\zeta \to z = \zeta^{1/n} = ((w-u)/(w-v))^{1/n}$. This maps the whole of the n-sheeted Riemann surface $\mathcal{R}_n$ to the z-plane $\mathbf{C}$. Now consider the holomorphic component of the stress tensor $T(w)$. This is related to the transformed stress tensor $T(z)$ by [16]

$$T(w) = (\mathrm{d}z/\mathrm{d}w)^2\, T(z) + \frac{c}{12}\{z,w\}, \tag{3.1}$$

where $\{z,w\}$ is the Schwartzian derivative $(z'''z' - (3/2)z''^2)/z'^2$. In particular, taking the expectation value of this, and using $\langle T(z)\rangle_{\mathbf{C}} = 0$ by translational and rotational invariance, we find

$$\langle T(w)\rangle_{\mathcal{R}_n} = \frac{c}{12}\{z,w\} = \frac{c(1-(1/n)^2)}{24}\frac{(v-u)^2}{(w-u)^2(w-v)^2}. \tag{3.2}$$

This is to be compared with the standard form [16] of the correlator of T with two primary operators $\Phi_n(u)$ and $\Phi_{-n}(v)$ which have the same complex scaling dimensions $\Delta_n = \overline{\Delta}_n = (c/24)(1-(1/n)^2)$:

$$\langle T(w)\Phi_n(u)\Phi_{-n}(v)\rangle_{\mathbf{C}} = \frac{\Delta_n}{(w-u)^2(w-v)^2(v-u)^{2\Delta_n-2}(\bar{v}-\bar{u})^{2\overline{\Delta}_n}}, \tag{3.3}$$

where $\Phi_{\pm n}$ are normalized so that $\langle \Phi_n(u)\Phi_{-n}(v)\rangle_{\mathbf{C}} = |v-u|^{-2\Delta_n-2\overline{\Delta}_n}$. Equation (3.3) is equivalent to the conformal Ward identity [16]

$$\langle T(w)\Phi_n(u)\Phi_{-n}(v)\rangle_{\mathbf{C}} = \left(\frac{\Delta_n}{(w-u)^2} + \frac{\Delta_n}{(w-v)^2} + \frac{1}{w-u}\frac{\partial}{\partial u} + \frac{1}{w-v}\frac{\partial}{\partial v}\right)\langle \Phi_n(u)\Phi_{-n}(v)\rangle_{\mathbf{C}}. \tag{3.4}$$

In writing the above, we are assuming that w is a complex coordinate on a single sheet $\mathbf{C}$, which is now decoupled from the others. We have therefore shown that

$$\langle T(w)\rangle_{\mathcal{R}_n} \equiv \frac{\int [\mathrm{d}\phi]\, T(w)\mathrm{e}^{-S_{\mathrm{E}}(\mathcal{R}_n)}}{\int [\mathrm{d}\phi]\, \mathrm{e}^{-S_{\mathrm{E}}(\mathcal{R}_n)}} = \frac{\langle T(w)\Phi_n(u)\Phi_{-n}(v)\rangle_{\mathbf{C}}}{\langle \Phi_n(u)\Phi_{-n}(v)\rangle_{\mathbf{C}}}. \tag{3.5}$$

Now consider the effect of an infinitesimal conformal transformation $w \to w' = w + \alpha(w)$ of $\mathbf{C}$ which acts identically on all the sheets of $\mathcal{R}_n$. The effect of this is to insert a factor

$$\frac{1}{2\pi\mathrm{i}}\int_C \alpha(w)T(w)\,\mathrm{d}w - \frac{1}{2\pi\mathrm{i}}\int_C \overline{\alpha(w)}\overline{T}(\bar{w})\,\mathrm{d}\bar{w} \tag{3.6}$$

into the path integral, where the contour C encircles the points u and v. The insertion of $T(w)$ is given by (3.3). Since this is to be inserted on each sheet, the right-hand side gets multiplied by a factor n.

Since the Ward identity (3.4) determines all the properties under conformal transformations, we conclude that the renormalized $Z_n(A)/Z^n \propto \operatorname{Tr}\rho_{\mathrm{A}}^n$ behaves (apart from a possible overall constant) under scale and conformal transformations identically to the nth power of two-point function of a primary operator Φ_n with $\Delta_n = \overline{\Delta}_n = (c/24)(1-(1/n)^2)$. In particular, this means that

$$\operatorname{Tr}\rho_{\mathrm{A}}^n = c_n((v-u)/a)^{-(c/6)(n-1/n)}, \tag{3.7}$$

where the exponent is just $4n\Delta_n$. The power of a (corresponding to the renormalization constant $\mathcal{Z}$) has been inserted so as the make the final result dimensionless, as it should be. The constants c_n are not determined by this method. However c_1 must be unity. Differentiating with respect to n and setting $n=1$, we recover the result of Holzhey *et al*.

The fact that $\operatorname{Tr}\rho_{\mathrm{A}}^n$ transforms under a general conformal transformation as a two-point function of primary operators $\Phi_{\pm n}$ means that it is simple to compute in other geometries, obtained by a conformal mapping $z \to z' = w(z)$, using the formula [16]

$$\langle\Phi(z_1,\bar{z}_1)\Phi(z_2,\bar{z}_2)\cdots\rangle = \prod_j |w'(z_j)|^{2\Delta_n}\langle\Phi(w_1,\bar{w}_1)\Phi(w_2,\bar{w}_2)\cdots\rangle. \tag{3.8}$$

For example, the transformation $w \to w' = (\beta/2\pi)\log w$ maps each sheet in the w-plane into an infinitely long cylinder of circumference β. The sheets are now sewn together along a branch cut joining the images of the points u and v. By arranging this to lie parallel to the axis of the cylinder, we get an expression for $\operatorname{Tr}\rho_{\mathrm{A}}^n$ in a thermal mixed state at finite temperature β^{-1}. This leads to the result for the entropy

$$S_{\mathrm{A}}(\beta) \sim (c/3)\log((\beta/\pi a)\sinh(\pi\ell/\beta)) + c_1'. \tag{3.9}$$

For $\ell \ll \beta$ we find $S_{\mathrm{A}} \sim (c/3)\log(\ell/a)$ as before, while, in the opposite limit $\ell \gg \beta$, $S_{\mathrm{A}} \sim (\pi c/3)(\ell/\beta)$. In this limit, the von Neumann entropy is extensive, and its density agrees with that of the Gibbs entropy of an isolated system of length ℓ, as obtained from the standard CFT expression [12, 13] $\beta F \sim -(\pi c/6)(\ell/\beta)$ for its free energy.

Alternatively, we may orient the branch cut perpendicular to the axis of the cylinder, which, with the replacement $\beta \to L$, corresponds to the entropy of a subsystem of length ℓ in a finite 1D system of length L, with periodic boundary conditions, in its ground state. This gives

$$S_{\mathrm{A}} \sim (c/3)\log((L/\pi a)\sin(\pi\ell/L)) + c_1'. \tag{3.10}$$

Note that this expression is symmetric under $\ell \to L-\ell$. It is maximal when $\ell = L/2$.

3.2. Finite system with a boundary

Next consider the case when the 1D system is a semi-infinite line, say $[0,\infty)$, and the subsystem A is the finite interval $[0,\ell)$. The n-sheeted Riemann surface then consists of n copies of the half-plane $x \geq 0$, sewn together along $0 \leq x < \ell, \tau = 0$. Once again, we work initially at zero temperature. It is convenient to use the complex variable $w = \tau + \mathrm{i}x$. The uniformizing transformation is now $z = ((w-\mathrm{i}\ell)/(w+\mathrm{i}\ell))^{1/n}$, which maps the whole

Riemann surface to the unit disc $|z| \leq 1$. In this geometry, $\langle T(z)\rangle = 0$ by rotational invariance, so, using (3.1), we find

$$\langle T(w)\rangle_{\mathcal{R}_n} = \frac{\Delta_n (2\ell)^2}{(w - \mathrm{i}\ell)^2 (w + \mathrm{i}\ell)^2}, \tag{3.11}$$

where $\Delta_n = (c/24)(1 - n^{-2})$ as before. Note that in the half-plane, T and $\overline{T}$ are related by analytic continuation: $\overline{T}(\bar{w}) = T(w)^*$. Equation (3.11) has the same form as $\langle T(w)\Phi_n(\mathrm{i}\ell)\rangle$, which follows from the Ward identities of boundary CFT [17], with the normalization $\langle \Phi_n(\mathrm{i}\ell)\rangle = (2\ell)^{-\Delta_n}$.

The analysis then proceeds in analogy with the previous case. We find

$$\operatorname{Tr} \rho_{\mathrm{A}}^n \sim \tilde{c}_n (2\ell/a)^{(c/12)(n-1/n)}, \tag{3.12}$$

so $S_{\mathrm{A}} \sim (c/6)\log(2\ell/a) + \tilde{c}_1'$.

