Lattice Supersymmetry and Order-Disorder Coexistence in the Tricritical Ising Model

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We introduce and analyze a quantum spin or Majorana chain with a tricritical Ising point separating a critical phase from a gapped phase with order-disorder coexistence. We show that supersymmetry is not only an emergent property of the scaling limit but also manifests itself on the lattice. Namely, we find explicit lattice expressions for the supersymmetry generators and currents. Writing the Hamiltonian in terms of these generators allows us to find the ground states exactly at a frustration-free coupling. These confirm the coexistence between two (topologically) ordered ground states and a disordered one in the gapped phase. Deforming the model by including explicit chiral symmetry breaking, we find the phases persist up to an unusual chiral phase transition where the supersymmetry becomes exact even on the lattice.

Introduction.—The tricritical Ising model in two dimensions exhibits a host of remarkable features. Some known since the 1970s include a line of phase coexistence, a tricritical point where the line ends, and a nontrivial renormalization-group (RG) flow from it to the Ising critical point [1]. Another feature understood in the 1980s includes a description of the scaling limit of the tricritical point in terms of a conformal field theory (CFT). Even more strikingly, this strongly interacting CFT possesses supersymmetry, somewhat optimistically viewed as the first such manifestation in nature [2]. The supersymmetry is unbroken in the scaling limit of the phase coexistence line, while it is spontaneously broken along the flow to Ising. A marvelous consequence of the latter is that the ensuing Goldstinos are precisely the Majorana fermions of the Ising CFT [3]. The corresponding field theories are integrable and their quasiparticle excitations and their scattering matrices can be found exactly [4–7].

A host of lattice models reproduces the universal physics of the transition line. The classic example is the Blume-Capel model, best thought of as the Ising model with vacant sites allowed. Numerics and an RG analysis indicate when the fugacity for the vacancies is large enough; the usual order-disorder Ising transition line indeed converts to a first-order line [1]. The lines meet at a nontrivial tricritical point. However, couplings must be fine-tuned to find the transition line, as no lattice symmetry protects it. In an integrable model in the same universality class, interacting hard squares [8,9], the coexistence line can be located precisely, and the vanishing of the bulk ground-state energy characteristic of a supersymmetric theory derived exactly. Again, however, fine-tuning is required to obtain the desired physics.

Motivated both by the presence of supersymmetry and by the interesting topological features of Majorana

fermions, several groups recently revisited this story [10–12]. They introduced quantum spin chains with a self-duality that fixes the transition line exactly, and gave convincing numerical checks (extended further in [13]) that the models display the expected universal physics.

However, in these lattice models the supersymmetry is seen only indirectly, as an emergent property of the scaling limit, i.e., it only appears in the field-theory description valid at and near the tricritical point. Since a number of spin chains with explicit supersymmetry become supersymmetric field theories [14–17] in the scaling limit, it is rather curious that no such model for the tricritical Ising model was known. The purpose of this paper is to introduce and analyze a quantum spin chain in the same univerality class, but where the connection to supersymmetry is manifest on the lattice.

Our chain retains the desirable characteristics of earlier models, and includes several new ones. Its self-duality means no fine-tuning is needed to find the phase where the effective field theory is supersymmetric. It describes phase coexistence of order with disorder, or, in the fermionic picture [18], topologically ordered ground states that admit edge fermionic zero modes with a nontopological ground state that does not. This type of coexistence occurs without including extra degrees of freedom such as the vacancies in Blume-Capel or a second Majorana chain in Ref. [10]. Moreover, our Hamiltonian has the tricritical point at couplings of order 1, whereas that of Ref. [11] requires the interaction terms to be ~250 times larger than the noninteracting ones.

The most prominent new feature of our model is that the Hamiltonian can be written as a sum of the squares of two fermionic operators, just like a supersymmetric field theory familiar from particle physics. We use them to find lattice analogs of the supersymmetry generators, i.e., lattice

operators that in the scaling limit have the same properties as the corresponding fields. This form allows for a chiral deformation of the model to a line of couplings where supersymmetry becomes exact even on the lattice. We show that this line describes an unusual chiral phase transition out of the supersymmetric phase. Yet another advantage of our Hamiltonian is that it exhibits a "frustration-free" point, where the exact ground states can be found explicitly. These make the order-disorder coexistence precise and rigorous.