Once again, this result can be conformally transformed into a number of other cases. At finite temperature β^{-1} we find

$$S_{\mathrm{A}}(\beta) \sim (c/6)\log((\beta/\pi a)\sinh(2\pi\ell/\beta)) + \tilde{c}_1'. \tag{3.13}$$

By taking the limit when $\ell \gg \beta$ we find the same extensive entropy as before. However, we can now identify $\tilde{c}_1' - c_1'$ as the boundary entropy g, first discussed by Affleck and Ludwig [9].

For a completely finite 1D system, of length L, at zero temperature, divided into two pieces of lengths ℓ and $L - \ell$, we similarly find

$$S_{\mathrm{A}} = (c/6)\log((2L/\pi a)\sin(\pi\ell/L)) + g + c_1'. \tag{3.14}$$

3.3. General case

For general N, the algebra is more complicated, but the method is the same. The uniformizing transformation now has the form $z = \prod_i (w - w_i)^{\alpha_i}$, with $\sum_i \alpha_i = 0$ (so there is no singularity at infinity). Here w_i can be u_j, v_j, or $\pm \mathrm{i}u_j$ in the case of a boundary. In our case, we have $\alpha_i = \pm 1/n$, but it is interesting to consider the more general transformation, and the notation is simpler. Once again we have

$$\langle T(w)\rangle = \frac{c}{12}\{z, w\} = \frac{c}{12}\frac{z'''z' - \frac{3}{2}z''^2}{z'^2}. \tag{3.15}$$

Consider $\{z, w\}/z$. As a function of w, this is meromorphic, has a double pole at each $w = w_i$, and is $\mathrm{O}(w^{-2})$ as $w \to \infty$. Hence it has the form

$$\{z, w\}/z = \sum_i \frac{A_i}{(w - w_i)^2} + \sum_i \frac{B_i}{w - w_i}, \tag{3.16}$$

where $\sum_i B_i = 0$. In order to determine A_i and B_i, we need to compute z', etc, to second

order in their singularities at w_i. After some algebra, we find

$$z' = \left[\frac{\alpha_i}{w-w_i} + \sum_{j\neq i}\frac{\alpha_j}{w-w_j}\right] z; \tag{3.17}$$

$$z'' = \left[\frac{-\alpha_i+\alpha_i^2}{(w-w_i)^2} + 2\frac{\alpha_i}{w-w_i}\sum_{j\neq i}\frac{\alpha_j}{w_i-w_j} + \cdots\right] z; \tag{3.18}$$

$$z''' = \left[\frac{2\alpha_i - 3\alpha_i^2+\alpha_i^3}{(w-w_i)^3} + \frac{-3\alpha_i+3\alpha_i^2}{(w-w_i)^2}\sum_{j\neq i}\frac{\alpha_j}{w_i-w_j} + \cdots\right] z. \tag{3.19}$$

Let

$$C_i \equiv \sum_{j\neq i}\frac{\alpha_j}{w_i-w_j}. \tag{3.20}$$

Then the coefficient of $(w-w_i)^{-2}$ is

$$\{[\alpha_i(\alpha_i-1)(\alpha_i-2)+3\alpha_i(\alpha_i-1)C_i(w-w_i)+\cdots][\alpha_i+C_i(w-w_i)+\cdots] - \tfrac{3}{2}[\alpha_i(\alpha_i-1)+2\alpha_iC_i(w-w_i)+\cdots]^2\}/[\alpha_i+C_i(w-w_i)+\cdots]^2,$$

from which we find after a little more algebra that

$$A_i = \tfrac{1}{2}(1-\alpha_i^2); \tag{3.21}$$

$$B_i = C_i\frac{1-\alpha_i^2}{\alpha_i}. \tag{3.22}$$

Thus we have shown that

$$\langle T(w)\rangle = \frac{c}{12}\sum_i\left[\frac{\frac{1}{2}(1-\alpha_i^2)}{(w-w_i)^2} + \left(\frac{(1-\alpha_i^2)}{\alpha_i}\sum_{j\neq i}\frac{\alpha_j}{w_i-w_j}\right)\frac{1}{w-w_i}\right]. \tag{3.23}$$

This is to be compared with the conformal Ward identity

$$\left\langle T(w)\prod_i\Phi_i(w_i)\right\rangle = \sum_i\left[\frac{\Delta_i}{(w-w_i)^2} + \frac{1}{w-w_i}\frac{\partial}{\partial w_i}\right]\left\langle\prod_k\Phi_k(w_k)\right\rangle. \tag{3.24}$$

For these to be equivalent, we must have $\Delta_i = (1/2)(1-\alpha_i^2)$ and

$$\frac{(1-\alpha_i^2)}{\alpha_i}\sum_{j\neq i}\frac{\alpha_j}{w_i-w_j} = \frac{\partial}{\partial w_i}\log\left\langle\prod_k\Phi_k(w_k)\right\rangle. \tag{3.25}$$

A necessary and sufficient condition for this is that

$$\frac{\partial}{\partial w_k}\left(\frac{1-\alpha_i^2}{\alpha_i}\sum_{j\neq i}\frac{\alpha_j}{w_i-w_j}\right) = \frac{\partial}{\partial w_i}\left(\frac{1-\alpha_k^2}{\alpha_k}\sum_{j\neq k}\frac{\alpha_j}{w_k-w_j}\right), \tag{3.26}$$

for each pair (i,k). This reduces to

$$-\frac{1-\alpha_i^2}{\alpha_i}\frac{\alpha_k}{(w_k-w_i)^2} = -\frac{1-\alpha_k^2}{\alpha_k}\frac{\alpha_i}{(w_i-w_k)^2}, \tag{3.27}$$

that is, $\alpha_i = \pm\alpha_k$ for each pair (i,k). Since $\sum_i \alpha_i = 0$ the only way to satisfy this is to have $\alpha_i = \alpha\sigma_i$, with $\sigma_i = \pm 1$, and half the $\sigma_i = +1$ and the remainder -1. Interestingly enough, this is precisely the case we need, with $\alpha = 1/n$.

If these conditions are satisfied,

$$\frac{\partial}{\partial w_i} \log\left\langle \prod_j \Phi_j(w_j) \right\rangle = \frac{c}{12}(1-\alpha^2) \sum_{k\neq i} \frac{\sigma_i\sigma_k}{w_i - w_k}, \tag{3.28}$$

so

$$\log\left\langle \prod_j \Phi_j(w_j) \right\rangle = \frac{c}{12}(1-\alpha^2) \sum_{k<i} \sigma_i\sigma_k \log(w_i - w_k) + E, \tag{3.29}$$

where E is independent of all the w_i. In the case with no boundary, E can depend however on the $\bar{w}_i$. A similar calculation with $\overline{T}$ then gives a similar dependence. We conclude that $\operatorname{Tr}\rho_{\mathrm{A}}^n$ behaves under conformal transformations in the same way as

$$\left\langle \prod_j \Phi_j \right\rangle \propto \prod_{j<k} (w_i - w_k)^{(c/12)(1-\alpha^2)\sigma_i\sigma_k} (\bar{w}_i - \bar{w}_k)^{(c/12)(1-\alpha^2)\bar{\sigma}_i\bar{\sigma}_k}. \tag{3.30}$$

Taking now $\alpha = 1/n$, and $\sigma = \pm 1$ according to whether $w_i = u_j$ or v_j, we find

$$\operatorname{Tr}\rho_{\mathrm{A}}^n \sim c_n^N \left(\frac{\prod_{j<k}(u_k - u_j)(v_k - v_j)}{\prod_{j\leq k}(v_k - u_j)} \right)^{(c/6)(n-1/n)}. \tag{3.31}$$

The overall constant is fixed in terms of the previously defined c_n by taking the intervals to be far apart from each other, in comparison to their lengths.

Differentiating with respect to n and setting $n = 1$, we find our main result of this section:

$$S_{\mathrm{A}} = \frac{c}{3}\left(\sum_{j\leq k} \log((v_k - u_j)/a) - \sum_{j<k} \log((u_k - u_j)/a) - \sum_{j<k} \log((v_k - v_j)/a) \right) + N c_1'. \tag{3.32}$$

A similar expression holds in the case of a boundary, with half of the w_i corresponding to the image points.

Finally we comment on the recent result of Casini and Huerta [6], which corresponds to $N = 2$. In fact, it may be generalized to the ratio of Tsallis entropies: from (3.31) we find

$$\frac{S_n(A)S_n(B)}{S_n(A\cap B)S_n(A\cup B)} = \left(\frac{(v_1 - u_1)(v_2 - u_2)}{(u_2 - u_1)(v_2 - v_1)} \right)^{(c/6)(n-1/n)}, \tag{3.33}$$

where $S_n(A) \equiv \operatorname{Tr}\rho_{\mathrm{A}}^n$, and the expression in parentheses is the cross-ratio η of the four points. Notice that in this expression the dependence on the ultraviolet cut-off a disappeared, and so have the non-universal numbers c_n. Differentiating with respect to n gives the result of Casini and Huerta [6], who however *assumed* that the result should depend only on η.

4. Entropy in non-critical 1 + 1-dimensional models

4.1. Massive field theory—general case

In this section we consider an infinite non-critical model in $1+1$ dimensions, in the scaling limit where the lattice spacing $a \to 0$ with the correlation length (inverse mass) fixed. This corresponds to a massive relativistic QFT. We first consider the case when the subset A is the negative real axis, so that the appropriate Riemann surface has branch points of order n at 0 and infinity. However, for the non-critical case, the branch point at infinity is unimportant: we should arrive at the same expression by considering a finite system whose length L is much greater than ξ.

Our argument parallels that of Zamolodchikov [14] for the proof of his famous c-theorem. Let us consider the expectation value of the stress tensor $T_{\mu\nu}$ of a massive Euclidean QFT on such a Riemann surface. In complex coordinates, there are three non-zero components: $T \equiv T_{zz}$, $\overline{T} \equiv T_{\bar{z}\bar{z}}$, and the trace $\Theta = 4T_{z\bar{z}} = 4T_{\bar{z}z}$. These are related by the conservation equations

$$\partial_{\bar{z}} T + \tfrac{1}{4}\partial_z \Theta = 0; \tag{4.1}$$

$$\partial_z \overline{T} + \tfrac{1}{4}\partial_{\bar{z}} \Theta = 0. \tag{4.2}$$

Consider the expectation values of these components. In the single-sheeted geometry, $\langle T \rangle$ and $\langle \overline{T} \rangle$ both vanish, but $\langle \Theta \rangle$ is constant and non-vanishing: it measures the explicit breaking of scale invariance in the non-critical system. In the n-sheeted geometry, however, they all acquire a non-trivial spatial dependence. By rotational invariance about the origin, they have the form

$$\langle T(z,\bar{z}) \rangle = F_n(z\bar{z})/z^2; \tag{4.3}$$

$$\langle \Theta(z,\bar{z}) \rangle - \langle \Theta \rangle_1 = G_n(z\bar{z})/(z\bar{z}); \tag{4.4}$$

$$\langle \overline{T}(z,\bar{z}) \rangle = F_n(z\bar{z})/\bar{z}^2. \tag{4.5}$$

From the conservation conditions (4.1) we have

$$(z\bar{z})(F_n' + \tfrac{1}{4}G_n') = \tfrac{1}{4}G_n. \tag{4.6}$$

Now we expect that F_n and G_n both approach zero exponentially fast for $|z| \gg \xi$, while in the opposite limit, on distance scales $\ll \xi$, they approach the CFT values (see previous section) $F_n \to (c/24)(1 - n^{-2})$, $G_n \to 0$.

This means that if we define an effective C-function

$$C_n(R^2) \equiv (F(R^2) + \tfrac{1}{4}G(R^2)), \tag{4.7}$$

then

$$R^2 \frac{\partial}{\partial(R^2)} C_n(R^2) = \tfrac{1}{4}G_n(R^2). \tag{4.8}$$

If we were able to argue that $G_n \leq 0$, that is $\langle\Theta\rangle_n \leq \langle\Theta\rangle_1$, we would have found alternative formulation of the c-theorem. However, we can still derive an integrated form of the c-theorem, using the boundary conditions[6]:

$$\int_0^\infty \frac{G_n(R^2)}{R^2}\,\mathrm{d}(R^2) = -(c/6)(1-n^{-2}), \tag{4.9}$$

or equivalently

$$\int (\langle\Theta\rangle_n - \langle\Theta\rangle_1)\,\mathrm{d}^2R = -\pi n(c/6)(1-n^{-2}), \tag{4.10}$$

where the integral is over the whole of the n-sheeted surface. Now this integral (multiplied by a factor $1/2\pi$ corresponding to the conventional normalization of the stress tensor) measures the response of the free energy $-\log Z$ to a scale transformation, i.e. to a change in the mass m, since this is the only dimensionful parameter of the renormalized theory. Thus the left-hand side is equal to

$$-(2\pi)m(\partial/\partial m)\,[\log Z_n - n\log Z], \tag{4.11}$$

giving finally

$$\frac{Z_n}{Z^n} = c_n(ma)^{(c/12)(n-1/n)}, \tag{4.12}$$

where c_n is a constant (with however $c_1 = 1$), and we have inserted a power of a, corresponding to the renormalization constant $\mathcal{Z}$ discussed earlier, to make the result dimensionless.

This shows that the $(n-1/n)$ dependence for the exponent of the Tsallis entropy is a general property of the continuum theory. Differentiating at $n=1$, we find the main result of this section:

$$S_\mathrm{A} \sim -(c/6)\log(ma) = (c/6)\log(\xi/a), \tag{4.13}$$

where ξ is the correlation length. We re-emphasize that this result was obtained only for the scaling limit $\xi \gg a$. However, for lattice integrable models, we shall show how it is possible to obtain the full dependence without this restriction.

So far we have considered the simplest geometry in which the set A and its complement B are semi-infinite intervals. The more general case, when A is a union of disjoint intervals, is more difficult in the massive case. However, it is still true that the entropy can be expressed in terms of the derivative at $n=1$ of correlators of operators Φ_n. The above calculation can be thought of in terms of the one-point function $\langle\Phi_n\rangle$. In any quantum field theory a more general correlator $\langle\prod_{i=1}^k \Phi_{\pm n}(w_i)\rangle$ should obey cluster decomposition: that is, for separations $|w_i - w_j|$ all $\gg\xi$, it should approach $\langle\Phi_n\rangle^k$. This suggests that, in this limit, the entropy should behave as $S_\mathrm{A} \sim \mathcal{A}(c/6)\log(\xi/a)$, where $\mathcal{A} = k$ is the number of boundary points between A and its complement. This would be the 1D version of the area law [10]. When the interval lengths are of the order of ξ, we expect to see a complicated but universal scaling form for the crossover.

[6] We have assumed that theory is trivial in the infrared. If the RG flow is towards a non-trivial theory, c should be replaced by c_UV–c_IR.

4.2. Free bosonic field theory

In this subsection we verify equation (4.13) by an explicit calculation for a massive free field theory (Gaussian model). The action

$$\mathcal{S} = \int \tfrac{1}{2}\left((\partial_\mu \varphi)^2 + m^2\varphi^2\right) \mathrm{d}^2 r, \tag{4.14}$$

is, as before, considered on a n-sheeted Riemann surface with one cut, which we arbitrarily fix on the real negative axis.

To obtain the entanglement entropy we should know the ratio Z_n/Z^n, where Z_n is the partition function in the n-sheeted geometry. There are several equivalent ways to calculate such partition function. In the following, we find easier to use the identity[7]

$$\frac{\partial}{\partial m^2} \log Z_n = -\frac{1}{2}\int G_n(\mathbf{r},\mathbf{r})\, \mathrm{d}^2 r, \tag{4.15}$$

where $G_n(\mathbf{r},\mathbf{r}')$ is the two-point correlation function in the n-sheeted geometry. Thus we need the combination $G_n - nG_1$. G_n obeys

$$(-\nabla^2_{\mathbf{r}} + m^2) G_n(\mathbf{r},\mathbf{r}') = \delta^2(\mathbf{r}-\mathbf{r}'). \tag{4.16}$$

Its solution (see the appendix) may be expressed in polar coordinates as (here $0 < r, r' < \infty$ and $0 \le \theta, \theta' < 2\pi n$)

$$G_n(r,\theta,r',\theta') = \frac{1}{2\pi n}\sum_{k=0}^{\infty} d_k \int_0^\infty \lambda\, \mathrm{d}\lambda\, \frac{J_{k/n}(\lambda r) J_{k/n}(\lambda r')}{\lambda^2 + m^2} \mathcal{C}_k(\theta,\theta'), \tag{4.17}$$

where $\mathcal{C}_k(\theta,\theta') = \cos(k\theta/n)\cos(k\theta'/n) + \sin(k\theta/n)\sin(k\theta'/n)$, $d_0 = 1$, $d_{k>0} = 2$, and $J_k(x)$ are the Bessel functions of the first kind. At coincident points (i.e. $r = r'$, $\theta = \theta'$), and after integrating over θ and λ, we have

$$G_n(r) \equiv G_n(\mathbf{r},\mathbf{r}) = \sum_{k=0}^{\infty} d_k I_{k/n}(mr) K_{k/n}(mr), \tag{4.18}$$

where $I_k(x)$ and $K_k(x)$ are the modified Bessel functions of the first and second kind respectively [18].

The sum over k in (4.18) is UV divergent. This reflects the usual short-distance divergence which would occur even in the plane. However, if we formally exchange the order of the sum and integration we find

$$-\frac{\partial}{\partial m^2}\log Z_n = \frac{1}{2}\int \mathrm{d}^2 r\, G_n(r) = \frac{1}{2}\sum_k d_k \int_0^\infty I_{k/n}(mr) K_{k/n}(mr) r\, \mathrm{d}r = \frac{1}{4nm^2}\sum_k d_k k. \tag{4.19}$$

Interpreting the last sum as $2\zeta(-1) = -1/6$, we obtain the correct result, which we now derive more systematically.

Let us first regularize each sum over k by inserting a function $F(k/\Lambda_n)$: F is chosen so that $F(0) = 1$, and all its derivatives at the origin vanish: however, it goes to zero

[7] This holds only for non-interacting theories: in the presence of interactions the sum of all the zero-point diagrams has to be taken into account.

sufficiently fast at infinity. Since k/n is conjugate to the angle θ, we should think of this cut-off as being equivalent to a discretization $\delta\theta$. Thus we should choose $\Lambda_n = \Lambda \cdot n$, where $\Lambda \sim (\delta\theta)^{-1}$.