The model.—We study throughout the Ising or Majorana quantum spin chain with additional interactions. The Hilbert space is comprised of L two-state systems, with Hermitian operators acting on it given by $\sigma_j^r = 1 \otimes 1 \otimes 1 \cdots \sigma^r \otimes 1 \cdots 1$, with σ^r the Pauli matrix with $r = x, y, z$ acting on the two-state system at the jth site. Another extremely useful basis of operators is given by the Jordan-Wigner transformation to Majorana fermions

$$\gamma_{2j-1} = \sigma_j^z \prod_{k=1}^{j-1} \sigma_k^x, \qquad \gamma_{2j} = -i\sigma_j^x \gamma_{2j-1}. \qquad (1)$$

These Majorana operators are Hermitian and obey $\{\gamma_a, \gamma_{a'}\} = 2\delta_{a,a'}$. We consider only $\mathbb{Z}_2$-symmetric Hamiltonians commuting with the spin-flip (or fermion-number parity) operator $\mathcal{F} = \prod_j \sigma_j^x$. The simplest one is that of the Ising chain at its self-dual critical point:

$$H_I = -\sum_j (\sigma_j^x + \sigma_j^z \sigma_{j+1}^z) = i\sum_a \gamma_a \gamma_{a+1}, \qquad (2)$$

where we ignore subtleties with the boundary conditions. Roughly speaking, self-duality amounts to invariance of H under the shift $\gamma_a \to \gamma_{a+1}$; a more thorough discussion can be found in [19]. The scaling limit of this Hamiltonian is given by the Ising CFT [20].

To give a direct connection to supersymmetry, we add to the Hamiltonian the three-spin interaction

$$H_3 = \sum_j (\sigma_j^x \sigma_{j+1}^z \sigma_{j+2}^z + \sigma_j^z \sigma_{j+1}^z \sigma_{j+2}^x)$$
$$= -\sum_a \gamma_{a-2} \gamma_{a-1} \gamma_{a+1} \gamma_{a+2}. \qquad (3)$$

The Hamiltonian then can be rewritten as the sum of the squares of two fermionic Hermitian operators:

$$H = (Q^+)^2 + (Q^-)^2 = 2\lambda_I H_I + \lambda_3 H_3 + E_0, \qquad (4)$$

where $\lambda_3 \geq 0$, $E_0 = L(\lambda_I^2 + \lambda_3^2)/\lambda_3$, and

$$Q^\pm = \frac{1}{2\sqrt{\lambda_3}} \sum_a (\pm 1)^a (\lambda_I \gamma_a \pm i\lambda_3 \gamma_{a-1} \gamma_a \gamma_{a+1}). \qquad (5)$$

Under duality, $Q^\pm \to \pm Q^\pm$, so H remains self-dual. Since Q^+ and Q^- do not commute, supersymmetry here is not exact on the lattice. However, we show below that not only are these operators explicit lattice analogs of the super-symmetry generators, but they also lead to analogs of the supersymmetry currents.

The interaction H_3 is an irrelevant perturbation of the Ising fixed point, as there are no relevant self-dual operators in the Ising CFT. The least-irrelevant self-dual operator there is $T\bar{T}$, the product of left- and right-moving components of the energy-momentum tensor [3]. The four-fermi terms in H_3 differ from those ($\gamma_{a-2} \gamma_{a-1} \gamma_a \gamma_{a+1}$) analyzed in [11,12], but both are expected to renormalize onto $T\bar{T}$ in the Ising CFT. Both perturbations thus should lead to the same effective field theory. The Hamiltonian with both H_I and H_3 present is not integrable, but it is worth noting that H_3 by itself is not only integrable and supersymmetric but also has a hidden free-fermion structure [21].

Exact ground states with order-disorder coexistence.— We can find the ground states exactly when $\lambda_I = \lambda_3$. Since all energies found from (4) are non-negative, any states annihilated by both Q^+ and Q^- are necessarily ground states of H with zero energy. At $\lambda_I = \lambda_3 = 1$, $Q^\pm$ each can be rewritten in two useful ways [22]:

$$Q^\pm|_{\lambda_3=\lambda_I} = \sum_j (\gamma_{2j-2} \pm \gamma_{2j+1}) \left(\frac{1}{2} + \frac{i}{2} \gamma_{2j-1} \gamma_{2j} \right)$$
$$= \sum_j (\gamma_{2j+2} \pm \gamma_{2j-1}) \left(\frac{1}{2} + \frac{i}{2} \gamma_{2j} \gamma_{2j+1} \right). \qquad (6)$$