To perform the sum over k, we use the Euler–MacLaurin (EML) sum formula [18]

$$\frac{1}{2}\sum_{k=0}^{\infty} d_k f(k) = \int_0^\infty f(k)\,\mathrm{d}k - \frac{1}{12}f'(0) - \sum_{j=2}^{\infty}\frac{B_{2j}}{(2j)!}f^{(2j-1)}(0), \tag{4.20}$$

where B_{2n} are the Bernoulli numbers [18]. Using standard identities of the Bessel function (namely $\partial_k K_k(x)|_{k=0} = 0$ and $\partial_k I_k(x)|_{k=0} = -K_0(k)$ [18]), we obtain

$$\begin{aligned} G_n(r) = 2\int_0^\infty \mathrm{d}k\, I_{k/n}(mr)K_{k/n}(mr)F(k/n\Lambda) \\ + \frac{1}{6n}K_0^2(mr) + \int_0^\infty r\,\mathrm{d}r \sum_{j\geq 1}\frac{B_{2j}}{(2j)!n^{2j+1}}D_{2j+1}(r), \end{aligned} \tag{4.21}$$

where we define $D_i(x) = \partial^i(I_k(x)K_k(x))/\partial k^i|_{k=0}$. In the last term in equation (4.21) the order of the integral, derivative, and sum can be exchanged and each term in the sum is

$$\frac{\partial^i}{\partial k^i}\int_0^\infty x\,\mathrm{d}x\, I_k(x)K_k(x) = -\frac{\partial^i}{\partial k^i}\frac{k}{2} = 0 \qquad \text{for } i = 2j+1 \geq 2, \tag{4.22}$$

i.e. the sum in equation (4.21) is vanishing.

Thus, still with the regulator in place,

$$\frac{\partial}{\partial m^2}\log Z_n = -\int_0^\infty r\,\mathrm{d}r\int_0^\infty \mathrm{d}k\, I_{k/n}(mr)K_{k/n}(mr)F(k/n\Lambda) - \frac{1}{24nm^2}, \tag{4.23}$$

where we have used $\int_0^\infty rK_0^2(mr)\,\mathrm{d}r = 1/(2m^2)$.

Now the point is that in the integral over k the factor of n can be scaled out by letting $k \to nk$. Thus this potentially divergent term cancels in the required combination $G_n - nG_1$. Having taken this combination, we may now remove the regulator to find the main result

$$\frac{\partial}{\partial m^2}\frac{\log Z_n}{Z^n} = \frac{1}{24m^2}\left(n - \frac{1}{n}\right). \tag{4.24}$$

The integration of the last expression wrt m^2 (made following the recipe for the integration limits given in the previous section) gives

$$\log \mathrm{Tr}\,\rho^n = \log\frac{Z_n}{Z^n} = \frac{\log a^2m^2}{24}\left(n - \frac{1}{n}\right), \tag{4.25}$$

and finally the entanglement entropy

$$S = -\mathrm{Tr}\,\rho\log\rho = -\frac{\partial}{\partial n}\mathrm{Tr}\,\rho^n\Big|_{n=1} = -\frac{\partial}{\partial n}(m^2a^2)^{(1/24)(n-1/n)}\Big|_{n=1} = -\frac{1}{12}\log m^2a^2, \tag{4.26}$$

that agrees with the general formula we derive (4.13), with $c = 1$ and $m = \xi^{-1}$.

4.3. Integrable models and the corner transfer matrix

In this subsection we verify (4.13) for the transverse Ising chain and the uniaxial XXZ Heisenberg model. These results are also an independent check of the uniform convergence of the derivative wrt n of ρ^n when $n \to 1$, since for these models we can compute the eigenvalues of ρ_A exactly.

Although these systems are integrable and their ground state is known, a direct calculation of ρ is difficult. The difficulties arising in a direct calculation can be avoided mapping the quantum chains onto two-dimensional classical spin systems. As first pointed out by Nishino *et al* [19] the density matrix of the quantum chain is the partition function of a two-dimensional strip with a cut perpendicular to it. In fact the ground state of a quantum chain described by a Hamiltonian H is also eigenstate of the transfer matrix T of a classical system satisfying $[H, T] = 0$. Therefore the reduced density matrix of a subsystem A of the chain (defined in the introduction as $\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|$, with B the complement of A) is the partition function of two half-infinite strips, one extending from $-\infty$ to 0 and the other from $+\infty$ to 0, with the spins in B identified.

This partition function is the product of four Baxter corner transfer matrices (CTMs) [20] $\hat{A}$. If the lattice is chosen in a clever way (i.e. rotated by $\pi/4$ with respect to the cut), one ends up in the infinite length limit with [21]

$$\hat{\rho}_A = \hat{A}^4 = \mathrm{e}^{-\hat{H}_{CTM}}, \tag{4.27}$$

where $\hat{H}_{CTM}$ is an effective Hamiltonian, which, for the models under consideration, may be diagonalized by means of fermionization (see for more details about this equivalence [21] and references therein). Note that $\mathrm{Tr}\,\hat{\rho}_A \neq 1$; thus the usual density matrix is $\rho_A = \hat{\rho}_A / \mathrm{Tr}\,\hat{\rho}_A$.

The method just outlined is very general. However, for the integrable chains under consideration (and indeed for any model satisfying suitable Yang–Baxter equations [20]) it is possible to write $H_{CTM} = \epsilon \hat{O}$, with ϵ the scale giving the distance between the energy levels, and $\hat{O}$ is an operator with integer eigenvalues (for the Ising and XXZ models it is expressed in terms of free fermions). Using this property the entropy is given by[8]

$$S = -\mathrm{Tr}\,\rho_A \log \rho_A = -\mathrm{Tr}\,\frac{\hat{\rho}_A \log \hat{\rho}_A}{\mathrm{Tr}\,\hat{\rho}_A} + \log \mathrm{Tr}\,\hat{\rho}_A = -\epsilon \frac{\partial \log Z}{\partial \epsilon} + \log Z, \tag{4.28}$$

where we defined $Z = \mathrm{Tr}\,\hat{\rho}_A = \mathrm{Tr}\,\mathrm{e}^{-\hat{H}_{CTM}}$.

The Ising model in a transverse field can be described by the one-dimensional Hamiltonian

$$H_I = -\sum_{n=1}^{L-1} \sigma_n^x - \lambda \sum_{n=1}^{L-1} \sigma_n^z \sigma_{n+1}^z, \tag{4.29}$$

where σ_n are the Pauli matrices at the site n, and we normalize the Hamiltonian (following [21]) by imposing the transverse field in the x direction to be one; thus the transition is driven by the parameter λ. The classical equivalent of (4.29) is a two-dimensional Ising model. For $\lambda = 0$ the ground state of the Hamiltonian (4.29) is a quantum 'paramagnet' with all the spins aligned with the magnetic field in the x direction,

[8] In the rest of this section, all logarithms are assumed taken to base e.

and $\langle\sigma_n^z\rangle = 0$. In the opposite limit $\lambda = \infty$ the magnetic field is negligible and the ground state is ferromagnetic with $\langle\sigma_n^z\rangle = \pm 1$. The (second-order) transition between these two regimes happens at $\lambda = 1$. The exponent characterizing the divergence of the correlation length is $\nu = 1$, i.e. $\xi \simeq |\lambda - 1|^{-1}$.

The CTM of the Ising model may be diagonalized in terms of fermionic operators. The CTM Hamiltonian, written in terms of the fermion occupation number $\hat{n}_i$ (with eigenvalues 0 and 1), is [21]

$$\hat{H}_{\mathrm{CTM}} = \sum_{j=0}^{\infty} \epsilon_j \hat{n}_j. \tag{4.30}$$

The energy levels are

$$\epsilon_j = \begin{cases} (2j+1)\epsilon & \text{for } \lambda < 1, \\ 2j\epsilon & \text{for } \lambda > 1, \end{cases} \qquad \text{with } \epsilon = \pi \frac{K(\sqrt{1-k^2})}{K(k)}, \tag{4.31}$$

where $K(k)$ is the complete elliptic integral of the first kind [18], and $k = \min[\lambda, \lambda^{-1}]$.

For $\lambda < 1$,

$$Z = \mathrm{Tr}\, \mathrm{e}^{-\hat{H}_{\mathrm{CTM}}} = \prod_{j=0}^{\infty} \left[1 + \mathrm{e}^{-\epsilon(2j+1)}\right], \tag{4.32}$$

and the entropy from equation (4.28) is

$$S = \epsilon \sum_{j=0}^{\infty} \frac{2j+1}{1 + \mathrm{e}^{(2j+1)\epsilon}} + \sum_{j=0}^{\infty} \log(1 + \mathrm{e}^{-(2j+1)\epsilon}). \tag{4.33}$$

Analogously, in the quantum ferromagnetic phase ($\lambda > 1$),

$$S = \epsilon \sum_{j=0}^{\infty} \frac{2j}{1 + \mathrm{e}^{2j\epsilon}} + \sum_{j=0}^{\infty} \log(1 + \mathrm{e}^{-2j\epsilon}). \tag{4.34}$$

Figure 1 shows a plot of the entropy as a function of λ, characterized by a divergence at the quantum critical point $\lambda = 1$. Note that $S(0) = 0$ and $S(\infty) = \log 2$, in agreement with the expectation that the pure ferromagnetic ground state ($\lambda = \infty$) has two possible accessible configurations with opposite sign of magnetization (i.e. $S(\infty) = \log 2$) and the pure quantum paramagnetic ground state ($\lambda = 0$) has only one configuration available with all the spins aligned in the direction of the magnetic field x and the resulting entropy is zero.

Let us analyse in detail the behaviour at the critical point. For $\lambda \to 1$, $\epsilon \to 0$ in both the phases; thus the sums in equations (4.33) and (4.34) can be approximated by the integral

$$S \simeq \int_0^{\infty} \mathrm{d}x \left(\frac{2x\epsilon}{1 + \mathrm{e}^{2x\epsilon}} + \log(1 + \mathrm{e}^{-2x\epsilon}) \right) = \frac{\pi^2}{12} \frac{1}{\epsilon} \tag{4.35}$$

(by $\simeq$ we mean in the critical region). The λ dependence (we use $K(0) = \pi/2$, $K(x) = -1/2 \log(1-x) + \mathrm{O}((1-x)^0)$ [18])

$$S \simeq \frac{\pi^2}{12} \frac{1}{\epsilon} \simeq -\frac{1}{12} \log(1-k) = \frac{1}{12} \log \xi + C_1, \tag{4.36}$$

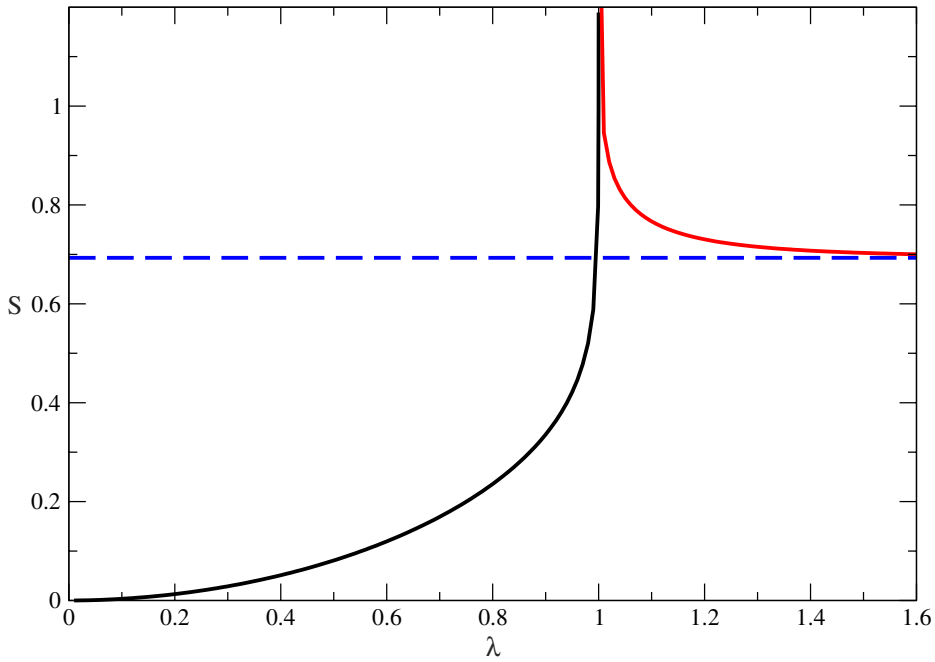

Figure 1. Entanglement entropy for the 1D Ising chain as a function of λ. The dashed line is the limit for $\lambda \to \infty$, i.e. $\log 2$.

where in the last equality $\xi \propto |1-k|^{-1}$ has been used. This agrees with (4.13), with $c = 1/2$ for the Ising model. The constant C_1 is not universal.

Another model whose density matrix has been derived by using the CTM is the XXZ model

$$H_{XXZ} = \sum_n (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z), \tag{4.37}$$

for $\Delta > 1$ (whose classical equivalent is the Baxter six-vertex model [20]). This model has an Ising-like ferromagnetic state for $\Delta > 1$ and a planar XX ferromagnetic ground state at $0 < \Delta < 1$ (the case $\Delta < 0$ describes the antiferromagnetic regime, in which we are not interested). At $\Delta = 1$ the Hamiltonian is isotropic (XXX) and approaching such a point from large Δ values the correlation length diverges as (see e.g. [20])

$$\log \frac{\xi}{a} \simeq \frac{\pi^2}{2\sqrt{2}\sqrt{\Delta - 1}}, \tag{4.38}$$

because at $\Delta = 1$ a massless excitation (Goldstone mode) is present in the spectrum.

The CTM Hamiltonian $\hat{H}_{\mathrm{CTM}}$ for this model has been obtained in [21]:

$$\hat{H}_{\mathrm{CTM}} = \sum_{j=0}^{\infty} 2j\epsilon n_j, \tag{4.39}$$

with $\epsilon = \operatorname{arccosh} \Delta$, for $\Delta > 1$. The entropy is given by equation (4.34). Close to the isotropic point ($\Delta = 1$) it holds that $\epsilon \simeq \sqrt{2}\sqrt{\Delta - 1}$ and (see equation (4.35))

$$S \simeq \frac{\pi^2}{12}\frac{1}{\epsilon} \simeq \frac{\pi^2}{12\sqrt{2}}\frac{1}{\sqrt{\Delta - 1}}. \tag{4.40}$$

Using equation (4.38) to write Δ in terms of the correlation length, we have

$$S \simeq \frac{1}{6} \log \frac{\xi}{a}, \tag{4.41}$$

in agreement with (4.13), with $c = 1$. Again, in the limit $\Delta \to \infty$, $S = \log 2$, since the ferromagnetic ground state is Ising-like.

The method used here for the transverse Ising model and the XXZ model can in principle be applied to all those integrable models whose weights satisfy Yang–Baxter equations.

5. Finite size effects

So far, we have studied the entropy either at the critical point, for a finite subsystem, or away from criticality in an infinite system. These two regimes may be linked by a generalization of finite size scaling theory. This would assert, for example, that the entropy of a subsystem A which forms e.g. the left half of a finite non-critical system of length $2L$ should have the form

$$S_{\rm A}(L, \xi) = \frac{c}{6} \left(\log L + s_{\rm FSS}(L/\xi) \right), \tag{5.1}$$

with $s_{\rm FSS}(0) = 0$ (this because we are referring to the scaling part; the constant term found before is not universal and it defines an overall additive normalization of S) and $s_{\rm FSS}(x) \sim -\log x$, for large x, so as to recover $S = (c/6) \log \xi$ in the infinite length limit. $s_{\rm FSS}(x)$ should admit a small x expansion:

$$s_{\rm FSS}(x) = \sum_{j \geq 1} s_j x^{2j}, \tag{5.2}$$

with s_j *universal* coefficients, and a large x expansion:

$$s_{\rm FSS}(x) = -\log x + \sum_{j \geq 0} \frac{s_j^\infty}{x^j}, \tag{5.3}$$

with s_j^∞ also universal (in special situations logarithmic corrections could also be generated). In the next section we test this hypothesis for the free massive field theory.

5.1. FSS in the Gaussian model

Once again we consider the n-sheeted surface, but it now consists of n discs of finite radius L, sewn together along the negative real axis from $-L$ to 0. The Gaussian two-point function in a finite geometry has been calculated in the appendix: equation (A.2). Setting $\theta = \theta'$, $r = r'$ and integrating over θ', the propagator at coincident point is

$$G_n(r) = \sum_{k=0}^{\infty} d_k \sum_{i=1}^{\infty} \frac{2/L^2}{J_{k/n+1}^2(\alpha_{k/n,i})} \frac{J_{k/n}^2(\alpha_{k/n,i} r/L)}{\alpha_{k/n,i}^2/L^2 + m^2}, \tag{5.4}$$

where $\alpha_{\nu,i}$ denotes the ith zero of $J_\nu(x)$. $G(r)$ is the analogue of (4.18) in a finite geometry.

As we showed in the previous section, the right order in which to proceed to get a sensitive Z_n is first to perform the sum over i, then sum over k, and finally integrate

over r. This is really hard, requiring a sum of Bessel functions for generic argument, over the zeros of different Bessel functions. What one can do is invert the order of the sums and integrations and try to understand what happens. From the previous exercise we know that this operation has to be done with care.

However, the formal result is

$$\log Z_n = \frac{1}{2}\sum_k d_k \sum_i \log \frac{\alpha_{k/n,i}^2/L^2 + m^2}{\alpha_{k/n,i}^2/L^2 + a^{-2}}. \tag{5.5}$$

Since the large i behaviour of the zeros of the Bessel functions is $\alpha_{\nu,i} \sim \pi(i + \nu/2 - 1/4)$ (see e.g. [18]), this sum diverges.