Since $\gamma_{2j-1} \gamma_{2j} = i\sigma_j^x$, the first form of (6) requires that any state where all σ_j^x have eigenvalue 1 is annihilated by both $Q^\pm$ and so is a ground state. The unique such state is the equal amplitude sum over all basis states s:

$$|\mathcal{G}_0\rangle = 2^{-L/2} \sum_s |s\rangle. \qquad (7)$$

Likewise, since $\gamma_{2j} \gamma_{2j+1} = i\sigma_j^z \sigma_{j+1}^z$, the second form of (6) requires that any states where all σ_j^z have the same eigenvalue are also ground states. Letting $\sigma^z|\uparrow\rangle = |\uparrow\rangle$ and $\sigma^z|\downarrow\rangle = -|\downarrow\rangle$, these ground states are

$$|\mathcal{G}_\uparrow\rangle = |\uparrow\uparrow\uparrow \cdots \uparrow\rangle, \qquad |\mathcal{G}_\downarrow\rangle = |\downarrow\downarrow\downarrow \cdots \downarrow\rangle. \qquad (8)$$

We prove in the Supplemental Material [23] that these three are the only ground states for $\lambda_I = \lambda_3$. The lowest-lying excitations are comprised of domain walls between the ground states, in accordance with exact field-theory results [4,5].

The ground state $|\mathcal{G}_0\rangle$ is completely disordered, while the perfectly ordered states $|\mathcal{G}_\uparrow\rangle$ and $|\mathcal{G}_\downarrow\rangle$ spontaneously break the $\mathbb{Z}_2$ symmetry. The degeneracy is a consequence of the

self-duality: if any of these states is a ground state, they all must be. Indeed, under duality, $|\mathcal{G}_\downarrow\rangle \to \sqrt{2}|\mathcal{G}_0\rangle$ and $|\mathcal{G}_\uparrow\rangle \to \sqrt{2}|\mathcal{G}_0\rangle$, while $|\mathcal{G}_0\rangle \to |\mathcal{G}_\downarrow\rangle + |\mathcal{G}_\uparrow\rangle$, as shown in Ref. [19]. Upon breaking the self-duality, this coexistence line becomes a first-order transition separating ordered and disordered phases.

The phase diagram.—At the frustration-free coupling $\lambda_I = \lambda_3$, our Hamiltonian (4) displays the physics of the gapped self-dual phase, with one $\mathbb{Z}_2$ symmetry-preserving ground state coexisting with two symmetry-broken ones. Doing perturbation theory around it gives an energy splitting proportional to $|\lambda_I - \lambda_3|^{L/2}$, and so exponentially small. The order-disorder coexistence therefore persists throughout a gapped phase, as expected [1,9]. The frustration-free point is presumably the basin for the RG flows in this phase. An additional piece of evidence for such is that the exact solution [21] of the gapless pure four-Fermi point $\lambda_I = 0$ indicates that the dynamical critical exponent there is $z = 3/2$. This suggests the emergent free fermions there are of dimension $1/4$ and that a fermion bilinear like the Ising perturbation is relevant, so all flows starting at $\lambda_3 > \lambda_I$ flow to the basin.

Because the Ising critical point is also stable under RG flows along the self-dual line, an unstable tricritical point must separate the critical Ising phase from the gapped coexistence phase at some value of λ_3 less than λ_I. Thus our results yield the phase diagram in Fig. 1, precisely that expected along the self-dual line.

To confirm this picture and locate the tricritical point, we complete extensive numerics. A key tool is that ratios of energy differences are universal at a critical point described by a CFT [24,25]: $(E_\alpha - E_\beta)/(E_\gamma - E_\delta) = (\Delta_\alpha - \Delta_\beta)/(\Delta_\gamma - \Delta_\delta)$, where Δ_α is the dimension of the CFT operator that creates an excited state with energy E_α. Since the spin-flip operator $\mathcal{F}$ commutes with H and obeys $\mathcal{F}^2 = 1$, we split the Hilbert space into sectors with eigenvalues ± 1 of $\mathcal{F}$. We label the corresponding energies of H with periodic and antiperiodic boundary conditions as $P_r^\pm$ and $A_r^\pm$, respectively, with $r = 0, 1, \ldots$ (antiperiodic means we flip the sign of all terms in H involving $\sigma_L^z \sigma_1^z$). Some of the CFT values (see e.g., [2,11]) and the central charge c are as follows:

CFT	c	$R_1 = (A_0^- - P_0^+)/$ $(P_1^+ - P_0^+)$	$R_2 = (P_0^- - P_0^+)/$ $(P_1^+ - P_0^+)$	$R_3 = (P_1^- - P_0^+)/$ $(P_1^+ - P_0^+)$
Ising	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{9}{8}$
TCI	$\frac{7}{10}$	$\frac{7}{2}$	$\frac{3}{8}$	$\frac{35}{8}$

We computed these ratios numerically using the Density-Matrix Renormalization Group (DMRG) with the ITensor C++ library [26]. Our results are given in Fig. 2. For $\lambda_3/\lambda_I \lesssim 0.856$ the ratios tend to the values predicted by the Ising CFT as L is increased, while at $\lambda_3/\lambda_I \approx 0.856$ the
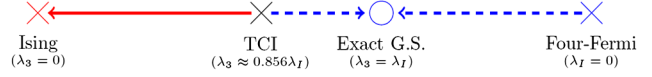

FIG. 1. Phase diagram and RG flows of (4).

ratios converge very nicely to the values predicted by the tricritical Ising (TCI) CFT. We also computed the central charges from the coefficient of the log term in the entanglement entropy [27] and found the appropriate values $c = 1/2$ and $c = 7/10$, respectively. We thus confirm that the non-trivial critical point in H_3 is indeed in the tricritical Ising universality class. We checked numerically that as λ_3/λ_I is increased to the tricritical point, the bulk ground-state energy decreases as $|\lambda_3 - .856\lambda_I|^\nu$, with $\nu \approx 5/2$, as predicted by perturbed CFT [3]. For $\lambda_3 > .856\lambda_I$, we found the differences between P_0^+, P_1^+, and P_0^- all go to zero exponentially fast as L is increased, while all other differences indicate a gap. We thus confirm there are three ground states throughout a gapped phase with order-disorder coexistence. In the Supplemental Material [23] we consider the effect of adding other couplings expected to be similarly relevant to H_3 and see that the tricritical transition survives [23].

Supersymmetry currents on the lattice.—We have given strong evidence that the scaling limit of H is indeed the supersymmetric field theory of [3–5]. We now turn to direct manifestations of supersymmetry on the lattice. The Hamiltonian of a $1 + 1$-dimensional supersymmetric field theory is

$$H_{\text{field theory}} = \left(\int dx \, G\right)^2 + \left(\int dx \, \bar{G}\right)^2, \quad (9)$$

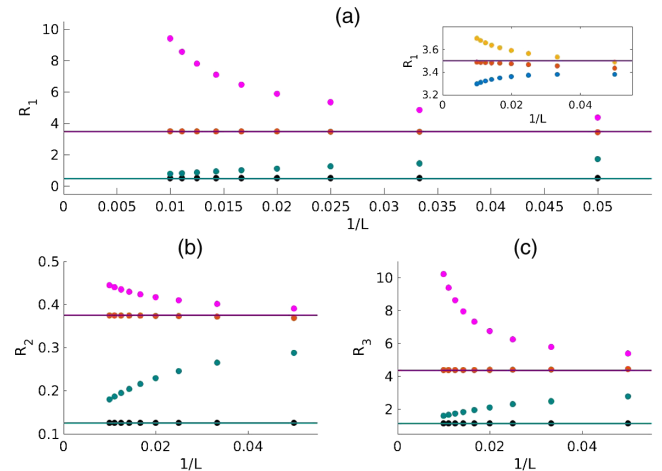where the integrals are over space with G and $\bar{G}$ the components of the supersymmetry current in complex

FIG. 2. The ratios R_1, R_2, and R_3 from the DMRG. Black, green, red, and mauve points are $\lambda_3/\lambda_I = 0$, 0.8, 0.856, 0.87, respectively, while blue and yellow in the inset are 0.855 and 0.857. Green and purple lines denote the theoretical values for the Ising and TCI CFTs, respectively.

coordinates. Comparing this Hamiltonian to our lattice one (4), (5) makes rather obvious the identification $Q^+ \to \int G$, $Q^- \to \int \bar{G}$ in the scaling limit. A further check is that along the phase coexistence line, $Q^\pm$ act on the domain walls just as the supersymmetry generators do in the corresponding integrable field theory [4,7].