A first, qualitatively correct expression for the universal function $s_{\mathrm{FSS}}(x)$ can be obtained from equation (5.5), truncating the EML formula equation (4.20) in the variable k at the first order in the derivative. This approximation gives a correct form of the result because it takes into account the complete infinite volume result (whose truncated expression is exact). The integral term in the EML is divergent, but it cancels as before in the ratio Z_n/Z^n if the cut-off in the angular modes is properly chosen as in the infinite volume case. The EML approximation at the first order is (we do not write the integral)

$$\log Z_n = \frac{\pi L^2(a^{-2} - m^2)}{12n} \sum_i \frac{\alpha_{0,i}}{(\alpha_{0,i}^2 + m^2L^2)(\alpha_{0,i}^2 + L^2/a^2)} \frac{Y_0(\alpha_{0,i})}{J_1(\alpha_{0,i})} \equiv \frac{F_1(mL, L/a)}{n}, \tag{5.6}$$

where we used

$$\left.\frac{\partial \alpha_{k,i}}{\partial k}\right|_{k=0} = \frac{\pi}{2}\frac{Y_0(\alpha_{0,i})}{J_1(\alpha_{0,i})}, \tag{5.7}$$

which can be derived using the formulae [18] for the derivative of the Bessel functions with respect to the order. The remaining sum over i (the zeros of the Bessel function) is now finite.

This gives the entropy as

$$S = -\frac{\partial}{\partial n} \operatorname{Tr} \rho^n \Big|_{n=1} = 2F_1(mL, L/a), \tag{5.8}$$

which reproduces the correct limits for $m = 0$ and $L = \infty$.

From this formula we may compare with the FSS ansatz (5.1), with $c = 1$:

$$s_{\mathrm{FSS}}^{(1)}(mL) = 12(F_1(mL, L/a) - F_1(0, L/a)) = -\pi \sum_i \frac{Y_0(\alpha_{0,i})}{J_1(\alpha_{0,i})} \frac{m^2L^2}{\alpha_{0,i}(\alpha_{0,i}^2 + m^2L^2)}, \tag{5.9}$$

that, as expected, is a function only of the product mL, and all the dependence upon a disappears after the subtraction. This agrees with $s_{\mathrm{FSS}}(0) = 0$ and $s_{\mathrm{FSS}}(0) = -\log x$, for large x (it can be easily shown, since in this limit the sum can be replaced by an integral). The superscript (1) is there to remind us that we made a first-order approximation in EML.

Equation (5.9) characterizes completely the crossover between the mass dominated (non-critical) and the geometry dominated (critical) regimes.

A plot of $s_{\mathrm{FSS}}^{(1)}$ is shown in figure 2 (calculated as the sum of the first 1000 zeros of J_0). From this figure we see that an optimum approximation for $x \geq 2$ of $s_{\mathrm{FSS}}^{(1)}(x)$ is

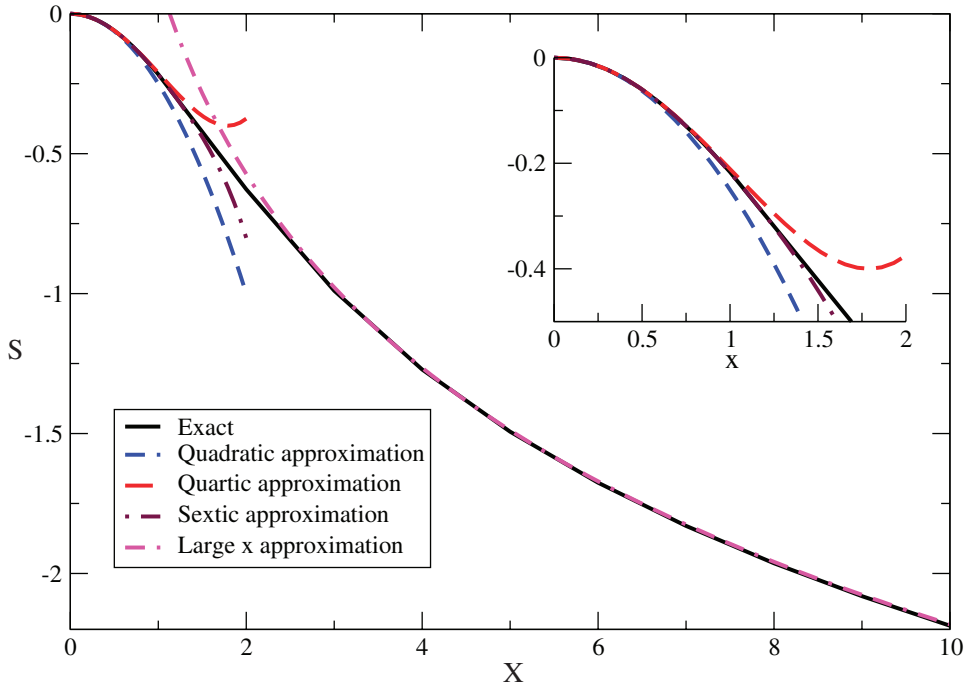

Figure 2. $s^{(1)}_{\rm FSS}(x)$: the exact expression obtained as numerical sums over the first 1000 zeros of the Bessel functions compared with small (quadratic, quartic, and sextic) and large x ($\log x + s_0^\infty$) approximations. Even the use of quadratic and large x approximation may reproduce the right formula over the whole range. Inset: the comparison of several small x approximants for $x < 2$.

$s^{\rm asy}_{\rm FSS}(x) = -\log x + s_0^\infty$, with $s_0^\infty = 0.120\,912$ (this value of s_0^∞ is a fit). We can also calculate analytically all the universal coefficients of the small mL expansion [22]:

$$s_1^{(1)} = -\pi \sum_{i=1}^{\infty} \frac{Y_0(\alpha_{0,i})}{J_1(\alpha_{0,i})} \frac{1}{\alpha_{0,i}^3} = -\frac{1}{4}, \tag{5.10}$$

$$s_2^{(1)} = \pi \sum_{i=1}^{\infty} \frac{Y_0(\alpha_{0,i})}{J_1(\alpha_{0,i})} \frac{1}{\alpha_{0,i}^5} = \frac{5}{128}, \tag{5.11}$$

$$s_3^{(1)} = -\pi \sum_{i=1}^{\infty} \frac{Y_0(\alpha_{0,i})}{J_1(\alpha_{0,i})} \frac{1}{\alpha_{0,i}^7} = -\frac{23}{3456}, \tag{5.12}$$

$$s_4^{(1)} \simeq 1.15 \times 10^{-3}, \qquad s_5^{(1)} \simeq -1.98 \times 10^{-4}, \qquad s_6^{(1)} \simeq 3.42 \times 10^{-5}. \tag{5.13}$$

Figure 2 (see also the inset) shows a plot of $s^{(1)}_{\rm FSS}(x)$, compared with quadratic, quartic, and sextic approximations. The agreement is excellent in the region $x \leq 2$.

Before starting the full calculation of the function $s_{\rm FSS}(x)$ let us summarize what we can learn from the first-order approximation in the EML expansion: a rather good approximation of the full function may be obtained by matching only the first-order small argument expansion with the large x behaviour $\sim \log x$ (which is exact and does not depend upon the approximation). Thus in the following we show how to calculate the coefficients s_j without any approximation.

Our starting point is again equation (5.5), but this time we will not use EML sum formula. For this reason we have to be careful since the sum is divergent.

The interesting object is $\log Z_n - n \log Z$. In particular we can write an FSS ansatz also for this 'free energy' and so subtract the $m = 0$ part, obtaining the universal function $f(x = mL)$

$$f(x) = 3 \sum_k d_k \sum_i \left[n \log \left(1 + \frac{x^2}{\alpha_{k,i}^2} \right) - \log \left(1 + \frac{x^2}{\alpha_{k/n,i}^2} \right) \right], \tag{5.14}$$

form which $s_{\rm FSS}(x) = -(\partial f(x)/\partial n)|_{n=1}$. This formula has to be intended in a formal way: in fact it is the difference of two diverging sums. To make this difference sensible the recipe for the cut-off explained in the previous section has to be used. Anyway, from the computational point of view it is simpler to make the calculation without being careful about the cut-off and to adjust the result only at the end. The numerical sum over the zeros of the Bessel functions of generic order cannot be done as before, but the small x expansion of $f(x) = \sum f_j x^{2j}$ with coefficients

$$f_j = \frac{(-1)^{j+1} 3}{j} \sum_k d_k \sum_i \left[\frac{n}{\alpha_{k,i}^{2j}} - \frac{1}{\alpha_{k/n,i}^{2j}} \right], \tag{5.15}$$

can be calculated analytically. For example

$$s_1^{\rm wrong} = -\frac{\partial}{\partial n} f_1 \Big|_{n=1} = 3 \sum_k d_k \sum_i \frac{1}{\alpha_{k,i}^2} \left[-1 + \frac{2k}{\alpha_{k,i}} \frac{\partial \alpha_{k,i}}{\partial k} \right] = -3 \sum_k d_k \frac{\partial}{\partial k} k \sum_i \frac{1}{\alpha_{k,i}^2}, \tag{5.16}$$

which using [22]

$$\sum_{i=1}^{\infty} \frac{1}{\alpha_{k,i}^2} = \frac{1}{4(k+1)}, \tag{5.17}$$

leads to

$$s_1^{\rm wrong} = -3 \sum_k d_k \frac{\partial}{\partial k} \frac{k}{4(k+1)} = -\frac{3}{4} \sum_k d_k \frac{1}{(k+1)^2} = -\frac{3}{4} \left(2 \frac{\pi^2}{6} - 1 \right). \tag{5.18}$$

We use the superscript 'wrong' because it is not strictly correct to make the calculation in this way, since we are implicitly using the same cut-off for Z and Z_n. In fact, applying the EML to the sum in (5.18), one has

$$s_1^{\rm wrong} = -\frac{3}{4} \sum_k d_k \frac{1}{(k+1)^2} = -\frac{3}{4} \left(2 \int_0^{\infty} \frac{{\rm d}k}{(k+1)^2} + \frac{1}{6} 2 + \cdots \right), \tag{5.19}$$

that has a first-order term $s_1^{(1)} = -1/4$ (in agreement with what was previously found) but the integral is not vanishing. This is a finite difference between the two divergent expressions with the wrong cut-off. The value of the integral $(2 \int {\rm d}k (k+1)^{-2} = 2)$ must be properly subtracted to give the right result:

$$s_1 = -\frac{3}{4} \left(2 \frac{\pi^2}{6} - 3 \right) = -0.217\,401 \ldots. \tag{5.20}$$

This is close to the first-order EML result $-1/4$, signalling that such approximation not only reproduces the qualitative physics but is also quantitatively reliable (at the level of 10%). In the same manner one can calculate all the coefficients s_j, using the more complicated expressions for the sum of higher negative powers of zeros of Bessel functions reported in the literature [22].

6. Higher dimensions: scaling of entropy and area law

The scaling hypothesis plays a fundamental role in understanding classical phase transitions. Crudely speaking, it asserts that the microscopic length scale a does not enter explicitly into thermodynamic relations near the critical point, as long as the various thermodynamic variables are suitably normalized and allowed to scale with their non-trivial scaling dimensions, related to the various universal critical exponents.

In fact, from the scaling of the singular part of the free energy density (here $t = |T - T_c|/T_c$, $\xi \propto t^{-\nu}$ is the correlation length, h the external magnetic field, $f_\pm(x)$ a universal function, the subscript $\pm$ refers to the two phases, and y_h is the scaling dimension of h),

$$f_{\rm sing}(t, h) = \xi^{-d} f_\pm(h\xi^{-y_h}), \tag{6.1}$$

the critical behaviour of all the thermodynamic observables and in particular the scaling laws may be derived. Note that $f_{\rm sing}$ is not the total free energy density: there is another non-universal piece which has an explicit a^{-d} dependence on the microscopic cut-off. However, this term is analytic in the thermodynamic variables.

As argued by Srednicki [10], for $d > 2$ the entropy $S_{\rm A}$ is proportional to the surface area $\mathcal{A}$ of the subsystem A. Thus we should discuss the entropy per unit area $s = S_{\rm A}/\mathcal{A}$. In analogy with the classical case, we may conjecture a scaling form for the singular part of the entropy per unit area near a quantum phase transition:

$$s_{\rm sing}(g, h, T) = \xi^{-(d-1)} s_\pm(h\xi^{-y_h}, T\xi^{-z}), \tag{6.2}$$

where $y_h > 0$ describes the relevant effect of the field conjugated to the order parameter, $z > 0$ the relevance of the temperature close to the quantum phase transition (in all the examples considered up to now, $z = 1$), and all other neglected operators are supposed to be irrelevant. $s_\pm$ should be a universal function apart from the normalizations of its arguments. The relation between ξ and g (the parameter driving the transition) is $\xi = |g - g_c|^{-\nu}$. As for the free energy, $s_{\rm sing}$ is not the total entropy: we also expect to find explicitly a-dependent pieces which are, however, analytic in g.

From the scaling of the entropy all the scaling laws can be obtained; e.g. the specific heat goes like

$$C = T\frac{\partial S}{\partial T} \sim T\xi^{-(d-1+z)}. \tag{6.3}$$

Similar identities can be derived for other observables.

This scaling can be explicitly checked in the case of the Gaussian model in all dimensions for $h = T = 0$ and $z = 1$. To calculate the entropy of a d-dimensional system close to a quantum critical point, one has to consider a $d + 1$ field theory. We consider the geometry where $d - 1$ dimensions are translationally invariant in a domain

delimited by a hypersurface of area $\mathcal{A}$, and on the remaining two-dimensional plane there is a branch cut going from 0 to ∞ in an arbitrary direction. The choice of a different geometry is not expected to change the main results.

The two-point function is translationally invariant in $d-2$ directions and coincides with equation (4.17) on the cut plane, i.e.

$$G(r_\perp; r, \theta, r', \theta') = \int \frac{\mathrm{d}^{d-1}k_\perp}{(2\pi)^{d-1}} \mathrm{e}^{\mathrm{i}k_\perp r_\perp} \frac{1}{2\pi n} \sum_{k=0}^{\infty} d_k \int_0^\infty \lambda \,\mathrm{d}\lambda \frac{J_{k/n}(\lambda r) J_{k/n}(\lambda r')}{\lambda^2 + m^2 + k_\perp^2} \mathcal{C}_k(\theta, \theta'), \tag{6.4}$$

where $r_\perp$ is the vector between the two points considered in the $d-1$-dimensional subspace.

The calculation proceeds as in the one-dimensional case with the substitution $m^2 \to m^2 + k_\perp^2$, and the integration over $k_\perp$ in front of all the equations. The space integration is obviously over $\mathrm{d}^{d-1}r'_\perp r \,\mathrm{d}r \,\mathrm{d}\theta$ (we use $r'_\perp$ to stress that it is different from that appearing in equation (6.4) that instead is $r_\perp = 0$, since the two-point function at coincident points has to be considered). Thus the analogue of equation (4.25) is

$$\log \mathrm{Tr}\, \rho^n = \int \mathrm{d}^{d-1}r'_\perp \int \frac{\mathrm{d}^{d-1}k_\perp}{(2\pi)^{d-1}} \frac{\log(k_\perp^2 + m^2)}{24} \left(n - \frac{1}{n}\right). \tag{6.5}$$

To have a finite result, both the integrations are in a finite region. This means that the system must live in a finite box with $\int \mathrm{d}^{d-1}r'_\perp = \mathcal{A}$ and a cut-off proportional to a^{-1} must be understood for the integration over $k_\perp$.

In this way the entropy is

$$S = -\frac{\mathcal{A}}{12} \int \frac{\mathrm{d}^{d-1}k_\perp}{(2\pi)^{d-1}} \log \frac{k_\perp^2 + m^2}{k_\perp^2 + a^{-2}}. \tag{6.6}$$

In general, the integral diverges like $a^{-(d-1)}$. However, the coefficient of this divergence (and other sub-leading divergences which can occur for sufficiently large d) are in general analytic in m. There is, however, a finite piece which behaves as m^{d-1} in agreement with (6.2). Note however that as $d \to 1$ this singular term combines with the non-singular piece $\propto a^{-(d-1)}$ to give the previously found result $\propto \log(ma)$.

Just as for the classical free energy, one may also conjecture a finite size scaling form for the entropy of the form

$$s_{\mathrm{sing}}(L, g, h, T) = L^{-(d-1)} s_{\mathrm{FSS}}(L^{1/\nu}|g - g_c|, L^{y_h} h, L^z T). \tag{6.7}$$

For $h = T = 0$ this reads

$$s_{\mathrm{sing}}(L, g, h = T = 0) = A_s L^{-(d-1)} s_{\mathrm{FSS}}(L/\xi), \tag{6.8}$$

with $s_{\mathrm{FSS}}(L/\xi)$ a function which satisfies $s_{\mathrm{FSS}}(0) = 1$ and $s_{\mathrm{FSS}}(x) \propto 1/x$ for large x, in order to reproduce the infinite volume limit, and A_s is a non-universal constant fixing the normalization of the entropy. In one dimension, this relation again does not make sense. The $\log L$ behaviour may be seen as resulting from a cancellation between the $L^{-(d-1)}$ behaviour of s_{sing} against a constant term coming from the regular part, with $A_s = \mathrm{O}((d-1)^{-1})$ as $d \to 1$. This gives the anticipated form

$$S_{\mathrm{sing}}(L, \xi) = \frac{c}{6} \left(\log L + s_{\mathrm{FSS}}(L/\xi)\right), \tag{6.9}$$

that has been explicitly checked for the Gaussian model in section 5.