We go further to find lattice analogs of the supersymmetry currents themselves, i.e., define lattice fermionic operators G_j whose correlation functions become those of G in the continuum limit, while keeping $Q^+ = \sum_j G_j$. Examining (5) suggests the simple, but not correct, guess for G_j as $\lambda_I \gamma_{2j} + i\lambda_3 \gamma_{2j-1} \gamma_{2j} \gamma_{2j+1}$. The complication is that in the tricritical Ising CFT, there occur other fermionic fields ψ and $\bar{\psi}$ of (left, right) dimensions $(3/5, 1/10)$ and $(1/10, 3/5)$, respectively. In a two-dimensional superconformal field theory, current conservation becomes $\partial_{\bar{z}} G = 0 = \partial_z \bar{G}$ so that $G(z)$ and $\bar{G}(\bar{z})$ are holomorphic and antiholomorphic fields of scaling dimensions $(3/2, 0)$ and $(0, 3/2)$, respectively [2]. Since all of these fields have the same conformal spin mod 1, lattice operators typically scale to mixtures of the fields, as happens for the lattice parafermions in the three-state Potts model [28].

The key to finding the correct lattice analogs is to understand the action of duality on the fields [28]. Under duality, G, Q^+, and $\bar{\psi}$ are even, while $\bar{G}$, Q^-, and ψ are odd [3,29]. Mimicking this behavior with local operators then suggests

$$G_j = \lambda_I(\gamma_{2j-1} + \gamma_{2j}) + i\lambda_3(\gamma_{2j-2} + \gamma_{2j+1})\gamma_{2j-1}\gamma_{2j},$$
$$\bar{G}_j = \lambda_I(\gamma_{2j-1} - \gamma_{2j}) + i\lambda_3(\gamma_{2j+1} - \gamma_{2j-2})\gamma_{2j-1}\gamma_{2j},$$
$$\psi_j = \lambda_I(\gamma_{2j-1} - \gamma_{2j}) + i\lambda_3(\gamma_{2j-2} - \gamma_{2j+1})\gamma_{2j-1}\gamma_{2j},$$
$$\bar{\psi}_j = \lambda_I(\gamma_{2j-1} + \gamma_{2j}) - i\lambda_3(\gamma_{2j+1} + \gamma_{2j-2})\gamma_{2j-1}\gamma_{2j}. \tag{10}$$

We checked these identifications by computing the two-point functions at the tricritical point, although we expect that our identifications hold throughout. The two-point functions of the operators in (10) in the ground state at the tricritical point $\lambda_3/\lambda_I = 0.856$ were computed for $L = 400$ and are shown in Fig. 3. We fitted them to the scaling form $A|j - k|^{-2\Delta_\Phi}$, where Δ_Φ is the sum of the left and right scaling dimensions. As can be seen, the correlators converge very nicely to the expected values of $\Delta_G = \Delta_{\bar{G}} = 3/2$ and $\Delta_\psi = \Delta_{\bar{\psi}} = 7/10$. These numerics thus not only provide another compelling check of the identification of the tricritical point but also that the lattice operators (10) indeed become the appropriate supersymmetry currents and fermions in the continuum limit.

Explicit lattice supersymmetry in the chiral model.—We now break chiral symmetry by varying the coefficients in front of the $(Q^\pm)^2$ terms, and find that it leads to an unusual phase transition with explicit lattice supersymmetry. Explicitly,
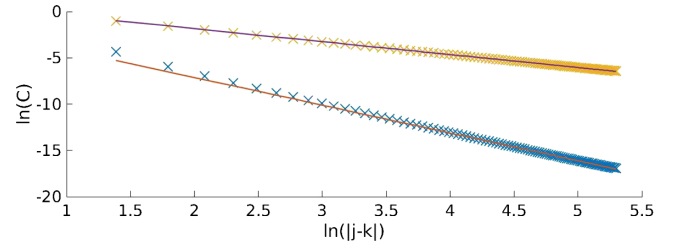

FIG. 3. Dependence of $C_{jk} = \langle \psi_j \psi_k \rangle$ (yellow crosses) and $\langle G_j G_k \rangle$ (blue crosses) on $|j - k|$ at the tricritical point $\lambda_3/\lambda_I = .856$. The purple and red lines correspond to $A|j - k|^{-1.4}$ and $B|j - k|^{-3}$ respectively, where A and B are fitting parameters.

$$\tilde{H} = \left(1 + \frac{\lambda_c}{\lambda_I}\right)(Q^+)^2 + \left(1 - \frac{\lambda_c}{\lambda_I}\right)(Q^-)^2$$
$$= 2\lambda_I H_I + \lambda_3 H_3 + \lambda_c H_c + E_0, \tag{11}$$

where the chiral part of the Hamiltonian is

$$H_c = \sum_j (\sigma_j^y \sigma_{j+1}^z - \sigma_j^z \sigma_{j+1}^y) = -i\sum_a \gamma_a \gamma_{a+2}. \tag{12}$$

It is straightforward to see that the phases remain intact as long as $|\lambda_c| < \lambda_I$. Along the Ising line, including H_c corresponds to a chiral and marginal perturbation of the CFT by $\int(T - \bar{T})$. This amounts to deforming spacetime by a conformal transformation, and so leaves the model conformally invariant. Indeed, since $[H_c, H_I] = 0$, this statement is exact when $\lambda_3 = 0$. In the gapped phase, the form of (11) means that Q^+ and Q^- still annihilate the three ground states for $|\lambda_c| < \lambda_I$. Our numerics confirm this picture, with ground states in the gapped phase at different values of λ_c but the same λ_3 and λ_I having overlap $> 99\%$. The tricritical transition persists as well, with its location depending very weakly on λ_c, changing only by about 5% up to $\lambda_c = \lambda_I$.

For $|\lambda_c| > \lambda_I$ the coefficient of one of the $(Q^\pm)^2$ terms in $\tilde{H}$ becomes negative. Some states with nonzero momentum become negative in energy, and so we expect phase transitions along the lines $\lambda_c = \pm\lambda_I$ to an incommensurate phase. In the Supplemental Material [23] we verify these phase transitions analytically for the Ising line $\lambda_3 = 0$ and numerically for nonzero λ_3.

Along these chiral phase transition lines, the Hamiltonians obey $\tilde{H}^\pm = 2(Q^\pm)^2$, and so have exact lattice supersymmetry as in Ref. [14]. Since $[Q^\pm, \tilde{H}^\pm] = 0$ along with $\{\mathcal{F}, Q^\pm\} = 0$, the spectra in the sectors with eigenvalues 1 and -1 of $\mathcal{F} = 1$ are necessarily identical. These Hamiltonians are Majorana analogs [22] of the Nicolai model [30]. Along these lines, a rigorous proof [22] shows that indeed supersymmetry is spontaneously broken for sufficiently small λ_3, and that more ground states than three occur in the frustration-free case $\lambda_3 = \lambda_I$. Even more strikingly, there occur excitations with dispersion $E \propto p^3$ along the transition line [22]. Our analysis thus suggests that these arise
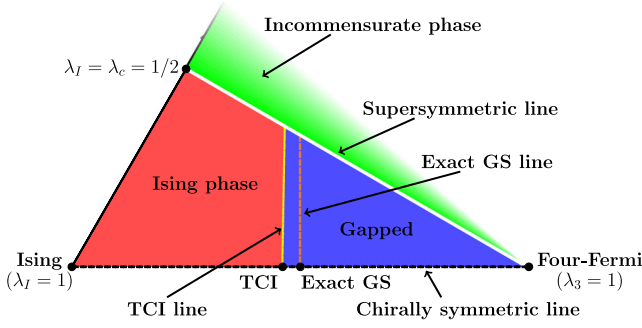
FIG. 4. Phase diagram including chiral interactions, with $\lambda_I + \lambda_3 + \lambda_c = 1$. The diagram is invariant under sending $\lambda_c \to -\lambda_c$.

because of the phase transition to the incommensurate phase. We have checked numerically that the tricritical transition stops at the supersymmetric line. However, numerical roadblocks (in particular, the presence of the p^3-dispersing excitations) have stopped us from further understanding the multicritical point where the two transition lines meet. We plot the full phase diagram in Fig. 4.

Conclusion.—We have introduced and analyzed a lattice Hamiltonian in the universality class of the tricritical Ising CFT and its self-dual perturbations. We found a frustration-free point with degenerate ground states, as well as lattice analogs of the supersymmetry currents. A fascinating open question is to characterize the $\lambda_c = \lambda_I$ line more thoroughly. At least three interesting phenomena occur there: the tricritical transition line terminates, there is a chiral phase transition with gapless p^3-dispersing modes, and the lattice supersymmetry becomes exact.

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