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Appendix: the two-point function in an N-sheeted Riemann geometry: infinite and finite volume results

The Green function $G(\mathbf{r},\mathbf{r}')$ of the Helmholtz differential equation $(-\nabla^2_{\mathbf{r}}+m^2)f(\mathbf{r})=0$, with a specific boundary condition, is the solution of

$$(-\nabla^2_{\mathbf{r}}+m^2)G(\mathbf{r},\mathbf{r}')=\delta^d(\mathbf{r}-\mathbf{r}').$$

$G(\mathbf{r},\mathbf{r}')$ admits an eigenfunction expansion

$$G(\mathbf{r},\mathbf{r}')=\sum_k N_k\phi_k(\mathbf{r})\phi_k(\mathbf{r}'),$$

in terms of $\phi_k(\mathbf{r})$, eigenfunctions of the Helmholtz differential operator $(-\nabla^2_{\mathbf{r}}+m^2)$. N_k is a normalization constant that should be derived from the orthonormality requirement of the eigenfunctions:

$$N_m\int\phi_m(\mathbf{r})\phi_n(\mathbf{r}')\,\mathrm{d}^dr=\delta_{nm}.$$

In the case of an n-sheeted Riemann surface that we are interested in, the eigenvalue problem is solved in polar coordinates $\mathbf{r}=(x,y)=(r\cos\theta,r\sin\theta)$. A complete set of eigenfunctions is

$$\phi^a_{\nu,i}=\cos(\mathrm{i}\nu\theta)J_\nu(\lambda_i r),\qquad \phi^b_{\nu,i}=\sin(\mathrm{i}\nu\theta)J_\nu(\lambda_i r),\tag{A.1}$$

with $J_\nu(x)$ Bessel functions of the first kind. Note that the Bessel functions of the second kind $Y_\nu(x)$, that are also eigenfunctions of the same differential operator, do not enter in the expansion since we require regularity at $r=0$.

Let us consider first the solution in a finite geometry. The infinite volume limit is recovered by taking the limit $L\to\infty$ in the sense of distributions. Imposing the $2\pi n$ periodicity boundary condition we have that ν is an integer multiple of $1/n$, i.e. $\nu=k/n$. Constraining the eigenfunctions to vanish at $r=L$, the eigenvalues are $\lambda_i L=\alpha_{\nu,i}$, with $\alpha_{\nu,i}$ the ith zero of the Bessel function.

Using the orthogonality relation of the Bessel functions (see e.g. [18])

$$\int_0^L r\,\mathrm{d}r\,J_\nu(\alpha_{\nu,i}r/L)J_{\nu'}(\alpha_{\nu',i}r/L)=\frac{L^2}{2}J^2_{\nu+1}(\alpha_{\nu,i})\delta_{\nu\nu'},$$

we get from the orthonormality requirement

$$N_{i,k}=\frac{d_k}{2\pi n}\frac{2/L^2}{J^2_{k/n+1}(\alpha_{k/n,i})}.$$

That gives the Green function

$$G(r,\theta,r',\theta') = \frac{1}{2\pi n}\sum_{k=0}^{\infty} d_k \sum_{i=1}^{\infty} \frac{2/L^2}{J^2_{k/n+1}(\alpha_{k/n,i})} \frac{J_{k/n}(\alpha_{k/n,i}r/L)J_{k/n}(\alpha_{k/n,i}r'/L)}{\alpha^2_{k/n,i}/L^2 + m^2} \mathcal{C}_k(\theta,\theta'). \tag{A.2}$$

The factor d_k comes from $\int_0^{2\pi} 1\,\mathrm{d}x = 2\pi$ and $\int_0^{2\pi} \mathrm{d}x \cos^2 x = \int_0^{2\pi} \mathrm{d}x \sin^2 x = \pi$.

In the limit $L \to \infty$, the index i becomes continuous, $\alpha_{\nu,i}/L \to \lambda$, and the $\delta_{\nu\nu'}$ in the orthonormality condition is replaced by $\delta(\lambda - \lambda')$. The limit of the normalization factor is $N_k(\lambda) = d_k\lambda/(2\pi n)$, which leads to the two-point function reported in the text (4.17). Note that a slightly different form of this two-point function (satisfying Dirichlet boundary conditions) was used to investigate the critical behaviour at an edge [23].

References

[1] Bennett C H, Bernstein H J, Popescu S and Schumacher B, *Concentrating partial entanglement by local operations*, 1996 *Phys. Rev.* A **53** 2046 [quant-ph/9511030]

[2] Osterloh A, Amico L, Falci G and Fazio R, *Scaling of entanglement close to a quantum phase transitions*, 2002 *Nature* **416** 608
Osborne T J and Nielsen M A, *Entanglement in a simple quantum phase transition*, 2002 *Phys. Rev.* A **66** 32110 [quant-ph/0202162]
Barnum H, Knill E, Ortiz G, Somma R and Viola L, *A subsystem-independent generalization of entanglement*, 2004 *Phys. Rev. Lett.* **92** 107902 [quant-ph/0305023]
Verstraete F, Martin-Delgado M A and Cirac J I, *Diverging entanglement length in gapped quantum spin systems*, 2004 *Phys. Rev. Lett.* **92** 087201 [quant-ph/0311087]

[3] Vidal G, Latorre J I, Rico E and Kitaev A, *Entanglement in quantum critical phenomena*, 2003 *Phys. Rev. Lett.* **90** 227902
Latorre J I, Rico E and Vidal G, *Ground state entanglement in quantum spin chains*, 2004 *Quant. Inf. Comput.* **4** 048 [quant-ph/0304098]
Latorre J I, Lutken C A, Rico E and Vidal G, *Fine-grained entanglement loss along renormalization group flows*, 2004 *Preprint* quant-ph/0404120

[4] Jin B-Q and Korepin V E, *Quantum spin chain, Toeplitz determinants and Fisher–Hartwig conjecture*, 2004 *J. Stat. Phys.* **116** 79 [quant-ph/0304108]

[5] Lambert N, Emary C and Brandes T, *Entanglement and the phase transition in single mode super-radiance*, 2004 *Phys. Rev. Lett.* **92** 073602 [quant-ph/0309027]

[6] Casini H and Huerta M, *A finite entanglement entropy and the c-theorem*, 2004 Preprint hep-th/0405111

[7] Holzhey C, Larsen F and Wilczek F, *Geometric and renormalized entropy in conformal field theory*, 1994 *Nucl. Phys.* B **424** 44 [hep-th/9403108]

[8] Cardy J and Peschel I, *Finite-size dependence of the free energy in two-dimensional critical systems*, 1988 *Nucl. Phys.* B **300** 377

[9] Affleck I and Ludwig A W W, *Universal non-integer 'ground-state degeneracy' in critical quantum systems*, 1991 *Phys. Rev. Lett.* **67** 161

[10] Srednicki M, *Entropy and area*, 1993 *Phys. Rev. Lett.* **71** 666 [hep-th/9303048]

[11] Korepin V E, *Universality of entropy scaling in 1D gap-less models*, 2004 *Phys. Rev. Lett.* **92** 096402 [cond-mat/0311056]

[12] Blöte H W J, Cardy J and Nightingale M P, *Conformal invariance, the central charge, and universal finite-size amplitudes at criticality*, 1986 *Phys. Rev. Lett.* **56** 742

[13] Affleck I, *Universal term in the free energy at a critical point and the conformal anomaly*, 1986 *Phys. Rev. Lett.* **56** 746

[14] Zamolodchikov A B, 1986 *Pis. Zh. Eksp. Teor. Fiz.* **43** 565
Zamolodchikov A B, *Irreversibility of the flux of the renormalization group in a 2D field theory*, 1986 *JETP Lett.* **43** 731 (translation)

[15] Tsallis C, *Possible generalization of Boltzmann–Gibbs statistics*, 1988 *J. Stat. Phys.* **52** 479

[16] Belavin A A, Polyakov A M and Zamolodchikov A B, *Infinite conformal symmetry in two-dimensional quantum field theory*, 1984 *Nucl. Phys.* B **241** 333

[17] Cardy J, *Conformal invariance and surface critical behavior*, 1984 *Nucl. Phys.* B **240** 514
[18] Abramowitz M and Stegun I A, 1972 *Handbook of Mathematical Functions* (Washington, DC: National Bureau of Standards) (tenth printing)
[19] Nishino T, *Density matrix renormalization group method for 2D classical models*, 1995 *J. Phys. Soc. Japan* **74** 3598 [cond-mat/9508111]
Nishino T and Okunishi K, *Density matrix and renormalization for classical lattice models*, 1997 *Strongly Correlated Magnetic and Superconducting Systems* (*Lecture Notes in Physics* vol 478) (Berlin: Springer) p 167 [cond-mat/9610107]
[20] Baxter R J, 1982 *Exactly Solved Models in Statistical Mechanics* (San Diego, CA: Academic)
[21] Peschel I, Kaulke M and Legeza O, *Density-matrix spectra for integrable models*, 1999 *Ann. Phys. Lpz.* **8** 153 [cond-mat/9810174]
[22] Watson G N, 1944 *A Treatise on the Theory of Bessel Functions* (Cambridge: Cambridge University Press)
Sneddon I N, *On some infinite series involving the zeros of Bessel functions of the first kind*, 1960 *Proc. Glasgow Math. Assoc.* **4** 144
[23] Cardy J, *Critical behaviour at an edge*, 1983 *J. Phys. A: Math. Gen.* **16** 3617